NON-STANDARD PARTIAL INTEGRATION: IMPLICATIONS TO MAXWELL AND KORN INEQUALITIES

CMAM-7 – Jyväskylä

COMPUTATIONAL METHODS IN APPLIED MATHEMATICS

MS 6: RECENT DEVELOPMENTS ON COMPUTATIONAL ELECTROMAGNETISM AND RELATED APPLICATIONS (ORGANIZED BY ULRICH LANGER & JUN ZOU)

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Open-Minded :-)

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OVERVIEW

NON-STANDARD INTEGRATION BY PARTS

Proposition (Grisvard's book and older...)

Let $\Omega \subset \mathbb{R}^N$ be piecewise C^2. Then for all $v \in \mathring{\mathrm{C}}^\infty(\overline{\Omega})$

$$|\operatorname{div} v|^2_{\mathrm{L}^2(\Omega)} + |\operatorname{rot} v|^2_{\mathrm{L}^2(\Omega)} - |\nabla v|^2_{\mathrm{L}^2(\Omega)} = \sum_{\ell=1}^{L} \int_{\Gamma_\ell} \underbrace{\big(\operatorname{div} \nu \, |v_{\mathrm{n}}|^2 + ((\nabla \nu)\, v_{\mathrm{t}}) \cdot v_{\mathrm{t}}\big)}_{\text{curvature, sign!}} + \sum_{\ell=1}^{L} \int_{\Gamma_\ell} \underbrace{\big(v_{\mathrm{n}} \operatorname{div}_\Gamma v_{\mathrm{t}} - v_{\mathrm{t}} \cdot \nabla_\Gamma v_{\mathrm{n}}\big)}_{\text{boundary conditions, no sign!}}$$

and for all $v \in \mathring{\mathrm{C}}^\infty_{\mathrm{t,n}}(\Omega)$

$$|\operatorname{div} v|^2_{\mathrm{L}^2(\Omega)} + |\operatorname{rot} v|^2_{\mathrm{L}^2(\Omega)} - |\nabla v|^2_{\mathrm{L}^2(\Omega)} = \sum_{\ell=1}^{L} \int_{\Gamma_\ell} \big(\operatorname{div} \nu \, |v_{\mathrm{n}}|^2 + ((\nabla \nu)\, v_{\mathrm{t}}) \cdot v_{\mathrm{t}}\big) \begin{cases} \leq 0 & \text{, if } \Omega \text{ is piecewise } \mathrm{C}^2\text{-concave,} \\ = 0 & \text{, if } \Omega \text{ is a polyhedron,} \\ \geq 0 & \text{, if } \Omega \text{ is piecewise } \mathrm{C}^2\text{-convex.} \end{cases}$$

TWO MAXWELL INEQUALITIES

$\Omega \subset \mathbb{R}^3$ bounded, weak Lipschitz (even weaker possible)

$$\Rightarrow \qquad \mathring{\mathsf{R}}(\Omega) \cap \operatorname{rot} \mathsf{R}(\Omega) \hookrightarrow \mathsf{L}^2(\Omega) \quad \Leftrightarrow \quad \mathsf{R}(\Omega) \cap \operatorname{rot} \mathring{\mathsf{R}}(\Omega) \hookrightarrow \mathsf{L}^2(\Omega)$$

$\Rightarrow$ Maxwell estimates:

$$\exists\, \mathring{c}_{\mathsf{m}} > 0 \qquad \forall\, E \in \mathring{\mathsf{R}}(\Omega) \cap \operatorname{rot} \mathsf{R}(\Omega) \qquad |E|_{\mathsf{L}^2(\Omega)} \leq \mathring{c}_{\mathsf{m}} |\operatorname{rot} E|_{\mathsf{L}^2(\Omega)}$$

$$\exists\, c_{\mathsf{m}} > 0 \qquad \forall\, H \in \mathsf{R}(\Omega) \cap \operatorname{rot} \mathring{\mathsf{R}}(\Omega) \qquad |H|_{\mathsf{L}^2(\Omega)} \leq c_{\mathsf{m}} |\operatorname{rot} H|_{\mathsf{L}^2(\Omega)}$$

note: best constants

$$\frac{1}{\mathring{c}_{\mathsf{m}}} = \inf_{0 \neq E \in \mathring{\mathsf{R}}(\Omega) \cap \operatorname{rot} \mathsf{R}(\Omega)} \frac{|\operatorname{rot} E|_{\mathsf{L}^2(\Omega)}}{|E|_{\mathsf{L}^2(\Omega)}}, \quad \frac{1}{c_{\mathsf{m}}} = \inf_{0 \neq H \in \mathsf{R}(\Omega) \cap \operatorname{rot} \mathring{\mathsf{R}}(\Omega)} \frac{|\operatorname{rot} H|_{\mathsf{L}^2(\Omega)}}{|H|_{\mathsf{L}^2(\Omega)}}$$

Theorem

(i) $\mathring{c}_{\mathsf{m}} = c_{\mathsf{m}}$ **(ii)** Ω *convex* $\Rightarrow c_{\mathsf{m}} \leq c_{\mathsf{p}}$

Poincaré estimate: $\exists\, c_{\mathsf{p}} > 0 \qquad \forall\, u \in \mathsf{H}^1(\Omega) \cap \mathbb{R}^{\perp} \qquad |u|_{\mathsf{L}^2(\Omega)} \leq c_{\mathsf{p}} |\nabla u|_{\mathsf{L}^2(\Omega)}$

best constant: $\displaystyle \frac{1}{c_{\mathsf{p}}} = \inf_{0 \neq u \in \mathsf{H}^1(\Omega) \cap \mathbb{R}^{\perp}} \frac{|\nabla u|_{\mathsf{L}^2(\Omega)}}{|u|_{\mathsf{L}^2(\Omega)}}$

PROOF OF MAXWELL INEQUALITIES

step one: two lin., cl., dens. def. op. and their reduced op.

$$A : D(A) \subset X \to Y, \qquad \mathcal{A} : D(\mathcal{A}) := D(A) \cap R(A^*) \subset R(A^*) \to R(A),$$
$$A^* : D(A^*) \subset Y \to X, \qquad \mathcal{A}^* : D(\mathcal{A}^*) := D(A^*) \cap R(A) \subset R(A) \to R(A^*)$$

crucial assumption: $D(\mathcal{A}) \hookrightarrow X \; \big(\Leftrightarrow \; D(\mathcal{A}^*) \hookrightarrow Y\big)$

$\Downarrow$

gen. Poincaré estimates:

$$\exists\, c_A > 0 \qquad \forall\, x \in D(\mathcal{A}) \qquad |x| \leq c_A |Ax|$$
$$\exists\, c_{A^*} > 0 \qquad \forall\, y \in D(\mathcal{A}^*) \qquad |y| \leq c_{A^*} |A^* y|$$

note: best constants

$$\frac{1}{c_A} = \inf_{0 \neq x \in D(\mathcal{A})} \frac{|Ax|}{|x|}, \quad \frac{1}{c_{A^*}} = \inf_{0 \neq y \in D(\mathcal{A}^*)} \frac{|A^* y|}{|y|}$$

Theorem

$c_A = c_{A^*}$

PROOF OF MAXWELL INEQUALITIES

<u>step two:</u> two lin., cl., den. def. op. and their reduced op.

$$A : D(A) \subset X \to Y, \qquad \mathcal{A} : D(\mathcal{A}) := D(A) \cap R(A^*) \subset R(A^*) \to R(A),$$
$$A^* : D(A^*) \subset Y \to X, \qquad \mathcal{A}^* : D(\mathcal{A}^*) := D(A^*) \cap R(A) \subset R(A) \to R(A^*)$$

choose

$$A := \mathring{\mathrm{rot}} : \mathring{\mathrm{R}}(\Omega) \subset \mathrm{L}^2(\Omega) \to \mathrm{L}^2(\Omega), \qquad \mathring{rot} : \mathring{\mathrm{R}}(\Omega) \cap \mathrm{rot}\,\mathrm{R}(\Omega) \subset \mathrm{rot}\,\mathrm{R}(\Omega) \to \mathrm{rot}\,\mathring{\mathrm{R}}(\Omega),$$
$$\mathrm{rot} = \mathring{\mathrm{rot}}^* : \mathrm{R}(\Omega) \subset \mathrm{L}^2(\Omega) \to \mathrm{L}^2(\Omega), \quad rot = \mathring{rot}^* : \mathrm{R}(\Omega) \cap \mathrm{rot}\,\mathring{\mathrm{R}}(\Omega) \subset \mathrm{rot}\,\mathring{\mathrm{R}}(\Omega) \to \mathrm{rot}\,\mathrm{R}(\Omega)$$

crucial assumption: $\mathring{\mathrm{R}}(\Omega) \cap \mathrm{rot}\,\mathrm{R}(\Omega) \hookrightarrow \mathrm{L}^2(\Omega) \ \big(\Leftrightarrow \ \mathrm{R}(\Omega) \cap \mathrm{rot}\,\mathring{\mathrm{R}}(\Omega) \hookrightarrow \mathrm{L}^2(\Omega)\big)$

$\Downarrow$

gen. Poincaré estimates (Maxwell estimates):

$$\exists\, \mathring{c}_{\mathrm{m}} > 0 \qquad \forall\, E \in \mathring{\mathrm{R}}(\Omega) \cap \mathrm{rot}\,\mathrm{R}(\Omega) \qquad |E|_{\mathrm{L}^2(\Omega)} \leq \mathring{c}_{\mathrm{m}} |\,\mathrm{rot}\, E|_{\mathrm{L}^2(\Omega)}$$
$$\exists\, c_{\mathrm{m}} > 0 \qquad \forall\, H \in \mathrm{R}(\Omega) \cap \mathrm{rot}\,\mathring{\mathrm{R}}(\Omega) \qquad |H|_{\mathrm{L}^2(\Omega)} \leq c_{\mathrm{m}} |\,\mathrm{rot}\, H|_{\mathrm{L}^2(\Omega)}$$

Theorem

$\mathring{c}_{\mathrm{m}} = c_{\mathrm{m}}$

PROOF OF MAXWELL INEQUALITIES

step three:

Proposition (integration by parts (Grisvard's book and older...))

Let $\Omega \subset \mathbb{R}^3$ *be piecewise* C^2. *Then for all* $E \in \overset{\circ}{\mathrm{C}}{}^{\infty}(\overline{\Omega})$

$$|\operatorname{div} E|^2_{\mathrm{L}^2(\Omega)} + |\operatorname{rot} E|^2_{\mathrm{L}^2(\Omega)} - |\nabla E|^2_{\mathrm{L}^2(\Omega)}$$

$$= \sum_{\ell=1}^{L} \int_{\Gamma_\ell} \underbrace{\big(\operatorname{div} \nu \, |E_{\mathrm{n}}|^2 + ((\nabla \nu)\, E_{\mathrm{t}}) \cdot E_{\mathrm{t}}\big)}_{\substack{\text{curvature, sign!} \\ \ldots \geq 0, \text{ if } \Omega \text{ convex.}}} + \sum_{\ell=1}^{L} \int_{\Gamma_\ell} \underbrace{\big(E_{\mathrm{n}} \operatorname{div}_\Gamma E_{\mathrm{t}} - E_{\mathrm{t}} \cdot \nabla_\Gamma E_{\mathrm{n}}\big)}_{\text{boundary conditions, no sign!}}.$$

approx. convex Ω from inside by convex and smooth $(\Omega_k)_k \quad \Rightarrow$

Corollary (Gaffney's inequality)

Let $\Omega \subset \mathbb{R}^3$ *be convex and* $E \in \overset{\circ}{\mathrm{R}}(\Omega) \cap \mathrm{D}(\Omega)$ *or* $E \in \mathrm{R}(\Omega) \cap \overset{\circ}{\mathrm{D}}(\Omega)$.
Then $E \in \mathrm{H}^1(\Omega)$ *and*

$$|\operatorname{rot} E|^2_{\mathrm{L}^2(\Omega)} + |\operatorname{div} E|^2_{\mathrm{L}^2(\Omega)} - |\nabla E|^2_{\mathrm{L}^2(\Omega)} \geq 0.$$

PROOF OF MAXWELL INEQUALITIES

step four:

$$\text{(Poincaré)} \qquad \exists\, c_{\mathrm{p}} > 0 \quad \forall\, u \in \mathrm{H}^1(\Omega) \cap \mathbb{R}^{\perp} \qquad |u|_{\mathrm{L}^2(\Omega)} \leq c_{\mathrm{p}} |\nabla u|_{\mathrm{L}^2(\Omega)}$$

Let Ω be convex and $E \in \mathrm{R}(\Omega) \cap \mathring{\mathrm{D}}_0(\Omega)$. Note $\mathring{\mathrm{D}}_0(\Omega) = \operatorname{rot} \mathring{\mathrm{R}}(\Omega)$.

Cor. (Gaffney) $\Rightarrow$ $E \in \mathrm{H}^1(\Omega)$ and $E = \operatorname{rot} H$ with $H \in \mathring{\mathrm{R}}(\Omega)$.

$\Rightarrow$ $E \in \mathrm{H}^1(\Omega) \cap (\mathbb{R}^3)^{\perp} \cap \mathring{\mathrm{D}}_0(\Omega)$, since $\langle E, a \rangle_{\mathrm{L}^2(\Omega)} = \langle \operatorname{rot} H, a \rangle_{\mathrm{L}^2(\Omega)} = 0$ for $a \in \mathbb{R}^3$

$\Downarrow$

$$|E|_{\mathrm{L}^2(\Omega)} \leq c_{\mathrm{p}} |\nabla E|_{\mathrm{L}^2(\Omega)} \leq c_{\mathrm{p}} |\operatorname{rot} E|_{\mathrm{L}^2(\Omega)}$$

$\Downarrow$

$$c_{\mathrm{m}} \leq c_{\mathrm{p}}$$

□

Theorem

Ω convex $\Rightarrow$ $\mathring{c}_{\mathrm{m}} = c_{\mathrm{m}} \leq c_{\mathrm{p}}$

MATRICES

Let $A \in \mathbb{R}^{N \times N}$.

$$\begin{matrix}\mathrm{sym}\\ \mathrm{skw}\end{matrix} A := \frac{1}{2}(A \pm A^\top), \qquad \mathrm{id}_A := \frac{\mathrm{tr}\, A}{N}\,\mathrm{id}, \qquad \mathrm{tr}\, A := A \cdot \mathrm{id}, \qquad \mathrm{dev}\, A := A - \mathrm{id}_A$$

(pointwise orthogonality) $\Rightarrow$

$$|A|^2 = |\,\mathrm{dev}\, A|^2 + \frac{1}{N}|\,\mathrm{tr}\, A|^2, \quad |A|^2 = |\,\mathrm{sym}\, A|^2 + |\,\mathrm{skw}\, A|^2, \quad |\,\mathrm{sym}\, A|^2 = |\,\mathrm{dev}\,\mathrm{sym}\, A|^2 + \frac{1}{N}|\,\mathrm{tr}\, A|^2$$

$$\Rightarrow \quad |\,\mathrm{dev}\, A|, N^{-1/2}|\,\mathrm{tr}\, A|, |\,\mathrm{sym}\, A|, |\,\mathrm{skw}\, A| \leq |A|$$

$\Omega \subset \mathbb{R}^N$ and $A := \nabla v := J_v^\top$ for $v \in \mathsf{H}^1(\Omega)$ $\quad\Rightarrow\quad$ (pointwise)

$$|\,\mathrm{skw}\, \nabla v|^2 = \frac{1}{2}|\,\mathrm{rot}\, v|^2, \quad \mathrm{tr}\, \nabla v = \mathrm{div}\, v,$$

$$|\nabla v|^2 = |\,\mathrm{dev}\,\mathrm{sym}\, \nabla v|^2 + \frac{1}{N}|\,\mathrm{div}\, v|^2 + \frac{1}{2}|\,\mathrm{rot}\, v|^2 \tag{1}$$

Moreover

$$|\nabla v|^2 = |\,\mathrm{rot}\, v|^2 + \langle \nabla v, (\nabla v)^\top \rangle$$

since $\quad 2|\,\mathrm{skw}\, \nabla v|^2 = \frac{1}{2}|\nabla v - (\nabla v)^\top|^2 = |\nabla v|^2 - \langle \nabla v, (\nabla v)^\top \rangle.$

Korn's first inequality: standard boundary conditions

Lemma (Korn's first inequality: $\overset{\circ}{\mathrm{H}}{}^1$-version)

Let Ω be an open subset of $\mathbb{R}^N$ with $2 \leq N \in \mathbb{N}$. Then for all $v \in \overset{\circ}{\mathrm{H}}{}^1(\Omega)$

$$|\nabla v|^2_{\mathrm{L}^2(\Omega)} = 2|\operatorname{dev}\operatorname{sym}\nabla v|^2_{\mathrm{L}^2(\Omega)} + \frac{2-N}{N}|\operatorname{div} v|^2_{\mathrm{L}^2(\Omega)} \leq 2|\operatorname{dev}\operatorname{sym}\nabla v|^2_{\mathrm{L}^2(\Omega)}$$

and equality holds if and only if $\operatorname{div} v = 0$ *or* $N = 2$*.*

Proof.

note: $-\Delta = \operatorname{rot}^* \operatorname{rot} - \nabla \operatorname{div}$ (vector Laplacian)

$$\Rightarrow \quad \forall\, v \in \overset{\circ}{\mathrm{C}}{}^\infty(\Omega) \quad |\nabla v|^2_{\mathrm{L}^2(\Omega)} = |\operatorname{rot} v|^2_{\mathrm{L}^2(\Omega)} + |\operatorname{div} v|^2_{\mathrm{L}^2(\Omega)} \quad \text{(Gaffney's equality)} \quad (2)$$

(2) extends to all $v \in \overset{\circ}{\mathrm{H}}{}^1(\Omega)$ by continuity. Then

$$|\nabla v|^2_{\mathrm{L}^2(\Omega)} = |\operatorname{dev}\operatorname{sym}\nabla v|^2_{\mathrm{L}^2(\Omega)} + \frac{1}{2}|\nabla v|^2_{\mathrm{L}^2(\Omega)} + \frac{2-N}{2N}|\operatorname{div} v|^2_{\mathrm{L}^2(\Omega)}$$

follows by (1), i.e., $|\nabla v|^2 = |\operatorname{dev}\operatorname{sym}\nabla v|^2 + \frac{1}{N}|\operatorname{div} v|^2 + \frac{1}{2}|\operatorname{rot} v|^2$, and (2). □

KORN'S FIRST INEQUALITY: TANGENTIAL/NORMAL BOUNDARY CONDITIONS

main result:

Theorem (Korn's first inequality: tangential/normal version)

Let $\Omega \subset \mathbb{R}^N$ be piecewise C^2-concave and $v \in \overset{\circ}{\mathsf{H}}{}^1_{\mathsf{t,n}}(\Omega)$. Then Korn's first inequality

$$|\nabla v|_{\mathsf{L}^2(\Omega)} \leq \sqrt{2}|\operatorname{dev}\operatorname{sym}\nabla v|_{\mathsf{L}^2(\Omega)}$$

holds. If Ω is a polyhedron, even

$$|\nabla v|^2_{\mathsf{L}^2(\Omega)} = 2|\operatorname{dev}\operatorname{sym}\nabla v|^2_{\mathsf{L}^2(\Omega)} + \frac{2-N}{N}|\operatorname{div} v|^2_{\mathsf{L}^2(\Omega)} \leq 2|\operatorname{dev}\operatorname{sym}\nabla v|^2_{\mathsf{L}^2(\Omega)}$$

is true and equality holds if and only if $\operatorname{div} v = 0$ or $N = 2$.

KORN'S FIRST INEQUALITY: TANGENTIAL/NORMAL BOUNDARY CONDITIONS

tools:

Proposition (integration by parts (Grisvard's book and older...))

Let $\Omega \subset \mathbb{R}^N$ be piecewise C^2. Then

$$|\operatorname{div} v|^2_{\mathsf{L}^2(\Omega)} + |\operatorname{rot} v|^2_{\mathsf{L}^2(\Omega)} - |\nabla v|^2_{\mathsf{L}^2(\Omega)} = \sum_{\ell=1}^{L} \int_{\Gamma_\ell} \underbrace{\big(\operatorname{div}\nu\, |v_{\mathsf{n}}|^2 + ((\nabla\nu)\, v_{\mathsf{t}}) \cdot v_{\mathsf{t}}\big)}_{\text{curvature, sign!}} + \sum_{\ell=1}^{L} \int_{\Gamma_\ell} \underbrace{(v_{\mathsf{n}} \operatorname{div}_\Gamma v_{\mathsf{t}} - v_{\mathsf{t}} \cdot \nabla_\Gamma v_{\mathsf{n}})}_{\text{boundary conditions, no sign!}},$$

$$|\operatorname{div} v|^2_{\mathsf{L}^2(\Omega)} + |\operatorname{rot} v|^2_{\mathsf{L}^2(\Omega)} - |\nabla v|^2_{\mathsf{L}^2(\Omega)} = \sum_{\ell=1}^{L} \int_{\Gamma_\ell} \big(\operatorname{div}\nu\, |v_{\mathsf{n}}|^2 + ((\nabla\nu)\, v_{\mathsf{t}}) \cdot v_{\mathsf{t}}\big).$$

holds for all $v \in \overset{\circ}{\mathsf{C}}{}^\infty(\overline{\Omega})$ resp. $v \in \overset{\circ}{\mathsf{C}}{}^\infty_{\mathsf{t},\mathsf{n}}(\Omega)$.

Corollary (Gaffney's inequalities)

Let $\Omega \subset \mathbb{R}^N$ be piecewise C^2 and $v \in \overset{\circ}{\mathsf{H}}{}^1_{\mathsf{t},\mathsf{n}}(\Omega)$. Then

$$|\operatorname{rot} v|^2_{\mathsf{L}^2(\Omega)} + |\operatorname{div} v|^2_{\mathsf{L}^2(\Omega)} - |\nabla v|^2_{\mathsf{L}^2(\Omega)} \begin{cases} \leq 0 & \text{, if } \Omega \text{ is piecewise } \mathsf{C}^2\text{-concave,} \\ = 0 & \text{, if } \Omega \text{ is a polyhedron,} \\ \geq 0 & \text{, if } \Omega \text{ is piecewise } \mathsf{C}^2\text{-convex.} \end{cases}$$

KORN'S FIRST INEQUALITY: TANGENTIAL/NORMAL BOUNDARY CONDITIONS

Proof.

(1), i.e., $|\nabla v|^2 = |\operatorname{dev}\operatorname{sym}\nabla v|^2 + \frac{1}{N}|\operatorname{div} v|^2 + \frac{1}{2}|\operatorname{rot} v|^2$, and the corollary $\Rightarrow$

$$|\nabla v|^2_{\mathsf{L}^2(\Omega)} \leq |\operatorname{dev}\operatorname{sym}\nabla v|^2_{\mathsf{L}^2(\Omega)} + \frac{1}{2}|\nabla v|^2_{\mathsf{L}^2(\Omega)} + \frac{2-N}{2N}|\operatorname{div} v|^2_{\mathsf{L}^2(\Omega)}$$

$\Rightarrow$ first estimate

Ω polyhedron $\Rightarrow$ equality holds □

REFERENCES

- Pauly, D.: Zapiski POMI, (2014)
  On Constants in Maxwell Inequalities for Bounded and Convex Domains
- Pauly, D.: Discrete Contin. Dyn. Syst. Ser. S, (2015)
  On Maxwell's and Poincaré's Constants
- Pauly, D.: Math. Methods Appl. Sci., (2016)
  On the Maxwell Constants in 3D

- Bauer, S. and Pauly, D.: Math. Methods Appl. Sci., (2016)
  On Korn's First Inequality for Tangential or Normal Boundary Conditions with Explicit Constants
- Bauer, S. and Pauly, D.: Ann. Univ. Ferrara Sez. VII Sci. Mat., (2016)
  On Korn's First Inequality for Tangential or Normal Boundary Conditions

DISTURBING CONSEQUENCES FOR VILLANI'S WORK

- Desvillettes, L. and Villani, C.: ESAIM Control Optim. Calc. Var., (2002)
  On a variant of Korn's inequality arising in statistical mechanics. A tribute to J.L. Lions.
- Desvillettes, L. and Villani, C.: Invent. Math., (2005)
  On the trend to global equilibrium for spatially inhomogeneous kinetic systems: the Boltzmann equation

(convergence to Boltzmann equilibrium, Landau damping, ...)