Traces for Hilbert Complexes

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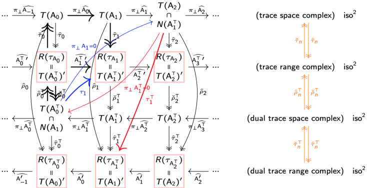
Traces for Hilbert Complexes

ToC

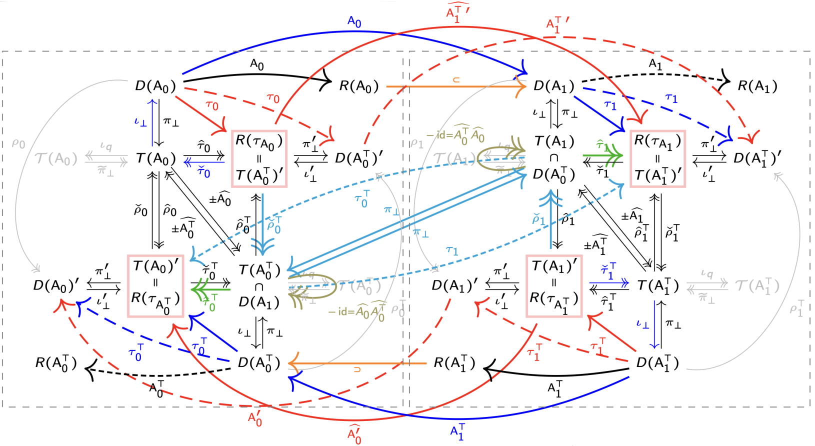


Traces for Hilbert Complexes

Question: Why are traces so complicated?

Question: What is $\widetilde{H}^{\pm 1/2}_{\|/\perp/0(0)}(\operatorname{div}_\Gamma \,/\, \operatorname{rot}_\Gamma, \Gamma)$?

. . . some answers below . . .



Traces for Hilbert Complexes

OVERVIEW and BASIC IDEAS

paper in JFA 2023:

R. Hiptmair, D. Pauly, and E. Schulz: *Traces for Hilbert Complexes*

Traces

? Traces ?

Traces without any regularity of the domain?

Is this even possible?

even better:

? Traces ?

Traces without domains (or boundaries)?

Traces

$\mathrm{A} : D(\mathrm{A}) \subset \mathrm{H}_0 \to \mathrm{H}_1$ ldddc: lin, dendef, cl

Traces for $D(\mathrm{A})$?

$\Omega \subset \mathbb{R}^N$ Lipschitz:

very classical

$D(\mathrm{A}) = \mathrm{H}^1$ or $\mathrm{W}^{1,p}$, scalar trace $u_{\mathrm{s}} = u|_\Gamma$

classical (we stay in Hilbert spaces)

$D(\mathrm{A}) = \mathrm{H}(\mathrm{rot})$ or $\mathrm{H}(\mathrm{div})$, tan or nor traces $v_{\mathrm{t}} = (\nu \times v \times \nu)\big|_\Gamma$, $v_{\mathrm{n}} = (\nu \cdot v)\big|_\Gamma$

more recent (BGG, zoo of complexes)

$$D(\mathrm{A}) = \mathrm{H}(\mathrm{Rot}^\top \mathrm{Rot}_{\mathbb{S}}), \mathrm{H}(\mathrm{div}\,\mathrm{Div}_{\mathbb{S}}), \mathrm{H}(\mathrm{sym}\,\mathrm{Rot}_{\mathbb{T}}), \dots$$
$$\dots \mathrm{H}(\mathrm{Rot}\,\mathrm{Rot}\,\mathrm{Rot}), \mathrm{H}(\mathrm{rot}\,\mathrm{Div}), \mathrm{H}(\mathrm{Grad}\,\mathrm{rot}) \dots$$

$\boxed{\text{traces?}}$

Traces

$\mathrm{A} : D(\mathrm{A}) \subset \mathrm{H}_0 \to \mathrm{H}_1$ lddc

Traces for $D(\mathrm{A})$?

$\Omega \subset \mathbb{R}^N$ Lipschitz

What if less regularity? What if

- $\boxed{\Omega \text{ just open / no regularity}}$ and $D(\mathrm{A}) = \mathrm{H}^1(\Omega), \mathrm{H}(\mathrm{rot}, \Omega), \mathrm{H}(\mathrm{div}, \Omega), \ldots$?
- $\boxed{\text{no } \Omega}$ at all, just $D(\mathrm{A})$?

Traces & Literature

Trace for $D(\mathrm{A})$ where $\mathrm{A} : D(\mathrm{A}) \subset \mathrm{H}_0 \to \mathrm{H}_1$ lddc

What is known?

well known/basic idea: $D(\tau_{\mathrm{A}})/N(\tau_{\mathrm{A}}) = D(\mathrm{A})/D(\mathring{\mathrm{A}}) = \mathcal{T}(\mathrm{A}) \cong T(\mathrm{A}) = D(\mathring{\mathrm{A}})^{\perp_{D(\mathrm{A})}}$

not so well known: $T(\mathrm{A})' \cong_{\hat{\rho}_{\mathrm{A}}} \boxed{T(\mathrm{A}) = N(\mathrm{A}^\top \mathrm{A} + 1)} \cong_{\hat{\tau}_{\mathrm{A}}} T(\mathrm{A}^\top)' = R(\tau_{\mathrm{A}}) = D(\mathrm{A}^*)^\circ$

- Friedrichs (Friedrichs systems): 1958 (S skew-sym, i.e., $\mathrm{S} + \mathrm{S}^\top = 0$)
- Ern, Guermond (Aubin): 2006, 2007 (S alm skew-sym, i.e., $|(\mathrm{S} + \mathrm{S}^\top)x|_{\mathrm{H}} \leq c|x|_{\mathrm{H}}$)
- Picard, Trostorff: 2024 (S skew-sym)

theory: bc ⇒ well-posedness / skew-selfadjoint extensions / accretive extensions

connection to our setting: $\mathrm{S} = \begin{bmatrix} 0 & -\mathrm{A}^\top \\ \mathrm{A} & 0 \end{bmatrix}$ but no complexes, $\mathrm{S}^2 = \begin{bmatrix} -\mathrm{A}^\top \mathrm{A} & 0 \\ 0 & -\mathrm{A}\,\mathrm{A}^\top \end{bmatrix}$

maybe not known (at all?):

connection to Hilbert complexes

connection to trace complexes

NOT! $\boxed{\tau_{\mathrm{A}_1} \mathrm{A}_0 = \mathrm{A}_0^\top{}' \tau_{\mathrm{A}_0}}$ BUT! $\boxed{\tau_{\mathrm{A}_1} \mathrm{A}_0 = -\mathrm{A}_1^\top{}' \tau_{\mathrm{A}_0}}$

surface differential operators

compact embeddings

. . . dual representations / regular potentials

Traces

$\mathrm{A}: D(\mathrm{A}) \subset \mathrm{H}_0 \to \mathrm{H}_1$	lddc
$\mathrm{A}^*: D(\mathrm{A}^*) \subset \mathrm{H}_1 \to \mathrm{H}_0$	lddc, Hilbert space adjoint

Traces for $D(\mathrm{A})$?

basic idea: integration by parts / extension of adjoints

$$\forall\, x \in D(\mathrm{A}) \quad \forall\, y \in D(\mathrm{A}^*) \qquad \boxed{\langle y, \mathrm{A}\, x\rangle_{\mathrm{H}_1} - \langle \mathrm{A}^*\, y, x\rangle_{\mathrm{H}_0} = 0}$$

think of $\mathrm{A} = \mathring{\operatorname{grad}} : D(\mathrm{A}) = \mathring{\mathrm{H}}^1 \subset \mathrm{L}^2 \to \mathrm{L}^2$
and $\mathrm{A}^* = -\operatorname{div} : D(\mathrm{A}^*) = \mathrm{H}(\operatorname{div}) \subset \mathrm{L}^2 \to \mathrm{L}^2$

$$\langle y, \mathring{\operatorname{grad}}\, x\rangle_{\mathrm{L}^2} + \langle \operatorname{div} y, x\rangle_{\mathrm{L}^2} = 0$$

Traces

$\mathring{A} \subset A$ lddc

$A^* \subset A^\top := \mathring{A}^*$ ($A^\top$ formal transpose of A) lddc, Hilbert space adjoints

Traces for $D(A)$?

basic idea and setting: integration by parts / extension of adjoints

$$\exists\, x \in D(A) \quad \exists\, y \in D(A^\top) \qquad \boxed{\langle y, A\,x\rangle_{H_1} - \langle A^\top y, x\rangle_{H_0} \neq 0}$$

think of $\mathring{\operatorname{grad}} = \mathring{A} \subset A = \operatorname{grad}$ $D(\mathring{\operatorname{grad}}) = \mathring{H}^1 \subset D(\operatorname{grad}) = H^1$

and $-\mathring{\operatorname{div}} = \operatorname{grad}^* = A^* \subset A^\top = \mathring{A}^* = \mathring{\operatorname{grad}}^* = -\operatorname{div}$ $D(\mathring{\operatorname{div}}) = \mathring{H}(\operatorname{div}) \subset D(\operatorname{div}) = H(\operatorname{div})$

$$\langle y, \operatorname{grad} x\rangle_{L^2(\Omega)} + \langle \operatorname{div} y, x\rangle_{L^2(\Omega)} = \underbrace{\langle y_n, x_s\rangle_{\text{“}L^2(\Gamma)\text{''}}}_{} \neq 0$$

$$= \text{“}\int_\Gamma y_n x_s\text{''} = \langle\!\langle y_n, x_s\rangle\!\rangle_{H^{-1/2}(\Gamma),H^{1/2}(\Gamma)}$$

for some $x \in H^1$, $y \in H(\operatorname{div})$

For simplicity of this talk: real Hilbert spaces

Traces

$\mathring{\mathrm{A}} \subset \mathrm{A}$ lddc

$\mathrm{A}^* \subset \mathrm{A}^\top = \mathring{\mathrm{A}}^*$ lddc, Hilbert space adjoints

$(\mathring{\mathrm{A}}, \mathrm{A}^*)$ pair "with" boundary conditions $(\mathrm{A}, \mathrm{A}^*)$, $(\mathring{\mathrm{A}}, \mathrm{A}^\top = \mathring{\mathrm{A}}^*)$

$(\mathrm{A}, \mathrm{A}^\top)$ pair "without" boundary conditions dual/adjoint pairs

Traces for $D(\mathrm{A})$?

basic idea and setting: integration by parts / extension of adjoints

bd trace

$$\tau_{\mathrm{A}} : D(\mathrm{A}) \to D(\mathrm{A}^\top)', \qquad \tau_{\mathrm{A}} x(y) := \langle y, \mathrm{A}\, x\rangle_{\mathrm{H}_1} - \langle \mathrm{A}^\top y, x\rangle_{\mathrm{H}_0}$$

$$x \mapsto \tau_{\mathrm{A}} x \qquad \boxed{x \in D(\mathrm{A}),\ y \in D(\mathrm{A}^\top)}$$

bd dual trace

$$\tau_{\mathrm{A}^\top} : D(\mathrm{A}^\top) \to D(\mathrm{A})', \qquad \tau_{\mathrm{A}^\top} y(x) := \langle x, \mathrm{A}^\top y\rangle_{\mathrm{H}_0} - \langle \mathrm{A}\, x, y\rangle_{\mathrm{H}_1}$$

$$y \mapsto \tau_{\mathrm{A}^\top} y$$

note $\boxed{\tau_{\mathrm{A}^\top} y(x) = -\tau_{\mathrm{A}} x(y)}$ skew-symmteric structure $\mathrm{S} = \begin{bmatrix} 0 & -\mathrm{A}^\top \\ \mathrm{A} & 0 \end{bmatrix}$

equivalently **bilinear form** on $D(\mathrm{A}) \times D(\mathrm{A}^\top)$ resp. $D(\mathrm{A}^\top) \times D(\mathrm{A})$

$$\langle\!\langle x, y\rangle\!\rangle := \tau_{\mathrm{A}} x(y) = -\tau_{\mathrm{A}^\top} y(x) = \langle y, \mathrm{A}\, x\rangle_{\mathrm{H}_1} - \langle \mathrm{A}^\top y, x\rangle_{\mathrm{H}_0} = \Big\langle \mathrm{S} \begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} x \\ y \end{bmatrix} \Big\rangle_{\mathrm{H}_0 \times \mathrm{H}_1}$$

Hilbert Complexes

Traces for Hilbert Complexes

- We give Traces for Hilbert Complexes.
- On the other hand Hilbert Complexes are necessary for Traces.

Traces for Single Operators

Traces for Single Operators

Traces for Single Operators (and Adjoints)

$\mathring{\mathrm{A}} \subset \mathrm{A}$ and $\mathrm{A}^* \subset \mathrm{A}^\top = \mathring{\mathrm{A}}^*$ lddc (Hilbert space adjoints)

Traces for $D(\mathrm{A})$ and $D(\mathrm{A}^\top)$ — traces come always in pairs

$$\boxed{\tau_{\mathrm{A}} x(y) = \langle y, \mathrm{A}\, x \rangle_{\mathrm{H}_1} - \langle \mathrm{A}^\top y, x \rangle_{\mathrm{H}_0}}$$

primal / dual traces $\tau_{\mathrm{A}} : D(\mathrm{A}) \to D(\mathrm{A}^\top)'$, $\tau_{\mathrm{A}^\top} : D(\mathrm{A}^\top) \to D(\mathrm{A})'$

primal / dual adjoint traces $\tau'_{\mathrm{A}} : D(\mathrm{A}^\top)'' \to D(\mathrm{A})'$, $\tau'_{\mathrm{A}^\top} : D(\mathrm{A})'' \to D(\mathrm{A}^\top)'$

note: Hilbert spaces $\mathrm{H} \overset{\text{here}}{=} D(\mathrm{A}) \vee D(\mathrm{A}^\top)$ are $\boxed{\text{self-dual}}$ (Riesz) and $\boxed{\text{reflexive}}$

$\Rightarrow$ isometric isomorphisms $\rho_{\mathrm{H}} : \mathrm{H} \to \mathrm{H}'$ and $\iota_d : \mathrm{H} \to \mathrm{H}''$

Theorem (kernels, boundedness, and adjoints)

- $N(\tau_{\mathrm{A}}) = D(\mathring{\mathrm{A}})$ *and* $N(\tau_{\mathrm{A}^\top}) = D(\mathrm{A}^*)$ *and* $\|\tau_{\mathrm{A}}\|, \|\tau_{\mathrm{A}^\top}\| \leq 1$
- $\tau'_{\mathrm{A}} \iota_d = -\tau_{\mathrm{A}^\top}$ *and* $\tau'_{\mathrm{A}^\top} \iota_d = -\tau_{\mathrm{A}}$

Remark

$\overline{R(\tau_{\mathrm{A}^\top})} = \overline{R(\tau'_{\mathrm{A}})} = N(\tau_{\mathrm{A}})^\circ = D(\mathring{\mathrm{A}})^\circ$ and $\overline{R(\tau_{\mathrm{A}})} = D(\mathrm{A}^*)^\circ$

Traces for Single Operators (Riesz Isometric Isometries)

$\mathring{\mathrm{A}} \subset \mathrm{A}$ and $\mathrm{A}^* \subset \mathrm{A}^\top = \mathring{\mathrm{A}}^*$ lddc (Hilbert space adjoints)

Let $x \in D(\mathrm{A})$. What is / solves

$$\check{y} := -\rho^{-1}_{D(\mathrm{A}^\top)}\tau_{\mathrm{A}} x \in D(\mathrm{A}^\top) \quad \text{and} \quad \check{x} := \mathrm{A}^\top \check{y}\,?$$

Lemma (extension / right inverse)

$$(\check{x},\check{y}) \in N(\mathrm{A}^\top \mathrm{A}+1) \times N(\mathrm{A}\,\mathrm{A}^\top+1) \quad \textit{and} \quad \check{x}-x \in D(\mathring{\mathrm{A}}) = N(\tau_{\mathrm{A}})$$

$$\Rightarrow \quad \boxed{\tau_{\mathrm{A}}\,\mathrm{A}^\top \check{y} = \tau_{\mathrm{A}}\check{x} = \tau_{\mathrm{A}} x}$$

$$\Rightarrow \quad \boxed{-\tau_{\mathrm{A}}\,\mathrm{A}^\top \rho^{-1}_{D(\mathrm{A}^\top)} = \mathrm{id}_{R(\tau_{\mathrm{A}})}} \quad \Rightarrow \quad -\mathrm{A}^\top \rho^{-1}_{D(\mathrm{A}^\top)} \textit{ right inverse of } \tau_{\mathrm{A}} \textit{ on } R(\tau_{\mathrm{A}})$$

note $\quad (\mathrm{S}+1)\begin{bmatrix}\check{x}\\ \check{y}\end{bmatrix} = 0 \quad$ with $\quad \mathrm{S} = \begin{bmatrix}0 & -\mathrm{A}^\top\\ \mathrm{A} & 0\end{bmatrix}$ (formally skew-symmetric)

Traces for Single Operators (Riesz Isometric Isometries)

$\mathring{\mathrm{A}} \subset \mathrm{A}$ and $\mathrm{A}^* \subset \mathrm{A}^\top = \mathring{\mathrm{A}}^*$ lddc (Hilbert space adjoints)

Lemma (extensions / right inverses)

$-\tau_{\mathrm{A}}\, \mathrm{A}^\top \rho^{-1}_{D(\mathrm{A}^\top)} = \mathrm{id}_{R(\tau_{\mathrm{A}})}$ *and* $\check{\tau}_{\mathrm{A}} := -\mathrm{A}^\top \rho^{-1}_{D(\mathrm{A}^\top)}$ *right inverse of* τ_{A} *on* $R(\tau_{\mathrm{A}})$

$-\tau_{\mathrm{A}^\top}\, \mathrm{A}\, \rho^{-1}_{D(\mathrm{A})} = \mathrm{id}_{R(\tau_{\mathrm{A}^\top})}$ *and* $\check{\tau}_{\mathrm{A}^\top} := -\mathrm{A}\, \rho^{-1}_{D(\mathrm{A})}$ *right inverse of* $\tau_{\mathrm{A}^\top}$ *on* $R(\tau_{\mathrm{A}^\top})$

Definition (extensions / right inverses)

Let $\phi \in R(\tau_{\mathrm{A}})$ and $\psi \in R(\tau_{\mathrm{A}^\top})$. We call:

- $\check{\phi} = -\rho^{-1}_{D(\mathrm{A}^\top)}\phi \in N(\mathrm{A}\,\mathrm{A}^\top + 1)$ harm Neumann ext of ϕ since $\tau_{\mathrm{A}}\, \mathrm{A}^\top \check{\phi} = \phi$
- $\check{\check{\phi}} = \mathrm{A}^\top \check{\phi} = -\mathrm{A}^\top \rho^{-1}_{D(\mathrm{A}^\top)}\phi \in N(\mathrm{A}^\top \mathrm{A} + 1)$ harm Dirichlet ext of ϕ since $\tau_{\mathrm{A}} \check{\check{\phi}} = \phi$
- $\check{\psi} = -\rho^{-1}_{D(\mathrm{A})}\psi \in N(\mathrm{A}^\top \mathrm{A} + 1)$ harm Neumann ext of ψ since $\tau_{\mathrm{A}^\top}\, \mathrm{A}\, \check{\psi} = \psi$
- $\check{\check{\psi}} = \mathrm{A}\, \check{\psi} = -\mathrm{A}\, \rho^{-1}_{D(\mathrm{A})}\psi \in N(\mathrm{A}\,\mathrm{A}^\top + 1)$ harm Dirichlet ext of ψ since $\tau_{\mathrm{A}^\top} \check{\check{\psi}} = \psi$

Traces for Single Operators $\mathring{\mathrm{A}} \subset \mathrm{A}$ and $\mathrm{A}^* \subset \mathrm{A}^\top = \mathring{\mathrm{A}}^*$ (lddc)

Theorem (kernels, ranges = annihilators)

- $N(\tau_{\mathrm{A}}) = D(\mathring{\mathrm{A}})$
- $R(\tau_{\mathrm{A}}) = D(\mathrm{A}^*)^\circ = \big\{\Phi \in D(\mathrm{A}^\top)' : D(\mathrm{A}^*) \subset N(\Phi)\big\}$
- $N(\tau_{\mathrm{A}^\top}) = D(\mathrm{A}^*)$
- $R(\tau_{\mathrm{A}^\top}) = D(\mathring{\mathrm{A}})^\circ = \big\{\Phi \in D(\mathrm{A})' : D(\mathring{\mathrm{A}}) \subset N(\Phi)\big\}$

In particular, the kernels and ranges are closed.

Definition and Lemma (trace spaces)

- $T(\mathrm{A}) := D(\mathring{\mathrm{A}})^{\perp_{D(\mathrm{A})}} = N(\mathrm{A}^\top \mathrm{A} + 1) \cong D(\tau_{\mathrm{A}})/N(\tau_{\mathrm{A}}) = D(\mathrm{A})/D(\mathring{\mathrm{A}}) =: \mathcal{T}(\mathrm{A})$
- $T(\mathrm{A}^\top) := D(\mathrm{A}^*)^{\perp_{D(\mathrm{A}^\top)}} = N(\mathrm{A}\,\mathrm{A}^\top + 1) \cong D(\tau_{\mathrm{A}^\top})/N(\tau_{\mathrm{A}^\top}) = D(\mathrm{A}^\top)/D(\mathrm{A}^*) =: \mathcal{T}(\mathrm{A}^\top)$

$\Rightarrow$ red traces $\boxed{\widehat{\tau}_{\mathrm{A}} := \tau_{\mathrm{A}}|_{T(\mathrm{A})} : T(\mathrm{A}) \to R(\tau_{\mathrm{A}})}$ $\boxed{\widehat{\tau}_{\mathrm{A}^\top} := \tau_{\mathrm{A}^\top}|_{T(\mathrm{A}^\top)} : T(\mathrm{A}^\top) \to R(\tau_{\mathrm{A}^\top})}$

Theorem (ranges and trace isometries)

$$R(\widehat{\tau}_{\mathrm{A}}) = R(\tau_{\mathrm{A}}) = D(\mathrm{A}^*)^\circ = \rho_{D(\mathrm{A}^\top)} T(\mathrm{A}^\top) = T(\mathrm{A}^\top)'$$
$$R(\widehat{\tau}_{\mathrm{A}^\top}) = R(\tau_{\mathrm{A}^\top}) = D(\mathring{\mathrm{A}})^\circ = \rho_{D(\mathrm{A})} T(\mathrm{A}) = T(\mathrm{A})'$$

The reduced traces are isometric isomorphisms.

Traces for Single Operators $\mathring{\mathrm{A}} \subset \mathrm{A}$ and $\mathrm{A}^* \subset \mathrm{A}^\top = \mathring{\mathrm{A}}^*$ (lddc)

Remark (trace /Riesz isometric isomorphisms $\twoheadrightarrow$)

$$\begin{aligned}
\tau_{\mathrm{A}} &: D(\mathrm{A}) \to R(\tau_{\mathrm{A}}) \subset D(\mathrm{A}^\top)', & \rho_{\mathrm{A}} &:= \rho_{D(\mathrm{A})} : D(\mathrm{A}) \twoheadrightarrow D(\mathrm{A})' \\
\tau_{\mathrm{A}^\top} &: D(\mathrm{A}^\top) \to R(\tau_{\mathrm{A}^\top}) \subset D(\mathrm{A})', & \rho_{\mathrm{A}^\top} &:= \rho_{D(\mathrm{A}^\top)} : D(\mathrm{A}^\top) \twoheadrightarrow D(\mathrm{A}^\top)' \\
\widehat{\tau}_{\mathrm{A}} = \tau_{\mathrm{A}}|_{T(\mathrm{A})} &: T(\mathrm{A}) \twoheadrightarrow R(\tau_{\mathrm{A}}) = T(\mathrm{A}^\top)', & \widehat{\rho}_{\mathrm{A}} &:= \rho_{\mathrm{A}}|_{T(\mathrm{A})} : T(\mathrm{A}) \twoheadrightarrow T(\mathrm{A})' \\
\widehat{\tau}_{\mathrm{A}^\top} = \tau_{\mathrm{A}^\top}|_{T(\mathrm{A}^\top)} &: T(\mathrm{A}^\top) \twoheadrightarrow R(\tau_{\mathrm{A}^\top}) = T(\mathrm{A})' & \widehat{\rho}_{\mathrm{A}^\top} &:= \rho_{\mathrm{A}^\top}|_{T(\mathrm{A}^\top)} : T(\mathrm{A}^\top) \twoheadrightarrow T(\mathrm{A}^\top)'
\end{aligned}$$

Lemma (trace /Riesz isometric isomorphisms $\twoheadrightarrow$)

$$\begin{aligned}
R(\tau_{\mathrm{A}}) = R(\widehat{\tau}_{\mathrm{A}}) = R(\widehat{\rho}_{\mathrm{A}^\top}) &= T(\mathrm{A}^\top)', & \widehat{\tau}_{\mathrm{A}} &: T(\mathrm{A}) \twoheadrightarrow T(\mathrm{A}^\top)', & \widehat{\rho}_{\mathrm{A}^\top} &: T(\mathrm{A}^\top) \twoheadrightarrow T(\mathrm{A}^\top)' \\
R(\tau_{\mathrm{A}^\top}) = R(\widehat{\tau}_{\mathrm{A}^\top}) = R(\widehat{\rho}_{\mathrm{A}}) &= T(\mathrm{A})', & \widehat{\tau}_{\mathrm{A}^\top} &: T(\mathrm{A}^\top) \twoheadrightarrow T(\mathrm{A})', & \widehat{\rho}_{\mathrm{A}} &: T(\mathrm{A}) \twoheadrightarrow T(\mathrm{A})'
\end{aligned}$$

Definition (inverses of trace /Riesz isometric isomorphisms $\twoheadrightarrow$)

$$\begin{aligned}
\check{\tau}_{\mathrm{A}} &:= \widehat{\tau}_{\mathrm{A}}^{-1} : T(\mathrm{A}^\top)' \twoheadrightarrow T(\mathrm{A}), & \check{\rho}_{\mathrm{A}^\top} &:= \widehat{\rho}_{\mathrm{A}^\top}^{-1} : T(\mathrm{A}^\top)' \twoheadrightarrow T(\mathrm{A}^\top) \\
\check{\tau}_{\mathrm{A}^\top} &:= \widehat{\tau}_{\mathrm{A}^\top}^{-1} : T(\mathrm{A})' \twoheadrightarrow T(\mathrm{A}^\top), & \check{\rho}_{\mathrm{A}} &:= \widehat{\rho}_{\mathrm{A}}^{-1} : T(\mathrm{A})' \twoheadrightarrow T(\mathrm{A})
\end{aligned}$$

Remark

Continuity of traces and extensions for free! (no ass on $R(\mathrm{A})$ or domains Ω)

⚠ EVEN ISOMETRIES ⚠

Traces for Single Operators $\mathring{\mathrm{A}} \subset \mathrm{A}$ and $\mathrm{A}^* \subset \mathrm{A}^\top = \mathring{\mathrm{A}}^*$ (Iddc)

Remark (trace /Riesz isometric isomorphisms $\twoheadrightarrow$)

Continuity/isometry of traces and extensions for free! (no ass on $R(\mathrm{A})$ or domains Ω)

⚠ NO compact embeddings or Friedrichs/Poincaré type estimates NEEDED ⚠

We do NOT need:

e.g.: $\Omega \subset \mathbb{R}^3$ bd, weak Lip (compact embeddings)
(Weck 1972/'74, Weber 1980, Picard 1984)

$$\mathrm{H}^1(\Omega) \hookrightarrow\!\!\!\rightarrow \mathrm{L}^2(\Omega) \quad \text{or} \quad \mathring{\mathrm{H}}(\operatorname{rot}, \Omega) \cap \mathrm{H}(\operatorname{div}, \Omega) \hookrightarrow\!\!\!\rightarrow \mathrm{L}^2(\Omega)$$

or even weaker (Friedrichs/Poincaré type estimate $\Leftrightarrow$ closed range)

Ω (weak Lip, bd in ONE direction) $\Rightarrow$ $R(\mathring{\operatorname{grad}})$ and $R(\operatorname{div})$ closed

Ω (weak Lip, bd in TWO directions) $\Rightarrow$ $R(\mathring{\operatorname{rot}})$ and $R(\operatorname{rot})$ closed

Ω (weak Lip, bd in THREE directions) $\Rightarrow$ $R(\mathring{\operatorname{div}})$ and $R(\operatorname{grad})$ closed

Traces for Single Operators $\mathring{\mathrm{A}} \subset \mathrm{A}$ and $\mathrm{A}^* \subset \mathrm{A}^\top = \mathring{\mathrm{A}}^*$ (lddc)

Theorem (trace /Riesz isometric isomorphisms $\twoheadrightarrow$)

$$T(\mathrm{A})' \cong_{\hat{\rho}_\mathrm{A}} T(\mathrm{A}) \cong_{\hat{\tau}_\mathrm{A}} T(\mathrm{A}^\top)', \qquad T(\mathrm{A}^\top)' \cong_{\hat{\rho}_{\mathrm{A}^\top}} T(\mathrm{A}^\top) \cong_{\hat{\tau}_{\mathrm{A}^\top}} T(\mathrm{A})'$$

bilinear (sesquilinear) forms on $T(\mathrm{A}) \times T(\mathrm{A}^\top)$ or $D(\mathrm{A}) \times D(\mathrm{A}^\top)$

$$\begin{aligned} \langle\langle x, y\rangle\rangle := \langle\langle x, y\rangle\rangle_\tau &:= \tau_\mathrm{A} x(y) = -\tau_{\mathrm{A}^\top} y(x) = \langle \mathrm{A}\,x, y\rangle_{\mathrm{H}_1} - \langle x, \mathrm{A}^\top y\rangle_{\mathrm{H}_0}, \\ \langle\langle x, y\rangle\rangle_\rho &:= \rho_\mathrm{A} x(y) = \langle x, y\rangle_{D(\mathrm{A})} = \langle x, y\rangle_{\mathrm{H}_0} + \langle \mathrm{A}\,x, \mathrm{A}\,y\rangle_{\mathrm{H}_1} \end{aligned}$$

Corollary ("integration by parts")

$$\langle \mathrm{A}\,x, y\rangle_{\mathrm{H}_1} = \langle x, \mathrm{A}^\top y\rangle_{\mathrm{H}_0} + \langle\langle x, y\rangle\rangle$$

Traces for Single Operators $\mathring{\mathrm{A}} \subset \mathrm{A}$ and $\mathrm{A}^* \subset \mathrm{A}^\top = \mathring{\mathrm{A}}^*$ (lddc)

Isometric Isomorphisms ($\twoheadrightarrow$)

$$\begin{tikzcd}
 & D(\mathrm{A}) \arrow[d, "\pi_\perp"'] & & \\
\mathcal{T}(\mathrm{A}) & T(\mathrm{A}) \arrow[l, two heads, "\iota_q"'] \arrow[r, two heads, "\widehat{\tau}_\mathrm{A}"] \arrow[d, two heads, "\widehat{\rho}_\mathrm{A}"'] & R(\tau_\mathrm{A}) = T(\mathrm{A}^\top)' & \\
 & T(\mathrm{A})' = R(\tau_{\mathrm{A}^\top}) & T(\mathrm{A}^\top) \arrow[l, two heads, "\widehat{\tau}_{\mathrm{A}^\top}"] \arrow[u, two heads, "\widehat{\rho}_{\mathrm{A}^\top}"'] \arrow[r, two heads, "\iota_q"'] & \mathcal{T}(\mathrm{A}^\top) \\
 & & D(\mathrm{A}^\top) \arrow[u, "\pi_\perp"'] &
\end{tikzcd}$$

$D(\mathrm{A}) = D(\mathring{\mathrm{A}}) \oplus_{D(\mathrm{A})} T(\mathrm{A})$

$[x_\perp] = [x]$ and $\tau_\mathrm{A} x_\perp = \tau_\mathrm{A} x = \tau_\mathrm{A}[x]$

Traces for Single Operators $\mathring{A} \subset A$ and $A^* \subset A^\top = \mathring{A}^*$ (Iddc)

Isometric Isomorphisms ($\twoheadrightarrow$)

$$\begin{tikzcd}
 & D(A) \arrow[rr, "A"] \arrow[d, "\pi_\perp"'] \arrow[dr, "\tau_A"] & & R(A) \\
\mathcal{T}(A) & T(A) \arrow[l, two heads, "\iota_q"'] \arrow[r, two heads, "\hat{\tau}_A"] \arrow[dr, two heads, "\hat{A}"] \arrow[d, two heads, "\hat{\rho}_A"'] & R(\tau_A) = T(A^\top)' & \\
 & T(A)' = R(\tau_{A^\top}) & T(A^\top) \arrow[l, two heads, "\hat{\tau}_{A^\top}"] \arrow[u, two heads, "\hat{\rho}_{A^\top}"'] \arrow[r, two heads, "\iota_q"'] & \mathcal{T}(A^\top) \\
R(A^\top) & & D(A^\top) \arrow[ll, "A^\top"] \arrow[u, "\pi_\perp"'] \arrow[ul, "\tau_{A^\top}"] &
\end{tikzcd}$$

$\hat{A} := A|_{T(A)}$

Traces for Single Operators $\mathring{A} \subset A$ and $A^* \subset A^\top = \mathring{A}^*$ (Iddc)

Isometric Isomorphisms ($\twoheadrightarrow$)

$$\begin{tikzcd}
 & D(A) \arrow[rr, bend left, "A"] \arrow[dr, "\tau_A"] \arrow[drr, bend left, "\tau_A"] \arrow[d, shift left, "\pi_\perp=\iota_\perp^*"] \arrow[ddl, two heads, bend right=60, "\rho_A"'] & & R(A) \\
\mathcal{T}(A) \arrow[r, shift right, two heads, "\widetilde{\pi}_\perp"'] & T(A) \arrow[u, shift left, "\iota_\perp"] \arrow[l, shift right, two heads, "\iota_q"'] \arrow[r, shift left, two heads, "\widehat{\tau}_A"] \arrow[d, shift left, two heads, "\widehat{\rho}_A"] \arrow[dr, shift left, two heads, "\pm\widehat{A}"] & R(\tau_A) = T(A^\top)' \arrow[l, shift left, two heads, "\check{\tau}_A"] \arrow[r, shift left, "\pi'_\perp"] \arrow[d, shift left, two heads, "\check{\rho}_{A^\top}"] & D(A^\top)' \arrow[l, shift left, "\iota'_\perp"] \\
D(A)' \arrow[r, shift right, "\iota'_\perp"'] & T(A)' = R(\tau_{A^\top}) \arrow[l, shift right, "\pi'_\perp"'] \arrow[u, shift left, two heads, "\check{\rho}_A"] \arrow[r, shift left, two heads, "\check{\tau}_{A^\top}"] & T(A^\top) \arrow[l, shift left, two heads, "\widehat{\tau}_{A^\top}"] \arrow[u, shift left, two heads, "\widehat{\rho}_{A^\top}"] \arrow[ul, shift left, two heads, "\pm\widehat{A^\top}"] \arrow[r, shift left, two heads, "\iota_q"] \arrow[d, shift left, "\iota_\perp"] & \mathcal{T}(A^\top) \arrow[l, shift left, two heads, "\widetilde{\pi}_\perp"] \\
R(A^\top) & & D(A^\top) \arrow[u, shift left, "\pi_\perp=\iota_\perp^*"] \arrow[ul, "\tau_{A^\top}"] \arrow[ull, bend left, "\tau_{A^\top}"] \arrow[ll, bend left, "A^\top"] \arrow[uur, two heads, bend right=60, "\rho_{A^\top}"'] &
\end{tikzcd}$$

"on $T(A) = N(A^\top A + 1)$ and $T(A^\top) = N(A\,A^\top + 1)$":

$$\check{\tau}_A = -\widehat{A}^\top \check{\rho}_{A^\top} \qquad \widehat{\tau}_A = \widehat{\rho}_{A^\top}\widehat{A} \qquad \widehat{A}^{-1} = -\widehat{A}^\top \qquad \widehat{A} = A|_{T(A)}$$

$$\check{\tau}_{A^\top} = -\widehat{A}\check{\rho}_A \qquad \widehat{\tau}_{A^\top} = \widehat{\rho}_A\widehat{A}^\top \qquad (\widehat{A}^\top)^{-1} = -\widehat{A} \qquad \widehat{A}^\top = A^\top|_{T(A^\top)}$$

Traces for Single Operators $\quad \mathring{A} \subset A \quad$ and $\quad A^* \subset A^\top = \mathring{A}^* \quad$ (Iddc)

Theorem (kernels and ranges of traces / isometric isomorphisms)

- $N(\tau_A) = D(\mathring{A})$
- $R(\tau_A) = R(\hat{\tau}_A) = D(A^*)^\circ = R(\hat{\rho}_{A^\top}) = T(A^\top)'$
- $T(A) = D(\mathring{A})^{\perp_{D(A)}} = N(A^\top A + 1)$
- $T(A) \cong \mathcal{T}(A) = D(A)/D(\mathring{A})$
- $N(\tau_{A^\top}) = D(A^*)$
- $R(\tau_{A^\top}) = R(\hat{\tau}_{A^\top}) = D(\mathring{A})^\circ = R(\hat{\rho}_A) = T(A)'$
- $T(A^\top) = D(A^*)^{\perp_{D(A^\top)}} = N(A\,A^\top + 1)$
- $T(A^\top) \cong \mathcal{T}(A^\top) = D(A^\top)/D(A^*)$

Remark (summary)

- trace ranges are annihilators of trace kernels
- trace ranges are duals of reduced transpose trace spaces
- trace spaces are kernels $N(A^\top A + 1)$ and $N(A\,A^\top + 1)$ ("harmonic fields")
- trace spaces are orthogonal complements of trace kernels (—)
- trace spaces are minimal norm extensions (—)
- trace spaces are quotient spaces of trace kernels

note:

- elements of the trace spaces are "smooth"
- regularity is never a problem (regularity not a good term)
- integrability is always the problem

Traces for Hilbert Complexes

so far NO (Hilbert) complexes

Traces for Hilbert Complexes

Traces for Hilbert Complexes

Traces for Hilbert Complexes

Different Hilbert Complexes

$$\begin{tikzcd}
\cdots \arrow[r, shift left, "\cdots"] & \mathrm{H}_0 \arrow[l, shift left, "\cdots"] \arrow[r, shift left, "\mathring{\mathrm{A}}_0"] & \mathrm{H}_1 \arrow[l, shift left, "\mathrm{A}_0^{\top}=\mathring{\mathrm{A}}_0^{*}"] \arrow[r, shift left, "\mathring{\mathrm{A}}_1"] & \mathrm{H}_2 \arrow[l, shift left, "\mathrm{A}_1^{\top}=\mathring{\mathrm{A}}_1^{*}"] \arrow[r, shift left, "\cdots"] & \cdots \arrow[l, shift left, "\cdots"]
\end{tikzcd}$$
(unbd prim/dual HilComs)

$$\begin{tikzcd}
\cdots \arrow[r, shift left, "\cdots"] & \mathrm{H}_0 \arrow[l, shift left, "\cdots"] \arrow[r, shift left, "\mathrm{A}_0"] & \mathrm{H}_1 \arrow[l, shift left, "\mathrm{A}_0^{*}"] \arrow[r, shift left, "\mathrm{A}_1"] & \mathrm{H}_2 \arrow[l, shift left, "\mathrm{A}_1^{*}"] \arrow[r, shift left, "\cdots"] & \cdots \arrow[l, shift left, "\cdots"]
\end{tikzcd}$$
(unbd prim/dual HilComs)

$$\begin{tikzcd}
\cdots \arrow[r, "\cdots"] & D(\mathrm{A}_0) \arrow[r, "\mathrm{A}_0"] & D(\mathrm{A}_1) \arrow[r, "\mathrm{A}_1"] & D(\mathrm{A}_2) \arrow[r, "\cdots"] & \cdots \\
\cdots & D(\mathrm{A}_0)' \arrow[l, "\cdots"] & D(\mathrm{A}_1)' \arrow[l, "\mathrm{A}_0'"] & D(\mathrm{A}_2)' \arrow[l, "\mathrm{A}_1'"] & \cdots \arrow[l, "\cdots"]
\end{tikzcd}$$
(bd DomCom)
(bd adjoint DomCom)

$$\begin{tikzcd}
\cdots & D(\mathrm{A}_{-1}^{\top}) \arrow[l, "\cdots"'] & D(\mathrm{A}_0^{\top}) \arrow[l, "\mathrm{A}_0^{\top}"'] & D(\mathrm{A}_1^{\top}) \arrow[l, "\mathrm{A}_1^{\top}"'] & \cdots \arrow[l, "\cdots"'] \\
\cdots \arrow[r, "\cdots"'] & D(\mathrm{A}_{-1}^{\top})' \arrow[r, "\mathrm{A}_0^{\top}{}'"'] & D(\mathrm{A}_0^{\top})' \arrow[r, "\mathrm{A}_1^{\top}{}'"'] & D(\mathrm{A}_1^{\top})' \arrow[r, "\cdots"'] & \cdots
\end{tikzcd}$$
(bd DomCom)
(bd adjoint DomCom)

setting

- $\mathring{\mathrm{A}}_\ell \subset \mathrm{A}_\ell$ and $\mathrm{A}_\ell^{*} \subset \mathrm{A}_\ell^{\top} = \mathring{\mathrm{A}}_\ell^{*}$ (lddc)
- $R(\mathring{\mathrm{A}}_0) \subset N(\mathring{\mathrm{A}}_1)$, $R(\mathrm{A}_0) \subset N(\mathrm{A}_1)$ (prim HilComs)
- $R(\mathrm{A}_1^{*}) \subset N(\mathrm{A}_0^{*})$, $R(\mathrm{A}_1^{\top}) \subset N(\mathrm{A}_0^{\top})$ (dual HilComs)
- $R(\mathrm{A}_1') \subset N(\mathrm{A}_0')$, $R(\mathrm{A}_0^{\top}{}') \subset N(\mathrm{A}_1^{\top}{}')$ (adjoint HilComs)

note $\mathrm{A}_n^{\top}{}' : D(\mathrm{A}_{n-1}^{\top})' \to D(\mathrm{A}_n^{\top})'$

$\Rightarrow$ $\mathrm{A}_n^{\top}{}' \;\widehat{\ncong}\; \mathrm{A}_n : D(\mathrm{A}_n) \to D(\mathrm{A}_{n+1})$ but $\mathrm{A}_n^{\top}{}' \;\widehat{\cong}\; \mathrm{A}_{n-1} : D(\mathrm{A}_{n-1}) \to D(\mathrm{A}_n)$ (index shift)

Traces for Hilbert Complexes $\mathring{\mathrm{A}}_\ell \subset \mathrm{A}_\ell$ and $R(\dots) \subset N(\dots)$

$\mathrm{A}_0 : D(\mathrm{A}_0) \to D(\mathrm{A}_1)$	$\mathrm{A}_1^\top : D(\mathrm{A}_1^\top) \to D(\mathrm{A}_0^\top)$	(vol diff ops)
$\tau_{\mathrm{A}_0} : D(\mathrm{A}_0) \to D(\mathrm{A}_0^\top)'$	$\tau_{\mathrm{A}_1^\top} : D(\mathrm{A}_1^\top) \to D(\mathrm{A}_1)'$	(trace ops)
$\mathrm{A}_1^{\top\,\prime} : D(\mathrm{A}_0^\top)' \to D(\mathrm{A}_1^\top)'$	$\mathrm{A}_0' : D(\mathrm{A}_1)' \to D(\mathrm{A}_0)'$	(surf diff ops)

Theorem (surface differential operators / commutators with traces)

$$\tau_{\mathrm{A}_1}\, \mathrm{A}_0 = -\,\mathrm{A}_1^{\top\,\prime} \tau_{\mathrm{A}_0} \qquad \textit{and} \qquad \tau_{\mathrm{A}_0^\top}\, \mathrm{A}_1^\top = -\,\mathrm{A}_0'\, \tau_{\mathrm{A}_1^\top}$$

Theorem (integration by parts . . .)

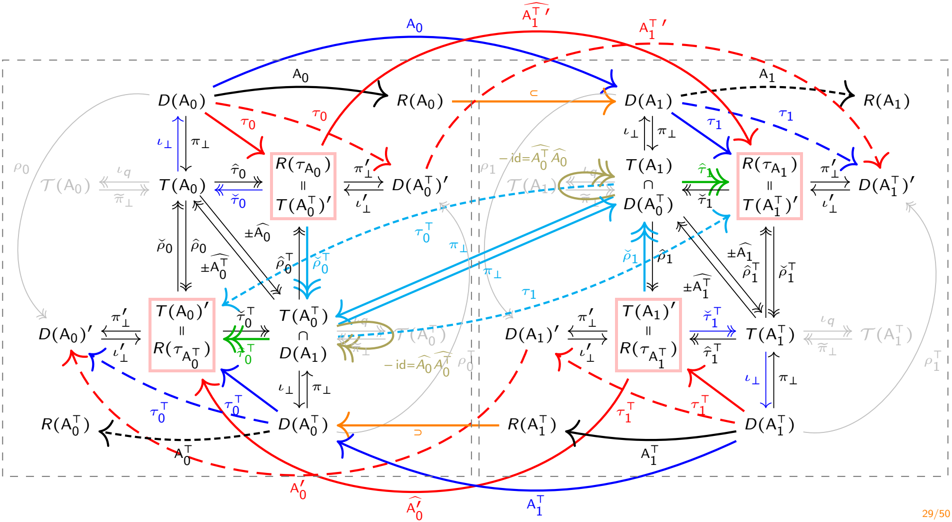
- *. . . on domains:* $x \in D(\mathrm{A})$, $y \in D(\mathrm{A}^\top)$ *or* $x \in T(\mathrm{A})$, $y \in T(\mathrm{A}^\top)$ $\Rightarrow$ $\langle \mathrm{A}\,x, y\rangle_{\mathrm{H}_1} = \langle x, \mathrm{A}^\top y\rangle_{\mathrm{H}_0} + \langle\!\langle x, y\rangle\!\rangle$
- *. . . on trace domains* $\boxed{\tau_{\mathrm{A}_1}\, \mathrm{A}_0 = -\,\mathrm{A}_1^{\top\,\prime} \tau_{\mathrm{A}_0}}$ $x \in D(\mathrm{A}_0)$, $z \in D(\mathrm{A}_1^\top)$ $\Rightarrow$
  $$\langle\!\langle \mathrm{A}_0\, x, z\rangle\!\rangle_1 = \tau_{\mathrm{A}_1}(\mathrm{A}_0\, x)(z) = -\tau_{\mathrm{A}_0}(x)(\mathrm{A}_1^\top z) = -\langle\!\langle x, \mathrm{A}_1^\top z\rangle\!\rangle_0$$
- *. . . on trace spaces* $\boxed{\hat{\tau}_{\mathrm{A}_1}\, \pi_\perp \widehat{\mathrm{A}_0} = -\widehat{\mathrm{A}_1^{\top\,\prime}}\, \pi_\perp'\, \hat{\tau}_{\mathrm{A}_0}}$ $x \in T(\mathrm{A}_0)$, $z \in T(\mathrm{A}_1^\top)$ $\Rightarrow$
  $$\langle\!\langle \pi_\perp \widehat{\mathrm{A}_0} x, z\rangle\!\rangle_1 = \hat{\tau}_{\mathrm{A}_1}(\pi_\perp \widehat{\mathrm{A}_0} x)(z) = -\hat{\tau}_{\mathrm{A}_0}(x)(\pi_\perp \widehat{\mathrm{A}_1^\top} z) = -\langle\!\langle x, \pi_\perp \widehat{\mathrm{A}_1^\top} z\rangle\!\rangle_0$$
- *. . . on trace ranges* $\boxed{\widehat{\mathrm{A}_1^{\top\,\prime}} = -\hat{\tau}_{\mathrm{A}_1}\, \iota_d^{-1} (\widehat{\mathrm{A}_0'})'\, \iota_d\, \check{\tau}_{\mathrm{A}_0}}$ $\varphi \in R(\tau_{\mathrm{A}_0})$, $\psi \in R(\tau_{\mathrm{A}_1^\top})$ $\Rightarrow$
  $$\langle\!\langle\!\langle \widehat{\mathrm{A}_1^{\top\,\prime}} \varphi, \psi\rangle\!\rangle\!\rangle_1 = \langle\!\langle \check{\tau}_{\mathrm{A}_1} \widehat{\mathrm{A}_1^{\top\,\prime}} \varphi, \check{\tau}_{\mathrm{A}_1^\top} \psi\rangle\!\rangle_1 = -\langle\!\langle \check{\tau}_{\mathrm{A}_0} \varphi, \check{\tau}_{\mathrm{A}_0^\top} \widehat{\mathrm{A}_0'} \psi\rangle\!\rangle_0 = -\langle\!\langle\!\langle \varphi, \widehat{\mathrm{A}_0'} \psi\rangle\!\rangle\!\rangle_0$$

Traces for Hilbert Complexes $\mathring{A}_\ell \subset A_\ell$ and $R(\dots) \subset N(\dots)$

$A_0 : D(A_0) \to D(A_1)$	(vol diff ops)	$A_1^\top : D(A_1^\top) \to D(A_0^\top)$
$A_1^\top{}' : D(A_0^\top)' \to D(A_1^\top)'$	(surf diff ops)	$A_0' : D(A_1)' \to D(A_0)'$

⚠ CRAZY ⚠

simple idea ⇒ amazing complexity



Traces for Hilbert Complexes $\mathring{A}_\ell \subset A_\ell$ and $R(\dots) \subset N(\dots)$

$$\begin{tikzcd}
\cdots \arrow[r, "\pi_\perp \widehat{A_{-1}}"] & T(A_0) \arrow[r, "\pi_\perp \widehat{A_0}"] \arrow[d, shift left, two heads, "\hat{\tau}_0"] \arrow[ddd, bend right=40, two heads, "\hat{\rho}_0"'] & T(A_1) \arrow[r, "\pi_\perp \widehat{A_1}"] \arrow[d, two heads, "\hat{\tau}_1"] \arrow[ddd, bend right=40, two heads, "\hat{\rho}_1"'] & \begin{matrix} T(A_2) \\ \cap \\ N(A_1^\top) \end{matrix} \arrow[r, "\pi_\perp \widehat{A_2}"] \arrow[d, two heads, "\hat{\tau}_2"] \arrow[ddd, bend left=40, two heads, "\hat{\rho}_2"] \arrow[ddll, bend left, "\pi_\perp A_1^\top = 0"] \arrow[dddl, "\tau_1^\top"] & \cdots \\
\cdots \arrow[r, "\widehat{A_0^\top}'"] & \boxed{\begin{matrix} R(\tau_{A_0}) \\ \shortparallel \\ T(A_0^\top)' \end{matrix}} \arrow[u, shift left, two heads, "\check{\tau}_0"] \arrow[r, "\widehat{A_1^\top}'"] \arrow[d, shift left, two heads, "\check{\rho}_0^\top"] & \boxed{\begin{matrix} R(\tau_{A_1}) \\ \shortparallel \\ T(A_1^\top)' \end{matrix}} \arrow[r, "\widehat{A_2^\top}'"'] \arrow[d, two heads, "\check{\rho}_1^\top"] & \boxed{\begin{matrix} R(\tau_{A_2}) \\ \shortparallel \\ T(A_2^\top)' \end{matrix}} \arrow[r, "\widehat{A_3^\top}'"] \arrow[d, two heads, "\check{\rho}_2^\top"] & \cdots \\
\cdots & \begin{matrix} T(A_0^\top) \\ \cap \\ N(A_1) \end{matrix} \arrow[l, "\pi_\perp \widehat{A_0^\top}"] \arrow[u, shift left, two heads, "\hat{\rho}_0^\top"] \arrow[d, two heads, "\hat{\tau}_0^\top"] \arrow[ur, "\tau_1"'] \arrow[uurr, bend left, "\pi_\perp A_1 = 0"] & T(A_1^\top) \arrow[l, "\pi_\perp \widehat{A_1^\top}"] \arrow[d, two heads, "\hat{\tau}_1^\top"] & T(A_2^\top) \arrow[l, "\pi_\perp \widehat{A_2^\top}"] \arrow[d, two heads, "\hat{\tau}_2^\top"] & \cdots \arrow[l, "\pi_\perp \widehat{A_3^\top}"] \\
\cdots & \boxed{\begin{matrix} R(\tau_{A_0^\top}) \\ \shortparallel \\ T(A_0)' \end{matrix}} \arrow[l, "\widehat{A'_{-1}}"] & \boxed{\begin{matrix} R(\tau_{A_1^\top}) \\ \shortparallel \\ T(A_1)' \end{matrix}} \arrow[l, "\widehat{A'_0}"] & \boxed{\begin{matrix} R(\tau_{A_2^\top}) \\ \shortparallel \\ T(A_2)' \end{matrix}} \arrow[l, "\widehat{A'_1}"] & \cdots \arrow[l, "\widehat{A'_2}"]
\end{tikzcd}$$

$$\begin{tikzcd}
\text{(trace space complex)} \quad \text{iso}^2 \arrow[d, shift left, two heads, "\hat{\tau}_n"] \\
\text{(trace range complex)} \quad \text{iso}^2 \arrow[u, shift left, two heads, "\check{\tau}_n"] \arrow[d, shift left, two heads, "\check{\rho}_n^\top"] \\
\text{(dual trace space complex)} \quad \text{iso}^2 \arrow[u, shift left, two heads, "\hat{\rho}_n^\top"] \arrow[d, shift left, two heads, "\hat{\tau}_n^\top"] \\
\text{(dual trace range complex)} \quad \text{iso}^2 \arrow[u, shift left, two heads, "\check{\tau}_n^\top"]
\end{tikzcd}$$

. . . Examples of Traces. . .

Examples

Example 1

De Rham Complex – ∇ / div

$\mathring{A} \subset A$	lddc
$A^* \subset A^\top = \mathring{A}^*$	$(\mathring{A}, A^*)$ pair "with" and $(A, A^\top)$ pair "without" bc
bd trace $\tau_A : D(A) \to D(A^\top)'$	with $\langle\!\langle x, y \rangle\!\rangle = \tau_A x(y) = \langle y, A\, x \rangle_{H_1} - \langle A^\top y, x \rangle_{H_0} = -\tau_{A^\top} y(x)$
classical ibp	$\langle\!\langle u, E \rangle\!\rangle = \tau_\nabla u(E) = \langle \nabla u, E \rangle_{L^2(\Omega)} - \langle u, -\operatorname{div} E \rangle_{L^2(\Omega)} = \langle u\|_\Gamma, \nu \cdot E\|_\Gamma \rangle_{L^2(\Gamma)} = -\tau_{-\operatorname{div}} E(u)$

$$\begin{aligned}
\mathring{\nabla} : D(\mathring{\nabla}) = \mathring{H}^1(\Omega) \subset L^2(\Omega) \to L^2(\Omega) &\quad \text{(minimal gradient)} \quad \mathring{\nabla}^* = -\operatorname{div}\\
\nabla : D(\nabla) = H^1(\Omega) \subset L^2(\Omega) \to L^2(\Omega) &\quad \text{(maximal gradient)} \quad \nabla^* = -\mathring{\operatorname{div}}\\
\mathring{\operatorname{div}} : D(\mathring{\operatorname{div}}) = \mathring{H}(\operatorname{div}, \Omega) \subset L^2(\Omega) \to L^2(\Omega) &\quad \text{(minimal divergence)} \quad \mathring{\operatorname{div}}^* = -\nabla\\
\operatorname{div} : D(\operatorname{div}) = H(\operatorname{div}, \Omega) \subset L^2(\Omega) \to L^2(\Omega) &\quad \text{(maximal divergence)} \quad \operatorname{div}^* = -\mathring{\nabla}
\end{aligned}$$

$$\mathring{\nabla} = \mathring{A} \subset A = \nabla$$

$$-\mathring{\operatorname{div}} = \nabla^* = A^* \subset A^\top = \mathring{A}^* = \mathring{\nabla}^* = -\operatorname{div}$$

Example 1

De Rham Complex – ∇ / div

$\mathring{\mathrm{A}} \subset \mathrm{A}$	lddc
$\mathrm{A}^* \subset \mathrm{A}^\top = \mathring{\mathrm{A}}^*$	$(\mathring{\mathrm{A}}, \mathrm{A}^*)$ pair “with” and $(\mathrm{A}, \mathrm{A}^\top)$ pair “without” bc
bd trace $\tau_{\mathrm{A}} : D(\mathrm{A}) \to D(\mathrm{A}^\top)'$	with $\langle\!\langle x, y \rangle\!\rangle = \tau_{\mathrm{A}} x(y) = \langle y, \mathrm{A}\, x \rangle_{\mathrm{H}_1} - \langle \mathrm{A}^\top y, x \rangle_{\mathrm{H}_0} = -\tau_{\mathrm{A}^\top} y(x)$
classical ibp	$\langle\!\langle u, E \rangle\!\rangle = \tau_\nabla u(E) = \langle \nabla\, u, E \rangle_{\mathrm{L}^2(\Omega)} - \langle u, -\operatorname{div} E \rangle_{\mathrm{L}^2(\Omega)} = \langle u\vert_\Gamma, \nu \cdot E\vert_\Gamma \rangle_{\mathrm{L}^2(\Gamma)} = -\tau_{-\operatorname{div}} E(u)$

$$\mathring{\nabla} = \mathring{\mathrm{A}} \subset \mathrm{A} = \nabla \qquad -\mathring{\operatorname{div}} = \mathrm{A}^* \subset \mathrm{A}^\top = -\operatorname{div}$$

$$\tau_\nabla : \mathrm{H}^1(\Omega) \to \mathrm{H}(\operatorname{div}, \Omega)', \qquad \tau_\nabla u(E) \overset{\text{class}}{=} \langle u\vert_\Gamma, \nu \cdot E\vert_\Gamma \rangle_{\mathrm{L}^2(\Gamma)}$$

$$\tau_{-\operatorname{div}} : \mathrm{H}(\operatorname{div}, \Omega) \to \mathrm{H}^1(\Omega)', \qquad \tau_{-\operatorname{div}} E(u) \overset{\text{class}}{=} -\langle \nu \cdot E\vert_\Gamma, u\vert_\Gamma \rangle_{\mathrm{L}^2(\Gamma)}$$

$$\mathrm{H}(\operatorname{div}, \Omega)' \ni \tau_\nabla u(\,\cdot\,) = \langle\!\langle u, \,\cdot\, \rangle\!\rangle \overset{\text{class}}{=} \langle u\vert_\Gamma, \nu \cdot (\,\cdot\,)\vert_\Gamma \rangle_{\mathrm{L}^2(\Gamma)}$$

$\tau_\nabla u \rightsquigarrow u\vert_\Gamma$ class scalar trace

$$\mathrm{H}^1(\Omega)' \ni \tau_{-\operatorname{div}} E(\,\cdot\,) = -\langle\!\langle \,\cdot\,, E \rangle\!\rangle \overset{\text{class}}{=} -\langle \nu \cdot E\vert_\Gamma, \,\cdot\, \rangle_{\mathrm{L}^2(\Gamma)}$$

class normal trace

$$\boxed{N(-\nabla \operatorname{div} + 1)' = \mathring{\mathrm{H}}(\operatorname{div}, \Omega)^\circ = R(\tau_\nabla) \ni \tau_\nabla u \overset{\text{class}}{\in} \mathrm{H}^{-1/2}(\Gamma)' \overset{\iota_d}{=} \mathrm{H}^{1/2}(\Gamma)}$$

$$\boxed{N(-\operatorname{div} \nabla + 1)' = \mathring{\mathrm{H}}^1(\Omega)^\circ = R(\tau_{\operatorname{div}}) \ni \tau_{\operatorname{div}} E \overset{\text{class}}{\in} \mathrm{H}^{1/2}(\Gamma)' = \mathrm{H}^{-1/2}(\Gamma)}$$

Example 2

De Rham Complex – rot

$\mathring{\mathrm{A}} \subset \mathrm{A}$	lddc
$\mathrm{A}^* \subset \mathrm{A}^\top = \mathring{\mathrm{A}}^*$	$(\mathring{\mathrm{A}}, \mathrm{A}^*)$ pair "with" and $(\mathrm{A}, \mathrm{A}^\top)$ pair "without" bc
bd trace $\tau_{\mathrm{A}} : D(\mathrm{A}) \to D(\mathrm{A}^\top)'$	with $\langle\langle x, y\rangle\rangle = \tau_{\mathrm{A}} x(y) = \langle y, \mathrm{A}\, x\rangle_{\mathrm{H}_1} - \langle \mathrm{A}^\top y, x\rangle_{\mathrm{H}_0} = -\tau_{\mathrm{A}^\top} y(x)$

classical ibp $\langle\langle E, H\rangle\rangle = \tau_{\mathrm{rot}} E(H) = \langle \mathrm{rot}\, E, H\rangle_{\mathrm{L}^2(\Omega)} - \langle E, \mathrm{rot}\, H\rangle_{\mathrm{L}^2(\Omega)} = \langle \nu \times \nu \times E|_\Gamma, \nu \times H|_\Gamma\rangle_{\mathrm{L}^2(\Gamma)} = -\tau_{\mathrm{rot}} H(E)$

$$\mathring{\mathrm{rot}} : D(\mathring{\mathrm{rot}}) = \mathring{\mathrm{H}}(\mathrm{rot}, \Omega) \subset \mathrm{L}^2(\Omega) \to \mathrm{L}^2(\Omega) \qquad \text{(minimal rotation)} \qquad \mathring{\mathrm{rot}}^* = \mathrm{rot}$$

$$\mathrm{rot} : D(\mathrm{rot}) = \mathrm{H}(\mathrm{rot}, \Omega) \subset \mathrm{L}^2(\Omega) \to \mathrm{L}^2(\Omega) \qquad \text{(maximal rotation)} \qquad \mathrm{rot}^* = \mathring{\mathrm{rot}}$$

$$\mathring{\mathrm{rot}} = \mathring{\mathrm{A}} \subset \mathrm{A} = \mathrm{rot}$$

$$\mathring{\mathrm{rot}} = \mathrm{rot}^* = \mathrm{A}^* \subset \mathrm{A}^\top = \mathring{\mathrm{A}}^* = \mathring{\mathrm{rot}}^* = \mathrm{rot}$$

Example 2

De Rham Complex – rot

$\mathring{\mathrm{A}} \subset \mathrm{A}$ lddc

$\mathrm{A}^* \subset \mathrm{A}^\top = \mathring{\mathrm{A}}^*$ $(\mathring{\mathrm{A}}, \mathrm{A}^*)$ pair "with" and $(\mathrm{A}, \mathrm{A}^\top)$ pair "without" bc

bd trace $\tau_{\mathrm{A}} : D(\mathrm{A}) \to D(\mathrm{A}^\top)'$ with $\langle\!\langle x, y \rangle\!\rangle = \tau_{\mathrm{A}} x(y) = \langle y, \mathrm{A}\, x \rangle_{\mathrm{H}_1} - \langle \mathrm{A}^\top y, x \rangle_{\mathrm{H}_0} = -\tau_{\mathrm{A}^\top} y(x)$

classical ibp $\langle\!\langle E, H \rangle\!\rangle = \tau_{\mathrm{rot}} E(H) = \langle \mathrm{rot}\, E, H \rangle_{\mathrm{L}^2(\Omega)} - \langle E, \mathrm{rot}\, H \rangle_{\mathrm{L}^2(\Omega)} = \langle \nu \times \nu \times E|_\Gamma, \nu \times H|_\Gamma \rangle_{\mathrm{L}^2(\Gamma)} = -\tau_{\mathrm{rot}} H(E)$

$\mathring{\mathrm{rot}} = \mathring{\mathrm{A}} \subset \mathrm{A} = \mathrm{rot}$ $\mathring{\mathrm{rot}} = \mathrm{A}^* \subset \mathrm{A}^\top = \mathrm{rot}$

$$\tau_{\mathrm{rot}} : \mathrm{H}(\mathrm{rot}, \Omega) \to \mathrm{H}(\mathrm{rot}, \Omega)', \quad \tau_{\mathrm{rot}} E(H) \overset{\text{class}}{=} \langle \nu \times \nu \times E|_\Gamma, \nu \times H|_\Gamma \rangle_{\mathrm{L}^2(\Gamma)} \overset{\text{class}}{=} -\tau_{\mathrm{rot}} H(E)$$

$$\mathrm{H}(\mathrm{rot}, \Omega)' \ni \tau_{\mathrm{rot}} E(\,\cdot\,) = \langle\!\langle E, \cdot \rangle\!\rangle \overset{\text{class}}{=} \langle \nu \times \nu \times E|_\Gamma, \nu \times (\,\cdot\,)|_\Gamma \rangle_{\mathrm{L}^2(\Gamma)}$$

class tangential trace

$$\mathrm{H}(\mathrm{rot}, \Omega)' \ni \tau_{\mathrm{rot}} H(\,\cdot\,) = -\langle\!\langle \cdot, H \rangle\!\rangle \overset{\text{class}}{=} \langle \nu \times H|_\Gamma, \nu \times (\,\cdot\,)|_\Gamma \times \nu \rangle_{\mathrm{L}^2(\Gamma)}$$

cl twisted tan trace

$$\boxed{N(\mathrm{rot}\,\mathrm{rot} + 1)' = \mathring{\mathrm{H}}(\mathrm{rot}, \Omega)^\circ = R(\tau_{\mathrm{rot}}) \ni \tau_{\mathrm{rot}} E \overset{\text{class}}{\in} \mathrm{H}^{-1/2}(\mathrm{rot}_\Gamma, \Gamma) \overset{\iota_d}{=} \mathrm{H}_\times^{-1/2}(\mathrm{div}_\Gamma, \Gamma)'}$$

$$\boxed{N(\mathrm{rot}\,\mathrm{rot} + 1)' = \mathring{\mathrm{H}}(\mathrm{rot}, \Omega)^\circ = R(\tau_{\mathrm{rot}}) \ni \tau_{\mathrm{rot}} H \overset{\text{class}}{\in} \mathrm{H}_\times^{-1/2}(\mathrm{div}_\Gamma, \Gamma) = \mathrm{H}^{-1/2}(\mathrm{rot}_\Gamma, \Gamma)'}$$

Example 3

De Rham Complex – mixed bc

$\mathring{\mathrm{A}} \subset \mathrm{A}$ lddc

$\mathrm{A}^* \subset \mathrm{A}^\top = \mathring{\mathrm{A}}^*$ $(\mathring{\mathrm{A}}, \mathrm{A}^*)$ pair "with" and $(\mathrm{A}, \mathrm{A}^\top)$ pair "without" bc

bd trace $\tau_{\mathrm{A}} : D(\mathrm{A}) \to D(\mathrm{A}^\top)'$ with $\langle\langle x, y\rangle\rangle = \tau_{\mathrm{A}} x(y) = \langle y, \mathrm{A}\, x\rangle_{\mathrm{H}_1} - \langle \mathrm{A}^\top y, x\rangle_{\mathrm{H}_0} = -\tau_{\mathrm{A}^\top} y(x)$

classical ibp $\langle\langle u, E\rangle\rangle = \tau_\nabla u(E) = \langle \nabla\, u, E\rangle_{\mathrm{L}^2(\Omega)} - \langle u, -\operatorname{div} E\rangle_{\mathrm{L}^2(\Omega)} = \langle u|_\Gamma, \nu \cdot E|_\Gamma\rangle_{\mathrm{L}^2(\Gamma)} = -\tau_{-\operatorname{div}} E(u)$

$\langle\langle E, H\rangle\rangle = \tau_{\operatorname{rot}} E(H) = \langle \operatorname{rot} E, H\rangle_{\mathrm{L}^2(\Omega)} - \langle E, \operatorname{rot} H\rangle_{\mathrm{L}^2(\Omega)} = \langle \nu \times \nu \times E|_\Gamma, \nu \times H|_\Gamma\rangle_{\mathrm{L}^2(\Gamma)} = -\tau_{\operatorname{rot}} H(E)$

"$\Gamma = \Gamma_0 \cup \Gamma_1$"

$$\nabla_{\Gamma_0} : D(\nabla_{\Gamma_0}) = \mathrm{H}^1_{\Gamma_0}(\Omega) \subset \mathrm{L}^2(\Omega) \to \mathrm{L}^2(\Omega) \quad \text{(gradient)} \quad \nabla^*_{\Gamma_0} = -\operatorname{div}_{\Gamma_1}$$

$$\operatorname{div}_{\Gamma_0} : D(\operatorname{div}_{\Gamma_0}) = \mathrm{H}_{\Gamma_0}(\operatorname{div}, \Omega) \subset \mathrm{L}^2(\Omega) \to \mathrm{L}^2(\Omega) \quad \text{(divergence)} \quad \operatorname{div}^*_{\Gamma_0} = -\nabla_{\Gamma_1}$$

$$\operatorname{rot}_{\Gamma_0} : D(\operatorname{rot}_{\Gamma_0}) = \mathrm{H}_{\Gamma_0}(\operatorname{rot}, \Omega) \subset \mathrm{L}^2(\Omega) \to \mathrm{L}^2(\Omega) \quad \text{(rotation)} \quad \operatorname{rot}^*_{\Gamma_0} = \operatorname{rot}_{\Gamma_1}$$

$$\nabla_{\Gamma_0} = \mathring{\mathrm{A}} \subset \mathrm{A} = \nabla$$

$$-\mathring{\operatorname{div}} = \nabla^* = \mathrm{A}^* \subset \mathrm{A}^\top = \mathring{\mathrm{A}}^* = \nabla^*_{\Gamma_0} = -\operatorname{div}_{\Gamma_1}$$

$\tau_\nabla \widehat{=} (\,\cdot\,)\big|_{\Gamma_0}$: partial scalar trace on Γ_0

$$\mathring{\nabla} = \mathring{\mathrm{A}} \subset \mathrm{A} = \nabla_{\Gamma_0}$$

$$-\operatorname{div}_{\Gamma_1} = \nabla^*_{\Gamma_0} = \mathrm{A}^* \subset \mathrm{A}^\top = \mathring{\mathrm{A}}^* = \mathring{\nabla}^* = -\operatorname{div}$$

$\tau_{\operatorname{div}} \widehat{=} \nu \cdot (\,\cdot\,)\big|_{\Gamma_1}$: partial normal trace on Γ_1

$$\operatorname{rot}_{\Gamma_0} = \mathring{\mathrm{A}} \subset \mathrm{A} = \operatorname{rot}$$

$$\mathring{\operatorname{rot}} = \operatorname{rot}^* = \mathrm{A}^* \subset \mathrm{A}^\top = \mathring{\mathrm{A}}^* = \operatorname{rot}^*_{\Gamma_0} = \operatorname{rot}_{\Gamma_1}$$

$\tau_{\operatorname{rot}} \widehat{=} \nu \times (\,\cdot\,)\big|_{\Gamma_0}$: partial tangential trace on Γ_0

Example 4

De Rham Complex – Laplacians

$\mathring{\mathrm{A}} \subset \mathrm{A}$	lddc
$\mathrm{A}^* \subset \mathrm{A}^\top = \mathring{\mathrm{A}}^*$	$(\mathring{\mathrm{A}}, \mathrm{A}^*)$ pair “with” and $(\mathrm{A}, \mathrm{A}^\top)$ pair “without” bc
bd trace $\tau_{\mathrm{A}} : D(\mathrm{A}) \to D(\mathrm{A}^\top)'$	with $\langle\!\langle x, y \rangle\!\rangle = \tau_{\mathrm{A}} x(y) = \langle y, \mathrm{A}\, x \rangle_{\mathrm{H}_1} - \langle \mathrm{A}^\top y, x \rangle_{\mathrm{H}_0} = -\tau_{\mathrm{A}^\top} y(x)$

cl ibp $\langle\!\langle u, v \rangle\!\rangle = \tau_\Delta u(v) = \langle \Delta u, v \rangle_{\mathrm{L}^2(\Omega)} - \langle u, \Delta v \rangle_{\mathrm{L}^2(\Omega)} = \langle \partial_\nu\, u|_\Gamma, v|_\Gamma \rangle_{\mathrm{L}^2(\Gamma)} - \langle u|_\Gamma, \partial_\nu\, v|_\Gamma \rangle_{\mathrm{L}^2(\Gamma)} = -\tau_\Delta v(u)$

$\overset{\circ\circ}{\Delta} : D(\overset{\circ\circ}{\Delta}) = \mathring{\mathrm{H}}^2(\Omega) \subset \mathrm{L}^2(\Omega) \to \mathrm{L}^2(\Omega)$	(minimal Laplacian)	$\overset{\circ\circ}{\Delta}{}^* = \Delta$
$\mathring{\Delta}_{\mathtt{D}} = \operatorname{div} \mathring{\nabla} : D(\mathring{\Delta}_{\mathtt{D}}) \subset \mathrm{L}^2(\Omega) \to \mathrm{L}^2(\Omega)$	(Dirichlet Laplacian, s.a.)	$\mathring{\Delta}_{\mathtt{D}}^* = \mathring{\Delta}_{\mathtt{D}}$
$\mathring{\Delta}_{\mathtt{N}} = \mathring{\operatorname{div}} \nabla : D(\mathring{\Delta}_{\mathtt{N}}) \subset \mathrm{L}^2(\Omega) \to \mathrm{L}^2(\Omega)$	(Neumann Laplacian, s.a.)	$\mathring{\Delta}_{\mathtt{N}}^* = \mathring{\Delta}_{\mathtt{N}}$
$\Delta : D(\Delta) = \mathrm{H}(\Delta, \Omega) \subset \mathrm{L}^2(\Omega) \to \mathrm{L}^2(\Omega)$	(maximal Laplacian)	$\Delta^* = \overset{\circ\circ}{\Delta}$

$$\overset{\circ\circ}{\Delta} = \mathring{\mathrm{A}} \subset \mathrm{A} = \Delta$$
$$\overset{\circ\circ}{\Delta} = \Delta^* = \mathrm{A}^* \subset \mathrm{A}^\top = \mathring{\mathrm{A}}^* = \overset{\circ\circ}{\Delta}{}^* = \Delta$$

$$\mathring{\Delta}_{\mathtt{D}} = \mathring{\mathrm{A}} \subset \mathrm{A} = \Delta$$
$$\overset{\circ\circ}{\Delta} = \Delta^* = \mathrm{A}^* \subset \mathrm{A}^\top = \mathring{\mathrm{A}}^* = \mathring{\Delta}_{\mathtt{D}}^* = \mathring{\Delta}_{\mathtt{D}}$$

$$\overset{\circ\circ}{\Delta} = \mathring{\mathrm{A}} \subset \mathrm{A} = \mathring{\Delta}_{\mathtt{D/N}}$$
$$\mathring{\Delta}_{\mathtt{D/N}} = \mathring{\Delta}_{\mathtt{D/N}}^* = \mathrm{A}^* \subset \mathrm{A}^\top = \mathring{\mathrm{A}}^* = \overset{\circ\circ}{\Delta}{}^* = \Delta$$

$$\mathring{\Delta}_{\mathtt{N}} = \mathring{\mathrm{A}} \subset \mathrm{A} = \Delta$$
$$\overset{\circ\circ}{\Delta} = \Delta^* = \mathrm{A}^* \subset \mathrm{A}^\top = \mathring{\mathrm{A}}^* = \mathring{\Delta}_{\mathtt{N}}^* = \mathring{\Delta}_{\mathtt{N}}$$

Example 4

De Rham Complex – Laplacians

$\mathring{\mathrm{A}} \subset \mathrm{A}$ lddc

$\mathrm{A}^* \subset \mathrm{A}^\top = \mathring{\mathrm{A}}^*$ $(\mathring{\mathrm{A}}, \mathrm{A}^*)$ pair “with” and $(\mathrm{A}, \mathrm{A}^\top)$ pair “without” bc

bd trace $\tau_{\mathrm{A}} : D(\mathrm{A}) \to D(\mathrm{A}^\top)'$ with $\langle\!\langle x, y \rangle\!\rangle = \tau_{\mathrm{A}} x(y) = \langle y, \mathrm{A}\, x \rangle_{\mathrm{H}_1} - \langle \mathrm{A}^\top y, x \rangle_{\mathrm{H}_0} = -\tau_{\mathrm{A}^\top} y(x)$

cl ibp $\langle\!\langle u, v \rangle\!\rangle = \tau_\Delta u(v) = \langle \Delta u, v \rangle_{\mathrm{L}^2(\Omega)} - \langle u, \Delta v \rangle_{\mathrm{L}^2(\Omega)} = \langle \partial_\nu\, u|_\Gamma, v|_\Gamma \rangle_{\mathrm{L}^2(\Gamma)} - \langle u|_\Gamma, \partial_\nu\, v|_\Gamma \rangle_{\mathrm{L}^2(\Gamma)} = -\tau_\Delta v(u)$

$$\overset{\circ\circ}{\Delta} = \mathring{\mathrm{A}} \subset \mathrm{A} = \Delta$$
$$\overset{\circ\circ}{\Delta} = \mathrm{A}^* \subset \mathrm{A}^\top = \Delta$$

$$\mathring{\Delta}_{\mathrm{D}} = \mathring{\mathrm{A}} \subset \mathrm{A} = \Delta$$
$$\overset{\circ\circ}{\Delta} = \mathrm{A}^* \subset \mathrm{A}^\top = \mathring{\Delta}_{\mathrm{D}}$$

$$\mathring{\Delta}_{\mathrm{N}} = \mathring{\mathrm{A}} \subset \mathrm{A} = \Delta$$
$$\overset{\circ\circ}{\Delta} = \mathrm{A}^* \subset \mathrm{A}^\top = \mathring{\Delta}_{\mathrm{N}}$$

$$\tau_\Delta : \mathrm{H}(\Delta, \Omega) \to \mathrm{H}(\Delta, \Omega)', \qquad \tau_\Delta u(v) \overset{\text{class}}{=} \langle \partial_\nu\, u|_\Gamma, v|_\Gamma \rangle_{\mathrm{L}^2(\Gamma)} - \langle u|_\Gamma, \partial_\nu\, v|_\Gamma \rangle_{\mathrm{L}^2(\Gamma)}$$
$$\tau_\Delta : \mathrm{H}(\Delta, \Omega) \to D(\mathring{\Delta}_{\mathrm{D}})', \qquad \tau_\Delta u(v) \overset{\text{class}}{=} -\langle u|_\Gamma, \partial_\nu\, v|_\Gamma \rangle_{\mathrm{L}^2(\Gamma)}, \qquad v \in \mathring{\mathrm{H}}^1(\Omega)$$
$$\tau_\Delta : \mathrm{H}(\Delta, \Omega) \to D(\mathring{\Delta}_{\mathrm{N}})', \qquad \tau_\Delta u(v) \overset{\text{class}}{=} \langle \partial_\nu\, u|_\Gamma, v|_\Gamma \rangle_{\mathrm{L}^2(\Gamma)}, \qquad \nabla\, v \in \mathring{\mathrm{H}}(\mathrm{div}, \Omega)$$

$$\tau_{\mathring{\Delta}_{\mathrm{D}}} : D(\mathring{\Delta}_{\mathrm{D}}) \to \mathrm{H}(\Delta, \Omega)', \qquad \tau_{\mathring{\Delta}_{\mathrm{D}}} v(u) \overset{\text{class}}{=} \langle u|_\Gamma, \partial_\nu\, v|_\Gamma \rangle_{\mathrm{L}^2(\Gamma)}, \qquad v \in \mathring{\mathrm{H}}^1(\Omega)$$
$$\tau_{\mathring{\Delta}_{\mathrm{N}}} : D(\mathring{\Delta}_{\mathrm{N}}) \to \mathrm{H}(\Delta, \Omega)', \qquad \tau_{\mathring{\Delta}_{\mathrm{N}}} v(u) \overset{\text{class}}{=} -\langle \partial_\nu\, u|_\Gamma, v|_\Gamma \rangle_{\mathrm{L}^2(\Gamma)}, \qquad \nabla\, v \in \mathring{\mathrm{H}}(\mathrm{div}, \Omega)$$

Example 4

De Rham Complex – Laplacians

$\mathring{\mathrm{A}} \subset \mathrm{A}$	lddc
$\mathrm{A}^* \subset \mathrm{A}^\top = \mathring{\mathrm{A}}^*$	$(\mathring{\mathrm{A}}, \mathrm{A}^*)$ pair "with" and $(\mathrm{A}, \mathrm{A}^\top)$ pair "without" bc
bd trace $\tau_{\mathrm{A}} : D(\mathrm{A}) \to D(\mathrm{A}^\top)'$	with $\langle\!\langle x, y \rangle\!\rangle = \tau_{\mathrm{A}} x(y) = \langle y, \mathrm{A}\, x \rangle_{\mathrm{H}_1} - \langle \mathrm{A}^\top y, x \rangle_{\mathrm{H}_0} = -\tau_{\mathrm{A}^\top} y(x)$

cl ibp $\langle\!\langle u, v \rangle\!\rangle = \tau_{\Delta} u(v) = \langle \Delta u, v \rangle_{\mathrm{L}^2(\Omega)} - \langle u, \Delta v \rangle_{\mathrm{L}^2(\Omega)} = \langle \partial_\nu\, u|_\Gamma, v|_\Gamma \rangle_{\mathrm{L}^2(\Gamma)} - \langle u|_\Gamma, \partial_\nu\, v|_\Gamma \rangle_{\mathrm{L}^2(\Gamma)} = -\tau_{\Delta} v(u)$

$$\overset{\circ\circ}{\Delta} = \mathring{\mathrm{A}} \subset \mathrm{A} = \Delta$$
$$\overset{\circ\circ}{\Delta} = \mathrm{A}^* \subset \mathrm{A}^\top = \Delta$$

$$\mathring{\Delta}_{\mathtt{D}} = \mathring{\mathrm{A}} \subset \mathrm{A} = \Delta$$
$$\overset{\circ\circ}{\Delta} = \mathrm{A}^* \subset \mathrm{A}^\top = \mathring{\Delta}_{\mathtt{D}}$$

$$\mathring{\Delta}_{\mathtt{N}} = \mathring{\mathrm{A}} \subset \mathrm{A} = \Delta$$
$$\overset{\circ\circ}{\Delta} = \mathrm{A}^* \subset \mathrm{A}^\top = \mathring{\Delta}_{\mathtt{N}}$$

$$\tau_{\Delta} : \mathrm{H}(\Delta, \Omega) \to \mathrm{H}(\Delta, \Omega)', \qquad \tau_{\Delta} u(v) \overset{\text{class}}{=} \langle \partial_\nu\, u|_\Gamma, v|_\Gamma \rangle_{\mathrm{L}^2(\Gamma)} - \langle u|_\Gamma, \partial_\nu\, v|_\Gamma \rangle_{\mathrm{L}^2(\Gamma)}$$
$$\tau_{\Delta} : \mathrm{H}(\Delta, \Omega) \to D(\mathring{\Delta}_{\mathtt{D}})', \qquad \tau_{\Delta} u(v) \overset{\text{class}}{=} -\langle u|_\Gamma, \partial_\nu\, v|_\Gamma \rangle_{\mathrm{L}^2(\Gamma)}, \qquad v \in \mathring{\mathrm{H}}^1(\Omega)$$
$$\tau_{\Delta} : \mathrm{H}(\Delta, \Omega) \to D(\mathring{\Delta}_{\mathtt{N}})', \qquad \tau_{\Delta} u(v) \overset{\text{class}}{=} \langle \partial_\nu\, u|_\Gamma, v|_\Gamma \rangle_{\mathrm{L}^2(\Gamma)}, \qquad \nabla\, v \in \mathring{\mathrm{H}}(\mathrm{div}, \Omega)$$

$\mathrm{H}(\Delta, \Omega)' \ni \tau_{\Delta} u(\,\cdot\,) = \langle\!\langle u, \cdot \rangle\!\rangle \overset{\text{class}}{=} \langle \partial_\nu\, u|_\Gamma, \cdot|_\Gamma \rangle_{\mathrm{L}^2(\Gamma)} - \langle u|_\Gamma, (\partial_\nu\, \cdot)|_\Gamma \rangle_{\mathrm{L}^2(\Gamma)}$ — Dir and Neu trace as sum

$D(\mathring{\Delta}_{\mathtt{D}})' \ni \tau_{\Delta} u(\,\cdot\,) = \langle\!\langle u, \cdot \rangle\!\rangle \overset{\text{class}}{=} -\langle u|_\Gamma, (\partial_\nu\, \cdot)|_\Gamma \rangle_{\mathrm{L}^2(\Gamma)}$ — non-standard Dirichlet trace

$D(\mathring{\Delta}_{\mathtt{N}})' \ni \tau_{\Delta} u(\,\cdot\,) = \langle\!\langle u, \cdot \rangle\!\rangle \overset{\text{class}}{=} \langle \partial_\nu\, u|_\Gamma, \cdot|_\Gamma \rangle_{\mathrm{L}^2(\Gamma)}$ — non-standard Neumann trace

Example 4 De Rham Complex – Laplacians

$\mathring{\mathrm{A}} \subset \mathrm{A}$ lddc

$\mathrm{A}^* \subset \mathrm{A}^\top = \mathring{\mathrm{A}}^*$ $(\mathring{\mathrm{A}}, \mathrm{A}^*)$ pair “with” and $(\mathrm{A}, \mathrm{A}^\top)$ pair “without” bc

bd trace $\tau_{\mathrm{A}} : D(\mathrm{A}) \to D(\mathrm{A}^\top)'$ with $\langle\!\langle x, y\rangle\!\rangle = \tau_{\mathrm{A}} x(y) = \langle y, \mathrm{A}\, x\rangle_{\mathrm{H}_1} - \langle \mathrm{A}^\top y, x\rangle_{\mathrm{H}_0} = -\tau_{\mathrm{A}^\top} y(x)$

cl ibp $\langle\!\langle u, v\rangle\!\rangle = \tau_\Delta u(v) = \langle \Delta u, v\rangle_{\mathrm{L}^2(\Omega)} - \langle u, \Delta v\rangle_{\mathrm{L}^2(\Omega)} = \langle \partial_\nu\, u|_\Gamma, v|_\Gamma\rangle_{\mathrm{L}^2(\Gamma)} - \langle u|_\Gamma, \partial_\nu\, v|_\Gamma\rangle_{\mathrm{L}^2(\Gamma)} = -\tau_\Delta v(u)$

$$\tau_{\color{red}\Delta} : \mathrm{H}(\Delta, \Omega) \to \mathrm{H}(\Delta, \Omega)', \qquad \tau_{\color{red}\Delta} u(v) \overset{\text{class}}{=} \langle \partial_\nu\, u|_\Gamma, v|_\Gamma\rangle_{\mathrm{L}^2(\Gamma)} - \langle u|_\Gamma, \partial_\nu\, v|_\Gamma\rangle_{\mathrm{L}^2(\Gamma)}$$

$$\tau_{\color{orange}\Delta} : \mathrm{H}(\Delta, \Omega) \to D(\mathring{\Delta}_{\mathrm{D}})', \qquad \tau_{\color{orange}\Delta} u(v) \overset{\text{class}}{=} -\langle u|_\Gamma, \partial_\nu\, v|_\Gamma\rangle_{\mathrm{L}^2(\Gamma)}, \qquad v \in \mathring{\mathrm{H}}^1(\Omega)$$

$$\tau_{\color{blue}\Delta} : \mathrm{H}(\Delta, \Omega) \to D(\mathring{\Delta}_{\mathrm{N}})', \qquad \tau_{\color{blue}\Delta} u(v) \overset{\text{class}}{=} \langle \partial_\nu\, u|_\Gamma, v|_\Gamma\rangle_{\mathrm{L}^2(\Gamma)}, \qquad \nabla\, v \in \mathring{\mathrm{H}}(\operatorname{div}, \Omega)$$

$\mathrm{H}(\Delta, \Omega)' \ni \tau_{\color{red}\Delta} u(\,\cdot\,) = \langle\!\langle u, \cdot\rangle\!\rangle \overset{\text{class}}{=} \langle \partial_\nu\, u|_\Gamma, \cdot|_\Gamma\rangle_{\mathrm{L}^2(\Gamma)} - \langle u|_\Gamma, (\partial_\nu\, \cdot)|_\Gamma\rangle_{\mathrm{L}^2(\Gamma)}$ Dir and Neu trace as sum

$D(\mathring{\Delta}_{\mathrm{D}})' \ni \tau_{\color{orange}\Delta} u(\,\cdot\,) = \langle\!\langle u, \cdot\rangle\!\rangle \overset{\text{class}}{=} -\langle u|_\Gamma, (\partial_\nu\, \cdot)|_\Gamma\rangle_{\mathrm{L}^2(\Gamma)}$ non-standard Dirichlet trace

$D(\mathring{\Delta}_{\mathrm{N}})' \ni \tau_{\color{blue}\Delta} u(\,\cdot\,) = \langle\!\langle u, \cdot\rangle\!\rangle \overset{\text{class}}{=} \langle \partial_\nu\, u|_\Gamma, \cdot|_\Gamma\rangle_{\mathrm{L}^2(\Gamma)}$ non-standard Neumann trace

$$N(\Delta\Delta + 1)' = \mathring{\mathrm{H}}^2(\Omega)^\circ = R(\tau_{\color{red}\Delta}) \ni \tau_{\color{red}\Delta} u \overset{\text{class}}{\in} {\color{red}?}$$

$$N(\Delta\Delta + 1)' = \mathring{\mathrm{H}}^2(\Omega)^\circ = R(\tau_{\color{orange}\Delta}) \ni \tau_{\color{orange}\Delta} u \overset{\text{class}}{\in} \mathrm{H}^{-1/2}(\Gamma)' \overset{\iota_d}{=} \mathrm{H}^{1/2}(\Gamma)$$

$$N(\Delta\Delta + 1)' = \mathring{\mathrm{H}}^2(\Omega)^\circ = R(\tau_{\color{blue}\Delta}) \ni \tau_{\color{blue}\Delta} u \overset{\text{class}}{\in} \mathrm{H}^{1/2}(\Gamma)' = \mathrm{H}^{-1/2}(\Gamma)$$

Grisvard trace (Lions/Magenes)

. . . to be continued . . .

Appendix

Regular Subspaces and Duals

"Regular Subspaces" and Duals

Regular Subspaces and Duals

$$\boxed{A_0' : D(A_1)' \to D(A_0)'}$$

- $H_1^+ \subset D(A_1) \subset H_1$ (bd dense embs of reg subsps)
- $D(A_1) = H_1^+ + A_0\, H_0^+$ (bd reg deco ops)
- $H_0^+(A_0) = \{x \in H_0^+ : A_0\, x \in H_1^+\} \subset H_0^+ \subset D(A_0) \subset H_0$ (bd dense embs)
- $\mathring{H}_0^- = {H_0^+}'$ (duals)
- $H_1^+ \subset D(A_1) \cap D(A_0^\top) \subset H_1$ (bd dense embs of reg subsps)

note: $H_0^+(A_0) \subset H_0^+ \subset D(A_0) \subset H_0$ and $H_0' \subset D(A_0)' \subset \mathring{H}_0^- \subset H_0^+(A_0)'$

$\Rightarrow$ extend A_0' to $\boxed{A_0' : \mathring{H}_1^- \to H_0^+(A_0)'}$ by

$\forall\, x \in H_0^+(A_0) \quad A_0'\,\psi(x) := \psi(A_0\, x)$

- $H_1' \subset D(A_1)' \overset{!}{=} \boxed{\mathring{H}_1^-(A_0') := \{\psi \in \mathring{H}_1^- : A_0'\,\psi \in \mathring{H}_0^-\}} \subset \mathring{H}_1^- = {H_1^+}' \subset H_1^+(A_1)'$

$\subset$: $H_1^+ \subset D(A_1) \quad \Rightarrow \quad \psi \in D(A_1)' \subset \mathring{H}_1^-$ and $A_0'\,\psi \in D(A_0)' \subset \mathring{H}_0^-$

$\supset$: $\psi \in \mathring{H}_1^-(A_0')$ and $D(A_1) \ni y = y_1 + A_0\, y_0 \in H_1^+ + A_0\, H_0^+$ (reg deco)

$\Rightarrow \quad \psi y := \psi y_1 + (A_0'\,\psi) y_0$ and $|\psi y| \le c|\psi|_{\mathring{H}_1^-(A_0')}|y|_{D(A_1)} \quad \Rightarrow \quad \psi \in D(A_1)'$

Characterisation of Dual Spaces by Regular Subspaces

Characterisation of Dual Spaces by “Regular Subspaces”

Characterisation of Dual Spaces by Regular Subspaces

Theorem (Characterisation of Dual Spaces by Regular Subspaces)

$$D(\mathrm{A}_1)' = \mathring{\mathrm{H}}_1^{-}(\mathrm{A}_0') = \{\psi \in \mathring{\mathrm{H}}_1^{-} : \mathrm{A}_0'\,\psi \in \mathring{\mathrm{H}}_0^{-}\}$$
$$D(\mathrm{A}_0^{\top})' = \mathring{\mathrm{H}}_1^{-}(\mathrm{A}_1^{\top}{}') = \{\psi \in \mathring{\mathrm{H}}_1^{-} : \mathrm{A}_1^{\top}{}'\psi \in \mathring{\mathrm{H}}_2^{-}\}$$

with equivalent norms.

Theorem (Characterisation of Dual Spaces by Regular Subspaces)

$$D(\mathring{\mathrm{A}}_1)' = \mathrm{H}_1^{-}(\mathring{\mathrm{A}}_0') = \{\psi \in \mathrm{H}_1^{-} : \mathring{\mathrm{A}}_0'\,\psi \in \mathrm{H}_0^{-}\}$$
$$D(\mathrm{A}_0^{*})' = \mathrm{H}_1^{-}(\mathrm{A}_1^{*}{}') = \{\psi \in \mathrm{H}_1^{-} : \mathrm{A}_1^{*}{}'\psi \in \mathrm{H}_2^{-}\}$$

with equivalent norms.

Characterisation of Trace Ranges by Regular Subspaces

Characterisation of Trace Ranges by “Regular Subspaces”

Characterisation of Trace Ranges by Regular Subspaces

recall traces: $\tau_{\mathrm{A}_0} : D(\mathrm{A}_0) \to D(\mathrm{A}_0^\top)'$, $\qquad \tau_{\mathrm{A}_1^\top} : D(\mathrm{A}_1^\top) \to D(\mathrm{A}_1)'$

- $N(\tau_{\mathrm{A}_1^\top}) = D(\mathrm{A}_1^*)$
- $R(\tau_{\mathrm{A}_1^\top}) = D(\mathring{\mathrm{A}}_1)^\circ = \big\{\psi \in D(\mathrm{A}_1)' : \psi|_{D(\mathring{\mathrm{A}}_1)} = 0\big\}$
- $N(\tau_{\mathrm{A}_0}) = D(\mathring{\mathrm{A}}_0)$
- $R(\tau_{\mathrm{A}_0}) = D(\mathrm{A}_0^*)^\circ = \big\{\psi \in D(\mathrm{A}_0^\top)' : \psi|_{D(\mathrm{A}_0^*)} = 0\big\}$

density of $\quad \mathring{\mathrm{H}}_1^+ \subset D(\mathring{\mathrm{A}}_1)$ and $\overset{*}{\mathrm{H}}{}_1^+ \subset D(\mathrm{A}_0^*) \quad \Rightarrow$

- $R(\tau_{\mathrm{A}_1^\top}) = \mathring{\mathrm{H}}_1^{+\circ}$ as closed subspace of $D(\mathrm{A}_1)'$
- $R(\tau_{\mathrm{A}_0}) = \overset{*}{\mathrm{H}}{}_1^{+\circ}$ as closed subspace of $D(\mathrm{A}_0^\top)'$

$\Rightarrow$ more detailed

Theorem (Characterisation of Trace Ranges by Regular Subspaces)

$$R(\tau_{\mathrm{A}_1^\top}) = D(\mathrm{A}_1)' \cap D(\mathring{\mathrm{A}}_1)^\circ = \mathring{\mathrm{H}}_1^-(\mathrm{A}_0') \cap \mathring{\mathrm{H}}_1^{+\circ} = \{\psi \in \mathring{\mathrm{H}}_1^- : \mathrm{A}_0'\,\psi \in \mathring{\mathrm{H}}_0^- \ \wedge \ \psi|_{\mathring{\mathrm{H}}_1^+} = 0\}$$

$$R(\tau_{\mathrm{A}_0}) = D(\mathrm{A}_0^\top)' \cap D(\mathrm{A}_0^*)^\circ = \mathring{\mathrm{H}}_1^-(\mathrm{A}_1^{\top}{}') \cap \overset{*}{\mathrm{H}}{}_1^{+\circ} = \{\psi \in \mathring{\mathrm{H}}_1^- : \mathrm{A}_1^{\top}{}'\psi \in \mathring{\mathrm{H}}_2^- \ \wedge \ \psi|_{\overset{*}{\mathrm{H}}{}_1^+} = 0\}$$

with equivalent norms.

Trace Hilbert Complexes

Hilbert Complexes of Traces and Trace Spaces

Trace Hilbert Complexes

Hilbert Complexes of Traces and Trace Spaces

. . . to be continued . . .

- different unbounded versions of "surface differential operators"

$$\begin{tikzcd}
\cdots \arrow[r, "\cdots"] & D(\mathrm{A}_0^\top)' \arrow[r, "\mathrm{A}_1^\top{}'"] & D(\mathrm{A}_1^\top)' \arrow[r, "\mathrm{A}_2^\top{}'"] & D(\mathrm{A}_2^\top)' \arrow[r, "\cdots"] & \cdots \\
\cdots & D(\mathrm{A}_1)' \arrow[l, "\cdots"'] & D(\mathrm{A}_2)' \arrow[l, "\mathrm{A}_1'"'] & D(\mathrm{A}_3)' \arrow[l, "\mathrm{A}_2'"'] & \cdots \arrow[l, "\cdots"']
\end{tikzcd}$$

$$\begin{tikzcd}
\cdots \arrow[r, "\cdots"] & \overset{*}{\mathrm{H}}{}_1^{+\circ} \arrow[r, "\mathrm{A}_1^\top{}'"] & \overset{*}{\mathrm{H}}{}_2^{+\circ} \arrow[r, "\mathrm{A}_2^\top{}'"] & \overset{*}{\mathrm{H}}{}_3^{+\circ} \arrow[r, "\cdots"] & \cdots \\
\cdots & \mathring{\mathrm{H}}{}_1^{+\circ} \arrow[l, "\cdots"'] & \mathring{\mathrm{H}}{}_2^{+\circ} \arrow[l, "\mathrm{A}_1'"'] & \mathring{\mathrm{H}}{}_3^{+\circ} \arrow[l, "\mathrm{A}_2'"'] & \cdots \arrow[l, "\cdots"']
\end{tikzcd}$$

$$\begin{tikzcd}
\cdots \arrow[r, shift left, "\cdots"] & \mathring{\mathrm{H}}{}_1^{-} \arrow[l, shift left, "\cdots"] \arrow[r, shift left, "\mathrm{A}_1^\top{}'"] & \mathring{\mathrm{H}}{}_2^{-} \arrow[l, shift left, "\mathrm{A}_1'"] \arrow[r, shift left, "\mathrm{A}_2^\top{}'"] & \mathring{\mathrm{H}}{}_3^{-} \arrow[l, shift left, "\mathrm{A}_2'"] \arrow[r, shift left, "\cdots"] & \cdots \arrow[l, shift left, "\cdots"]
\end{tikzcd}$$

three interpretations

- $D(\mathrm{A}_{n+1}^\top{}') = R(\tau_{\mathrm{A}_n}) \subset D(\mathrm{A}_n^\top)'$
- $D(\mathrm{A}_n') = R(\tau_{\mathrm{A}_{n+1}^\top}) \subset D(\mathrm{A}_{n+1})'$
- $R(\tau_{\mathrm{A}_0}) = \overset{*}{\mathrm{H}}{}_1^{+\circ}$ cl sbsp of both $D(\mathrm{A}_0^\top)' \subset \mathring{\mathrm{H}}{}_1^{-}$
- $\mathrm{A}_1^\top{}' : R(\tau_{\mathrm{A}_0}) \to R(\tau_{\mathrm{A}_1})$
- $R(\tau_{\mathrm{A}_n}) \subset \overset{*}{\mathrm{H}}{}_{n+1}^{+\circ} \subset \mathring{\mathrm{H}}{}_{n+1}^{-}$
- $R(\tau_{\mathrm{A}_0}) = D(\mathrm{A}_0^\top)' \cap D(\overset{*}{\mathrm{A}}_0)^\circ$

$$= \mathring{\mathrm{H}}{}_1^{-}(\mathrm{A}_1^\top{}') \cap \overset{*}{\mathrm{H}}{}_1^{+\circ}$$

$$= \big\{\psi \in \mathring{\mathrm{H}}{}_1^{-} : \mathrm{A}_1^\top{}'\psi \in \mathring{\mathrm{H}}{}_2^{-} \;\wedge\; \psi|_{\overset{*}{\mathrm{H}}{}_1^{+}} = 0\big\}$$

$$= \underbrace{\big\{\psi \in \overset{*}{\mathrm{H}}{}_1^{+\circ} : \mathrm{A}_1^\top{}'\psi \in \overset{*}{\mathrm{H}}{}_2^{+\circ}\big\}}_{=:\overset{*}{\mathrm{H}}{}_1^{+\circ}(\mathrm{A}_1^\top{}')}$$

- compact embeddings for trace Hilbert complexes
- boundary value problems on trace Hilbert complexes

. . . to be continued . . .

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