Poincaré meets Korn via Maxwell: Extending Korn's First Inequality to H(Curl)-Tensor Fields and Applications to Gradient Plasticity with Plastic Spin

Dirk Pauly

Universität Essen, Jyväskylän Yliopisto

joint work with

Patrizio Neff, **Karl-Josef Witsch**, Essen
Jan Valdman, Ostrava
Immanuel Anjam, Jyväskylä

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UNIVERSITÄT DUISBURG ESSEN

JYVÄSKYLÄN YLIOPISTO
University of Jyväskylä

Tiny Motivation: Gradient Plasticity / Special Micromorphic Model

NEW MODEL (gradient plasticity for finite deformations)
Neff ('06) for (non-symmetric!) plastic deformation (distortion) tensor (PDT) P:

let: u classical displacement, $G := \nabla u$ classical deformation, displacement gradient
plasticity: decomposition $G = E + P$ in elastic and plastic distortion
plastic spin: P non-symmetric

MINIMIZATION PROBLEM ❶ (formulation for the plastic distortion P)
Find PDT-field $\hat{P} : \Omega \subset \mathbb{R}^3 \to \mathbb{R}^{3\times 3}$ with $\tau\hat{P} = \tau G$ on $\Gamma_t(\text{open}, \neq \emptyset) \subset \Gamma := \partial\Omega$ minimizing

$$\min_P \tilde{\mathcal{E}}(P) = \tilde{\mathcal{E}}(\hat{P}),$$

where $\hat{P}, P \in \mathsf{H}(\operatorname{Curl};\Omega)$ and (free thermodynamic energy functional)

$$\tilde{\mathcal{E}}(P) := \underbrace{\mu\|\operatorname{sym}\underbrace{(G-P)}_{=E}\|^2_{\mathsf{L}^2(\Omega)}}_{\text{elastic energy}} + \underbrace{\mu\|\operatorname{sym} P\|^2_{\mathsf{L}^2(\Omega)}}_{\text{linear kinematic energy}} + \underbrace{\lambda\|\operatorname{Curl} P\|^2_{\mathsf{L}^2(\Omega)}}_{\text{dislocation energy density}} + \underbrace{\kappa\|\operatorname{tr} P\|^2_{\mathsf{L}^2(\Omega)}}_{\text{trace free!}}$$

with $\mu, \lambda > 0$, $\kappa \geq 0$.

$\Leftrightarrow E = G - P$:

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MINIMIZATION PROBLEM ❷ (formulation for the elastic distortion T)

Find tensor field $\hat{T} \in \overset{\circ}{\mathrm{H}}(\mathrm{Curl}; \Gamma_t, \Omega)$, i.e., $\tau \hat{T} = 0$ on Γ_t, minimizing

$$\min_T \mathcal{E}(T) = \mathcal{E}(\hat{T}),$$

where $\hat{T}, T \in \overset{\circ}{\mathrm{H}}(\mathrm{Curl}; \Gamma_t, \Omega)$ and (energy functional)

$$\mathcal{E}(T) := \mu \|\mathrm{sym}\, T\|^2_{\mathrm{L}^2(\Omega)} + \mu \|\mathrm{sym}(T - G)\|^2_{\mathrm{L}^2(\Omega)} + \lambda \|\mathrm{Curl}\, T\|^2_{\mathrm{L}^2(\Omega)} + \kappa \|\mathrm{tr}(T - G)\|^2_{\mathrm{L}^2(\Omega)}.$$

note: $T = \nabla v$ with $v \in \overset{\circ}{\mathrm{H}}{}^1(\Gamma_t; \Omega) \Rightarrow$ standard linear elasticity, i.e.,

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Tiny Motivation: Variational Principle

VARIATIONAL PROBLEM Find $\hat{T}$ in $\overset{\circ}{\mathrm{H}}(\mathrm{Curl};\Gamma_t,\Omega)$ such that

$$\forall\, T \in \overset{\circ}{\mathrm{H}}(\mathrm{Curl};\Gamma_t,\Omega) \qquad b(\hat{T},T) = f(T),$$

where

$$b(S,T) := 2\mu\,\langle \operatorname{sym} S, \operatorname{sym} T\rangle_{\mathrm{L}^2(\Omega)} + \lambda\,\langle \operatorname{Curl} S, \operatorname{Curl} T\rangle_{\mathrm{L}^2(\Omega)} + \kappa\,\langle \operatorname{tr} S, \operatorname{tr} T\rangle_{\mathrm{L}^2(\Omega)},$$

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STRONG PROBLEM Find $\hat{T} \in \overset{\circ}{\mathrm{H}}(\mathrm{Curl};\Gamma_t,\Omega)$ with $\operatorname{Curl}\hat{T} \in \overset{\circ}{\mathrm{H}}(\mathrm{Curl};\Gamma_n,\Omega)$ such that

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note: $\operatorname{sym}\hat{T}$ inst. of $\hat{T}$ strongly couples the system, otherwise 3 decoupled coercive Maxwell problems for each row of $\hat{T}$.

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for all $T \in \overset{\circ}{\mathrm{H}}(\mathrm{Curl};\Gamma_t,\Omega)$.

OPEN PROBLEMS ('06-'11) well defined problem?,
right Hilbert space (tangential trace)?, b coercive?, unique solution?

ANSWER (Xmas '10) new estimate $\Rightarrow$ unique solution by Lax-Milgram and ... $\surd$

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A New Look at Korn's First Inequality

- $\Omega \subset \mathbb{R}^N$ bounded domain with Lipschitz boundary $\Gamma := \partial\Omega$ (think of $N = 2, 3$)
- $\emptyset \neq \Gamma_t \subset \Gamma$ relatively open, separated from $\Gamma_n := \Gamma \setminus \overline{\Gamma_t}$ by Lipschitz curve
- tensor-$\mathrm{H}(\mathrm{Curl};\Omega)$ = row-wise-vector-$\mathrm{H}(\mathrm{curl};\Omega)$

note: $v \in \overset{\circ}{\mathrm{H}}{}^1(\Gamma_t;\Omega) \Rightarrow \nabla v \in \overset{\circ}{\mathrm{H}}(\mathrm{Curl}_0;\Gamma_t,\Omega)$, i.e., $v = 0$ on $\Gamma_t \Rightarrow \tau\nabla v = 0$ on Γ_t

Theorem (Korn's First Inequality, Tangential Neumann Version)

$$\exists\, c > 0 \quad \forall\, v \in \mathrm{H}^1(\Omega) \textit{ with } \nabla v \in \overset{\circ}{\mathrm{H}}(\mathrm{Curl}_0;\Gamma_t,\Omega)$$
$$\|\nabla v\|_{\mathrm{L}^2(\Omega)} \leq c\, \|\mathrm{sym}\, \nabla v\|_{\mathrm{L}^2(\Omega)}$$

$\Uparrow$ $(\mathrm{Curl}\, T = 0 \Leftrightarrow T = \nabla v$ if Ω simply connected$)$ $\Downarrow$

Theorem (Korn's First Inequality, Irrotational Version)

$$\exists\, c > 0 \quad \forall\, T \in \overset{\circ}{\mathrm{H}}(\mathrm{Curl}_0;\Gamma_t,\Omega) \qquad \|T\|_{\mathrm{L}^2(\Omega)} \leq c\, \|\mathrm{sym}\, T\|_{\mathrm{L}^2(\Omega)}$$

now: replace $\overset{\circ}{\mathrm{H}}(\mathrm{Curl}_0;\Gamma_t,\Omega)$ by $\overset{\circ}{\mathrm{H}}(\mathrm{Curl};\Gamma_t,\Omega)$, i.e., from irrotational $\mathrm{H}(\mathrm{Curl}_0;\Omega)$ to $\mathrm{H}(\mathrm{Curl};\Omega)$

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now: replace $\overset{\circ}{\mathrm{H}}(\mathrm{Curl}_0; \Gamma_t, \Omega)$ by $\overset{\circ}{\mathrm{H}}(\mathrm{Curl}; \Gamma_t, \Omega)$, i.e., from irrotational $\mathrm{H}(\mathrm{Curl}_0; \Omega)$ to $\mathrm{H}(\mathrm{Curl}; \Omega)$

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$$\exists\, c > 0 \quad \forall\, T \in \overset{\circ}{\mathrm{H}}(\mathrm{Curl};\Gamma_t,\Omega) \qquad c\,\|T\|^2_{\mathrm{L}^2(\Omega)} \leq |||T|||^2 = \|\operatorname{sym} T\|^2_{\mathrm{L}^2(\Omega)} + \|\operatorname{Curl} T\|^2_{\mathrm{L}^2(\Omega)}$$

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$|||\cdot|||$ is a norm on $\overset{\circ}{\mathrm{H}}(\mathrm{Curl};\Gamma_t,\Omega)$ equivalent to the $\|\cdot\|_{\mathrm{H}(\mathrm{Curl};\Omega)}$-norm, i.e.:

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JYVÄSKYLÄN YLIOPISTO University of Jyväskylä

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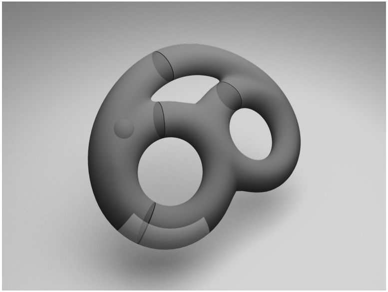
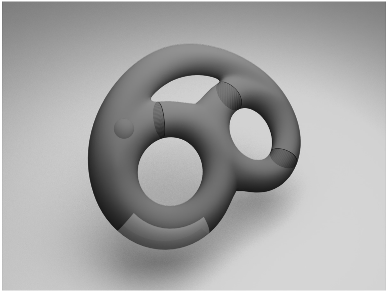
Sliceable Domains

Two ways to cut a sliceable domain:





(Thanks to Kostas Pamfilos for the pictures!)

Interesting Mathematical Consequences

The three fundamental inequalities are implied by two!

1. Maxwell
$|v| \leq c_{\mathrm{m}}(|\operatorname{curl} v| + |\operatorname{div} v|)$

2. Poincaré
$|u| \leq c_{\mathrm{p}} |\nabla u|$

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$|\nabla v| \leq c_{\mathrm{k}} |\operatorname{sym} \nabla v|$

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$|E| \leq c_{\mathrm{pf},q}(|\mathrm{d}\, E| + |\delta\, E|)$

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I. → 1.: $q=1$, v vector field, $\mathrm{d}=\operatorname{curl}$, $\delta=\operatorname{div}$

I. → 2.: $q=0$, u function, $\mathrm{d}=\nabla$, $\delta=0$

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$$c_{\mathrm{p}} = c_{\mathrm{pf},0}, \quad c_{\mathrm{m}} = c_{\mathrm{pf},1}, \quad c_{\mathrm{k}}, c_{\mathrm{p}} \leq c$$

q	0	1	2	3
d	grad $= \nabla$	curl $= \nabla\times$	div $= \nabla\cdot$	0
δ	0	div	$-$ curl	grad
$\overset{\circ}{\mathrm{D}}{}^q(\Gamma_t,\Omega)$	$\overset{\circ}{\mathrm{H}}(\mathrm{grad};\Gamma_t,\Omega) = \overset{\circ}{\mathrm{H}}{}^1(\Gamma_t;\Omega)$	$\overset{\circ}{\mathrm{H}}(\mathrm{curl};\Gamma_t,\Omega)$	$\overset{\circ}{\mathrm{H}}(\mathrm{div};\Gamma_t,\Omega)$	$\mathrm{L}^2(\Omega)$
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identification table for
q-forms and vector proxies in $\mathbb{R}^3$

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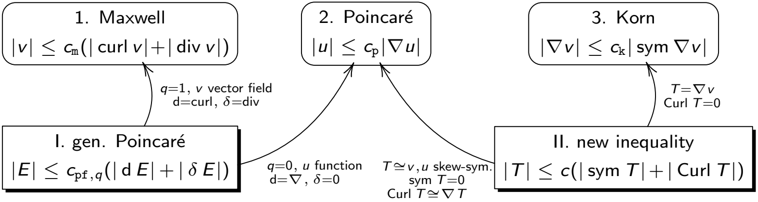

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Proof of Main Inequality: 3 Tools & 1 Trick

combination of techniques from

- electro-magnetic theory (static Maxwell equations with mixed bc)
- linear elasticity theory

three crucial tools:

(HD) Helmholtz' decomposition for tensor fields, i.e.,

$$\mathrm{L}^2(\Omega) = \overset{\circ}{\mathrm{H}}(\mathrm{Curl}_0; \Gamma_t, \Omega) \oplus \mathrm{Curl}\,\overset{\circ}{\mathrm{H}}(\mathrm{Curl}; \Gamma_n, \Omega)$$

(MI) the Maxwell inequality for solenoidal tensor fields, i.e.,

$$\|T\|_{\mathrm{L}^2(\Omega)} \leq c_{\mathrm{m}} \|\mathrm{Curl}\, T\|_{\mathrm{L}^2(\Omega)}$$

for all $T \in \overset{\circ}{\mathrm{H}}(\mathrm{Curl}; \Gamma_t, \Omega) \cap \mathrm{Curl}\,\overset{\circ}{\mathrm{H}}(\mathrm{Curl}; \Gamma_n, \Omega)$

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$$\|T\|_{\mathrm{L}^2(\Omega)} \leq c_{\mathrm{k}} \|\mathrm{sym}\, T\|_{\mathrm{L}^2(\Omega)}$$

and one trick:

(SD) sliceable domains to get KI

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(SD) sliceable domains to get KI

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combination of techniques from

- electro-magnetic theory (static Maxwell equations with mixed bc)
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three crucial tools:

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First Tool: HD - The Helmholtz Decomposition

generally:
H_n: Hilbert spaces, $A : D(A) \subset H_1 \to H_2$ linear, closed, densely defined operator
projection theorem $\Rightarrow$

$$H_1 = \underbrace{N(A)}_{\text{kernel}} \oplus \underbrace{\overline{R(A^*)}}_{\text{range}}, \qquad H_2 = \underbrace{N(A^*)}_{\text{kernel}} \oplus \underbrace{\overline{R(A)}}_{\text{range}}$$

choose: H_n as $\mathrm{L}^2(\Omega)$-spaces and

$$A := \mathrm{curl}, \qquad D(A) := \overset{\circ}{\mathrm{H}}(\mathrm{curl}; \Gamma_n, \Omega) \qquad R(A) = A\, D(A)$$

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Second Tool: MI - The Maxwell Inequality

generally:
H_n: Hilbert spaces, $A : D(A) \subset H_1 \to H_2$ linear, closed, densely defined operator with $D(A) \hookrightarrow H_1$ compact (graph norm)
standard indirect argument $\Rightarrow$

$$\exists\, c > 0 \quad \forall\, u \in D(A) \cap N(A)^{\perp} \qquad |u|_{H_1} \leq c|Au|_{H_2}$$

example:

H_n scalar/vector $\mathrm{L}^2(\Omega)$-spaces, $A = \nabla$, $D(A) = \mathrm{H}^1(\Omega)$ or $D(A) = \overset{\circ}{\mathrm{H}}{}^1(\Omega)$, compact embedding $D(A) \hookrightarrow \mathrm{L}^2(\Omega)$ (Rellich's selection theorem) $\Rightarrow$

$$\exists\, c_{\mathrm{p}} > 0 \quad \forall\, u \in D(A) \cap N(A)^{\perp} \quad \|u\|_{\mathrm{L}^2(\Omega)} \leq c_{\mathrm{p}} \|\nabla u\|_{\mathrm{L}^2(\Omega)} \quad \text{(Poincaré's inequality)}$$

Maxwell Inequality:
analogously, H_n (product) vector $\mathrm{L}^2(\Omega)$-spaces, $A = (\operatorname{curl}, \operatorname{div})$,
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Second Tool: MI - The Maxwell Inequality Continued

note: MCP $\Rightarrow$ $\dim N(A) < \infty$ for space of Dirichlet-Neumann fields

$$\begin{aligned} N(A) &= \overset{\circ}{\mathrm{H}}(\mathrm{curl}_0; \Gamma_t, \Omega) \cap \overset{\circ}{\mathrm{H}}(\mathrm{div}_0; \Gamma_n, \Omega) \\ &= \{E \in \overset{\circ}{\mathrm{H}}(\mathrm{curl}; \Gamma_t, \Omega) \cap \overset{\circ}{\mathrm{H}}(\mathrm{div}; \Gamma_n, \Omega) \ : \ \mathrm{curl}\, E = 0,\ \mathrm{div}\, E = 0\} \\ &= \{E \in \mathrm{L}^2(\Omega) \ : \ \mathrm{curl}\, E = 0,\ \mathrm{div}\, E = 0,\ \nu \times E|_{\Gamma_t} = 0,\ \nu \cdot E|_{\Gamma_n} = 0\} \quad \text{(classical)} \end{aligned}$$

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Third Tool & Trick: KI - The Generalized Korn's First Inequality

Korn's 1. Inequality: $\exists\, c_k > 0 \quad \forall\, v \in \overset{\circ}{\mathrm{H}}{}^1(\Gamma_t;\Omega)$

$$\|\nabla v\|_{\mathrm{L}^2(\Omega)} \leq c_k \, \|\operatorname{sym} \nabla v\|_{\mathrm{L}^2(\Omega)}$$

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extension to irrotational tensor fields: $\exists\, c_k > 0 \quad \forall\, T \in \overset{\circ}{\mathrm{H}}(\operatorname{Curl}_0;\Gamma_t,\Omega)$

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problem: generally $\nabla\overset{\circ}{\mathrm{H}}{}^1(\Gamma_t;\Omega) \subsetneq \overset{\circ}{\mathrm{H}}(\operatorname{Curl}_0;\Gamma_t,\Omega)$ if Ω not simply connected

solution: sliceable-trick

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problem: generally $\nabla \overset{\circ}{\mathrm{H}}{}^1(\Gamma_t;\Omega) \subsetneq \overset{\circ}{\mathrm{H}}(\mathrm{Curl}_0;\Gamma_t,\Omega)$ if Ω not simply connected

solution: sliceable-trick

Proof of Main Inequality

$T \in \overset{\circ}{\mathrm{H}}(\mathrm{Curl};\Gamma_t,\Omega)$

- HD $\Rightarrow$ $T = R + S \in \overset{\circ}{\mathrm{H}}(\mathrm{Curl}_0;\Gamma_t,\Omega) \oplus \big(\overset{\circ}{\mathrm{H}}(\mathrm{Curl};\Gamma_t,\Omega) \cap \mathrm{Curl}\,\overset{\circ}{\mathrm{H}}(\mathrm{Curl};\Gamma_n,\Omega)\big)$
  and $\mathrm{Curl}\, S = \mathrm{Curl}\, T$
- MI $\Rightarrow$ $\|S\|_{\mathrm{L}^2(\Omega)} \le c_{\mathrm{m}} \|\mathrm{Curl}\, T\|_{\mathrm{L}^2(\Omega)} \qquad (*)$
- KI and $(*)$ $\Rightarrow$

$$\|T\|^2_{\mathrm{L}^2(\Omega)} = \|R\|^2_{\mathrm{L}^2(\Omega)} + \|S\|^2_{\mathrm{L}^2(\Omega)}$$
$$\le c_{\mathrm{k}}^2 \|\mathrm{sym}\, R\|^2_{\mathrm{L}^2(\Omega)} + \|S\|^2_{\mathrm{L}^2(\Omega)} \le 2c_{\mathrm{k}}^2 \|\mathrm{sym}\, T\|^2_{\mathrm{L}^2(\Omega)} + (1+2c_{\mathrm{k}}^2)\|S\|^2_{\mathrm{L}^2(\Omega)}$$

- $\Rightarrow$ $\|T\|^2_{\mathrm{L}^2(\Omega)} \le c^2 \|\!|T|\!\|^2$ $\square$

note: $c = \max\{\sqrt{2}c_{\mathrm{k}}, c_{\mathrm{m}}\sqrt{1+2c_{\mathrm{k}}^2}\}$

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Almost Last Slide!

FIRST PAPERS

[1, 2] $\Omega \subset \mathbb{R}^3$, $\Gamma_t = \Gamma$ connected
[3] $\Omega \subset \mathbb{R}^N$, $\Gamma_t = \Gamma$ connected (differential forms, curl := d ...)
[4] $\Omega \subset \mathbb{R}^3$, $\Gamma_t \subset \Gamma$ (this talk!)
[5] $\Omega \subset \mathbb{R}^N$, $\Gamma_t \subset \Gamma$ (differential forms)

ONGOING WORK

exterior domains, non-homogeneous tangential traces, L^p, inhomogeneous media ...
(already done, needs to be LaTexed ...)

NUMERICS/COMPUTATIONS/SIMULATIONS

with Jan Valdman (Ostrava) and Immanuel Anjam (Jyväskylä)
new tensor-H(Curl)-FE with homogeneous restricted tangential traces

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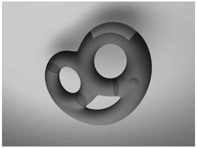

...now some numerics by Jan and Immanuel...

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