ON KORN'S FIRST INEQUALITY FOR TANGENTIAL OR NORMAL BOUNDARY CONDITIONS WITH EXPLICIT CONSTANTS — OR — HOW ONE CAN NOT APPLY THE CLOSED GRAPH THEOREM

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Sebastian Bauer & Dirk Pauly
Universität Duisburg-Essen

UNIVERSITÄT DUISBURG ESSEN
Open-Minded :-)

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OVERVIEW

MATRICES

Let $A \in \mathbb{R}^{N\times N}$.

$$\begin{matrix}\text{sym}\\ \text{skw}\end{matrix} A := \frac{1}{2}(A \pm A^\top), \qquad \text{id}_A := \frac{\text{tr}\, A}{N}\,\text{id}, \qquad \text{tr}\, A := A \cdot \text{id}, \qquad \text{dev}\, A := A - \text{id}_A$$

(pointwise orthogonality) $\Rightarrow$

$$|A|^2 = |\,\text{dev}\, A|^2 + \frac{1}{N}|\,\text{tr}\, A|^2, \quad |A|^2 = |\,\text{sym}\, A|^2 + |\,\text{skw}\, A|^2, \quad |\,\text{sym}\, A|^2 = |\,\text{dev}\,\text{sym}\, A|^2 + \frac{1}{N}|\,\text{tr}\, A|^2$$

$$\Rightarrow \quad |\,\text{dev}\, A|, N^{-1/2}|\,\text{tr}\, A|, |\,\text{sym}\, A|, |\,\text{skw}\, A| \leq |A|$$

$A := \nabla v := J_v^\top$ for $v \in \mathsf{H}^1(\Omega)$ $\quad\Rightarrow\quad$ (pointwise)

$$|\,\text{skw}\, \nabla v|^2 = \frac{1}{2}|\,\text{rot}\, v|^2, \quad \text{tr}\, \nabla v = \text{div}\, v,$$

$$|\nabla v|^2 = |\,\text{dev}\,\text{sym}\, \nabla v|^2 + \frac{1}{N}|\,\text{div}\, v|^2 + \frac{1}{2}|\,\text{rot}\, v|^2 \tag{1}$$

Moreover

$$|\nabla v|^2 = |\,\text{rot}\, v|^2 + \langle \nabla v, (\nabla v)^\top \rangle \tag{2}$$

since

$$2|\,\text{skw}\, \nabla v|^2 = \frac{1}{2}|\nabla v - (\nabla v)^\top|^2 = |\nabla v|^2 - \langle \nabla v, (\nabla v)^\top \rangle.$$

KORN'S FIRST INEQUALITY: STANDARD BOUNDARY CONDITIONS

Lemma (Korn's first inequality: $\overset{\circ}{\mathrm{H}}{}^1$-version)

Let Ω be an open subset of $\mathbb{R}^N$ with $2 \leq N \in \mathbb{N}$. Then for all $v \in \overset{\circ}{\mathrm{H}}{}^1(\Omega)$

$$|\nabla v|^2_{\mathrm{L}^2(\Omega)} = 2|\operatorname{dev}\operatorname{sym}\nabla v|^2_{\mathrm{L}^2(\Omega)} + \frac{2-N}{N}|\operatorname{div} v|^2_{\mathrm{L}^2(\Omega)} \leq 2|\operatorname{dev}\operatorname{sym}\nabla v|^2_{\mathrm{L}^2(\Omega)}$$

and equality holds if and only if $\operatorname{div} v = 0$ *or* $N = 2$*.*

Proof.

note: $-\Delta = \operatorname{rot}^* \operatorname{rot} - \nabla \operatorname{div}$ (vector Laplacian)

$$\Rightarrow \quad \forall\, v \in \overset{\circ}{\mathrm{C}}{}^\infty(\Omega) \quad |\nabla v|^2_{\mathrm{L}^2(\Omega)} = |\operatorname{rot} v|^2_{\mathrm{L}^2(\Omega)} + |\operatorname{div} v|^2_{\mathrm{L}^2(\Omega)} \quad \text{(Gaffney's equality)} \quad (3)$$

(3) extends to all $v \in \overset{\circ}{\mathrm{H}}{}^1(\Omega)$ by continuity. By (1)

$$|\nabla v|^2_{\mathrm{L}^2(\Omega)} = |\operatorname{dev}\operatorname{sym}\nabla v|^2_{\mathrm{L}^2(\Omega)} + \frac{1}{2}|\nabla v|^2_{\mathrm{L}^2(\Omega)} + \frac{2-N}{2N}|\operatorname{div} v|^2_{\mathrm{L}^2(\Omega)}. \quad (4)$$

□

KORN'S FIRST INEQUALITY: TANGENTIAL/NORMAL BOUNDARY CONDITIONS

main result:

Theorem (Korn's first inequality: tangential/normal version)

Let $\Omega \subset \mathbb{R}^N$ *be piecewise* C^2*-concave and* $v \in \overset{\circ}{\mathsf{H}}{}^1_{\mathsf{t},\mathsf{n}}(\Omega)$*. Then Korn's first inequality*

$$|\nabla v|_{\mathsf{L}^2(\Omega)} \leq \sqrt{2}|\operatorname{dev}\operatorname{sym}\nabla v|_{\mathsf{L}^2(\Omega)}$$

holds. If Ω *is a polyhedron, even*

$$|\nabla v|^2_{\mathsf{L}^2(\Omega)} = 2|\operatorname{dev}\operatorname{sym}\nabla v|^2_{\mathsf{L}^2(\Omega)} + \frac{2-N}{N}|\operatorname{div} v|^2_{\mathsf{L}^2(\Omega)} \leq 2|\operatorname{dev}\operatorname{sym}\nabla v|^2_{\mathsf{L}^2(\Omega)}$$

is true and equality holds if and only if $\operatorname{div} v = 0$ *or* $N = 2$*.*

KORN'S FIRST INEQUALITY: TANGENTIAL/NORMAL BOUNDARY CONDITIONS

Tools:

Proposition (integration by parts (Grisvard's book and older...))

Let $\Omega \subset \mathbb{R}^N$ *be piecewise* C^2*. Then*

$$|\operatorname{div} v|^2_{\mathrm{L}^2(\Omega)} + |\operatorname{rot} v|^2_{\mathrm{L}^2(\Omega)} - |\nabla v|^2_{\mathrm{L}^2(\Omega)} = \int_{\Gamma_1} \big(\operatorname{div} \nu \, |v_{\mathrm{n}}|^2 + ((\nabla \nu)\, v_{\mathrm{t}}) \cdot v_{\mathrm{t}}\big) + \int_{\Gamma_1} (v_{\mathrm{n}} \operatorname{div}_\Gamma v_{\mathrm{t}} - v_{\mathrm{t}} \cdot \nabla_\Gamma v_{\mathrm{n}}),$$

$$|\operatorname{div} v|^2_{\mathrm{L}^2(\Omega)} + |\operatorname{rot} v|^2_{\mathrm{L}^2(\Omega)} - |\nabla v|^2_{\mathrm{L}^2(\Omega)} = \int_{\Gamma_1} \big(\operatorname{div} \nu \, |v_{\mathrm{n}}|^2 + ((\nabla \nu)\, v_{\mathrm{t}}) \cdot v_{\mathrm{t}}\big).$$

holds for all $v \in \mathring{\mathrm{C}}^\infty(\overline{\Omega})$ *resp.* $v \in \mathring{\mathrm{C}}^\infty_{\mathrm{t,n}}(\Omega)$*.*

Corollary (Gaffney's inequalities)

Let $\Omega \subset \mathbb{R}^N$ *be piecewise* C^2 *and* $v \in \mathring{\mathrm{H}}^1_{\mathrm{t,n}}(\Omega)$*. Then*

$$|\operatorname{rot} v|^2_{\mathrm{L}^2(\Omega)} + |\operatorname{div} v|^2_{\mathrm{L}^2(\Omega)} - |\nabla v|^2_{\mathrm{L}^2(\Omega)} \begin{cases} \leq 0 & \text{, if } \Omega \text{ is piecewise } \mathrm{C}^2\text{-concave,} \\ = 0 & \text{, if } \Omega \text{ is a polyhedron,} \\ \geq 0 & \text{, if } \Omega \text{ is piecewise } \mathrm{C}^2\text{-convex.} \end{cases}$$

KORN'S FIRST INEQUALITY: TANGENTIAL/NORMAL BOUNDARY CONDITIONS

Proof.
(1) and the corollary $\Rightarrow$

$$|\nabla v|^2_{L^2(\Omega)} \leq |\operatorname{dev}\operatorname{sym}\nabla v|^2_{L^2(\Omega)} + \frac{1}{2}|\nabla v|^2_{L^2(\Omega)} + \frac{2-N}{2N}|\operatorname{div} v|^2_{L^2(\Omega)},$$

$\Rightarrow$ first estimate

Ω polyhedron $\Rightarrow$ equality holds □

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  On the trend to global equilibrium for spatially inhomogeneous kinetic systems: the Boltzmann equation

CITATIONS

- Desvillettes, L. and Villani, C.: ESAIM Control Optim. Calc. Var., (2002)
  On a variant of Korn's inequality arising in statistical mechanics.
  A tribute to J.L. Lions.
    - page 607
    - page 608
    - page 609
    - Proposition 5
    - (end of) Theorem 3 (continued)
    - page 609 (closed graph theorem)

- Desvillettes, L. and Villani, C.: Invent. Math., (2005)
  On the trend to global equilibrium for spatially inhomogeneous kinetic systems:
  the Boltzmann equation
    - page 306