The Closed Range Property of the De Rham Complex in Unbounded Domains

(joint work with **Marcus Waurick**[@TUBAFG])

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TECHNISCHE UNIVERSITÄT DRESDEN

ESI Thematic Programme (April 20 – June 5, 2026)

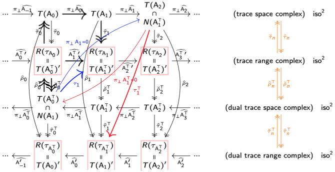
Differential Complexes: Theory, Discretization, and Applications

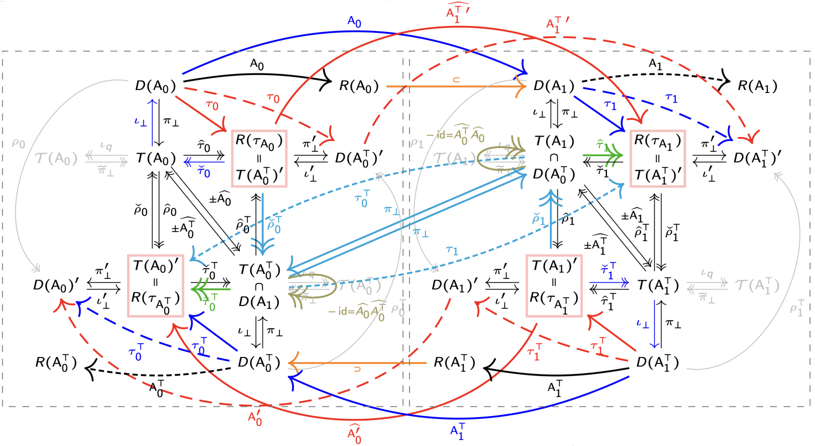
hosts:	Andreas Cap	(U Wien)
	Ralf Hiptmair	(ETH Zürich)
	Kaibo Hu	(U Oxford)
	Joachim Schöberl	(TU Wien)

May 5 @ Erwin Schrödinger Institut

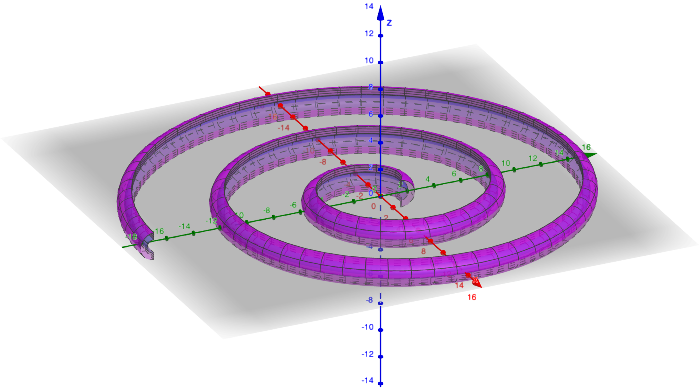
I will not talk about traces...

...but I love this pic!





Does curl have closed range in the infinitely growing half snail shell?



my educated guess one year ago: probably NOT!

now: YES! – paper in the birthday book **Rainer Picard 80**
DP and M. Waurick: *Gaffney's inequality and the Closed Range Property of the de Rham Complex in Unbounded Domains*

ESI Wien

May 5, 2026

The Closed Range Property of the De Rham Complex in Unbounded Domains

Hilbert complex / De Rham complex

$$H_0 = L^2 \;\underset{A_0^* = -\operatorname{div}}{\overset{A_0 = \mathring{\nabla}}{\rightleftarrows}}\; H_1 = L^2 \;\underset{A_1^* = \operatorname{rot}}{\overset{A_1 = \mathring{\operatorname{rot}}}{\rightleftarrows}}\; H_2 = L^2 \;\underset{A_2^* = -\nabla}{\overset{A_2 = \mathring{\operatorname{div}}}{\rightleftarrows}}\; H_3 = L^2$$

H_n Hilbert spaces, A_n, A_n^* ldde

fundamental property: $R(A_n)$ closed

closed range thm $\Rightarrow$ $R(A_n)$ cl $\iff$ $R(A_n^*)$ cl

here: $R(\mathring{\nabla})$ cl $\iff$ $R(\operatorname{div})$ cl

$R(\mathring{\operatorname{rot}})$ cl $\iff$ $R(\operatorname{rot})$ cl

$R(\mathring{\operatorname{div}})$ cl $\iff$ $R(\nabla)$ cl

applications

- solution theory / bd courses
- spectral gap at 0
- low frequency asymptotics (Neumann series)
- exponential stability of the Cauchy problem

reduction step

$A : D(A) \subset H_0 \to H_1$ $\qquad$ $A^* : D(A^*) \subset H_1 \to H_0$

$\mathcal{A} := A|_{N(A)^\perp}$ $\qquad$ $\mathcal{A}^* := A^*|_{N(A^*)^\perp}$ (indeed!)

$D(\mathcal{A}) = D(A) \cap N(A)^\perp$

$N(A)^\perp = \overline{R(A^*)}$ $\qquad$ (projection theo/ Helmholtz) $\qquad$ $N(A^*)^\perp = \overline{R(A)}$

$\Rightarrow$ reduced ops

$\mathcal{A} : D(\mathcal{A}) \subset N(A)^\perp = \overline{R(A^*)} \longrightarrow \overline{R(A)} = \tilde{H}_1$ $\qquad$ 1e1d1c

$\mathcal{A}^* : D(\mathcal{A}^*) \subset N(A^*)^\perp = \overline{R(A)} \longrightarrow \overline{R(A^*)} = \tilde{H}_0$

$H_0 = N(A) \oplus \overline{R(A^*)}$ $\qquad$ $H_1 = N(A^*) \oplus \overline{R(A)}$

$\Rightarrow D(A) = N(A) \oplus D(\mathcal{A})$ $\qquad$ $D(A^*) = N(A^*) \oplus D(\mathcal{A}^*)$

$\Rightarrow R(A) = R(\mathcal{A})$ (reg. for free) $\qquad$ $R(A^*) = R(\mathcal{A}^*)$

$\Rightarrow$ $\mathcal{A}^{-1} : R(A) \to D(\mathcal{A})$, $\qquad$ $(\mathcal{A}^*)^{-1} : R(A^*) \to D(\mathcal{A}^*)$

exist, might be unbd.

fundamental lemma 1 $\qquad$ (*)

(i) $\mathcal{A}^{-1} : R(A) \to D(\mathcal{A})$ bd $\qquad$ $|\mathcal{A}^{-1}| = |(\mathcal{A}^*)^{-1}|$

(i') $(\mathcal{A}^*)^{-1} : R(A^*) \to D(\mathcal{A}^*)$ bd

(ii) $\exists\, c_A > 0 \;\forall x \in D(\mathcal{A})$ $\quad |x| \le c_A |Ax|$ $\qquad$ (Friedrichs)

(ii') $\exists\, c_{A^*} > 0 \;\forall y \in D(\mathcal{A}^*)$ $\quad |y| \le c_{A^*} |A^* y|$

(iii) $R(A) = R(\mathcal{A})$ closed

(iii') $R(A^*) = R(\mathcal{A}^*)$ closed

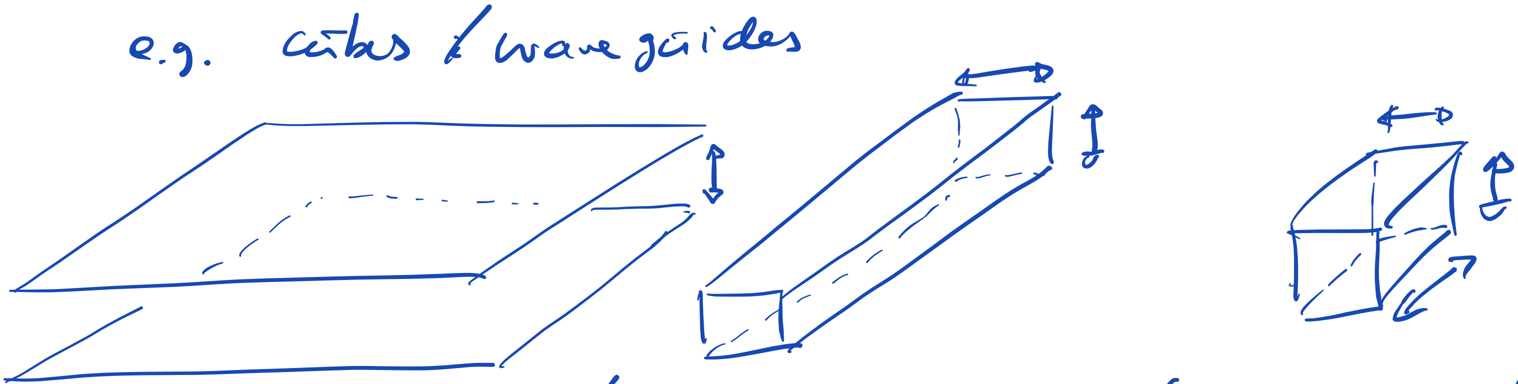
lemma 2 $\quad D(\mathcal{A}) \hookrightarrow\hookrightarrow H_0 \iff D(\mathcal{A}^*) \hookrightarrow\hookrightarrow H_1$

$\Rightarrow$ assertions of lemma 1 hold.

typical situations

- $\Omega \subset \mathbb{R}^N$ bd Lip dom / Riemannian manifold ⇒ cpt emb realistic ⇒ closed ranges
- $\mathbb{R}^N \setminus \overline{\Omega}$ bd ⇒ cpt emb. fails ⇒ maybe no closed ranges
- our solution: use polynomially weighted Sobolev spaces
- another question: partially bd domains

  e.g. cubes / waveguides



bd in 1 direction / bd in 2 directions / bd in 3 directions

well known

- $\mathring{\nabla} : \mathring{H}^1 \subset L^2 \to L^2$, $\Omega \subset \mathbb{R}^3$ bd in 1 dir

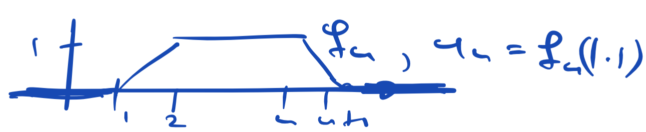
  ⇒ $R(\mathring{\nabla})$ closed (fund. theo $u(x_3) = \underbrace{u(0)}_{=0} + \int_0^{x_3} \partial_3 u$)

  ⇔ $R(\mathrm{div})$ closed
  cl. ran

- $\nabla : H^1 \subset L^2 \to L^2$, $\Omega \subset \mathbb{R}^3$

$R(\nabla)$ closed ⇔ $\Omega \subset \mathbb{R}^3$ bd in 3 dir ⇔ Ω bd

⇔ $R(\mathring{\mathrm{div}})$ closed
cl. ran



⇒ $|u_n|^2_{L^2} \sim n^3$, $|\nabla u_n|^2_{L^2} \sim n^2$

main result Ω (cube) $\subset \mathbb{R}^3$

- $R(\mathring{\nabla})$ cl $\Leftrightarrow$ $R(\operatorname{div})$ cl $\Leftrightarrow$ Ω bd in 1 dir
- $R(\mathring{\operatorname{rot}})$ cl $\Leftrightarrow$ $R(\operatorname{rot})$ cl $\Leftrightarrow$ Ω bd in 2 dir
- $R(\nabla)$ cl $\Leftrightarrow$ $R(\mathring{\operatorname{div}})$ cl $\Leftrightarrow$ Ω bd in 3 dir

moreover $\phi : \Theta \to \Omega$ bi-Lipschitz trafo
($\det \phi' \geq c > 0$)

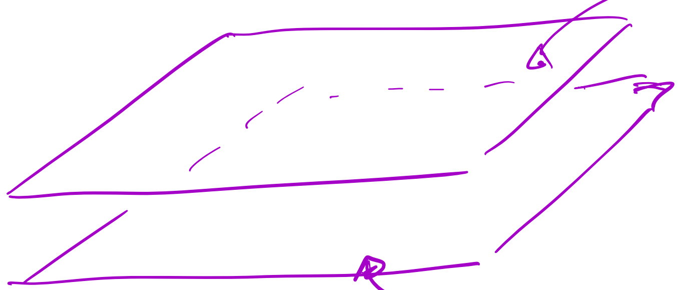
$\Rightarrow$ $R(\mathring{\nabla}_\Omega)$ cl $\Leftrightarrow$ $R(\mathring{\nabla}_\Theta)$ cl

$R(\mathring{\operatorname{rot}}_\Omega)$ cl $\Leftrightarrow$ $R(\mathring{\operatorname{rot}}_\Theta)$ cl

$R(\mathring{\operatorname{div}}_\Omega)$ cl $\Leftrightarrow$ $R(\mathring{\operatorname{div}}_\Theta)$ cl

n.b. $R(\operatorname{rot}_{\Gamma_t})$ cl $\not\Leftarrow$ Ω bd in 1 dir

mixed bc



$n \cdot E|_{\Gamma_n} = 0$ (normal)

$n \times E \times n|_{\Gamma_t} = 0$ (tangential)

sketch of proof (Ω unbd cube )

① show Friedrichs/Poincaré type estimate

$$\forall E \in D(\mathring{\mathrm{rot}}) \cap N(\mathring{\mathrm{rot}})^{\perp} \qquad |E|_{L^2} \leq c\, |\mathrm{rot}\, E|_{L^2}$$

② use complex property

$$N(\mathring{\mathrm{rot}})^{\perp} = \overline{R(\mathrm{rot})} \subset N(\mathrm{div}) \subset D(\mathrm{div})$$

$$\Rightarrow D(\mathring{\mathrm{rot}}) \cap N(\mathring{\mathrm{rot}})^{\perp} \subset D(\mathring{\mathrm{rot}}) \cap N(\mathrm{div})$$

$$\subset D(\mathring{\mathrm{rot}}) \cap D(\mathrm{div}) = \mathring{H}(\mathrm{rot}) \cap H(\mathrm{div})$$

③ show/use 1-Gaffney on (unbounded cubes)

$$E \in D(\mathring{\mathrm{rot}}) \cap D(\mathrm{div}) \Rightarrow E \in H^1 \cap D(\mathring{\mathrm{rot}})$$

$$\wedge \; |\nabla E|^2_{L^2} \leq |\mathrm{rot}\, E|^2_{L^2} + |\mathrm{div}\, E|^2_{L^2}$$

(even =)

④ $u \in H^1$ with suitable homogeneous Dirichlet bc on each component E_n (zero on one face of the cube)

$\Rightarrow$ Friedrichs estimate

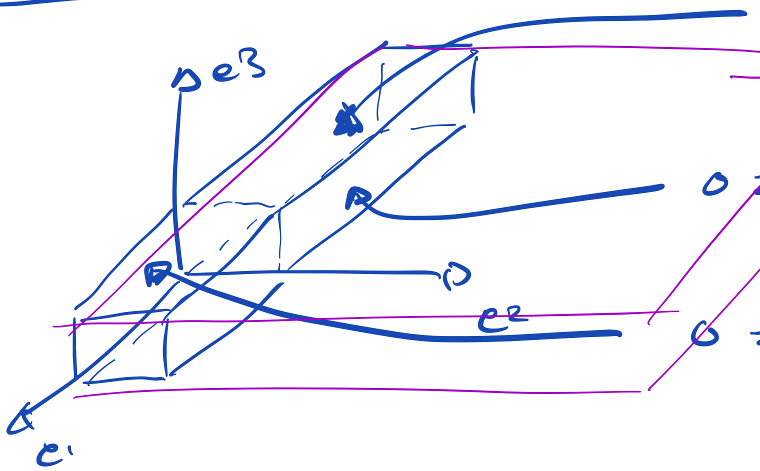
$$|u|_{L^2} \leq c_F\, |\nabla u|_{L^2}$$

⑤ ③ ∧ ④ $\Rightarrow$

$$|E|^2_{L^2} = \sum_n |E_n|^2_{L^2} \leq c_F^2 \sum_n |\nabla E_n|^2_{L^2} = c_F^2 |\nabla E|^2_{L^2}$$

$$\leq c_F^2\, |\mathrm{rot}\, E|^2_{L^2} \quad \text{as } \mathrm{div}\, E = 0$$

observation $E \in H^1 \cap \mathring{H}(\mathrm{rot})$



$0 = u \cdot E\big|_{x_3=1} = E_3$

$0 = u \times E \times u\big|_{x_3=0} = \begin{bmatrix} E_1 \\ E_2 \\ 0 \end{bmatrix}$

$0 = u \times E \times u\big|_{x_2=0} \quad \begin{bmatrix} E_1 \\ 0 \\ E_3 \end{bmatrix}$

$\Rightarrow$ $E_1, E_2 = 0$ on $x_3 = 0$; $E_1, E_3 = 0$ on $x_2 = 0$ $\Rightarrow$ ✓

$E_1, E_2 = 0$ on $x_3 = 0$; $E_3 = 0$ on $x_3 = 1$ $\Rightarrow$ ✓

additional candies (nice and simple proofs)

- Gaffney estimate / integration by parts
- Lipschitz trafo theo
- applications for waveguides