Functional A Posteriori Error Estimates for First Order Systems

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CMAM 6
International Conference on Computational Methods in Applied Mathematics

September 29 2014
Strobl/St. Wolfgang, Austria

Model Problem: Electro-Static Maxwell Equations

setting: Hilbert/Sobolev spaces (L^2-based)
geometry: $\Omega \subset \mathbb{R}^3$ bounded domain with Lipschitz (or weaker) boundary $\Gamma = \partial\Omega$

$$\operatorname{rot} E = F \quad \text{in } \Omega \tag{1}$$

$$-\operatorname{div} \varepsilon E = G \quad \text{in } \Omega \tag{2}$$

$$\nu \times E = 0 \quad \text{at } \Gamma \tag{3}$$

non-trivial kernel: $\mathcal{H}_{\mathtt{D},\varepsilon} = \{E \in \mathsf{L}^2 : \operatorname{rot} E = 0,\ \operatorname{div} \varepsilon E = 0,\ \nu \times E|_\Gamma = 0\}$
additional condition:

$$\pi_{\mathtt{D}} E = H \in \mathcal{H}_{\mathtt{D},\varepsilon} \tag{4}$$

well known:
(1)-(4) uniquely solvable
by Helmholtz decompositions and Poincaré/Maxwell estimates
for certain right hand sides F, G, H

Underlying Structure of the Model Problem

exact sequence:

$$0 \underset{0}{\overset{0}{\rightleftarrows}} \mathsf{L}^2 \underset{-\operatorname{div}\varepsilon}{\overset{\mathring{\nabla}}{\rightleftarrows}} \mathsf{L}^2_\varepsilon \underset{\varepsilon^{-1}\operatorname{rot}}{\overset{\mathring{\operatorname{rot}}}{\rightleftarrows}} \mathsf{L}^2 \underset{-\nabla}{\overset{\mathring{\operatorname{div}}}{\rightleftarrows}} \mathsf{L}^2 \underset{0}{\overset{0}{\rightleftarrows}} 0$$

unbounded, densely defined, closed, linear operators with adjoints

$$\mathring{\nabla} : \mathring{\mathsf{H}}^1 \subset \mathsf{L}^2 \to \mathsf{L}^2_\varepsilon, \quad -\operatorname{div}\varepsilon = (\mathring{\nabla})^* : \varepsilon^{-1}\mathsf{D} \subset \mathsf{L}^2_\varepsilon \to \mathsf{L}^2$$

$$\mathring{\operatorname{rot}} : \mathring{\mathsf{R}} \subset \mathsf{L}^2_\varepsilon \to \mathsf{L}^2, \quad \varepsilon^{-1}\operatorname{rot} = (\mathring{\operatorname{rot}})^* : \mathsf{R} \subset \mathsf{L}^2 \to \mathsf{L}^2_\varepsilon$$

$$\mathring{\operatorname{div}} : \mathring{\mathsf{D}} \subset \mathsf{L}^2 \to \mathsf{L}^2, \quad -\nabla = (\mathring{\operatorname{div}})^* : \mathsf{H}^1 \subset \mathsf{L}^2 \to \mathsf{L}^2$$

sometimes:

$$\mathsf{R} = H(\operatorname{rot}) = H(\operatorname{curl})$$

$$\mathsf{D} = H(\operatorname{div})$$

exact: 'range ⊂ kernel' ($\operatorname{rot}\nabla = 0$, $\operatorname{div}\operatorname{rot} = 0$)

$$\mathring{\nabla}\mathring{\mathsf{H}}^1 = R(\mathring{\nabla}) \subset N(\mathring{\operatorname{rot}}) = \mathring{\mathsf{R}}_0, \quad -\operatorname{div}\varepsilon\varepsilon^{-1}\mathsf{D} = R(-\operatorname{div}\varepsilon) \subset N(0) = \mathsf{L}^2$$

$$\mathring{\operatorname{rot}}\,\mathring{\mathsf{R}} = R(\mathring{\operatorname{rot}}) \subset N(\mathring{\operatorname{div}}) = \mathring{\mathsf{D}}_0, \quad \varepsilon^{-1}\operatorname{rot}\mathsf{R} = R(\varepsilon^{-1}\operatorname{rot}) \subset N(-\operatorname{div}\varepsilon) = \varepsilon^{-1}\mathsf{D}_0$$

$$\mathring{\operatorname{div}}\mathring{\mathsf{D}} = R(\mathring{\operatorname{div}}) \subset N(0) = \mathsf{L}^2, \quad -\nabla\mathsf{H}^1 = R(-\nabla) \subset N(\varepsilon^{-1}\operatorname{rot}) = \mathsf{R}_0$$

crucial: compact embeddings (Rellich's selection theorem, Maxwell cpt property)

$$\mathsf{H}^1 \hookrightarrow \mathsf{L}^2, \quad \mathring{\mathsf{R}} \cap \varepsilon^{-1}\mathsf{D}, \quad \mathsf{R} \cap \varepsilon^{-1}\mathring{\mathsf{D}} \hookrightarrow \mathsf{L}^2$$

$\Rightarrow$ Helmholtz decompositions and Poincaré/Maxwell estimates $\surd$

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Abstract Formulation

$$\begin{aligned} \operatorname{rot} E &= F && \text{in } \Omega \\ -\operatorname{div} \varepsilon E &= G && \text{in } \Omega \\ \nu \times E &= 0 && \text{at } \Gamma \\ \pi_{\mathrm{D}} E &= H \in \mathcal{H}_{\mathrm{D},\varepsilon} \end{aligned}$$

$$\wr$$

$$\begin{aligned} \mathring{\operatorname{rot}} E &= F \\ -\operatorname{div} \varepsilon E &= G \\ \pi_{\mathrm{D}} E &= H \in \mathcal{H}_{\mathrm{D},\varepsilon} \end{aligned}$$

$$(\mathrm{A}_i := \mathring{\operatorname{rot}}, \quad \mathrm{A}_i^* = \varepsilon^{-1} \operatorname{rot}) \qquad \wr \; (x := E) \qquad (\mathrm{A}_{i-1} := \mathring{\nabla}, \quad \mathrm{A}_{i-1}^* = -\operatorname{div} \varepsilon)$$

$$\begin{aligned} \mathrm{A}_i x &= f \\ \mathrm{A}_{i-1}^* x &= g \\ \pi_i x &= h \in \mathcal{H}_i := N(\mathrm{A}_i) \cap N(\mathrm{A}_{i-1}^*) \end{aligned}$$

General or Abstract Problem

setting: unbounded, densely defined, closed, linear operators with adjoints

$$\mathrm{A}_i : D(\mathrm{A}_i) \subset \mathsf{H}_i \to \mathsf{H}_{i+1}, \quad \mathrm{A}_i^* : D(\mathrm{A}_i^*) \subset \mathsf{H}_{i+1} \to \mathsf{H}_i, \quad i \in \mathbb{Z}$$

exact sequence:

$$\cdots \rightleftarrows \mathsf{H}_{i-2} \underset{\mathrm{A}_{i-2}^*}{\overset{\mathrm{A}_{i-2}}{\rightleftarrows}} \boxed{\mathsf{H}_{i-1} \underset{\mathrm{A}_{i-1}^*}{\overset{\mathrm{A}_{i-1}}{\rightleftarrows}} \boxed{\mathsf{H}_i} \underset{\mathrm{A}_i^*}{\overset{\mathrm{A}_i}{\rightleftarrows}} \mathsf{H}_{i+1}} \underset{\mathrm{A}_{i+1}^*}{\overset{\mathrm{A}_{i+1}}{\rightleftarrows}} \mathsf{H}_{i+2} \rightleftarrows \ldots$$

exact: 'range $\subset$ kernel' ($\mathrm{A}_i \mathrm{A}_{i-1} = 0$, $\mathrm{A}_{i-1}^* \mathrm{A}_i^* = 0$)

$$R(\mathrm{A}_{i-1}) \subset N(\mathrm{A}_i), \quad R(\mathrm{A}_i^*) \subset N(\mathrm{A}_{i-1}^*)$$

<u>problem:</u> find $x \in D(\mathrm{A}_i) \cap D(\mathrm{A}_{i-1}^*)$ s.t.

$$\boxed{\mathrm{A}_i x = f, \quad \mathrm{A}_{i-1}^* x = g, \quad \pi_i x = h,}$$

where $f \in R(\mathrm{A}_i)$, $g \in R(\mathrm{A}_{i-1}^*)$ and $h \in \mathcal{H}_i$ with kernel $\mathcal{H}_i := N(\mathrm{A}_i) \cap N(\mathrm{A}_{i-1}^*)$

tool box

Hodge/Helmholtz/Weyl decompositions:

$$\mathsf{H}_i = N(\mathrm{A}_i) \oplus_{\mathsf{H}_i} \overline{R(\mathrm{A}_i^*)}, \qquad \mathsf{H}_{i+1} = N(\mathrm{A}_i^*) \oplus_{\mathsf{H}_{i+1}} \overline{R(\mathrm{A}_i)}$$

$\Rightarrow$ reduced (injective) operators

$$\mathcal{A}_i : D(\mathcal{A}_i) := D(\mathrm{A}_i) \cap \overline{R(\mathrm{A}_i^*)} \subset \overline{R(\mathrm{A}_i^*)} \to \overline{R(\mathrm{A}_i)}, \qquad \big(\mathrm{A}_i : D(\mathrm{A}_i) \subset \mathsf{H}_i \to \mathsf{H}_{i+1}\big)$$
$$\mathcal{A}_i^* : D(\mathcal{A}_i^*) := D(\mathrm{A}_i^*) \cap \overline{R(\mathrm{A}_i)} \subset \overline{R(\mathrm{A}_i)} \to \overline{R(\mathrm{A}_i^*)}, \qquad \big(\mathrm{A}_i^* : D(\mathrm{A}_i^*) \subset \mathsf{H}_{i+1} \to \mathsf{H}_i\big)$$

$\Rightarrow$ $\mathcal{A}_i^{-1}$, $(\mathcal{A}_i^*)^{-1}$ exist, exact sequence for $\mathcal{A}_i$, $\mathcal{A}_i^*$ $\surd$
crucial: compact embeddings

$$D(\mathcal{A}_i) \hookrightarrow \mathsf{H}_i \quad \Leftrightarrow \quad \big(D(\mathcal{A}_i^*) \hookrightarrow \mathsf{H}_{i+1}\big)$$

$$\Rightarrow \begin{cases} \text{(general) Poincaré estimates (Poincaré, Friedrichs, Maxwell, ...)} \\ \text{closed ranges} \\ \text{continuous and compact invers operators} \\ \text{Helmholtz decompositions} \end{cases}$$

tool box (Poincaré estimates, closed ranges, compact inverses)

compact embedding $D(\mathcal{A}_i) \hookrightarrow \mathsf{H}_i \Rightarrow$

- $\forall\, \varphi \in D(\mathcal{A}_i) \quad |\varphi|_{\mathsf{H}_i} \le c_{\mathrm{A}_i} |\mathrm{A}_i \varphi|_{\mathsf{H}_{i+1}}$
- $\forall\, \psi \in D(\mathcal{A}_i^*) \quad |\psi|_{\mathsf{H}_{i+1}} \le c_{\mathrm{A}_i^*} |\mathrm{A}_i^* \psi|_{\mathsf{H}_i}$
- $R(\mathrm{A}_i) = R(\mathcal{A}_i)$, $R(\mathrm{A}_i^*) = R(\mathcal{A}_i^*)$ closed

$\Rightarrow$ reduced operators

$$\mathcal{A}_i : D(\mathcal{A}_i) := D(\mathrm{A}_i) \cap R(\mathrm{A}_i^*) \subset R(\mathrm{A}_i^*) \to R(\mathrm{A}_i), \qquad \big(\mathrm{A}_i : D(\mathrm{A}_i) \subset \mathsf{H}_i \to \mathsf{H}_{i+1}\big)$$
$$\mathcal{A}_i^* : D(\mathcal{A}_i^*) := D(\mathrm{A}_i^*) \cap R(\mathrm{A}_i) \subset R(\mathrm{A}_i) \to R(\mathrm{A}_i^*), \qquad \big(\mathrm{A}_i^* : D(\mathrm{A}_i^*) \subset \mathsf{H}_{i+1} \to \mathsf{H}_i\big)$$

- $\mathcal{A}_i^{-1} : R(\mathrm{A}_i) \to D(\mathcal{A}_i)$ cont., $\mathcal{A}_i^{-1} : R(\mathrm{A}_i) \to R(\mathcal{A}_i^*)$ cpt., $\|\mathcal{A}_i^{-1}\| = c_{\mathrm{A}_i}$
- $(\mathcal{A}_i^*)^{-1} : R(\mathrm{A}_i^*) \to D(\mathcal{A}_i^*)$ cont., $(\mathcal{A}_i^*)^{-1} : R(\mathrm{A}_i^*) \to R(\mathcal{A}_i)$ cpt., $\|(\mathcal{A}_i^*)^{-1}\| = c_{\mathrm{A}_i^*}$

note: 'best' constants c_{A_i} and $c_{\mathrm{A}_i^*}$ satisfy

$$\frac{1}{c_{\mathrm{A}_i}} = \inf_{\varphi \in D(\mathcal{A}_i)} \frac{|\mathrm{A}_i \varphi|_{\mathsf{H}_{i+1}}}{|\varphi|_{\mathsf{H}_i}} = \inf_{\psi \in D(\mathcal{A}_i^*)} \frac{|\mathrm{A}_i^* \psi|_{\mathsf{H}_i}}{|\psi|_{\mathsf{H}_{i+1}}} = \frac{1}{c_{\mathrm{A}_i^*}} \quad \Rightarrow \quad \boxed{c_i := c_{\mathrm{A}_i} = c_{\mathrm{A}_i^*}}$$

tool box (Hodge/Helmholtz/Weyl decompositions)

$$\mathsf{H}_i = N(\mathrm{A}_i) \oplus_{\mathsf{H}_i} R(\mathrm{A}_i^*), \qquad \mathsf{H}_i = R(\mathrm{A}_{i-1}) \oplus_{\mathsf{H}_i} N(\mathrm{A}_{i-1}^*)$$

$$D(\mathrm{A}_i) = N(\mathrm{A}_i) \oplus_{\mathsf{H}_i} D(\mathcal{A}_i), \qquad D(\mathrm{A}_{i-1}^*) = D(\mathcal{A}_{i-1}^*) \oplus_{\mathsf{H}_i} N(\mathrm{A}_{i-1}^*)$$

exact sequence: $R(\mathrm{A}_{i-1}) \subset N(\mathrm{A}_i)$, $R(\mathrm{A}_i^*) \subset N(\mathrm{A}_{i-1}^*)$ $\Rightarrow$

$$N(\mathrm{A}_{i-1}^*) = \mathcal{H}_i \oplus_{\mathsf{H}_i} R(\mathrm{A}_i^*), \quad N(\mathrm{A}_i) = R(\mathrm{A}_{i-1}) \oplus_{\mathsf{H}_i} \mathcal{H}_i, \quad \mathcal{H}_i = N(\mathrm{A}_i) \cap N(\mathrm{A}_{i-1}^*)$$

$\Rightarrow$ refined Helmholtz decomposition

$$\begin{aligned} \mathsf{H}_i &= R(\mathrm{A}_{i-1}) \oplus_{\mathsf{H}_i} \mathcal{H}_i \oplus_{\mathsf{H}_i} R(\mathrm{A}_i^*) \\ D(\mathrm{A}_i) &= R(\mathrm{A}_{i-1}) \oplus_{\mathsf{H}_i} \mathcal{H}_i \oplus_{\mathsf{H}_i} D(\mathcal{A}_i) \\ D(\mathrm{A}_{i-1}^*) &= D(\mathcal{A}_{i-1}^*) \oplus_{\mathsf{H}_i} \mathcal{H}_i \oplus_{\mathsf{H}_i} R(\mathrm{A}_i^*) \end{aligned}$$

with orthonormal projectors

$$\begin{aligned} &\pi_{\mathrm{A}_{i-1}} : \mathsf{H}_i \to R(\mathrm{A}_{i-1}), && \forall\, \psi \in D(\mathrm{A}_{i-1}^*) && \pi_{\mathrm{A}_{i-1}}\psi \in D(\mathcal{A}_{i-1}^*) && \wedge && \mathrm{A}_{i-1}^*\pi_{\mathrm{A}_{i-1}}\psi = \mathrm{A}_{i-1}^*\psi \\ &\pi_{\mathrm{A}_i^*} : \mathsf{H}_i \to R(\mathrm{A}_i^*), && \forall\, \varphi \in D(\mathrm{A}_i) && \pi_{\mathrm{A}_i^*}\varphi \in D(\mathcal{A}_i) && \wedge && \mathrm{A}_i\pi_{\mathrm{A}_i^*}\varphi = \mathrm{A}_i\varphi \\ &\pi_i : \mathsf{H}_i \to \mathcal{H}_i \end{aligned}$$

Abstract Problem and Goal

problem: find $x \in D(\mathrm{A}_i) \cap D(\mathrm{A}_{i-1}^*)$ s.t.

$$\mathrm{A}_i x = f$$
$$\mathrm{A}_{i-1}^* x = g$$
$$\pi_i x = h$$

Theorem (solution theory)

unique solution (dpd. cont. on data) $\Leftrightarrow$ $f \in R(\mathrm{A}_i)$, $g \in R(\mathrm{A}_{i-1}^*)$ *and* $h \in \mathcal{H}_i$

Proof.

$x = \mathcal{A}_i^{-1} f + (\mathcal{A}_{i-1}^*)^{-1} g + h$ □

goal: functional a posteriori error estimates 'in the spirit of Sergey Repin'

for $\tilde{x} \in \mathrm{H}_i$ (very non-conforming!) estimate $|x - \tilde{x}|_{\mathrm{H}_i}$ in terms of $\tilde{x}$, f, g, h

Solution Theory by Variational Methods

unique solution $x = \mathcal{A}_i^{-1} f + (\mathcal{A}_{i-1}^*)^{-1} g + h \in D(\mathrm{A}_i) \cap D(\mathrm{A}_{i-1}^*)$ of

$$\boxed{\mathrm{A}_i x = f, \quad \mathrm{A}_{i-1}^* x = g, \quad \pi_i x = h}$$

can be found by variational techniques (Lax-Milgram)

- for $\mathcal{A}_i^{-1} f$ we solve $\mathrm{A}_i \mathrm{A}_i^* \psi = f$: find $\psi \in D(\mathcal{A}_i^*)$ with

$$\forall\, \varphi \in D(\mathcal{A}_i^*) \qquad \langle \mathrm{A}_i^* \psi, \mathrm{A}_i^* \varphi \rangle_{\mathrm{H}_i} = \langle f, \varphi \rangle_{\mathrm{H}_{i+1}} \tag{5}$$

  $f \in R(\mathrm{A}_i) \Rightarrow$ (5) holds for all $\varphi \in D(\mathrm{A}_i^*)$
  $\Rightarrow x_{\mathrm{A}_i} := \mathrm{A}_i^* \psi \in D(\mathcal{A}_i)$ and $\mathrm{A}_i x_{\mathrm{A}_i} = f$
  $\Rightarrow x_{\mathrm{A}_i} = \mathcal{A}_i^{-1} f \in D(\mathcal{A}_i)$ and $|x_{\mathrm{A}_i}|_{\mathrm{H}_i} \le c_i |f|_{\mathrm{H}_{i+1}}$
  note: $D(\mathcal{A}_i^*) = D(\mathrm{A}_i^*) \cap R(\mathrm{A}_i)$ and $R(\mathrm{A}_i) = N(\mathrm{A}_i^*)^{\perp_{\mathrm{H}_{i+1}}}$
  $\Rightarrow$ (5) is equivalent to the saddle point problem: find $\psi \in D(\mathrm{A}_i^*)$ with

$$\begin{aligned} &\forall\, \varphi \in D(\mathrm{A}_i^*) \qquad && \langle \mathrm{A}_i^* \psi, \mathrm{A}_i^* \varphi \rangle_{\mathrm{H}_i} = \langle f, \varphi \rangle_{\mathrm{H}_{i+1}}, \\ &\forall\, \phi \in N(\mathrm{A}_i^*) && \langle \psi, \phi \rangle_{\mathrm{H}_{i+1}} = 0 \end{aligned}$$

Solution Theory by Variational Methods

unique solution $x = \mathcal{A}_i^{-1} f + (\mathcal{A}_{i-1}^*)^{-1} g + h \in D(\mathrm{A}_i) \cap D(\mathrm{A}_{i-1}^*)$ of

$$\boxed{\mathrm{A}_i x = f, \quad \mathrm{A}_{i-1}^* x = g, \quad \pi_i x = h}$$

can be found by variational techniques (Lax-Milgram)

- for $(\mathcal{A}_{i-1}^*)^{-1} g$ we solve $\mathrm{A}_{i-1}^* \mathrm{A}_{i-1} \psi = f$: find $\psi \in D(\mathcal{A}_{i-1})$ with

$$\forall\, \varphi \in D(\mathcal{A}_{i-1}) \qquad \langle \mathrm{A}_{i-1}\psi, \mathrm{A}_{i-1}\varphi \rangle_{\mathrm{H}_i} = \langle g, \varphi \rangle_{\mathrm{H}_{i-1}} \tag{6}$$

$g \in R(\mathrm{A}_{i-1}^*) \Rightarrow$ (6) holds for all $\varphi \in D(\mathrm{A}_{i-1})$
$\Rightarrow x_{\mathrm{A}_{i-1}^*} := \mathrm{A}_{i-1}\psi \in D(\mathcal{A}_{i-1}^*)$ and $\mathrm{A}_{i-1}^* x_{\mathrm{A}_{i-1}^*} = g$
$\Rightarrow x_{\mathrm{A}_{i-1}^*} = (\mathcal{A}_{i-1}^*)^{-1} g \in D(\mathcal{A}_{i-1}^*)$ and $|x_{\mathrm{A}_{i-1}^*}|_{\mathrm{H}_i} \le c_{i-1} |g|_{\mathrm{H}_{i-1}}$
note: $D(\mathcal{A}_{i-1}) = D(\mathrm{A}_{i-1}) \cap R(\mathrm{A}_{i-1}^*)$ and $R(\mathrm{A}_{i-1}^*) = N(\mathrm{A}_{i-1})^{\perp_{\mathrm{H}_{i-1}}}$
$\Rightarrow$ (6) is equivalent to the saddle point problem: find $\psi \in D(\mathrm{A}_{i-1})$ with

$$\forall\, \varphi \in D(\mathrm{A}_{i-1}) \qquad \langle \mathrm{A}_{i-1}\psi, \mathrm{A}_{i-1}\varphi \rangle_{\mathrm{H}_i} = \langle g, \varphi \rangle_{\mathrm{H}_{i-1}},$$

$$\forall\, \phi \in N(\mathrm{A}_{i-1}) \qquad \langle \psi, \phi \rangle_{\mathrm{H}_{i-1}} = 0$$

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Upper Bounds

problem: find $x \in D(\mathrm{A}_i) \cap D(\mathrm{A}_{i-1}^*)$ s.t. $\mathrm{A}_i x = f, \quad \mathrm{A}_{i-1}^* x = g, \quad \pi_i x = h$

'very' non-conforming 'approximation' of x: $\tilde{x} \in \mathrm{H}_i$

define error $e := x - \tilde{x}$ and decompose

$$e = \pi_{\mathrm{A}_{i-1}} e + \pi_i e + \pi_{\mathrm{A}_i^*} e \in \mathrm{H}_i = R(\mathrm{A}_{i-1}) \oplus_{\mathrm{H}_i} \mathcal{H}_i \oplus_{\mathrm{H}_i} R(\mathrm{A}_i^*)$$

Theorem (sharp upper bounds I)

Let $\tilde{x} \in \mathrm{H}_i$ and $e := x - \tilde{x}$. Then

$$|e|_{\mathrm{H}_i}^2 = |\pi_{\mathrm{A}_{i-1}} e|_{\mathrm{H}_i}^2 + |\pi_i e|_{\mathrm{H}_i}^2 + |\pi_{\mathrm{A}_i^*} e|_{\mathrm{H}_i}^2,$$

$$|\pi_{\mathrm{A}_{i-1}} e|_{\mathrm{H}_i} = \min_{\phi \in D(\mathrm{A}_{i-1}^*)} \Big(c_{i-1} |g - \mathrm{A}_{i-1}^* \phi|_{\mathrm{H}_{i-1}} + |\pi_{\mathrm{A}_{i-1}} (\phi - \tilde{x})|_{\mathrm{H}_i} \Big),$$

$$|\pi_{\mathrm{A}_i^*} e|_{\mathrm{H}_i} = \min_{\varphi \in D(\mathrm{A}_i)} \Big(c_i |f - \mathrm{A}_i \varphi|_{\mathrm{H}_{i+1}} + |\pi_{\mathrm{A}_i^*} (\varphi - \tilde{x})|_{\mathrm{H}_i} \Big),$$

$$\pi_i e = h - \pi_i \tilde{x}.$$

The minima are attained at $\phi = \varphi = x$.

Upper Bounds

Theorem (sharp upper bounds II)

Let $\tilde{x} \in \mathrm{H}_i$ *and* $e := x - \tilde{x}$. *Then*

$$|e|_{\mathrm{H}_i}^2 = |\pi_{\mathrm{A}_{i-1}} e|_{\mathrm{H}_i}^2 + |\pi_i e|_{\mathrm{H}_i}^2 + |\pi_{\mathrm{A}_i^*} e|_{\mathrm{H}_i}^2,$$

$$|\pi_{\mathrm{A}_{i-1}} e|_{\mathrm{H}_i} = \min_{\phi \in D(\mathrm{A}_{i-1}^*)} \big(c_{i-1} |g - \mathrm{A}_{i-1}^* \phi|_{\mathrm{H}_{i-1}} + |\pi_{\mathrm{A}_{i-1}} (\phi - \tilde{x})|_{\mathrm{H}_i} \big),$$

$$|\pi_{\mathrm{A}_i^*} e|_{\mathrm{H}_i} = \min_{\varphi \in D(\mathrm{A}_i)} \big(c_i |f - \mathrm{A}_i \varphi|_{\mathrm{H}_{i+1}} + |\pi_{\mathrm{A}_i^*} (\varphi - \tilde{x})|_{\mathrm{H}_i} \big),$$

$$\pi_i e = h - \pi_i \tilde{x},$$

$$|\pi_{\mathrm{A}_{i-1}} (\phi - \tilde{x})|_{\mathrm{H}_i} = \min_{\xi \in N(\mathrm{A}_{i-1}^*)} |\phi - \tilde{x} - \xi|_{\mathrm{H}_i},$$

$$|\pi_{\mathrm{A}_i^*} (\varphi - \tilde{x})|_{\mathrm{H}_i} = \min_{\zeta \in N(\mathrm{A}_i)} |\varphi - \tilde{x} - \zeta|_{\mathrm{H}_i},$$

$$|h - \pi_i \tilde{x}|_{\mathrm{H}_i} = \min_{\substack{\tau \in D(\mathrm{A}_{i-1}), \\ \sigma \in D(\mathrm{A}_i^*)}} |h - \tilde{x} - \mathrm{A}_{i-1} \tau - \mathrm{A}_i^* \sigma|_{\mathrm{H}_i}.$$

The minima are attained at $\phi = \varphi = x$ *as well as*
$\xi = (1 - \pi_{\mathrm{A}_{i-1}})(\phi - \tilde{x})$ *and* $\zeta = (1 - \pi_{\mathrm{A}_i^*})(\varphi - \tilde{x})$ *and* $\mathrm{A}_{i-1}\tau + \mathrm{A}_i^* \sigma = (\pi_i - 1)\tilde{x}$.

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Upper Bounds

Corollary (upper bounds III)

Let $\tilde{x} \in \mathrm{H}_i$ *and* $e := x - \tilde{x}$. *Then*

$$\begin{aligned}
|e|_{\mathrm{H}_i}^2 &= |\pi_{\mathrm{A}_{i-1}} e|_{\mathrm{H}_i}^2 + |\pi_i e|_{\mathrm{H}_i}^2 + |\pi_{\mathrm{A}_i^*} e|_{\mathrm{H}_i}^2 \\
&= \min_{\phi \in D(\mathrm{A}_{i-1}^*)} \Big(c_{i-1} |g - \mathrm{A}_{i-1}^* \phi|_{\mathrm{H}_{i-1}} + \underbrace{\min_{\xi \in N(\mathrm{A}_{i-1}^*)} |\phi - \tilde{x} - \xi|_{\mathrm{H}_i}}_{= |\pi_{\mathrm{A}_{i-1}}(\phi - \tilde{x})|_{\mathrm{H}_i} \leq |\phi - \tilde{x}|_{\mathrm{H}_i} \quad (\xi = 0)} \Big)^2 \\
&\quad + \min_{\varphi \in D(\mathrm{A}_i)} \Big(c_i |f - \mathrm{A}_i \varphi|_{\mathrm{H}_{i+1}} + \underbrace{\min_{\zeta \in N(\mathrm{A}_i)} |\varphi - \tilde{x} - \zeta|_{\mathrm{H}_i}}_{= |\pi_{\mathrm{A}_i^*}(\varphi - \tilde{x})|_{\mathrm{H}_i} \leq |\varphi - \tilde{x}|_{\mathrm{H}_i} \quad (\zeta = 0)} \Big)^2 \\
&\quad + \underbrace{\min_{\substack{\tau \in D(\mathrm{A}_{i-1}), \\ \sigma \in D(\mathrm{A}_i^*)}} |h - \tilde{x} - \mathrm{A}_{i-1}\tau - \mathrm{A}_i^* \sigma|_{\mathrm{H}_i}^2}_{= |\pi_i e|_{\mathrm{H}_i} = |h - \pi_i \tilde{x}|_{\mathrm{H}_i} \quad (\mathrm{A}_{i-1}\tau + \mathrm{A}_i^* \sigma = (\pi_i - 1)\tilde{x})} \\
&\leq 3 |e|_{\mathrm{H}_i}^2 .
\end{aligned}$$

put $\phi = \varphi = x$

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Upper Bounds (Proof)

$$|e|_{\mathrm{H}_i}^2 = |\pi_{\mathrm{A}_{i-1}} e|_{\mathrm{H}_i}^2 + |\pi_i e|_{\mathrm{H}_i}^2 + |\pi_{\mathrm{A}_i^*} e|_{\mathrm{H}_i}^2$$

- $\pi_{\mathrm{A}_{i-1}} e = \mathrm{A}_{i-1} u$ with $u \in D(\mathcal{A}_{i-1})$
  for $\psi \in D(\mathcal{A}_{i-1})$, $\phi \in D(\mathrm{A}_{i-1}^*)$

$$\begin{aligned}
\langle \mathrm{A}_{i-1} u, \mathrm{A}_{i-1}\psi\rangle_{\mathrm{H}_i} &= \langle \pi_{\mathrm{A}_{i-1}} e, \mathrm{A}_{i-1}\psi\rangle_{\mathrm{H}_i} = \langle e, \mathrm{A}_{i-1}\psi\rangle_{\mathrm{H}_i} \\
&= \langle x - \phi, \mathrm{A}_{i-1}\psi\rangle_{\mathrm{H}_i} + \langle \phi - \tilde{x}, \mathrm{A}_{i-1}\psi\rangle_{\mathrm{H}_i} \\
&= \langle g - \mathrm{A}_{i-1}^*\phi, \psi\rangle_{\mathrm{H}_{i-1}} + \langle \pi_{\mathrm{A}_{i-1}}(\phi - \tilde{x}), \mathrm{A}_{i-1}\psi\rangle_{\mathrm{H}_i} \\
&\le \big(c_{i-1}|g - \mathrm{A}_{i-1}^*\phi|_{\mathrm{H}_{i-1}} + |\pi_{\mathrm{A}_{i-1}}(\phi - \tilde{x})|_{\mathrm{H}_i}\big)|\mathrm{A}_{i-1}\psi|_{\mathrm{H}_i}
\end{aligned}$$

note: $\forall\, \psi \in D(\mathcal{A}_{i-1}) \quad |\psi|_{\mathrm{H}_{i-1}} \le c_{i-1}|\mathrm{A}_{i-1}\psi|_{\mathrm{H}_i}$

$$\psi := u \Rightarrow \boxed{|\pi_{\mathrm{A}_{i-1}} e|_{\mathrm{H}_i} = |\mathrm{A}_{i-1} u|_{\mathrm{H}_i} \le c_{i-1}|g - \mathrm{A}_{i-1}^*\phi|_{\mathrm{H}_{i-1}} + |\pi_{\mathrm{A}_{i-1}}(\phi - \tilde{x})|_{\mathrm{H}_i}}$$

Upper Bounds (Proof)

$$|e|_{\mathrm{H}_i}^2 = |\pi_{\mathrm{A}_{i-1}} e|_{\mathrm{H}_i}^2 + |\pi_i e|_{\mathrm{H}_i}^2 + |\pi_{\mathrm{A}_i^*} e|_{\mathrm{H}_i}^2$$

- $\pi_{\mathrm{A}_i^*} e = \mathrm{A}_i^* u$ with $u \in D(\mathcal{A}_i^*)$
  for $\psi \in D(\mathcal{A}_i^*)$, $\varphi \in D(\mathrm{A}_i)$

$$\begin{aligned}\langle \mathrm{A}_i^* u, \mathrm{A}_i^* \psi\rangle_{\mathrm{H}_i} &= \langle \pi_{\mathrm{A}_i^*} e, \mathrm{A}_i^* \psi\rangle_{\mathrm{H}_i} = \langle e, \mathrm{A}_i^* \psi\rangle_{\mathrm{H}_i} \\ &= \langle x - \varphi, \mathrm{A}_i^* \psi\rangle_{\mathrm{H}_i} + \langle \varphi - \tilde{x}, \mathrm{A}_i^* \psi\rangle_{\mathrm{H}_i} \\ &= \langle f - \mathrm{A}_i \varphi, \psi\rangle_{\mathrm{H}_{i+1}} + \langle \pi_{\mathrm{A}_i^*}(\varphi - \tilde{x}), \mathrm{A}_i^* \psi\rangle_{\mathrm{H}_i} \\ &\leq \big(c_i |f - \mathrm{A}_i \varphi|_{\mathrm{H}_{i+1}} + |\pi_{\mathrm{A}_i^*}(\varphi - \tilde{x})|_{\mathrm{H}_i} \big) |\mathrm{A}_i^* \psi|_{\mathrm{H}_i}\end{aligned}$$

note: $\forall\, \psi \in D(\mathcal{A}_i^*) \quad |\psi|_{\mathrm{H}_{i+1}} \leq c_i |\mathrm{A}_i^* \psi|_{\mathrm{H}_i}$

$\psi := u \Rightarrow$ $\boxed{|\pi_{\mathrm{A}_i^*} e|_{\mathrm{H}_i} = |\mathrm{A}_i^* u|_{\mathrm{H}_i} \leq c_i |f - \mathrm{A}_i \varphi|_{\mathrm{H}_{i+1}} + |\pi_{\mathrm{A}_i^*}(\varphi - \tilde{x})|_{\mathrm{H}_i}}$

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Upper Bounds (Proof)

recall

$$|e|^2_{\mathrm{H}_i} = |\pi_{\mathrm{A}_{i-1}} e|^2_{\mathrm{H}_i} + |\pi_i e|^2_{\mathrm{H}_i} + |\pi_{\mathrm{A}_i^*} e|^2_{\mathrm{H}_i}$$

$$|\pi_{\mathrm{A}_{i-1}} e|_{\mathrm{H}_i} = \min_{\phi \in D(\mathrm{A}^*_{i-1})} \big(c_{i-1} |g - \mathrm{A}^*_{i-1}\phi|_{\mathrm{H}_{i-1}} + |\pi_{\mathrm{A}_{i-1}}(\phi - \tilde{x})|_{\mathrm{H}_i} \big)$$

$$|\pi_{\mathrm{A}_i^*} e|_{\mathrm{H}_i} = \min_{\varphi \in D(\mathrm{A}_i)} \big(c_i |f - \mathrm{A}_i \varphi|_{\mathrm{H}_{i+1}} + |\pi_{\mathrm{A}_i^*}(\varphi - \tilde{x})|_{\mathrm{H}_i} \big)$$

$$\pi_i e = h - \pi_i \tilde{x}$$

$\Rightarrow$ for all $\xi \in N(\mathrm{A}^*_{i-1})$ and $\zeta \in N(\mathrm{A}_i)$

$$|\pi_{\mathrm{A}_{i-1}}(\phi - \tilde{x})|^2_{\mathrm{H}_i} = \langle \phi - \tilde{x} - \xi, \pi_{\mathrm{A}_{i-1}}(\phi - \tilde{x}) \rangle_{\mathrm{H}_i} \quad \Rightarrow \quad |\pi_{\mathrm{A}_{i-1}}(\phi - \tilde{x})|_{\mathrm{H}_i} \leq |\phi - \tilde{x} - \xi|_{\mathrm{H}_i}$$

$$|\pi_{\mathrm{A}_i^*}(\varphi - \tilde{x})|^2_{\mathrm{H}_i} = \langle \varphi - \tilde{x} - \zeta, \pi_{\mathrm{A}_i^*}(\varphi - \tilde{x}) \rangle_{\mathrm{H}_i} \quad \Rightarrow \quad |\pi_{\mathrm{A}_i^*}(\varphi - \tilde{x})|_{\mathrm{H}_i} \leq |\varphi - \tilde{x} - \zeta|_{\mathrm{H}_i}$$

for all $\tau \in D(\mathrm{A}_{i-1})$, $\sigma \in D(\mathrm{A}_i^*)$

$$|h - \pi_i \tilde{x}|^2_{\mathrm{H}_i} = \langle h - \pi_i \tilde{x} - \mathrm{A}_{i-1}\tau - \mathrm{A}_i^*\sigma, h - \pi_i \tilde{x} \rangle_{\mathrm{H}_i}$$

$$\Rightarrow \quad |h - \pi_i \tilde{x}|_{\mathrm{H}_i} \leq |h - \pi_i \tilde{x} - \mathrm{A}_{i-1}\tau - \mathrm{A}_i^*\sigma|_{\mathrm{H}_i}$$

Lower Bounds

recall problem: find $x \in D(\mathrm{A}_i) \cap D(\mathrm{A}_{i-1}^*)$ s.t. $\mathrm{A}_i x = f$, $\mathrm{A}_{i-1}^* x = g$, $\pi_i x = h$

'very' non-conforming 'approximation' of x: $\tilde{x} \in \mathrm{H}_i$

error $e := x - \tilde{x}$ with $e = \pi_{\mathrm{A}_{i-1}} e + \pi_i e + \pi_{\mathrm{A}_i^*} e \in \mathrm{H}_i = R(\mathrm{A}_{i-1}) \oplus_{\mathrm{H}_i} \mathcal{H}_i \oplus_{\mathrm{H}_i} R(\mathrm{A}_i^*)$

Theorem (sharp lower bounds)

Let $\tilde{x} \in \mathrm{H}_i$ and $e := x - \tilde{x}$. Then

$$
\begin{aligned}
|e|_{\mathrm{H}_i}^2 &= |\pi_{\mathrm{A}_{i-1}} e|_{\mathrm{H}_i}^2 + |\pi_i e|_{\mathrm{H}_i}^2 + |\pi_{\mathrm{A}_i^*} e|_{\mathrm{H}_i}^2,\\
|\pi_{\mathrm{A}_{i-1}} e|_{\mathrm{H}_i}^2 &= \max_{\phi \in D(\mathrm{A}_{i-1})} \big(2\langle g, \phi\rangle_{\mathrm{H}_{i-1}} - \langle 2\tilde{x} + \mathrm{A}_{i-1}\phi, \mathrm{A}_{i-1}\phi\rangle_{\mathrm{H}_i} \big),\\
|\pi_{\mathrm{A}_i^*} e|_{\mathrm{H}_i}^2 &= \max_{\varphi \in D(\mathrm{A}_i^*)} \big(2\langle f, \varphi\rangle_{\mathrm{H}_{i+1}} - \langle 2\tilde{x} + \mathrm{A}_i^*\varphi, \mathrm{A}_i^*\varphi\rangle_{\mathrm{H}_i} \big),\\
\pi_i e &= h - \pi_i \tilde{x}.
\end{aligned}
$$

The maxima are attained at $\phi \in D(\mathrm{A}_{i-1})$ with $\mathrm{A}_{i-1}\phi = \pi_{\mathrm{A}_{i-1}} e$ and $\varphi \in D(\mathrm{A}_i^*)$ with $\mathrm{A}_i^*\varphi = \pi_{\mathrm{A}_i^*} e$.

Lower Bounds (Proof)

$$|e|^2_{H_i} = |\pi_{A_{i-1}} e|^2_{H_i} + |\pi_i e|^2_{H_i} + |\pi_{A_i^*} e|^2_{H_i}$$

note: $|u|^2 = \max_v \left(2\langle u, v\rangle - |v|^2\right)$ (max at $v = u$)

$\Rightarrow$ for all $\phi \in D(A_{i-1})$ and $\varphi \in D(A_i^*)$ and with $\pi_{A_{i-1}} e \in R(A_{i-1})$ and $\pi_{A_i^*} e \in R(A_i^*)$

$$\begin{aligned}
|\pi_{A_{i-1}} e|^2_{H_i} &= \max_{\phi \in D(A_{i-1})} \Big(2 \underbrace{\langle \pi_{A_{i-1}} e, A_{i-1}\phi\rangle_{H_i}}_{=\langle e, A_{i-1}\phi\rangle_{H_i}} - |A_{i-1}\phi|^2_{H_i}\Big) \\
&= \max_{\phi \in D(A_{i-1})} \Big(2\langle g, \phi\rangle_{H_{i-1}} - 2\langle \tilde{x}, A_{i-1}^*\phi\rangle_{H_i} - |A_{i-1}\phi|^2_{H_i}\Big) \\
&= \max_{\phi \in D(A_{i-1})} \Big(2\langle g, \phi\rangle_{H_{i-1}} - \langle 2\tilde{x} + A_{i-1}\phi, A_{i-1}\phi\rangle_{H_i}\Big) \\
|\pi_{A_i^*} e|^2_{H_i} &= \max_{\varphi \in D(A_i^*)} \Big(2 \underbrace{\langle \pi_{A_i^*} e, A_i^*\varphi\rangle_{H_i}}_{=\langle e, A_i^*\varphi\rangle_{H_i}} - |A_i^*\varphi|^2_{H_i}\Big) \\
&= \max_{\varphi \in D(A_i^*)} \Big(2\langle f, \varphi\rangle_{H_{i+1}} - 2\langle \tilde{x}, A_i^*\varphi\rangle_{H_i}\Big) - \langle A_i^*\varphi, A_i^*\varphi\rangle_{H_i} \\
&= \max_{\varphi \in D(A_i^*)} \Big(2\langle f, \varphi\rangle_{H_{i+1}} - \langle 2\tilde{x} + A_i^*\varphi, A_i^*\varphi\rangle_{H_i}\Big)
\end{aligned}$$

Abstract Problem and Goal

problem: find $x \in D(\mathrm{A}_i^* \mathrm{A}_i) \cap D(\mathrm{A}_{i-1}^*)$ s.t.

$$\mathrm{A}_i^* \mathrm{A}_i x = f$$
$$\mathrm{A}_{i-1}^* x = g$$
$$\pi_i x = h$$

equivalent mixed formulation ($y := \mathrm{A}_i x$):
find pair $(x, y) \in \big(D(\mathrm{A}_i) \cap D(\mathrm{A}_{i-1}^*)\big) \times \big(\underbrace{D(\mathrm{A}_i^*) \times R(\mathrm{A}_i)}_{=D(\mathcal{A}_i^*)}\big)$ s.t.

$$\begin{aligned} \mathrm{A}_i x &= y, & \mathrm{A}_{i+1} y &= 0 \\ \mathrm{A}_{i-1}^* x &= g, & \mathrm{A}_i^* y &= f \\ \pi_i x &= h, & \pi_{i+1} y &= 0 \end{aligned}$$

cont. solution theory $\surd$: $x = \mathcal{A}_i^{-1} y + (\mathcal{A}_{i-1}^*)^{-1} g + h$ and $y = (\mathcal{A}_i^*)^{-1} f$
goal: functional a posteriori error estimates ‘in the spirit of Sergey Repin’

for $(\tilde{x}, \tilde{y}) \in \mathrm{H}_i \times \mathrm{H}_{i+1}$ (very non-conforming!)

estimate $|(x, y) - (\tilde{x}, \tilde{y})|_{\mathrm{H}_i \times \mathrm{H}_{i+1}}$ in terms of $\tilde{x}$, $\tilde{y}$, f, g, h

Upper Bounds

problem: find $(x, y) \in \big(D(\mathrm{A}_i) \cap D(\mathrm{A}_{i-1}^*)\big) \times D(\mathcal{A}_i^*)$ s.t.

$$\mathrm{A}_i^* y = f, \quad \mathrm{A}_i x = y, \quad \mathrm{A}_{i-1}^* x = g, \quad \pi_i x = h$$

non-conforming 'approximation' of x: $(\tilde{x}, \tilde{y}) \in \mathrm{H}_i \times \mathrm{H}_{i+1}$

define errors $e_x := x - \tilde{x}$ and $e_y := y - \tilde{y}$ and decompose

$$e_x = \pi_{\mathrm{A}_{i-1}} e_x + \pi_i e_x + \pi_{\mathrm{A}_i^*} e_x \in \mathrm{H}_i = R(\mathrm{A}_{i-1}) \oplus_{\mathrm{H}_i} \mathcal{H}_i \oplus_{\mathrm{H}_i} R(\mathrm{A}_i^*)$$

$$e_y = \underbrace{\pi_{\mathrm{A}_i} e_y}_{= y - \pi_{\mathrm{A}_i} \tilde{y}} + \underbrace{\pi_{i+1} e_y}_{= -\pi_{i+1} \tilde{y}} + \underbrace{\pi_{\mathrm{A}_{i+1}^*} e_y}_{= -\pi_{\mathrm{A}_{i+1}^*} \tilde{y}} \in \mathrm{H}_{i+1} = R(\mathrm{A}_i) \oplus_{\mathrm{H}_{i+1}} \mathcal{H}_{i+1} \oplus_{\mathrm{H}_{i+1}} R(\mathrm{A}_{i+1}^*)$$

$$\Rightarrow (1 - \pi_{\mathrm{A}_i}) e_y = -(\pi_{i+1} + \pi_{\mathrm{A}_{i+1}^*}) \tilde{y} = -(1 - \pi_{\mathrm{A}_i}) \tilde{y}$$

$\Downarrow$

$$|e_x|_{\mathrm{H}_i}^2 = |\pi_{\mathrm{A}_{i-1}} e_x|_{\mathrm{H}_i}^2 + |\pi_i e_x|_{\mathrm{H}_i}^2 + |\pi_{\mathrm{A}_i^*} e_x|_{\mathrm{H}_i}^2$$

$$|e_y|_{\mathrm{H}_{i+1}}^2 = |\pi_{\mathrm{A}_i} e_y|_{\mathrm{H}_{i+1}}^2 + |(1 - \pi_{\mathrm{A}_i}) \tilde{y}|_{\mathrm{H}_{i+1}}^2$$

Upper Bounds

$(\tilde{x},\tilde{y}) \in \mathrm{H}_i \times \mathrm{H}_{i+1}$ and $e = (x,y) - (\tilde{x},\tilde{y}) \in \mathrm{H}_i \times \mathrm{H}_{i+1}$
$\Downarrow$

$$\pi_i e_x = h - \pi_i \tilde{x}, \quad (1 - \pi_{\mathrm{A}_i}) e_y = -(1 - \pi_{\mathrm{A}_i})\tilde{y}$$

and

$$|\pi_{\mathrm{A}_{i-1}} e_x|_{\mathrm{H}_i} = \min_{\phi \in D(\mathrm{A}^*_{i-1})} \left(c_{i-1} |g - \mathrm{A}^*_{i-1}\phi|_{\mathrm{H}_{i-1}} + |\pi_{\mathrm{A}_{i-1}}(\phi - \tilde{x})|_{\mathrm{H}_i} \right)$$

$$|\pi_{\mathrm{A}^*_i} e_x|_{\mathrm{H}_i} = \min_{\varphi \in D(\mathrm{A}_i)} \left(c_i |y - \mathrm{A}_i \varphi|_{\mathrm{H}_{i+1}} + |\pi_{\mathrm{A}^*_i}(\varphi - \tilde{x})|_{\mathrm{H}_i} \right)$$

$$|\pi_{\mathrm{A}_i} e_y|_{\mathrm{H}_{i+1}} = \min_{\phi \in D(\mathrm{A}^*_i)} \left(c_i |f - \mathrm{A}^*_i \phi|_{\mathrm{H}_i} + |\pi_{\mathrm{A}_i}(\phi - \tilde{y})|_{\mathrm{H}_{i+1}} \right)$$

'$y, \tilde{\tilde{y}} = \mathrm{A}_i \varphi \in R(\mathrm{A}_i)$' $\Rightarrow \pi_{\mathrm{A}_i}(y - \mathrm{A}_i\varphi) = y - \mathrm{A}_i\varphi$, $\pi_{\mathrm{A}^*_{i+1}}(y - \mathrm{A}_i\varphi) = 0$, $\pi_{i+1}(y - \mathrm{A}_i\varphi) = 0$

$\Downarrow$

$$|y - \mathrm{A}_i\varphi|_{\mathrm{H}_{i+1}} = |\pi_{\mathrm{A}_i}(y - \mathrm{A}_i\varphi)|_{\mathrm{H}_{i+1}} = \min_{\phi \in D(\mathrm{A}^*_i)} \left(c_i |f - \mathrm{A}^*_i \phi|_{\mathrm{H}_i} + |\pi_{\mathrm{A}_i}\phi - \mathrm{A}_i\varphi|_{\mathrm{H}_{i+1}} \right)$$

Upper Bounds

Theorem (sharp upper bounds)

Let $(\tilde{x},\tilde{y}) \in \mathrm{H}_i \times \mathrm{H}_{i+1}$ *and* $e := (x,y) - (\tilde{x},\tilde{y}) \in \mathrm{H}_i \times \mathrm{H}_{i+1}$.
Then $\pi_i e_x = h - \pi_i \tilde{x}$ *and* $(1-\pi_{\mathrm{A}_i})e_y = -(1-\pi_{\mathrm{A}_i})\tilde{y}$ *as well as*

$$|\pi_{\mathrm{A}_i} e_y|_{\mathrm{H}_{i+1}} = \min_{\xi \in D(\mathrm{A}_i^*)} \Big(c_i |f - \mathrm{A}_i^* \xi|_{\mathrm{H}_i} + |\pi_{\mathrm{A}_i}(\xi - \tilde{y})|_{\mathrm{H}_{i+1}} \Big),$$

$$|(1-\pi_{\mathrm{A}_i})e_y|_{\mathrm{H}_{i+1}} = |(1-\pi_{\mathrm{A}_i})\tilde{y}|_{\mathrm{H}_{i+1}} = \min_{\zeta \in D(\mathrm{A}_i)} |\tilde{y} - \mathrm{A}_i \zeta|_{\mathrm{H}_{i+1}},$$

$$|\pi_{\mathrm{A}_{i-1}} e_x|_{\mathrm{H}_i} = \min_{\phi \in D(\mathrm{A}_{i-1}^*)} \Big(c_{i-1} |g - \mathrm{A}_{i-1}^* \phi|_{\mathrm{H}_{i-1}} + |\pi_{\mathrm{A}_{i-1}}(\phi - \tilde{x})|_{\mathrm{H}_i} \Big),$$

$$|\pi_{\mathrm{A}_i^*} e_x|_{\mathrm{H}_i} = \min_{\varphi \in D(\mathrm{A}_i)} \Big(|\pi_{\mathrm{A}_i^*}(\varphi - \tilde{x})|_{\mathrm{H}_i} + c_i \min_{\psi \in D(\mathrm{A}_i^*)} \big(c_i |f - \mathrm{A}_i^* \psi|_{\mathrm{H}_i} + |\pi_{\mathrm{A}_i}\psi - \mathrm{A}_i \varphi|_{\mathrm{H}_{i+1}} \big) \Big).$$

The projectors can be computed and sharply estimates as before.

recall

$$|e_x|_{\mathrm{H}_i}^2 = |\pi_{\mathrm{A}_{i-1}} e_x|_{\mathrm{H}_i}^2 + |\pi_i e_x|_{\mathrm{H}_i}^2 + |\pi_{\mathrm{A}_i^*} e_x|_{\mathrm{H}_i}^2,$$

$$|e_y|_{\mathrm{H}_{i+1}}^2 = |\pi_{\mathrm{A}_i} e_y|_{\mathrm{H}_{i+1}}^2 + |(1-\pi_{\mathrm{A}_i})e_y|_{\mathrm{H}_{i+1}}^2$$

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Lower Bounds

...

Electro-Static Maxwell

$\Omega \subset \mathbb{R}^3$ bounded domain with Lipschitz (or weaker) boundary $\Gamma = \partial\Omega$

$$\begin{aligned} \operatorname{rot} E &= F \in \operatorname{rot} \mathring{\mathrm{R}} && \text{in } \Omega \\ -\operatorname{div} \varepsilon E &= G \in \operatorname{div} \mathrm{D} = \mathrm{L}^2 && \text{in } \Omega \\ \nu \times E &= 0 && \text{at } \Gamma \end{aligned}$$

$$\pi_{\mathrm{D}} E = D \in \mathcal{H}_{\mathrm{D},\varepsilon} = \mathring{\mathrm{R}}_0 \cap \varepsilon^{-1}\mathrm{D}_0$$

$\Rightarrow E \in \mathring{\mathrm{R}} \cap \varepsilon^{-1}\mathrm{D}$
set $i := 1$

$$\mathrm{A}_{i-1} := \mathring{\nabla} : \mathring{\mathrm{H}}^1 \subset \mathrm{L}^2 \to \mathrm{L}^2_\varepsilon, \qquad \boxed{\mathrm{A}_i := \mathring{\operatorname{rot}}} : \mathring{\mathrm{R}} \subset \mathrm{L}^2_\varepsilon \to \mathrm{L}^2$$

$$\boxed{\mathrm{A}^*_{i-1} = -\operatorname{div} \varepsilon} : \varepsilon^{-1}\mathrm{D} \subset \mathrm{L}^2_\varepsilon \to \mathrm{L}^2, \qquad \mathrm{A}^*_i = \varepsilon^{-1} \operatorname{rot} : \mathrm{R} \subset \mathrm{L}^2 \to \mathrm{L}^2_\varepsilon$$

Electro-Static Maxwell

compact embeddings:

$$D(\mathcal{A}_{i-1}) \hookrightarrow \mathrm{H}_{i-1} \quad \Leftrightarrow \quad \mathring{\mathrm{H}}^1 \hookrightarrow \mathrm{L}^2 \quad \text{(Rellich's selection theorem)}$$

$$D(\mathcal{A}_{i}) \hookrightarrow \mathrm{H}_{i} \quad \Leftrightarrow \quad \mathring{\mathrm{R}} \cap \varepsilon^{-1} \operatorname{rot} \mathrm{R} \hookrightarrow \mathrm{L}^2_\varepsilon \quad \text{(tan. Maxwell cpt property)}$$

$c_{i-1} = c_{\mathtt{p},\circ}$ (Poincaré/Friedrichs constant) and $c_i = c_{\mathtt{m},\mathtt{t}}$ (tangential Maxwell constant)

$$\forall\, \varphi \in D(\mathcal{A}_{i-1}) \quad |\varphi|_{\mathrm{H}_{i-1}} \le c_{i-1} |\mathrm{A}_{i-1}\varphi|_{\mathrm{H}_i} \quad \Leftrightarrow \quad \forall\, \varphi \in \mathring{\mathrm{H}}^1 \quad |\varphi|_{\mathrm{L}^2} \le c_{\mathtt{p},\circ} |\mathring{\nabla}\varphi|_{\mathrm{L}^2_\varepsilon}$$

$$\forall\, \phi \in D(\mathcal{A}^*_{i-1}) \quad |\phi|_{\mathrm{H}_{i}} \le c_{i-1} |\mathrm{A}^*_{i-1}\phi|_{\mathrm{H}_{i-1}} \quad \Leftrightarrow \quad \forall\, \Phi \in \varepsilon^{-1}\mathrm{D} \cap \nabla\mathring{\mathrm{H}}^1 \quad |\Phi|_{\mathrm{L}^2_\varepsilon} \le c_{\mathtt{p},\circ} |\operatorname{div} \varepsilon\Phi|_{\mathrm{L}^2}$$

$$\forall\, \varphi \in D(\mathcal{A}_{i}) \quad |\varphi|_{\mathrm{H}_{i}} \le c_{i} |\mathrm{A}_{i}\varphi|_{\mathrm{H}_{i+1}} \quad \Leftrightarrow \quad \forall\, \Phi \in \mathring{\mathrm{R}} \cap \varepsilon^{-1} \operatorname{rot} \mathrm{R} \quad |\Phi|_{\mathrm{L}^2_\varepsilon} \le c_{\mathtt{m},\mathtt{t}} |\mathring{\operatorname{rot}}\, \Phi|_{\mathrm{L}^2}$$

$$\forall\, \psi \in D(\mathcal{A}^*_{i}) \quad |\psi|_{\mathrm{H}_{i+1}} \le c_{i} |\mathrm{A}^*_{i}\psi|_{\mathrm{H}_{i}} \quad \Leftrightarrow \quad \forall\, \Psi \in \mathrm{R} \cap \operatorname{rot} \mathring{\mathrm{R}} \quad |\Psi|_{\mathrm{L}^2} \le c_{\mathtt{m},\mathtt{t}} |\operatorname{rot} \Psi|_{\mathrm{L}^2_\varepsilon}$$

Helmholtz decomposition:

$$\mathrm{H}_i = R(\mathrm{A}_{i-1}) \oplus_{\mathrm{H}_i} \mathcal{H}_i \oplus_{\mathrm{H}_i} R(\mathrm{A}^*_i) \quad \Leftrightarrow \quad \mathrm{L}^2_\varepsilon = \nabla\mathring{\mathrm{H}}^1 \oplus_{\mathrm{L}^2_\varepsilon} \mathcal{H}_{\mathrm{D},\varepsilon} \oplus_{\mathrm{L}^2_\varepsilon} \varepsilon^{-1} \operatorname{rot} \mathrm{R}$$

orthonormal projectors:

$$\pi_{\mathrm{A}_{i-1}} : \mathrm{H}_i \to R(\mathrm{A}_{i-1}), \qquad \pi_{\mathrm{A}^*_i} : \mathrm{H}_i \to R(\mathrm{A}^*_i), \qquad \pi_i : \mathrm{H}_i \to \mathcal{H}_i$$

$$\Leftrightarrow \qquad \pi_{\mathring{\nabla}} : \mathrm{L}^2_\varepsilon \to \nabla\mathring{\mathrm{H}}^1, \qquad \pi_{\varepsilon^{-1}\operatorname{rot}} : \mathrm{L}^2_\varepsilon \to \varepsilon^{-1} \operatorname{rot} \mathrm{R}, \qquad \pi_{\mathrm{D}} : \mathrm{L}^2_\varepsilon \to \mathcal{H}_{\mathrm{D},\varepsilon}$$

Electro-Static Maxwell: Upper Bounds

Theorem (sharp upper bounds I)

Let $\tilde{E} \in L^2_\varepsilon$ *(very non-conforming!) and* $e := E - \tilde{E}$. *Then*

$$|e|^2_{L^2_\varepsilon} = |\pi_{\overset{\circ}{\nabla}} e|^2_{L^2_\varepsilon} + |\pi_{\varepsilon^{-1}\operatorname{rot}} e|^2_{L^2_\varepsilon} + |\pi_D e|^2_{L^2_\varepsilon}$$

$$= \min_{\Phi \in \varepsilon^{-1} D} \Big(c_{p,\circ} |G + \operatorname{div} \varepsilon \Phi|_{L^2} + \underbrace{\min_{\Psi \in \varepsilon^{-1} D_0} |\Phi - \tilde{E} - \Psi|_{L^2_\varepsilon}}_{= |\pi_{\overset{\circ}{\nabla}} (\Phi - \tilde{E})|_{L^2_\varepsilon} \le |\Phi - \tilde{E}|_{L^2_\varepsilon} \quad (\Psi = 0)} \Big)^2$$

$$+ \min_{\Phi \in \overset{\circ}{R}} \Big(c_{m,t} |F - \operatorname{rot} \Phi|_{L^2} + \underbrace{\min_{\Psi \in \overset{\circ}{R}_0} |\Phi - \tilde{E} - \Psi|_{L^2_\varepsilon}}_{= |\pi_{\varepsilon^{-1}\operatorname{rot}} (\Phi - \tilde{E})|_{L^2_\varepsilon} \le |\Phi - \tilde{E}|_{L^2_\varepsilon} \quad (\Psi = 0)} \Big)^2$$

$$+ \underbrace{\min_{\phi \in \overset{\circ}{H}{}^1, \Psi \in R} |D - \tilde{E} - \nabla \phi - \varepsilon^{-1} \operatorname{rot} \Psi|^2_{L^2_\varepsilon}}_{= |\pi_D e|_{L^2_\varepsilon} = |D - \pi_D \tilde{E}|_{L^2_\varepsilon} \quad (\nabla\phi + \varepsilon^{-1} \operatorname{rot} \Psi = (\pi_D - 1)\tilde{E})}$$

$$\le 3 |e|^2_{L^2_\varepsilon}.$$

put $\Phi = E$; note: Γ connected $\Rightarrow \pi_D = 0$ and $\overset{\circ}{R}_0 = \nabla \overset{\circ}{H}{}^1$ and $D_0 = \operatorname{rot} R$

note: $\boxed{\Omega \text{ convex} \overset{\varepsilon = \mu = 1}{\Rightarrow} c_{p,\circ} \le c_{m,t} \le \frac{\operatorname{diam}_\Omega}{\pi}}$ $\Rightarrow$ everything is computable!

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Electro-Static Maxwell: Upper Bounds

Corollary (sharp upper bounds II)

Let Γ *be connected and* $\tilde{E} \in \mathsf{L}^2_\varepsilon$ *(non-conforming!) and* $e := E - \tilde{E}$*. Then*

$$
\begin{aligned}
|e|^2_{\mathsf{L}^2_\varepsilon} &= |\pi_{\overset{\circ}{\nabla}} e|^2_{\mathsf{L}^2_\varepsilon} + |\pi_{\varepsilon^{-1}\,\mathrm{rot}} e|^2_{\mathsf{L}^2_\varepsilon} \\
&= \min_{\Phi \in \varepsilon^{-1}\mathsf{D}} \Big(c_{\mathtt{p},\circ} |G + \operatorname{div} \varepsilon\Phi|_{\mathsf{L}^2} + \underbrace{\min_{\psi \in \mathsf{R}} |\Phi - \tilde{E} - \varepsilon^{-1} \operatorname{rot} \psi|_{\mathsf{L}^2_\varepsilon}}_{= |\pi_{\overset{\circ}{\nabla}} (\Phi - \tilde{E})|_{\mathsf{L}^2_\varepsilon} \le |\Phi - \tilde{E}|_{\mathsf{L}^2_\varepsilon} \quad (\psi = 0)} \Big)^2 \\
&\quad + \min_{\Phi \in \overset{\circ}{\mathsf{R}}} \Big(c_{\mathtt{m},\mathtt{t}} |F - \operatorname{rot} \Phi|_{\mathsf{L}^2} + \underbrace{\min_{\psi \in \overset{\circ}{\mathsf{H}}{}^1} |\Phi - \tilde{E} - \nabla \psi|_{\mathsf{L}^2_\varepsilon}}_{= |\pi_{\varepsilon^{-1}\,\mathrm{rot}} (\Phi - \tilde{E})|_{\mathsf{L}^2_\varepsilon} \le |\Phi - \tilde{E}|_{\mathsf{L}^2_\varepsilon} \quad (\psi = 0)} \Big)^2 \\
&\le 2|e|^2_{\mathsf{L}^2_\varepsilon}.
\end{aligned}
$$

put $\Phi = E$

note: $\boxed{\Omega \text{ convex} \overset{\varepsilon=\mu=1}{\Rightarrow} c_{\mathtt{p},\circ} \le c_{\mathtt{m},\mathtt{t}} \le \frac{\operatorname{diam}_\Omega}{\pi}}$ $\Rightarrow$ everything is computable!

Magneto-Static Maxwell

$\Omega \subset \mathbb{R}^3$ bounded domain with Lipschitz (or weaker) boundary $\Gamma = \partial\Omega$

$$\begin{aligned} \operatorname{rot} H &= F \in \operatorname{rot} \mathrm{R} && \text{in } \Omega \\ -\operatorname{div} \varepsilon H &= G \in \operatorname{div} \mathring{\mathrm{D}} = \mathrm{L}^2 \cap \mathbb{R}^\perp && \text{in } \Omega \\ \nu \cdot \varepsilon H &= 0 && \text{at } \Gamma \\ \pi_{\mathrm{N}} H &= N \in \mathcal{H}_{\mathrm{N},\varepsilon} = \mathrm{R}_0 \cap \varepsilon^{-1} \mathring{\mathrm{D}}_0 \end{aligned}$$

$\Rightarrow H \in \mathrm{R} \cap \varepsilon^{-1} \mathring{\mathrm{D}}$
set $i := 1$

$$\mathrm{A}_{i-1} := \nabla : \mathrm{H}^1 \subset \mathrm{L}^2 \to \mathrm{L}^2_\varepsilon, \qquad \boxed{\mathrm{A}_i := \operatorname{rot}} : \mathrm{R} \subset \mathrm{L}^2_\varepsilon \to \mathrm{L}^2$$

$$\boxed{\mathrm{A}^*_{i-1} = -\mathring{\operatorname{div}} \varepsilon} : \varepsilon^{-1} \mathring{\mathrm{D}} \subset \mathrm{L}^2_\varepsilon \to \mathrm{L}^2, \qquad \mathrm{A}^*_i = \varepsilon^{-1} \mathring{\operatorname{rot}} : \mathring{\mathrm{R}} \subset \mathrm{L}^2 \to \mathrm{L}^2_\varepsilon$$

Magneto-Static Maxwell

compact embeddings:

$$N(\mathcal{A}_{i-1}) \hookrightarrow \mathrm{H}_{i-1} \quad \Leftrightarrow \quad \mathrm{H}^1 \hookrightarrow \mathrm{L}^2 \quad \text{(Rellich's selection theorem)}$$

$$N(\mathcal{A}_{i}) \hookrightarrow \mathrm{H}_{i} \quad \Leftrightarrow \quad \mathrm{R} \cap \varepsilon^{-1} \operatorname{rot} \mathring{\mathrm{R}} \hookrightarrow \mathrm{L}^2_\varepsilon \quad \text{(nor. Maxwell cpt property)}$$

$c_{i-1} = c_\mathrm{p}$ (Poincaré/Friedrichs constant) and $c_i = c_{\mathtt{m},\mathtt{n}}$ (normal Maxwell constant)

$$\forall\, \varphi \in N(\mathcal{A}_{i-1}) \quad |\varphi|_{\mathrm{H}_{i-1}} \le c_{i-1} |\mathrm{A}_{i-1}\varphi|_{\mathrm{H}_i} \quad \Leftrightarrow \quad \forall\, \varphi \in \mathrm{H}^1 \quad |\varphi|_{\mathrm{L}^2} \le c_\mathrm{p} |\nabla\varphi|_{\mathrm{L}^2_\varepsilon}$$

$$\forall\, \phi \in N(\mathcal{A}^*_{i-1}) \quad |\phi|_{\mathrm{H}_{i}} \le c_{i-1} |\mathrm{A}^*_{i-1}\phi|_{\mathrm{H}_{i-1}} \quad \Leftrightarrow \quad \forall\, \Phi \in \varepsilon^{-1}\mathring{\mathrm{D}} \cap \nabla\mathrm{H}^1 \quad |\Phi|_{\mathrm{L}^2_\varepsilon} \le c_\mathrm{p} |\operatorname{\mathring{div}} \varepsilon\Phi|_{\mathrm{L}^2}$$

$$\forall\, \varphi \in N(\mathcal{A}_{i}) \quad |\varphi|_{\mathrm{H}_{i}} \le c_{i} |\mathrm{A}_{i}\varphi|_{\mathrm{H}_{i+1}} \quad \Leftrightarrow \quad \forall\, \Phi \in \mathrm{R} \cap \varepsilon^{-1} \operatorname{rot} \mathring{\mathrm{R}} \quad |\Phi|_{\mathrm{L}^2_\varepsilon} \le c_{\mathtt{m},\mathtt{n}} |\operatorname{rot}\Phi|_{\mathrm{L}^2}$$

$$\forall\, \psi \in N(\mathcal{A}^*_{i}) \quad |\psi|_{\mathrm{H}_{i+1}} \le c_{i} |\mathrm{A}^*_{i}\psi|_{\mathrm{H}_{i}} \quad \Leftrightarrow \quad \forall\, \Psi \in \mathring{\mathrm{R}} \cap \operatorname{rot} \mathrm{R} \quad |\Psi|_{\mathrm{L}^2} \le c_{\mathtt{m},\mathtt{n}} |\operatorname{\mathring{rot}}\Psi|_{\mathrm{L}^2_\varepsilon}$$

Helmholtz decomposition:

$$\mathrm{H}_i = R(\mathrm{A}_{i-1}) \oplus_{\mathrm{H}_i} \mathcal{H}_i \oplus_{\mathrm{H}_i} R(\mathrm{A}^*_i) \quad \Leftrightarrow \quad \mathrm{L}^2_\varepsilon = \nabla\mathrm{H}^1 \oplus_{\mathrm{L}^2_\varepsilon} \mathcal{H}_{\mathtt{N},\varepsilon} \oplus_{\mathrm{L}^2_\varepsilon} \varepsilon^{-1} \operatorname{rot} \mathring{\mathrm{R}}$$

orthonormal projectors:

$$\pi_{\mathrm{A}_{i-1}} : \mathrm{H}_i \to R(\mathrm{A}_{i-1}), \qquad \pi_{\mathrm{A}^*_i} : \mathrm{H}_i \to R(\mathrm{A}^*_i), \qquad \pi_i : \mathrm{H}_i \to \mathcal{H}_i$$

$$\Leftrightarrow \qquad \pi_\nabla : \mathrm{L}^2_\varepsilon \to \nabla\mathrm{H}^1, \qquad \pi_{\varepsilon^{-1}\operatorname{\mathring{rot}}} : \mathrm{L}^2_\varepsilon \to \varepsilon^{-1}\operatorname{rot}\mathring{\mathrm{R}}, \qquad \pi_\mathtt{N} : \mathrm{L}^2_\varepsilon \to \mathcal{H}_{\mathtt{N},\varepsilon}$$

Magneto-Static Maxwell: Upper Bounds

Theorem (sharp upper bounds I)

Let $\tilde{H} \in \mathsf{L}^2_\varepsilon$ *(very non-conforming!) and* $e := H - \tilde{H}$. *Then*

$$
\begin{aligned}
|e|^2_{\mathsf{L}^2_\varepsilon} &= |\pi_\nabla e|^2_{\mathsf{L}^2_\varepsilon} + |\pi_{\varepsilon^{-1}\overset{\circ}{\mathrm{rot}}} e|^2_{\mathsf{L}^2_\varepsilon} + |\pi_{\mathrm{N}} e|^2_{\mathsf{L}^2_\varepsilon} \\
&= \min_{\Phi \in \varepsilon^{-1}\overset{\circ}{\mathsf{D}}} \Big(c_{\mathrm{p}} |G + \operatorname{div} \varepsilon\Phi|_{\mathsf{L}^2} + \underbrace{\min_{\Psi \in \varepsilon^{-1}\overset{\circ}{\mathsf{D}}_0} |\Phi - \tilde{H} - \Psi|_{\mathsf{L}^2_\varepsilon}}_{= |\pi_\nabla(\Phi - \tilde{H})|_{\mathsf{L}^2_\varepsilon} \leq |\Phi - \tilde{H}|_{\mathsf{L}^2_\varepsilon} \quad (\Psi = 0)} \Big)^2 \\
&\quad + \min_{\Phi \in \mathsf{R}} \Big(c_{\mathrm{m,n}} |F - \operatorname{rot} \Phi|_{\mathsf{L}^2} + \underbrace{\min_{\Psi \in \mathsf{R}_0} |\Phi - \tilde{H} - \Psi|_{\mathsf{L}^2_\varepsilon}}_{= |\pi_{\varepsilon^{-1}\overset{\circ}{\mathrm{rot}}}(\Phi - \tilde{H})|_{\mathsf{L}^2_\varepsilon} \leq |\Phi - \tilde{H}|_{\mathsf{L}^2_\varepsilon} \quad (\Psi = 0)} \Big)^2 \\
&\quad + \underbrace{\min_{\phi \in \mathsf{H}^1, \Psi \in \overset{\circ}{\mathsf{R}}} |N - \tilde{H} - \nabla\phi - \varepsilon^{-1} \operatorname{rot} \Psi|^2_{\mathsf{L}^2_\varepsilon}}_{= |\pi_{\mathrm{N}} e|_{\mathsf{L}^2_\varepsilon} = |N - \pi_{\mathrm{N}} \tilde{H}|_{\mathsf{L}^2_\varepsilon} \quad (\nabla\phi + \varepsilon^{-1} \operatorname{rot} \Psi = (\pi_{\mathrm{N}} - 1)\tilde{H})} \\
&\leq 3|e|^2_{\mathsf{L}^2_\varepsilon}.
\end{aligned}
$$

put $\Phi = H$; note: Ω simply connected $\Rightarrow \pi_{\mathrm{N}} = 0$ and $\mathsf{R}_0 = \nabla \mathsf{H}^1$ and $\overset{\circ}{\mathsf{D}}_0 = \operatorname{rot} \overset{\circ}{\mathsf{R}}$

note: $\boxed{\Omega \text{ convex} \overset{\varepsilon=\mu=1}{\Rightarrow} c_{\mathrm{m,n}} \leq c_{\mathrm{p}} \leq \frac{\operatorname{diam}_\Omega}{\pi}}$ $\Rightarrow$ everything is computable!

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Magneto-Static Maxwell: Upper Bounds

Corollary (sharp upper bounds II)

Let Ω *be simply connected and* $\tilde{H} \in \mathsf{L}^2_\varepsilon$ *(non-conforming!) and* $e := H - \tilde{H}$. *Then*

$$|e|^2_{\mathsf{L}^2_\varepsilon} = |\pi_\nabla e|^2_{\mathsf{L}^2_\varepsilon} + |\pi_{\varepsilon^{-1}\overset{\circ}{\mathrm{rot}}} e|^2_{\mathsf{L}^2_\varepsilon}$$

$$= \min_{\Phi \in \varepsilon^{-1}\overset{\circ}{\mathsf{D}}} \Big(c_{\mathrm{p}} |G + \operatorname{div} \varepsilon\Phi|_{\mathsf{L}^2} + \underbrace{\min_{\psi \in \overset{\circ}{\mathsf{R}}} |\Phi - \tilde{H} - \varepsilon^{-1} \operatorname{rot} \psi|_{\mathsf{L}^2_\varepsilon}}_{= |\pi_\nabla (\Phi - \tilde{H})|_{\mathsf{L}^2_\varepsilon} \le |\Phi - \tilde{H}|_{\mathsf{L}^2_\varepsilon} \quad (\psi = 0)} \Big)^2$$

$$+ \min_{\Phi \in \mathsf{R}} \Big(c_{\mathrm{m,n}} |F - \operatorname{rot} \Phi|_{\mathsf{L}^2} + \underbrace{\min_{\psi \in \mathsf{H}^1} |\Phi - \tilde{H} - \nabla\psi|_{\mathsf{L}^2_\varepsilon}}_{= |\pi_{\varepsilon^{-1}\overset{\circ}{\mathrm{rot}}} (\Phi - \tilde{H})|_{\mathsf{L}^2_\varepsilon} \le |\Phi - \tilde{H}|_{\mathsf{L}^2_\varepsilon} \quad (\psi = 0)} \Big)^2$$

$$\le 2|e|^2_{\mathsf{L}^2_\varepsilon}.$$

put $\Phi = H$

note: $\boxed{\Omega \text{ convex} \overset{\varepsilon=\mu=1}{\Rightarrow} c_{\mathrm{m,n}} \le c_{\mathrm{p}} \le \frac{\operatorname{diam}_\Omega}{\pi}}$ $\Rightarrow$ everything is computable!

Dirichlet Laplace

$\Omega \subset \mathbb{R}^3$ bounded domain with Lipschitz (or weaker) boundary $\Gamma = \partial\Omega$

$$-\operatorname{div}\varepsilon\nabla u = f \in \mathrm{L}^2 \quad \text{in } \Omega$$
$$u = 0 \quad \text{at } \Gamma$$

$$\Leftrightarrow \quad \nabla u = E \in \nabla\mathring{\mathrm{H}}^1 \qquad \operatorname{rot} E = 0 \quad \text{in } \Omega$$
$$-\operatorname{div}\varepsilon E = f \in \mathrm{L}^2 \quad \text{in } \Omega$$
$$u = 0 \qquad \nu \times E = 0 \quad \text{at } \Gamma$$
$$\pi_{\mathrm{D}} E = 0 \in \mathcal{H}_{\mathrm{D},\varepsilon}$$

$\Rightarrow (u, E) \in \mathring{\mathrm{H}}^1 \times \left(\varepsilon^{-1}\mathrm{D} \cap \nabla\mathring{\mathrm{H}}^1\right)$
set $i := 0$

$$\boxed{\mathrm{A}_i := \mathring{\nabla}} : \mathring{\mathrm{H}}^1 \subset \mathrm{L}^2 \to \mathrm{L}^2_\varepsilon, \qquad \mathrm{A}_{i+1} := \mathring{\operatorname{rot}} : \mathring{\mathrm{R}} \subset \mathrm{L}^2_\varepsilon \to \mathrm{L}^2$$
$$\boxed{\mathrm{A}_i^* = -\operatorname{div}\varepsilon} : \varepsilon^{-1}\mathrm{D} \subset \mathrm{L}^2_\varepsilon \to \mathrm{L}^2, \qquad \mathrm{A}_{i+1}^* = \varepsilon^{-1}\operatorname{rot} : \mathrm{R} \subset \mathrm{L}^2 \to \mathrm{L}^2_\varepsilon$$

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Dirichlet Laplace: Upper Bounds

Theorem (sharp upper bounds)

Let $(\tilde{u}, \tilde{E}) \in \mathrm{L}^2 \times \mathrm{L}^2_\varepsilon$ *(very non-conforming!) and* $e := (u, E) - (\tilde{u}, \tilde{E}) \in \mathrm{L}^2 \times \mathrm{L}^2_\varepsilon$.
Then $\pi_i = 0$, $\pi_{-\operatorname{div}\varepsilon} = \mathrm{id}$ *and* $(1 - \pi_{\mathring{\nabla}})e_E = -(1 - \pi_{\mathring{\nabla}})\tilde{E}$ *and*

$$|\pi_{\mathring{\nabla}} e_E|_{\mathrm{L}^2_\varepsilon} = \min_{\Phi \in \varepsilon^{-1}\mathrm{D}} \Big(c_{\mathrm{p},\circ}|f + \operatorname{div}\varepsilon\Phi|_{\mathrm{L}^2} + \underbrace{\min_{\Psi \in \varepsilon^{-1}\mathrm{D}_0} |\Phi - \tilde{E} - \Psi|_{\mathrm{L}^2_\varepsilon}}_{= |\pi_{\mathring{\nabla}}(\Phi - \tilde{E})|_{\mathrm{L}^2_\varepsilon}} \Big),$$

$$|(1 - \pi_{\mathring{\nabla}})e_E|_{\mathrm{L}^2_\varepsilon} = |(1 - \pi_{\mathring{\nabla}})\tilde{E}|_{\mathrm{L}^2_\varepsilon} = \min_{\varphi \in \mathring{\mathrm{H}}^1} |\tilde{E} - \nabla\varphi|_{\mathrm{L}^2_\varepsilon},$$

$$|e_u|_{\mathrm{L}^2} = \min_{\varphi \in \mathring{\mathrm{H}}^1} \Big(|\varphi - \tilde{u}|_{\mathrm{L}^2} + c_{\mathrm{p},\circ} \min_{\Phi \in \varepsilon^{-1}\mathrm{D}} \big(c_{\mathrm{p},\circ}|f + \operatorname{div}\varepsilon\Phi|_{\mathrm{L}^2} + \underbrace{\min_{\Psi \in \varepsilon^{-1}\mathrm{D}_0} |\Phi - \mathring{\nabla}\varphi - \Psi|_{\mathrm{L}^2_\varepsilon}}_{= |\pi_{\mathring{\nabla}}\Phi - \mathring{\nabla}\varphi|_{\mathrm{L}^2_\varepsilon}} \big) \Big).$$

recall

$$|e_E|^2_{\mathrm{L}^2_\varepsilon} = |\pi_{\mathring{\nabla}} e_E|^2_{\mathrm{L}^2_\varepsilon} + |(1 - \pi_{\mathring{\nabla}})e_E|^2_{\mathrm{L}^2_\varepsilon}$$

note: $\tilde{E} \in \mathrm{L}^2_\varepsilon$ approx. of $\nabla u \Rightarrow$ applicable to any DG-method

Neumann Laplace

$\Omega \subset \mathbb{R}^3$ bounded domain with Lipschitz (or weaker) boundary $\Gamma = \partial\Omega$

$$\begin{aligned} -\operatorname{div}\varepsilon\nabla u &= f \in \mathrm{L}^2 && \text{in } \Omega \\ \nu\cdot\varepsilon\nabla u &= 0 && \text{at } \Gamma \\ \pi_{\mathbb{R}} u &= \alpha \in \mathbb{R} \end{aligned}$$

$$\Leftrightarrow \quad \begin{aligned} \nabla u &= H \in \nabla\mathrm{H}^1 & \operatorname{rot} H &= 0 && \text{in } \Omega \\ & & -\operatorname{div}\varepsilon H &= f \in \mathrm{L}^2 && \text{in } \Omega \\ & & \nu\cdot\varepsilon H &= 0 && \text{at } \Gamma \\ \pi_{\mathbb{R}} u &= \alpha \in \mathbb{R} & \pi_{\mathtt{N}} H &= 0 \in \mathcal{H}_{\mathtt{N},\varepsilon} \end{aligned}$$

$\Rightarrow (u, H) \in \mathrm{H}^1 \times \left(\varepsilon^{-1}\mathring{\mathrm{D}} \cap \nabla\mathrm{H}^1\right)$

set $i := 0$

$$\boxed{\mathrm{A}_i := \nabla} : \mathrm{H}^1 \subset \mathrm{L}^2 \to \mathrm{L}^2_\varepsilon, \qquad \mathrm{A}_{i+1} := \operatorname{rot} : \mathrm{R} \subset \mathrm{L}^2_\varepsilon \to \mathrm{L}^2$$

$$\boxed{\mathrm{A}_i^* = -\mathring{\operatorname{div}}\varepsilon} : \varepsilon^{-1}\mathring{\mathrm{D}} \subset \mathrm{L}^2_\varepsilon \to \mathrm{L}^2, \qquad \mathrm{A}_{i+1}^* = \varepsilon^{-1}\mathring{\operatorname{rot}} : \mathring{\mathrm{R}} \subset \mathrm{L}^2 \to \mathrm{L}^2_\varepsilon$$

Neumann Laplace: Upper Bounds

Theorem (sharp upper bounds)

Let $(\tilde{u}, \tilde{H}) \in \mathsf{L}^2 \times \mathsf{L}^2_\varepsilon$ *(very non-conforming!) and* $e := (u, H) - (\tilde{u}, \tilde{H}) \in \mathsf{L}^2 \times \mathsf{L}^2_\varepsilon$*. Then*

$$|\pi_\nabla e_H|_{\mathsf{L}^2_\varepsilon} = \min_{\Phi \in \varepsilon^{-1}\mathring{\mathsf{D}}} \Big(c_{\mathrm{p}} |f + \operatorname{div} \varepsilon \Phi|_{\mathsf{L}^2} + \underbrace{\min_{\Psi \in \varepsilon^{-1}\mathring{\mathsf{D}}_0} |\Phi - \tilde{H} - \Psi|_{\mathsf{L}^2_\varepsilon}}_{= |\pi_\nabla (\Phi - \tilde{H})|_{\mathsf{L}^2_\varepsilon}} \Big),$$

$$|(1 - \pi_\nabla) e_H|_{\mathsf{L}^2_\varepsilon} = |(1 - \pi_\nabla) \tilde{H}|_{\mathsf{L}^2_\varepsilon} = \min_{\varphi \in \mathsf{H}^1} |\tilde{H} - \nabla \varphi|_{\mathsf{L}^2_\varepsilon}.$$

Again, also estimate for $|e_u|_{\mathsf{L}^2}$.

recall

$$|e_H|^2_{\mathsf{L}^2_\varepsilon} = |\pi_\nabla e_H|^2_{\mathsf{L}^2_\varepsilon} + |(1 - \pi_\nabla) e_H|^2_{\mathsf{L}^2_\varepsilon}$$

note: $\tilde{H} \in \mathsf{L}^2_\varepsilon$ approx. of ∇u $\Rightarrow$ applicable to any DG-method

First Order Systems

$\Omega \subset \mathbb{R}^3$ bounded domain with Lipschitz (or weaker) boundary $\Gamma = \partial\Omega$

Electro/Magneto-Static Maxwell with mixed boundary conditions:

$$\begin{aligned} \operatorname{rot} E &= F && \text{in } \Omega \\ -\operatorname{div} \varepsilon E &= G && \text{in } \Omega \\ \nu \times E &= 0 && \text{at } \Gamma_t \\ \nu \cdot \varepsilon E &= 0 && \text{at } \Gamma_n \\ \pi_{\mathtt{D},\mathtt{N}} E &= D && \end{aligned}$$

First Order Systems

$\Omega \subset \mathbb{R}^3$ bounded differentiable Riemannian manifold with Lipschitz boundary $\Gamma = \partial\Omega$

Electro-Static Maxwell:

$$\begin{aligned} \operatorname{rot}_\Omega E &= F && \text{on } \Omega \\ -\operatorname{div}_\Omega \varepsilon E &= G && \text{on } \Omega \\ \tau E &= 0 && \text{at } \Gamma \\ \pi_{\mathtt{D}} E &= D \in \mathcal{H}_{\mathtt{D},\varepsilon} \end{aligned}$$

Magneto-Static Maxwell:

$$\begin{aligned} \operatorname{rot}_\Omega H &= F && \text{on } \Omega \\ -\operatorname{div}_\Omega \varepsilon H &= G && \text{on } \Omega \\ \nu \varepsilon H &= 0 && \text{at } \Gamma \\ \pi_{\mathtt{N}} H &= N \in \mathcal{H}_{\mathtt{N},\varepsilon} \end{aligned}$$

First Order Systems

Ω differentiable Riemannian manifold with cpt closure and Lipschitz boundary $\Gamma = \partial\Omega$

Generalized Electro-Static Maxwell:

$$\begin{aligned} \mathrm{d}\,E &= F && \text{on } \Omega \\ -\delta\,\varepsilon E &= G && \text{on } \Omega \\ \tau E &= 0 && \text{on } \Gamma \\ \pi_{\mathtt{D}} E &= D \in \mathcal{H}_{\mathtt{D},\varepsilon} && \end{aligned}$$

Generalized Magneto-Static Maxwell:

$$\begin{aligned} \mathrm{d}\,H &= F && \text{on } \Omega \\ -\delta\,\varepsilon H &= G && \text{on } \Omega \\ \nu\varepsilon H &= 0 && \text{on } \Gamma \\ \pi_{\mathtt{N}} H &= N \in \mathcal{H}_{\mathtt{N},\varepsilon} && \end{aligned}$$

Second Order Systems

$\Omega \subset \mathbb{R}^n$ bounded domain with Lipschitz (or weaker) boundary $\Gamma = \partial\Omega$

Dirichlet Laplace:

$$\begin{aligned} -\operatorname{div} \varepsilon \nabla u &= f && \text{in } \Omega \\ u &= 0 && \text{at } \Gamma \end{aligned}$$

Neumann Laplace:

$$\begin{aligned} -\operatorname{div} \varepsilon \nabla u &= f && \text{in } \Omega \\ \nu \cdot \varepsilon \nabla u &= 0 && \text{at } \Gamma \\ \pi_{\mathbb{R}} u &= \alpha \end{aligned}$$

Dirichlet/Neumann Laplace with mixed boundary conditions:

$$\begin{aligned} -\operatorname{div} \varepsilon \nabla u &= f && \text{in } \Omega \\ u &= 0 && \text{at } \Gamma_t \\ \nu \cdot \varepsilon \nabla u &= 0 && \text{at } \Gamma_n \\ \pi_{\mathbb{R}} u &= \alpha \quad (\text{if } \Gamma_t = \emptyset) \end{aligned}$$

Second Order Systems

$\Omega \subset \mathbb{R}^n$ bounded differentiable Riemannian manifold
with Lipschitz (or weaker) boundary $\Gamma = \partial\Omega$

Dirichlet/Neumann Laplace:

$$-\operatorname{div}_\Omega \varepsilon \nabla_\Omega u = f \qquad \text{on } \Omega$$

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Second Order Systems

$\Omega \subset \mathbb{R}^3$ bounded domain with Lipschitz (or weaker) boundary $\Gamma = \partial\Omega$

Electro-Static double-rot:

$$\begin{aligned} \operatorname{rot} \mu^{-1} \operatorname{rot} E &= F && \text{in } \Omega \\ -\operatorname{div} \varepsilon E &= G && \text{in } \Omega \\ \nu \times E &= 0 && \text{at } \Gamma \\ \pi_{\mathtt{D}} E &= D \in \mathcal{H}_{\mathtt{D},\varepsilon} && \end{aligned}$$

Magneto-Static double-rot:

$$\begin{aligned} \operatorname{rot} \varepsilon^{-1} \operatorname{rot} H &= F && \text{in } \Omega \\ -\operatorname{div} \mu H &= G && \text{in } \Omega \\ \nu \cdot \mu H &= 0 && \text{at } \Gamma \\ \pi_{\mathtt{N}} H &= N \in \mathcal{H}_{\mathtt{N},\varepsilon} && \end{aligned}$$

Second Order Systems

Ω bounded differentiable Riemannian manifold
with Lipschitz (or weaker) boundary $\Gamma = \partial\Omega$

Generalized Electro-Magneto-Static:

$$\begin{aligned} -\delta\,\mu\,\mathrm{d}\,E &= F && \text{on } \Omega \\ -\delta\,\varepsilon E &= G && \text{on } \Omega \\ \tau E = 0 \quad \text{or} \quad \nu\varepsilon E &= 0 && \text{on } \Gamma \\ \pi_{\mathrm{D}} E = D \quad \text{or} \quad \pi_{\mathrm{N}} E &= N \end{aligned}$$

The End

more results:

- Stokes $\surd$
- unbounded like exterior domains $\Rightarrow$ estimates in polynomially weighted normes $\surd$
- mixed boundary conditions $\surd$
- inhomogeneous boundary conditions $\surd$

Thank You

Computation of Projections

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