Chapter 8
On an Extension of the First Korn Inequality to Incompatible Tensor Fields on Domains of Arbitrary Dimensions

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Abstract For a bounded domain Ω in $\mathbb{R}^N$ with Lipschitz boundary $\Gamma = \partial\Omega$ and a relatively open and non-empty subset Γ_t of Γ, we prove the existence of a positive constant c such that inequality $c\|T\|_{\mathsf{L}^2(\Omega,\mathbb{R}^{N\times N})} \leq \|\operatorname{sym} T\|_{\mathsf{L}^2(\Omega,\mathbb{R}^{N\times N})} + \|\operatorname{Curl} T\|_{\mathsf{L}^2(\Omega,\mathbb{R}^{N\times N(N-1)/2})}$ holds for all tensor fields $T \in \overset{\circ}{\mathsf{H}}(\operatorname{Curl}; \Gamma_t, \Omega, \mathbb{R}^{N\times N})$, this is, for all $T : \Omega \to \mathbb{R}^{N\times N}$ which are square-integrable and possess a row-wise square-integrable rotation tensor field $\operatorname{Curl} T : \Omega \to \mathbb{R}^{N\times N(N-1)/2}$ and vanishing row-wise tangential trace on Γ_t. For compatible tensor fields $T = \nabla v$ with $v \in \mathsf{H}^1(\Omega, \mathbb{R}^N)$ having vanishing tangential Neumann trace on Γ_t the inequality reduces to a non-standard variant of the first Korn inequality since $\operatorname{Curl} T = 0$, while for skew-symmetric tensor fields T the Poincaré inequality is recovered. If $\Gamma_t = \emptyset$, our estimate still holds at least for simply connected Ω and for all tensor fields $T \in \mathsf{H}(\operatorname{Curl}; \Omega, \mathbb{R}^{N\times N})$ which are $\mathsf{L}^2(\Omega, \mathbb{R}^{N\times N})$-perpendicular to $\mathfrak{so}(N)$, i.e., to all skew-symmetric constant tensors.

8.1 Introduction and Main Results

We extend the Korn-type inequalities from [15] presented earlier in less general settings in [11–14] to the N-dimensional case. For this, let $N \in \mathbb{N}$ and Ω be a bounded domain in $\mathbb{R}^N$ as well as Γ_t be an open and non-empty subset of its boundary $\Gamma := \partial\Omega$. Our main result reads:

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Theorem 8.1 (Main theorem) *Let the pair* (Ω, Γ_t) *be admissible.*[1] *There exists a constant* $\hat{c} > 0$ *such that the inequality*

$$\|T\|_{\mathsf{L}^2(\Omega)} \leq \hat{c}(\|\operatorname{sym} T\|_{\mathsf{L}^2(\Omega)} + \|\operatorname{Curl} T\|_{\mathsf{L}^2(\Omega)})$$

holds for all tensor fields $T \in \overset{\circ}{\mathsf{H}}(\mathrm{Curl}; \Gamma_t, \Omega)$. *In other words, on* $\overset{\circ}{\mathsf{H}}(\mathrm{Curl}; \Gamma_t, \Omega)$ *the right-hand side defines a norm equivalent to the standard norm in* $\mathsf{H}(\mathrm{Curl}; \Omega)$.

Remark 8.1 Here, the differential operator Curl denotes the row-wise application of the standard curl in $\mathbb{R}^N$ and a tensor field T belongs to the Hilbert Sobolev-type space $\overset{\circ}{\mathsf{H}}(\mathrm{Curl}; \Gamma_t, \Omega)$ if T and its distributional Curl T belong both to the standard Lebesgue spaces $\mathsf{L}^2(\Omega)$ and the row-wise weak tangential trace of T vanishes at the boundary part Γ_t. Exact definitions of all spaces and operators used will be given in Sect. 8.2. The constant $\hat{c}$ is given by (8.8) and depends in a simply algebraic way only on the constants $c_{\mathtt{k}}$, $c_{\mathtt{m}}$ in the first Korn and Maxwell inequality.

For the proof of Theorem 8.1 we follow in close lines the proofs from [15]. Therefore, again we need to combine three crucial tools, namely

- a Maxwell estimate, Corollary 8.6;
- a Helmholtz decomposition, Corollary 8.7;
- a generalized version of the first Korn inequality, Lemma 8.5.

Our assumptions on the domain Ω and the part of the boundary Γ_t, i.e., on the pair (Ω, Γ_t), are precisely made for this three major tools to hold. We will present these assumptions in Sect. 8.2 and a pair (Ω, Γ_t) satisfying those will be called admissible.

Theorem 8.1 can be looked at as a common generalization and formulation of two well-known and very important classical inequalities, namely the first Korn and Poincaré inequality. This is, taking irrotational tensor fields T, i.e., Curl $T = 0$, then a non-standard version of the first Korn inequality

$$\|T\|_{\mathsf{L}^2(\Omega)} \leq \hat{c}\|\operatorname{sym} T\|_{\mathsf{L}^2(\Omega)}$$

holds for all $T \in \overset{\circ}{\mathsf{H}}(\mathrm{Curl}_0; \Gamma_t, \Omega)$. Another, less general choice, is $T = \nabla v$ yielding

$$\|\nabla v\|_{\mathsf{L}^2(\Omega)} \leq \hat{c}\|\operatorname{sym} \nabla v\|_{\mathsf{L}^2(\Omega)}$$

with some vector field v belonging to $\overset{\circ}{\mathsf{H}}{}^1(\Gamma_t; \Omega)$ or just to $\mathsf{H}^1(\Omega)$ with ∇v_n, $n = 1, \ldots, N$, normal at Γ_t. Note that

$$\nabla\overset{\circ}{\mathsf{H}}{}^1(\Gamma_t; \Omega), \quad \nabla\{v \in \mathsf{H}^1(\Omega) \mid \nabla v_n \text{ normal at } \Gamma_t\ \forall n = 1, \ldots, N\} \subset \overset{\circ}{\mathsf{H}}(\mathrm{Curl}_0; \Gamma_t, \Omega).$$

[1] The precise meaning of 'admissible' will be given in Definition 8.5.

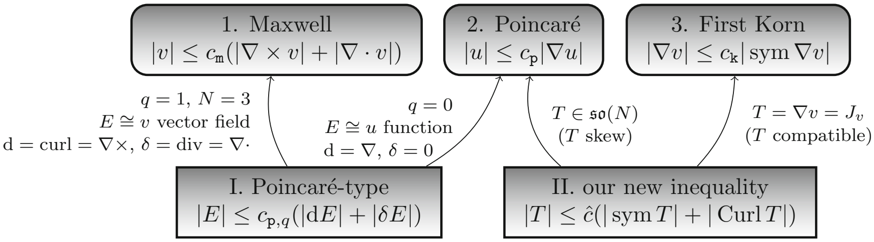

Fig. 8.1 The three fundamental inequalities are implied by two. For the constants we have $c_{\mathtt{p}} = c_{\mathtt{p},0}$, $c_{\mathtt{m}} = c_{\mathtt{p},1}$ and $c_{\mathtt{k}}, c_{\mathtt{p}} \le \hat{c}$

On the other hand, taking a skew-symmetric tensor field T, i.e., $\operatorname{sym} T = 0$, then the Poincaré inequality in disguise

$$\|T\|_{\mathsf{L}^2(\Omega)} \le \hat{c}\|\operatorname{Curl} T\|_{\mathsf{L}^2(\Omega)}$$

appears. We note that since T can be identified with a vector field v and the Curl is as good as the gradient ∇ on v we have

$$\|v\|_{\mathsf{L}^2(\Omega)} \le c\|\nabla v\|_{\mathsf{L}^2(\Omega)}.$$

These connections between the first Korn and Poincaré inequalities and also to the Maxwell inequalities and the more general Poincaré-type inequalities are illustrated in Fig. 8.1.

8.2 Definitions and Preliminaries

As mentioned before, let generally $N \in \mathbb{N}$ and Ω be a bounded domain in $\mathbb{R}^N$ as well as $\emptyset \neq \Gamma_t$ be an open subset of the boundary $\Gamma = \partial\Omega$. We will use the notations from our earlier papers [11–15].

8.2.1 Differential Forms

In particular, we denote the Lebesgue spaces of square-integrable q-forms[2] by $\mathsf{L}^{2,q}(\Omega)$. Moreover, we have the standard Sobolev-type spaces for the exterior derivative d and co-derivative $\delta := (-1)^{(q-1)N} * \mathrm{d}*$ (acting on q-forms)

$$\mathsf{D}^q(\Omega) := \{E \in \mathsf{L}^{2,q}(\Omega) \mid \mathrm{d}E \in \mathsf{L}^{2,q+1}(\Omega)\},$$
$$\Delta^q(\Omega) := \{H \in \mathsf{L}^{2,q}(\Omega) \mid \delta H \in \mathsf{L}^{2,q-1}(\Omega)\},$$

[2] alternating differential forms of rank $q \in \{0, \ldots, N\}$

where as usual $*$ denotes the Hodge star isomorphism. $\overset{\circ}{\mathsf{C}}{}^{\infty,q}(\Omega)$ is the space of smooth and compactly supported q-forms on Ω, often called test space. Due to the more complex geometry and topology of the domain Ω and its boundary parts Γ, Γ_t we need some more test spaces

$$\overset{\circ}{\mathsf{C}}{}^{\infty,q}(\Gamma_t,\Omega) := \big\{E \in \mathsf{C}^{\infty,q}(\overline{\Omega}) \mid \operatorname{dist}(\operatorname{supp} E, \Gamma_t) > 0\big\},$$
$$\mathsf{C}^{\infty,q}(\overline{\Omega}) := \big\{E|_\Omega \mid E \in \overset{\circ}{\mathsf{C}}{}^{\infty,q}(\mathbb{R}^N)\big\}.$$

Then, we define

$$\overset{\circ}{\mathsf{D}}{}^q(\Gamma_t,\Omega) := \overline{\overset{\circ}{\mathsf{C}}{}^{\infty,q}(\Gamma_t,\Omega)}$$

taking the closure in $\mathsf{D}^q(\Omega)$ and note that a q-form in $\overset{\circ}{\mathsf{D}}{}^q(\Gamma_t,\Omega)$ has generalized vanishing tangential trace on Γ_t, which can be seen easily by the Stokes theorem. If $\Gamma_t = \Gamma$ we can identify $\overset{\circ}{\mathsf{C}}{}^{\infty,q}(\Gamma_t,\Omega)$ with $\overset{\circ}{\mathsf{C}}{}^{\infty,q}(\Omega)$ and write

$$\overset{\circ}{\mathsf{D}}{}^q(\Gamma_t,\Omega) = \overline{\overset{\circ}{\mathsf{C}}{}^{\infty,q}(\Gamma_t,\Omega)} = \overline{\overset{\circ}{\mathsf{C}}{}^{\infty,q}(\Omega)} =: \overset{\circ}{\mathsf{D}}{}^q(\Omega).$$

An index 0 at the lower right corner indicates vanishing derivatives, e.g.,

$$\overset{\circ}{\mathsf{D}}{}^q_0(\Gamma_t,\Omega) := \big\{E \in \overset{\circ}{\mathsf{D}}{}^q(\Gamma_t,\Omega) \mid \mathrm{d}E = 0\big\}.$$

Analogously, we introduce the corresponding Sobolev-type spaces for the co-derivative δ which are usually assigned to the boundary complement $\Gamma_n := \Gamma \setminus \overline{\Gamma_t}$ of Γ_t. We have, e.g.,

$$\Delta^q_0(\Omega) = \{H \in \Delta^q(\Omega) \mid \delta H = 0\}, \quad \overset{\circ}{\Delta}{}^q(\Gamma_n,\Omega), \quad \overset{\circ}{\Delta}{}^q_0(\Gamma_n,\Omega),$$

where in the latter spaces a vanishing normal trace on Γ_n is generalized. Moreover, we define the spaces of so-called 'harmonic Dirichlet-Neumann forms'

$$\mathscr{H}^q(\Omega) := \overset{\circ}{\mathsf{D}}{}^q_0(\Gamma_t,\Omega) \cap \overset{\circ}{\Delta}{}^q_0(\Gamma_n,\Omega). \tag{8.1}$$

We note that in classical terms a harmonic Dirichlet-Neumann q-form E satisfies

$$\mathrm{d}E = 0, \quad \delta E = 0, \quad \iota^* E|_{\Gamma_t} = 0, \quad \iota^* * E|_{\Gamma_n} = 0,$$

where ι^* denotes the pullback of the canonical embedding $\iota : \Gamma \hookrightarrow \overline{\Omega}$ and the restrictions to Γ_t and Γ_n should be understood as pullbacks as well. Equipped with their natural graph norms all these spaces are Hilbert spaces.

Now, we can begin to introduce our regularity assumptions on the boundary Γ and the interface $\gamma := \overline{\Gamma_t} \cap \overline{\Gamma_n}$. We start with the following:

Definition 8.1 The pair (Ω, Γ_t) has the 'Maxwell compactness property' (MCP), if for all q the embeddings

$$\overset{\circ}{\mathsf{D}}^q(\Gamma_t, \Omega) \cap \overset{\circ}{\Delta}^q(\Gamma_n, \Omega) \hookrightarrow \mathsf{L}^2(\Omega)$$

are compact.

Remark 8.2 1. There exists a substantial amount of literature and different proofs for the MCP. See, for example, the papers and books of Costabel, Kuhn, Leis, Pauly, Picard, Saranen, Weber, Weck, Witsch [2, 7–10, 16–25, 27–31]. All these papers are concerned with the special cases $\Gamma_t = \Gamma$ resp. $\Gamma_t = \emptyset$. For the case $N = 3$, $q = 1$, i.e., $\Omega \subset \mathbb{R}^3$, we refer to [2, 8–10, 21, 23, 25, 27–29, 31], whereas for the general case, i.e., $\Omega \subset \mathbb{R}^N$ or even Ω a Riemannian manifold, we correspond to [7, 16–20, 22, 24, 30]. We note that even weaker regularity of Γ than Lipschitz is sufficient for the MCP to hold. The first proof of the MCP for non-smooth domains and even for smooth Riemannian manifolds with non-smooth boundaries (cone property) was given in 1974 by Weck in [30]. To the best of our knowledge, the strongest result so far can be found in [25]. See also our discussion in [15]. An interesting proof has been given by Costabel in [2]. He made the detour of showing more fractional Sobolev regularity for the vector fields. More precisely, he was able to prove that for Lipschitz domains $\Omega \subset \mathbb{R}^3$ and $q = 1$ the embedding

$$\overset{\circ}{\mathsf{D}}^q(\Omega) \cap \Delta^q(\Omega) \hookrightarrow \mathsf{H}^{1/2}(\Omega) \tag{8.2}$$

is continuous. Then, for all $0 \leq k < 1/2$ the embeddings

$$\overset{\circ}{\mathsf{D}}^q(\Omega) \cap \Delta^q(\Omega) \hookrightarrow \mathsf{H}^k(\Omega)$$

are compact, especially for $k = 0$, where $\mathsf{H}^k(\Omega) = \mathsf{L}^2(\Omega)$ holds.

2. For the general case $\emptyset \subset \Gamma_t \subset \Gamma$ with possibly $\emptyset \subsetneq \Gamma_t \subsetneq \Gamma$, Jochmann gave a proof for the MCP in [5], where he considered the special case of a bounded domain $\Omega \subset \mathbb{R}^3$. He can admit Ω to be Lipschitz and γ to be a Lipschitz interface. Generalizing the ideas of Weck in [30], Kuhn showed in his dissertation [6] that the MCP holds for smooth domains $\Omega \subset \mathbb{R}^N$ or even for smooth Riemannian manifolds Ω with smooth boundary and admissible interface γ. See also our discussion in [15].

A result, which meets our needs, has been proved quite recently by M. Mitrea and his collaborators. More precisely, we will use results by Gol'dshtein and Mitrea (I. & M.) from [3]. In the language of this paper we assume Ω to be a weakly Lipschitz domain, this is, Ω is a Lipschitz manifold with boundary, see [3, Definition 3.6], and $\Gamma_t \subset \Gamma$ to be an admissible patch (yielding γ to be an admissible path), i.e., Γ_t is a Lipschitz submanifold with boundary, see [3, Definition 3.7]. Roughly speaking, Ω and Γ_t are defined by Lipschitz functions. Here, the main point in proving the MCP, i.e., [3, Proposition 4.4, (4.21)], is that then Ω is locally Lipschitz diffeomorphic to a 'creased domain' in $\mathbb{R}^N$, first introduced by Brown in [1]. See [3, Sect. 3.6] for more details and to find the statement 'Informally speaking, the pieces in which the boundary is partitioned are admissible patches which meet at an angle $< \pi$. In particular, creased domains are inherently non-smooth'. Whereas in [3] everything is defined in the more general framework of manifolds, in [4] the MCP is proved by Jakab and Mitrea (I. & M.) for creased domains $\Omega \subset \mathbb{R}^N$. By the Lipschitz diffeomorphisms, the MCP holds then for general manifolds/domains Ω as well. In [4] the authors follow and generalize the idea (8.2) of Costabel from [2]. Particularly, in [4, (1.2), Theorem 1.1, (1.9)] the following regularity result has been proved: For all q the embeddings

$$\mathring{\mathsf{D}}^q(\Gamma_t, \Omega) \cap \mathring{\Delta}^q(\Gamma_n, \Omega) \hookrightarrow \mathsf{H}^{1/2}(\Omega)$$

are continuous. Therefore, as before, for all q and for all $0 \leq k < 1/2$ the embeddings

$$\mathring{\mathsf{D}}^q(\Gamma_t, \Omega) \cap \mathring{\Delta}^q(\Gamma_n, \Omega) \hookrightarrow \mathsf{H}^k(\Omega)$$

are compact, giving the MCP for $k = 0$.

By [3, Proposition 4.4, (4.21)] and Remark 8.2 we have:

Theorem 8.2 *Let Ω be a weakly Lipschitz domain and Γ_t be an admissible patch, i.e., let Ω be a (weakly) Lipschitz domain and Γ_t be an Lipschitz patch of Γ. Then the pair (Ω, Γ_t) has the MCP.*

Corollary 8.1 *Let the pair (Ω, Γ_t) have the MCP. Then, for all q the spaces of harmonic Dirichlet-Neumann forms $\mathscr{H}^q(\Omega)$ are finite dimensional.*

Proof The MCP implies immediately that the unit ball in $\mathscr{H}^q(\Omega)$ is compact.

For details about the particular dimensions see [21] or [3]. We note that the dimensions of $\mathscr{H}^q(\Omega)$ depend only on topological properties of the pair (Ω, Γ_t).

Lemma 8.1 (Poincaré-type estimate for differential forms) *Let the pair (Ω, Γ_t) have the MCP. Then, for all q there exist positive constants $c_{\mathrm{p},q}$, such that*

$$\|E\|_{\mathsf{L}^{2,q}(\Omega)} \leq c_{\mathrm{p},q} \left(\|\mathrm{d}E\|^2_{\mathsf{L}^{2,q+1}(\Omega)} + \|\delta E\|^2_{\mathsf{L}^{2,q-1}(\Omega)} \right)^{1/2}$$

holds for all $E \in \mathring{\mathsf{D}}^q(\Gamma_t, \Omega) \cap \mathring{\Delta}^q(\Gamma_n, \Omega) \cap \mathscr{H}^q(\Omega)^{\perp}$. Moreover,

$$\left\| \left(\mathrm{id} - \pi_q\right) E \right\|_{\mathsf{L}^{2,q}(\Omega)} \leq c_{\mathrm{p},q} \left(\|\mathrm{d}E\|^2_{\mathsf{L}^{2,q+1}(\Omega)} + \|\delta E\|^2_{\mathsf{L}^{2,q-1}(\Omega)} \right)^{1/2}$$

holds for all $E \in \mathring{\mathsf{D}}^q(\Gamma_t, \Omega) \cap \mathring{\Delta}^q(\Gamma_n, \Omega)$, *where* $\pi_q : \mathsf{L}^{2,q}(\Omega) \to \mathscr{H}^q(\Omega)$ *denotes the* $\mathsf{L}^{2,q}(\Omega)$*-orthogonal projection onto the Dirichlet-Neumann forms* $\mathscr{H}^q(\Omega)$.

Here and throughout the paper, $\perp$ denotes orthogonality in $\mathsf{L}^{2,q}(\Omega)$.

Proof A standard indirect argument utilizing the MCP yields the desired estimates.

By the Stokes theorem and approximation always

$$\mathring{\mathsf{D}}^q_0(\Gamma_t, \Omega) \subset \Big(\delta\mathring{\Delta}^{q+1}(\Gamma_n, \Omega)\Big)^{\perp}, \quad \mathring{\Delta}^q_0(\Gamma_n, \Omega) \subset \Big(\mathrm{d}\mathring{\mathsf{D}}^{q-1}(\Gamma_t, \Omega)\Big)^{\perp}$$

hold. Equality in the latter relations is not clear and needs another assumption on the pair (Ω, Γ_t).

Definition 8.2 The pair (Ω, Γ_t) has the 'Maxwell approximation property' (MAP), if for all q

$$\mathring{\mathsf{D}}^q_0(\Gamma_t, \Omega) = \Big(\delta\mathring{\Delta}^{q+1}(\Gamma_n, \Omega)\Big)^{\perp}, \quad \mathring{\Delta}^q_0(\Gamma_n, \Omega) = \Big(\mathrm{d}\mathring{\mathsf{D}}^{q-1}(\Gamma_t, \Omega)\Big)^{\perp}.$$

Remark 8.3 By $*$-duality the pair (Ω, Γ_t) has the MAP, if and only if the pair (Ω, Γ_n) has the MAP, i.e., if and only if for all q

$$\mathring{\mathsf{D}}^q_0(\Gamma_n, \Omega) = \Big(\delta\mathring{\Delta}^{q+1}(\Gamma_t, \Omega)\Big)^{\perp}, \quad \mathring{\Delta}^q_0(\Gamma_t, \Omega) = \Big(\mathrm{d}\mathring{\mathsf{D}}^{q-1}(\Gamma_n, \Omega)\Big)^{\perp}.$$

Remark 8.4 If $\Gamma_t = \Gamma$ or $\Gamma_t = \emptyset$, the MAP is simply given by the projection theorem in Hilbert spaces and by the definitions of the respective Sobolev spaces. For the general case $\emptyset \subset \Gamma_t \subset \Gamma$ with possibly $\emptyset \subsetneq \Gamma_t \subsetneq \Gamma$, Jochmann proved the MAP in [5] considering the special case of a bounded domain $\Omega \subset \mathbb{R}^3$. As in Remark 8.2 he needs Ω to be Lipschitz and γ to be a Lipschitz interface. Kuhn showed the MAP in [6] for smooth domains $\Omega \subset \mathbb{R}^N$ or even for smooth Riemannian manifolds Ω with smooth boundary and admissible interface γ. Again, a sufficient result for us has been given recently by Gol'dshtein and Mitrea (I. & M.) in [3, Theorem 4.3, (4.16)]. Like in Remark 8.2, for this Ω has to be a weakly Lipschitz domain and $\Gamma_t \subset \Gamma$ to be an admissible patch.

By [3, Theorem 4.3, (4.16)] and Remark 8.4 we have:

Theorem 8.3 *Let* Ω *be a weakly Lipschitz domain and* Γ_t *be an admissible patch, i.e., let* Ω *be a (weakly) Lipschitz domain and* Γ_t *be an Lipschitz patch of* Γ. *Then the pair* (Ω, Γ_t) *has the MAP.*

Lemma 8.2 [Hodge-Helmholtz decomposition for differential forms] *Let the pair* (Ω, Γ_t) *have the MAP. Then, the orthogonal decompositions*

$$\begin{aligned}\mathsf{L}^{2,q}(\Omega) &= \overline{\mathrm{d}\mathring{\mathsf{D}}^{q-1}(\Gamma_t,\Omega)} \oplus \mathring{\Delta}_0^q(\Gamma_n,\Omega)\\ &= \mathring{\mathsf{D}}_0^q(\Gamma_t,\Omega) \oplus \overline{\delta\mathring{\Delta}^{q+1}(\Gamma_n,\Omega)}\\ &= \overline{\mathrm{d}\mathring{\mathsf{D}}^{q-1}(\Gamma_t,\Omega)} \oplus \mathscr{H}^q(\Omega) \oplus \overline{\delta\mathring{\Delta}^{q+1}(\Gamma_n,\Omega)}\end{aligned}$$

hold. If the pair (Ω,Γ_t) has additionally the MCP, then

$$\begin{aligned}\mathrm{d}\mathring{\mathsf{D}}^{q-1}(\Gamma_t,\Omega) &= \mathrm{d}\left(\mathring{\mathsf{D}}^{q-1}(\Gamma_t,\Omega)\cap\delta\mathring{\Delta}^q(\Gamma_n,\Omega)\right) = \mathring{\mathsf{D}}_0^q(\Gamma_t,\Omega)\cap\mathscr{H}^q(\Omega)^\perp,\\ \delta\mathring{\Delta}^{q+1}(\Gamma_n,\Omega) &= \delta\left(\mathring{\Delta}^{q+1}(\Gamma_n,\Omega)\cap\mathrm{d}\mathring{\mathsf{D}}^q(\Gamma_t,\Omega)\right) = \mathring{\Delta}_0^q(\Gamma_n,\Omega)\cap\mathscr{H}^q(\Omega)^\perp\end{aligned}$$

and these are closed subspaces of $\mathsf{L}^{2,q}(\Omega)$. Moreover, then the orthogonal decompositions

$$\begin{aligned}\mathsf{L}^{2,q}(\Omega) &= \mathrm{d}\mathring{\mathsf{D}}^{q-1}(\Gamma_t,\Omega) \oplus \mathring{\Delta}_0^q(\Gamma_n,\Omega)\\ &= \mathring{\mathsf{D}}_0^q(\Gamma_t,\Omega) \oplus \delta\mathring{\Delta}^{q+1}(\Gamma_n,\Omega)\\ &= \mathrm{d}\mathring{\mathsf{D}}^{q-1}(\Gamma_t,\Omega) \oplus \mathscr{H}^q(\Omega) \oplus \delta\mathring{\Delta}^{q+1}(\Gamma_n,\Omega)\end{aligned}$$

hold.

Here, $\oplus$ denotes the $\mathsf{L}^{2,q}(\Omega)$-orthogonal sum and all closures are taken in $\mathsf{L}^{2,q}(\Omega)$.

Proof By the projection theorem in Hilbert space and the MAP we obtain immediately the two $\mathsf{L}^{2,q}(\Omega)$-orthogonal decompositions

$$\overline{\mathrm{d}\mathring{\mathsf{D}}^{q-1}(\Gamma_t,\Omega)} \oplus \mathring{\Delta}_0^q(\Gamma_n,\Omega) = \mathsf{L}^{2,q}(\Omega) = \mathring{\mathsf{D}}_0^q(\Gamma_t,\Omega) \oplus \overline{\delta\mathring{\Delta}^{q+1}(\Gamma_n,\Omega)},$$

where the closures are taken in $\mathsf{L}^{2,q}(\Omega)$. Since

$$\mathrm{d}\mathring{\mathsf{D}}^{q-1}(\Gamma_t,\Omega)\subset\mathring{\mathsf{D}}_0^q(\Gamma_t,\Omega),\quad \delta\mathring{\Delta}^{q+1}(\Gamma_n,\Omega)\subset\mathring{\Delta}_0^q(\Gamma_n,\Omega)$$

and applying the latter decompositions separately to $\mathring{\Delta}_0^q(\Gamma_n,\Omega)$ or $\mathring{\mathsf{D}}_0^q(\Gamma_t,\Omega)$ we get a refined decomposition, namely

$$\mathsf{L}^{2,q}(\Omega) = \overline{\mathrm{d}\mathring{\mathsf{D}}^{q-1}(\Gamma_t,\Omega)} \oplus \mathscr{H}^q(\Omega) \oplus \overline{\delta\mathring{\Delta}^{q+1}(\Gamma_n,\Omega)}.$$

Applying this decomposition to $\mathring{\mathsf{D}}^{q-1}(\Gamma_t,\Omega)$ and $\mathring{\Delta}^{q+1}(\Gamma_n,\Omega)$ yields also

$$\mathrm{d}\mathring{\mathsf{D}}^{q-1}(\Gamma_t,\Omega)=\mathrm{d}\left(\mathring{\mathsf{D}}^{q-1}(\Gamma_t,\Omega)\cap\overline{\delta\mathring{\Delta}^{q}(\Gamma_n,\Omega)}\right),$$
$$\delta\mathring{\Delta}^{q+1}(\Gamma_n,\Omega)=\delta\left(\mathring{\Delta}^{q+1}(\Gamma_n,\Omega)\cap\overline{\mathrm{d}\mathring{\mathsf{D}}^{q}(\Gamma_t,\Omega)}\right).$$

Now, Lemma 8.1 shows that $\mathrm{d}\mathring{\mathsf{D}}^{q-1}(\Gamma_t,\Omega)$ and $\delta\mathring{\Delta}^{q+1}(\Gamma_n,\Omega)$ are even closed subspaces of $\mathsf{L}^{2,q}(\Omega)$. Hence, we obtain the asserted Hodge-Helmholtz decompositions of $\mathsf{L}^{2,q}(\Omega)$.

8.2.2 Functions and Vector Fields

We turn to the special case $q=1$, the case of vector fields, and use the notations and identifications from [11, 13–15]. Especially, $\mathsf{L}^{2,q}(\Omega)$ can be identified with the usual Lebesgue spaces of square integrable functions or vector fields on Ω with values in $\mathbb{R}^n$, $n:=n_{N,q}:=\binom{N}{q}$, and will be denoted by $\mathsf{L}^2(\Omega):=\mathsf{L}^2(\Omega,\mathbb{R}^n)$. We have the standard Sobolev spaces

$$\mathsf{H}(\mathrm{grad};\Omega):=\left\{u\in\mathsf{L}^2(\Omega,\mathbb{R})\mid\operatorname{grad}u\in\mathsf{L}^2\left(\Omega,\mathbb{R}^N\right)\right\},$$
$$\mathsf{H}(\mathrm{div};\Omega):=\left\{v\in\mathsf{L}^2\left(\Omega,\mathbb{R}^N\right)\mid\operatorname{div}v\in\mathsf{L}^2(\Omega,\mathbb{R})\right\},$$
$$\mathsf{H}(\mathrm{curl};\Omega):=\left\{v\in\mathsf{L}^2\left(\Omega,\mathbb{R}^N\right)\mid\operatorname{curl}v\in\mathsf{L}^2\left(\Omega,\mathbb{R}^{N(N-1)/2}\right)\right\}$$

and by natural isomorphic identification

$$\mathsf{D}^0(\Omega)\cong\mathsf{H}(\mathrm{grad};\Omega),\quad\Delta^1(\Omega)\cong\mathsf{H}(\mathrm{div};\Omega),\quad\mathsf{D}^1(\Omega)\cong\mathsf{H}(\mathrm{curl};\Omega).$$

Generally, $\mathsf{D}^q(\Omega)\cong\Delta^{N-q}(\Omega)$ holds by the Hodge star duality. For $v\in\mathsf{C}^\infty(\Omega)$ and $N=3,4$

$$\operatorname{curl}v=\begin{bmatrix}\partial_2v_3-\partial_3v_2\\\partial_3v_1-\partial_1v_3\\\partial_1v_2-\partial_2v_1\end{bmatrix}\in\mathbb{R}^3,\quad\operatorname{curl}v=\begin{bmatrix}\partial_1v_2-\partial_2v_1\\\partial_1v_3-\partial_3v_1\\\partial_1v_4-\partial_4v_1\\\partial_2v_3-\partial_3v_2\\\partial_2v_4-\partial_4v_2\\\partial_3v_4-\partial_4v_3\end{bmatrix}\in\mathbb{R}^6$$

hold, whereas $\operatorname{curl}v=\partial_1v_2-\partial_2v_1\in\mathbb{R}$ or $\operatorname{curl}v\in\mathbb{R}^{10}$ for $N=2$ or $N=5$, respectively (Table 8.1).

Table 8.1 Identification table for q-forms and vector proxies in $\mathbb{R}^3$

q	0	1	2	3
d	grad	curl	div	0
δ	0	div	$-$ curl	grad
$\mathring{\mathsf{D}}^q(\Gamma_t, \Omega)$	$\mathring{\mathsf{H}}(\text{grad}; \Gamma_t, \Omega)$	$\mathring{\mathsf{H}}(\text{curl}; \Gamma_t, \Omega)$	$\mathring{\mathsf{H}}(\text{div}; \Gamma_t, \Omega)$	$\mathsf{L}^2(\Omega)$
$\mathring{\Delta}^q(\Gamma_n, \Omega)$	$\mathsf{L}^2(\Omega)$	$\mathring{\mathsf{H}}(\text{div}; \Gamma_n, \Omega)$	$\mathring{\mathsf{H}}(\text{curl}; \Gamma_n, \Omega)$	$\mathring{\mathsf{H}}(\text{grad}; \Gamma_n, \Omega)$
$\iota^*_{\Gamma_t} E$	$E\|_{\Gamma_t}$	$\nu \times E\|_{\Gamma_t}$	$\nu \cdot E\|_{\Gamma_t}$	0
$\circledast\iota^*_{\Gamma_n} * E$	0	$\nu \cdot E\|_{\Gamma_n}$	$-\nu \times (\nu \times E)\|_{\Gamma_n}$	$E\|_{\Gamma_n}$

Moreover, we have the closed subspaces

$$\mathring{\mathsf{H}}(\text{grad}; \Gamma_t, \Omega), \quad \mathring{\mathsf{H}}(\text{curl}; \Gamma_t, \Omega), \quad \mathring{\mathsf{H}}(\text{div}; \Gamma_n, \Omega),$$

in which the homogeneous scalar, tangential and normal boundary conditions

$$u|_{\Gamma_t} = 0, \quad \nu \times v|_{\Gamma_t} = 0, \quad \nu \cdot v|_{\Gamma_n} = 0$$

are generalized, as reincarnations of $\mathring{\mathsf{D}}^0(\Gamma_t, \Omega)$, $\mathring{\mathsf{D}}^1(\Gamma_t, \Omega)$ and $\mathring{\Delta}^1(\Gamma_n, \Omega)$, respectively. Here ν denotes the outer unit normal at Γ. If $\Gamma_t = \Gamma$ (and $\Gamma_n = \emptyset$) we obtain the usual Sobolev spaces

$$\mathring{\mathsf{H}}(\text{grad}; \Omega), \quad \mathring{\mathsf{H}}(\text{curl}; \Omega), \quad \mathsf{H}(\text{div}; \Omega).$$

We note that $\mathsf{H}(\text{grad}; \Omega)$ and $\mathring{\mathsf{H}}(\text{grad}; \Omega)$ coincide with the usual standard Sobolev spaces $\mathsf{H}^1(\Omega)$ and $\mathring{\mathsf{H}}^1(\Omega)$, respectively. As before, the index 0, now attached to the symbols curl or div, indicates vanishing curl or div, e.g.,

$$\mathring{\mathsf{H}}(\text{curl}_0; \Gamma_t, \Omega) = \left\{ v \in \mathring{\mathsf{H}}(\text{curl}; \Gamma_t, \Omega) \mid \text{curl}\, v = 0 \right\},$$

$$\mathsf{H}(\text{div}_0; \Omega) = \{ v \in \mathsf{H}(\text{div}; \Omega) \mid \text{div}\, v = 0 \}.$$

Finally, we denote the 'harmonic Dirichlet-Neumann fields' by

$$\mathscr{H}^1(\Omega) \cong \mathscr{H}(\Omega) := \mathring{\mathsf{H}}(\text{curl}_0; \Gamma_t, \Omega) \cap \mathring{\mathsf{H}}(\text{div}_0; \Gamma_n, \Omega).$$

Assuming the MCP for the pair (Ω, Γ_t), then $\mathscr{H}(\Omega)$ is finite dimensional by Corollary 8.1 and we have the two (out of four) compact embeddings

$$\overset{\circ}{\mathsf{H}}(\mathrm{grad}; \Gamma_t, \Omega) \hookrightarrow \mathsf{L}^2(\Omega), \tag{8.3}$$

$$\overset{\circ}{\mathsf{H}}(\mathrm{curl}; \Gamma_t, \Omega) \cap \overset{\circ}{\mathsf{H}}(\mathrm{div}; \Gamma_n, \Omega) \hookrightarrow \mathsf{L}^2(\Omega), \tag{8.4}$$

i.e., the Rellich selection theorem ($q = 0$) and the vectorial Maxwell compactness property ($q = 1$). Moreover, by Lemma 8.1 we get the following Poincaré and Maxwell estimates:

Corollary 8.2 [Poincaré estimate for functions] *Let the pair* (Ω, Γ_t) *have the MCP and* $c_{\mathtt{p}} := c_{\mathtt{p},0}$. *Then*

$$\|u\|_{\mathsf{L}^2(\Omega)} \leq c_{\mathtt{p}} \|\,\mathrm{grad}\, u\|_{\mathsf{L}^2(\Omega)}$$

holds for all $u \in \overset{\circ}{\mathsf{H}}(\mathrm{grad}; \Gamma_t, \Omega)$.

We note that $\mathscr{H}^0(\Omega) = \{0\}$.

Corollary 8.3 [Maxwell estimate for vector fields] *Let the pair* (Ω, Γ_t) *have the MCP and* $c_{\mathtt{m}} := c_{\mathtt{p},1}$. *Then*

$$\|v\|_{\mathsf{L}^2(\Omega)} \leq c_{\mathtt{m}} \left(\|\,\mathrm{curl}\, v\|^2_{\mathsf{L}^2(\Omega)} + \|\,\mathrm{div}\, v\|^2_{\mathsf{L}^2(\Omega)}\right)^{1/2}$$

holds for all $v \in \overset{\circ}{\mathsf{H}}(\mathrm{curl}; \Gamma_t, \Omega) \cap \overset{\circ}{\mathsf{H}}(\mathrm{div}; \Gamma_n, \Omega) \cap \mathscr{H}(\Omega)^{\perp}$ *as well as*

$$\|(\mathrm{id} - \pi_1)v\|_{\mathsf{L}^2(\Omega)} \leq c_{\mathtt{m}} \left(\|\,\mathrm{curl}\, v\|^2_{\mathsf{L}^2(\Omega)} + \|\,\mathrm{div}\, v\|^2_{\mathsf{L}^2(\Omega)}\right)^{1/2}$$

holds for all $v \in \overset{\circ}{\mathsf{H}}(\mathrm{curl}; \Gamma_t, \Omega) \cap \overset{\circ}{\mathsf{H}}(\mathrm{div}; \Gamma_n, \Omega)$, *where again* $\pi_1 : \mathsf{L}^2(\Omega) \to \mathscr{H}(\Omega)$ *denotes the* $\mathsf{L}^2(\Omega)$*-orthogonal projection onto the Dirichlet-Neumann fields* $\mathscr{H}(\Omega)$.

Lemma 8.2 yields:

Corollary 8.4 [Helmholtz decompositions for vector fields] *Let the pair* (Ω, Γ_t) *have the MCP and the MAP.*

Then, the orthogonal decompositions

$$\begin{aligned} \mathsf{L}^2(\Omega) &= \mathrm{grad}\, \overset{\circ}{\mathsf{H}}(\mathrm{grad}; \Gamma_t, \Omega) \oplus \overset{\circ}{\mathsf{H}}(\mathrm{div}_0; \Gamma_n, \Omega) \\ &= \overset{\circ}{\mathsf{H}}(\mathrm{curl}_0; \Gamma_t, \Omega) \oplus \left(\overset{\circ}{\mathsf{H}}(\mathrm{div}_0; \Gamma_n, \Omega) \cap \mathscr{H}(\Omega)^{\perp}\right) \end{aligned}$$

hold.

8.2.3 Tensor Fields

Next, we extend our calculus to tensor fields, i.e., matrix fields. For vector fields v with components in $\mathsf{H}(\mathrm{grad};\Omega)$ and tensor fields T with rows in $\mathsf{H}(\mathrm{curl};\Omega)$ resp. $\mathsf{H}(\mathrm{div};\Omega)$, i.e.,

$$v = \begin{bmatrix} v_1 \\ \vdots \\ v_N \end{bmatrix}, \quad v_n \in \mathsf{H}(\mathrm{grad};\Omega), \quad T = \begin{bmatrix} T_1^\top \\ \vdots \\ T_N^\top \end{bmatrix}, \quad T_n \in \mathsf{H}(\mathrm{curl};\Omega) \text{ resp. } \mathsf{H}(\mathrm{div};\Omega)$$

for $n = 1, \ldots, N$ we define (in the weak sense)

$$\mathrm{Grad}\, v := \begin{bmatrix} \mathrm{grad}^\top v_1 \\ \vdots \\ \mathrm{grad}^\top v_N \end{bmatrix} = J_v, \quad \mathrm{Curl}\, T := \begin{bmatrix} \mathrm{curl}^\top T_1 \\ \vdots \\ \mathrm{curl}^\top T_N \end{bmatrix}, \quad \mathrm{Div}\, T := \begin{bmatrix} \mathrm{div}\, T_1 \\ \vdots \\ \mathrm{div}\, T_N \end{bmatrix},$$

where J_v[3] denotes the Jacobian of v and $^\top$ the transpose. We note that v and $\mathrm{Div}\, T$ are N-vector fields, T and $\mathrm{Grad}\, v$ are $(N \times N)$-tensor fields, whereas $\mathrm{Curl}\, T$ is a $(N \times N(N-1)/2)$-tensor field. The corresponding Sobolev spaces will be denoted by

$$\mathsf{H}(\mathrm{Grad};\Omega), \quad \mathsf{H}(\mathrm{Curl};\Omega), \quad \mathsf{H}(\mathrm{Curl}_0;\Omega), \quad \mathsf{H}(\mathrm{Div};\Omega), \quad \mathsf{H}(\mathrm{Div}_0;\Omega)$$

and

$$\overset{\circ}{\mathsf{H}}(\mathrm{Grad};\Gamma_t,\Omega), \quad \overset{\circ}{\mathsf{H}}(\mathrm{Curl};\Gamma_t,\Omega), \quad \overset{\circ}{\mathsf{H}}(\mathrm{Curl}_0;\Gamma_t,\Omega), \quad \overset{\circ}{\mathsf{H}}(\mathrm{Div};\Gamma_n,\Omega), \quad \overset{\circ}{\mathsf{H}}(\mathrm{Div}_0;\Gamma_n,\Omega),$$

again with the usual notations if $\Gamma_t = \Gamma$.

From Corollaries 8.2, 8.3, and 8.4 we obtain immediately:

Corollary 8.5 [Poincaré estimate for vector fields] *Let the pair (Ω, Γ_t) have the MCP. Then*

$$\|v\|_{\mathsf{L}^2(\Omega)} \leq c_{\mathtt{p}} \|\,\mathrm{Grad}\, v\|_{\mathsf{L}^2(\Omega)}$$

holds for all $v \in \overset{\circ}{\mathsf{H}}(\mathrm{Grad};\Gamma_t,\Omega)$.

Corollary 8.6 [Maxwell estimate for tensor fields] *Let the pair (Ω, Γ_t) have the MCP. Then*

$$\|T\|_{\mathsf{L}^2(\Omega)} \leq c_{\mathtt{m}} \Big(\|\,\mathrm{Curl}\, T\|^2_{\mathsf{L}^2(\Omega)} + \|\,\mathrm{Div}\, T\|^2_{\mathsf{L}^2(\Omega)}\Big)^{1/2}$$

holds for all $T \in \overset{\circ}{\mathsf{H}}(\mathrm{Curl};\Gamma_t,\Omega) \cap \overset{\circ}{\mathsf{H}}(\mathrm{Div};\Gamma_n,\Omega) \cap (\mathscr{H}(\Omega)^N)^\perp$ as well as

---

[3] Sometimes, the Jacobian J_v is also denoted by ∇v.

$$\left\| \left(\mathrm{id} - \pi_1^N\right) T \right\|_{\mathsf{L}^2(\Omega)} \leq c_{\mathrm{m}} \left(\| \operatorname{Curl} T \|^2_{\mathsf{L}^2(\Omega)} + \| \operatorname{Div} T \|^2_{\mathsf{L}^2(\Omega)} \right)^{1/2}$$

holds for all $T \in \overset{\circ}{\mathsf{H}}(\mathrm{Curl}; \Gamma_t, \Omega) \cap \overset{\circ}{\mathsf{H}}(\mathrm{Div}; \Gamma_n, \Omega)$*, where* $\pi_1^N : \mathsf{L}^2(\Omega) \to \mathscr{H}(\Omega)^N$ *denotes the* $\mathsf{L}^2(\Omega)$*-orthogonal projection onto the (*N*-times)-Dirichlet-Neumann fields* $\mathscr{H}(\Omega)^N$.

Corollary 8.7 [Helmholtz decompositions for tensor fields] *Let the pair* (Ω, Γ_t) *have the MCP and the MAP.*

Then, the orthogonal decompositions

$$\begin{aligned} \mathsf{L}^2(\Omega) &= \operatorname{Grad} \overset{\circ}{\mathsf{H}}(\mathrm{Grad}; \Gamma_t, \Omega) \oplus \overset{\circ}{\mathsf{H}}(\mathrm{Div}_0; \Gamma_n, \Omega) \\ &= \overset{\circ}{\mathsf{H}}(\mathrm{Curl}_0; \Gamma_t, \Omega) \oplus \left(\overset{\circ}{\mathsf{H}}(\mathrm{Div}_0; \Gamma_n, \Omega) \cap (\mathscr{H}(\Omega)^N)^{\perp} \right) \end{aligned}$$

hold.

We also need the first Korn inequality.

Definition 8.3 [Second Korn inequality] The domain Ω has the 'Korn property' (KP), if

(i) the second Korn inequality holds, this is, there exists a constant $c > 0$ such that for all vector fields $v \in \mathsf{H}(\mathrm{Grad}; \Omega)$

$$c \| \operatorname{Grad} v \|_{\mathsf{L}^2(\Omega)} \leq \| v \|_{\mathsf{L}^2(\Omega)} + \| \operatorname{sym} \operatorname{Grad} v \|_{\mathsf{L}^2(\Omega)},$$

(ii) and the Rellich selection theorem holds for $\mathsf{H}(\mathrm{grad}; \Omega)$, this is, the embedding $\mathsf{H}(\mathrm{grad}; \Omega) \hookrightarrow \mathsf{L}^2(\Omega)$ is compact.

Here, we introduce the symmetric and skew-symmetric parts

$$\operatorname{sym} T := \frac{1}{2}\left(T + T^{\top}\right), \quad \operatorname{skew} T := T - \operatorname{sym} T = \frac{1}{2}\left(T - T^{\top}\right)$$

of a tensor field $T = \operatorname{skew} T + \operatorname{sym} T$.[4]

Remark 8.5 There exists a rich amount of literature for the KP, which we do not intend to list here. We refer to our overview on the Korn inequalities in [15].

Theorem 8.4 *The second Korn inequality holds for domains* Ω *having the strict cone property. For domains* Ω *with the segment property, the Rellich selection theorem for* $\mathsf{H}(\mathrm{grad}; \Omega)$ *is valid. Thus, e.g., Lipschitz domains* Ω *possess the KP.*

[4] Note that $\operatorname{sym} T$ and $\operatorname{skew} T$ are point-wise orthogonal with respect to the standard inner product in $\mathbb{R}^{N \times N}$.

Proof Book of Leis [10].

By a standard indirect argument we immediately obtain:

Corollary 8.8 [First Korn inequality: standard version] *Let Ω have the KP. Then, there exists a constant $c_{\mathrm{k},s} > 0$ such that*

$$\left(1 + c_{\mathrm{p}}^2\right)^{-1/2} \|v\|_{\mathsf{H}^1(\Omega)} \leq \|\operatorname{Grad} v\|_{\mathsf{L}^2(\Omega)} \leq c_{\mathrm{k},s} \|\operatorname{sym}\operatorname{Grad} v\|_{\mathsf{L}^2(\Omega)} \tag{8.5}$$

holds for all vector fields $v \in \overset{\circ}{\mathsf{H}}(\mathrm{Grad}; \Gamma_t, \Omega)$.

8.2.4 Sliceable and Admissible Domains

The essential tools to prove our main result Theorem 8.1 are

1. the Maxwell estimate for tensor fields (Corollary 8.6),
2. the Helmholtz decomposition for tensor fields (Corollary 8.7),
3. and a generalized version of the first Korn inequality (Corollary 8.8).

For the first two tools the pair (Ω, Γ_t) needs to have the MCP and the MAP. The third tool will be provided in Lemma 8.5 and needs at least the KP. As already pointed out, these three properties hold, e.g., for Lipschitz domains Ω and admissible boundary patches Γ_t. Moreover, we will make use of the fact that any irrotational vector field is already a gradient if the underlying domain is simply connected. For this, we present a trick, the concept of sliceable domains, which we have used already in [15] (Fig. 8.2).

Definition 8.4 The pair (Ω, Γ_t) is called 'sliceable', if there exist $J \in \mathbb{N}$ and $\Omega_j \subset \Omega$, $j = 1, \ldots, J$, such that $\Omega \setminus (\Omega_1 \cup \ldots \cup \Omega_J)$ has zero Lebesgue-measure and for $j = 1, \ldots, J$

(i) Ω_j are open, disjoint and simply connected subdomains of Ω having the KP,
(ii) $\Gamma_{t,j} := \operatorname{int}_{\mathsf{rel}}(\overline{\Omega_j} \cap \Gamma_t) \neq \emptyset$.

Here, $\operatorname{int}_{\mathsf{rel}}$ denotes the interior with respect to the topology on Γ.

Remark 8.6 From a practical point of view, *all* domains considered in applications are sliceable, but it is unclear whether every Lipschitz pair (Ω, Γ_t) is already sliceable.

Now, we can introduce our general assumptions on the domain and its boundary parts.

Definition 8.5 The pair (Ω, Γ_t) is called 'admissible', if

(i) the pair (Ω, Γ_t) possesses the MCP and the MAP, and
(ii) the pair (Ω, Γ_t) is sliceable.

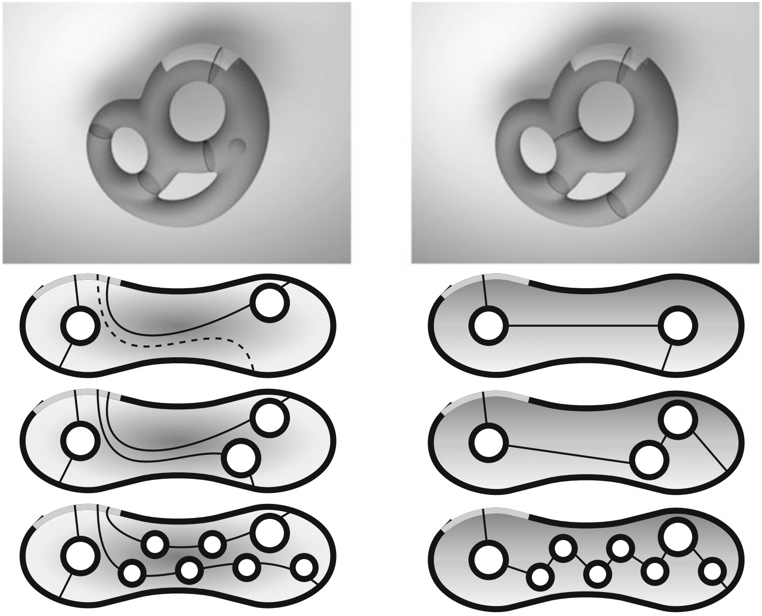

Fig. 8.2 Some ways to 'cut' sliceable domains Ω in $\mathbb{R}^3$ and $\mathbb{R}^2$ into two ($J = 2$) or more ($J = 3, 4$) 'pieces'. The boundary part Γ_t is colored in light gray. Roughly speaking, a domain is sliceable if it can be cut into finitely many simply connected Lipschitz pieces Ω_j, i.e., any closed curve inside some piece Ω_j is homotop to a point, this is, one has to cut all 'handles'. In three and higher dimensions, holes inside Ω are permitted, but this is forbidden in the two-dimensional case. Note that, in these examples it is always possible to slice Ω into two ($J = 2$) pieces

Remark 8.7 In particular, the pair (Ω, Γ_t) is admissible if

- Ω has a Lipschitz boundary Γ,
- Γ_t is a Lipschitz patch,
- (Ω, Γ_t) is sliceable.

8.3 Proofs

Let the pair (Ω, Γ_t) be *admissible*. On our way to prove our main result we follow in close lines the arguments of [15, Sect. 3]. First we prove a non-standard version of the first Korn inequality Corollary 8.8, which will be presented as Lemma 8.5. Then, we prove our main result. Although, all subsequent proofs are very similar to the ones given in [15, Lemmas 8, 9, 12, Theorem 14], we will repeat them here for the convenience of the reader.

Lemma 8.3 *Let* $u \in \mathsf{H}(\mathrm{grad}; \Omega)$ *with* $\mathrm{grad}\, u \in \mathring{\mathsf{H}}(\mathrm{curl}_0; \Gamma_t, \Omega)$. *Then, u is constant on any connected component of* Γ_t.

Proof Let $x \in \Gamma_t$ and $B_{2r} := B_{2r}(x)$ be the open ball of radius $2r > 0$ around x such that B_{2r} is covered by a Lipschitz-chart domain and $\Gamma \cap B_{2r} \subset \Gamma_t$. Moreover, we pick a cut-off function $\varphi \in \mathring{\mathsf{C}}^\infty(B_{2r})$ with $\varphi|_{B_r} = 1$. Then, $\varphi \operatorname{grad} u \in \mathring{\mathsf{H}}(\mathrm{curl}; \Omega \cap B_{2r})$. Thus, the extension by zero v of $\varphi \operatorname{grad} u$ to B_{2r} belongs to $\mathsf{H}(\mathrm{curl}; B_{2r})$. Hence, $v|_{B_r} \in \mathsf{H}(\mathrm{curl}_0; B_r)$. Since B_r is simply connected, there exists a $\tilde{u} \in \mathsf{H}(\mathrm{grad}; B_r)$ with $\operatorname{grad} \tilde{u} = v$ in B_r. In $B_r \setminus \overline{\Omega}$ we have $v = 0$. Therefore, $\tilde{u}|_{B_r \setminus \overline{\Omega}} = \tilde{c}$ with some $\tilde{c} \in \mathbb{R}$. Moreover, $\operatorname{grad} u = v = \operatorname{grad} \tilde{u}$ holds in $B_r \cap \Omega$, which yields $u = \tilde{u} + c$ in $B_r \cap \Omega$ with some $c \in \mathbb{R}$. Finally, $u|_{B_r \cap \Gamma_t} = \tilde{c} + c$ is constant. Therefore, u is locally constant and hence the assertion follows.

Lemma 8.4 [First Korn inequality: tangential version] *There exists a constant* $c_{\mathrm{k},t} \geq c_{\mathrm{k},s}$, *such that*

$$\| \operatorname{Grad} v\|_{\mathsf{L}^2(\Omega)} \leq c_{\mathrm{k},t} \| \operatorname{sym} \operatorname{Grad} v\|_{\mathsf{L}^2(\Omega)}$$

holds for all $v \in \mathsf{H}(\mathrm{Grad}; \Omega)$ *with* $\operatorname{Grad} v \in \mathring{\mathsf{H}}(\mathrm{Curl}_0; \Gamma_t, \Omega)$.

In classical terms, $\operatorname{Grad} v \in \mathring{\mathsf{H}}(\mathrm{Curl}_0; \Gamma_t, \Omega)$ means that $\operatorname{grad} v_n = \nabla v_n$, $n = 1, \ldots, N$, are normal at Γ_t.

Proof We pick a relatively open connected component $\tilde{\Gamma} \neq \emptyset$ of Γ_t. Then, there exists a constant vector $c_v \in \mathbb{R}^3$ such that $v - c_v$ belongs to $\mathring{\mathsf{H}}(\mathrm{Grad}; \tilde{\Gamma}, \Omega)$ by Lemma 8.3 applied to each component of v. Corollary 8.8 (i) (with $\Gamma_t = \tilde{\Gamma}$ and a possibly larger $c_{\mathrm{k},t}$) completes the proof.

Now, we extend the first Korn inequality from gradient to merely irrotational tensor fields.

Lemma 8.5 (First Korn inequality: irrotational version) *There exists* $c_{\mathrm{k}} \geq c_{\mathrm{k},t} > 0$, *such that for all tensor fields* $T \in \mathring{\mathsf{H}}(\mathrm{Curl}_0; \Gamma_t, \Omega)$

$$\|T\|_{\mathsf{L}^2(\Omega)} \leq c_{\mathrm{k}} \| \operatorname{sym} T\|_{\mathsf{L}^2(\Omega)}. \tag{8.6}$$

Again we note that in classical terms a tensor $T \in \mathring{\mathsf{H}}(\mathrm{Curl}_0; \Gamma_t, \Omega)$ is irrotational and the vector field $T\tau|_{\Gamma_t}$ vanishes for all tangential vector fields τ at Γ. Moreover, the sliceability of (Ω, Γ_t) is precisely needed for Lemma 8.5 to hold.

Proof Let $T \in \mathring{\mathsf{H}}(\mathrm{Curl}_0; \Gamma_t, \Omega)$. We choose a sequence $(T^\ell) \subset \mathring{\mathsf{C}}^\infty(\Gamma_t; \Omega)$ converging to T in $\mathsf{H}(\mathrm{Curl}; \Omega)$. According to Definition 8.4 we decompose Ω into $\Omega_1 \cup \ldots \cup \Omega_J$ and pick some $1 \leq j \leq J$. Then, the restriction $T_j := T|_{\Omega_j}$ belongs to $\mathsf{H}(\mathrm{Curl}_0; \Omega_j)$ and $(T^\ell|_{\overline{\Omega_j}}) \subset \mathring{\mathsf{C}}^\infty(\Gamma_{t,j}; \Omega)$ converges to T_j in $\mathsf{H}(\mathrm{Curl}; \Omega_j)$. Thus, $T_j \in \mathring{\mathsf{H}}(\mathrm{Curl}_0; \Gamma_{t,j}, \Omega_j)$. Since Ω_j is simply connected, there exists a potential vector field v_j in $\mathsf{H}(\mathrm{Grad}; \Omega_j)$ with $\operatorname{Grad} v_j = T_j$ and Lemma 8.4 yields

$$\|T_j\|_{\mathsf{L}^2(\Omega_j)} \leq c_{\mathrm{k},t,j}\|\operatorname{sym} T_j\|_{\mathsf{L}^2(\Omega_j)}, \quad c_{\mathrm{k},t,j} > 0.$$

This can be done for each j. Summing up, we obtain

$$\|T\|_{\mathsf{L}^2(\Omega)} \leq c_{\mathrm{k}}\|\operatorname{sym} T\|_{\mathsf{L}^2(\Omega)}, \quad c_{\mathrm{k}} := \max_{j=1,\dots,J} c_{\mathrm{k},t,j},$$

which completes the proof.

We are ready to prove our main theorem.

Proof (Theorem 8.1) Let $T \in \overset{\circ}{\mathsf{H}}(\mathrm{Curl}; \Gamma_t, \Omega)$. By Corollary 8.7 we have the orthogonal decomposition

$$T = R + S \in \overset{\circ}{\mathsf{H}}(\mathrm{Curl}_0; \Gamma_t, \Omega) \oplus \Big(\overset{\circ}{\mathsf{H}}(\mathrm{Div}_0; \Gamma_n, \Omega) \cap (\mathscr{H}(\Omega)^N)^{\perp}\Big).$$

Moreover, by Corollary 8.6 we obtain

$$\|S\|_{\mathsf{L}^2(\Omega)} \leq c_{\mathrm{m}}\|\operatorname{Curl} T\|_{\mathsf{L}^2(\Omega)} \tag{8.7}$$

since $\operatorname{Curl} S = \operatorname{Curl} T$ and $S \in \overset{\circ}{\mathsf{H}}(\mathrm{Curl}; \Gamma_t, \Omega) \cap \overset{\circ}{\mathsf{H}}(\mathrm{Div}_0; \Gamma_n, \Omega) \cap (\mathscr{H}(\Omega)^N)^{\perp}$.
Then, by orthogonality, Lemma 8.5 (i) for R and (8.7)

$$\begin{aligned}
\|T\|^2_{\mathsf{L}^2(\Omega)} = \|R\|^2_{\mathsf{L}^2(\Omega)} + \|S\|^2_{\mathsf{L}^2(\Omega)} &\leq c_{\mathrm{k}}^2\|\operatorname{sym} R\|^2_{\mathsf{L}^2(\Omega)} + \|S\|^2_{\mathsf{L}^2(\Omega)} \\
&\leq 2c_{\mathrm{k}}^2\|\operatorname{sym} T\|^2_{\mathsf{L}^2(\Omega)} + \left(1 + 2c_{\mathrm{k}}^2\right)\|S\|^2_{\mathsf{L}^2(\Omega)} \\
&\leq \hat{c}^2\left(\|\operatorname{sym} T\|^2_{\mathsf{L}^2(\Omega)} + \|\operatorname{Curl} T\|^2_{\mathsf{L}^2(\Omega)}\right)
\end{aligned}$$

with

$$\hat{c} := \max\left\{\sqrt{2}c_{\mathrm{k}}, c_{\mathrm{m}}\sqrt{1 + 2c_{\mathrm{k}}^2}\right\} \tag{8.8}$$

holds.

8.4 One Additional Result

As in [15, Sect. 3.4] we can prove a generalization for media with structural changes. To apply the main result from [26], let $\mu \in \mathsf{C}^0(\overline{\Omega})$ be a $(N \times N)$-matrix field satisfying $\det \mu \geq \hat{\mu} > 0$.

Corollary 8.9 *Let the pair (Ω, Γ_t) be admissible. Then there exists $c > 0$ such that*

$$c\|T\|_{\mathsf{L}^2(\Omega)} \leq \|\operatorname{sym}(\mu T)\|_{\mathsf{L}^2(\Omega)} + \|\operatorname{Curl} T\|_{\mathsf{L}^2(\Omega)}$$

holds for all tensor fields $T \in \overset{\circ}{\mathsf{H}}(\mathrm{Curl}; \Gamma_t, \Omega)$*. In other words, on* $\overset{\circ}{\mathsf{H}}(\mathrm{Curl}; \Gamma_t, \Omega)$ *the right-hand side defines a norm equivalent to the standard norm in* $\mathsf{H}(\mathrm{Curl}; \Omega)$.

Acknowledgments We heartily thank Kostas Pamfilos for the beautiful pictures of 3D sliceable domains.

Appendix: Construction of Hodge-Helmholtz Projections

We want to point out how to compute the projections in the Hodge-Helmholtz decompositions in Lemma 8.2. Recalling from Lemma 8.2 the orthogonal decompositions

$$\begin{aligned}\mathsf{L}^{2,q}(\Omega) &= \mathrm{d}\overset{\circ}{\mathsf{D}}{}^{q-1}(\Gamma_t,\Omega) \oplus \overset{\circ}{\Delta}{}^q_0(\Gamma_n,\Omega)\\ &= \overset{\circ}{\mathsf{D}}{}^q_0(\Gamma_t,\Omega) \oplus \delta\overset{\circ}{\Delta}{}^{q+1}(\Gamma_n,\Omega)\\ &= \mathrm{d}\overset{\circ}{\mathsf{D}}{}^{q-1}(\Gamma_t,\Omega) \oplus \mathscr{H}^q(\Omega) \oplus \delta\overset{\circ}{\Delta}{}^{q+1}(\Gamma_n,\Omega),\end{aligned}$$

we denote the corresponding $\mathsf{L}^{2,q}(\Omega)$-orthogonal projections by π_{d}, π_δ and $\pi_{\mathscr{H}}$. Then, we have

$$\pi_{\mathscr{H}} = \mathrm{id} - \pi_{\mathrm{d}} - \pi_\delta$$

and

$$\begin{aligned}\pi_{\mathrm{d}}\mathsf{L}^{2,q}(\Omega) &= \mathrm{d}\overset{\circ}{\mathsf{D}}{}^{q-1}(\Gamma_t,\Omega) = \mathrm{d}\mathsf{X}^{q-1}(\Omega), \quad \mathsf{X}^{q-1}(\Omega) := \overset{\circ}{\mathsf{D}}{}^{q-1}(\Gamma_t,\Omega) \cap \delta\overset{\circ}{\Delta}{}^q(\Gamma_n,\Omega),\\ \pi_\delta\mathsf{L}^{2,q}(\Omega) &= \delta\overset{\circ}{\Delta}{}^{q+1}(\Gamma_n,\Omega) = \delta\mathsf{Y}^{q+1}(\Omega), \quad \mathsf{Y}^{q+1}(\Omega) := \overset{\circ}{\Delta}{}^{q+1}(\Gamma_n,\Omega) \cap \mathrm{d}\overset{\circ}{\mathsf{D}}{}^q(\Gamma_t,\Omega),\\ \pi_{\mathscr{H}}\mathsf{L}^{2,q}(\Omega) &= \mathscr{H}^q(\Omega).\end{aligned}$$

By the Poincaré estimate, i.e., Lemma 8.1, we have

$$\forall E \in \mathsf{X}^{q-1}(\Omega) \qquad \|E\|_{\mathsf{L}^{2,q-1}(\Omega)} \le c_{\mathrm{p},q-1}\|\mathrm{d}E\|_{\mathsf{L}^{2,q}(\Omega)}, \tag{8.9}$$

$$\forall H \in \mathsf{Y}^{q+1}(\Omega) \qquad \|H\|_{\mathsf{L}^{2,q+1}(\Omega)} \le c_{\mathrm{p},q+1}\|\delta H\|_{\mathsf{L}^{2,q}(\Omega)}. \tag{8.10}$$

Hence, the bilinear forms

$$\left(\tilde{E}, E\right) \mapsto \left\langle \mathrm{d}\tilde{E}, \mathrm{d}E\right\rangle_{\mathsf{L}^{2,q}(\Omega)}, \quad \left(\tilde{H}, H\right) \mapsto \left\langle \delta\tilde{H}, \delta H\right\rangle_{\mathsf{L}^{2,q}(\Omega)}$$

are continuous and coercive over $\mathsf{X}^{q-1}(\Omega)$ and $\mathsf{Y}^{q+1}(\Omega)$, respectively. Moreover, for any $F \in \mathsf{L}^{2,q}(\Omega)$ the linear functionals

$$E \mapsto \langle F, \mathrm{d}E\rangle_{\mathsf{L}^{2,q}(\Omega)}, \quad H \mapsto \langle F, \delta H\rangle_{\mathsf{L}^{2,q}(\Omega)}$$

are continuous over $\mathsf{X}^{q-1}(\Omega)$, respectively $\mathsf{Y}^{q+1}(\Omega)$. Thus, by the Lax-Milgram theorem we get unique solutions $E_{\mathrm{d}} \in \mathsf{X}^{q-1}(\Omega)$ and $H_\delta \in \mathsf{Y}^{q+1}(\Omega)$ of the two variational problems

$$\langle \mathrm{d}E_{\mathrm{d}}, \mathrm{d}E\rangle_{\mathsf{L}^{2,q}(\Omega)} = \langle F, \mathrm{d}E\rangle_{\mathsf{L}^{2,q}(\Omega)} \qquad \forall E \in \mathsf{X}^{q-1}(\Omega), \tag{8.11}$$

$$\langle \delta H_\delta, \delta H\rangle_{\mathsf{L}^{2,q}(\Omega)} = \langle F, \delta H\rangle_{\mathsf{L}^{2,q}(\Omega)} \qquad \forall H \in \mathsf{Y}^{q+1}(\Omega) \tag{8.12}$$

and the corresponding solution operators, mapping F to E_{d} and H_δ, respectively, are continuous. In fact, we have as usual

$$\|\mathrm{d}E_{\mathrm{d}}\|_{\mathsf{L}^{2,q}(\Omega)} \le \|F\|_{\mathsf{L}^{2,q}(\Omega)}, \quad \|\delta H_\delta\|_{\mathsf{L}^{2,q}(\Omega)} \le \|F\|_{\mathsf{L}^{2,q}(\Omega)},$$

respectively, and therefore together with (8.9) and (8.10)

$$\|E_{\mathrm{d}}\|_{\mathsf{X}^{q-1}(\Omega)} = \|E_{\mathrm{d}}\|_{\mathsf{D}^{q-1}(\Omega)} \le \sqrt{1 + c_{\mathrm{p},q-1}^2}\,\|F\|_{\mathsf{L}^{2,q}(\Omega)},$$

$$\|H_\delta\|_{\mathsf{Y}^{q+1}(\Omega)} = \|H_\delta\|_{\Delta^{q+1}(\Omega)} \le \sqrt{1 + c_{\mathrm{p},q+1}^2}\,\|F\|_{\mathsf{L}^{2,q}(\Omega)}.$$

Since $\mathrm{d}\mathring{\mathsf{D}}^{q-1}(\Gamma_t, \Omega) = \mathrm{d}\mathsf{X}^{q-1}(\Omega)$ and $\delta\mathring{\Delta}^{q+1}(\Gamma_n, \Omega) = \delta\mathsf{Y}^{q+1}(\Omega)$ we see that (8.11) and (8.12) hold also for $E \in \mathring{\mathsf{D}}^{q-1}(\Gamma_t, \Omega)$ and $H \in \mathring{\Delta}^{q+1}(\Gamma_n, \Omega)$, respectively, and that

$$F - \mathrm{d}E_{\mathrm{d}} \in \big(\mathrm{d}\mathsf{X}^{q-1}(\Omega)\big)^\perp = \big(\mathrm{d}\mathring{\mathsf{D}}^{q-1}(\Gamma_t, \Omega)\big)^\perp = \mathring{\Delta}_0^q(\Gamma_n, \Omega),$$

$$F - \delta H_\delta \in \big(\delta\mathsf{Y}^{q+1}(\Omega)\big)^\perp = \big(\delta\mathring{\Delta}^{q+1}(\Gamma_n, \Omega)\big)^\perp = \mathring{\mathsf{D}}_0^q(\Gamma_t, \Omega).$$

Hence, we have found our projections since

$$\pi_{\mathrm{d}}F := \mathrm{d}E_{\mathrm{d}} \in \mathrm{d}\mathsf{X}^{q-1}(\Omega) \subset \mathring{\mathsf{D}}_0^q(\Gamma_t, \Omega),$$

$$\pi_\delta F := \delta H_\delta \in \delta\mathsf{Y}^{q+1}(\Omega) \subset \mathring{\Delta}_0^q(\Gamma_n, \Omega)$$

and

$$\pi_{\mathscr{H}}F := F - \mathrm{d}E_{\mathrm{d}} - \delta H_\delta \in \mathring{\mathsf{D}}_0^q(\Gamma_t, \Omega) \cap \mathring{\Delta}_0^q(\Gamma_n, \Omega) = \mathscr{H}^q(\Omega).$$

Explicit formulas for the dimensions of $\mathscr{H}^q(\Omega)$ or explicit constructions of bases of $\mathscr{H}^q(\Omega)$ depending on the topology of the pair (Ω, Γ_t) can be found, e.g., in [21] for the case $\Gamma_t = \Gamma$ or $\Gamma_t = \emptyset$, or in [3] for the general case.

References

1. Brown R (1994) The mixed problem for Laplace's equation in a class of Lipschitz domains. Comm Partial Differ Equ 19(7–8):1217–1233
2. Costabel M (1990) A remark on the regularity of solutions of Maxwell's equations on Lipschitz domains. Math Methods Appl Sci 12(4):365–368
3. Gol'dshtein V, Mitrea I, Mitrea M (2011) Hodge decompositions with mixed boundary conditions and applications to partial differential equations on Lipschitz manifolds. J Math Sci 172(3):347–400
4. Jakab T, Mitrea I, Mitrea M (2009) On the regularity of differential forms satisfying mixed boundary conditions in a class of Lipschitz domains. Indiana Univ Math J 58(5):2043–2071
5. Jochmann F (1997) A compactness result for vector fields with divergence and curl in $L^q(\Omega)$ involving mixed boundary conditions. Appl Anal 66(1–2):189–203
6. Kuhn P (2000) Die Maxwellgleichung mit wechselnden Randbedingungen. Ph.D. thesis, Universität Essen, Aachen. ArXiv:1108.2028.
7. Kuhn P, Pauly D (2010) Regularity results for generalized electro-magnetic problems. Analysis (Munich) 30(3):225–252
8. Leis R (1968) Zur Theorie elektromagnetischer Schwingungen in anisotropen inhomogenen Medien. Math Z 106:213–224
9. Leis R (1971) Zur Theorie der zeitunabhängigen Maxwellschen Gleichungen. Gesellschaft für Mathematik und Datenverarbeitung, Bonn
10. Leis R (1986) Initial-boundary value problems in mathematical physics. Wiley, Stuttgart/Chichester
11. Neff P, Pauly D, Witsch KJ (2011) A canonical extension of Korn's first inequality to H(Curl) motivated by gradient plasticity with plastic spin. C R Acad Sci Paris 349(23–24):1251–1254
12. Neff P, Pauly D, Witsch KJ (2011) A Korn's inequality for incompatible tensor fields. Proc Appl Math Mech (PAMM) 11(1):683–684
13. Neff P, Pauly D, Witsch KJ (2012) Maxwell meets Korn: A new coercive inequality for tensor fields in $\mathbb{R}^{N\times N}$ with square-integrable exterior derivative. Math Methods Appl Sci 35(1):65–71
14. Neff P, Pauly D, Witsch KJ (2012) On a canonical extension of Korn's first and Poincaré's inequalities to H(Curl). J Math Sci (NY) 185(5):721–727
15. Neff P, Pauly D, Witsch KJ (2012) Poincaré meets Korn via Maxwell: extending Korn's first inequality to incompatible tensor fields. J Diff Eqn. arXiv:1203.2744
16. Pauly D (2006) Low frequency asymptotics for time-harmonic generalized Maxwell's equations in nonsmooth exterior domains. Adv Math Sci Appl 16(2):591–622
17. Pauly D (2007) Generalized electro-magneto statics in nonsmooth exterior domains. Analysis (Munich) 27(4):425–464
18. Pauly D (2008) Complete low frequency asymptotics for time-harmonic generalized Maxwell equations in nonsmooth exterior domains. Asymptot Anal 60(3–4):125–184
19. Pauly D (2008) Hodge-Helmholtz decompositions of weighted Sobolev spaces in irregular exterior domains with inhomogeneous and anisotropic media. Math Methods Appl Sci 31(13):1509–1543
20. Picard R (1981) Randwertaufgaben in der verallgemeinerten Potentialtheorie. Math Methods Appl Sci 3(1):218–228
21. Picard R (1982) On the boundary value problems of electro- and magnetostatics. Proc R Soc Edinburgh Sect A 92(1–2):165–174
22. Picard R (1984) An elementary proof for a compact imbedding result in generalized electromagnetic theory. Math Z 187(2):151–164
23. Picard R (1984) On the low frequency asymptotics in electromagnetic theory. J Reine Angew Math 354:50–73
24. Picard R (1990) Some decomposition theorems and their application to non-linear potential theory and Hodge theory. Math Methods Appl Sci 12(1):35–52
25. Picard R, Weck N, Witsch KJ (2001) Time-harmonic Maxwell equations in the exterior of perfectly conducting, irregular obstacles. Analysis (Munich) 21(3):231–263

26. Pompe W (2003) Korn's first inequality with variable coefficients and its generalization. Comment Math Univ Carolin 44(1):57–70
27. Saranen J (1980) Über das Verhalten der Lösungen der Maxwellschen Randwertaufgabe in Gebieten mit Kegelspitzen. Math Methods Appl Sci 2(2):235–250
28. Saranen J (1981) Über das Verhalten der Lösungen der Maxwellschen Randwertaufgabe in einigen nichtglatten Gebieten. Ann Acad Sci Fenn Ser A I Math 6(1):15–28
29. Weber C (1980) A local compactness theorem for Maxwell's equations. Math Methods Appl Sci 2(1):12–25
30. Weck N (1974) Maxwell's boundary value problem on Riemannian manifolds with nonsmooth boundaries. J Math Anal Appl 46:410–437
31. Witsch KJ (1993) A remark on a compactness result in electromagnetic theory. Math Methods Appl Sci 16(2):123–129