Functional A Posteriori Error Estimates for Static Maxwell Problems

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Introduction: Static Maxwell Problem

- $\Omega \subset \mathbb{R}^3$ bounded domain with Lipschitz boundary $\Gamma = \partial\Omega$
- $\varepsilon, \mu : \Omega \to \mathbb{R}^{3\times3}$ medium properties: bd., sym., unif. pos. def. matrices
- F given right hand side (current), G given boundary data (boundary current)
- E electric field, $H := \mu^{-1} \operatorname{curl} E$ magnetic field
- τ tangential trace, i.e., $\tau E = \nu \times E|_\Gamma$
- $\perp$ orthogonality w.r.t. $\mathsf{L}^2(\Omega)$-scalar product $\langle E, H\rangle_\Omega := \int_\Omega E \cdot H$
- $\mathcal{H}_\varepsilon(\Omega)$ Dirichlet fields; $H \in \mathcal{H}_\varepsilon(\Omega)$, iff $\operatorname{curl} H = 0$ and $\operatorname{div} \varepsilon H = 0$ and $\tau H = 0$

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- goal: error estimates for $e := E - \tilde{E}$ and $h := \mu H - \tilde{H}$, where $\tilde{E}, \tilde{H}$ approx. of $E, \mu H$
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Introduction: Sobolev Spaces

spaces:

$$\mathsf{H}(\mathrm{curl};\Omega) := \{E \in \mathsf{L}^2(\Omega) \, : \, \mathrm{curl}\, E \in \mathsf{L}^2(\Omega)\}$$

$$\mathsf{H}(\mathrm{curl}_0;\Omega) := \{E \in \mathsf{H}(\mathrm{curl};\Omega) \, : \, \mathrm{curl}\, E = 0\}$$

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Variational Formulation

testing $\operatorname{curl}\mu^{-1}\operatorname{curl}E = F$ with $\Phi \in \overset{\circ}{\mathrm{H}}(\operatorname{curl};\Omega)$

$$\varphi(\Phi) := \langle F, \Phi\rangle_\Omega = \langle \mu^{-1}\operatorname{curl}E, \operatorname{curl}\Phi\rangle_\Omega =: b(E,H)$$

unfortunately: $\overset{\circ}{\mathrm{H}}(\operatorname{curl};\Omega)$ is not the proper Hilbert space! (kernel of curl)

Poincaré-Friedrichs inequality: $\exists\, c_{\mathrm{PF}} > 0 \quad \forall\, E \in \mathrm{H}(\operatorname{curl};\Omega) \cap \varepsilon^{-1}\mathrm{H}(\operatorname{div};\Omega)$

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special case: $\forall\, E \in \mathbb{H} := \overset{\circ}{\mathrm{H}}(\operatorname{curl};\Omega) \cap \varepsilon^{-1}\mathrm{H}(\operatorname{div}_0;\Omega) \cap \mathcal{H}_\varepsilon(\Omega)^{\perp_\varepsilon}$

$$\|E\|_{\mathrm{L}^2(\Omega)} \leq c_{\mathrm{PF}} \, \|\operatorname{curl} E\|_{\mathrm{L}^2(\Omega)}$$

$\Rightarrow$ b bilinear, continuous and <u>coercive</u> over $\mathbb{H}$, φ linear and continuous over $\mathbb{H}$

Lax-Milgram $\Rightarrow$ unique solution $E \in \mathbb{H} + \check{\tau} G$ with proper tang. ext. operator $\check{\tau}$

key tool: compact embedding of $\overset{\circ}{\mathrm{H}}(\operatorname{curl};\Omega) \cap \mathrm{H}(\operatorname{div};\Omega)$ into $\mathrm{L}^2(\Omega)$

Upper and Lower Bounds for Non-Conforming Approximations

$\tilde{H} \in \mathsf{L}^2(\Omega)$ approximation of $\operatorname{curl} E = \mu H$ (first, only approximation of magnetic field)

Theorem 1 *For all $\tilde{H} \in \mathsf{L}^2(\Omega)$ the estimates*

$$\left\| \mu H - \tilde{H} \right\|_{\mathsf{L}^2(\Omega)} \leq \inf_{Y \in \mathsf{H}(\operatorname{curl};\Omega)} M_+(\tilde{H}; Y, \operatorname{curl} Y) + \inf_{\substack{X \in \mathsf{H}(\operatorname{curl};\Omega) \\ \tau X = G}} m_+(\tilde{H}; \operatorname{curl} X),$$

$$\left\| \mu H - \tilde{H} \right\|^2_{\mathsf{L}^2(\Omega)} \geq \sup_{X \in \overset{\circ}{\mathsf{H}}(\operatorname{curl};\Omega)} M_-(\tilde{H}; X, \operatorname{curl} X) + \sup_{\substack{\mu^{-1} Y, Z \in \mathsf{H}(\operatorname{curl};\Omega) \\ \operatorname{curl} \mu^{-1} Y = 0 \\ \tau Z = G}} m_-(\tilde{H}; Y, \operatorname{curl} Z)$$

hold. Here,

$$M_+(\tilde{H}; Y, \operatorname{curl} Y) := c_{\mu,1} \big(c_{\mathrm{PF}} \| F - \operatorname{curl} Y \|_{\mathsf{L}^2(\Omega)} + \left\| \mu^{-1} \tilde{H} - Y \right\|_{\mathsf{L}^2(\Omega)} \big),$$

$$m_+(\tilde{H}; \operatorname{curl} X) := c_{\mu,2} \left\| \operatorname{curl} X - \tilde{H} \right\|_{\mathsf{L}^2(\Omega)},$$

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Upper and Lower Bounds for (Very) Conforming Approximations

$\tilde{E} \in \mathrm{H}(\mathrm{curl};\Omega)$ approx. of E and $\tilde{H} := \mathrm{curl}\,\tilde{E} \in \mathrm{L}^2(\Omega)$ approx. of $\mathrm{curl}\,E = \mu H$

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Corollary 2 *For all $\tilde{E} \in \mathrm{H}(\mathrm{curl};\Omega)$ with $\tau\tilde{E} = G$, i.e., $E-\tilde{E} \in \overset{\circ}{\mathrm{H}}(\mathrm{curl};\Omega)$*

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Norm Estimates

- norm estimates for $h = \mu H - \tilde{H}$ $\quad \surd$
- $E \mapsto \|\operatorname{curl} E\|_{\mathrm{L}^2(\Omega)}$ semi-norm but not norm on $\mathrm{H}(\operatorname{curl};\Omega)$ or $\overset{\circ}{\mathrm{H}}(\operatorname{curl};\Omega)$ (not controlling $\|\operatorname{div} E\|_{\mathrm{L}^2(\Omega)}$ and projection on Dirichlet fields)
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  is well defined and a <u>norm</u> on $\overset{\circ}{\mathrm{H}}(\operatorname{curl};\Omega) \cap \varepsilon^{-1}\mathrm{H}(\operatorname{div};\Omega)$
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Constants and Sharpness

typical features of functional a posteriori error estimates

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Proofs (Helmholtz Decomposition)

main tools: standard techniques and Helmholtz decomposition

$\tilde{H}$ approximation of $\mu H = \operatorname{curl} E$, $h := \mu H - \tilde{H}$ error
simplicity $\varepsilon = \mu = \mathrm{id}$

$$\mathrm{L}^2(\Omega) \ni h = \operatorname{curl} E_c \oplus H_d$$

$$\operatorname{curl} E_c \in \operatorname{curl} \overset{\circ}{\mathrm{H}}(\operatorname{curl};\Omega) = \operatorname{curl} \mathbb{H}$$

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recall $\mathbb{H} = \overset{\circ}{\mathrm{H}}(\operatorname{curl};\Omega) \cap \mathrm{H}(\operatorname{div}_0;\Omega) \cap \mathcal{H}(\Omega)^{\perp}$

$\operatorname{curl} E_c$ regular/conforming error

H_d non-conforming error (boundary error)

orthogonality $\Rightarrow$ $\|h\|^2_{\mathrm{L}^2(\Omega)} = \|\operatorname{curl} E_c\|^2_{\mathrm{L}^2(\Omega)} + \|H_d\|^2_{\mathrm{L}^2(\Omega)}$

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$$H_{\mathrm{d}} \in \mathrm{H}(\operatorname{curl}_0; \Omega)$$

recall $\quad \mathbb{H} = \overset{\circ}{\mathrm{H}}(\operatorname{curl}; \Omega) \cap \mathrm{H}(\operatorname{div}_0; \Omega) \cap \mathcal{H}(\Omega)^{\perp}$

$\operatorname{curl} E_{\mathrm{c}}$ regular/conforming error

H_{d} non-conforming error (boundary error)

orthogonality $\Rightarrow \quad \|h\|^2_{\mathrm{L}^2(\Omega)} = \|\operatorname{curl} E_{\mathrm{c}}\|^2_{\mathrm{L}^2(\Omega)} + \|H_{\mathrm{d}}\|^2_{\mathrm{L}^2(\Omega)}$

Proofs (Helmholtz Decomposition)

main tools: standard techniques and Helmholtz decomposition

$\tilde{H}$ approximation of $\mu H = \operatorname{curl} E$, $h := \mu H - \tilde{H}$ error
simplicity $\varepsilon = \mu = \mathrm{id}$

$$\mathrm{L}^2(\Omega) \ni h = \operatorname{curl} E_{\mathrm{c}} \oplus H_{\mathrm{d}}$$

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$$\text{orthogonality} \quad \Rightarrow \quad \|h\|^2_{\mathsf{L}^2(\Omega)} = \|\operatorname{curl} E_{\mathsf{c}}\|^2_{\mathsf{L}^2(\Omega)} + \|H_{\mathsf{d}}\|^2_{\mathsf{L}^2(\Omega)}$$

Proofs (Upper Bounds)

recall $h = H - \tilde{H} = \operatorname{curl} E_c \oplus H_d \in \operatorname{curl} \mathbb{H} \oplus \mathsf{H}(\operatorname{curl}_0; \Omega)$

standard argument for $\operatorname{curl} E_c$: for all $\Phi \in \mathbb{H}$

$$\begin{aligned}\langle \operatorname{curl} E_c, \operatorname{curl} \Phi \rangle_{\mathsf{L}^2(\Omega)} &= \langle h, \operatorname{curl} \Phi \rangle_{\mathsf{L}^2(\Omega)} \quad (H_d \perp \operatorname{curl} \Phi) \\ &= \langle F, \Phi \rangle_{\mathsf{L}^2(\Omega)} - \left\langle \tilde{H}, \operatorname{curl} \Phi \right\rangle_{\mathsf{L}^2(\Omega)} \\ &= \langle F - \operatorname{curl} Y, \Phi \rangle_{\mathsf{L}^2(\Omega)} - \left\langle \tilde{H} - Y, \operatorname{curl} \Phi \right\rangle_{\mathsf{L}^2(\Omega)}\end{aligned}$$

since $\langle \operatorname{curl} Y, \Phi \rangle_{\mathsf{L}^2(\Omega)} = \langle Y, \operatorname{curl} \Phi \rangle_{\mathsf{L}^2(\Omega)}$ for all $Y \in \mathsf{H}(\operatorname{curl}; \Omega)$; note $\mathbb{H} \subset \overset{\circ}{\mathsf{H}}(\operatorname{curl}; \Omega)$.

Cauchy-Schwarz and Poincaré-Friedrichs and $\Phi := E_c \in \mathbb{H} \quad \Rightarrow$

$$\|\operatorname{curl} E_c\|_{\mathsf{L}^2(\Omega)} \leq c_{\mathrm{PF}} \|F - \operatorname{curl} Y\|_{\mathsf{L}^2(\Omega)} + \left\|\tilde{H} - Y\right\|_{\mathsf{L}^2(\Omega)} = M_+(\tilde{H}; Y, \operatorname{curl} Y)$$

Proofs (Upper Bounds)

recall $h = H - \tilde{H} = \operatorname{curl} E_{\mathrm{c}} \oplus H_{\mathrm{d}} \in \operatorname{curl} \mathbb{H} \oplus \mathrm{H}(\operatorname{curl}_0; \Omega)$

standard argument for $\operatorname{curl} E_{\mathrm{c}}$: for all $\Phi \in \mathbb{H}$

$$\begin{aligned}\langle \operatorname{curl} E_{\mathrm{c}}, \operatorname{curl} \Phi \rangle_{\mathrm{L}^2(\Omega)} &= \langle h, \operatorname{curl} \Phi \rangle_{\mathrm{L}^2(\Omega)} \quad (H_{\mathrm{d}} \perp \operatorname{curl} \Phi) \\ &= \langle F, \Phi \rangle_{\mathrm{L}^2(\Omega)} - \left\langle \tilde{H}, \operatorname{curl} \Phi \right\rangle_{\mathrm{L}^2(\Omega)} \\ &= \langle F - \operatorname{curl} Y, \Phi \rangle_{\mathrm{L}^2(\Omega)} - \left\langle \tilde{H} - Y, \operatorname{curl} \Phi \right\rangle_{\mathrm{L}^2(\Omega)}\end{aligned}$$

since $\langle \operatorname{curl} Y, \Phi \rangle_{\mathrm{L}^2(\Omega)} = \langle Y, \operatorname{curl} \Phi \rangle_{\mathrm{L}^2(\Omega)}$ for all $Y \in \mathrm{H}(\operatorname{curl}; \Omega)$; note $\mathbb{H} \subset \overset{\circ}{\mathrm{H}}(\operatorname{curl}; \Omega)$.

Cauchy-Schwarz and Poincaré-Friedrichs and $\Phi := E_{\mathrm{c}} \in \mathbb{H} \quad \Rightarrow$

$$\|\operatorname{curl} E_{\mathrm{c}}\|_{\mathrm{L}^2(\Omega)} \leq c_{\mathrm{PF}} \|F - \operatorname{curl} Y\|_{\mathrm{L}^2(\Omega)} + \left\| \tilde{H} - Y \right\|_{\mathrm{L}^2(\Omega)} = M_+(\tilde{H}; Y, \operatorname{curl} Y)$$

Proofs (Upper Bounds)

recall $h = H - \tilde{H} = \operatorname{curl} E_c \oplus H_d \in \operatorname{curl} \mathbb{H} \oplus \mathsf{H}(\operatorname{curl}_0; \Omega)$

standard argument for $\operatorname{curl} E_c$: for all $\Phi \in \mathbb{H}$

$$\begin{aligned}\langle \operatorname{curl} E_c, \operatorname{curl} \Phi \rangle_{\mathsf{L}^2(\Omega)} &= \langle h, \operatorname{curl} \Phi \rangle_{\mathsf{L}^2(\Omega)} \quad (H_d \perp \operatorname{curl} \Phi)\\ &= \langle F, \Phi \rangle_{\mathsf{L}^2(\Omega)} - \left\langle \tilde{H}, \operatorname{curl} \Phi \right\rangle_{\mathsf{L}^2(\Omega)}\\ &= \langle F - \operatorname{curl} Y, \Phi \rangle_{\mathsf{L}^2(\Omega)} - \left\langle \tilde{H} - Y, \operatorname{curl} \Phi \right\rangle_{\mathsf{L}^2(\Omega)}\end{aligned}$$

since $\langle \operatorname{curl} Y, \Phi \rangle_{\mathsf{L}^2(\Omega)} = \langle Y, \operatorname{curl} \Phi \rangle_{\mathsf{L}^2(\Omega)}$ for all $Y \in \mathsf{H}(\operatorname{curl}; \Omega)$; note $\mathbb{H} \subset \overset{\circ}{\mathsf{H}}(\operatorname{curl}; \Omega)$.

Cauchy-Schwarz and Poincaré-Friedrichs and $\Phi := E_c \in \mathbb{H} \quad \Rightarrow$

$$\|\operatorname{curl} E_c\|_{\mathsf{L}^2(\Omega)} \leq c_{PF} \|F - \operatorname{curl} Y\|_{\mathsf{L}^2(\Omega)} + \left\|\tilde{H} - Y\right\|_{\mathsf{L}^2(\Omega)} = M_+(\tilde{H}; Y, \operatorname{curl} Y)$$

Proofs (Upper Bounds)

recall $h = H - \tilde{H} = \operatorname{curl} E_{\mathrm{c}} \oplus H_{\mathrm{d}} \in \operatorname{curl} \mathbb{H} \oplus \mathsf{H}(\operatorname{curl}_0; \Omega)$

standard argument for curl E_{c}: for all $\Phi \in \mathbb{H}$

$$
\begin{aligned}
\langle \operatorname{curl} E_{\mathrm{c}}, \operatorname{curl} \Phi \rangle_{\mathsf{L}^2(\Omega)} &= \langle h, \operatorname{curl} \Phi \rangle_{\mathsf{L}^2(\Omega)} \quad (H_{\mathrm{d}} \perp \operatorname{curl} \Phi) \\
&= \langle F, \Phi \rangle_{\mathsf{L}^2(\Omega)} - \left\langle \tilde{H}, \operatorname{curl} \Phi \right\rangle_{\mathsf{L}^2(\Omega)} \\
&= \langle F - \operatorname{curl} Y, \Phi \rangle_{\mathsf{L}^2(\Omega)} - \left\langle \tilde{H} - Y, \operatorname{curl} \Phi \right\rangle_{\mathsf{L}^2(\Omega)}
\end{aligned}
$$

since $\langle \operatorname{curl} Y, \Phi \rangle_{\mathsf{L}^2(\Omega)} = \langle Y, \operatorname{curl} \Phi \rangle_{\mathsf{L}^2(\Omega)}$ for all $Y \in \mathsf{H}(\operatorname{curl}; \Omega)$; note $\mathbb{H} \subset \overset{\circ}{\mathsf{H}}(\operatorname{curl}; \Omega)$.

Cauchy-Schwarz and Poincaré-Friedrichs and $\Phi := E_{\mathrm{c}} \in \mathbb{H} \quad \Rightarrow$

$$
\| \operatorname{curl} E_{\mathrm{c}} \|_{\mathsf{L}^2(\Omega)} \leq c_{\mathrm{PF}} \| F - \operatorname{curl} Y \|_{\mathsf{L}^2(\Omega)} + \left\| \tilde{H} - Y \right\|_{\mathsf{L}^2(\Omega)} = M_+(\tilde{H}; Y, \operatorname{curl} Y)
$$

Proofs (Upper Bounds)

recall $h = H - \tilde{H} = \operatorname{curl} E_{\mathrm{c}} \oplus H_{\mathrm{d}} \in \operatorname{curl} \mathbb{H} \oplus \mathrm{H}(\operatorname{curl}_0; \Omega)$

standard argument for $\operatorname{curl} E_{\mathrm{c}}$: for all $\Phi \in \mathbb{H}$

$$\begin{aligned} \langle \operatorname{curl} E_{\mathrm{c}}, \operatorname{curl} \Phi \rangle_{\mathrm{L}^2(\Omega)} &= \langle h, \operatorname{curl} \Phi \rangle_{\mathrm{L}^2(\Omega)} \quad (H_{\mathrm{d}} \perp \operatorname{curl} \Phi) \\ &= \langle F, \Phi \rangle_{\mathrm{L}^2(\Omega)} - \left\langle \tilde{H}, \operatorname{curl} \Phi \right\rangle_{\mathrm{L}^2(\Omega)} \\ &= \langle F - \operatorname{curl} Y, \Phi \rangle_{\mathrm{L}^2(\Omega)} - \left\langle \tilde{H} - Y, \operatorname{curl} \Phi \right\rangle_{\mathrm{L}^2(\Omega)} \end{aligned}$$

since $\langle \operatorname{curl} Y, \Phi \rangle_{\mathrm{L}^2(\Omega)} = \langle Y, \operatorname{curl} \Phi \rangle_{\mathrm{L}^2(\Omega)}$ for all $Y \in \mathrm{H}(\operatorname{curl}; \Omega)$; note $\mathbb{H} \subset \overset{\circ}{\mathrm{H}}(\operatorname{curl}; \Omega)$.

Cauchy-Schwarz and Poincaré-Friedrichs and $\Phi := E_{\mathrm{c}} \in \mathbb{H} \quad \Rightarrow$

$$\|\operatorname{curl} E_{\mathrm{c}}\|_{\mathrm{L}^2(\Omega)} \leq c_{\mathrm{PF}} \|F - \operatorname{curl} Y\|_{\mathrm{L}^2(\Omega)} + \left\| \tilde{H} - Y \right\|_{\mathrm{L}^2(\Omega)} = M_+(\tilde{H}; Y, \operatorname{curl} Y)$$

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Proofs (Upper Bounds)

recall $h = H - \tilde{H} = \operatorname{curl} E_{c} \oplus H_{d} \in \operatorname{curl} \mathbb{H} \oplus \mathsf{H}(\operatorname{curl}_0; \Omega)$

standard argument for $\operatorname{curl} E_c$: for all $\Phi \in \mathbb{H}$

$$\begin{aligned}
\langle \operatorname{curl} E_c, \operatorname{curl} \Phi \rangle_{\mathsf{L}^2(\Omega)} &= \langle h, \operatorname{curl} \Phi \rangle_{\mathsf{L}^2(\Omega)} \quad (H_d \perp \operatorname{curl} \Phi) \\
&= \langle F, \Phi \rangle_{\mathsf{L}^2(\Omega)} - \left\langle \tilde{H}, \operatorname{curl} \Phi \right\rangle_{\mathsf{L}^2(\Omega)} \\
&= \langle F - \operatorname{curl} Y, \Phi \rangle_{\mathsf{L}^2(\Omega)} - \left\langle \tilde{H} - Y, \operatorname{curl} \Phi \right\rangle_{\mathsf{L}^2(\Omega)}
\end{aligned}$$

since $\langle \operatorname{curl} Y, \Phi \rangle_{\mathsf{L}^2(\Omega)} = \langle Y, \operatorname{curl} \Phi \rangle_{\mathsf{L}^2(\Omega)}$ for all $Y \in \mathsf{H}(\operatorname{curl}; \Omega)$; note $\mathbb{H} \subset \overset{\circ}{\mathsf{H}}(\operatorname{curl}; \Omega)$.

Cauchy-Schwarz and Poincaré-Friedrichs and $\Phi := E_c \in \mathbb{H} \quad \Rightarrow$

$$\|\operatorname{curl} E_c\|_{\mathsf{L}^2(\Omega)} \leq c_{\mathrm{PF}} \, \|F - \operatorname{curl} Y\|_{\mathsf{L}^2(\Omega)} + \left\| \tilde{H} - Y \right\|_{\mathsf{L}^2(\Omega)} = M_+(\tilde{H}; Y, \operatorname{curl} Y)$$

Proofs (Upper Bounds)

recall $h = H - \tilde{H} = \operatorname{curl} E_{\mathrm{c}} \oplus H_{\mathrm{d}} \in \operatorname{curl} \mathbb{H} \oplus \mathrm{H}(\operatorname{curl}_0; \Omega)$

standard argument for $\operatorname{curl} E_{\mathrm{c}}$: for all $\Phi \in \mathbb{H}$

$$\begin{aligned}
\langle \operatorname{curl} E_{\mathrm{c}}, \operatorname{curl} \Phi \rangle_{\mathrm{L}^2(\Omega)} &= \langle h, \operatorname{curl} \Phi \rangle_{\mathrm{L}^2(\Omega)} \quad (H_{\mathrm{d}} \perp \operatorname{curl} \Phi) \\
&= \langle F, \Phi \rangle_{\mathrm{L}^2(\Omega)} - \left\langle \tilde{H}, \operatorname{curl} \Phi \right\rangle_{\mathrm{L}^2(\Omega)} \\
&= \langle F - \operatorname{curl} Y, \Phi \rangle_{\mathrm{L}^2(\Omega)} - \left\langle \tilde{H} - Y, \operatorname{curl} \Phi \right\rangle_{\mathrm{L}^2(\Omega)}
\end{aligned}$$

since $\langle \operatorname{curl} Y, \Phi \rangle_{\mathrm{L}^2(\Omega)} = \langle Y, \operatorname{curl} \Phi \rangle_{\mathrm{L}^2(\Omega)}$ for all $Y \in \mathrm{H}(\operatorname{curl}; \Omega)$; note $\mathbb{H} \subset \mathring{\mathrm{H}}(\operatorname{curl}; \Omega)$.

Cauchy-Schwarz and Poincaré-Friedrichs and $\Phi := E_{\mathrm{c}} \in \mathbb{H} \quad \Rightarrow$

$$\|\operatorname{curl} E_{\mathrm{c}}\|_{\mathrm{L}^2(\Omega)} \leq c_{\mathrm{PF}} \, \|F - \operatorname{curl} Y\|_{\mathrm{L}^2(\Omega)} + \left\| \tilde{H} - Y \right\|_{\mathrm{L}^2(\Omega)} = M_+(\tilde{H}; Y, \operatorname{curl} Y)$$

Proofs (Upper Bounds Continued)

recall $h = H - \tilde{H} = \operatorname{curl} E_{\mathrm{c}} \oplus H_{\mathrm{d}} \in \operatorname{curl} \mathbb{H} \oplus \mathsf{H}(\operatorname{curl}_0;\Omega)$

argument for H_{d}: for all $\Psi \in \mathsf{H}(\operatorname{curl}_0;\Omega)$

$$\langle H_{\mathrm{d}}, \Psi\rangle_{\mathsf{L}^2(\Omega)} = \left\langle \operatorname{curl} X - \tilde{H}, \Psi \right\rangle_{\mathsf{L}^2(\Omega)} \quad (H_{\mathrm{d}} + \tilde{H} - \operatorname{curl} X = \operatorname{curl}(\underbrace{E - X - E_{\mathrm{c}}}_{\in \overset{\circ}{\mathsf{H}}(\operatorname{curl};\Omega)}) \perp \Psi)$$

for all $X \in \mathsf{H}(\operatorname{curl};\Omega)$ with $\tau X = G$.

Cauchy-Schwarz and $\Psi := H_{\mathrm{d}}$ $\Rightarrow$

$$\|H_{\mathrm{d}}\|_{\mathsf{L}^2(\Omega)} \leq \left\|\operatorname{curl} X - \tilde{H}\right\|_{\mathsf{L}^2(\Omega)} = m_+(\tilde{H}; \operatorname{curl} X)$$

$\Rightarrow$ finally

$$\|h\|^2_{\mathsf{L}^2(\Omega)} = \|\operatorname{curl} E_{\mathrm{c}}\|^2_{\mathsf{L}^2(\Omega)} + \|H_{\mathrm{d}}\|^2_{\mathsf{L}^2(\Omega)} \leq M_+^2(\tilde{H}; Y, \operatorname{curl} Y) + m_+^2(\tilde{H}; \operatorname{curl} X)$$

for all $Y \in \mathsf{H}(\operatorname{curl};\Omega)$ and all $X \in \mathsf{H}(\operatorname{curl};\Omega)$ with $\tau X = G$.

$$\Rightarrow \quad m_+(\tilde{H}; \operatorname{curl} X) = \left\|\operatorname{curl}(X - \tilde{E})\right\|_{\mathsf{L}^2(\Omega)} = \left\|\operatorname{curl} \check{\tau}(G - \tau\tilde{E})\right\|_{\mathsf{L}^2(\Omega)} \leq c_\tau \left\|G - \tau\tilde{E}\right\|_{\mathtt{trace}}$$

if $\tilde{H} = \operatorname{curl} \tilde{E}$ with $\tilde{E} \in \mathsf{H}(\operatorname{curl};\Omega)$
and we choose $X := \tilde{E} - \check{\tau}\tau\tilde{E} + \check{\tau}G \in \mathsf{H}(\operatorname{curl};\Omega)$
(then $\tau X = G$)

Proofs (Upper Bounds Continued)

recall $h = H - \tilde{H} = \operatorname{curl} E_c \oplus H_d \in \operatorname{curl} \mathbb{H} \oplus \mathsf{H}(\operatorname{curl}_0; \Omega)$

argument for H_d: for all $\Psi \in \mathsf{H}(\operatorname{curl}_0; \Omega)$

$$\langle H_d, \Psi \rangle_{\mathsf{L}^2(\Omega)} = \left\langle \operatorname{curl} X - \tilde{H}, \Psi \right\rangle_{\mathsf{L}^2(\Omega)} \quad (H_d + \tilde{H} - \operatorname{curl} X = \operatorname{curl}(\underbrace{E - X - E_c}_{\in \mathring{\mathsf{H}}(\operatorname{curl};\Omega)}) \perp \Psi)$$

for all $X \in \mathsf{H}(\operatorname{curl}; \Omega)$ with $\tau X = G$.

Cauchy-Schwarz and $\Psi := H_d \quad \Rightarrow$

$$\|H_d\|_{\mathsf{L}^2(\Omega)} \leq \left\| \operatorname{curl} X - \tilde{H} \right\|_{\mathsf{L}^2(\Omega)} = m_+(\tilde{H}; \operatorname{curl} X)$$

$\Rightarrow$ finally

$$\|h\|^2_{\mathsf{L}^2(\Omega)} = \|\operatorname{curl} E_c\|^2_{\mathsf{L}^2(\Omega)} + \|H_d\|^2_{\mathsf{L}^2(\Omega)} \leq M_+^2(\tilde{H}; Y, \operatorname{curl} Y) + m_+^2(\tilde{H}; \operatorname{curl} X)$$

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if $\tilde{H} = \operatorname{curl} \tilde{E}$ with $\tilde{E} \in \mathsf{H}(\operatorname{curl}; \Omega)$
and we choose $X := \tilde{E} - \check{\tau}\tau\tilde{E} + \check{\tau} G \in \mathsf{H}(\operatorname{curl}; \Omega)$
(then $\tau X = G$)

Proofs (Upper Bounds Continued)

recall $h = H - \tilde{H} = \operatorname{curl} E_{\mathrm{c}} \oplus H_{\mathrm{d}} \in \operatorname{curl} \mathbb{H} \oplus \mathrm{H}(\operatorname{curl}_0; \Omega)$

argument for H_{d}: for all $\Psi \in \mathrm{H}(\operatorname{curl}_0; \Omega)$

$$\langle H_{\mathrm{d}}, \Psi \rangle_{\mathrm{L}^2(\Omega)} = \left\langle \operatorname{curl} X - \tilde{H}, \Psi \right\rangle_{\mathrm{L}^2(\Omega)} \quad (H_{\mathrm{d}} + \tilde{H} - \operatorname{curl} X = \operatorname{curl}(\underbrace{E - X - E_{\mathrm{c}}}_{\in \mathring{\mathrm{H}}(\operatorname{curl};\Omega)}) \perp \Psi)$$

for all $X \in \mathrm{H}(\operatorname{curl}; \Omega)$ with $\tau X = G$.

Cauchy-Schwarz and $\Psi := H_{\mathrm{d}} \quad \Rightarrow$

$$\|H_{\mathrm{d}}\|_{\mathrm{L}^2(\Omega)} \leq \left\| \operatorname{curl} X - \tilde{H} \right\|_{\mathrm{L}^2(\Omega)} = m_+(\tilde{H}; \operatorname{curl} X)$$

$\Rightarrow$ finally

$$\|h\|^2_{\mathrm{L}^2(\Omega)} = \|\operatorname{curl} E_{\mathrm{c}}\|^2_{\mathrm{L}^2(\Omega)} + \|H_{\mathrm{d}}\|^2_{\mathrm{L}^2(\Omega)} \leq M^2_+(\tilde{H}; Y, \operatorname{curl} Y) + m^2_+(\tilde{H}; \operatorname{curl} X)$$

for all $Y \in \mathrm{H}(\operatorname{curl}; \Omega)$ and all $X \in \mathrm{H}(\operatorname{curl}; \Omega)$ with $\tau X = G$.

$$\Rightarrow \quad m_+(\tilde{H}; \operatorname{curl} X) = \left\| \operatorname{curl}(X - \tilde{E}) \right\|_{\mathrm{L}^2(\Omega)} = \left\| \operatorname{curl} \check{\tau}(G - \tau \tilde{E}) \right\|_{\mathrm{L}^2(\Omega)} \leq c_\tau \left\| G - \tau \tilde{E} \right\|_{\mathtt{trace}}$$

if $\tilde{H} = \operatorname{curl} \tilde{E}$ with $\tilde{E} \in \mathrm{H}(\operatorname{curl}; \Omega)$
and we choose $X := \tilde{E} - \check{\tau}\tau\tilde{E} + \check{\tau} G \in \mathrm{H}(\operatorname{curl}; \Omega)$
(then $\tau X = G$)

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Proofs (Upper Bounds Continued)

recall $h = H - \tilde{H} = \operatorname{curl} E_c \oplus H_d \in \operatorname{curl} \mathbb{H} \oplus \mathsf{H}(\operatorname{curl}_0; \Omega)$

argument for H_d: for all $\Psi \in \mathsf{H}(\operatorname{curl}_0; \Omega)$

$$\langle H_d, \Psi \rangle_{\mathsf{L}^2(\Omega)} = \left\langle \operatorname{curl} X - \tilde{H}, \Psi \right\rangle_{\mathsf{L}^2(\Omega)} \quad (H_d + \tilde{H} - \operatorname{curl} X = \operatorname{curl}\underbrace{(E - X - E_c)}_{\in \overset{\circ}{\mathsf{H}}(\operatorname{curl};\Omega)} \perp \Psi)$$

for all $X \in \mathsf{H}(\operatorname{curl}; \Omega)$ with $\tau X = G$.

Cauchy-Schwarz and $\Psi := H_d \quad \Rightarrow$

$$\|H_d\|_{\mathsf{L}^2(\Omega)} \leq \left\| \operatorname{curl} X - \tilde{H} \right\|_{\mathsf{L}^2(\Omega)} = m_+(\tilde{H}; \operatorname{curl} X)$$

$\Rightarrow$ finally

$$\|h\|^2_{\mathsf{L}^2(\Omega)} = \|\operatorname{curl} E_c\|^2_{\mathsf{L}^2(\Omega)} + \|H_d\|^2_{\mathsf{L}^2(\Omega)} \leq M^2_+(\tilde{H}; Y, \operatorname{curl} Y) + m^2_+(\tilde{H}; \operatorname{curl} X)$$

for all $Y \in \mathsf{H}(\operatorname{curl}; \Omega)$ and all $X \in \mathsf{H}(\operatorname{curl}; \Omega)$ with $\tau X = G$.

$$\Rightarrow \quad m_+(\tilde{H}; \operatorname{curl} X) = \left\| \operatorname{curl}(X - \tilde{E}) \right\|_{\mathsf{L}^2(\Omega)} = \left\| \operatorname{curl} \check{\tau}(G - \tau \tilde{E}) \right\|_{\mathsf{L}^2(\Omega)} \leq c_\tau \left\| G - \tau \tilde{E} \right\|_{\mathrm{trace}}$$

if $\tilde{H} = \operatorname{curl} \tilde{E}$ with $\tilde{E} \in \mathsf{H}(\operatorname{curl}; \Omega)$
and we choose $X := \tilde{E} - \check{\tau}\tau\tilde{E} + \check{\tau}G \in \mathsf{H}(\operatorname{curl}; \Omega)$
(then $\tau X = G$)

Proofs (Upper Bounds Continued)

recall $h = H - \tilde{H} = \operatorname{curl} E_c \oplus H_d \in \operatorname{curl} \mathbb{H} \oplus \mathsf{H}(\operatorname{curl}_0;\Omega)$

argument for H_d: for all $\Psi \in \mathsf{H}(\operatorname{curl}_0;\Omega)$

$$\langle H_d, \Psi \rangle_{\mathsf{L}^2(\Omega)} = \left\langle \operatorname{curl} X - \tilde{H}, \Psi \right\rangle_{\mathsf{L}^2(\Omega)} \quad (H_d + \tilde{H} - \operatorname{curl} X = \operatorname{curl}(\underbrace{E - X - E_c}_{\in \overset{\circ}{\mathsf{H}}(\operatorname{curl};\Omega)}) \perp \Psi)$$

for all $X \in \mathsf{H}(\operatorname{curl};\Omega)$ with $\tau X = G$.

Cauchy-Schwarz and $\Psi := H_d \quad \Rightarrow$

$$\|H_d\|_{\mathsf{L}^2(\Omega)} \leq \left\| \operatorname{curl} X - \tilde{H} \right\|_{\mathsf{L}^2(\Omega)} = m_+(\tilde{H}; \operatorname{curl} X)$$

$\Rightarrow$ finally

$$\|h\|^2_{\mathsf{L}^2(\Omega)} = \|\operatorname{curl} E_c\|^2_{\mathsf{L}^2(\Omega)} + \|H_d\|^2_{\mathsf{L}^2(\Omega)} \leq M_+^2(\tilde{H}; Y, \operatorname{curl} Y) + m_+^2(\tilde{H}; \operatorname{curl} X)$$

for all $Y \in \mathsf{H}(\operatorname{curl};\Omega)$ and all $X \in \mathsf{H}(\operatorname{curl};\Omega)$ with $\tau X = G$.

$$\Rightarrow \quad m_+(\tilde{H}; \operatorname{curl} X) = \left\| \operatorname{curl}(X - \tilde{E}) \right\|_{\mathsf{L}^2(\Omega)} = \left\| \operatorname{curl} \check{\tau}(G - \tau\tilde{E}) \right\|_{\mathsf{L}^2(\Omega)} \leq c_\tau \left\| G - \tau\tilde{E} \right\|_{\mathrm{trace}}$$

if $\tilde{H} = \operatorname{curl} \tilde{E}$ with $\tilde{E} \in \mathsf{H}(\operatorname{curl};\Omega)$
and we choose $X := \tilde{E} - \check{\tau}\tau\tilde{E} + \check{\tau}G \in \mathsf{H}(\operatorname{curl};\Omega)$
(then $\tau X = G$)

Proofs (Upper Bounds Continued)

recall $h = H - \tilde{H} = \operatorname{curl} E_{\mathrm{c}} \oplus H_{\mathrm{d}} \in \operatorname{curl} \mathbb{H} \oplus \mathrm{H}(\operatorname{curl}_0;\Omega)$

argument for H_{d}: for all $\Psi \in \mathrm{H}(\operatorname{curl}_0;\Omega)$

$$\langle H_{\mathrm{d}}, \Psi\rangle_{\mathrm{L}^2(\Omega)} = \left\langle \operatorname{curl} X - \tilde{H}, \Psi \right\rangle_{\mathrm{L}^2(\Omega)} \qquad (H_{\mathrm{d}} + \tilde{H} - \operatorname{curl} X = \operatorname{curl}(\underbrace{E - X - E_{\mathrm{c}}}_{\in \overset{\circ}{\mathrm{H}}(\operatorname{curl};\Omega)}) \perp \Psi)$$

for all $X \in \mathrm{H}(\operatorname{curl};\Omega)$ with $\tau X = G$.

Cauchy-Schwarz and $\Psi := H_{\mathrm{d}} \quad \Rightarrow$

$$\|H_{\mathrm{d}}\|_{\mathrm{L}^2(\Omega)} \leq \left\|\operatorname{curl} X - \tilde{H}\right\|_{\mathrm{L}^2(\Omega)} = m_+(\tilde{H}; \operatorname{curl} X)$$

$\Rightarrow$ finally

$$\|h\|^2_{\mathrm{L}^2(\Omega)} = \|\operatorname{curl} E_{\mathrm{c}}\|^2_{\mathrm{L}^2(\Omega)} + \|H_{\mathrm{d}}\|^2_{\mathrm{L}^2(\Omega)} \leq M_+^2(\tilde{H}; Y, \operatorname{curl} Y) + m_+^2(\tilde{H}; \operatorname{curl} X)$$

for all $Y \in \mathrm{H}(\operatorname{curl};\Omega)$ and all $X \in \mathrm{H}(\operatorname{curl};\Omega)$ with $\tau X = G$.

$$\Rightarrow \quad m_+(\tilde{H}; \operatorname{curl} X) = \left\|\operatorname{curl}(X - \tilde{E})\right\|_{\mathrm{L}^2(\Omega)} = \left\|\operatorname{curl} \check{\tau}(G - \tau\tilde{E})\right\|_{\mathrm{L}^2(\Omega)} \leq c_\tau \left\|G - \tau\tilde{E}\right\|_{\mathtt{trace}}$$

if $\tilde{H} = \operatorname{curl} \tilde{E}$ with $\tilde{E} \in \mathrm{H}(\operatorname{curl};\Omega)$
and we choose $X := \tilde{E} - \check{\tau}\tau\tilde{E} + \check{\tau}G \in \mathrm{H}(\operatorname{curl};\Omega)$
(then $\tau X = G$)

Proofs (Upper Bounds Continued)

recall $h = H - \tilde{H} = \operatorname{curl} E_{\mathrm{c}} \oplus H_{\mathrm{d}} \in \operatorname{curl} \mathbb{H} \oplus \mathrm{H}(\operatorname{curl}_0; \Omega)$

argument for H_{d}: for all $\Psi \in \mathrm{H}(\operatorname{curl}_0; \Omega)$

$$\langle H_{\mathrm{d}}, \Psi \rangle_{\mathrm{L}^2(\Omega)} = \left\langle \operatorname{curl} X - \tilde{H}, \Psi \right\rangle_{\mathrm{L}^2(\Omega)} \quad (H_{\mathrm{d}} + \tilde{H} - \operatorname{curl} X = \operatorname{curl}(\underbrace{E - X - E_{\mathrm{c}}}_{\in \overset{\circ}{\mathrm{H}}(\operatorname{curl};\Omega)}) \perp \Psi)$$

for all $X \in \mathrm{H}(\operatorname{curl}; \Omega)$ with $\tau X = G$.

Cauchy-Schwarz and $\Psi := H_{\mathrm{d}}$ $\Rightarrow$

$$\|H_{\mathrm{d}}\|_{\mathrm{L}^2(\Omega)} \leq \left\| \operatorname{curl} X - \tilde{H} \right\|_{\mathrm{L}^2(\Omega)} = m_+(\tilde{H}; \operatorname{curl} X)$$

$\Rightarrow$ finally

$$\|h\|^2_{\mathrm{L}^2(\Omega)} = \|\operatorname{curl} E_{\mathrm{c}}\|^2_{\mathrm{L}^2(\Omega)} + \|H_{\mathrm{d}}\|^2_{\mathrm{L}^2(\Omega)} \leq M^2_+(\tilde{H}; Y, \operatorname{curl} Y) + m^2_+(\tilde{H}; \operatorname{curl} X)$$

for all $Y \in \mathrm{H}(\operatorname{curl}; \Omega)$ and all $X \in \mathrm{H}(\operatorname{curl}; \Omega)$ with $\tau X = G$.

$$\Rightarrow \quad m_+(\tilde{H}; \operatorname{curl} X) = \left\| \operatorname{curl}(X - \tilde{E}) \right\|_{\mathrm{L}^2(\Omega)} = \left\| \operatorname{curl} \check{\tau}(G - \tau \tilde{E}) \right\|_{\mathrm{L}^2(\Omega)} \leq c_\tau \left\| G - \tau \tilde{E} \right\|_{\mathtt{trace}}$$

if $\tilde{H} = \operatorname{curl} \tilde{E}$ with $\tilde{E} \in \mathrm{H}(\operatorname{curl}; \Omega)$
and we choose $X := \tilde{E} - \check{\tau}\tau\tilde{E} + \check{\tau} G \in \mathrm{H}(\operatorname{curl}; \Omega)$
(then $\tau X = G$)

Proofs (Lower Bounds)

recall $h = H - \tilde{H} = \operatorname{curl} E_{\mathrm{c}} \oplus H_{\mathrm{d}} \in \operatorname{curl} \mathbb{H} \oplus \mathrm{H}(\operatorname{curl}_0;\Omega)$

standard argument for $\operatorname{curl} E_{\mathrm{c}}$: for all $X \in \overset{\circ}{\mathrm{H}}(\operatorname{curl};\Omega)$

$$\|\operatorname{curl} E_{\mathrm{c}}\|^2_{\mathrm{L}^2(\Omega)} \geq 2 \underbrace{\langle \operatorname{curl} E_{\mathrm{c}}, \operatorname{curl} X\rangle_{\mathrm{L}^2(\Omega)}}_{=\langle h, \operatorname{curl} X\rangle_{\mathrm{L}^2(\Omega)}} - \|\operatorname{curl} X\|^2_{\mathrm{L}^2(\Omega)} \qquad (H_{\mathrm{d}} \perp \operatorname{curl} X)$$

$$= 2\langle F, \Phi\rangle_{\mathrm{L}^2(\Omega)} - \left\langle \operatorname{curl} X + 2\tilde{H}, \operatorname{curl} X\right\rangle_{\mathrm{L}^2(\Omega)} = M_-(\tilde{H}; X, \operatorname{curl} X)$$

similar for H_{d}: for all $Y \in \mathrm{H}(\operatorname{curl}_0;\Omega)$ and for all $Z \in \mathrm{H}(\operatorname{curl};\Omega)$ with $\tau Z = G$

$$\|H_{\mathrm{d}}\|^2_{\mathrm{L}^2(\Omega)} \geq 2 \underbrace{\langle H_{\mathrm{d}}, Y\rangle_{\mathrm{L}^2(\Omega)}}_{=\langle h, Y\rangle_{\mathrm{L}^2(\Omega)}} - \|Y\|^2_{\mathrm{L}^2(\Omega)} \qquad (\operatorname{curl} E_{\mathrm{c}} \perp Y)$$

$$= 2\left\langle \operatorname{curl} Z - \tilde{H}, Y\right\rangle_{\mathrm{L}^2(\Omega)} - \|Y\|^2_{\mathrm{L}^2(\Omega)} \qquad (\operatorname{curl} \underbrace{(E - Z)}_{\in \overset{\circ}{\mathrm{H}}(\operatorname{curl};\Omega)} \perp Y)$$

$$= m_-(\tilde{H}; Y, \operatorname{curl} Z)$$

Proofs (Lower Bounds)

recall $h = H - \tilde{H} = \operatorname{curl} E_c \oplus H_d \in \operatorname{curl} \mathbb{H} \oplus \mathsf{H}(\operatorname{curl}_0; \Omega)$

standard argument for curl E_c: for all $X \in \overset{\circ}{\mathsf{H}}(\operatorname{curl}; \Omega)$

$$\|\operatorname{curl} E_c\|^2_{\mathsf{L}^2(\Omega)} \geq 2 \underbrace{\langle \operatorname{curl} E_c, \operatorname{curl} X \rangle_{\mathsf{L}^2(\Omega)}}_{=\langle h, \operatorname{curl} X\rangle_{\mathsf{L}^2(\Omega)}} - \|\operatorname{curl} X\|^2_{\mathsf{L}^2(\Omega)} \qquad (H_d \perp \operatorname{curl} X)$$

$$= 2 \langle F, \Phi \rangle_{\mathsf{L}^2(\Omega)} - \left\langle \operatorname{curl} X + 2\tilde{H}, \operatorname{curl} X \right\rangle_{\mathsf{L}^2(\Omega)} = M_-(\tilde{H}; X, \operatorname{curl} X)$$

similar for H_d: for all $Y \in \mathsf{H}(\operatorname{curl}_0; \Omega)$ and for all $Z \in \mathsf{H}(\operatorname{curl}; \Omega)$ with $\tau Z = G$

$$\|H_d\|^2_{\mathsf{L}^2(\Omega)} \geq 2 \underbrace{\langle H_d, Y \rangle_{\mathsf{L}^2(\Omega)}}_{=\langle h, Y\rangle_{\mathsf{L}^2(\Omega)}} - \|Y\|^2_{\mathsf{L}^2(\Omega)} \qquad (\operatorname{curl} E_c \perp Y)$$

$$= 2 \left\langle \operatorname{curl} Z - \tilde{H}, Y \right\rangle_{\mathsf{L}^2(\Omega)} - \|Y\|^2_{\mathsf{L}^2(\Omega)} \qquad (\operatorname{curl} \underbrace{(E - Z)}_{\in \overset{\circ}{\mathsf{H}}(\operatorname{curl};\Omega)} \perp Y)$$

$$= m_-(\tilde{H}; Y, \operatorname{curl} Z)$$

Proofs (Lower Bounds)

recall $h = H - \tilde{H} = \operatorname{curl} E_{\mathrm{c}} \oplus H_{\mathrm{d}} \in \operatorname{curl} \mathbb{H} \oplus \mathrm{H}(\operatorname{curl}_0; \Omega)$

standard argument for curl E_{c}: for all $X \in \mathring{\mathrm{H}}(\operatorname{curl}; \Omega)$

$$\|\operatorname{curl} E_{\mathrm{c}}\|^2_{\mathrm{L}^2(\Omega)} \geq 2 \underbrace{\langle \operatorname{curl} E_{\mathrm{c}}, \operatorname{curl} X \rangle_{\mathrm{L}^2(\Omega)}}_{= \langle h, \operatorname{curl} X \rangle_{\mathrm{L}^2(\Omega)}} - \|\operatorname{curl} X\|^2_{\mathrm{L}^2(\Omega)} \qquad (H_{\mathrm{d}} \perp \operatorname{curl} X)$$

$$= 2 \langle F, \Phi \rangle_{\mathrm{L}^2(\Omega)} - \left\langle \operatorname{curl} X + 2\tilde{H}, \operatorname{curl} X \right\rangle_{\mathrm{L}^2(\Omega)} = M_-(\tilde{H}; X, \operatorname{curl} X)$$

similar for H_{d}: for all $Y \in \mathrm{H}(\operatorname{curl}_0; \Omega)$ and for all $Z \in \mathrm{H}(\operatorname{curl}; \Omega)$ with $\tau Z = G$

$$\|H_{\mathrm{d}}\|^2_{\mathrm{L}^2(\Omega)} \geq 2 \underbrace{\langle H_{\mathrm{d}}, Y \rangle_{\mathrm{L}^2(\Omega)}}_{= \langle h, Y \rangle_{\mathrm{L}^2(\Omega)}} - \|Y\|^2_{\mathrm{L}^2(\Omega)} \qquad (\operatorname{curl} E_{\mathrm{c}} \perp Y)$$

$$= 2 \left\langle \operatorname{curl} Z - \tilde{H}, Y \right\rangle_{\mathrm{L}^2(\Omega)} - \|Y\|^2_{\mathrm{L}^2(\Omega)} \qquad (\operatorname{curl} \underbrace{(E - Z)}_{\in \mathring{\mathrm{H}}(\operatorname{curl}; \Omega)} \perp Y)$$

$$= m_-(\tilde{H}; Y, \operatorname{curl} Z)$$

Proofs (Lower Bounds)

recall $h = H - \tilde{H} = \operatorname{curl} E_c \oplus H_d \in \operatorname{curl} \mathbb{H} \oplus \mathsf{H}(\operatorname{curl}_0; \Omega)$

standard argument for $\operatorname{curl} E_c$: for all $X \in \overset{\circ}{\mathsf{H}}(\operatorname{curl}; \Omega)$

$$\|\operatorname{curl} E_c\|^2_{\mathsf{L}^2(\Omega)} \geq 2 \underbrace{\langle \operatorname{curl} E_c, \operatorname{curl} X \rangle_{\mathsf{L}^2(\Omega)}}_{=\langle h, \operatorname{curl} X\rangle_{\mathsf{L}^2(\Omega)}} - \|\operatorname{curl} X\|^2_{\mathsf{L}^2(\Omega)} \qquad (H_d \perp \operatorname{curl} X)$$

$$= 2\langle F, \Phi\rangle_{\mathsf{L}^2(\Omega)} - \left\langle \operatorname{curl} X + 2\tilde{H}, \operatorname{curl} X \right\rangle_{\mathsf{L}^2(\Omega)} = M_-(\tilde{H}; X, \operatorname{curl} X)$$

similar for H_d: for all $Y \in \mathsf{H}(\operatorname{curl}_0; \Omega)$ and for all $Z \in \mathsf{H}(\operatorname{curl}; \Omega)$ with $\tau Z = G$

$$\|H_d\|^2_{\mathsf{L}^2(\Omega)} \geq 2 \underbrace{\langle H_d, Y \rangle_{\mathsf{L}^2(\Omega)}}_{=\langle h, Y\rangle_{\mathsf{L}^2(\Omega)}} - \|Y\|^2_{\mathsf{L}^2(\Omega)} \qquad (\operatorname{curl} E_c \perp Y)$$

$$= 2\left\langle \operatorname{curl} Z - \tilde{H}, Y \right\rangle_{\mathsf{L}^2(\Omega)} - \|Y\|^2_{\mathsf{L}^2(\Omega)} \qquad (\operatorname{curl} \underbrace{(E - Z)}_{\in \overset{\circ}{\mathsf{H}}(\operatorname{curl}; \Omega)} \perp Y)$$

$$= m_-(\tilde{H}; Y, \operatorname{curl} Z)$$

Proofs (Lower Bounds)

recall $h = H - \tilde{H} = \operatorname{curl} E_c \oplus H_d \in \operatorname{curl} \mathbb{H} \oplus \mathsf{H}(\operatorname{curl}_0; \Omega)$

standard argument for curl E_c: for all $X \in \overset{\circ}{\mathsf{H}}(\operatorname{curl}; \Omega)$

$$\|\operatorname{curl} E_c\|^2_{\mathsf{L}^2(\Omega)} \geq 2 \underbrace{\langle \operatorname{curl} E_c, \operatorname{curl} X \rangle_{\mathsf{L}^2(\Omega)}}_{=\langle h, \operatorname{curl} X\rangle_{\mathsf{L}^2(\Omega)}} - \|\operatorname{curl} X\|^2_{\mathsf{L}^2(\Omega)} \qquad (H_d \perp \operatorname{curl} X)$$

$$= 2 \langle F, \Phi \rangle_{\mathsf{L}^2(\Omega)} - \left\langle \operatorname{curl} X + 2\tilde{H}, \operatorname{curl} X \right\rangle_{\mathsf{L}^2(\Omega)} = M_-(\tilde{H}; X, \operatorname{curl} X)$$

similar for H_d: for all $Y \in \mathsf{H}(\operatorname{curl}_0; \Omega)$ and for all $Z \in \mathsf{H}(\operatorname{curl}; \Omega)$ with $\tau Z = G$

$$\|H_d\|^2_{\mathsf{L}^2(\Omega)} \geq 2 \underbrace{\langle H_d, Y \rangle_{\mathsf{L}^2(\Omega)}}_{=\langle h, Y\rangle_{\mathsf{L}^2(\Omega)}} - \|Y\|^2_{\mathsf{L}^2(\Omega)} \qquad (\operatorname{curl} E_c \perp Y)$$

$$= 2 \left\langle \operatorname{curl} Z - \tilde{H}, Y \right\rangle_{\mathsf{L}^2(\Omega)} - \|Y\|^2_{\mathsf{L}^2(\Omega)} \qquad (\operatorname{curl} \underbrace{(E - Z)}_{\in \overset{\circ}{\mathsf{H}}(\operatorname{curl}; \Omega)} \perp Y)$$

$$= m_-(\tilde{H}; Y, \operatorname{curl} Z)$$

Proofs (Lower Bounds)

recall $h = H - \tilde{H} = \operatorname{curl} E_c \oplus H_d \in \operatorname{curl} \mathbb{H} \oplus \mathrm{H}(\operatorname{curl}_0; \Omega)$

standard argument for curl E_c: for all $X \in \overset{\circ}{\mathrm{H}}(\operatorname{curl}; \Omega)$

$$\|\operatorname{curl} E_c\|^2_{\mathrm{L}^2(\Omega)} \geq 2 \underbrace{\langle \operatorname{curl} E_c, \operatorname{curl} X\rangle_{\mathrm{L}^2(\Omega)}}_{=\langle h, \operatorname{curl} X\rangle_{\mathrm{L}^2(\Omega)}} - \|\operatorname{curl} X\|^2_{\mathrm{L}^2(\Omega)} \qquad (H_d \perp \operatorname{curl} X)$$

$$= 2\langle F, \Phi\rangle_{\mathrm{L}^2(\Omega)} - \left\langle \operatorname{curl} X + 2\tilde{H}, \operatorname{curl} X \right\rangle_{\mathrm{L}^2(\Omega)} = M_-(\tilde{H}; X, \operatorname{curl} X)$$

similar for H_d: for all $Y \in \mathrm{H}(\operatorname{curl}_0; \Omega)$ and for all $Z \in \mathrm{H}(\operatorname{curl}; \Omega)$ with $\tau Z = G$

$$\|H_d\|^2_{\mathrm{L}^2(\Omega)} \geq 2 \underbrace{\langle H_d, Y\rangle_{\mathrm{L}^2(\Omega)}}_{=\langle h, Y\rangle_{\mathrm{L}^2(\Omega)}} - \|Y\|^2_{\mathrm{L}^2(\Omega)} \qquad (\operatorname{curl} E_c \perp Y)$$

$$= 2\left\langle \operatorname{curl} Z - \tilde{H}, Y \right\rangle_{\mathrm{L}^2(\Omega)} - \|Y\|^2_{\mathrm{L}^2(\Omega)} \qquad (\operatorname{curl} \underbrace{(E - Z)}_{\in \overset{\circ}{\mathrm{H}}(\operatorname{curl};\Omega)} \perp Y)$$

$$= m_-(\tilde{H}; Y, \operatorname{curl} Z)$$

Proofs (Lower Bounds)

recall $h = H - \tilde{H} = \operatorname{curl} E_{\mathtt{c}} \oplus H_{\mathtt{d}} \in \operatorname{curl} \mathbb{H} \oplus \mathsf{H}(\operatorname{curl}_0;\Omega)$

standard argument for curl $E_{\mathtt{c}}$: for all $X \in \overset{\circ}{\mathsf{H}}(\operatorname{curl};\Omega)$

$$\|\operatorname{curl} E_{\mathtt{c}}\|^2_{\mathsf{L}^2(\Omega)} \geq 2\underbrace{\langle \operatorname{curl} E_{\mathtt{c}}, \operatorname{curl} X\rangle_{\mathsf{L}^2(\Omega)}}_{=\langle h,\operatorname{curl} X\rangle_{\mathsf{L}^2(\Omega)}} - \|\operatorname{curl} X\|^2_{\mathsf{L}^2(\Omega)} \qquad (H_{\mathtt{d}} \perp \operatorname{curl} X)$$

$$= 2\langle F, \Phi\rangle_{\mathsf{L}^2(\Omega)} - \left\langle \operatorname{curl} X + 2\tilde{H}, \operatorname{curl} X\right\rangle_{\mathsf{L}^2(\Omega)} = M_{-}(\tilde{H}; X, \operatorname{curl} X)$$

similar for $H_{\mathtt{d}}$: for all $Y \in \mathsf{H}(\operatorname{curl}_0;\Omega)$ and for all $Z \in \mathsf{H}(\operatorname{curl};\Omega)$ with $\tau Z = G$

$$\|H_{\mathtt{d}}\|^2_{\mathsf{L}^2(\Omega)} \geq 2\underbrace{\langle H_{\mathtt{d}}, Y\rangle_{\mathsf{L}^2(\Omega)}}_{=\langle h,Y\rangle_{\mathsf{L}^2(\Omega)}} - \|Y\|^2_{\mathsf{L}^2(\Omega)} \qquad (\operatorname{curl} E_{\mathtt{c}} \perp Y)$$

$$= 2\left\langle \operatorname{curl} Z - \tilde{H}, Y\right\rangle_{\mathsf{L}^2(\Omega)} - \|Y\|^2_{\mathsf{L}^2(\Omega)} \qquad (\operatorname{curl}\underbrace{(E - Z)}_{\in \overset{\circ}{\mathsf{H}}(\operatorname{curl};\Omega)} \perp Y)$$

$$= m_{-}(\tilde{H}; Y, \operatorname{curl} Z)$$

Proofs (Lower Bounds)

recall $h = H - \tilde{H} = \operatorname{curl} E_c \oplus H_d \in \operatorname{curl} \mathbb{H} \oplus \mathsf{H}(\operatorname{curl}_0;\Omega)$

standard argument for $\operatorname{curl} E_c$: for all $X \in \overset{\circ}{\mathsf{H}}(\operatorname{curl};\Omega)$

$$\|\operatorname{curl} E_c\|^2_{\mathsf{L}^2(\Omega)} \geq 2\underbrace{\langle \operatorname{curl} E_c, \operatorname{curl} X\rangle_{\mathsf{L}^2(\Omega)}}_{=\langle h,\operatorname{curl} X\rangle_{\mathsf{L}^2(\Omega)}} - \|\operatorname{curl} X\|^2_{\mathsf{L}^2(\Omega)} \qquad (H_d \perp \operatorname{curl} X)$$

$$= 2\langle F, \Phi\rangle_{\mathsf{L}^2(\Omega)} - \left\langle \operatorname{curl} X + 2\tilde{H}, \operatorname{curl} X\right\rangle_{\mathsf{L}^2(\Omega)} = M_-(\tilde{H}; X, \operatorname{curl} X)$$

similar for H_d: for all $Y \in \mathsf{H}(\operatorname{curl}_0;\Omega)$ and for all $Z \in \mathsf{H}(\operatorname{curl};\Omega)$ with $\tau Z = G$

$$\|H_d\|^2_{\mathsf{L}^2(\Omega)} \geq 2\underbrace{\langle H_d, Y\rangle_{\mathsf{L}^2(\Omega)}}_{=\langle h,Y\rangle_{\mathsf{L}^2(\Omega)}} - \|Y\|^2_{\mathsf{L}^2(\Omega)} \qquad (\operatorname{curl} E_c \perp Y)$$

$$= 2\left\langle \operatorname{curl} Z - \tilde{H}, Y\right\rangle_{\mathsf{L}^2(\Omega)} - \|Y\|^2_{\mathsf{L}^2(\Omega)} \qquad (\operatorname{curl}\underbrace{(E - Z)}_{\in\overset{\circ}{\mathsf{H}}(\operatorname{curl};\Omega)} \perp Y)$$

$$= m_-(\tilde{H}; Y, \operatorname{curl} Z)$$

Proofs (Lower Bounds)

recall $h = H - \tilde{H} = \operatorname{curl} E_c \oplus H_d \in \operatorname{curl} \mathbb{H} \oplus \mathrm{H}(\operatorname{curl}_0; \Omega)$

standard argument for $\operatorname{curl} E_c$: for all $X \in \overset{\circ}{\mathrm{H}}(\operatorname{curl}; \Omega)$

$$\|\operatorname{curl} E_c\|^2_{\mathrm{L}^2(\Omega)} \geq 2 \underbrace{\langle \operatorname{curl} E_c, \operatorname{curl} X \rangle_{\mathrm{L}^2(\Omega)}}_{=\langle h, \operatorname{curl} X\rangle_{\mathrm{L}^2(\Omega)}} - \|\operatorname{curl} X\|^2_{\mathrm{L}^2(\Omega)} \quad (H_d \perp \operatorname{curl} X)$$

$$= 2 \langle F, \Phi \rangle_{\mathrm{L}^2(\Omega)} - \left\langle \operatorname{curl} X + 2\tilde{H}, \operatorname{curl} X \right\rangle_{\mathrm{L}^2(\Omega)} = M_-(\tilde{H}; X, \operatorname{curl} X)$$

similar for H_d: for all $Y \in \mathrm{H}(\operatorname{curl}_0; \Omega)$ and for all $Z \in \mathrm{H}(\operatorname{curl}; \Omega)$ with $\tau Z = G$

$$\|H_d\|^2_{\mathrm{L}^2(\Omega)} \geq 2 \underbrace{\langle H_d, Y \rangle_{\mathrm{L}^2(\Omega)}}_{=\langle h, Y\rangle_{\mathrm{L}^2(\Omega)}} - \|Y\|^2_{\mathrm{L}^2(\Omega)} \quad (\operatorname{curl} E_c \perp Y)$$

$$= 2 \left\langle \operatorname{curl} Z - \tilde{H}, Y \right\rangle_{\mathrm{L}^2(\Omega)} - \|Y\|^2_{\mathrm{L}^2(\Omega)} \quad (\operatorname{curl} \underbrace{(E - Z)}_{\in \overset{\circ}{\mathrm{H}}(\operatorname{curl}; \Omega)} \perp Y)$$

$$= m_-(\tilde{H}; Y, \operatorname{curl} Z)$$

Proofs (Lower Bounds)

recall $h = H - \tilde{H} = \operatorname{curl} E_c \oplus H_d \in \operatorname{curl} \mathbb{H} \oplus \mathsf{H}(\operatorname{curl}_0;\Omega)$

standard argument for curl E_c: for all $X \in \overset{\circ}{\mathsf{H}}(\operatorname{curl};\Omega)$

$$\|\operatorname{curl} E_c\|^2_{\mathsf{L}^2(\Omega)} \geq 2 \underbrace{\langle \operatorname{curl} E_c, \operatorname{curl} X\rangle_{\mathsf{L}^2(\Omega)}}_{=\langle h, \operatorname{curl} X\rangle_{\mathsf{L}^2(\Omega)}} - \|\operatorname{curl} X\|^2_{\mathsf{L}^2(\Omega)} \qquad (H_d \perp \operatorname{curl} X)$$

$$= 2\langle F, \Phi\rangle_{\mathsf{L}^2(\Omega)} - \left\langle \operatorname{curl} X + 2\tilde{H}, \operatorname{curl} X \right\rangle_{\mathsf{L}^2(\Omega)} = M_-(\tilde{H}; X, \operatorname{curl} X)$$

similar for H_d: for all $Y \in \mathsf{H}(\operatorname{curl}_0;\Omega)$ and for all $Z \in \mathsf{H}(\operatorname{curl};\Omega)$ with $\tau Z = G$

$$\|H_d\|^2_{\mathsf{L}^2(\Omega)} \geq 2 \underbrace{\langle H_d, Y\rangle_{\mathsf{L}^2(\Omega)}}_{=\langle h, Y\rangle_{\mathsf{L}^2(\Omega)}} - \|Y\|^2_{\mathsf{L}^2(\Omega)} \qquad (\operatorname{curl} E_c \perp Y)$$

$$= 2\left\langle \operatorname{curl} Z - \tilde{H}, Y \right\rangle_{\mathsf{L}^2(\Omega)} - \|Y\|^2_{\mathsf{L}^2(\Omega)} \qquad (\operatorname{curl} \underbrace{(E - Z)}_{\in \overset{\circ}{\mathsf{H}}(\operatorname{curl};\Omega)} \perp Y)$$

$$= m_-(\tilde{H}; Y, \operatorname{curl} Z)$$

Proofs (Lower Bounds Continued)

$\Rightarrow$ finally

$$\|h\|^2_{\mathrm{L}^2(\Omega)} = \|\operatorname{curl} E_c\|^2_{\mathrm{L}^2(\Omega)} + \|H_d\|^2_{\mathrm{L}^2(\Omega)} \geq M_-(\tilde{H}; X, \operatorname{curl} X) + m_-(\tilde{H}; Y, \operatorname{curl} Z)$$

for all $Y \in \mathrm{H}(\operatorname{curl}_0; \Omega)$ and all $Z \in \mathrm{H}(\operatorname{curl}; \Omega)$ with $\tau Z = G$.

if $\tilde{H} = \operatorname{curl} \tilde{E}$ with $\tilde{E} \in \mathrm{H}(\operatorname{curl}; \Omega) \Rightarrow$

$$m_-(\tilde{H}; Y, \operatorname{curl} Z) = 2 \left\langle \operatorname{curl}(Z - \tilde{E}), Y \right\rangle_{\mathrm{L}^2(\Omega)} - \|Y\|^2_{\mathrm{L}^2(\Omega)}$$

$$= 2 \underbrace{\left\langle G - \tau\tilde{E}, \tau_n Y \right\rangle_{\mathtt{trace}}}_{\text{"} = \int_\Gamma (G - \nu \times \tilde{E}) Y \text{"}} - \|Y\|^2_{\mathrm{L}^2(\Omega)}$$

Proofs (Lower Bounds Continued)

$\Rightarrow$ finally

$$\|h\|^2_{\mathrm{L}^2(\Omega)} = \|\operatorname{curl} E_{\mathrm{c}}\|^2_{\mathrm{L}^2(\Omega)} + \|H_{\mathrm{d}}\|^2_{\mathrm{L}^2(\Omega)} \geq M_-(\tilde{H}; X, \operatorname{curl} X) + m_-(\tilde{H}; Y, \operatorname{curl} Z)$$

for all $Y \in \mathrm{H}(\operatorname{curl}_0; \Omega)$ and all $Z \in \mathrm{H}(\operatorname{curl}; \Omega)$ with $\tau Z = G$.
if $\tilde{H} = \operatorname{curl} \tilde{E}$ with $\tilde{E} \in \mathrm{H}(\operatorname{curl}; \Omega) \Rightarrow$

$$\begin{aligned} m_-(\tilde{H}; Y, \operatorname{curl} Z) &= 2 \left\langle \operatorname{curl}(Z - \tilde{E}), Y \right\rangle_{\mathrm{L}^2(\Omega)} - \|Y\|^2_{\mathrm{L}^2(\Omega)} \\ &= 2 \underbrace{\left\langle G - \tau \tilde{E}, \tau_n Y \right\rangle_{\mathtt{trace}}}_{\text{"} = \int_\Gamma (G - \nu \times \tilde{E}) Y \text{"}} - \|Y\|^2_{\mathrm{L}^2(\Omega)} \end{aligned}$$

Remarks

- Ω exterior domain or bounded in one direction
- differential forms, $\mathbb{R}^N$, Riemannian manifolds
- hyperbolic problems

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