# Functional A Posteriori Error Estimates for Static Maxwell Problems 

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(4th ANMPDEA)
4th Workshop on Advanced Numerical Methods for Partial Differential Equation Analysis for Junior Scientists

Euler Institute, St. Petersburg
August 22, 2011

## Introduction: Static Maxwell Problem

n $\Omega \subset \mathbb{R}^{3}$ bounded domain with Lipschitz boundary $\Gamma=\partial \Omega$

- $\varepsilon, \mu: \Omega \rightarrow \mathbb{R}^{3 \times 3}$ medium properties:
- $F$ given right hand side (current),
- E electric field,
- $\tau$ tangential trace, i.e., $\tau E=\nu \times\left. E\right|_{r}$
$=1$ orthogonality w.r.t. $L^{2}(\Omega)$-scalar product $\langle E, H\rangle_{\Omega}:=\int_{\Omega} E \cdot H$
- $\mathcal{H}_{\varepsilon}(\Omega)$ Dirichlet fields; $H \in \mathcal{H}_{\varepsilon}(\Omega)$, iff curl $H=0$ and $\operatorname{div} \varepsilon H=0$ and $\tau H=0$
electro-magneto static problem


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1. goal: error estimates for e:=E - \tilde{E}\mathrm{ and h:= }HH-\tilde{H}\mathrm{ , where }\tilde{E},\tilde{H}\mathrm{ approx. of}
    E,\muH
|- method: pioneering work of Sergey Repin since late 1990's
    similar estimates (elliptic, elastic, ...)
owvesmitr
    DEU'ISENRG
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\begin{aligned}
\mathrm{H}(\text { curl } ; \Omega)) & :=\left\{E \in \mathrm{~L}^{2}(\Omega): \operatorname{curl} E \in \mathrm{~L}^{2}(\Omega)\right\} \\
\mathrm{H}\left(\text { curl }_{0} ; \Omega\right) & :=\{E \in \mathrm{H}(\text { curl } ; \Omega): \operatorname{curl} E=0\} \\
\stackrel{\circ}{\mathrm{H}}(\text { curl } \Omega) & :=\{E \in \mathrm{H}(\text { curl } ; \Omega): \tau E=0\}=\stackrel{\circ}{\mathrm{C}} \infty(\Omega) \\
\mathrm{H}\left(\text { curl }_{0} ; \Omega\right) & :=\stackrel{\circ}{\mathrm{H}}(\text { curl } ; \Omega) \cap \mathrm{H}\left(\text { curl }_{0} ; \Omega\right)
\end{aligned}
\]
analogously:
```

    \(H(\operatorname{div} ; \Omega):=\left\{E \in L^{2}(\Omega): \operatorname{div} E \in L^{2}(\Omega)\right\}\)
    \(H\left(\operatorname{div}_{0} ; \Omega\right):=\{E \in H(\operatorname{div} ; \Omega): \operatorname{div} E=0\}\)
    $E \in \varepsilon^{-1} \mathrm{H}\left(\operatorname{div}_{0} ; \Omega\right) \Leftrightarrow \varepsilon E \in \mathrm{H}\left(\operatorname{div}_{0} ; \Omega\right)$
and:

```
```

$\mathcal{H}_{\varepsilon}(\Omega):=\mathrm{H}\left(\operatorname{curl}_{0} ; \Omega\right) \cap \varepsilon^{-1} \mathrm{H}\left(\operatorname{div}_{0} ; \Omega\right)$
$=\left\{E \in \mathrm{~L}^{2}(\Omega): \operatorname{curl} E=0, \operatorname{div} \varepsilon E=0, \tau E=0\right\}$

```

\section*{Introduction: Sobolev Spaces}

\section*{spaces:}
\(\mathrm{H}(\) curl \(; \Omega):=\left\{E \in \mathrm{~L}^{2}(\Omega):\right.\) curl \(\left.E \in \mathrm{~L}^{2}(\Omega)\right\}\)
\(H(\) curlo \(; \Omega):=\{E \in H(\) curl \(; \Omega):\) curl \(E=0\}\)
\(\stackrel{\circ}{\mathrm{H}}(\) curl \(; \Omega):=\{E \in \mathrm{H}(\) curl \(; \Omega): \tau E=0\}=\bar{\circ}{ }^{\circ}(\Omega)\)
\(H(\) curlo: \(\Omega):=H(\) curl: \(\Omega) \cap \mathrm{H}(\) curlo: \(\Omega)\)
analogously:
\(H(\operatorname{div} ; \Omega):=\left\{E \in L^{2}(\Omega): \operatorname{div} E \in L^{2}(\Omega)\right\}\)
\(H\left(\operatorname{div}_{0} ; \Omega\right):=\{E \in \mathrm{H}(\operatorname{div} ; \Omega): \operatorname{div} E=0\}\)
\(E \in \varepsilon^{-1} \mathrm{H}\left(\operatorname{div}_{n} ; \Omega\right) \Leftrightarrow \varepsilon E \in \mathrm{H}\left(\operatorname{div}_{n} ; \Omega\right)\)
and:
\(\mathcal{H}_{\varepsilon}(\Omega):=\dot{H}\left(\operatorname{curl}_{0} ; \Omega\right) \cap \varepsilon^{-1} \mathrm{H}\left(\operatorname{div}_{0} ; \Omega\right)\)

\section*{Introduction: Sobolev Spaces}

\section*{spaces:}
\[
\begin{aligned}
\mathrm{H}(\operatorname{curl} ; \Omega) & :=\left\{E \in \mathrm{~L}^{2}(\Omega): \operatorname{curl} E \in \mathrm{~L}^{2}(\Omega)\right\} \\
\mathrm{H}\left(\text { curl }_{0} ; \Omega\right) & :=\{E \in \mathrm{H}(\operatorname{curl} ; \Omega): \operatorname{curl} E=0\}
\end{aligned}
\]
\(\mathrm{H}(\) curl \(; \Omega):=\{E \in \mathrm{H}(\) curl \(; \Omega): \tau E=0\}=\mathrm{C}^{\infty}(\Omega)\)
\(H(\) curlo: \(\Omega):-H(\) curl \(; \Omega) \cap H(\) curl \(0: \Omega)\)
analogously:
\(H(\operatorname{div} ; \Omega):=\left\{E \in L^{2}(\Omega): \operatorname{div} E \in L^{2}(\Omega)\right\}\)
\(\mathrm{H}\left(\operatorname{div}_{0} ; \Omega\right):=\{E \in \mathrm{H}(\operatorname{div} ; \Omega): \operatorname{div} E=0\}\)
\(E \in \varepsilon^{-1} \mathrm{H}\left(\operatorname{div}_{0} ; \Omega\right) \Leftrightarrow \varepsilon E \in \mathrm{H}\left(\operatorname{div}_{0} ; \Omega\right)\)
and:
\(\mathcal{H}_{\varepsilon}(\Omega):=\mathrm{H}\left(\operatorname{curl}_{0} ; \Omega\right) \cap \varepsilon^{-1} \mathrm{H}\left(\operatorname{div}_{0} ; \Omega\right)\)

\section*{Introduction: Sobolev Spaces}
spaces:
\[
\begin{aligned}
\mathrm{H}(\operatorname{curl} ; \Omega) & :=\left\{E \in \mathrm{~L}^{2}(\Omega): \operatorname{curl} E \in \mathrm{~L}^{2}(\Omega)\right\} \\
\mathrm{H}\left(\operatorname{curl}_{0} ; \Omega\right) & :=\{E \in \mathrm{H}(\operatorname{curl} ; \Omega): \operatorname{curl} E=0\} \\
\stackrel{\circ}{\mathrm{H}}(\operatorname{curl} ; \Omega) & :=\{E \in \mathrm{H}(\operatorname{curl} ; \Omega): \tau E=0\}=\bar{\circ} \stackrel{\circ}{C}^{\circ}(\Omega)
\end{aligned}
\]

\section*{Introduction: Sobolev Spaces}
spaces:
\[
\begin{aligned}
\mathrm{H}(\text { curl } ; \Omega) & :=\left\{E \in \mathrm{~L}^{2}(\Omega): \text { curl } E \in \mathrm{~L}^{2}(\Omega)\right\} \\
\mathrm{H}\left(\text { curl }_{0} ; \Omega\right) & :=\{E \in \mathrm{H}(\text { curl } ; \Omega): \text { curl } E=0\} \\
\mathrm{H}(\text { curl } ; \Omega) & :=\{E \in \mathrm{H}(\text { curl } ; \Omega): \tau E=0\}=\overline{\mathrm{C}^{\circ} \infty(\Omega)} \mathrm{H}(\text { curl } ; \Omega)
\end{aligned}
\]
(Gauß' theorem)
analogously:
\(\mathrm{H}(\operatorname{div} ; \Omega):=\left\{E \in \mathrm{~L}^{2}(\Omega): \operatorname{div} E \in \mathrm{~L}^{2}(\Omega)\right\}\) \(\mathrm{H}\left(\operatorname{div}_{0} ; \Omega\right):=\{E \in \mathrm{H}(\operatorname{div} ; \Omega): \operatorname{div} E=0\}\) \(E \in \varepsilon^{-1} H\left(\operatorname{div}_{0} ; \Omega\right) \Leftrightarrow \varepsilon E \in H\left(\operatorname{div}_{0} ; \Omega\right)\)
and
\(\mathcal{H}_{\varepsilon}(\Omega):=\mathrm{H}\left(\operatorname{curl}_{0} ; \Omega\right) \cap \varepsilon^{-1} \mathrm{H}\left(\operatorname{div}_{0} ; \Omega\right)\) \(=\left\{E \in \mathrm{~L}^{2}(\Omega): \operatorname{curl} E=0, \operatorname{div} \varepsilon E=0, \tau E=0\right\}\)

\section*{Introduction: Sobolev Spaces}
spaces:
\[
\begin{aligned}
& \mathrm{H}(\operatorname{curl} ; \Omega):=\left\{E \in \mathrm{~L}^{2}(\Omega): \operatorname{curl} E \in \mathrm{~L}^{2}(\Omega)\right\} \\
& \mathrm{H}\left(\text { curl }_{0} ; \Omega\right):=\{E \in \mathrm{H}(\text { curl } ; \Omega): \operatorname{curl} E=0\} \\
& \stackrel{\circ}{\mathrm{H}}(\operatorname{curl} ; \Omega):=\{E \in \mathrm{H}(\operatorname{curl} ; \Omega): \tau E=0\}=\stackrel{\circ}{\mathrm{C}} \infty(\Omega) \\
& \mathrm{H}\left(\text { curl }^{\prime} \Omega\right) \\
& \mathrm{H}\left(\operatorname{curl}_{0} ; \Omega\right):=\stackrel{\circ}{\mathrm{H}}(\operatorname{curl} ; \Omega) \cap \mathrm{H}\left(\operatorname{curl}_{0} ; \Omega\right)
\end{aligned}
\]
analogously:
\[
\begin{aligned}
\mathrm{H}(\operatorname{div} ; \Omega) & :=\left\{E \in \mathrm{~L}^{2}(\Omega): \operatorname{div} E \in \mathrm{~L}^{2}(\Omega)\right\} \\
\mathrm{H}\left(\operatorname{div}_{0} ; \Omega\right) & :=\{E \in \mathrm{H}(\operatorname{div} ; \Omega): \operatorname{div} E=0\}
\end{aligned}
\]
\(E \in \varepsilon^{-1} H\left(\operatorname{div}_{0} ; \Omega\right) \Leftrightarrow \varepsilon E \in H\left(\operatorname{div}_{0} ; \Omega\right)\)
and
\(\mathcal{H}_{\varepsilon}(\Omega):=\mathrm{H}\left(\right.\) curl \(\left._{0} ; \Omega\right) \cap \varepsilon^{-1} \mathrm{H}\left(\operatorname{div}_{0} ; \Omega\right)\) \(=\left\{E \in \mathrm{~L}^{2}(\Omega): \operatorname{curl} E=0, \operatorname{div} \varepsilon E=0, \tau E=0\right\}\)

\section*{Introduction: Sobolev Spaces}
spaces:
\[
\begin{aligned}
\mathrm{H}(\text { curl } ; \Omega) & :=\left\{E \in \mathrm{~L}^{2}(\Omega): \operatorname{curl} E \in \mathrm{~L}^{2}(\Omega)\right\} \\
\mathrm{H}\left(\text { curl }_{0} ; \Omega\right) & :=\{E \in \mathrm{H}(\text { curl } ; \Omega): \operatorname{curl} E=0\} \\
\mathrm{H}(\text { curl } ; \Omega) & :=\{E \in \mathrm{H}(\text { curl } ; \Omega): \tau E=0\}=\overline{\mathrm{C}^{\infty}(\Omega)} \mathrm{H}(\text { curl } ; \Omega)
\end{aligned}
\]
(Gauß' theorem)
analogously:
\[
\begin{aligned}
\mathrm{H}(\operatorname{div} \Omega) & :=\left\{E \in \mathrm{~L}^{2}(\Omega): \operatorname{div} E \in \mathrm{~L}^{2}(\Omega)\right\} \\
\mathrm{H}\left(\operatorname{div}_{0} ; \Omega\right) & :=\{E \in \mathrm{H}(\operatorname{div} ; \Omega): \operatorname{div} E=0\} \\
E \in \varepsilon^{-1} \mathrm{H}\left(\operatorname{div}_{0} ; \Omega\right) & \Leftrightarrow \varepsilon E \in \mathrm{H}\left(\operatorname{div}_{0} ; \Omega\right)
\end{aligned}
\]
and:

\title{
\(\mathcal{H}_{\varepsilon}(\Omega):=\mathrm{H}\left(\operatorname{curl}_{0} ; \Omega\right) \cap \varepsilon^{-1} \mathrm{H}\left(\operatorname{div}_{0} ; \Omega\right)\)
}

\section*{Introduction: Sobolev Spaces}
spaces:
\[
\begin{aligned}
\mathrm{H}(\operatorname{curl} ; \Omega) & :=\left\{E \in \mathrm{~L}^{2}(\Omega): \operatorname{curl} E \in \mathrm{~L}^{2}(\Omega)\right\} \\
\mathrm{H}\left(\text { curl }_{0} ; \Omega\right) & :=\{E \in \mathrm{H}(\text { curl } ; \Omega): \operatorname{curl} E=0\} \\
\mathrm{O}(\operatorname{curl} ; \Omega) & :=\{E \in \mathrm{H}(\text { curl } ; \Omega): \tau E=0\}=\overline{\mathrm{C}^{\circ} \infty(\Omega)} \mathrm{H}(\text { curl } ; \Omega) \\
\mathrm{H}\left(\operatorname{curl}_{0} ; \Omega\right) & :=\stackrel{\circ}{\mathrm{H}}\left(\operatorname{curl}^{\prime} ; \Omega\right) \cap \mathrm{H}\left(\operatorname{curl}_{0} ; \Omega\right)
\end{aligned}
\]
(Gauß' theorem)
analogously:
\[
\begin{aligned}
& \mathrm{H}(\operatorname{div} \Omega):=\left\{E \in \mathrm{~L}^{2}(\Omega): \operatorname{div} E \in \mathrm{~L}^{2}(\Omega)\right\} \\
& \mathrm{H}\left(\operatorname{div}_{0} ; \Omega\right):=\{E \in \mathrm{H}(\operatorname{div} ; \Omega): \operatorname{div} E=0\} \\
& E \in \varepsilon^{-1} \mathrm{H}\left(\operatorname{div}_{0} ; \Omega\right) \Leftrightarrow \varepsilon E \in \mathrm{H}\left(\operatorname{div}_{0} ; \Omega\right) \\
& \text { and: }
\end{aligned}
\]
\[
\begin{aligned}
\mathcal{H}_{\varepsilon}(\Omega) & :=\stackrel{\circ}{\mathrm{H}}\left(\operatorname{curl}_{0} ; \Omega\right) \cap \varepsilon^{-1} \mathrm{H}\left(\operatorname{div}_{0} ; \Omega\right) \\
& =\left\{E \in \mathrm{~L}^{2}(\Omega): \operatorname{curl} E=0, \operatorname{div} \varepsilon E=0, \tau E=0\right\}
\end{aligned}
\]
(finite dimension)

\section*{Variational Formulation}
testing curl \(\mu^{-1}\) curl \(E=F\) with \(\Phi \in \stackrel{\circ}{\mathrm{H}}(\operatorname{curl} ; \Omega)\)
\[
\varphi(\Phi):=\langle F, \Phi\rangle_{\Omega}=\left\langle\mu^{-1} \operatorname{curl} E, \operatorname{curl} \Phi\right\rangle_{\Omega}=: b(E, H)
\]
unfortunately: H (curl; \(\Omega\) ) is not the proper Hilbert space! (kernel of curl)


special case: \(\forall E \in \mathbb{H}:=\mathrm{H}(\operatorname{curl} ; \Omega) \cap \varepsilon^{-1} \mathrm{H}\left(\operatorname{div}_{0} ; \Omega\right) \cap \mathcal{H}_{\varepsilon}(\Omega)^{\perp_{\varepsilon}}\)

\(\Rightarrow b\) bilinear, continuous and coercive over \(\mathbb{H}, \varphi\) linear and continuous over \(\mathbb{H}\)
Lax-Mitgram \(\Rightarrow\) unique solution \(E \in \mathbb{H}+\check{\tau} G\) with proper tang. ext. operator \(\asymp\)
key tool: compact embedding of \(H(\operatorname{curl} ; \Omega) \cap H(\operatorname{div} ; \Omega)\) into \(L^{2}(\Omega)\)

\section*{Variational Formulation}
testing curl \(\mu^{-1}\) curl \(E=F\) with \(\Phi \in \stackrel{\circ}{\mathrm{H}}(\) curl \(; \Omega)\)
\[
\varphi(\Phi):=\langle F, \Phi\rangle_{\Omega}=\left\langle\mu^{-1} \operatorname{curl} E, \operatorname{curl} \Phi\right\rangle_{\Omega}=: b(E, H)
\]
unfortunately: H (curl; \(\Omega\) ) is not the proper Hilbert space! (kernel of curl)
Poincaré-Friedrichs inequality: \(\exists \mathrm{cpp}>0 \quad \forall E \in H(c u r i ; \Omega) \cap \varepsilon^{-1} H(d i v ; \Omega)\)

special case: \(\forall E \in \mathbb{H}:=\mathrm{H}(\operatorname{curl} ; \Omega) \cap \varepsilon^{-1} \mathrm{H}\left(\operatorname{div}_{0} ; \Omega\right) \cap \mathcal{H}_{\varepsilon}(\Omega)^{\perp_{\varepsilon}}\)
\(\qquad\)
\(\Rightarrow b\) bilinear, continuous and coercive over \(\mathbb{H}, \varphi\) linear and continuous over \(\mathbb{H}\)
Lax-Milgram \(\Rightarrow\) unique solution \(E \in \pi \Pi+\varkappa G\) with proper tang. ext. operator \(\nsucc\)
key tool: compact embedding of \(\mathrm{H}(\operatorname{curl} ; \Omega) \cap \mathrm{H}(\operatorname{div} ; \Omega)\) into \(\mathrm{L}^{2}(\Omega)\)

\section*{Variational Formulation}
testing curl \(\mu^{-1}\) curl \(E=F\) with \(\Phi \in \stackrel{\circ}{\mathrm{H}}(\operatorname{curl} ; \Omega)\)
\[
\varphi(\Phi):=\langle F, \Phi\rangle_{\Omega}=\left\langle\mu^{-1} \text { curl } E, \operatorname{curl} \Phi\right\rangle_{\Omega}=: b(E, H)
\]
unfortunately: \(\stackrel{\circ}{\mathrm{H}}(\operatorname{curl} ; \Omega)\) is not the proper Hilbert space! (kernel of curl)
Poincaré-Friedrichs inequality: \(\exists C_{\mathrm{PF}}>0 \quad \forall E \in H(\operatorname{curl} ; \Omega) \cap \varepsilon^{-1} H(\operatorname{div} ; \Omega)\)

special case: \(\forall E \in \mathbb{H}:=\mathrm{H}(\operatorname{curl} ; \Omega) \cap \varepsilon^{-1} \mathrm{H}\left(\operatorname{div}_{0} ; \Omega\right) \cap \mathcal{H}_{\varepsilon}(\Omega)^{\perp \varepsilon}\)
\(\square\)
\(\Rightarrow b\) bilinear, continuous and coercive over \(\mathbb{H}, \varphi\) linear and continuous over \(\mathbb{H}\)
Lax-Milgram \(\Rightarrow\) unique solution \(E \in \mathbb{H}+千 G\) with proper tang. ext. operator \(\not \subset\)
key tool: compact embedding of \(\mathrm{H}(\operatorname{curl} ; \Omega) \cap \mathrm{H}(\operatorname{div} ; \Omega)\) into \(\mathrm{L}^{2}(\Omega)\)

\section*{Variational Formulation}
testing curl \(\mu^{-1}\) curl \(E=F\) with \(\Phi \in \stackrel{\circ}{\mathrm{H}}(\operatorname{curl} ; \Omega)\)
\[
\varphi(\Phi):=\langle F, \Phi\rangle_{\Omega}=\left\langle\mu^{-1} \text { curl } E, \operatorname{curl} \Phi\right\rangle_{\Omega}=: b(E, H)
\]
unfortunately: \(\stackrel{\circ}{\mathrm{H}}(\operatorname{curl} ; \Omega)\) is not the proper Hilbert space! (kernel of curl)
Poincaré-Friedrichs inequality: \(\exists C_{p F}>0 \quad \forall E \in H(\) curl \(; \Omega) \cap \varepsilon^{-1} H(\operatorname{div} ; \Omega)\)

special case: \(\forall E \in \mathbb{H}:=\mathrm{H}(\operatorname{curl} ; \Omega) \cap \varepsilon^{-1} \mathrm{H}\left(\operatorname{div}_{0} ; \Omega\right) \cap \mathcal{H}_{\varepsilon}(\Omega)^{\perp_{\varepsilon}}\) \(\|E\|_{L^{2}(\Omega)} \leq C_{P F}\|c u r \mid E\|_{L^{2}(\Omega)}\)
\(\Rightarrow b\) bilinear, continuous and coercive over \(\mathbb{H}, \varphi\) linear and continuous over \(\mathbb{H}\)
Lax-Milgram \(\Rightarrow\) unique solution \(E \in \mathbb{H}+千 G\) with proper tang. ext. operator \(\not \subset\)
key tool: compact embedding of \(\mathrm{H}(\operatorname{curl} ; \Omega) \cap \mathrm{H}(\operatorname{div} ; \Omega)\) into \(\mathrm{L}^{2}(\Omega)\)

\section*{Variational Formulation}
testing curl \(\mu^{-1}\) curl \(E=F\) with \(\Phi \in \stackrel{\circ}{\mathrm{H}}(\) curl \(; \Omega)\)
\[
\varphi(\Phi):=\langle F, \Phi\rangle_{\Omega}=\left\langle\mu^{-1} \text { curl } E, \text { curl } \Phi\right\rangle_{\Omega}=: b(E, H)
\]
unfortunately: \(\stackrel{\circ}{\mathrm{H}}\) (curl; \(\Omega\) ) is not the proper Hilbert space! (kernel of curl)
Poincaré-Friedrichs inequality: \(\exists c_{\mathrm{PF}}>0 \quad \forall E \in \mathrm{H}(\operatorname{curl} ; \Omega) \cap \varepsilon^{-1} \mathrm{H}(\operatorname{div} ; \Omega)\)
\[
c_{\mathrm{PF}}^{-1}\|E\|_{\mathrm{L}^{2}(\Omega)} \leq\|\operatorname{curl} E\|_{\mathrm{L}^{2}(\Omega)}+\|\operatorname{div} \varepsilon E\|_{\mathrm{L}^{2}(\Omega)}+\|\tau E\|_{\text {trace }}+\sum_{\ell \text { finite }}\left|\left\langle\varepsilon E, E_{\ell}\right\rangle_{\Omega}\right|
\]
special case: \(\forall E \in \mathbb{H}:=\mathrm{H}\left(\operatorname{curl}_{1} ; \Omega\right) \cap \varepsilon^{-1} \mathrm{H}\left(\operatorname{div}_{0} ; \Omega\right) \cap \mathcal{H}_{\varepsilon}(\Omega)^{\perp_{\varepsilon}}\)

\section*{Variational Formulation}
testing curl \(\mu^{-1}\) curl \(E=F\) with \(\Phi \in \stackrel{\circ}{\mathrm{H}}(\) curl \(; \Omega)\)
\[
\varphi(\Phi):=\langle F, \Phi\rangle_{\Omega}=\left\langle\mu^{-1} \text { curl } E, \text { curl } \Phi\right\rangle_{\Omega}=: b(E, H)
\]
unfortunately: \(\stackrel{\circ}{\mathrm{H}}\) (curl; \(\Omega\) ) is not the proper Hilbert space! (kernel of curl)
Poincaré-Friedrichs inequality: \(\exists c_{\mathrm{PF}}>0 \quad \forall E \in \mathrm{H}(\operatorname{curl} ; \Omega) \cap \varepsilon^{-1} \mathrm{H}(\operatorname{div} ; \Omega)\)
\[
c_{\mathrm{PF}}^{-1}\|E\|_{\mathrm{L}^{2}(\Omega)} \leq\|\operatorname{curl} E\|_{L^{2}(\Omega)}+\|\operatorname{div} \varepsilon E\|_{L^{2}(\Omega)}+\|\tau E\|_{\text {trace }}+\sum_{\ell \text { finite }}\left|\left\langle\varepsilon E, E_{\ell}\right\rangle_{\Omega}\right|
\]
special case: \(\forall E \in \mathbb{H}:=\stackrel{\circ}{\mathrm{H}}(\operatorname{curl} ; \Omega) \cap \varepsilon^{-1} \mathrm{H}\left(\operatorname{div}_{0} ; \Omega\right) \cap \mathcal{H}_{\varepsilon}(\Omega)^{\perp_{\varepsilon}}\)
\[
\|E\|_{L^{2}(\Omega)} \leq c_{\mathrm{PF}}\|\operatorname{curl} E\|_{\mathrm{L}^{2}(\Omega)}
\]
\(\Rightarrow b\) bilinear, continuous and coercive over \(\mathbb{H}, \varphi\) linear and continuous over \(\mathbb{H}\)
Lax-Milgram \(\Rightarrow\) unique solution \(E \in \mathbb{H}+\check{\tau} G\) with proper tang. ext. operator \(\check{\tau}\)
key tool: compact embedding of \(\mathrm{H}(\operatorname{curl} ; \Omega) \cap \mathrm{H}(\operatorname{div} ; \Omega)\) into \(\mathrm{L}^{2}(\Omega)\)

\section*{Variational Formulation}
testing curl \(\mu^{-1}\) curl \(E=F\) with \(\Phi \in \stackrel{\circ}{\mathrm{H}}(\) curl \(; \Omega)\)
\[
\varphi(\Phi):=\langle F, \Phi\rangle_{\Omega}=\left\langle\mu^{-1} \text { curl } E, \text { curl } \Phi\right\rangle_{\Omega}=: b(E, H)
\]
unfortunately: \(\stackrel{\circ}{\mathrm{H}}\) (curl; \(\Omega\) ) is not the proper Hilbert space! (kernel of curl)
Poincaré-Friedrichs inequality: \(\exists c_{\mathrm{PF}}>0 \quad \forall E \in \mathrm{H}(\operatorname{curl} ; \Omega) \cap \varepsilon^{-1} \mathrm{H}(\operatorname{div} ; \Omega)\)
\[
c_{\mathrm{PF}}^{-1}\|E\|_{\mathrm{L}^{2}(\Omega)} \leq\|\operatorname{curl} E\|_{L^{2}(\Omega)}+\|\operatorname{div} \varepsilon E\|_{L^{2}(\Omega)}+\|\tau E\|_{\text {trace }}+\sum_{\ell \text { finite }}\left|\left\langle\varepsilon E, E_{\ell}\right\rangle_{\Omega}\right|
\]
special case: \(\forall E \in \mathbb{H}:=\stackrel{\circ}{\mathrm{H}}(\operatorname{curl} ; \Omega) \cap \varepsilon^{-1} \mathrm{H}\left(\operatorname{div}_{0} ; \Omega\right) \cap \mathcal{H}_{\varepsilon}(\Omega)^{\perp_{\varepsilon}}\)
\[
\|E\|_{L^{2}(\Omega)} \leq c_{\mathrm{PF}}\|\operatorname{curl} E\|_{\mathrm{L}^{2}(\Omega)}
\]
\(\Rightarrow b\) bilinear, continuous and coercive over \(\mathbb{H}, \varphi\) linear and continuous over \(\mathbb{H}\)
Lax-Milgram \(\Rightarrow\) unique solution \(E \in \mathbb{H}+\check{\tau} G\) with proper tang. ext. operator \(\check{\tau}\)
key tool: compact embedding of \(\mathrm{H}(\) curl \(; \Omega) \cap \mathrm{H}(\operatorname{div} ; \Omega)\) into \(\mathrm{L}^{2}(\Omega)\)

\section*{Variational Formulation}
testing curl \(\mu^{-1}\) curl \(E=F\) with \(\Phi \in \stackrel{\circ}{\mathrm{H}}(\) curl \(; \Omega)\)
\[
\varphi(\Phi):=\langle F, \Phi\rangle_{\Omega}=\left\langle\mu^{-1} \text { curl } E, \text { curl } \Phi\right\rangle_{\Omega}=: b(E, H)
\]
unfortunately: \(\stackrel{\circ}{\mathrm{H}}\) (curl; \(\Omega\) ) is not the proper Hilbert space! (kernel of curl)
Poincaré-Friedrichs inequality: \(\exists c_{\mathrm{PF}}>0 \quad \forall E \in \mathrm{H}(\operatorname{curl} ; \Omega) \cap \varepsilon^{-1} \mathrm{H}(\operatorname{div} ; \Omega)\)
\[
c_{\mathrm{PF}}^{-1}\|E\|_{\mathrm{L}^{2}(\Omega)} \leq\|\operatorname{curl} E\|_{L^{2}(\Omega)}+\|\operatorname{div} \varepsilon E\|_{L^{2}(\Omega)}+\|\tau E\|_{\text {trace }}+\sum_{\ell \text { finite }}\left|\left\langle\varepsilon E, E_{\ell}\right\rangle_{\Omega}\right|
\]
special case: \(\forall E \in \mathbb{H}:=\stackrel{\circ}{\mathrm{H}}(\operatorname{curl} ; \Omega) \cap \varepsilon^{-1} \mathrm{H}\left(\operatorname{div}_{0} ; \Omega\right) \cap \mathcal{H}_{\varepsilon}(\Omega)^{\perp_{\varepsilon}}\)
\[
\|E\|_{L^{2}(\Omega)} \leq c_{\mathrm{PF}}\|\operatorname{curl} E\|_{\mathrm{L}^{2}(\Omega)}
\]
\(\Rightarrow b\) bilinear, continuous and coercive over \(\mathbb{H}, \varphi\) linear and continuous over \(\mathbb{H}\)
Lax-Milgram \(\Rightarrow\) unique solution \(E \in \mathbb{H}+\check{\tau} G\) with proper tang. ext. operator \(\check{\tau}\)
key tool: compact embedding of \(\mathrm{H}(\mathrm{curl} ; \Omega) \cap \mathrm{H}(\operatorname{div} ; \Omega)\) into \(\mathrm{L}^{2}(\Omega)\)

\section*{Variational Formulation}
testing curl \(\mu^{-1}\) curl \(E=F\) with \(\Phi \in \stackrel{\circ}{\mathrm{H}}(\) curl \(; \Omega)\)
\[
\varphi(\Phi):=\langle F, \Phi\rangle_{\Omega}=\left\langle\mu^{-1} \text { curl } E, \text { curl } \Phi\right\rangle_{\Omega}=: b(E, H)
\]
unfortunately: \(\stackrel{\circ}{\mathrm{H}}(\mathrm{curl} ; \Omega)\) is not the proper Hilbert space! (kernel of curl)
Poincaré-Friedrichs inequality: \(\exists c_{\mathrm{PF}}>0 \quad \forall E \in \mathrm{H}(\mathrm{curl} ; \Omega) \cap \varepsilon^{-1} \mathrm{H}(\operatorname{div} ; \Omega)\)
\[
c_{\mathrm{PF}}^{-1}\|E\|_{\mathrm{L}^{2}(\Omega)} \leq\|\operatorname{curl} E\|_{\mathrm{L}^{2}(\Omega)}+\|\operatorname{div} \varepsilon E\|_{\mathrm{L}^{2}(\Omega)}+\|\tau E\|_{\text {trace }}+\sum_{\ell \text { finite }}\left|\left\langle\varepsilon E, E_{\ell}\right\rangle_{\Omega}\right|
\]
special case: \(\forall E \in \mathbb{H}:=\stackrel{\circ}{\mathrm{H}}(\operatorname{curl} ; \Omega) \cap \varepsilon^{-1} \mathrm{H}\left(\operatorname{div}_{0} ; \Omega\right) \cap \mathcal{H}_{\varepsilon}(\Omega)^{\perp_{\varepsilon}}\)
\[
\|E\|_{L^{2}(\Omega)} \leq c_{\mathrm{PF}}\|\operatorname{curl} E\|_{\mathrm{L}^{2}(\Omega)}
\]
\(\Rightarrow b\) bilinear, continuous and coercive over \(\mathbb{H}, \varphi\) linear and continuous over \(\mathbb{H}\)
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key tool: compact embedding of \(\mathrm{H}(\operatorname{curl} ; \Omega) \cap \mathrm{H}(\operatorname{div} ; \Omega)\) into \(\mathrm{L}^{2}(\Omega)\)
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\section*{Upper and Lower Bounds for Non-Conforming Approximations}
\(\tilde{H} \in \mathrm{~L}^{2}(\Omega)\) approximation of curl \(E=\mu H\) (first, only approximation of magnetic field)
Theorem 1 For all \(H \in L^{2}(\Omega)\) the estimates


\section*{hold. Here,}
\[
\begin{aligned}
& M_{+}(\tilde{H} ; Y, \operatorname{curl} Y):=c_{\mu, 1}\left(c_{\mathrm{PF}}\|F-\operatorname{curl} Y\|_{\mathrm{L}^{2}(\Omega)}+\left\|\mu^{-1} \tilde{H}-Y\right\|_{L^{2}(\Omega)}\right. \\
& m_{+}(\tilde{H} ; \operatorname{curl} X):=c_{\mu, 2}\|\operatorname{curl} X-\tilde{H}\|_{\mathrm{L}^{2}(\Omega)} \\
& M_{-}(\tilde{H}: X, \operatorname{curl} X):=2\langle F, X\rangle_{L^{2}(\Omega)}-\left\langle\mu^{-1}(\operatorname{curl} X+2 \tilde{H}), \operatorname{curl} X\right\rangle_{L^{2}(\Omega)} \\
& m_{-}(\tilde{H} ; Y, \operatorname{curl} Z):=2\left\langle\mu^{-1}(\operatorname{curl} Z-\tilde{H}), Y\right\rangle_{\mathrm{L}^{2}(\Omega)}-\|Y\|_{\mathrm{L}^{2}(\Omega)}^{2} \\
& \text { Niversitat } \\
& D_{E} U_{S} S_{S E} S_{N} \text { URG }
\end{aligned}
\]

\section*{Upper and Lower Bounds for Non-Conforming Approximations}
\(\tilde{H} \in \mathrm{~L}^{2}(\Omega)\) approximation of curl \(E=\mu H\) (first, only approximation of magnetic field)
Theorem 1 For all \(\tilde{H} \in \mathrm{~L}^{2}(\Omega)\) the estimates
\[
\begin{aligned}
& \|\mu H-\tilde{H}\|_{L^{2}(\Omega)} \leq \inf _{Y \in \mathrm{H}(\operatorname{curl} ; \Omega)} M_{+}(\tilde{H} ; Y, \operatorname{curl} Y)+\inf _{\substack{X \in \mathrm{H}(\operatorname{curl} ; \Omega) \\
\tau X=G}} m_{+}(\tilde{H} ; \operatorname{curl} X), \\
& \|\mu H-\tilde{H}\|_{L^{2}(\Omega)}^{2} \geq \sup _{\substack{\circ \\
X \in \mathrm{H}(\operatorname{curl} ; \Omega)}} M_{-}(\tilde{H} ; X, \operatorname{curl} X)+\sup _{\substack{\mu^{-1} Y, Z \in \mathrm{H}(\operatorname{curl} ; \Omega) \\
\operatorname{curl} \mu^{-1} Y=0 \\
\tau Z=G}} m_{-}(\tilde{H} ; Y, \operatorname{curl} Z)
\end{aligned}
\]
hold.
Here,
\(\square\)

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M_{+}(\tilde{H} ; Y, \operatorname{curl} Y) & :=c_{\mu, 1}\left(c_{\mathrm{PF}}\|F-\operatorname{curl} Y\|_{\mathrm{L}^{2}(\Omega)}+\left\|\mu^{-1} \tilde{H}-Y\right\|_{\mathrm{L}^{2}(\Omega)}\right), \\
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\section*{Upper and Lower Bounds for (Very) Conforming Approximations}
\(\tilde{E} \in \mathrm{H}(\) curl \(; \Omega)\) approx. of \(E\) and \(\tilde{H}:=\operatorname{curl} \tilde{E} \in \mathrm{~L}^{2}(\Omega)\) approx. of curl \(E=\mu H\)
Corollary \(\mathbf{1}\) For all \(\tilde{E} \in H(c u r l ; \Omega)\)


Corollary 2 For all \(\tilde{E} \in \mathrm{H}(\operatorname{curl} ; \Omega)\) with \(\tau \tilde{E}=G\), i.e., \(E-\tilde{E} \in \mathrm{H}(\) curl \(; \Omega)\)
sup
\(X \in H(\) curl \(; \Omega)\)

universitat


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Corollary 1 For all \(\tilde{E} \in \mathrm{H}(\) curl \(; \Omega)\)
\[
\begin{aligned}
\|\operatorname{curl}(E-\tilde{E})\|_{L^{2}(\Omega)} \leq & \inf _{Y \in \mathrm{H}(\operatorname{curl} ; \Omega)} M_{+}(\tilde{H} ; Y, \operatorname{curl} Y)+c_{\mu, 2} c_{\tau}\|G-\tau \tilde{E}\|_{\text {trace }} \\
\|\operatorname{curl}(E-\tilde{E})\|_{L^{2}(\Omega)}^{2} \geq & \sup _{X \in \mathrm{H}(\operatorname{curl} ; \Omega)} M_{-}(\tilde{H} ; X, \operatorname{curl} X) \\
& \quad+\sup _{Y \in \mu \mathrm{H}\left(\operatorname{curl}_{0} ; \Omega\right)}\left(2\left\langle G-\tau \tilde{E}, \mu^{-1} Y\right\rangle_{\text {trace }}-\|Y\|_{L^{2}(\Omega)}^{2}\right) .
\end{aligned}
\]

\section*{Corollary 2 For all \(\tilde{E} \in \mathrm{H}(\operatorname{curl} ; \Omega)\) with \(\tau \tilde{E}=G\), i.e., \(E-\tilde{E} \in \mathrm{H}(\) curl \(; \Omega)\)}

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\[
\sup _{X \in \mathrm{H}(\operatorname{curl} ; \Omega)} M_{-}(\tilde{H} ; X, \operatorname{curl} X) \leq\|\operatorname{curl}(E-\tilde{E})\|_{L^{2}(\Omega)}^{2} \leq \inf _{Y \in H(\operatorname{curl} ; \Omega)} M_{+}^{2}(\tilde{H} ; Y, \operatorname{curl} Y) .
\]

\section*{Norm Estimates}
- norm estimates for \(h=\mu H-\tilde{H}\)
- \(E \mapsto \|\) curl \(E \|_{L^{2}(\Omega)}\) semi-norm but not norm on \(H(\) curl \(; \Omega)\) or \(H(c u r l ; \Omega)\) (not controlling \(\|\operatorname{div} E\|_{L^{2}(\Omega)}\) and projection on Dirichlet fields)
- for conforming anproximations \(E \in \mathrm{H}(\) curl \(; \Omega) \cap \varepsilon^{-1} \mathrm{H}(\) div \(; \Omega)\) the semi-norm

is well defined and a norm on \(\mathrm{H}(\) curl \(; \Omega) \cap \varepsilon^{-1} \mathrm{H}(\operatorname{div} ; \Omega)\)
- \(\Rightarrow\) norm estimates for \(e=E-\tilde{E}\)

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- for conforming approximations \(E \in H(\operatorname{curl} ; \Omega) \cap \varepsilon^{-1} H(\operatorname{div} ; \Omega)\) the semi-norm

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- for conforming approximations \(E \in \mathrm{H}(\operatorname{curl} ; \Omega) \cap \varepsilon^{-1} \mathrm{H}(\operatorname{div} ; \Omega)\) the semi-norm
\[
E \mapsto\|E\|:=\|\operatorname{curl} E\|_{L^{2}(\Omega)}+\|\operatorname{div} \varepsilon E\|_{L^{2}(\Omega)}+\sum_{\ell=1}^{d}\left|\left\langle\varepsilon E, E_{\ell}\right\rangle_{L^{2}(\Omega)}\right|
\]
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\]
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Corollary 3 For all \(\tilde{E} \in H(\operatorname{curl} ; \Omega) \cap \varepsilon^{-1} \mathrm{H}(\operatorname{div} ; \Omega)\) with \(\tau \tilde{E}=G\) and \(\operatorname{div} \varepsilon \tilde{E}=0\) and \(\varepsilon \tilde{E} \perp \mathcal{H}_{\varepsilon}(\Omega)\), i.e., \(E-\tilde{E} \in \mathbb{H}\),
\(X \in \mathrm{H}(\) curl \(; \Omega)\)


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- \(\Rightarrow\) norm estimates for \(e=E-\tilde{E}\), e.g.,

Corollary 3 For all \(\tilde{E} \in \mathrm{H}(\tilde{\operatorname{curl}} ; \Omega) \cap \varepsilon^{-1} \mathrm{H}(\operatorname{div} ; \Omega)\) with \(\tau \tilde{E}=G\) and \(\operatorname{div} \varepsilon \tilde{E}=0\) and \(\varepsilon \tilde{E} \perp \mathcal{H}_{\varepsilon}(\Omega)\), i.e., \(E-\tilde{E} \in \mathbb{H}\),
\[
\sup _{X \in \stackrel{H}{(c u r l} ; \Omega)} M_{-}(\tilde{H} ; X, \operatorname{curl} X) \leq\|E-\tilde{E}\|^{2} \leq \inf _{Y \in H(\operatorname{curl} ; \Omega)} M_{+}^{2}(\tilde{H} ; Y, \text { curl } Y)
\]

\section*{Constants and Sharpness}
typical features of functional a posteriori error estimates
- estimates for errors: basic (integral) relations, constants for embedding inequalities \(C_{\text {PF }}, c_{\mu, i}, c_{\tau}\)
- recall e.g.: For \(\tilde{E} \in H(\operatorname{curl} ; \Omega) \cap \varepsilon^{-1} H\left(\operatorname{div}_{0} ; \Omega\right) \cap \mathcal{H}_{\varepsilon}(\Omega)^{\perp_{\varepsilon}}\) with \(\tau \tilde{E}=G\) \(\|E-\tilde{E}\| \leq \inf _{Y \in H(\operatorname{curl} ; \Omega)} M_{+}(\tilde{H} ; Y, \operatorname{curl} Y)\)


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\[
\begin{aligned}
\|E-\tilde{E}\| & \leq \inf _{Y \in \mathrm{H}(\operatorname{curl} ; \Omega)} M_{+}(\tilde{H} ; Y, \operatorname{curl} Y) \\
& =\inf _{Y \in \mathrm{H}(\operatorname{curl} ; \Omega)} c_{\mu, 1}\left(\operatorname{cpF}\|F-\operatorname{curl} Y\|_{L^{2}(\Omega)}+\left\|\mu^{-1} \tilde{H}-Y\right\|_{L^{2}(\Omega)}\right) .
\end{aligned}
\]
\(\Rightarrow \quad M_{+}(\tilde{H} ; Y, \operatorname{curl} Y)=0 \quad \Leftrightarrow \quad \mu Y=H=\tilde{H}=\operatorname{curl} \tilde{E} \wedge \tilde{E}=E \quad\) sharp

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\end{aligned}
\]

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\section*{Proofs (Helmholtz Decomposition)}
main tools: standard techniques and Helmholtz decomposition
\(\tilde{H}\) approximation of \(\mu H=\operatorname{curl} E, h:=\mu H-\tilde{H}\) error simplicity \(\varepsilon=\mu=\mathrm{id}\)
\[
\begin{aligned}
\mathrm{L}^{2}(\Omega) \ni h & =\operatorname{curl} E_{\mathrm{c}} \oplus H_{d} \\
\text { curl } E_{\mathrm{c}} & \in \operatorname{curl} \stackrel{\circ}{\mathrm{H}}(\mathrm{curl} ; \Omega) \\
H_{\mathrm{d}} & \in \mathrm{H}\left(\text { curl }_{0} ; \Omega\right)
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H_{\mathrm{d}} & \in \mathrm{H}\left(\operatorname{curl}_{0} ; \Omega\right)
\end{aligned}
\]
recall \(\mathbb{H}=H(\) curl \(; \Omega) \cap H\left(\operatorname{div}_{0} ; \Omega\right) \cap \mathcal{H}(\Omega)^{\perp}\) curl \(E_{c} \quad\) regular/conforming error \(H_{d}\) non-conforming error (bounclary error) orthogonality \(\Rightarrow\|h\|_{\mathrm{L}^{2}(\Omega)}^{2}=\|\) curl \(E_{\mathrm{c}}\left\|_{\mathrm{L}^{2}(\Omega)}^{2}+\right\| H_{d} \|_{\mathrm{L}^{2}(\Omega)}^{2}\)

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H_{\mathrm{d}} & \in \mathrm{H}\left(\operatorname{curl}_{0} ; \Omega\right) \\
\text { recall } \mathbb{H} & =\stackrel{\circ}{\mathrm{H}}(\operatorname{curl} ; \Omega) \cap \mathrm{H}\left(\operatorname{div}_{0} ; \Omega\right) \cap \mathcal{H}(\Omega)^{\perp}
\end{aligned}
\]
curl \(E_{c} \quad\) regular/conforming error
\(H_{\mathrm{d}}\) non-conforming error (boundary error)
orthogonality \(\Rightarrow\left\|h_{L^{2}(\Omega)}^{\|^{2}}=\right\| \operatorname{curl} E_{C}\left\|_{L^{2}(\Omega)}^{2}+\right\| \boldsymbol{H}_{\mathrm{d}} \|_{L^{2}(\Omega)}^{2}\)

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\text { curl } E_{\mathrm{c}} \quad & \text { regular/conforming error }
\end{aligned}
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orthogonality \(\Rightarrow\|h\|_{\mathrm{L}^{2}(\Omega)}^{2}=\|\) curl \(E_{\mathrm{c}}\left\|_{\mathrm{L}^{2}(\Omega)}^{2}+\right\| H_{\mathrm{d}} \|_{\mathrm{L}^{2}(\Omega)}^{2}\)

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\text { recall } \mathbb{H} & =\stackrel{\circ}{\mathrm{H}}(\operatorname{curl} ; \Omega) \cap \mathrm{H}\left(\operatorname{div}_{0} ; \Omega\right) \cap \mathcal{H}(\Omega)^{\perp} \\
\operatorname{curl} E_{\mathrm{c}} & \text { regular/conforming error } \\
H_{\mathrm{d}} & \text { non-conforming error (boundary error) } \\
\text { orthogonality } \Rightarrow h_{L^{2}(\Omega)}^{2} & =\| \text { curl } E_{\mathrm{c}}\left\|_{L^{2}(\Omega)}^{2}+\right\| H_{\mathrm{d}} \|_{L^{2}(\Omega)}^{2}
\end{aligned}
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& H_{\mathrm{d}} \in \mathrm{H}\left(\operatorname{curl}_{0} ; \Omega\right) \\
& \text { recall } \quad \mathbb{H}=\stackrel{\circ}{\mathrm{H}}(\operatorname{curl} ; \Omega) \cap \mathrm{H}\left(\operatorname{div}_{0} ; \Omega\right) \cap \mathcal{H}(\Omega)^{\perp} \\
& \operatorname{curl} E_{\mathrm{c}} \text { regular/conforming error } \\
& H_{\mathrm{d}} \quad \text { non-conforming error (boundary error) } \\
& \text { orthogonality } \Rightarrow \quad\|h\|_{\mathrm{L}^{2}(\Omega)}^{2}=\| \text { curl } E_{\mathrm{c}}\left\|_{\mathrm{L}^{2}(\Omega)}^{2}+\right\| H_{\mathrm{d}} \|_{\mathrm{L}^{2}(\Omega)}^{2}
\end{aligned}
\end{gathered}
\]

\section*{Proofs (Upper Bounds)}
recall \(h=H-\tilde{H}=\) curl \(E_{\mathrm{c}} \oplus H_{\mathrm{d}} \in \operatorname{curl} \mathbb{H} \oplus \mathrm{H}\left(\right.\) curl \(\left._{0} ; \Omega\right)\)
standard argument for curl \(E_{c}\) : for all \(\Phi \in \mathbb{H}\)

since \(\langle\text { curl } Y, \Phi\rangle_{L^{2}(\Omega)}=\langle Y, \operatorname{curl} \Phi\rangle_{L^{2}(\Omega)}\) for all \(Y \in \mathrm{H}(\) curl; \(\Omega)\); note \(\mathbb{H} \subset \mathrm{H}(\) curl \(; \Omega)\).
Cauchy-Schwarz and Poincaré-Friedrichs and \(\Phi:=E_{c} \in \mathbb{H} \quad \Rightarrow\)


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recall \(h=H-\tilde{H}=\) curl \(E_{\mathrm{c}} \oplus H_{\mathrm{d}} \in \operatorname{curl} \mathbb{H} \oplus \mathrm{H}\left(\right.\) curl \(\left._{0} ; \Omega\right)\)
standard argument for curl \(E_{\mathrm{c}}\) :

since \(\langle\text { curl } Y, \Phi\rangle_{L^{2}(\Omega)}=\langle Y, \operatorname{curl} \Phi\rangle_{L^{2}(\Omega)}\) for all \(Y \in \mathrm{H}(\) curl; \(\Omega)\); note \(\mathbb{H} \subset \mathrm{H}(\) curl \(; \Omega)\).
Cauchy-Schwarz and Poincaré-Friedrichs and \(\Phi:=E_{c} \in \mathbb{H} \quad \Rightarrow\)


\section*{Proofs (Upper Bounds)}
recall \(h=H-\tilde{H}=\) curl \(E_{\mathrm{c}} \oplus H_{\mathrm{d}} \in \operatorname{curl} \mathbb{H} \oplus \mathrm{H}\left(\right.\) curl \(\left._{0} ; \Omega\right)\)
standard argument for curl \(E_{\mathrm{c}}\) : for all \(\Phi \in \mathbb{H}\)
\[
\left\langle\operatorname{curl} E_{\mathrm{c}}, \operatorname{curl} \Phi\right\rangle_{\mathrm{L}^{2}(\Omega)}=\langle h, \operatorname{curl} \Phi\rangle_{L^{2}(\Omega)} \quad\left(H_{d} \perp \operatorname{curl} \Phi\right)
\]

since \(\langle\text { curl } Y, \Phi\rangle_{L^{2}(\Omega)}=\langle Y, \operatorname{curl} \Phi\rangle_{L^{2}(\Omega)}\) for all \(Y \in \mathrm{H}(\) curl \(; \Omega)\); note \(\mathbb{H} \subset \mathrm{H}(\) curl \(; \Omega)\)
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since \(\langle\text { curl } Y, \Phi\rangle_{L^{2}(\Omega)}=\langle Y, \operatorname{curl} \Phi\rangle_{L^{2}(\Omega)}\) for all \(Y \in \mathrm{H}(\) curl \(; \Omega)\); note \(\mathbb{H} \subset \mathrm{H}(\) curl \(; \Omega)\).
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standard argument for curl \(E_{\mathrm{c}}\) : for all \(\Phi \in \mathbb{H}\)
\[
\begin{aligned}
\left\langle\operatorname{curl} E_{\mathrm{c}}, \operatorname{curl} \Phi\right\rangle_{\mathrm{L}^{2}(\Omega)} & =\langle h, \operatorname{curl} \Phi\rangle_{\mathrm{L}^{2}(\Omega)} \quad\left(H_{\mathrm{d}} \perp \operatorname{curl} \Phi\right) \\
& =\langle F, \Phi\rangle_{\mathrm{L}^{2}(\Omega)}-\langle\tilde{H}, \operatorname{curl} \Phi\rangle_{\mathrm{L}^{2}(\Omega)}
\end{aligned}
\]

since \(\langle\text { curl } Y, \Phi\rangle_{L^{2}(\Omega)}=\langle Y, \operatorname{curl} \Phi\rangle_{L^{2}(\Omega)}\) for all \(Y \in H(\operatorname{curl} ; \Omega)\); note \(\mathbb{H} \subset H(\) curl \(; \Omega)\).
Cauchy Schwarz and Poincaré Friedrichs and \(\infty:=E_{\mathrm{c}} \in \mathbb{H} \quad \rightarrow\)
\(\left\|\operatorname{curl} E_{c}\right\|_{L^{2}(\Omega)} \leq c_{\text {PF }}\|F-\operatorname{curl} Y\|_{L^{2}(\Omega)}+\|\tilde{H}-Y\|_{L^{2}(\Omega)}=M_{+}(\tilde{H} ; Y, \operatorname{curl} Y)\)

\section*{Proofs (Upper Bounds)}
recall \(h=H-\tilde{H}=\) curl \(E_{\mathrm{c}} \oplus H_{\mathrm{d}} \in \operatorname{curl} \mathbb{H} \oplus \mathrm{H}\left(\right.\) curl \(\left._{0} ; \Omega\right)\)
standard argument for curl \(E_{\mathrm{c}}\) : for all \(\Phi \in \mathbb{H}\)
\[
\begin{aligned}
\left\langle\operatorname{curl} E_{\mathrm{c}}, \operatorname{curl} \Phi\right\rangle_{\mathrm{L}^{2}(\Omega)} & =\langle h, \operatorname{curl} \Phi\rangle_{\mathrm{L}^{2}(\Omega)} \quad\left(H_{\mathrm{d}} \perp \operatorname{curl} \Phi\right) \\
& =\langle F, \Phi\rangle_{\mathrm{L}^{2}(\Omega)}-\langle\tilde{H}, \operatorname{curl} \Phi\rangle_{\mathrm{L}^{2}(\Omega)} \\
& =\langle F-\operatorname{curl} Y, \Phi\rangle_{\mathrm{L}^{2}(\Omega)}-\langle\tilde{H}-Y, \operatorname{curl} \Phi\rangle_{\mathrm{L}^{2}(\Omega)}
\end{aligned}
\]
since \(\langle\text { curl } Y, \Phi\rangle_{L^{2}(\Omega)}=\langle Y, \operatorname{curl} \Phi\rangle_{L^{2}(\Omega)}\) for all \(Y \in \mathrm{H}(\) curl \(; \Omega) ;\) note \(\mathbb{H} \subset \stackrel{\circ}{\mathrm{H}}(\) curl \(; \Omega)\).
Cauchy-Schwarz and Poincaré-Friedrichs and \(\Phi:=E_{c} \in \mathbb{H} \quad \Rightarrow\)
\[
\| \text { curl } E_{\mathrm{c}}\left\|_{L^{2}(\Omega)} \leq c_{\mathrm{PF}}\right\| F-\operatorname{curl} Y\left\|_{L^{2}(\Omega)}+\right\| \tilde{H}-Y \|_{L^{2}(\Omega)}=M_{+}(\tilde{H} ; Y, \text { curl } Y)
\]

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\section*{Proofs (Upper Bounds)}
recall \(h=H-\tilde{H}=\) curl \(E_{\mathrm{c}} \oplus H_{\mathrm{d}} \in \operatorname{curl} \mathbb{H} \oplus \mathrm{H}\left(\right.\) curl \(\left._{0} ; \Omega\right)\)
standard argument for curl \(E_{\mathrm{c}}\) : for all \(\Phi \in \mathbb{H}\)
\[
\begin{aligned}
\left\langle\operatorname{curl} E_{\mathrm{c}}, \operatorname{curl} \Phi\right\rangle_{\mathrm{L}^{2}(\Omega)} & =\langle h, \operatorname{curl} \Phi\rangle_{\mathrm{L}^{2}(\Omega)} \quad\left(H_{\mathrm{d}} \perp \operatorname{curl} \Phi\right) \\
& =\langle F, \Phi\rangle_{\mathrm{L}^{2}(\Omega)}-\langle\tilde{H}, \operatorname{curl} \Phi\rangle_{\mathrm{L}^{2}(\Omega)} \\
& =\langle F-\operatorname{curl} Y, \Phi\rangle_{\mathrm{L}^{2}(\Omega)}-\langle\tilde{H}-Y, \operatorname{curl} \Phi\rangle_{\mathrm{L}^{2}(\Omega)}
\end{aligned}
\]
since \(\langle\text { curl } Y, \Phi\rangle_{L^{2}(\Omega)}=\langle Y, \operatorname{curl} \Phi\rangle_{L^{2}(\Omega)}\) for all \(Y \in \mathrm{H}(\) curl \(; \Omega)\); note \(\mathbb{H} \subset \stackrel{\circ}{\mathrm{H}}(\) curl \(; \Omega)\).
Cauchy-Schwarz and Poincaré-Friedrichs and \(\Phi:=E_{c} \in \mathbb{H} \quad \Rightarrow\)
\[
\left\|\operatorname{curl} E_{\mathrm{c}}\right\|_{\mathrm{L}^{2}(\Omega)} \leq c_{\mathrm{PF}}\|F-\operatorname{curl} Y\|_{\mathrm{L}^{2}(\Omega)}+\|\tilde{H}-Y\|_{\mathrm{L}^{2}(\Omega)}=M_{+}(\tilde{H} ; Y, \operatorname{curl} Y)
\]

\section*{Proofs (Upper Bounds Continued)}
recall \(h=H-\tilde{H}=\operatorname{curl} E_{\mathrm{c}} \oplus H_{\mathrm{d}} \in \operatorname{curl} \mathbb{H} \oplus \mathrm{H}\left(\right.\) curl \(\left._{0} ; \Omega\right)\)
argument for \(H_{d}\) : for all \(\psi \in H(\) curlo; \(\Omega)\)

\(\in \mathrm{H}(\) curl \(; \Omega)\)
for all \(X \in \mathrm{H}(\) curl \(; \Omega)\) with \(\tau X=G\).
Cauchy-Schwarz and \(\psi:=H_{d} \Rightarrow\)
\[
\left\|H_{\mathrm{d}}\right\|_{\mathrm{L}^{2}(\Omega)} \leq\|\operatorname{curl} X-\tilde{H}\|_{\mathrm{L}^{2}(\Omega)}=m_{+}(\tilde{H} ; \operatorname{curl} X)
\]
\(\Rightarrow\) finally
\[
\left\|h_{L^{2}(\Omega)}^{2}=\right\| \operatorname{cur} \mid E_{c}\left\|_{L^{2}(\Omega)}^{2}+\right\| H_{d} \|_{L^{2}(\Omega)}^{2} \leq M_{+}^{2}(\tilde{H} ; Y, \operatorname{curl} Y)+m^{2}(\tilde{H} ; \operatorname{curl} X)
\]
for all \(Y \in H(\operatorname{curl} ; \Omega)\) and all \(X \in H(\operatorname{curl} ; \Omega)\) with \(\tau X=G\).

if \(\tilde{H}=\operatorname{curl} \tilde{E}\) with \(\tilde{E} \in \mathrm{H}(\operatorname{curl} ; \Omega)\)

(then \(\tau X=G\) )

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\section*{Proofs (Upper Bounds Continued)}
recall \(h=H-\tilde{H}=\operatorname{curl} E_{\mathrm{c}} \oplus H_{\mathrm{d}} \in \operatorname{curl} \mathbb{H} \oplus \mathrm{H}\left(\right.\) curl \(\left._{0} ; \Omega\right)\)
argument for \(H_{d}\) : for all \(\psi \in H(\) curlo; \(\Omega)\)

\(\in H(\) curl \(; \Omega)\)
for all \(X \in \mathrm{H}(\) curl \(; \Omega)\) with \(\tau X=G\).
Cauchy-Schwarz and \(\psi:=H_{d} \Rightarrow\)
\[
\left\|H_{\mathrm{d}}\right\|_{\mathrm{L}^{2}(\Omega)} \leq\|\operatorname{curl} X-\tilde{H}\|_{\mathrm{L}^{2}(\Omega)}=m_{+}(\tilde{H} ; \operatorname{curl} X)
\]
\(\Rightarrow\) finally
\[
\|h\|_{L^{2}(\Omega)}^{2}=\left\|\operatorname{curl} E_{c}\right\|_{L^{2}(\Omega)}^{2}+\left\|H_{d}\right\|_{L^{2}(\Omega)}^{2} \leq M_{+}^{2}(\tilde{H} ; Y, \operatorname{curl} Y)+m^{2}(\tilde{H} ; \operatorname{curl} X)
\]
for all \(Y \in H(\operatorname{curl} ; \Omega)\) and all \(X \in H(\operatorname{curl} ; \Omega)\) with \(\tau X=G\).

if \(\tilde{H}=\operatorname{curl} \tilde{E}\) with \(\tilde{E} \in \mathrm{H}(\operatorname{curl} ; \Omega)\)

(then \(\tau X=G\) )

\section*{Proofs (Upper Bounds Continued)}
recall \(h=H-\tilde{H}=\operatorname{curl} E_{c} \oplus H_{\mathrm{d}} \in \operatorname{curl} \mathbb{H} \oplus \mathrm{H}\left(\right.\) curl \(\left._{0} ; \Omega\right)\)
argument for \(H_{d}\) : for all \(\Psi \in \mathrm{H}\left(\right.\) curl \(\left._{0} ; \Omega\right)\)
\(\left\langle H_{\mathrm{d}}, \Psi\right\rangle_{\mathrm{L}^{2}(\Omega)}\)

for all \(X \in \mathrm{H}(\) curl \(; \Omega)\) with \(\tau X=G\).
Cauchy-Schmarz and \(\Psi:=H_{a} \quad \Rightarrow\)
\[
\left\|H_{d}\right\|_{L^{2}(\Omega)} \leq\|\operatorname{curl} X-\tilde{H}\|_{L^{2}(\Omega)}=m_{+}(\tilde{H} ; \operatorname{curl} X)
\]
\(\Rightarrow\) finally
\[
\|h\|_{L^{2}(\Omega)}^{2}=\| \text { curl } E_{d}\left\|_{L^{2}(\Omega)}^{2}+\right\| H_{d} \|_{L^{2}(\Omega)}^{2} \leq M_{+}^{2}(\tilde{H} ; Y, \text { curl } Y)+m^{2}(\tilde{H} ; \text { curl } X)
\]
for all \(Y \in H(c u r l ; \Omega)\) and all \(X \in H(\) curl \(; \Omega)\) with \(\tau X=G\).

if \(\tilde{H}=\operatorname{curl} \tilde{E}\) with \(\tilde{E} \in \mathrm{H}(\operatorname{curl} ; \Omega)\)
and we choose \(X:=\tilde{E}-\check{\tau} \tau \tilde{E}+\check{\tau} G \in H(c u r l ; \Omega)\)
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(then \(\tau X=G\) )


\section*{Proofs (Upper Bounds Continued)}
recall \(h=H-\tilde{H}=\operatorname{curl} E_{\mathrm{c}} \oplus H_{\mathrm{d}} \in \operatorname{curl} \mathbb{H} \oplus \mathrm{H}\left(\right.\) curl \(\left._{0} ; \Omega\right)\)
argument for \(H_{d}\) : for all \(\Psi \in \mathrm{H}\left(\right.\) curl \(\left._{0} ; \Omega\right)\)
\[
\left\langle H_{\mathrm{d}}, \Psi\right\rangle_{\mathrm{L}^{2}(\Omega)}=\langle\operatorname{curl} X-\tilde{H}, \Psi\rangle_{\mathrm{L}^{2}(\Omega)} \quad(H_{\mathrm{d}}+\tilde{H}-\operatorname{curl} X=\operatorname{curl}(\underbrace{E-X-E_{\mathrm{c}}}_{\in \dot{\mathrm{H}}(\mathrm{curl} ; \Omega)}) \perp \Psi)
\]
for all \(X \in \mathrm{H}(\) curl \(; \Omega)\) with \(\tau X=G\).
Cauchy-Schwarz and \(\psi:=H_{d}\)
\[
\left\|H_{d}\right\|_{L^{2}(\Omega)} \leq\|\operatorname{curl} X-\tilde{H}\|_{L^{2}(\Omega)}=m_{+}(\tilde{H} ; \operatorname{curl} X)
\]
\(\Rightarrow\) finally
\[
\|h\|_{L^{2}(\Omega)}^{2}=\| \text { curl } E_{d}\left\|_{L^{2}(\Omega)}^{2}+\right\| H_{d} \|_{L^{2}(\Omega)}^{2} \leq M_{+}^{2}(\tilde{H} ; Y, \text { curl } Y)+m^{2}(\tilde{H} ; \text { curl } X)
\]
for all \(Y \in H(c u r l ; \Omega)\) and all \(X \in H(c u r l ; \Omega)\) with \(\tau X=G\).
\(\square\) \(m_{+}(\tilde{H} ; \operatorname{curl} X)=\|\operatorname{curl}(X-\tilde{E})\|_{L^{2}(\Omega)}=\|\operatorname{curl} \check{\tau}(G-\tau \tilde{E})\|_{L^{2}(\Omega)} \leq c_{\tau}\|G-\tau \tilde{E}\|\) trace
if \(\tilde{H}=\operatorname{curl} \tilde{E}\) with \(\tilde{E} \in \mathrm{H}(\operatorname{curl} ; \Omega)\)
and we choose \(X:=\tilde{E}-\check{\tau} \tau \tilde{E}+\check{\tau} G \in H(c u r l ; \Omega)\)
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(then \(\tau X=G\) )

\section*{Proofs (Upper Bounds Continued)}
recall \(h=H-\tilde{H}=\operatorname{curl} E_{\mathrm{c}} \oplus H_{\mathrm{d}} \in \operatorname{curl} \mathbb{H} \oplus \mathrm{H}\left(\right.\) curl \(\left._{0} ; \Omega\right)\)
argument for \(H_{d}\) : for all \(\Psi \in \mathrm{H}\left(\right.\) curl \(\left._{0} ; \Omega\right)\)
\[
\left\langle H_{\mathrm{d}}, \Psi\right\rangle_{\mathrm{L}^{2}(\Omega)}=\langle\operatorname{curl} X-\tilde{H}, \Psi\rangle_{\mathrm{L}^{2}(\Omega)} \quad(H_{\mathrm{d}}+\tilde{H}-\operatorname{curl} X=\operatorname{curl}(\underbrace{E-X-E_{\mathrm{c}}}_{\in \mathrm{H}(\mathrm{curl} ; \Omega)}) \perp \Psi)
\]
for all \(X \in \mathrm{H}(\) curl \(; \Omega)\) with \(\tau X=G\).
Cauchy-Schwarz and \(\Psi:=H_{d} \quad \Rightarrow\)
\[
\left\|H_{d}\right\|_{L^{2}(\Omega)} \leq\|\operatorname{curl} X-\tilde{H}\|_{L^{2}(\Omega)}=m_{+}(\tilde{H} ; \operatorname{curl} X)
\]
\(\Rightarrow\) finally
\[
\|h\|_{L^{2}(\Omega)}^{2}=\| \text { curl } E_{d}\left\|_{L^{2}(\Omega)}^{2}+\right\| H_{d} \|_{L^{2}(\Omega)}^{2} \leq M_{+}^{2}(\tilde{H} ; Y, \text { curl } Y)+m^{2}(\tilde{H} ; \text { curl } X)
\]
for all \(Y \in H(c u r l ; \Omega)\) and all \(X \in H(c u r l ; \Omega)\) with \(\tau X=G\)
\(\square\) \(m_{+}(\tilde{H} ; \operatorname{curl} X)=\|\operatorname{curl}(X-\tilde{E})\|_{L^{2}(\Omega)}=\|\operatorname{curl} \check{\tau}(G-\tau \tilde{E})\|_{L^{2}(\Omega)} \leq c_{\tau}\|G-\tau \tilde{E}\|\) trace
if \(\tilde{H}=\) curl \(\tilde{E}\) with \(\tilde{E} \in \mathrm{H}(\) curl \(\Omega)\)
and we choose \(X:=\tilde{E}-\check{\tau} \tau \tilde{E}+\check{\tau} G \in H(\operatorname{curl} ; \Omega)\)
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(then \(\tau X=G\) )

\section*{Proofs (Upper Bounds Continued)}
recall \(h=H-\tilde{H}=\operatorname{curl} E_{\mathrm{c}} \oplus H_{\mathrm{d}} \in \operatorname{curl} \mathbb{H} \oplus \mathrm{H}\left(\right.\) curl \(\left._{0} ; \Omega\right)\)
argument for \(H_{d}\) : for all \(\Psi \in \mathrm{H}\left(\right.\) curl \(\left._{0} ; \Omega\right)\)
\[
\left\langle H_{\mathrm{d}}, \Psi\right\rangle_{\mathrm{L}^{2}(\Omega)}=\langle\operatorname{curl} X-\tilde{H}, \Psi\rangle_{\mathrm{L}^{2}(\Omega)} \quad(H_{\mathrm{d}}+\tilde{H}-\operatorname{curl} X=\operatorname{curl}(\underbrace{E-X-E_{\mathrm{c}}}_{\in \dot{\mathrm{H}}(\mathrm{curl} ; \Omega)}) \perp \Psi)
\]
for all \(X \in \mathrm{H}(\) curl \(; \Omega)\) with \(\tau X=G\).
Cauchy-Schwarz and \(\Psi:=H_{d} \quad \Rightarrow\)
\[
\left\|H_{d}\right\|_{L^{2}(\Omega)} \leq\|\operatorname{curl} X-\tilde{H}\|_{L^{2}(\Omega)}=m_{+}(\tilde{H} ; \operatorname{curl} X)
\]
\(\Rightarrow\) finally
\[
\|h\|_{L^{2}(\Omega)}^{2}=\left\|\operatorname{curl} E_{\mathrm{c}}\right\|_{\mathrm{L}^{2}(\Omega)}^{2}+\left\|H_{d}\right\|_{\mathrm{L}^{2}(\Omega)}^{2} \leq M_{+}^{2}(\tilde{H} ; Y, \operatorname{curl} Y)+m_{+}^{2}(\tilde{H} ; \operatorname{curl} X)
\]
for all \(Y \in \mathrm{H}(\operatorname{curl} ; \Omega)\) and all \(X \in \mathrm{H}(\operatorname{curl} ; \Omega)\) with \(\tau X=G\).

\section*{Proofs (Upper Bounds Continued)}
recall \(h=H-\tilde{H}=\operatorname{curl} E_{\mathrm{c}} \oplus H_{\mathrm{d}} \in \operatorname{curl} \mathbb{H} \oplus \mathrm{H}\left(\right.\) curl \(\left._{0} ; \Omega\right)\)
argument for \(H_{d}\) : for all \(\Psi \in \mathrm{H}\left(\right.\) curl \(\left._{0} ; \Omega\right)\)
\[
\left\langle H_{\mathrm{d}}, \Psi\right\rangle_{\mathrm{L}^{2}(\Omega)}=\langle\operatorname{curl} X-\tilde{H}, \Psi\rangle_{\mathrm{L}^{2}(\Omega)} \quad(H_{\mathrm{d}}+\tilde{H}-\operatorname{curl} X=\operatorname{curl}(\underbrace{E-X-E_{\mathrm{c}}}_{\in \dot{\mathrm{H}}(\mathrm{curl} ; \Omega)}) \perp \Psi)
\]
for all \(X \in \mathrm{H}(\) curl \(; \Omega)\) with \(\tau X=G\).
Cauchy-Schwarz and \(\Psi:=H_{d} \quad \Rightarrow\)
\[
\left\|H_{d}\right\|_{L^{2}(\Omega)} \leq\|\operatorname{curl} X-\tilde{H}\|_{L^{2}(\Omega)}=m_{+}(\tilde{H} ; \operatorname{curl} X)
\]
\(\Rightarrow\) finally
\[
\|h\|_{L^{2}(\Omega)}^{2}=\left\|\operatorname{curl} E_{\mathrm{c}}\right\|_{\mathrm{L}^{2}(\Omega)}^{2}+\left\|H_{d}\right\|_{\mathrm{L}^{2}(\Omega)}^{2} \leq M_{+}^{2}(\tilde{H} ; Y, \operatorname{curl} Y)+m_{+}^{2}(\tilde{H} ; \operatorname{curl} X)
\]
for all \(Y \in \mathrm{H}(\operatorname{curl} ; \Omega)\) and all \(X \in \mathrm{H}(\operatorname{curl} ; \Omega)\) with \(\tau X=G\).
\(\Rightarrow \quad m_{+}(\tilde{H} ; \operatorname{curl} X)=\|\operatorname{curl}(X-\tilde{E})\|_{L^{2}(\Omega)}=\|\operatorname{curl} \check{\tau}(G-\tau \tilde{E})\|_{L^{2}(\Omega)} \leq c_{\tau}\|G-\tau \tilde{E}\|_{\text {trace }}\)
if \(\tilde{H}=\operatorname{curl} \tilde{E}\) with \(\tilde{E} \in \mathrm{H}(\) curl \(; \Omega)\)
and we choose \(X:=\tilde{E}-\check{\tau} \tau \tilde{E}+\check{\tau} G \in \mathrm{H}(\) curl \(; \Omega)\)
(then \(\tau X=G\) )

\section*{Proofs (Lower Bounds)}
recall \(h=H-\tilde{H}=\operatorname{curl} E_{\mathrm{c}} \oplus H_{\mathrm{d}} \in \operatorname{curl} \mathbb{H} \oplus \mathrm{H}\left(\right.\) curl \(\left._{0} ; \Omega\right)\)
standard argument for curl \(E_{c}\) : for all \(X \in \mathrm{H}(\) curl \(; \Omega)\)
\(\left\|\operatorname{cur}\left|E_{C}\left\|_{L^{2}(\Omega)}^{2} \geq 2\langle\operatorname{cur}| E_{C}, \operatorname{cur}|X\rangle_{L^{2}(\Omega)}-\right\| \operatorname{cur}\right| X\right\|_{L^{2}(\Omega)}^{2} \quad\left(H_{d} \perp \operatorname{cur} \mid X\right)\) \(=\langle h, \operatorname{curl} X\rangle_{L^{2}(\Omega)}\)
\(=2\langle F, \Phi\rangle_{L^{2}(\Omega)}-\langle\operatorname{curl} X+2 \tilde{H}, \operatorname{curl} X\rangle_{L^{2}(\Omega)}=M_{-}(\tilde{H} ; X, \operatorname{curl} X)\)
similar for \(H_{d}\) : for all \(Y \in H\left(\right.\) curl \(\left._{0} ; \Omega\right)\) and for all \(Z \in H(\operatorname{curl} ; \Omega)\) with \(\tau Z=G\)

\(=m_{-}(\tilde{H} ; Y, \operatorname{curl} Z)\)

\section*{Proofs (Lower Bounds)}
recall \(h=H-\tilde{H}=\operatorname{curl} E_{\mathrm{c}} \oplus H_{\mathrm{d}} \in \operatorname{curl} \mathbb{H} \oplus \mathrm{H}\left(\right.\) curl \(\left._{0} ; \Omega\right)\)
standard argument for curl \(E_{\mathrm{c}}\) : for all \(X \in H(\) curl \(; \Omega)\)
\(\left\|\operatorname{curl} E_{\mathrm{C}}\right\|_{\mathrm{L}^{2}(\Omega)}^{2} \geq 2\left\langle\operatorname{curl} E_{\mathrm{C}}, \operatorname{curl} X\right\rangle_{\mathrm{L}^{2}(\Omega)}-\|\operatorname{curl} X\|_{L^{2}(\Omega)}^{2} \quad\left(H_{\mathrm{d}} \perp \operatorname{curl} X\right)\)

similar for \(H_{d}:\) for all \(Y \in H\left(\right.\) curl \(\left._{0} ; \Omega\right)\) and for all \(Z \in H(\operatorname{curl} ; \Omega)\) with \(\tau Z=G\)


\section*{Proofs (Lower Bounds)}
recall \(h=H-\tilde{H}=\operatorname{curl} E_{\mathrm{c}} \oplus H_{\mathrm{d}} \in \operatorname{curl} \mathbb{H} \oplus \mathrm{H}\left(\right.\) curl \(\left._{0} ; \Omega\right)\)
standard argument for curl \(E_{\mathrm{c}}\) : for all \(X \in H(\) curl \(; \Omega)\) \(\|\) curl \(E_{c} \|_{L^{2}(\Omega)}^{2} \geq 2\left\langle\text { curl } E_{\mathrm{c}}, \text { curl } X\right\rangle_{L^{2}(\Omega)}-\|\) curl \(X \|_{L^{2}(\Omega)}^{2} \quad\left(H_{d} \perp\right.\) curl \(\left.X\right)\) \(=2\langle F, \Phi\rangle_{L^{2}(\Omega)}-\langle\operatorname{curl} X+2 \tilde{H}, \operatorname{curl} X\rangle_{L^{2}(\Omega)}=M_{-}(\tilde{H} ; X, \operatorname{curl} X)\)
similar for \(H_{d}\) : for all \(\gamma \in H(c u r l o ; \Omega)\) and for all \(Z \in H(c u r i ; \Omega)\) with \(\tau Z=G\)

\(=m_{-}(\tilde{H} ; Y, \operatorname{curl} Z)\)

\section*{Proofs (Lower Bounds)}
recall \(h=H-\tilde{H}=\operatorname{curl} E_{\mathrm{c}} \oplus H_{\mathrm{d}} \in \operatorname{curl} \mathbb{H} \oplus \mathrm{H}\left(\right.\) curl \(\left._{0} ; \Omega\right)\)
standard argument for curl \(E_{c}\) : for all \(X \in \stackrel{\circ}{\mathrm{H}}(\) curl \(; \Omega)\)
\[
\left\|\operatorname{curl} E_{\mathrm{c}}\right\|_{\mathrm{L}^{2}(\Omega)}^{2} \geq 2 \underbrace{\left\langle\operatorname{curl} E_{\mathrm{c}}, \operatorname{curl} X\right\rangle_{\mathrm{L}^{2}(\Omega)}}_{=\langle h, \operatorname{curl} X\rangle_{\mathrm{L}^{2}(\Omega)}}-\|\operatorname{curl} X\|_{\mathrm{L}^{2}(\Omega)}^{2} \quad\left(H_{\mathrm{d}} \perp \operatorname{curl} X\right)
\]

similar for \(H_{d}\) : for all \(Y \in H\left(\right.\) curl \(\left._{0} ; \Omega\right)\) and for all \(Z \in H(\operatorname{curl} ; \Omega)\) with \(\tau Z=G\)

\(=m_{-}(\tilde{H} ; Y, \operatorname{curl} Z)\)
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\section*{Proofs (Lower Bounds)}
recall \(h=H-\tilde{H}=\operatorname{curl} E_{\mathrm{c}} \oplus H_{\mathrm{d}} \in \operatorname{curl} \mathbb{H} \oplus \mathrm{H}\left(\right.\) curl \(\left._{0} ; \Omega\right)\)
standard argument for curl \(E_{\mathrm{c}}\) : for all \(X \in \stackrel{\circ}{\mathrm{H}}(\) curl \(; \Omega)\)
\[
\begin{aligned}
\left\|\operatorname{curl} E_{\mathrm{c}}\right\|_{\mathrm{L}^{2}(\Omega)}^{2} & \geq 2 \underbrace{\left\langle\operatorname{curl} E_{\mathrm{c}}, \operatorname{curl} X\right\rangle_{\mathrm{L}^{2}(\Omega)}}_{=\langle h, \operatorname{curl} X\rangle_{\mathrm{L}^{2}(\Omega)}}-\|\operatorname{curl} X\|_{\mathrm{L}^{2}(\Omega)}^{2} \quad\left(H_{\mathrm{d}} \perp \operatorname{curl} X\right) \\
& =2\langle F, \Phi\rangle_{\mathrm{L}^{2}(\Omega)}-\langle\operatorname{curl} X+2 \tilde{H}, \operatorname{curl} X\rangle_{\mathrm{L}^{2}(\Omega)}=M_{-}(\tilde{H} ; X, \operatorname{curl} X)
\end{aligned}
\]
similar for \(H_{d}\) : for all \(Y \in H(\) curlo; \(\Omega)\) and for all \(Z \in H(c u r l ; \Omega)\) with \(\tau Z=G\)


\section*{Proofs (Lower Bounds)}
recall \(h=H-\tilde{H}=\operatorname{curl} E_{\mathrm{c}} \oplus H_{\mathrm{d}} \in \operatorname{curl} \mathbb{H} \oplus \mathrm{H}\left(\right.\) curl \(\left._{0} ; \Omega\right)\)
standard argument for curl \(E_{\mathrm{c}}\) : for all \(X \in \stackrel{\circ}{\mathrm{H}}(\) curl \(; \Omega)\)
\[
\begin{aligned}
\left\|\operatorname{curl} E_{\mathrm{c}}\right\|_{\mathrm{L}^{2}(\Omega)}^{2} & \geq 2 \underbrace{}_{=\langle h, \operatorname{curl} X\rangle_{\mathrm{L}^{2}(\Omega)}\left\langle\operatorname{curl} E_{\mathrm{c}}, \operatorname{curl} X\right\rangle_{\mathrm{L}^{2}(\Omega)}}-\|\operatorname{curl} X\|_{\mathrm{L}^{2}(\Omega)}^{2} \quad\left(H_{\mathrm{d}} \perp \operatorname{curl} X\right) \\
& =2\langle F, \Phi\rangle_{\mathrm{L}^{2}(\Omega)}-\langle\operatorname{curl} X+2 \tilde{H}, \operatorname{curl} X\rangle_{\mathrm{L}^{2}(\Omega)}=M_{-}(\tilde{H} ; X, \operatorname{curl} X)
\end{aligned}
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similar for \(H_{d}\) : for all \(Y \in H(\) curlo; \(\Omega)\) and for all \(Z \in H(\) curl \(; \Omega)\) with \(\tau Z=G\)


\section*{Proofs (Lower Bounds)}
recall \(h=H-\tilde{H}=\operatorname{curl} E_{\mathrm{c}} \oplus H_{\mathrm{d}} \in \operatorname{curl} \mathbb{H} \oplus \mathrm{H}\left(\right.\) curl \(\left._{0} ; \Omega\right)\)
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\[
\left\|H_{d}\right\|_{L^{2}(\Omega)}^{2} \geq 2 \underbrace{\left\langle H_{d}, Y\right\rangle_{L^{2}(\Omega)}}_{=\langle h, Y\rangle_{L^{2}(\Omega)}}-\|Y\|_{L^{2}(\Omega)}^{2}
\]
\[
\text { (curl } E_{c} \perp Y \text { ) }
\]


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\[
\begin{array}{rlr}
\left\|H_{d}\right\|_{L^{2}(\Omega)}^{2} & \geq 2 \underbrace{}_{=\langle h, Y\rangle_{L^{2}(\Omega)}\left\langle H_{d}, Y\right\rangle_{L^{2}(\Omega)}}-\|Y\|_{L^{2}(\Omega)}^{2} & \quad\left(\operatorname{curl} E_{c} \perp Y\right) \\
& =2\langle\operatorname{curl} Z-\tilde{H}, Y\rangle_{L^{2}(\Omega)}-\|Y\|_{L^{2}(\Omega)}^{2} \quad(\operatorname{curl} \underbrace{(E-Z)}_{\in \dot{H}(\text { curl } ; \Omega)} \perp Y)
\end{array}
\]

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\begin{aligned}
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& =2\langle\operatorname{curl} Z-\tilde{H}, Y\rangle_{L^{2}(\Omega)}-\|Y\|_{L^{2}(\Omega)}^{2} \\
& =m_{-}(\tilde{H} ; Y, \operatorname{curl} Z)
\end{aligned}
\]
\[
(\operatorname{curl} \underbrace{(E-Z)}_{\substack{\circ \\ \in(\operatorname{curl} ; \Omega)}} \perp Y)
\]

\section*{Proofs (Lower Bounds Continued)}
\(\Rightarrow\) finally
\[
\|h\|_{\mathrm{L}^{2}(\Omega)}^{2}=\left\|\operatorname{curl} E_{\mathrm{c}}\right\|_{\mathrm{L}^{2}(\Omega)}^{2}+\left\|H_{d}\right\|_{\mathrm{L}^{2}(\Omega)}^{2} \geq M_{-}(\tilde{H} ; X, \operatorname{curl} X)+m_{-}(\tilde{H} ; Y, \operatorname{curl} Z)
\]
for all \(Y \in \mathrm{H}\left(\right.\) curl \(\left._{0} ; \Omega\right)\) and all \(Z \in \mathrm{H}(\operatorname{curl} ; \Omega)\) with \(\tau Z=G\).
if \(\tilde{H}=\operatorname{curl} \tilde{E}\) with \(\tilde{E} \in H(c u r l ; \Omega) \Rightarrow\)


\section*{Proofs (Lower Bounds Continued)}
\(\Rightarrow\) finally
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\|h\|_{\mathrm{L}^{2}(\Omega)}^{2}=\left\|\operatorname{curl} E_{\mathrm{c}}\right\|_{\mathrm{L}^{2}(\Omega)}^{2}+\left\|H_{d}\right\|_{\mathrm{L}^{2}(\Omega)}^{2} \geq M_{-}(\tilde{H} ; X, \operatorname{curl} X)+m_{-}(\tilde{H} ; Y, \operatorname{curl} Z)
\]
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if \(\tilde{H}=\operatorname{curl} \tilde{E}\) with \(\tilde{E} \in \mathrm{H}(\) curl \(; \Omega) \Rightarrow\)
\[
\begin{aligned}
m_{-}(\tilde{H} ; Y, \operatorname{curl} Z)= & 2\langle\operatorname{curl}(Z-\tilde{E}), Y\rangle_{\mathrm{L}^{2}(\Omega)}-\|Y\|_{\mathrm{L}^{2}(\Omega)}^{2} \\
= & 2 \underbrace{\left\langle G-\tau \tilde{E}, \tau_{n} Y\right\rangle_{\text {trace }}}-\|Y\|_{\mathrm{L}^{2}(\Omega)}^{2} \\
& =\int_{\Gamma}(G-\nu \times \tilde{E}) Y "
\end{aligned}
\]

\section*{Remarks}
- \(\Omega\) exterior domain or bounded in one direction

\section*{- differential forms, \(\mathbb{R}^{N}\), Riemannian manifolds}
- hyperbolic problems

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