Hilbert Complexes and PDEs

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Austrian Numerical Analysis Day ~~2020~~ 2022

and Colloquium dedicated to **Ulrich Langer** and **Walter Zulehner**
on the occasion of their Retirement

RICAM, JKU-ICM
May 4-6 2022, Linz

Hosts: Herbert Egger, Clemens Hofreither, Stefan Takacs

May the Force . . .

Hilbert Complexes and PDEs

Some Hilbert Complexes

PDEs: de Rham complex 3D

grad-complex

$$\{0\} \underset{\pi_0}{\overset{\iota_0}{\rightleftarrows}} L^2 \underset{-\operatorname{div}}{\overset{\mathring{\operatorname{grad}}}{\rightleftarrows}} L^2 \underset{\operatorname{rot}}{\overset{\mathring{\operatorname{rot}}}{\rightleftarrows}} L^2 \underset{-\operatorname{grad}}{\overset{\mathring{\operatorname{div}}}{\rightleftarrows}} L^2 \underset{\iota_{\mathbb{R}}}{\overset{\pi_{\mathbb{R}}}{\rightleftarrows}} \mathbb{R}$$

PDEs

$$\partial_t^n - \underbrace{\operatorname{div}\mathring{\operatorname{grad}}}_{=\Delta_{\mathrm{D}}}, \qquad \partial_t^n - \underbrace{\mathring{\operatorname{div}}\operatorname{grad}}_{=\Delta_{\mathrm{N}}}, \qquad \partial_t^n + \underbrace{\operatorname{rot}\mathring{\operatorname{rot}}}_{=\vec{\square}_{\mathrm{t}}}, \qquad \partial_t^n + \underbrace{\operatorname{rot}\mathring{\operatorname{rot}} - \mathring{\operatorname{grad}}\operatorname{div}}_{=-\vec{\Delta}_{\mathrm{t}}}$$

elliptic ($n = 0$) / parabolic ($n = 1$) / hyperbolic ($n = 2$)

or FOSs

$$\partial_t^m - \underbrace{\begin{bmatrix} 0 & \operatorname{div} \\ \mathring{\operatorname{grad}} & 0 \end{bmatrix}}_{=\mathrm{Maxwell}_{\mathrm{D,acoustic}}}, \qquad \partial_t^m - \underbrace{\begin{bmatrix} 0 & -\operatorname{rot} \\ \mathring{\operatorname{rot}} & 0 \end{bmatrix}}_{=\mathrm{Maxwell}_{\mathrm{t,electromagnetic}}}$$

PDEs: de Rham complex ND / manifolds

d-complex (mother of all complexes)

$$\{0\} \underset{\pi_0}{\overset{\iota_0}{\rightleftarrows}} \cdots \quad \mathsf{L}^{2,q-1} \underset{-\,\delta_q}{\overset{\mathring{\mathsf{d}}_{q-1}}{\rightleftarrows}} \mathsf{L}^{2,q} \underset{-\,\delta_{q+1}}{\overset{\mathring{\mathsf{d}}_q}{\rightleftarrows}} \mathsf{L}^{2,q+1} \quad \cdots \underset{\iota_{*\mathbb{R}}}{\overset{\pi_{*\mathbb{R}}}{\rightleftarrows}} *\,\mathbb{R}$$

PDEs

$$\partial_t^n - \delta\,\mathring{\mathsf{d}}, \qquad \partial_t^n - \mathring{\delta}\,\mathsf{d}, \qquad \partial_t^n - \delta\,\mathring{\mathsf{d}}, \qquad \partial_t^n \underbrace{-\,\delta\,\mathring{\mathsf{d}} - \mathring{\mathsf{d}}\,\delta}_{=-\Delta_{\mathsf{t}}}$$

elliptic ($n = 0$) / parabolic ($n = 1$) / hyperbolic ($n = 2$)

or FOS $$\partial_t^m - \underbrace{\begin{bmatrix} 0 & \delta \\ \mathring{\mathsf{d}} & 0 \end{bmatrix}}_{=\mathrm{Maxwell}_{\mathsf{t}}}$$

PDEs: elasticity complex 3D

symGrad-complex

$$\{0\} \underset{\pi_0}{\overset{\iota_0}{\rightleftarrows}} \mathsf{L}^2 \underset{-\,\mathrm{Div}_{\mathbb{S}}}{\overset{\mathring{\mathrm{symGrad}}}{\rightleftarrows}} \mathsf{L}^2_{\mathbb{S}} \underset{\mathrm{RotRot}_{\mathbb{S}}^{\top}}{\overset{\mathring{\mathrm{RotRot}}_{\mathbb{S}}^{\top}}{\rightleftarrows}} \mathsf{L}^2_{\mathbb{S}} \underset{-\,\mathrm{symGrad}}{\overset{\mathring{\mathrm{Div}}_{\mathbb{S}}}{\rightleftarrows}} \mathsf{L}^2 \underset{\iota_{\mathbb{RM}}}{\overset{\pi_{\mathbb{RM}}}{\rightleftarrows}} \mathbb{RM}$$

PDEs

$$\partial_t^n - \underbrace{\mathrm{Div}_{\mathbb{S}}\,\mathring{\mathrm{symGrad}}}_{=\vec{\Delta}_{\mathbb{S},\mathrm{D}}}, \qquad \partial_t^n - \underbrace{\mathring{\mathrm{Div}}_{\mathbb{S}}\,\mathrm{symGrad}}_{=\vec{\Delta}_{\mathbb{S},\mathrm{N}}}, \qquad \partial_t^n + \mathrm{RotRot}_{\mathbb{S}}^{\top}\,\mathring{\mathrm{RotRot}}_{\mathbb{S}}^{\top},$$

$$\partial_t^n + \underbrace{\mathrm{RotRot}_{\mathbb{S}}^{\top}\,\mathring{\mathrm{RotRot}}_{\mathbb{S}}^{\top}}_{\text{4th order}} - \underbrace{\mathring{\mathrm{symGrad}}\,\mathrm{Div}_{\mathbb{S}}}_{\text{2nd order}}$$

(apparently mixed order type, but NOT compact perturbation!)

elliptic ($n = 0$) / parabolic ($n = 1$) / hyperbolic ($n = 2$)

PDEs: 1st and 2nd biharmonic / general relativity complexes 3D

Gradgrad-complex

$$\{0\} \underset{\pi_0}{\overset{\iota_0}{\rightleftarrows}} \mathsf{L}^2 \underset{\operatorname{divDiv}_{\mathbb{S}}}{\overset{\mathring{\operatorname{Gradgrad}}}{\rightleftarrows}} \mathsf{L}^2_{\mathbb{S}} \underset{\operatorname{symRot}_{\mathbb{T}}}{\overset{\mathring{\operatorname{Rot}}_{\mathbb{S}}}{\rightleftarrows}} \mathsf{L}^2_{\mathbb{T}} \underset{-\operatorname{devGrad}}{\overset{\mathring{\operatorname{Div}}_{\mathbb{T}}}{\rightleftarrows}} \mathsf{L}^2 \underset{\iota_{\mathbb{RT}}}{\overset{\pi_{\mathbb{RT}}}{\rightleftarrows}} \mathbb{RT}$$

devGrad-complex

$$\{0\} \underset{\pi_0}{\overset{\iota_0}{\rightleftarrows}} \mathsf{L}^2 \underset{-\operatorname{Div}_{\mathbb{T}}}{\overset{\mathring{\operatorname{devGrad}}}{\rightleftarrows}} \mathsf{L}^2_{\mathbb{T}} \underset{\operatorname{Rot}_{\mathbb{S}}}{\overset{\mathring{\operatorname{symRot}}_{\mathbb{T}}}{\rightleftarrows}} \mathsf{L}^2_{\mathbb{S}} \underset{\operatorname{Gradgrad}}{\overset{\mathring{\operatorname{divDiv}}_{\mathbb{S}}}{\rightleftarrows}} \mathsf{L}^2 \underset{\iota_{\mathbb{P}_1}}{\overset{\pi_{\mathbb{P}_1}}{\rightleftarrows}} \mathbb{P}_1$$

PDEs

$$\partial_t^n + \underbrace{\operatorname{divDiv}_{\mathbb{S}} \mathring{\operatorname{Gradgrad}}}_{=\Delta^2_{\mathsf{D}}}, \qquad \partial_t^n - \underbrace{\mathring{\operatorname{Div}}_{\mathbb{T}} \operatorname{devGrad}}_{=\vec{\Delta}_{\mathbb{T},\mathsf{N}}}, \qquad \partial_t^n + \operatorname{Rot}_{\mathbb{S}} \mathring{\operatorname{symRot}}_{\mathbb{T}},$$

$$\partial_t^n + \underbrace{\operatorname{symRot}_{\mathbb{T}} \mathring{\operatorname{Rot}}_{\mathbb{S}}}_{\text{2nd order}} + \underbrace{\mathring{\operatorname{Gradgrad}} \operatorname{divDiv}_{\mathbb{S}}}_{\text{4th order}}$$

(apparently mixed order type, but NOT compact perturbation!)

elliptic ($n = 0$) / parabolic ($n = 1$) / hyperbolic ($n = 2$)

BGG

whole zoo more ...

BGG

Eastwood, Arnold, Falk, Winther, ...

also important for FEM, DEC, ...
(construction and analysis)

General Complex / $\rightsquigarrow$ FA-ToolBox

densely defined and closed (unbounded) linear operators

$$\mathrm{A}_0 : D(\mathrm{A}_0) \subset \mathrm{H}_0 \to \mathrm{H}_1, \qquad \mathrm{A}_1 : D(\mathrm{A}_1) \subset \mathrm{H}_1 \to \mathrm{H}_2$$
$$\mathrm{A}_0^* : D(\mathrm{A}_0^*) \subset \mathrm{H}_1 \to \mathrm{H}_0, \qquad \mathrm{A}_1^* : D(\mathrm{A}_1^*) \subset \mathrm{H}_2 \to \mathrm{H}_1$$

general complex property $\boxed{\mathrm{A}_1\mathrm{A}_0 = 0}$, i.e., $R(\mathrm{A}_0) \subset N(\mathrm{A}_1)$ $\quad (\Leftrightarrow \; R(\mathrm{A}_1^*) \subset N(\mathrm{A}_0^*))$

Hilbert complex

$$N(\mathrm{A}_0) \underset{\pi_{N(\mathrm{A}_0)}}{\overset{\iota_{N(\mathrm{A}_0)}}{\rightleftarrows}} \cdots \; \mathrm{H}_0 \underset{\mathrm{A}_0^*}{\overset{\mathrm{A}_0}{\rightleftarrows}} \mathrm{H}_1 \underset{\mathrm{A}_1^*}{\overset{\mathrm{A}_1}{\rightleftarrows}} \mathrm{H}_2 \; \cdots \underset{\iota_{N(\mathrm{A}_n^*)}}{\overset{\pi_{N(\mathrm{A}_n^*)}}{\rightleftarrows}} N(\mathrm{A}_n^*)$$

General Complex $\rightsquigarrow$ FA-ToolBox

Hilbert complex

$$N(\mathrm{A}_0) \underset{\pi_{N(\mathrm{A}_0)}}{\overset{\iota_{N(\mathrm{A}_0)}}{\rightleftarrows}} \cdots \quad \mathrm{H}_0 \underset{\mathrm{A}_0^*}{\overset{\mathrm{A}_0}{\rightleftarrows}} \mathrm{H}_1 \underset{\mathrm{A}_1^*}{\overset{\mathrm{A}_1}{\rightleftarrows}} \mathrm{H}_2 \quad \cdots \underset{\iota_{N(\mathrm{A}_n^*)}}{\overset{\pi_{N(\mathrm{A}_n^*)}}{\rightleftarrows}} N(\mathrm{A}_n^*)$$

some equations

$$\partial_t^n + \mathrm{A}_0^*\mathrm{A}_0, \qquad \partial_t^n + \mathrm{A}_1\mathrm{A}_1^*, \qquad \partial_t^n + \mathrm{A}_1^*\mathrm{A}_1 + \mathrm{A}_0\mathrm{A}_0^*, \qquad \partial_t^m - \begin{bmatrix} 0 & -\mathrm{A}_1^* \\ \mathrm{A}_1 & 0 \end{bmatrix}, \qquad \begin{bmatrix} \partial_t^m & \mathrm{A}_1^* \\ -\mathrm{A}_1 & \partial_t^\ell \end{bmatrix}$$

elliptic ($n = 0$) / parabolic ($n = 1$) / hyperbolic ($n = 2$) $\qquad m, \ell \in \{0, 1\}$

ell ($m = \ell = 0$) / para ($m = 1, \ell = 0$ or $m = 0, \ell = 1$) / hyper ($m = \ell = 1$)

Getting in Touch with Walter and Ulrich

Hilbert Complexes and PDEs

Getting in Touch with Walter and Ulrich

Getting in Touch with Walter and Ulrich

August 2011	1st contact Ulrich	AANMPDE 4, Euler MI, St. Petersburg
October 2012	1st contact Walter	DK Statusseminar, Strobl
April 2013	1st longer discussion Walter	Korn's inequalities
November 2014	2nd longer discussion Walter	duals of $H_0(\text{div})$ and $H_0(\text{rot})$

note: known in 2D for simply connected domains

$$H_0(\text{div})' = H^{-1}(\text{rot}) \qquad \text{eq} \qquad H_0(\text{rot})' = H^{-1}(\text{div})$$

- What if the domain is not simply connected?
- What about ND?
- What about duals of H(div) and H(rot)?

$\Rightarrow$ 2014: starting point of long, fruitful, and wonderful cooperation with Walter

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2016	2nd project with Walter	kernel of divDiv

$$N(\text{divDiv}_{\mathbb{S}}) = R(\text{symRot}_{\mathbb{T}})$$

more precisely, if Ω topologically trivial (simply connected and connected boundary)
$S \in \mathsf{L}^2(\Omega, \mathbb{R}^{3\times 3})$ symmetric with $\text{divDiv}\, S = 0$
$\Rightarrow \quad \exists\ T\ \mathsf{L}^2(\Omega, \mathbb{R}^{3\times 3})$ trace-free with $\text{symRot}\, T = S$

$$N(\text{symRot}_{\mathbb{T}}) = R(\text{devGrad})$$

more precisely, if Ω topologically trivial (simply connected and connected boundary)
$T \in \mathsf{L}^2(\Omega, \mathbb{R}^{3\times 3})$ trace-free with $\text{symRot}\, T = 0$
$\Rightarrow \quad \exists\ v\ \mathsf{L}^2(\Omega, \mathbb{R}^{33})$ with $\text{devGrad}\, v = T$

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November 2014	1st project with Walter	duals of $H_0(\text{div})$ and $H_0(\text{rot})$
2016	2nd project with Walter	kernel of divDiv
2016	3rd project with Walter	1st biharmonic complex

$$\{0\} \underset{\pi_0}{\overset{\iota_0}{\rightleftarrows}} \mathsf{L}^2 \underset{\operatorname{divDiv}_{\mathbb{S}}}{\overset{\mathring{\operatorname{Gradgrad}}}{\rightleftarrows}} \mathsf{L}^2_{\mathbb{S}} \underset{\operatorname{symRot}_{\mathbb{T}}}{\overset{\mathring{\operatorname{Rot}}_{\mathbb{S}}}{\rightleftarrows}} \mathsf{L}^2_{\mathbb{T}} \underset{-\operatorname{devGrad}}{\overset{\mathring{\operatorname{Div}}_{\mathbb{T}}}{\rightleftarrows}} \mathsf{L}^2 \underset{\iota_{\mathbb{RT}}}{\overset{\pi_{\mathbb{RT}}}{\rightleftarrows}} \mathbb{RT}$$

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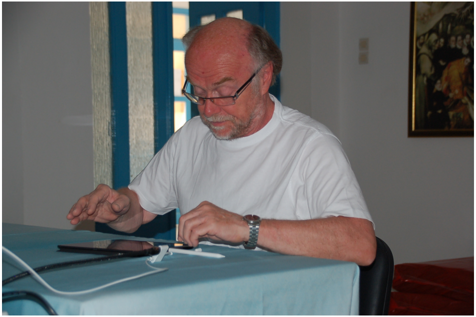
November 2014	1st project with Walter	duals of $H_0(\text{div})$ and $H_0(\text{rot})$
2016	2nd project with Walter	kernel of divDiv
2016	3rd project with Walter	biharmonic complex
2018	4th project with Walter	elasticity complex

$$\{0\} \underset{\pi_0}{\overset{\iota_0}{\rightleftarrows}} \mathsf{L}^2 \underset{-\operatorname{Div}_{\mathbb{S}}}{\overset{\mathring{\operatorname{symGrad}}}{\rightleftarrows}} \mathsf{L}^2_{\mathbb{S}} \underset{\operatorname{RotRot}^\top_{\mathbb{S}}}{\overset{\mathring{\operatorname{RotRot}}^\top_{\mathbb{S}}}{\rightleftarrows}} \mathsf{L}^2_{\mathbb{S}} \underset{-\operatorname{symGrad}}{\overset{\mathring{\operatorname{Div}}_{\mathbb{S}}}{\rightleftarrows}} \mathsf{L}^2 \underset{\iota_{\mathbb{RM}}}{\overset{\pi_{\mathbb{RM}}}{\rightleftarrows}} \mathbb{RM}$$

Hilbert Complexes and PDEs

Some Results from the Joint Work with Walter



mini FA-ToolBox

Theorem (compact complexes)

All the latter Hilbert complexes are compact, i.e., for all n

$$D(\mathrm{A}_n) \cap D(\mathrm{A}_{n-1}^*) \hookrightarrow \mathrm{H}_n \text{ compact.}$$

Corollary (closed complexes)

All the latter Hilbert complexes are closed, i.e., for all n

$$R(\mathrm{A}_n),\ R(\mathrm{A}_n^*) \text{ closed.}$$

Corollary (Friedrichs/Poincaré estimates)

All the latter operators admit Friedrichs/Poincaré type estimates.

mini FA-ToolBox

Corollary (compact resolvents)

All the latter corresponding inverses (of the reduced operators) are compact, i.e., for all n

$$\mathcal{A}_n^{-1},\ (\mathcal{A}_n^*)^{-1} \text{ compact.}$$

Corollary (spectra)

All the latter operators have discrete point spectra with finite eigenspaces, i.e., for all n

$$\sigma\Big(\begin{bmatrix} 0 & \mathrm{A}_n^* \\ \mathrm{A}_n & 0 \end{bmatrix}\Big) \setminus \{0\} = \pm\sqrt{\sigma(\mathrm{A}_n^*\mathrm{A}_n)} \setminus \{0\} = \pm\sqrt{\sigma(\mathrm{A}_n\mathrm{A}_n^*)} \setminus \{0\}$$

and $\sigma(\mathrm{A}_n^*\mathrm{A}_n) \setminus \{0\} = \{0 < \lambda_1^2 < \lambda_2^2 < \cdots < \lambda_\ell^2 \to \infty\}$ *and* λ_ℓ^2 *finite multiplicity.*

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Corollary (spectral theorems)

All the latter operators admit a spectral representation, i.e., for all n there exist orthonormal bases (ξ_n) *and* (ζ_n) *with, e.g.,*

$$x = \sum_\ell x_\ell \xi_\ell, \qquad A_n x = \sum_\ell \lambda_\ell x_\ell \zeta_\ell, \qquad A_n^* A_n x = \sum_\ell \lambda_\ell^2 x_\ell \xi_\ell,$$

$$y = \sum_\ell y_\ell \zeta_\ell, \qquad A_n^* y = \sum_\ell \lambda_\ell y_\ell \xi_\ell, \qquad A_n A_n^* y = \sum_\ell \lambda_\ell^2 y_\ell \zeta_\ell.$$

Corollary (Friedrichs/Poincaré estimates)

Friedrichs/Poincaré estimates type for higher eigenspaces.

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key ingredients

Lemma (bounded regular decompositions)

All the latter Hilbert complexes admit for all n bounded regular decompositions

$$D(\mathrm{A}_n) = \mathrm{H}_n^+ + \mathrm{A}_{n-1}\mathrm{H}_{n-1}^+.$$

Corollary (bounded regular potentials)

All the latter Hilbert complexes admit for all n bounded regular potentials

$$R(\mathrm{A}_n) = \mathrm{A}_n\mathrm{H}_n^+.$$

Corollary (bounded regular potential operators)

All the latter Hilbert complexes admit for all n bounded lin regular potential operators

$$P_{\mathrm{A}_n} : R(\mathrm{A}_n) \to \mathrm{H}_n^+ \quad \textit{with} \quad \mathrm{A}_n P_{\mathrm{A}_n} = \mathrm{id}_{R(\mathrm{A}_n)}.$$

mini FA-ToolBox

application: characterisation of duals

Theorem (characterisation of dual spaces)

Define regular dual spaces $\mathsf{H}_n^- := (\mathsf{H}_n^+)'$. *Then:*

regular decomposition	*dual space*
$D(\mathsf{A}_1) = \mathsf{H}_1^+ + \mathsf{A}_0 \mathsf{H}_0^+$	$D(\mathsf{A}_1)' = \{\Phi \in \mathsf{H}_1^- : \mathsf{A}_0' \Phi \in \mathsf{H}_0^-\}$
$D(\mathsf{A}_0^*) = \mathsf{H}_1^+ + \mathsf{A}_1^* \mathsf{H}_2^+$	$D(\mathsf{A}_1^*)' = \{\Phi \in \mathsf{H}_1^- : (\mathsf{A}_1^*)' \Phi \in \mathsf{H}_2^-\}$

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application: Dirichlet-Neumann fields

Theorem (Dirichlet-Neumann fields / cohomology groups)

The cohomology groups (generalised Dirichlet-Neumann fields) are independent of *metric* *and* *Sobolev order* *and admit a* *duality*. *In particular, they possess finite* C^∞*-smooth per-bases (bases after projection).*

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application: biharmonic split

biharmonic equation equivalent to 3 elliptic 2nd order problems

$$\Delta_{\mathrm{D}}^2 u = f \qquad \Leftrightarrow \qquad \operatorname{divDiv}_{\mathbb{S}} \operatorname{Gra\mathring{d}grad} u = f$$

$$\Leftrightarrow \qquad \begin{aligned} p &= \Delta_{\mathrm{D}}^{-1} f, \\ E &= (\mathring{\operatorname{Rot}}_{\mathbb{S}} \operatorname{symRot}_{\mathbb{T}})^{-1}_{\operatorname{Div}_{\mathbb{T}}=0} \operatorname{spn} \operatorname{grad} p, \\ u &= \Delta_{\mathrm{D}}^{-1}(3p + \operatorname{tr} \operatorname{symRot}_{\mathbb{T}} E) \end{aligned}$$

FE for $\operatorname{symRot}_{\mathbb{T}}$ needed!

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some recent literature

• DP and Walter Zulehner: Applicable Analysis 2020
The divDiv-Complex and Applications to Biharmonic Equations

• DP and Walter Zulehner: Applicable Analysis 2022
The Elasticity Complex: Compact Embeddings and Regular Decompositions

• DP and Michael Schomburg: Mathematical Methods in the Applied Sciences 2022
Hilbert Complexes with Mixed Boundary Conditions - Part 1: De Rham Complex

• DP and Michael Schomburg: Mathematical Methods in the Applied Sciences 2022
Hilbert Complexes with Mixed Boundary Conditions - Part 2: Elasticity Complex

• DP and Marcus Waurick: Mathematische Zeitschrift 2022
*The Index of some Mixed Order Dirac-Type Operators
and Generalised Dirichlet-Neumann Tensor Fields*

• Ralf Hiptmair, DP, and Erick Schulz: submitted 2022
Traces for Hilbert Complexes

. . . working on our complexes . . .

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Hilbert Complexes and PDEs

Appendix

more and more detailed results

mini FA-ToolBox

Theorem (mini FAT)

Let $D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) \hookrightarrow \mathrm{H}_1$ *be compact. Then:*

- **(i)** $R(\mathrm{A}_n) = R(\mathcal{A}_n)$, $R(\mathrm{A}_n^*) = R(\mathcal{A}_n^*)$, $n = 0, 1$, *are closed.* *(ranges closed)*
- **(ii)** $\mathcal{A}_n^{-1}$, $(\mathcal{A}_n^*)^{-1}$, $n = 0, 1$, *are compact.* *(inverse operators compact)*
- **(iii)** $\dim \big(N(\mathrm{A}_1) \cap N(\mathrm{A}_0^*)\big) < \infty$ *(cohomology group finite dim)*
- **(iv)** $\mathrm{H}_1 = R(\mathrm{A}_0) \oplus_{\mathrm{H}_1} N_{0,1} \oplus_{\mathrm{H}_1} R(\mathrm{A}_1^*)$ *(orthogonal Helmholtz-type deco)*
- **(v)** *For* $n = 0, 1$*:* $\exists\, c_n > 0$ *such that* *(Friedrichs/Poincaré-type est)*

$$\forall\, x \in D(\mathcal{A}_n) = D(\mathrm{A}_n) \cap N(\mathrm{A}_n)^{\perp_{\mathrm{H}_0}} = D(\mathrm{A}_n) \cap R(\mathrm{A}_n^*) \qquad |x|_{\mathrm{H}_n} \leq c_n |\mathrm{A}_n x|_{\mathrm{H}_{n+1}},$$

$$\forall\, y \in D(\mathcal{A}_n^*) = D(\mathrm{A}_n^*) \cap N(\mathrm{A}_n^*)^{\perp_{\mathrm{H}_{n+1}}} = D(\mathrm{A}_n^*) \cap R(\mathrm{A}_n) \qquad |y|_{\mathrm{H}_{n+1}} \leq c_n |\mathrm{A}_n^* y|_{\mathrm{H}_n}.$$

- **(v')** $\forall\, y \in D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) \cap N_{0,1}^{\perp_{\mathrm{H}_1}} \qquad |y|_{\mathrm{H}_1}^2 \leq c_0^2 |\mathrm{A}_0^* y|_{\mathrm{H}_0}^2 + c_1^2 |\mathrm{A}_1 y|_{\mathrm{H}_2}^2$

think of $\mathrm{A}_0 = \mathrm{grad}_{\gamma_t}$, $\mathrm{A}_0^* = -\mathrm{div}_{\gamma_n}$, $\mathrm{A}_1 = \mathrm{rot}_{\gamma_t}$, $\mathrm{A}_1^* = \mathrm{rot}_{\gamma_n}$ and

$$D(\mathrm{rot}_{\gamma_t}) \cap D(\mathrm{div}_{\gamma_n}) \hookrightarrow \mathrm{L}^2 \quad \text{compact}$$

mini FA-ToolBox

one key ingredient

bounded regular decomposition $\quad D(\mathrm{A}_1) = \mathrm{H}_1^+ + \mathrm{A}_0\mathrm{H}_0^+$

Lemma (regular potentials, regular decompositions, and compact embeddings)

Let $\mathrm{H}_n^+ \hookrightarrow \mathrm{H}_n$, $n = 0, 1$, *be compact embeddings, and let* $D(\mathrm{A}_1) = \mathrm{H}_1^+ + \mathrm{A}_0\mathrm{H}_0^+$ *be a bounded regular decomposition. Then:*

- **(i)** $D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) \hookrightarrow \mathrm{H}_1$ *is compact.* *(crucial compact embedding)*
- **(ii)** $R(\mathrm{A}_1) = \mathrm{A}_1\mathrm{H}_1^+$ *(bd reg pot representation)*
- **(ii')** $\exists\, P_{\mathrm{A}_1} : R(\mathrm{A}_1) \to \mathrm{H}_1^+$ *with* $\mathrm{A}_1 P_{\mathrm{A}_1} = \mathrm{id}_{R(\mathrm{A}_1)}$ *(bd lin reg pot operator)*

think of $\quad \mathrm{A}_0 = \mathrm{grad}_{\gamma_t}, \quad \mathrm{A}_0^* = -\,\mathrm{div}_{\gamma_n}, \quad \mathrm{A}_1 = \mathrm{rot}_{\gamma_t} \quad$ and

$$D(\mathrm{rot}_{\gamma_t}) = \mathrm{H}^1_{\gamma_t} + \mathrm{grad}_{\gamma_t} \mathrm{H}^1_{\gamma_t}$$

another key ingredient

bounded regular potentials ,,$\Rightarrow$" bounded regular decompositions

mini FA-ToolBox

simple idea of solving equations holds true

$$\mathrm{A}_0 x = f \in R(\mathrm{A}_0) \quad + \text{ condition on kernel} \quad \Leftrightarrow \quad x = \mathcal{A}_0^{-1} f$$

$$\mathrm{A}_0^* \mathrm{A}_0 x = f \in R(\mathrm{A}_0^*) \quad + \text{ condition on kernel} \quad \Leftrightarrow \quad x = \mathcal{A}_0^{-1} (\mathcal{A}_0^*)^{-1} f$$

with $\mathcal{A}_0^{-1}$ and $(\mathcal{A}_0^*)^{-1}$ bounded

think of

$$\operatorname{rot}_{\gamma_t} E = F \quad \Leftrightarrow \quad E = \operatorname{rot}_{\gamma_t}^{-1} F$$

$$-\Delta_{\epsilon,\gamma_t} u = -\operatorname{div}_{\gamma_n} \epsilon \operatorname{grad}_{\gamma_t} u = f \quad \Leftrightarrow \quad u = -\operatorname{grad}_{\gamma_t}^{-1} (\operatorname{div}_{\gamma_n} \epsilon)^{-1} f = -\Delta_{\epsilon,\gamma_t}^{-1} f$$

$$\Rightarrow \quad \Delta_{\epsilon,\gamma_t}^{-1} = \operatorname{grad}_{\gamma_t}^{-1} (\operatorname{div}_{\gamma_n} \epsilon)^{-1} = \operatorname{grad}_{\gamma_t}^{-1} \epsilon^{-1} \operatorname{div}_{\gamma_n}^{-1}$$

mini FA-ToolBox

solving		think of
$\mathrm{A}_0^*\mathrm{A}_0 x = f \in R(\mathrm{A}_0^*)$,	$x = \mathcal{A}_0^{-1}(\mathcal{A}_0^*)^{-1} f$, $\lvert x\rvert_{\mathrm{H}_0} \leq c_0^2 \lvert f\rvert_{\mathrm{H}_0}$	$-\operatorname{div}_{\gamma_n} \epsilon \operatorname{grad}_{\gamma_t} u = f$,
$\mathrm{A}_1 x = f \in R(\mathrm{A}_1)$, $\mathrm{A}_0^* x = g \in R(\mathrm{A}_0^*)$, $\pi_{N_{0,1}} x = h \in N_{0,1}$,	$x = \mathcal{A}_1^{-1} f + (\mathcal{A}_0^*)^{-1} g + h$, $\lvert x\rvert_{\mathrm{H}_1} \leq c_1 \lvert f\rvert_{\mathrm{H}_2} + c_0 \lvert g\rvert_{\mathrm{H}_0} + \lvert h\rvert_{\mathrm{H}_1}$,	$\operatorname{rot}_{\gamma_t} E = F$, $-\operatorname{div}_{\gamma_n} \varepsilon E = g$, $\pi_{\mathrm{DN}} E = H$
$\mathrm{A}_1^*\mathrm{A}_1 x = f \in R(\mathrm{A}_1^*)$, $\mathrm{A}_0^* x = g \in R(\mathrm{A}_0^*)$, $\pi_{N_{0,1}} x = h \in N_{0,1}$,	$x = \mathcal{A}_1^{-1}(\mathcal{A}_1^*)^{-1} f + (\mathcal{A}_0^*)^{-1} g + h$, $\lvert x\rvert_{\mathrm{H}_1} \leq c_1^2 \lvert f\rvert_{\mathrm{H}_1} + c_0 \lvert g\rvert_{\mathrm{H}_0} + \lvert h\rvert_{\mathrm{H}_1}$,	$\varepsilon^{-1} \operatorname{rot}_{\gamma_n} \mu^{-1} \operatorname{rot}_{\gamma_t} E = F$, $-\operatorname{div}_{\gamma_n} \varepsilon E = g$, $\pi_{\mathrm{DN}} E = H$
$\mathrm{A}_1^*\mathrm{A}_1 x = f \in R(\mathrm{A}_1^*)$, $\mathrm{A}_0\mathrm{A}_0^* x = g \in R(\mathrm{A}_0)$, $\pi_{N_{0,1}} x = h \in N_{0,1}$,	$x = \mathcal{A}_1^{-1}(\mathcal{A}_1^*)^{-1} f + (\mathcal{A}_0^*)^{-1}\mathcal{A}_0^{-1} g + h$, $\lvert x\rvert_{\mathrm{H}_1} \leq c_1^2 \lvert f\rvert_{\mathrm{H}_1} + c_0^2 \lvert g\rvert_{\mathrm{H}_1} + \lvert h\rvert_{\mathrm{H}_1}$,	$\varepsilon^{-1} \operatorname{rot}_{\gamma_n} \mu^{-1} \operatorname{rot}_{\gamma_t} E = F$, $-\operatorname{grad}_{\gamma_t} \operatorname{div}_{\gamma_n} \varepsilon E = G$, $\pi_{\mathrm{DN}} E = H$

mini FA-ToolBox

regular dual spaces: $\boxed{\mathsf{H}_n^- := (\mathsf{H}_n^+)'}$

Lemma (characterisation of dual spaces)

regular decomposition	*dual space*
$D(\mathsf{A}_1) = \mathsf{H}_1^+ + \mathsf{A}_0 \mathsf{H}_0^+$	$D(\mathsf{A}_1)' = \{\Phi \in \mathsf{H}_1^- : \mathsf{A}_0' \Phi \in \mathsf{H}_0^-\}$
$D(\mathsf{A}_0^*) = \mathsf{H}_1^+ + \mathsf{A}_1^* \mathsf{H}_2^+$	$D(\mathsf{A}_1^*)' = \{\Phi \in \mathsf{H}_1^- : (\mathsf{A}_1^*)' \Phi \in \mathsf{H}_2^-\}$

with $\mathsf{H}^{-1}_{\gamma_n} := (\mathsf{H}^1_{\gamma_t})'$ think of

regular decomposition	characterisation of dual space
$D(\mathrm{rot}_{\gamma_t}) = \mathsf{H}^1_{\gamma_t} + \mathrm{grad}_{\gamma_t} \mathsf{H}^1_{\gamma_t}$	$D(\mathrm{rot}_{\gamma_t})' = \{\Phi \in \mathsf{H}^{-1}_{\gamma_n} : \mathrm{div}\, \Phi \in \mathsf{H}^{-1}_{\gamma_n}\}$
$D(\mathrm{div}_{\gamma_n}) = \mathsf{H}^1_{\gamma_n} + \mathrm{rot}_{\gamma_n} \mathsf{H}^1_{\gamma_n}$	$D(\mathrm{div}_{\gamma_n})' = \{\Phi \in \mathsf{H}^{-1}_{\gamma_t} : \mathrm{rot}\, \Phi \in \mathsf{H}^{-1}_{\gamma_t}\}$

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number of Dirichlet-Neumann fields

$$\mathcal{H}_{\epsilon,\gamma_t,\gamma_n} := N(\mathrm{rot}_{\gamma_t}) \cap N(\mathrm{div}_{\gamma_n} \epsilon)$$

is independent of Sobolev order, i.e.,

Lemma (Dirichlet-Neumann fields / cohomology groups)

For all Sobolev order k

$$\dim N(\mathrm{rot}^k_{\gamma_t})/R(\mathrm{grad}^k_{\gamma_t}) = \dim N(\mathrm{div}^k_{\gamma_n})/R(\mathrm{rot}^k_{\gamma_n}) = \dim \mathcal{H}_{\epsilon,\gamma_t,\gamma_n} < \infty.$$

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More precisely:

Lemma (Dirichlet-Neumann fields / cohomology groups)

There exist smooth pre-bases of Dirichlet-Neumann fields

$$\mathcal{B}_{\mathrm{rot},\gamma_t} \subset N(\mathrm{rot}^{\infty}_{\gamma_t}) = \mathsf{C}^{\infty}_{\gamma_t}(\overline{\Omega}) \cap N(\mathrm{rot}), \qquad \textit{(finite set)}$$

$$\mathcal{B}_{\mathrm{div},\gamma_n} \subset N(\mathrm{div}^{\infty}_{\gamma_n}) = \mathsf{C}^{\infty}_{\gamma_n}(\overline{\Omega}) \cap N(\mathrm{div}), \qquad \textit{(finite set)}$$

such that

$$\mathcal{H}_{\epsilon,\gamma_t,\gamma_n} = \mathrm{lin}\,\pi_{N(\mathrm{div}_{\gamma_n}\,\epsilon)}\mathcal{B}_{\mathrm{rot},\gamma_t} = \mathrm{lin}\,\pi_{N(\mathrm{rot}_{\gamma_t})}\mathcal{B}_{\mathrm{div},\gamma_n}. \qquad \textit{(bases)}$$

Corollary (Dirichlet-Neumann fields / cohomology groups)

For all Sobolev order k

$$N(\mathrm{rot}^{k}_{\gamma_t})/R(\mathrm{grad}^{k}_{\gamma_t}) \cong \mathrm{lin}\,\mathcal{B}_{\mathrm{rot},\gamma_t} \cong \mathcal{H}_{\epsilon,\gamma_t,\gamma_n} \cong \mathrm{lin}\,\mathcal{B}_{\mathrm{div},\gamma_n} \cong N(\mathrm{div}^{k}_{\gamma_n})/R(\mathrm{rot}^{k}_{\gamma_n})$$

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Lemma (biharmonic equation)

The biharmonic Dirichlet equation splits equivalently into 3 elliptic (positive) 2nd order problems.
More precisely: 2 Dirichlet Laplace problems and 1 saddle point problem for $\mathring{\operatorname{Rot}}_{\mathbb{S}} \operatorname{symRot}_{\mathbb{T}}$.

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biharmonic split

biharmonic equation

$$\Delta_{\mathrm{D}}^2 u = f \qquad \Leftrightarrow \qquad \operatorname{divDiv}_{\mathbb{S}} \operatorname{Gr\mathring{a}dgrad} u = f$$

variant of devGrad-complex

$$\mathbb{RT} \xrightarrow{\iota_{\mathbb{RT}}} \mathsf{L}^2 \xrightarrow{\operatorname{devGrad}} \mathsf{L}^2_{\mathbb{T}} \xrightarrow{\operatorname{symRot}_{\mathbb{T}}} \mathsf{L}^2_{\mathbb{S}} \xrightarrow{\operatorname{divDiv}_{\mathbb{S}}} \mathsf{H}^{-1} \xrightarrow{\pi_0} \{0\}$$

with regular type decomposition

$$D(\operatorname{divDiv}_{\mathbb{S}}^{0,-1}) = \underbrace{D(\mathring{\operatorname{grad}})}_{=\mathsf{H}^1_0} \operatorname{id} \dot{+} N(\operatorname{divDiv}_{\mathbb{S}})$$

solve sequentially

$$\Delta_{\mathrm{D}} p = f \qquad \Leftrightarrow \qquad p = \Delta_{\mathrm{D}}^{-1} f,$$

$$\mathring{\operatorname{Rot}}_{\mathbb{S}} \operatorname{symRot}_{\mathbb{T}} E = -\mathring{\operatorname{Rot}}_{\mathbb{S}}(p \operatorname{id}) = \operatorname{spn} \operatorname{grad} p =: P \qquad \Leftrightarrow \qquad E = (\mathring{\operatorname{Rot}}_{\mathbb{S}} \operatorname{symRot}_{\mathbb{T}})^{-1}_{\operatorname{Div}_{\mathbb{T}}=0} P,$$

$$\operatorname{Div}_{\mathbb{T}} E = 0$$

$$\Delta_{\mathrm{D}} u = 3p + \operatorname{tr} \operatorname{symRot}_{\mathbb{T}} E =: g \qquad \Leftrightarrow \qquad u = \Delta_{\mathrm{D}}^{-1} g$$

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biharmonic split

$$\Delta_{\mathrm{D}}^2 u = f \quad \Leftrightarrow \quad \begin{bmatrix} 3 & \operatorname{tr}\operatorname{symRot}_{\mathbb{T}} & -\Delta_{\mathrm{D}} \\ \mathring{\operatorname{Rot}}_{\mathbb{S}}(\,\cdot\,\mathrm{id}) & (\mathring{\operatorname{Rot}}_{\mathbb{S}}\operatorname{symRot}_{\mathbb{T}})_{\operatorname{Div}_{\mathbb{T}}=0} & 0 \\ -\Delta_{\mathrm{D}} & 0 & 0 \end{bmatrix} \begin{bmatrix} p \\ E \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -f \end{bmatrix}$$

$$\Leftrightarrow \quad \begin{bmatrix} 3 & \operatorname{tr}\operatorname{symRot}_{\mathbb{T}} & 0 & -\Delta_{\mathrm{D}} \\ \mathring{\operatorname{Rot}}_{\mathbb{S}}(\,\cdot\,\mathrm{id}) & \mathring{\operatorname{Rot}}_{\mathbb{S}}\operatorname{symRot}_{\mathbb{T}} & \operatorname{devGrad}_{\mathbb{RT}^{\perp}} & 0 \\ 0 & -\mathring{\operatorname{Div}}_{\mathbb{T}} & 0 & 0 \\ -\Delta_{\mathrm{D}} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} p \\ E \\ v \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -f \end{bmatrix}$$

$$\Leftrightarrow \quad \begin{bmatrix} 3 & \operatorname{tr}\operatorname{symRot}_{\mathbb{T}} & 0 & 0 & -\Delta_{\mathrm{D}} \\ \mathring{\operatorname{Rot}}_{\mathbb{S}}(\,\cdot\,\mathrm{id}) & \mathring{\operatorname{Rot}}_{\mathbb{S}}\operatorname{symRot}_{\mathbb{T}} & \operatorname{devGrad} & 0 & 0 \\ 0 & -\mathring{\operatorname{Div}}_{\mathbb{T}} & 0 & \iota_{\mathbb{RT}} & 0 \\ 0 & 0 & \pi_{\mathbb{RT}} & 0 & 0 \\ -\Delta_{\mathrm{D}} & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} p \\ E \\ v \\ r \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -f \end{bmatrix}$$

FE for $\operatorname{symRot}_{\mathbb{T}}$ needed!