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10 Time-harmonic electro-magnetic scattering in exterior weak Lipschitz domains with mixed boundary conditions

Abstract: This paper treats the time-harmonic electro-magnetic scattering or radiation problem governed by Maxwell's equations, i. e.,

$$\begin{aligned} -\operatorname{rot} H + i\omega\varepsilon E &= F \quad \text{in } \Omega, & E \times \nu &= 0 \quad \text{on } \Gamma_1, \\ \operatorname{rot} E + i\omega\mu H &= G \quad \text{in } \Omega, & H \times \nu &= 0 \quad \text{on } \Gamma_2, \end{aligned}$$

where $\omega \in \mathbb{C} \setminus (0)$ and $\Omega \subset \mathbb{R}^3$ is an exterior weak Lipschitz domain with boundary Γ divided into two disjoint parts Γ_1 and Γ_2. We will present a solution theory using the framework of polynomially weighted Sobolev spaces for the rotation and divergence. For the physically interesting case $\omega \in \mathbb{R} \setminus (0)$, we will show a Fredholm alternative type result to hold using the principle of limiting absorption introduced by Eidus in the 1960s. The necessary a priori estimate and polynomial decay of eigenfunctions for the Maxwell equations will be obtained by transferring well-known results for the Helmholtz equation using a suitable decomposition of the fields E and H. The crucial point for existence is a local version of Weck's selection theorem, also called Maxwell compactness property.

Keywords: Maxwell equations, radiating solutions, exterior boundary value problems, polynomial decay, mixed boundary conditions, weighted Sobolev spaces, Hodge–Helmholtz decompositions

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1 Introduction

The equations that describe the behavior of electro-magnetic vector fields in some space-time domain $I \times \Omega \subset \mathbb{R} \times \mathbb{R}^3$, first completely formulated by J. C. Maxwell in 1864, are

$$\begin{aligned} -\operatorname{rot} H + \partial_t D &= J, & \operatorname{rot} E + \partial_t B &= 0, & \text{in } I \times \Omega, \\ \operatorname{div} D &= \rho, & \operatorname{div} B &= 0, & \text{in } I \times \Omega, \end{aligned}$$

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where E, H are the electric, respectively, magnetic field, D, B represent the displacement current and magnetic induction and J, ρ describe the current density, respectively, the charge density. Excluding, e. g., ferromagnetic, respectively, ferroelectric materials, the parameters linking E and H with D and B are often assumed to be of the linear form $D = \varepsilon E$ and $B = \mu H$, where ε and μ are matrix-valued functions describing the permittivity and permeability of the medium filling Ω. Here, we are especially interested in the case of an exterior domain $\Omega \subset \mathbb{R}^3$, i. e., a connected open subset with compact complement. Applying the divergence to the first two equations, we see that the latter two equations are implicitly included in the first two and may be omitted. Hence, neglecting the static case, Maxwell's equations reduce to

$$-\operatorname{rot} H + \partial_t(\varepsilon E) = F, \quad \operatorname{rot} E + \partial_t(\mu H) = G, \quad \text{in } I \times \Omega,$$

with arbitrary right-hand sides F, G. Among the wide range of phenomena described by these equations one important case is the discussion of *"time-harmonic"* electro-magnetic fields where all fields vary sinusoidally in time with frequency $\omega \in \mathbb{C} \setminus (0)$, i. e.,

$$E(t,x) = e^{i\omega t}E(x), \quad H(t,x) = e^{i\omega t}H(x), \quad G(t,x) = e^{i\omega t}G(x), \quad F(t,x) = e^{i\omega t}F(x)\,.$$

Substituting this ansatz into the equations (or using Fourier transformation in time) and assuming that ε and μ are time-independent we are lead to what is called *"time-harmonic Maxwell's equations"*:

$$\operatorname{rot} E + i\omega\mu H = G, \quad -\operatorname{rot} H + i\omega\varepsilon E = F, \quad \text{in } \Omega\,. \tag{1.1}$$

This system equipped with suitable boundary conditions describes, e. g., the scattering of time-harmonic electro-magnetic waves which is of high interest in many applications like geophysics, medicine, electrical engineering, biology and many others.

First existence results concerning boundary value problems for the time-harmonic Maxwell system in bounded and exterior domains have been given by Müller [13, 12]. He studied isotropic and homogeneous media and used integral equation methods. Using alternating differential forms, Weyl [29] investigated these equations on Riemannian manifolds of arbitrary dimension, while Werner [28] was able to transfer Müller's results to the case of inhomogeneous but isotropic media. However, for general inhomogeneous anisotropic media and arbitrary exterior domains, boundary integral methods are less useful since they heavily depend on the explicit knowledge of the fundamental solution and strong assumptions on boundary regularity. That is why Hilbert space methods are a promising alternative. Unfortunately, Maxwell's equations are nonelliptic, hence it is in general not possible to estimate all first derivatives of a solution. In [9], Leis could overcome this problem by transforming the boundary value problem for Maxwell's system into a boundary value problem for the Helmholtz equation, assuming that the medium filling Ω, is inhomogeneous and anisotropic within a bounded subset of Ω. Nevertheless, he still needed boundary regularity to gain equivalence of both problems. But also for nonsmooth boundaries Hilbert space methods are expedient. In fact, as shown by Leis [10], it is sufficient that Ω satisfies a certain selection theorem, later called *Weck's selection theorem* or *Maxwell compactness property*, which holds for a class of boundaries much larger than those accessible by the detour over H^1 (cf. Weck [24], Costabel [2] and Picard, Weck, Witsch [20]). See [11] for a detailed monograph and [1] for the most recent result and an overview. The most recent result regarding a solution theory is due to Pauly [16] (see also [14]) and in its structure comparable to the results of Picard [18] and Picard, Weck and Witsch [20]. While all these results above have been obtained for full boundary conditions, in the present paper we study the case of mixed boundary conditions. More precisely, we are interested in solving the system (1.1) for $\omega \in \mathbb{C} \setminus (0)$ in an exterior domain $\Omega \subset \mathbb{R}^3$, where we assume that $\Gamma := \partial\Omega$ is decomposed into two relatively open subsets Γ_1 and its complement $\Gamma_2 := \Gamma \setminus \overline{\Gamma}_1$ and impose homogeneous boundary conditions, which in classical terms can be written as

$$\nu \times E = 0 \text{ on } \Gamma_1, \qquad \nu \times H = 0 \text{ on } \Gamma_2, \qquad (\nu: \text{ outward unit normal}). \tag{1.2}$$

Conveniently, we can apply the same methods as in [15] (see also Picard, Weck and Witsch [20], Weck and Witsch [27, 25]) to construct a solution. Indeed, most of the proofs carry over practically verbatim. For $\omega \in \mathbb{C} \setminus \mathbb{R}$, the solution theory is obtained by standard Hilbert space methods as ω belongs to the resolvent set of the Maxwell operator. In the case of $\omega \in \mathbb{R} \setminus (0)$, i. e., ω is in the continuous spectrum of the Maxwell operator, we use the limiting absorption principle introduced by Eidus [4] and approximate solutions to $\omega \in \mathbb{R} \setminus (0)$ by solutions corresponding to $\omega \in \mathbb{C} \setminus \mathbb{R}$. This will be sufficient to show a generalized Fredholm alternative (cf. our main result, Theorem 3.10) to hold. The essential ingredients needed for the limit process are

- the polynomial decay of eigensolutions;
- an a priori estimate for solutions corresponding to nonreal frequencies;
- a Helmholtz-type decomposition;
- and *Weck's local selection theorem* (*WLST*), that is,

$$\mathbf{R}_{\Gamma_1}(\Omega) \cap \varepsilon^{-1}\mathbf{D}_{\Gamma_2}(\Omega) \longrightarrow \mathsf{L}^2_{\mathrm{loc}}(\overline{\Omega}) \text{ is compact.}$$

While the first two are obtained by transferring well-known results for the scalar Helmholtz equation to the time-harmonic Maxwell equations using a suitable decomposition of the fields E and H, Lemma 4.1, the last one is an assumption on the quality of the boundary. As we will see, WLST is an immediate consequence of *Weck's selection theorem* (*WST*), i. e.,

$$\mathbf{R}_{\Gamma_1}(\Theta) \cap \varepsilon^{-1}\mathbf{D}_{\Gamma_2}(\Theta) \longrightarrow \mathsf{L}^2(\Theta) \text{ is compact,}$$

which holds in bounded weak Lipschitz domains $\Theta \subset \mathbb{R}^3$, but fails in unbounded such as exterior domains (cf. Bauer, Pauly, Schomburg [1] and the references therein). For strong Lipschitz-domains, see Jochmann [7] and Fernandes, Gilardis [5].

2 Preliminaries and notation

Let $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ be the usual sets of integers, natural, real and complex numbers, respectively. Furthermore, let i be the imaginary unit, $\operatorname{Re} z$, $\operatorname{Im} z$ and $\overline{z}$ real part, imaginary part and complex conjugate of $z \in \mathbb{C}$, as well as

$$\mathbb{R}_+ := \{s \in \mathbb{R} \,|\, s > 0\}, \quad \mathbb{C}_+ := \{z \in \mathbb{C} \,|\, \operatorname{Im} z \geq 0\}, \quad \mathbb{I} := \big\{(2m+1)/2 \,\big|\, m \in \mathbb{Z} \setminus (0)\big\}\,.$$

For $x \in \mathbb{R}^n$ with $x \neq 0$ we set $r(x) := |x|$ and $\xi(x) := x/|x|$ ($|\cdot|$:Euclidean norm in $\mathbb{R}^n$). Moreover, $\mathrm{U}(\tilde{r})$, respectively, $\mathrm{B}(\tilde{r})$ indicate the open, respectively, closed ball of radius $\tilde{r}$ in $\mathbb{R}^n$ centered in the origin and we define

$$\mathrm{S}(\tilde{r}) := \mathrm{B}(\tilde{r}) \setminus \mathrm{U}(\tilde{r}), \quad \breve{\mathrm{U}}(\tilde{r}) := \mathbb{R}^3 \setminus \mathrm{B}(\tilde{r}), \quad \mathrm{G}(\tilde{r}, \hat{r}) := \breve{\mathrm{U}}(\tilde{r}) \cap \mathrm{U}(\hat{r})$$

with $\hat{r} > \tilde{r}$. If $f : \mathrm{X} \longrightarrow \mathrm{Y}$ is a function mapping X to Y the restriction of f to a subset $\mathrm{U} \subset \mathrm{X}$ will be marked with $f|_U$ and $\mathcal{D}(f)$, $\mathcal{N}(f)$, $\mathcal{R}(f)$, and $\operatorname{supp} f$ denote domain of definition, kernel, range, and support of f, respectively. For Banach or Hilbert spaces X and Y we denote by $\mathrm{L}(\mathsf{X}, \mathsf{Y})$ and $\mathrm{B}(\mathsf{X}, \mathsf{Y})$ the sets of linear respectively bounded linear operators mapping X to Y. For X, Y subspaces of a normed vector space V, $\mathrm{X} + \mathrm{Y}$, $\mathrm{X} \dotplus \mathrm{Y}$, and $\mathrm{X} \oplus \mathrm{Y}$ indicate the sum, the direct sum, and the orthogonal sum of X and Y, where in the last case we presume the existence of a scalar product $\langle \cdot, \cdot \rangle_{\mathrm{V}}$ on V. Moreover, $\langle \cdot, \cdot \rangle_{\mathrm{X}\times\mathrm{Y}}$, respectively, $\| \cdot \|_{\mathrm{X}\times\mathrm{Y}}$ denote the natural scalar product resp. induced norm on $\mathrm{X} \times \mathrm{Y}$. If $\mathrm{X} = \mathrm{Y}$, we often simply use the index X instead of $\mathrm{X} \times \mathrm{X}$.

2.1 General assumptions and weighted Sobolev spaces

Unless stated otherwise, from now on and throughout this paper, it is assumed that $\Omega \subset \mathbb{R}^3$ is an exterior weak Lipschitz domain with weak Lipschitz interface in the sense of [1, Definition 2.3, Definition 2.5], which in principle means that $\Gamma = \partial\Omega$ is a Lipschitz-manifold and Γ_1 respectively Γ_2 are Lipschitz-submanifolds of Γ. For later purposes, we fix $r_0 > 0$ such that $\mathbb{R}^3 \setminus \Omega \Subset \mathrm{U}(r_0)$ and define for arbitrary $\tilde{r} \geq r_0$,

$$\Omega(\tilde{r}) := \Omega \cap \mathrm{U}(\tilde{r})\,.$$

With $r_k := 2^k r_0$, $k \in \mathbb{N}$ and $\tilde{\eta} \in \mathsf{C}^\infty(\mathbb{R})$ such that

$$0 \leq \tilde{\eta} \leq 1, \quad \operatorname{supp} \tilde{\eta} \subset (-\infty, 2-\delta), \quad \tilde{\eta}|_{(-\infty,1+\delta)} = 1, \tag{2.1}$$

for some $0 < \delta < 1$, we define functions $\eta, \check{\eta}, \eta_k, \check{\eta}_k \in \mathsf{C}^\infty(\mathbb{R}^3)$ by

$$\eta(x) := \tilde{\eta}(r(x)/r_0), \quad \check{\eta}(x) := 1 - \eta(x), \quad \eta_k(x) := \tilde{\eta}\big(r(x)/r_k\big), \quad \text{respectively} \quad \check{\eta}_k(x) := 1 - \eta_k(x),$$

meaning

$$\begin{aligned} &\operatorname{supp}\eta \subset \mathrm{B}(r_1) \quad \text{with } \eta = 1 \text{ on } \mathrm{U}(r_0), & & \operatorname{supp}\eta_k \subset \mathrm{U}(r_{k+1}) \quad \text{with} \quad \eta_k = 1 \text{ on } \mathrm{U}(r_k), \\ &\operatorname{supp}\check{\eta} \subset \check{\mathrm{U}}(r_0) \quad \text{with } \check{\eta} = 1 \text{ on } \check{\mathrm{U}}(r_1), & \text{respectively} & \quad \operatorname{supp}\check{\eta}_k \subset \check{\mathrm{U}}(r_k) \quad \text{with} \quad \check{\eta}_k = 1 \text{ on } \check{\mathrm{U}}(r_{k+1}). \end{aligned}$$

These functions will later be utilized for particular cut-off procedures.

Next, we introduce our notation for Lebesgue and Sobolev spaces needed in the following discussion. Note that we will not indicate whether the elements of these spaces are scalar functions or vector fields. This will be always clear from the context. The example[1]

$$\begin{aligned} E &:= \nabla \ln(r) \in \mathsf{H}^1_{\mathrm{loc}}(\check{\mathrm{U}}(1)), & \operatorname{rot} E &= 0 \in \mathsf{L}^2(\check{\mathrm{U}}(1)), \\ \nu \times E|_{\mathrm{S}(1)} &= 0, & \operatorname{div} E &= r^{-2} \in \mathsf{L}^2(\check{\mathrm{U}}(1)) \end{aligned}$$

shows that a standard L^2-setting is not appropriate for exterior domains. Even for square-integrable right-hand sides, we cannot expect to find square-integrable solutions. Indeed, it turns out that we have to work in weighted Lebesgue and Sobolev spaces to develop a solution theory. With $\rho := (1 + r^2)^{1/2}$, we introduce for an arbitrary domain $\Omega \subset \mathbb{R}^3$, $t \in \mathbb{R}$, and $m \in \mathbb{N}$

$$\begin{aligned} \mathsf{L}^2_{\mathrm{t}}(\Omega) &:= \{w \in \mathsf{L}^2_{\mathrm{loc}}(\Omega) \,|\, \rho^t w \in \mathsf{L}^2(\Omega)\}, \\ \mathbf{H}^{\mathrm{m}}_{\mathrm{t}}(\Omega) &:= \{w \in \mathsf{L}^2_{\mathrm{t}}(\Omega) \,|\, \forall\, |\alpha| \le m : \ \partial^\alpha w \in \mathsf{L}^2_{\mathrm{t}}(\Omega)\}, \\ \mathsf{H}^{\mathrm{m}}_{\mathrm{t}}(\Omega) &:= \{w \in \mathsf{L}^2_{\mathrm{t}}(\Omega) \,|\, \forall\, |\alpha| \le m : \ \partial^\alpha w \in \mathsf{L}^2_{\mathrm{t}+|\alpha|}(\Omega)\}, \end{aligned}$$

$$\begin{aligned} \mathbf{R}_{\mathrm{t}}(\Omega) &:= \{E \in \mathsf{L}^2_{\mathrm{t}}(\Omega) \,|\, \operatorname{rot} E \in \mathsf{L}^2_{\mathrm{t}}(\Omega)\}, & \mathsf{R}_{\mathrm{t}}(\Omega) &:= \{E \in \mathsf{L}^2_{\mathrm{t}}(\Omega) \,|\, \operatorname{rot} E \in \mathsf{L}^2_{\mathrm{t}+1}(\Omega)\}, \\ \mathbf{D}_{\mathrm{t}}(\Omega) &:= \{H \in \mathsf{L}^2_{\mathrm{t}}(\Omega) \,|\, \operatorname{div} H \in \mathsf{L}^2_{\mathrm{t}}(\Omega)\}, & \mathsf{D}_{\mathrm{t}}(\Omega) &:= \{H \in \mathsf{L}^2_{\mathrm{t}}(\Omega) \,|\, \operatorname{div} H \in \mathsf{L}^2_{\mathrm{t}+1}(\Omega)\}, \end{aligned}$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}^3$ is a multi-index and $\partial^\alpha w := \partial_1^{\alpha_1} \partial_2^{\alpha_2} \partial_3^{\alpha_3} w$, $\operatorname{rot} E$, and $\operatorname{div} H$ are the usual distributional or weak derivatives. Equipped with the induced norms,

$$\begin{aligned} \|w\|^2_{\mathsf{L}^2_{\mathrm{t}}(\Omega)} &:= \left\|\rho^t w\right\|^2_{\mathsf{L}^2(\Omega)}, \\ \|w\|^2_{\mathbf{H}^{\mathrm{m}}_{\mathrm{t}}(\Omega)} &:= \sum_{|\alpha| \le m} \left\|\partial^\alpha w\right\|^2_{\mathsf{L}^2_{\mathrm{t}}(\Omega)}, \\ \|w\|^2_{\mathsf{H}^{\mathrm{m}}_{\mathrm{t}}(\Omega)} &:= \sum_{|\alpha| \le m} \left\|\partial^\alpha w\right\|^2_{\mathsf{L}^2_{\mathrm{t}+|\alpha|}(\Omega)}, \end{aligned}$$

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1 Although the right-hand sides 0 and r^{-2} are $\mathsf{L}^2(\check{\mathrm{U}}(1))$-functions, we have $E = \xi/r \notin \mathsf{L}^2(\check{\mathrm{U}}(1))$, but $E \in \mathsf{L}^2_{-1}(\check{\mathrm{U}}(1))$.

$$\|E\|^2_{\mathbf{R}_t(\Omega)} := \|E\|^2_{\mathsf{L}^2_t(\Omega)} + \|\operatorname{rot} E\|^2_{\mathsf{L}^2_t(\Omega)}\,, \qquad \|E\|^2_{\mathsf{R}_t(\Omega)} := \|E\|^2_{\mathsf{L}^2_t(\Omega)} + \|\operatorname{rot} E\|^2_{\mathsf{L}^2_{t+1}(\Omega)}\,,$$
$$\|H\|^2_{\mathbf{D}_t(\Omega)} := \|H\|^2_{\mathsf{L}^2_t(\Omega)} + \|\operatorname{div} H\|^2_{\mathsf{L}^2_t(\Omega)}\,, \qquad \|H\|^2_{\mathsf{D}_t(\Omega)} := \|H\|^2_{\mathsf{L}^2_t(\Omega)} + \|\operatorname{div} H\|^2_{\mathsf{L}^2_{t+1}(\Omega)}\,,$$

they become Hilbert spaces. As usual, the subscript “loc” respectively “vox” indicates local square-integrability respectively bounded support. Please note, that the bold spaces with weight $t = 0$ correspond to the classical Lebesque and Sobolev spaces and for bounded domains “nonweighted” and weighted spaces even coincide:

$$\Omega \subset \mathbb{R}^3 \text{ bounded} \implies \forall\, t \in \mathbb{R}: \begin{cases} \mathbf{H}^1_t(\Omega) = \mathsf{H}^1_t(\Omega) = \mathsf{H}^1_0(\Omega) = \mathsf{H}^1(\Omega) \\ \mathbf{R}_t(\Omega) = \mathsf{R}_t(\Omega) = \mathsf{R}_0(\Omega) = \mathsf{H}(\operatorname{rot},\Omega) \\ \mathbf{D}_t(\Omega) = \mathsf{D}_t(\Omega) = \mathsf{D}_0(\Omega) = \mathsf{H}(\operatorname{div},\Omega) \end{cases}$$

Besides the usual set $\mathring{\mathsf{C}}^\infty(\Omega)$ of test fields (resp., test functions), we introduce

$$\mathsf{C}^\infty_{\Gamma_i}(\Omega) := \left\{ \varphi|_\Omega \;\middle|\; \varphi \in \mathring{\mathsf{C}}^\infty(\mathbb{R}^3) \text{ and } \operatorname{dist}(\operatorname{supp}\varphi, \Gamma_i) > 0 \right\}, \quad \mathrm{i} = 1,2$$

to formulate boundary conditions in the weak sense:

$$\begin{aligned} \mathbf{H}^m_{t,\Gamma_i}(\Omega) &:= \overline{\mathsf{C}^\infty_{\Gamma_i}(\Omega)}^{\|\cdot\|_{\mathbf{H}^m_t(\Omega)}}, \quad \mathbf{R}_{t,\Gamma_i}(\Omega) := \overline{\mathsf{C}^\infty_{\Gamma_i}(\Omega)}^{\|\cdot\|_{\mathbf{R}_t(\Omega)}}, \quad \mathbf{D}_{t,\Gamma_i}(\Omega) := \overline{\mathsf{C}^\infty_{\Gamma_i}(\Omega)}^{\|\cdot\|_{\mathbf{D}_t(\Omega)}}, \\ \mathsf{H}^m_{t,\Gamma_i}(\Omega) &:= \overline{\mathsf{C}^\infty_{\Gamma_i}(\Omega)}^{\|\cdot\|_{\mathsf{H}^m_t(\Omega)}}, \quad \mathsf{R}_{t,\Gamma_i}(\Omega) := \overline{\mathsf{C}^\infty_{\Gamma_i}(\Omega)}^{\|\cdot\|_{\mathsf{R}_t(\Omega)}}, \quad \mathsf{D}_{t,\Gamma_i}(\Omega) := \overline{\mathsf{C}^\infty_{\Gamma_i}(\Omega)}^{\|\cdot\|_{\mathsf{D}_t(\Omega)}}. \end{aligned} \tag{2.2}$$

These spaces indeed generalize vanishing scalar, tangential and normal Dirichlet boundary conditions even and in particular to boundaries for which the notion of a normal vector may not make any sense. Moreover, 0 at the lower left corner denotes vanishing rotation respectively divergence, e. g.,

$${}_0\mathsf{R}_t(\Omega) := \{E \in \mathsf{R}_t(\Omega) \,|\, \operatorname{rot} E = 0\}, \quad {}_0\mathsf{D}_{t,\Gamma_1}(\Omega) := \{H \in \mathsf{D}_{t,\Gamma_1}(\Omega) \,|\, \operatorname{div} H = 0\}, \quad \ldots,$$

and if $t = 0$ in any of the definitions given above, we will skip the weight, e. g.,

$$\mathbf{H}^m(\Omega) = \mathbf{H}^m_0(\Omega), \quad \mathsf{R}_{\Gamma_1}(\Omega) = \mathsf{R}_{0,\Gamma_1}(\Omega), \quad \mathbf{D}_{\Gamma_1}(\Omega) = \mathbf{D}_{0,\Gamma_1}(\Omega), \quad \ldots.$$

Finally we set

$$\mathsf{X}_{<s} := \bigcap_{t<s} \mathsf{X}_t \quad \text{and} \quad \mathsf{X}_{>s} := \bigcup_{t>s} \mathsf{X}_t \quad (s \in \mathbb{R}),$$

for X_t being any of the spaces above. If $\Omega = \mathbb{R}^3$ we omit the space reference, e.g.,

$$\mathbf{H}^m_t := \mathbf{H}^m_t(\mathbb{R}^3), \quad \mathsf{R}_{t,\Gamma_1} := \mathsf{R}_{t,\Gamma_1}(\mathbb{R}^3), \quad \mathbf{D}_t := \mathbf{D}_t(\mathbb{R}^3), \quad \mathsf{H}^m_{t,\Gamma_2} := \mathsf{H}^m_{t,\Gamma_2}(\mathbb{R}^3), \quad \ldots \quad .$$

The material parameters ε and μ are assumed to be κ-admissible in the following sense.

Definition 2.1. Let $\kappa \geq 0$. We call a transformation γ κ-admissible, if
- $\gamma : \Omega \longrightarrow \mathbb{R}^{3\times 3}$ is an L^∞-matrix field,
- γ is symmetric, i. e.,

$$\forall\, E, H \in \mathsf{L}^2(\Omega): \quad \langle\, E\,, \gamma H\,\rangle_{\mathsf{L}^2(\Omega)} = \langle\, \gamma E\,, H\,\rangle_{\mathsf{L}^2(\Omega)}\,,$$

- γ is uniformly positive definite, i. e.,

$$\exists\, c > 0\; \forall\, E \in \mathsf{L}^2(\Omega): \quad \langle\, E\,, \gamma E\,\rangle_{\mathsf{L}^2(\Omega)} \geq c \cdot \|E\|^2_{\mathsf{L}^2(\Omega)}\,,$$

- γ is asymptotically a multiple of the identity, i. e.,

$$\gamma = \gamma_0 \cdot \mathbb{1} + \hat{\gamma} \quad \text{with} \quad \gamma_0 \in \mathbb{R}_+ \quad \text{and} \quad \hat{\gamma} = \mathcal{O}(r^{-\kappa}) \quad \text{as} \quad r \longrightarrow \infty\,.$$

Then ε, μ are pointwise invertible and ε^{-1}, μ^{-1} defined by

$$\varepsilon^{-1}(x) := \big(\varepsilon(x)\big)^{-1} \quad \text{and} \quad \mu^{-1}(x) := \big(\mu(x)\big)^{-1}, \quad x \in \Omega\,,$$

are also κ-admissible. Moreover,

$$\langle\,\cdot\,,\cdot\,\rangle_\varepsilon := \langle\, \varepsilon\,\cdot\,,\cdot\,\rangle_{\mathsf{L}^2(\Omega)} \quad \text{and} \quad \langle\,\cdot\,,\cdot\,\rangle_\mu := \langle\, \mu\,\cdot\,,\cdot\,\rangle_{\mathsf{L}^2(\Omega)}$$

define scalar products on $\mathsf{L}^2(\Omega)$ inducing norms equivalent to the standard ones. Consequently,

$$\mathsf{L}^2_\varepsilon(\Omega) := \big(\mathsf{L}^2(\Omega), \langle\,\cdot\,,\cdot\,\rangle_\varepsilon\big), \quad \mathsf{L}^2_\mu(\Omega) := \big(\mathsf{L}^2(\Omega), \langle\,\cdot\,,\cdot\,\rangle_\mu\big), \quad \text{and} \quad \mathsf{L}^2_\Lambda(\Omega) := \mathsf{L}^2_\varepsilon(\Omega) \times \mathsf{L}^2_\mu(\Omega)$$

are Hilbert spaces and we denote by

$$\|\cdot\|_\varepsilon\,,\; \|\cdot\|_\mu\; \|\cdot\|_\Lambda\,, \quad \oplus_\varepsilon,\, \oplus_\mu,\, \oplus_\Lambda, \quad \text{and} \quad \perp_\varepsilon,\, \perp_\mu,\, \perp_\Lambda$$

the norm, the orthogonal sum and the orthogonal complement in these spaces. For further simplification and to shorten notation, we also introduce for $\varepsilon = \varepsilon_0 \cdot \mathbb{1} + \hat{\varepsilon}$ and $\mu = \mu_0 \cdot \mathbb{1} + \hat{\mu}$ (recalling $\xi(x) = x/r(x)$) the formal matrix operators

$$\Lambda := \begin{bmatrix} \varepsilon & 0 \\ 0 & \mu \end{bmatrix}, \qquad \Lambda^{-1} := \begin{bmatrix} \varepsilon^{-1} & 0 \\ 0 & \mu^{-1} \end{bmatrix}, \qquad \hat{\Lambda} := \begin{bmatrix} \hat{\varepsilon} & 0 \\ 0 & \hat{\mu} \end{bmatrix},$$

$$\Lambda\,(E,H) = (\varepsilon E, \mu H), \qquad \Lambda^{-1}\,(E,H) = (\varepsilon^{-1}E, \mu^{-1}H), \qquad \hat{\Lambda}\,(E,H) = (\hat{\varepsilon}E, \hat{\mu}H),$$

$$\Lambda_0 := \begin{bmatrix} \varepsilon_0 & 0 \\ 0 & \mu_0 \end{bmatrix}, \qquad \widetilde{\Lambda}_0 := \begin{bmatrix} \mu_0 & 0 \\ 0 & \varepsilon_0 \end{bmatrix}, \qquad \Xi := \begin{bmatrix} 0 & -\xi\times \\ \xi\times & 0 \end{bmatrix},$$

$$\Lambda_0\,(E,H) = (\varepsilon_0 E, \mu_0 H), \qquad \widetilde{\Lambda}_0\,(E,H) = (\mu_0 E, \varepsilon_0 H), \qquad \Xi\,(E,H) = (-\xi \times H, \xi \times E),$$

$$\mathrm{Rot} := \begin{bmatrix} 0 & -\,\mathrm{rot} \\ \mathrm{rot} & 0 \end{bmatrix}, \qquad \mathrm{M} := i\Lambda^{-1}\,\mathrm{Rot} = \begin{bmatrix} 0 & -i\varepsilon^{-1}\,\mathrm{rot} \\ i\mu^{-1}\,\mathrm{rot} & 0 \end{bmatrix},$$

$$\operatorname{Rot}(E,H) = (-\operatorname{rot}H, \operatorname{rot}E), \qquad \mathrm{M}(E,H) = (-i\varepsilon^{-1}\operatorname{rot}H, i\mu^{-1}\operatorname{rot}E).$$

We end this section with a lemma, showing that the spaces defined in (2.2) indeed generalize vanishing scalar, tangential and normal boundary conditions.

Lemma 2.2. *For $t \in \mathbb{R}$ and $i \in (1,2)$, the following inclusions hold:*

(a) $\mathsf{H}^{m}_{t,\Gamma_i}(\Omega) \subset \mathbf{H}^{m}_{t,\Gamma_i}(\Omega)$, $\quad \mathsf{R}_{t,\Gamma_i}(\Omega) \subset \mathbf{R}_{t,\Gamma_i}(\Omega)$, $\quad \mathsf{D}_{t,\Gamma_i}(\Omega) \subset \mathbf{D}_{t,\Gamma_i}(\Omega)$

(b) $\nabla\mathbf{H}^{1}_{t,\Gamma_i}(\Omega) \subset {}_0\mathsf{R}_{t,\Gamma_i}(\Omega)$, $\quad \nabla\mathsf{H}^{1}_{t,\Gamma_i}(\Omega) \subset {}_0\mathsf{R}_{t+1,\Gamma_i}(\Omega)$

(c) $\operatorname{rot}\mathbf{R}_{t,\Gamma_i}(\Omega) \subset {}_0\mathsf{D}_{t,\Gamma_i}(\Omega)$, $\quad \operatorname{rot}\mathsf{R}_{t,\Gamma_i}(\Omega) \subset {}_0\mathsf{D}_{t+1,\Gamma_i}(\Omega)$

Additionally, we have for $i,j \in (1,2)$, $i \neq j$:

$$\mathbf{H}^{1}_{t,\Gamma_i}(\Omega) = \boldsymbol{\mathcal{H}}^{1}_{t,\Gamma_i}(\Omega) := \big\{ w \in \mathbf{H}^{1}_{t}(\Omega) \;\big|\; \forall\,\Phi \in \mathsf{C}^{\infty}_{\Gamma_j}(\Omega): \;\; \langle w, \operatorname{div}\Phi\rangle_{\mathsf{L}^2(\Omega)} = -\langle \nabla w, \Phi\rangle_{\mathsf{L}^2(\Omega)} \big\},$$
$$\mathbf{R}_{t,\Gamma_i}(\Omega) = \boldsymbol{\mathcal{R}}_{t,\Gamma_i}(\Omega) := \big\{ E \in \mathbf{R}_{t}(\Omega) \;\big|\; \forall\,\Phi \in \mathsf{C}^{\infty}_{\Gamma_j}(\Omega): \;\; \langle E, \operatorname{rot}\Phi\rangle_{\mathsf{L}^2(\Omega)} = \langle \operatorname{rot}E, \Phi\rangle_{\mathsf{L}^2(\Omega)} \big\},$$
$$\mathbf{D}_{t,\Gamma_i}(\Omega) = \boldsymbol{\mathcal{D}}_{t,\Gamma_i}(\Omega) := \big\{ H \in \mathbf{D}_{t}(\Omega) \;\big|\; \forall\,\phi \in \mathsf{C}^{\infty}_{\Gamma_j}(\Omega): \;\; \langle H, \nabla\phi\rangle_{\mathsf{L}^2(\Omega)} = -\langle \operatorname{div}H, \phi\rangle_{\mathsf{L}^2(\Omega)} \big\},$$

and

$$\mathsf{H}^{1}_{t,\Gamma_i}(\Omega) = \mathcal{H}^{1}_{t,\Gamma_i}(\Omega) := \big\{ w \in \mathsf{H}^{1}_{t}(\Omega) \;\big|\; \forall\,\Phi \in \mathsf{C}^{\infty}_{\Gamma_j}(\Omega): \;\; \langle w, \operatorname{div}\Phi\rangle_{\mathsf{L}^2(\Omega)} = -\langle \nabla w, \Phi\rangle_{\mathsf{L}^2(\Omega)} \big\},$$
$$\mathsf{R}_{t,\Gamma_i}(\Omega) = \mathcal{R}_{t,\Gamma_i}(\Omega) := \big\{ E \in \mathsf{R}_{t}(\Omega) \;\big|\; \forall\,\Phi \in \mathsf{C}^{\infty}_{\Gamma_j}(\Omega): \;\; \langle E, \operatorname{rot}\Phi\rangle_{\mathsf{L}^2(\Omega)} = \langle \operatorname{rot}E, \Phi\rangle_{\mathsf{L}^2(\Omega)} \big\},$$
$$\mathsf{D}_{t,\Gamma_i}(\Omega) = \mathcal{D}_{t,\Gamma_i}(\Omega) := \big\{ H \in \mathsf{D}_{t}(\Omega) \;\big|\; \forall\,\phi \in \mathsf{C}^{\infty}_{\Gamma_j}(\Omega): \;\; \langle H, \nabla\phi\rangle_{\mathsf{L}^2(\Omega)} = -\langle \operatorname{div}H, \phi\rangle_{\mathsf{L}^2(\Omega)} \big\},$$

where (by continuity of the L^2-scalar product) we may also replace $\mathsf{C}^{\infty}_{\Gamma_j}(\Omega)$ by

$$\mathbf{H}^{1}_{s,\Gamma_j}(\Omega),\ \mathbf{R}_{s,\Gamma_j}(\Omega),\ \mathbf{D}_{s,\Gamma_j}(\Omega) \quad \textit{resp.} \quad \mathsf{H}^{1}_{s,\Gamma_j}(\Omega),\ \mathsf{R}_{s,\Gamma_j}(\Omega),\ \mathsf{D}_{s,\Gamma_j}(\Omega),$$

with $s + t \geq 0$ resp. $s + t \geq -1$.

Proof. As representatives of the arguments, we show

$$\text{(i)}\ \operatorname{rot}\mathbf{R}_{t,\Gamma_2}(\Omega) \subset {}_0\mathsf{D}_{t,\Gamma_2}(\Omega) \quad \text{and} \quad \text{(ii)}\ \mathbf{R}_{t,\Gamma_1}(\Omega) = \boldsymbol{\mathcal{R}}_{t,\Gamma_1}(\Omega).$$

For $E \in \operatorname{rot}\mathbf{R}_{t,\Gamma_2}(\Omega)$, there exists a sequence $(\mathcal{E}_n)_{n\in\mathbb{N}} \subset \mathsf{C}^{\infty}_{\Gamma_2}(\Omega)$ such that $\operatorname{rot}\mathcal{E}_n \longrightarrow E$ in $\mathsf{L}^2_t(\Omega)$. Then

$$\forall\,\phi \in \mathring{\mathsf{C}}^{\infty}(\Omega): \quad \langle E, \nabla\phi \rangle_{\mathsf{L}^2(\Omega)} = \lim_{n\to\infty} \langle \operatorname{rot} \mathcal{E}_n, \nabla\phi \rangle_{\mathsf{L}^2(\Omega)} \\ = -\lim_{n\to\infty} \langle \operatorname{div}(\operatorname{rot} \mathcal{E}_n), \phi \rangle_{\mathsf{L}^2(\Omega)} = 0,$$

hence E has vanishing divergence and $(E_n)_{n\in\mathbb{N}}$ defined by $E_n := \operatorname{rot} \mathcal{E}_n$ satisfies

$$(E_n)_{n\in\mathbb{N}} \subset \mathsf{C}^{\infty}_{\Gamma_2}(\Omega), \quad E_n \xrightarrow{\mathsf{L}^2_t(\Omega)} E \quad \text{and} \quad \operatorname{div} E_n = \operatorname{div}(\operatorname{rot} \mathcal{E}_n) = 0 \xrightarrow{\mathsf{L}^2_{t+1}(\Omega)} 0 .$$

Thus $E \in {}_0\mathsf{D}_{t,\Gamma_2}(\Omega)$, showing (i). Let us show (ii). We have $\mathbf{R}_{t,\Gamma_1}(\Omega) \subset \mathcal{R}_{t,\Gamma_1}(\Omega)$. For the other direction, let $E \in \mathcal{R}_{t,\Gamma_1}(\Omega)$ and $\delta > 0$. Using the cut-off function from above we define $(E_k)_{k\in\mathbb{N}}$ by $E_k := \eta_k E$. Then $E_k \in \mathcal{R}_{\tilde{\Gamma}_1}(\Omega(2r_k))$, $\tilde{\Gamma}_1 := \Gamma_1 \cup \mathsf{S}(2r_k)$, since for $\Phi \in \mathsf{C}^{\infty}_{\Gamma_2}(\Omega(2r_k))$ it holds by $\eta_k\Phi \in \mathsf{C}^{\infty}_{\Gamma_2}(\Omega)$

$$\langle E_k, \operatorname{rot}\Phi \rangle_{\mathsf{L}^2(\Omega(2r_k))} = \langle \eta_k E, \operatorname{rot}\Phi \rangle_{\mathsf{L}^2(\Omega)} \\ = \langle E, \operatorname{rot}(\eta_k\Phi) \rangle_{\mathsf{L}^2(\Omega)} - \langle E, \nabla\eta_k \times \Phi \rangle_{\mathsf{L}^2(\Omega)} \\ = \langle \eta_k \operatorname{rot} E + \nabla\eta_k \times E, \Phi \rangle_{\mathsf{L}^2(\Omega(2r_k))} = \langle \operatorname{rot} E_k, \Phi \rangle_{\mathsf{L}^2(\Omega(2r_k))} .$$

By means of monotone convergence, we have[2]

$$\|E - E_k\|_{\mathbf{R}_t(\Omega)} = \|\check{\eta}_k E\|_{\mathbf{R}_t(\Omega)} \le c \cdot \Big(\|E\|_{\mathbf{R}_t(\check{\mathsf{U}}(r_k))} + \frac{1}{2^k} \cdot \|E\|_{\mathsf{L}^2_t(\Omega)} \Big) \to 0,$$

hence we can choose $\hat{k} > 0$ such that $\|E - E_{\hat{k}}\|_{\mathbf{R}_t(\Omega)} < \delta/2$. As $\Omega(2r_{\hat{k}}) = \Omega \cap \mathsf{U}(2r_{\hat{k}})$ is a bounded weak Lipschitz domain, we obtain $\mathcal{R}_{\tilde{\Gamma}_1}(\Omega(2r_{\hat{k}})) = \mathbf{R}_{\tilde{\Gamma}_1}(\Omega(2r_{\hat{k}}))$ by [1, Section 3.3], yielding the existence of some $\Psi \in \mathsf{C}^{\infty}_{\tilde{\Gamma}_1}(\Omega(2r_{\hat{k}}))$ such that

$$\|E_{\hat{k}} - \Psi\|_{\mathbf{R}_t(\Omega(2r_{\hat{k}}))} \le c \cdot \|E_{\hat{k}} - \Psi\|_{\mathbf{R}(\Omega(2r_{\hat{k}}))} < \delta/2 .$$

Extending Ψ by zero to Ω, we obtain (by abuse of notation) $\Psi \in \mathsf{C}^{\infty}_{\Gamma_1}(\Omega)$ with

$$\|E - \Psi\|_{\mathbf{R}_t(\Omega)} \le \|E - E_{\hat{k}}\|_{\mathbf{R}_t(\Omega)} + \|E_{\hat{k}} - \Psi\|_{\mathbf{R}_t(\Omega(2r_{\hat{k}}))} < \frac{\delta}{2} + \frac{\delta}{2} = \delta,$$

which completes the proof. □

2.2 Some functional analysis

Let H_1 and H_2 be Hilbert spaces and let $\mathrm{A} : \mathcal{D}(\mathrm{A}) \subset \mathsf{H}_1 \longrightarrow \mathsf{H}_2$ be a linear, densely defined, and closed linear operator with the adjoint $\mathrm{A}^* : \mathcal{D}(\mathrm{A}^*) \subset \mathsf{H}_2 \longrightarrow \mathsf{H}_1$, which is then linear, densely defined, and closed as well. Note that A^* is characterized by

$$\langle \mathrm{A}x, y \rangle_{\mathsf{H}_2} = \langle x, \mathrm{A}^* y \rangle_{\mathsf{H}_1} \quad \forall\, x \in \mathcal{D}(\mathrm{A}),\ y \in \mathcal{D}(\mathrm{A}^*) .$$

2 Here and hereafter, $c > 0$ denotes some generic constant.

By the projection theorem, we have the following Helmholtz-type decompositions:

$$\mathsf{H}_1 = \overline{\mathcal{R}(\mathrm{A}^*)} \oplus \mathcal{N}(\mathrm{A}), \quad \text{and} \quad \mathsf{H}_2 = \overline{\mathcal{R}(\mathrm{A})} \oplus \mathcal{N}(\mathrm{A}^*),$$

which propose the corresponding reduced operators $\mathcal{A} := \mathrm{A}|_{\mathcal{N}(\mathrm{A})^\perp}$, $\mathcal{A}^* := \mathrm{A}^*|_{\mathcal{N}(\mathrm{A}^*)^\perp}$, i. e.,

$$\begin{aligned} \mathcal{A} : \mathcal{D}(\mathcal{A}) \subset \overline{\mathcal{R}(\mathrm{A}^*)} &\longrightarrow \overline{\mathcal{R}(\mathrm{A})}, \\ \mathcal{D}(\mathcal{A}) = \mathcal{D}(\mathrm{A}) \cap \overline{\mathcal{R}(\mathrm{A}^*)}&, \end{aligned} \quad \text{resp.} \quad \begin{aligned} \mathcal{A}^* : \mathcal{D}(\mathcal{A}^*) \subset \overline{\mathcal{R}(\mathrm{A})} &\longrightarrow \overline{\mathcal{R}(\mathrm{A}^*)}, \\ \mathcal{D}(\mathcal{A}^*) = \mathcal{D}(\mathrm{A}^*) \cap \overline{\mathcal{R}(\mathrm{A})}&. \end{aligned}$$

These operators are also closed, densely defined and indeed adjoint to each other. Moreover, by definition $\mathcal{A}$ and $\mathcal{A}^*$ are injective and, therefore, the inverse operators

$$\mathcal{A}^{-1} : \mathcal{R}(\mathrm{A}) \longrightarrow \mathcal{D}(\mathcal{A}) \quad \text{and} \quad (\mathcal{A}^*)^{-1} : \mathcal{R}(\mathrm{A}^*) \longrightarrow \mathcal{D}(\mathcal{A}^*)$$

exist. The pair $(\mathcal{A}, \mathcal{A}^*)$ satisfies the following result of the so-called *Functional Analysis Toolbox* (see, e. g., [17, Section 2]), from which we will derive some Poincaré-type estimates for the time-harmonic Maxwell operator $(\mathcal{M} - \omega)$ (cf. Remark 3.11 and Remark 3.7).

Lemma 2.3. *The following assertions are equivalent:*
(1) $\exists\, c_{\mathrm{A}} \in (0,\infty) \;\; \forall\, x \in \mathcal{D}(\mathcal{A})$: $\quad \|x\|_{\mathsf{H}_1} \le c_{\mathrm{A}} \, \|\mathrm{A}\, x\|_{\mathsf{H}_2}$.
(1*) $\exists\, c_{\mathrm{A}^*} \in (0,\infty) \;\; \forall\, y \in \mathcal{D}(\mathcal{A}^*)$: $\quad \|y\|_{\mathsf{H}_2} \le c_{\mathrm{A}^*} \, \|\mathrm{A}^*\, y\|_{\mathsf{H}_1}$.
(2) $\mathcal{R}(\mathrm{A}) = \mathcal{R}(\mathcal{A})$ *is closed in* H_2.
(2*) $\mathcal{R}(\mathrm{A}^*) = \mathcal{R}(\mathcal{A}^*)$ *ist closed in* H_1.
(3) $\mathcal{A}^{-1} : \mathcal{R}(\mathrm{A}) \longrightarrow \mathcal{D}(\mathcal{A})$ *is continuous.*
(3*) $(\mathcal{A}^*)^{-1} : \mathcal{R}(\mathrm{A}^*) \longrightarrow \mathcal{D}(\mathcal{A}^*)$ *is continuous.*

Note that for the “best” constants c_{A} and c_{A^*} it holds

$$\left\|\mathcal{A}^{-1}\right\|_{\mathcal{R}(\mathrm{A}),\mathcal{R}(\mathrm{A}^*)} = c_{\mathrm{A}} = c_{\mathrm{A}^*} = \left\|(\mathcal{A}^*)^{-1}\right\|_{\mathcal{R}(\mathrm{A}^*),\mathcal{R}(\mathrm{A})} .$$

3 Solution theory for time-harmonic Maxwell equations

As mentioned above, we shall treat the time-harmonic Maxwell equations with mixed boundary conditions

$$\begin{aligned} -\operatorname{rot} H + i\omega\varepsilon E &= F \text{ in } \Omega, & E \times \nu &= 0 \quad \text{on } \Gamma_1, \\ \operatorname{rot} E + i\omega\mu H &= G \text{ in } \Omega, & H \times \nu &= 0 \quad \text{on } \Gamma_2, \end{aligned} \tag{3.1}$$

in an exterior weak Lipschitz domain $\Omega \subset \mathbb{R}^3$ and for frequencies $\omega \in \mathbb{C}\setminus(0)$. Moreover, we suppose that the material parameters ε and μ are κ-admissible with $\kappa \geq 0$. Using the abbreviations from above and rewriting

$$u := (E, H), \qquad f := i\Lambda^{-1}(-F, G),$$

the weak formulation of these boundary value problem reads:

$$\textit{For } f \in \mathsf{L}^2_{\mathrm{loc}}(\overline{\Omega}) \textit{ find } u \in \mathsf{R}_{\mathrm{loc},\Gamma_1}(\overline{\Omega}) \times \mathsf{R}_{\mathrm{loc},\Gamma_2}(\overline{\Omega}) \textit{ such that } (\mathrm{M} - \omega)\, u = f. \tag{3.2}$$

We shall solve this problem using polynomially weighted Hilbert spaces. In doing so, we avoid additional assumptions on boundary regularity for Ω, since only a compactness result comparable to Rellich's selection theorem is needed. More precisely, we will show that Ω satisfies *"Weck's (local) selection theorem"*, also called "(*local*) *Maxwell compactness property"*, which in fact is also an assumption on the quality of the boundary and in some sense supersedes assumptions on boundary regularity.

Definition 3.1. Let γ be κ-admissible with $\kappa \geq 0$ and let $\Omega \subset \mathbb{R}^3$ be open. Ω satisfies "Weck's local selection theorem" (WLST) (or has the "local Maxwell compactness property"), if the embedding

$$\mathbf{R}_{\Gamma_1}(\Omega) \cap \gamma^{-1}\mathbf{D}_{\Gamma_2}(\Omega) \hookrightarrow \mathsf{L}^2_{\mathrm{loc}}(\overline{\Omega}) \tag{3.3}$$

is compact. Ω satisfies "Weck's selection theorem" (WST) (or has the "Maxwell compactness property") if the embedding

$$\mathbf{R}_{\Gamma_1}(\Omega) \cap \gamma^{-1}\mathbf{D}_{\Gamma_2}(\Omega) \hookrightarrow \mathsf{L}^2(\Omega) \tag{3.4}$$

is compact.

Remark 3.2. Note that Weck's (local) selection theorem is essentially independent of γ meaning that a domain $\Omega \subset \mathbb{R}^3$ satisfies WST respectively WLST, if and only if the imbedding

$$\mathbf{R}_{\Gamma_1}(\Omega) \cap \mathbf{D}_{\Gamma_2}(\Omega) \hookrightarrow \mathsf{L}^2(\Omega) \quad \text{resp.} \quad \mathbf{R}_{\Gamma_1}(\Omega) \cap \mathbf{D}_{\Gamma_2}(\Omega) \hookrightarrow \mathsf{L}^2_{\mathrm{loc}}(\overline{\Omega})$$

is compact. The proof is practically identical with the one of [19, Lemma 2] (see also [24, 22]).

Lemma 3.3. *Let γ be κ-admissible with $\kappa \geq 0$ and let $\Omega \subset \mathbb{R}^3$ be an exterior domain. Then the following statements are equivalent:*
(a) *Ω satisfies WLST.*
(b) *For all $\tilde{r} > r_{0,}$ the imbedding*

$$\mathbf{R}_{\tilde{\Gamma}_1}(\Omega(\tilde{r})) \cap \gamma^{-1}\mathbf{D}_{\Gamma_2}(\Omega(\tilde{r})) \hookrightarrow \mathsf{L}^2(\Omega(\tilde{r}))$$

with $\tilde{\Gamma}_1 := \Gamma_1 \cup \mathrm{S}(\tilde{r})$ is compact, i.e., $\Omega(\tilde{r})$ satisfies WST.

(c) *For all $\tilde r > r_0$, the imbedding*

$$\mathsf{R}_{\Gamma_1}(\Omega(\tilde r)) \cap \gamma^{-1}\mathsf{D}_{\tilde\Gamma_2}(\Omega(\tilde r)) \hookrightarrow \mathsf{L}^2(\Omega(\tilde r))$$

with $\tilde\Gamma_2 := \Gamma_2 \cup \mathrm{S}(\tilde r)$ is compact, i.e., $\Omega(\tilde r)$ satisfies WST.

(d) *For all $s, t \in \mathbb{R}$ with $t < s$, the imbedding*

$$\mathsf{R}_{s,\Gamma_1}(\Omega) \cap \gamma^{-1}\mathsf{D}_{s,\Gamma_2}(\Omega) \hookrightarrow \mathsf{L}^2_t(\Omega)$$

is compact.

Proof. (a)⇒(b): Let $\tilde r > r_0$. By Remark 3.2, it is sufficient to show the compactness of

$$\mathsf{R}_{\tilde\Gamma_1}(\Omega(\tilde r)) \cap \mathsf{D}_{\Gamma_2}(\Omega(\tilde r)) \hookrightarrow \mathsf{L}^2(\Omega(\tilde r)) .$$

Therefore, let $(E_n)_{n\in\mathbb{N}} \subset \mathsf{R}_{\tilde\Gamma_1}(\Omega(\tilde r)) \cap \mathsf{D}_{\Gamma_2}(\Omega(\tilde r))$ be bounded, choose $r_0 < \hat r < \tilde r$ and a cut-off function $\chi \in \mathring{\mathsf{C}}^\infty(\mathbb{R}^3)$ with $\operatorname{supp}\chi \subset \mathrm{U}(\tilde r)$ and $\chi|_{\mathrm{B}(\hat r)} = 1$. Then, for every $n \in \mathbb{N}$ we have

$$E_n = \check E_n + \hat E_n := \chi E_n + (1-\chi)E_n , \quad \operatorname{supp}\check E_n \subset \Omega(\tilde r) , \quad \operatorname{supp}\hat E_n \subset \mathrm{G}(\hat r, \tilde r) ,$$

splitting $(E_n)_{n\in\mathbb{N}}$ into $(\check E_n)_{n\in\mathbb{N}}$ and $(\hat E_n)_{n\in\mathbb{N}}$. Extending $\check E_n$ respectively $\hat E_n$ by zero, we obtain (by abuse of notation) sequences

$$(\check E_n)_{n\in\mathbb{N}} \subset \mathsf{R}_{\Gamma_1}(\Omega) \cap \mathsf{D}_{\Gamma_2}(\Omega) \quad \text{and} \quad (\hat E_n)_{n\in\mathbb{N}} \subset \mathsf{R}_{\mathrm{S}(\tilde r)}(\mathrm{U}(\tilde r)) \cap \mathsf{D}(\mathrm{U}(\tilde r))$$

which are bounded in the respective spaces. Thus, using Weck's local selection theorem and Remark 3.2, we can choose a subsequence $(\check E_{\pi(n)})_{n\in\mathbb{N}}$ of $(\check E_n)_{n\in\mathbb{N}}$ converging in $\mathsf{L}^2_{\mathrm{loc}}(\overline{\Omega})$. The corresponding subsequence $(\hat E_{\pi(n)})_{n\in\mathbb{N}}$ is of course also bounded in $\mathsf{R}_{\mathrm{S}(\tilde r)}(\mathrm{U}(\tilde r)) \cap \mathsf{D}(\mathrm{U}(\tilde r))$ and by [23, Theorem 2.2], even in $\mathsf{H}^1(\mathrm{U}(\tilde r))$, hence (Rellich's selection theorem) has a subsequence $(\hat E_{\tilde\pi(n)})_{n\in\mathbb{N}}$ converging in $\mathsf{L}^2(\mathrm{U}(\tilde r))$. Thus

$$\begin{aligned}
&\|E_{\tilde\pi(n)} - E_{\tilde\pi(m)}\|_{\mathsf{L}^2(\Omega(\tilde r))} \\
&\quad\le c\cdot\Big(\|\chi(E_{\tilde\pi(n)} - E_{\tilde\pi(m)})\|_{\mathsf{L}^2(\Omega(\tilde r))} + \|(1-\chi)(E_{\tilde\pi(n)} - E_{\tilde\pi(m)})\|_{\mathsf{L}^2(\Omega(\tilde r))}\Big) \\
&\quad\le c\cdot\Big(\|\check E_{\tilde\pi(n)} - \check E_{\tilde\pi(m)}\|_{\mathsf{L}^2(\Omega(\tilde r))} + \|\hat E_{\tilde\pi(n)} - \hat E_{\tilde\pi(m)}\|_{\mathsf{L}^2(\mathrm{U}(\tilde r))}\Big) \xrightarrow{m,n\to\infty} 0 ,
\end{aligned}$$

meaning that $(E_{\tilde\pi(n)})_{n\in\mathbb{N}} \subset (E_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathsf{L}^2(\Omega(\tilde r))$.

(b)⇒(d): Let $s, t \in \mathbb{R}$ with $s > t$ and let $(E_n)_{n\in\mathbb{N}} \subset \mathsf{R}_{s,\Gamma_1}(\Omega) \cap \gamma^{-1}\mathsf{D}_{s,\Gamma_2}(\Omega)$ be bounded. Then there exists a subsequence $(E_{\pi(n)})_{n\in\mathbb{N}} \subset (E_n)_{n\in\mathbb{N}}$ which converges weakly in $\mathsf{R}_{s,\Gamma_1}(\Omega) \cap \gamma^{-1}\mathsf{D}_{s,\Gamma_2}(\Omega)$ to some vector field $E \in \mathsf{R}_{s,\Gamma_1}(\Omega) \cap \gamma^{-1}\mathsf{D}_{s,\Gamma_2}(\Omega)$. We now construct a subsequence $(E_{\tilde\pi(n)})_{n\in\mathbb{N}}$ of $(E_{\pi(n)})_{n\in\mathbb{N}}$ converging in $\mathsf{L}^2_{\mathrm{loc}}(\overline{\Omega})$ to the same limit E. For this, observe that

$$(E_{\pi(n),1})_{n\in\mathbb{N}} \quad \text{with} \quad E_{\pi(n),1} := \eta_1 E_{\pi(n)}$$

is bounded in $\mathsf{R}_{\tilde{\Gamma}_1}(\Omega(r_2)) \cap \gamma^{-1}\mathsf{D}_{\Gamma_2}(\Omega(r_2))$, $\tilde{\Gamma}_1 := \Gamma_1 \cup \mathrm{S}(r_2)$ such that by assumption there exists a subsequence $(E_{\pi_1(n),1})_{n\in\mathbb{N}}$ converging in $\mathsf{L}^2(\Omega(r_2))$. Then $(E_{\pi_1(n)})_{n\in\mathbb{N}} \subset (E_{\pi(n)})_{n\in\mathbb{N}}$ is converging in $\mathsf{L}^2(\Omega(r_1))$, and, as $(E_{\pi_1(n)})_{n\in\mathbb{N}}$ is also weakly convergent in $\mathsf{L}^2(\Omega(r_1))$, we have

$$E_{\pi_1(n)} \longrightarrow E \quad \text{in } \mathsf{L}^2(\Omega(r_1)) .$$

Multiplying $(E_{\pi_1(n)})_{n\in\mathbb{N}}$ with η_2, we obtain a sequence $(E_{\pi_1(n),2})_{n\in\mathbb{N}}$, $E_{\pi_1(n),2} := \eta_2 E_{\pi_1(n)}$ bounded in $\mathsf{R}_{\tilde{\Gamma}_1}(\Omega(r_3)) \cap \gamma^{-1}\mathsf{D}_{\Gamma_2}(\Omega(r_3))$, $\tilde{\Gamma}_1 := \Gamma_1 \cup \mathrm{S}(r_3)$, and, as before, we construct a subsequence $(E_{\pi_2(n),2})_{n\in\mathbb{N}}$ converging in $\mathsf{L}^2(\Omega(r_3))$, giving again a converging subsequence $(E_{\pi_2(n)})_{n\in\mathbb{N}} \subset (E_{\pi_1(n)})_{n\in\mathbb{N}}$ with

$$E_{\pi_2(n)} \longrightarrow E \quad \text{in } \mathsf{L}^2(\Omega(r_2)) .$$

Continuing like this, we successively construct converging subsequences $(E_{\pi_k(n)})_{n\in\mathbb{N}}$ with $E_{\pi_k(n)} \longrightarrow E$ in $\mathsf{L}^2(\Omega(r_k))$ and switching to the diagonal sequence we indeed end up with a sequence $(E_{\tilde{\pi}(n)})_{n\in\mathbb{N}}$, $\tilde{\pi}(n) := \pi_n(n)$, with $E_{\tilde{\pi}(n)} \longrightarrow E$ in $\mathsf{L}^2_{\mathrm{loc}}(\overline{\Omega})$. Now Lemma A.1 implies for arbitrary $\theta > 0$

$$\|E_{\tilde{\pi}(n)} - E\|_{\mathsf{L}^2_t(\Omega)} \leq c \cdot \|E_{\tilde{\pi}(n)} - E\|_{\mathsf{L}^2(\Omega(\delta))} + \theta ,$$

with $c, \delta \in (0, \infty)$ independent of $E_{\tilde{\pi}(n)}$. Hence

$$\limsup_{n\to\infty} \|E_{\tilde{\pi}(n)} - E\|_{\mathsf{L}^2_t(\Omega)} \leq \theta ,$$

and we obtain $E_{\tilde{\pi}(n)} \longrightarrow E$ in $\mathsf{L}^2_t(\Omega)$.

(d)⇒(a): For $(E_n)_{n\in\mathbb{N}}$ bounded in $\mathsf{R}_{\Gamma_1}(\Omega) \cap \gamma^{-1}\mathsf{D}_{\Gamma_2}(\Omega)$, assertion (c) implies the existence of a subsequence $(E_{\pi(n)})_{n\in\mathbb{N}}$ converging in $\mathsf{L}^2_{-1}(\Omega)$ to some $E \in \mathsf{L}^2_{-1}(\Omega)$. Then $E \in \mathsf{L}^2_{\mathrm{loc}}(\overline{\Omega})$ and as

$$\forall\, \tilde{r} > 0 : \quad \|E_{\pi(n)} - E\|_{\mathsf{L}^2(\Omega(\tilde{r}))} \leq (1 + \tilde{r})^{1/2} \cdot \|E_{\pi(n)} - E\|_{\mathsf{L}^2_{-1}(\Omega)} ,$$

we obtain $(E_{\pi(n)})_{n\in\mathbb{N}} \longrightarrow E$ in $\mathsf{L}^2_{\mathrm{loc}}(\overline{\Omega})$.

Similar arguments to those corresponding to (b) show the assertion for (c). □

As shown by Bauer, Pauly, and Schomburg [1, Theorem 4.7], bounded weak Lipschitz domains satisfy Weck's selection theorem and by Lemma 3.3 (a) this directly implies the following.

Theorem 3.4. *Exterior weak Lipschitz domains satisfy Weck's local selection theorem.*

Returning to our initial question, a first step to a solution theory for (3.2) is the following observation.

Theorem 3.5. *The* Maxwell operator

$$\mathcal{M} : \mathbf{R}_{\Gamma_1}(\Omega) \times \mathbf{R}_{\Gamma_2}(\Omega) \subset \mathsf{L}^2_\Lambda(\Omega) \longrightarrow \mathsf{L}^2_\Lambda(\Omega),\ u \longmapsto \mathrm{M}u,$$

is self-adjoint and reduced by the closure of its range

$$\overline{\mathcal{R}(\mathcal{M})} = \varepsilon^{-1}\overline{\operatorname{rot}\mathbf{R}_{\Gamma_2}(\Omega)} \times \mu^{-1}\overline{\operatorname{rot}\mathbf{R}_{\Gamma_1}(\Omega)}.$$

We note that here, in the case of an exterior domain Ω, the respective ranges are not closed.

Proof. The proof is straightforward using Lemma 2.2, i. e., the equivalence of the definition of weak and strong boundary conditions. □

Thus $\sigma(\mathcal{M}) \subset \mathbb{R}$, meaning that every $\omega \in \mathbb{C} \setminus \mathbb{R}$ is contained in the resolvent set of $\mathcal{M}$ and for given $f \in \mathsf{L}^2_\Lambda(\Omega)$ we obtain a unique solution of (3.2) by

$$u := (\mathcal{M} - \omega)^{-1} f \in \mathbf{R}_{\Gamma_1}(\Omega) \times \mathbf{R}_{\Gamma_2}(\Omega).$$

Moreover, using the resolvent estimate $\|(\mathcal{M} - \omega)^{-1}\| \leq |\operatorname{Im}\omega|^{-1}$ and the differential equation, we get

$$\|u\|_{\mathbf{R}(\Omega)} \leq c \cdot \Big(\|u\|_{\mathsf{L}^2_\Lambda(\Omega)} + \|f\|_{\mathsf{L}^2_\Lambda(\Omega)} + |\omega|\,\|u\|_{\mathsf{L}^2_\Lambda(\Omega)} \Big) \leq c \cdot \frac{1 + |\omega|}{|\operatorname{Im}\omega|} \cdot \|f\|_{\mathsf{L}^2_\Lambda(\Omega)}.$$

Theorem 3.6. *For $\omega \in \mathbb{C} \setminus \mathbb{R}$, the solution operator*

$$\mathcal{L}_\omega := (\mathcal{M} - \omega)^{-1} : \mathsf{L}^2_\Lambda(\Omega) \longrightarrow \mathbf{R}_{\Gamma_1}(\Omega) \times \mathbf{R}_{\Gamma_2}(\Omega)$$

is continuous with $\|\mathcal{L}_\omega\|_{\mathsf{L}^2_\Lambda(\Omega),\mathbf{R}(\Omega)} \leq c \cdot \frac{1+|\omega|}{|\operatorname{Im}\omega|}$, where c is independent of ω and f.

Remark 3.7. Let $\omega \in \mathbb{C} \setminus \mathbb{R}$. By Lemma 2.3, the following statements are equivalent to the boundedness of $\mathcal{L}_\omega$:

- (Friedrichs/Poincaré-type estimate) There exists $c > 0$ such that

$$\|u\|_{\mathbf{R}(\Omega)} \leq c \cdot \|(\mathrm{M} - \omega)u\|_{\mathsf{L}^2_\Lambda(\Omega)} \quad \forall\, u \in \mathbf{R}_{\Gamma_1}(\Omega) \times \mathbf{R}_{\Gamma_2}(\Omega).$$

- (Closed range) The range

$$\mathcal{R}(\mathcal{M} - \omega) = (\mathcal{M} - \omega)\big(\mathbf{R}_{\Gamma_1}(\Omega) \times \mathbf{R}_{\Gamma_2}(\Omega)\big)$$

  is closed in $\mathsf{L}^2_\Lambda(\Omega)$.

The case $\omega \in \mathbb{R} \setminus (0)$ is much more challenging, since we want to solve in the continuous spectrum of the Maxwell operator. Clearly, this cannot be done for every $f \in \mathsf{L}^2_\Lambda(\Omega)$, since otherwise we would have $\mathcal{R}(\mathcal{M} - \omega) = \mathsf{L}^2_\Lambda(\Omega)$ and, therefore, $(\mathcal{M} - \omega)^{-1}$ would be continuous (cf. Lemma 2.3) or in other words $\omega \notin \sigma(\mathcal{M})$. Thus we have to restrict ourselves to certain subspaces of L^2_Λ or generalize our solution concept. Actually, we will do both and show existence as well as uniqueness of weaker, so-called *"radiating solutions,"* by switching to data $f \in \mathsf{L}^2_s(\Omega)$ for some $s > 1/2$.

Definition 3.8. Let $\omega \in \mathbb{R} \setminus (0)$ and $f \in \mathsf{L}^2_{\text{loc}}(\Omega)$. We call u (radiating) solution of (3.2), if

$$u \in \mathbf{R}_{<-\frac{1}{2},\Gamma_1}(\Omega) \times \mathbf{R}_{<-\frac{1}{2},\Gamma_2}(\Omega)$$

and

$$(\mathrm{M} - \omega)u = f\,, \tag{3.5}$$

$$(\Lambda_0 + \sqrt{\varepsilon_0\mu_0}\,\Xi)u \in \mathsf{L}^2_{>-\frac{1}{2}}(\Omega)\,. \tag{3.6}$$

Remark 3.9. Since

$$(\Lambda_0 + \sqrt{\varepsilon_0\mu_0}\,\Xi)u = \Lambda_0\Big(E - \sqrt{\frac{\mu_0}{\varepsilon_0}}\,\xi \times H\,,\, H + \sqrt{\frac{\varepsilon_0}{\mu_0}}\xi \times E\Big)\,,$$

the last condition is just the classical Silver–Müller radiation condition which describes the behavior of the electro-magnetic field at infinity and is needed to distinguish outgoing from incoming waves (interchanging signs would yield incoming waves).

In order to construct such a radiating solution u, we use the *"limiting absorption principle"* introduced by Eidus and approximate u by solutions $(u_n)_{n\in\mathbb{N}}$ associated with frequencies $(\omega_n)_{n\in\mathbb{N}} \subset \mathbb{C} \setminus \mathbb{R}$ converging to $\omega \in \mathbb{R} \setminus (0)$. This leads to statement (4) of our main result Theorem 3.10, where the following abbreviations are used:

$$\begin{aligned}\mathcal{N}_{\text{gen}}(\mathcal{M} - \omega) &:= \{u \,|\, u \text{ is a radiating solution of } (\mathrm{M} - \omega)u = 0\}\\ &\qquad (\textit{generalized kernel of } \mathcal{M} - \omega)\,,\\ \sigma_{\text{gen}}(\mathcal{M}) &:= \{\omega \in \mathbb{C} \setminus (0) \,|\, \mathcal{N}_{\text{gen}}(\mathcal{M} - \omega) \neq (0)\}\\ &\qquad (\textit{generalized point spectrum of } \mathcal{M})\,.\end{aligned}$$

Theorem 3.10 (Fredholm alternative). *Let $\Omega \subset \mathbb{R}^3$ be an exterior weak Lipschitz domain with boundary Γ and weak Lipschitz boundary parts Γ_1 and $\Gamma_2 = \Gamma \setminus \overline{\Gamma}_1$. Furthermore, let $\omega \in \mathbb{R} \setminus (0)$ and ε, μ be κ-admissible with $\kappa > 1$. Then:*

(1) $\mathcal{N}_{\text{gen}}(\mathcal{M} - \omega) \subset \bigcap_{t\in\mathbb{R}} \big(\mathbf{R}_{t,\Gamma_1}(\Omega) \cap \varepsilon^{-1}{}_0\mathsf{D}_{t,\Gamma_2}(\Omega)\big) \times \big(\mathbf{R}_{t,\Gamma_2}(\Omega) \cap \mu^{-1}{}_0\mathsf{D}_{t,\Gamma_1}(\Omega)\big)$.[3]
(2) $\dim \mathcal{N}_{\text{gen}}(\mathcal{M} - \omega) < \infty$.
(3) *$\sigma_{\text{gen}}(\mathcal{M}) \subset \mathbb{R} \setminus (0)$ and $\sigma_{\text{gen}}(\mathcal{M})$ has no accumulation point in $\mathbb{R} \setminus (0)$.*
(4) *For all $f \in \mathsf{L}^2_{>\frac{1}{2}}(\Omega)$ there exists a radiating solution u of (3.2), if and only if*

$$\forall\, v \in \mathcal{N}_{\text{gen}}(\mathcal{M} - \omega): \quad \langle f, v\rangle_{\mathsf{L}^2_\Lambda(\Omega)} = 0\,. \tag{3.7}$$

3 We even have

$$\mathcal{N}_{\text{gen}}(\mathcal{M} - \omega) \subset \bigcap_{t\in\mathbb{R}} \big(\mathbf{R}_{t,\Gamma_1}(\Omega) \cap \varepsilon^{-1} \operatorname{rot} \mathbf{R}_{t,\Gamma_2}(\Omega)\big) \times \big(\mathbf{R}_{t,\Gamma_2}(\Omega) \cap \mu^{-1} \operatorname{rot} \mathbf{R}_{t,\Gamma_1}(\Omega)\big)\,.$$

Moreover, we can choose u such that

$$\forall v \in \mathcal{N}_{\mathrm{gen}}(\mathcal{M} - \omega): \quad \langle u, v \rangle_{\mathsf{L}^2_\Lambda(\Omega)} = 0. \tag{3.8}$$

Then u is uniquely determined.

(5) *For all* $s, -t > 1/2$*, the solution operator*

$$\mathcal{L}_\omega : \mathsf{L}^2_s(\Omega) \cap \mathcal{N}_{\mathrm{gen}}(\mathcal{M} - \omega)^{\perp_\Lambda} \longrightarrow \big(\mathbf{R}_{t,\Gamma_1}(\Omega) \times \mathbf{R}_{t,\Gamma_2}(\Omega)\big) \cap \mathcal{N}_{\mathrm{gen}}(\mathcal{M} - \omega)^{\perp_\Lambda}$$

defined by (4) *is continuous.*

Remark 3.11. Under the conditions of Theorem 3.10, the following statements are equivalent to the boundedness of $\mathcal{L}_\omega$ (cf. Lemma 2.3 and Remark 3.7):

– (Friedrichs/Poincaré-type estimate) For all $s,-t > 1/2$, there exists $c > 0$ such that

$$\|u\|_{\mathbf{R}_t(\Omega)} \leq c \cdot \big\|(\mathrm{M} - \omega)u\big\|_{\mathsf{L}^2_s(\Omega)}$$

holds for all $u \in \big(\mathbf{R}_{t,\Gamma_1}(\Omega) \times \mathbf{R}_{t,\Gamma_2}(\Omega)\big) \cap \mathcal{N}_{\mathrm{gen}}(\mathcal{M} - \omega)^{\perp_\Lambda}$ satisfying the radiation condition.

– (Closed range) For all $s, -t > 1/2$, the range

$$\mathcal{R}(\mathcal{M} - \omega) = (\mathcal{M} - \omega)\big(\mathbf{R}_{t,\Gamma_1}(\Omega) \times \mathbf{R}_{t,\Gamma_2}(\Omega)\big)$$

is closed in $\mathsf{L}^2_s(\Omega)$.

By the same indirect arguments as in [15, Corollary 3.9] (see also [14, Section 4.9]), we get even stronger estimates for the solution operator $\mathcal{L}_\omega$.

Corollary 3.12. *Let* $\Omega \subset \mathbb{R}^3$ *be an exterior weak Lipschitz domain with boundary* Γ *and weak Lipschitz boundary parts* Γ_1 *and* $\Gamma_2 := \Gamma \setminus \overline{\Gamma}_1$*. Furthermore, let* $s, -t > 1/2$*,* ε*,* μ *be* κ*-admissible with* $\kappa > 1$ *and* $K \Subset \mathbb{C}_+ \setminus (0)$ *with* $\overline{K} \cap \sigma_{\mathrm{gen}}(\mathcal{M}) = \emptyset$*. Then:*

(1) *There exist constants* $c > 0$ *and* $\hat{t} > -1/2$ *such that for all* $\omega \in \overline{K}$ *and* $f \in \mathsf{L}^2_s(\Omega)$

$$\|\mathcal{L}_\omega f\|_{\mathbf{R}_t(\Omega)} + \big\|(\Lambda_0 + \sqrt{\varepsilon_0\mu_0}\,\Xi)\mathcal{L}_\omega f\big\|_{\mathsf{L}^2_{\hat{t}}(\Omega)} \leq c \cdot \|f\|_{\mathsf{L}^2_s(\Omega)}$$

holds, implying that $\mathcal{L}_\omega : \mathsf{L}^2_s(\Omega) \longrightarrow \mathbf{R}_{t,\Gamma_1}(\Omega) \times \mathbf{R}_{t,\Gamma_2}(\Omega)$ *is equicontinuous w. r. t.* $\omega \in \overline{K}$*.*

(2) *The mapping*

$$\begin{array}{cccc} \mathcal{L} & : & \overline{K} & \longrightarrow & \mathrm{B}\big(\mathsf{L}^2_s(\Omega), \mathbf{R}_{t,\Gamma_1}(\Omega) \times \mathbf{R}_{t,\Gamma_2}(\Omega)\big) \\ & & \omega & \longmapsto & \mathcal{L}_\omega \end{array}$$

is uniformly continuous.

4 Polynomial decay and a priori estimate

As stated before, we will construct a solution u in the case of $\omega \in \mathbb{R} \setminus (0)$ by solving (3.2) for $\omega_n = \omega + i\sigma_n \in \mathbb{C}_+ \setminus \mathbb{R}$ and sending $\sigma_n \longrightarrow 0$ (using $(\omega_n)_{n\in\mathbb{N}} \in \mathbb{C}_- \setminus \mathbb{R}$ instead will lead to “incoming” solutions). The essential ingredients to generate convergence are the polynomial decay of eigensolutions, an a priori estimate for solutions corresponding to nonreal frequencies and Weck’s local selection theorem. While the latter one is already satisfied (cf. Theorem 3.4), we obtain the first two in the spirit of [27] using the following decomposition Lemma introduced in [14] (see also [15, 16]).

Lemma 4.1. *Let $\omega \in K \Subset \mathbb{C} \setminus (0)$, ε, μ be κ-admissible with $\kappa \geq 0$ and $s, t \in \mathbb{R}$ such that $0 \leq s \in \mathbb{R} \setminus \mathbb{I}$ and $t \leq s \leq t + \kappa$. Moreover, assume that $u \in \mathbf{R}_t(\Omega)$ satisfies the equation $(\mathrm{M} - \omega)\, u = f \in \mathsf{L}^2_s(\Omega)$. Then*

$$f_1 := (\mathsf{C}_{\mathrm{Rot},\check{\eta}} - i\omega\check{\eta}\hat{\Lambda})u - i\check{\eta}\Lambda f \in \mathsf{L}^2_s$$

and, by decomposing

$$f_1 = f_{\mathsf{R}} + f_{\mathsf{D}} + f_{\mathcal{S}} \in {}_0\mathsf{R}_s \dotplus {}_0\mathsf{D}_s \dotplus \mathcal{S}_s$$

according to [26, Theorem 4], *it holds*

$$f_2 := f_{\mathsf{D}} + \frac{i}{\omega}\, \tilde{\Lambda}_0^{-1}\ \mathrm{Rot} f_{\mathcal{S}} \in {}_0\mathsf{D}_s\,.$$

Additionally, u may be decomposed into

$$u = \eta u + u_1 + u_2 + u_3\,,$$

where

(1) $\eta u \in \mathsf{R}_{\mathrm{vox}}(\Omega)$ *and for all* $\hat{t} \in \mathbb{R}$

$$\|\eta u\|_{\mathsf{R}_{\hat{t}}(\Omega)} \leq c \cdot \Big(\|f\|_{\mathsf{L}^2_s(\Omega)} + \|u\|_{\mathsf{L}^2_{s-\kappa}(\Omega)} \Big)\,;$$

(2) $u_1 := -\frac{i}{\omega}\Lambda_0^{-1}(f_{\mathsf{R}} + f_{\mathcal{S}}) \in \mathsf{R}_s$ *and*

$$\|u_1\|_{\mathsf{R}_s} \leq c \cdot \|f_1\|_{\mathsf{L}^2_s}\,;$$

(3) $u_2 := \mathcal{F}^{-1}(\rho^{-2}(1 - ir\Xi)\mathcal{F}(f_2)) \in \mathbf{H}^1_s \cap {}_0\mathsf{D}_s$ *and*

$$\|u_2\|_{\mathbf{H}^1_s} \leq c \cdot \|f_2\|_{\mathsf{L}^2_s}\,;$$

(4) $u_3 := \tilde{u} - u_2 \in \mathbf{H}^2_t \cap {}_0\mathsf{D}_t$ *and for all* $\hat{t} \leq t$

$$\|u_3\|_{\mathbf{H}^2_{\hat{t}}} \leq c \cdot \Big(\|u_3\|_{\mathsf{L}^2_{\hat{t}}} + \|u_2\|_{\mathsf{H}^1_{\hat{t}}} \Big)\,,$$

where $\tilde{u} := i\omega^{-1}\Lambda_0^{-1}(\,\mathrm{Rot}\,\check{\eta}u - f_{\mathsf{D}}) \in \mathbf{H}^1_t \cap {}_0\mathsf{D}_t$

with constants $c \in (0,\infty)$ *independent of* u, f *or* ω. *These fields solve the following equations:*

$$(\mathrm{Rot} + i\omega\Lambda_0)\check{\eta}u = f_1\,, \quad (\mathrm{Rot} + i\omega\Lambda_0)\tilde{u} = f_2\,, \quad (\mathrm{Rot} + i\omega\Lambda_0)u_3 = (1 - \omega\Lambda_0)u_2\,,$$
$$(\Delta + \omega^2\varepsilon_0\mu_0)u_3 = (1 - i\omega\widetilde{\Lambda}_0)f_2 - (1 + \omega^2\varepsilon_0\mu_0)u_2\,.$$

Moreover, the following estimates hold for all $\hat{t} \le t$ *and uniformly w. r. t.* $\lambda \in K$, u *and* f*:*

- $\|f_2\|_{\mathsf{L}^2_s} \le c \cdot \|f_1\|_{\mathsf{L}^2_s} \le c \cdot \big(\, \|f\|_{\mathsf{L}^2_s(\Omega)} + \|u\|_{\mathsf{L}^2_{s-\kappa}(\Omega)} \big)$
- $\|u\|_{\mathbf{R}_{\hat{t}}(\Omega)} \le c \cdot \big(\, \|f\|_{\mathsf{L}^2_s(\Omega)} + \|u\|_{\mathsf{L}^2_{s-\kappa}(\Omega)} + \|u_3\|_{\mathsf{L}^2_{\hat{t}}} \big)$
- $\big\|(\Delta + \omega^2\varepsilon_0\mu_0)u_3\big\|_{\mathsf{L}^2_s} \le c \cdot \big(\, \|f\|_{\mathsf{L}^2_s(\Omega)} + \|u\|_{\mathsf{L}^2_{s-\kappa}(\Omega)} \big)$
- $\big\|(\mathrm{Rot} - i\lambda\sqrt{\varepsilon_0\mu_0}\,\Xi)u\big\|_{\mathsf{L}^2_{\hat{t}}} \le c \cdot \big(\, \|f\|_{\mathsf{L}^2_s(\Omega)} + \|u\|_{\mathsf{L}^2_{s-\kappa}(\Omega)} + \big\|(\mathrm{Rot} - i\lambda\sqrt{\varepsilon_0\mu_0}\,\Xi)u_3\big\|_{\mathsf{L}^2_{\hat{t}}} \big)$

Here, $\mathcal{S}_s$ *is a finite dimensional subspace of* $\mathring{\mathsf{C}}^\infty(\mathbb{R}^3)$, $\mathcal{F}$ *the Fourier transformation and*

$$\mathsf{C}_{A,B} := AB - BA$$

the commutator of A *and* B.

Basically, this lemma allows us to split u into two parts. One part (consisting of ηu, u_1 and u_2) has better integrability properties and the other part (consisting of u_3) is more regular and satisfies a Helmholtz equation in the whole of $\mathbb{R}^3$. Thus we can use well-known results from the theory for Helmholtz equation (cf. Appendix, Section B) to establish corresponding results for Maxwell's equations. We start with the polynomial decay of solutions, especially of eigensolutions, which will lead to assertions (1)–(3) of our main theorem. Moreover, this will also show, that the solution u we are going to construct, can be chosen to be perpendicular to the generalized kernel of the time-harmonic Maxwell operator. As in the proof of [16, Theorem 4.2], we obtain (see also Appendix, Section C) the following.

Lemma 4.2 (Polynomial decay of solutions). *Let* $J \subset \mathbb{R} \setminus (0)$ *be some interval,* $\omega \in J$, ε, μ *be* κ*-admissible with* $\kappa > 1$, *and* $s \in \mathbb{R} \setminus \mathbb{I}$ *with* $s > 1/2$. *If*

$$u \in \mathbf{R}_{>-\frac{1}{2}}(\Omega) \quad \textit{satisfies} \quad (\mathrm{M} - \omega)u =: f \in \mathsf{L}^2_s(\Omega)\,,$$

then

$$u \in \mathbf{R}_{s-1}(\Omega) \quad \textit{and} \quad \|u\|_{\mathbf{R}_{s-1}(\Omega)} \le c \cdot \Big(\, \|f\|_{\mathsf{L}^2_s(\Omega)} + \|u\|_{\mathsf{L}^2(\Omega(\delta))} \Big)\,,$$

with $c, \delta \in (0,\infty)$ *independent of* ω, u *and* f.

In short: If a solution u satisfies $u \in \mathbf{R}_t(\Omega)$ for some $t > -1/2$ and the right-hand side $f = (\mathrm{M} - \omega)u$ has better integrability properties, meaning $f \in \mathsf{L}^2_s(\Omega)$ for some $s > 1/2$, then also u is better integrable, i. e., $u \in \mathbf{R}_{s-1}(\Omega)$. Especially, if

$$u \in \mathbf{R}_{>-\frac{1}{2}}(\Omega) \quad \text{and} \quad f \in \mathsf{L}^2_s(\Omega) \quad \forall\, s \in \mathbb{R}\,,$$

then $u \in \mathbf{R}_s(\Omega)$ for all $s \in \mathbb{R}$, which is called *"polynomial decay."*

Corollary 4.3. *Let $\omega \in \mathbb{R} \setminus (0)$ and assume ε, μ to be κ-admissible with $\kappa > 1$ and*

$$u \in \mathbf{R}_{<-\frac{1}{2},\Gamma_1}(\Omega) \times \mathbf{R}_{<-\frac{1}{2},\Gamma_2}(\Omega)$$

to be a radiating solution (cf. Definition 3.8) of $(\mathcal{M} - \omega)u = 0$. Then:

$$u \in \bigcap_{t\in\mathbb{R}} \Big(\mathbf{R}_{t,\Gamma_1}(\Omega) \times \mathbf{R}_{t,\Gamma_2}(\Omega) \Big) .$$

Proof. According to Lemma 4.2, it suffices to show $u \in \mathbf{R}_t(\Omega)$ for some $t > -1/2$. Therefore, remember that u is a radiating solution, the radiation condition (3.6) holds and there exists $\hat{t} > -1/2$ such that

$$(\Lambda_0 + \sqrt{\varepsilon_0\mu_0}\,\Xi)u \in \mathsf{L}^2_{\hat{t}}(\Omega) . \tag{4.1}$$

On the other hand, we have

$$\begin{aligned} &\|(\Lambda_0 + \sqrt{\varepsilon_0\mu_0}\,\Xi)u\|^2_{\mathsf{L}^2_{\hat{t}}(\mathrm{G}(r_0,\tilde{r}))} \\ &= \|\Lambda_0 u\|^2_{\mathsf{L}^2_{\hat{t}}(\mathrm{G}(r_0,\tilde{r}))} + 2\sqrt{\varepsilon_0\mu_0}\ \mathrm{Re}\,\langle\, \Xi\, u\,, \Lambda_0 u\,\rangle_{\mathsf{L}^2_{\hat{t}}(\mathrm{G}(r_0,\tilde{r}))} + \varepsilon_0\mu_0\,\|\Xi\, u\|^2_{\mathsf{L}^2_{\hat{t}}(\mathrm{G}(r_0,\tilde{r}))} \end{aligned}$$

and using Lemma A.3 (cf. Appendix, Section A) with

$$\phi(s) := (1+s^2)^{\hat{t}}, \quad \Phi := \phi \circ r, \quad \psi(\sigma) = \int_{\max\{r_0,\sigma\}}^{\tilde{r}} \phi(\tau)\, d\tau, \quad \Psi = \psi \circ r ,$$

as well as the differential equation, we conclude

$$\begin{aligned} \mathrm{Re}\,\langle\, \Xi\, u\,, \Lambda_0 u\,\rangle_{\mathsf{L}^2_{\hat{t}}(\mathrm{G}(r_0,\tilde{r}))} &= \mathrm{Re}\,\langle\, \Phi\,\Xi\, u\,, \Lambda_0 u\,\rangle_{\mathsf{L}^2(\mathrm{G}(r_0,\tilde{r}))} \\ &= \mathrm{Re}\Big(\langle\, \Psi\, \mathrm{Rot}\, u\,, \Lambda_0 u\,\rangle_{\mathsf{L}^2(\Omega(\tilde{r}))} + \langle\, \Psi u\,, \widetilde{\Lambda}_0\, \mathrm{Rot}\, u\,\rangle_{\mathsf{L}^2(\Omega(\tilde{r}))} \Big) \\ &= \mathrm{Re}\Big(\langle\, -i\omega\Psi\Lambda u\,, \Lambda_0 u\,\rangle_{\mathsf{L}^2(\Omega(\tilde{r}))} + \langle\, \Psi u\,, -i\omega\widetilde{\Lambda}_0\Lambda u\,\rangle_{\mathsf{L}^2(\Omega(\tilde{r}))} \Big) \\ &= \mathrm{Re}\ \underbrace{i\omega\langle\, \Psi\Lambda u\,, (\widetilde{\Lambda}_0 - \Lambda_0)u\,\rangle_{\mathsf{L}^2(\Omega(\tilde{r}))}}_{\in i\mathbb{R}} = 0 , \end{aligned}$$

hence

$$\|u\|_{\mathsf{L}^2_{\hat{t}}(\mathrm{G}(r_0,\tilde{r}))} \le c \cdot \|(\Lambda_0 + \sqrt{\varepsilon_0\mu_0}\,\Xi)u\|_{\mathsf{L}^2_{\hat{t}}(\mathrm{G}(r_0,\tilde{r}))}$$

with $c \in (0, \infty)$ independent of $\tilde{r}$. Now the monotone convergence theorem and (4.1) show

$$\|u\|_{\mathsf{L}^2_{\hat{t}}(\check{\mathrm{U}}(r_0))} \le c \cdot \|(\Lambda_0 + \sqrt{\varepsilon_0\mu_0}\,\Xi)u\|_{\mathsf{L}^2_{\hat{t}}(\check{\mathrm{U}}(r_0))} < \infty ,$$

which already implies $u \in \mathsf{L}^2_{\hat{t}}(\Omega)$ and completes the proof. □

The next step is an a priori estimate for solutions corresponding to nonreal frequencies, which will later guarantee that our solution satisfies the radiation condition (3.6) and has the proper integrability. The proof of it is practically identical with the proof of [16, Lemma 6.3] (cf. Appendix, Section C).

Lemma 4.4 (A priori estimate for Maxwell's equations). *Let $J \Subset \mathbb{R} \setminus (0)$ be some interval, $-t, s > 1/2$ and ε, μ be κ-admissible with $\kappa > 1$. Then there exist constants $c, \delta \in (0, \infty)$ and some $\hat{t} > -1/2$, such that for all $\omega \in \mathbb{C}_+$ with $\omega^2 = \lambda^2 + i\lambda\sigma$, $\lambda \in J$, $\sigma \in (0, \sqrt{\varepsilon_0\mu_0}^{-1}]$ and $f \in \mathsf{L}^2_s(\Omega)$*

$$\|\mathcal{L}_\omega f\|_{\mathbf{R}_t(\Omega)} + \|(\Lambda_0 + \sqrt{\varepsilon_0\mu_0}\,\Xi)\mathcal{L}_\omega f\|_{\mathsf{L}^2_{\hat{t}}(\Omega)} \leq c \cdot \Big(\|f\|_{\mathsf{L}^2_s(\Omega)} + \|\mathcal{L}_\omega f\|_{\mathsf{L}^2(\Omega(\delta))} \Big) .$$

5 Proof of the main result

Before we start with the proof of Theorem 3.10, we provide some Helmholtz-type decompositions, which will be useful in the following. These are immediate consequences of the projection theorem and Lemma 2.2.

Lemma 5.1. *It holds*

$$\begin{aligned}
\mathsf{L}^2_\varepsilon(\Omega) &= \overline{\nabla \mathbf{H}^1_{\Gamma_1}(\Omega)} \oplus_\varepsilon \varepsilon^{-1}{}_0\mathbf{D}_{\Gamma_2}(\Omega)\,, \\
\mathsf{L}^2_\mu(\Omega) &= \overline{\nabla \mathbf{H}^1_{\Gamma_2}(\Omega)} \oplus_\mu \mu^{-1}{}_0\mathbf{D}_{\Gamma_1}(\Omega)\,, \\
\mathbf{R}_{\Gamma_1}(\Omega) &= \overline{\nabla \mathbf{H}^1_{\Gamma_1}(\Omega)} \oplus_\varepsilon \Big(\mathbf{R}_{\Gamma_1}(\Omega) \cap \varepsilon^{-1}{}_0\mathbf{D}_{\Gamma_2}(\Omega)\Big)\,, \\
\mathbf{R}_{\Gamma_2}(\Omega) &= \overline{\nabla \mathbf{H}^1_{\Gamma_2}(\Omega)} \oplus_\mu \Big(\mathbf{R}_{\Gamma_2}(\Omega) \cap \mu^{-1}{}_0\mathbf{D}_{\Gamma_1}(\Omega)\Big)\,,
\end{aligned}$$

where the closures are taken in $\mathsf{L}^2(\Omega)$.

Proof. Let $\gamma \in \{\varepsilon, \mu\}$ and $i, j \in \{1, 2\}$ with $i \neq j$. The linear operator

$$\nabla_i : \mathbf{H}^1_{\Gamma_i}(\Omega) \subset \mathsf{L}^2(\Omega) \longrightarrow \mathsf{L}^2_\gamma(\Omega)$$

is densely defined and closed with adjoint (cf. Lemma 2.2)

$$-\operatorname{div}_j \gamma : \gamma^{-1}\mathbf{D}_{\Gamma_j}(\Omega) \subset \mathsf{L}^2_\gamma(\Omega) \longrightarrow \mathsf{L}^2(\Omega)\,.$$

The projection theorem yields

$$\mathsf{L}^2_\gamma(\Omega) = \overline{\mathcal{R}(\nabla_i)} \oplus_\gamma \mathcal{N}(\operatorname{div}_j \gamma)\,.$$

The remaining assertion follows by $\nabla \mathbf{H}^1_{\Gamma_i}(\Omega) \subset \mathsf{R}_{\Gamma_i}(\Omega)$. □

Proof of Theorem 3.10. Let $\omega \in \mathbb{R} \setminus (0)$ and ε, μ be κ-admissible for some $\kappa > 1$.

(1): The assertion follows by Corollary 4.3 and the differential equation

$$(\mathrm{M} - \omega)\, u = 0 \iff u = i\omega^{-1}\Lambda^{-1}\,\mathrm{Rot}\, u\,,$$

using the fact that (cf. Lemma 2.2)

$$\operatorname{rot} \mathbf{R}_{\mathrm{t},\Gamma_1}(\Omega) \subset {}_0\mathsf{D}_{\mathrm{t},\Gamma_1}(\Omega) \quad \text{resp.} \quad \operatorname{rot} \mathbf{R}_{\mathrm{t},\Gamma_2}(\Omega) \subset {}_0\mathsf{D}_{\mathrm{t},\Gamma_2}(\Omega)\,.$$

(2): Assume $\dim \mathcal{N}_{\mathrm{gen}}(\,\mathcal{M} - \omega\,) = \infty$. Using (1) there exists a L^2_Λ-orthonormal sequence $(u_n)_{n\in\mathbb{N}} \subset \mathcal{N}_{\mathrm{gen}}(\,\mathcal{M} - \omega\,)$ converging weakly in $\mathsf{L}^2(\Omega)$ to 0. By the differential equation this sequence is bounded in

$$\big(\mathbf{R}_{\Gamma_1}(\Omega) \cap \varepsilon^{-1}{}_0\mathsf{D}_{\Gamma_2}(\Omega)\big) \times \big(\mathbf{R}_{\Gamma_2}(\Omega) \cap \mu^{-1}{}_0\mathsf{D}_{\Gamma_1}(\Omega)\big).$$

Hence, due to Weck's local selection theorem, we can choose a subsequence, $(u_{\pi(n)})_{n\in\mathbb{N}}$ converging to 0 in $\mathsf{L}^2_{\mathrm{loc}}(\overline{\Omega})$ ($(u_{\pi(n)})_{n\in\mathbb{N}}$ also converges weakly on every bounded subset). Now let $1 < s \in \mathbb{R} \setminus \mathbb{I}$. Then Lemma 4.2 guarantees the existence of $c, \delta \in (0, \infty)$ independent of $(u_{\pi(n)})_{n\in\mathbb{N}}$ such that

$$1 = \|u_{\pi(n)}\|_{\mathsf{L}^2_\Lambda(\Omega)} \leq c \cdot \|u_{\pi(n)}\|_{\mathbf{R}_{s-1}(\Omega)} \leq c \cdot \|u_{\pi(n)}\|_{\mathsf{L}^2(\Omega(\delta))} \xrightarrow{n\to\infty} 0$$

holds; a contradiction.

(3): $\mathcal{M}$ is a self-adjoint operator, hence we clearly have $\sigma_{\mathrm{gen}}(\mathcal{M}) \subset \mathbb{R}\setminus(0)$. Now assume $\widetilde{\omega} \in \mathbb{R}\setminus(0)$ is an accumulation point of $\sigma_{\mathrm{gen}}(\mathcal{M})$. Then we can choose a sequence $(\omega_n)_{n\in\mathbb{N}} \subset \mathbb{R} \setminus (0)$ with $\omega_n \neq \omega_m$ for $n \neq m$, $\omega_n \longrightarrow \widetilde{\omega}$ and a corresponding sequence $(u_n)_{n\in\mathbb{N}}$ with $u_n \in \mathcal{N}_{\mathrm{gen}}(\,\mathcal{M} - \omega_n\,) \setminus (0)$. As $\mathcal{M}$ is self-adjoint, eigenvectors associated to different eigenvalues are orthogonal provided they are well enough integrable (which is given by (1)), and thus by normalizing $(u_n)_{n\in\mathbb{N}}$ we end up with an L^2_Λ-orthonormal sequence. Continuing as in (2), we again obtain a contradiction.

(4): First of all, if a solution u satisfies (3.8), it is uniquely determined as for the homogeneous problem $u \in \mathcal{N}_{\mathrm{gen}}(\,\mathcal{M} - \omega\,)$ together with (1) and (3.8) implies $u = 0$. Moreover, using Lemma 2.2 and (1), we obtain

$$\langle f, v \rangle_{\mathsf{L}^2_\Lambda(\Omega)} = \langle (\mathrm{M} - \omega)\, u, v \rangle_{\mathsf{L}^2_\Lambda(\Omega)} = \langle u, (\mathrm{M} - \omega)\, v \rangle_{\mathsf{L}^2_\Lambda(\Omega)} = 0 \quad \forall\, v \in \mathcal{N}_{\mathrm{gen}}(\,\mathcal{M} - \omega\,),$$

meaning (3.7) is necessary. In order to show, that (3.7) is also sufficient, we use Eidus' principle of limiting absorption. Therefore, let $s > 1/2$ and $f \in \mathsf{L}^2_s(\Omega)$ satisfy (3.7). We take a sequence $(\sigma_n)_{n\in\mathbb{N}} \subset \mathbb{R}_+$ with $\sigma_n \longrightarrow 0$ and construct a sequence of frequencies

$$(\omega_n)_{n\in\mathbb{N}}, \quad \omega_n := \sqrt{\omega^2 + i\sigma_n\omega} \in \mathbb{C}_+ \setminus \mathbb{R}\,,$$

converging to ω. Since $\mathcal{M}$ is a self-adjoint operator we obtain (cf. Section 3) a corresponding sequence of solutions $(u_n)_{n\in\mathbb{N}}$, $u_n := \mathcal{L}_{\omega_n} f \in \mathbf{R}_{\Gamma_1}(\Omega) \times \mathbf{R}_{\Gamma_2}(\Omega)$ satisfying

$(\mathrm{M}-\omega_n)\,u_n = f$. Now our aim is to show that this sequence or at least a subsequence is converging to a solution u. By Lemma 5.1, we decompose

$$u_n = \hat{u}_n + \tilde{u}_n \quad \text{and} \quad f = \hat{f} + \tilde{f},$$

with

$$\begin{aligned} \hat{u}_n, \hat{f} &\in \overline{\nabla \mathbf{H}^1_{\Gamma_1}(\Omega)} \times \overline{\nabla \mathbf{H}^1_{\Gamma_2}(\Omega)} \subset {}_0\mathsf{R}_{\Gamma_1}(\Omega) \times {}_0\mathsf{R}_{\Gamma_2}(\Omega), \\ \tilde{u}_n, \tilde{f} &\in \Big(\mathbf{R}_{\Gamma_1}(\Omega) \cap \varepsilon^{-1}{}_0\mathsf{D}_{\Gamma_2}(\Omega)\Big) \times \Big(\mathbf{R}_{\Gamma_2}(\Omega) \cap \mu^{-1}{}_0\mathsf{D}_{\Gamma_1}(\Omega)\Big). \end{aligned} \tag{5.1}$$

Inserting these (orthogonal) decompositions in the differential equation, we end up with two equations

$$-\omega_n \hat{u}_n = \hat{f} \quad \text{and} \quad (\mathrm{M}-\omega_n)\,\tilde{u}_n = \tilde{f},$$

noting that the first one is trivial and implies L^2-convergence of $(\hat{u}_n)_{n\in\mathbb{N}}$. For dealing with the second equation, we need the following additional assumption on $(u_n)_{n\in\mathbb{N}}$, which we will prove in the end:

$$\forall\, t < -1/2 \quad \exists\, c \in (0,\infty) \quad \forall\, n \in \mathbb{N}: \qquad \|u_n\|_{\mathsf{L}^2_t(\Omega)} \le c \tag{5.2}$$

Let $\hat{t} < -1/2$ and $c \in (0,\infty)$ such that (5.2) holds. Then, by construction and $(5.1)_2$, the sequence $(\tilde{u}_n)_{n\in\mathbb{N}}$ is bounded in $(\mathbf{R}_{\hat{t},\Gamma_1}(\Omega) \cap \varepsilon^{-1}{}_0\mathsf{D}_{\hat{t},\Gamma_2}(\Omega)) \times (\mathbf{R}_{\hat{t},\Gamma_2}(\Omega) \cap \mu^{-1}{}_0\mathsf{D}_{\hat{t},\Gamma_1}(\Omega))$. Hence (Theorem 3.4 and Lemma 3.3), $(\tilde{u}_n)_{n\in\mathbb{N}}$ has a subsequence $(\tilde{u}_{\pi(n)})_{n\in\mathbb{N}}$ converging in $\mathsf{L}^2_{\tilde{t}}(\Omega)$ for some $\tilde{t} < \hat{t}$ and by the equation even in $\mathbf{R}_{\tilde{t},\Gamma_1}(\Omega) \times \mathbf{R}_{\tilde{t},\Gamma_2}(\Omega)$. Consequently, the entire sequence $(u_{\pi(n)})_{n\in\mathbb{N}}$ converges in $\mathbf{R}_{\tilde{t}}(\Omega)$ to some u satisfying

$$u \in \mathbf{R}_{\tilde{t},\Gamma_1}(\Omega) \times \mathbf{R}_{\tilde{t},\Gamma_2}(\Omega) \quad \text{and} \quad (\mathrm{M}-\omega)\,u = f.$$

Additionally, with Corollary 4.3 and Lemma 2.2 we obtain for $n \in \mathbb{N}$ and arbitrary $v \in \mathcal{N}_{\mathrm{gen}}(\mathcal{M}-\omega)$,

$$\begin{aligned} 0 = \langle f, v \rangle_{\mathsf{L}^2_\Lambda(\Omega)} &= \langle (\mathrm{M}-\omega_{\pi(n)})\,u_{\pi(n)}, v \rangle_{\mathsf{L}^2_\Lambda(\Omega)} \\ &= \langle u_{\pi(n)}, (\mathrm{M}-\overline{\omega}_{\pi(n)})\,v \rangle_{\mathsf{L}^2_\Lambda(\Omega)} = (\omega-\omega_{\pi(n)}) \cdot \langle u_{\pi(n)}, v \rangle_{\mathsf{L}^2_\Lambda(\Omega)}. \end{aligned}$$

Hence $\langle u_{\pi(n)}, v \rangle_{\mathsf{L}^2_\Lambda(\Omega)} = 0$ and as $\langle \,\cdot\,, v \rangle_{\mathsf{L}^2_\Lambda(\Omega)}$ is continuous on $\mathsf{L}^2_{\tilde{t}}(\Omega) \times \mathsf{L}^2_{\tilde{t}}(\Omega)$ by (1), we obtain

$$\langle u, v \rangle_{\mathsf{L}^2_\Lambda(\Omega)} = \lim_{n\to\infty} \langle u_{\pi(n)}, v \rangle_{\mathsf{L}^2_\Lambda(\Omega)} = 0.$$

Thus, up to now, we have constructed a vector field $u \in \mathcal{N}_{\mathrm{gen}}(\mathcal{M}-\omega)^{\perp_\Lambda}$, which has the right boundary conditions and satisfies the differential equation. But for being a radiating solution, it still remains to show, that $u \in \mathbf{R}_{<-\frac{1}{2}}(\Omega)$ and enjoys the radiation condition (3.6). For that, let $t < -1/2$. Then, by Lemma 4.4, there exist $c, \delta \in (0, \infty)$ and some $\check{t} > -1/2$, such that for $n \in \mathbb{N}$ large enough we obtain uniformly in $\sigma_{\pi(n)}$, $u_{\pi(n)}$, f and $\tilde{r} > 0$:

$$\|u_{\pi(n)}\|_{\mathbf{R}_t(\Omega(\tilde{r}))} + \|(\Lambda_0 + \sqrt{\varepsilon_0\mu_0}\,\Xi)u_{\pi(n)}\|_{\mathsf{L}^2_{\check{t}}(\Omega(\tilde{r}))} \leq c \cdot \Big(\|f\|_{\mathsf{L}^2_s(\Omega)} + \|u_{\pi(n)}\|_{\mathsf{L}^2(\Omega(\delta))} \Big).$$

Sending $n \longrightarrow \infty$ and afterwards $\tilde{r} \longrightarrow \infty$ (monotone convergence), we obtain

$$\|u\|_{\mathbf{R}_t(\Omega)} + \|(\Lambda_0 + \sqrt{\varepsilon_0\mu_0}\,\Xi)u\|_{\mathsf{L}^2_{\check{t}}(\Omega)} \leq c \cdot \Big(\|f\|_{\mathsf{L}^2_s(\Omega)} + \|u\|_{\mathsf{L}^2(\Omega(\delta))} \Big) < \infty\,, \tag{5.3}$$

yielding

$$u \in \mathbf{R}_{<-\frac{1}{2}}(\Omega) \quad \text{and} \quad (\Lambda_0 + \sqrt{\varepsilon_0\mu_0}\,\Xi)u \in \mathsf{L}^2_{>-\frac{1}{2}}(\Omega)\,.$$

This completes the proof of existence, if we can show (5.2). To this end, we assume it to be wrong, i. e., there exists $t < -1/2$ and a sequence $(u_n)_{n\in\mathbb{N}} \subset \mathbf{R}_{t,\Gamma_1}(\Omega) \times \mathbf{R}_{t,\Gamma_2}(\Omega)$, $u_n := \mathcal{L}_{\omega_n} f$ with $\|u_n\|_{\mathsf{L}^2_t(\Omega)} \longrightarrow \infty$ for $n \longrightarrow \infty$. Defining

$$\check{u}_n := \|u_n\|^{-1}_{\mathsf{L}^2_t(\Omega)} \cdot u_n \quad \text{and} \quad \check{f}_n := \|u_n\|^{-1}_{\mathsf{L}^2_t(\Omega)} \cdot f\,,$$

we have

$$\|\check{u}_n\|_{\mathsf{L}^2_t(\Omega)} = 1\,, \quad \check{f} \longrightarrow 0 \text{ in } \mathsf{L}^2_s(\Omega) \quad \text{and} \quad (\mathrm{M} - \omega_n)\,\check{u}_n = \check{f}_n\,.$$

Then, repeating the arguments from above, we obtain some $\check{t} < t$ and a subsequence $(\check{u}_{\pi(n)})_{n\in\mathbb{N}}$ converging in $\mathsf{L}^2_{\check{t}}(\Omega)$ to some $\check{u} \in \mathcal{N}_{\text{gen}}(\mathcal{M} - \omega) \cap \mathcal{N}_{\text{gen}}(\mathcal{M} - \omega)^{\perp_\Lambda}$, hence $\check{u} = 0$. But Lemma 4.4 ensures the existence of $c, \delta \in (0, \infty)$ (independent of $\sigma_{\pi(n)}$, $\check{u}_{\pi(n)}$ and $\check{f}_{\pi(n)}$) such that

$$1 = \|\check{u}_{\pi(n)}\|_{\mathsf{L}^2_t(\Omega)} \leq c \cdot \Big(\|\check{f}_{\pi(n)}\|_{\mathsf{L}^2_s(\Omega)} + \|\check{u}_{\pi(n)}\|_{\mathsf{L}^2(\Omega(\delta))}\Big) \xrightarrow{n\to\infty} 0$$

holds; a contradiction.

(5): Let $-t, s > 1/2$. By (4) the solution operator

$$\mathcal{L}_\omega : \underbrace{\mathsf{L}^2_s(\Omega) \cap \mathcal{N}_{\text{gen}}(\mathcal{M} - \omega)^{\perp_\Lambda}}_{=:\mathcal{D}(\mathcal{L}_\omega)} \longrightarrow \underbrace{\Big(\mathbf{R}_{t,\Gamma_1}(\Omega) \times \mathbf{R}_{t,\Gamma_2}(\Omega)\Big) \cap \mathcal{N}_{\text{gen}}(\mathcal{M} - \omega)^{\perp_\Lambda}}_{=:\mathcal{R}(\mathcal{L}_\omega)}$$

is well defined. Furthermore, due to the polynomial decay of eigensolutions, $\mathcal{D}(\mathcal{L}_\omega)$ is closed in $\mathsf{L}^2_s(\Omega)$. Thus, the assertion follows from the closed graph theorem, if we can show that $\mathcal{L}_\omega$ is closed. Therefore, take $(f_n)_{n\in\mathbb{N}} \subset \mathcal{D}(\mathcal{L}_\omega)$ with

$$f_n \longrightarrow f \text{ in } \mathsf{L}^2_s(\Omega) \quad \text{and} \quad u_n := \mathcal{L}_\omega f_n \longrightarrow u \text{ in } \mathbf{R}_{t,\Gamma_1}(\Omega) \times \mathbf{R}_{t,\Gamma_2}(\Omega)\,.$$

Then clearly $f \in \mathcal{D}(\mathcal{L}_\omega)$, $u \in \mathcal{R}(\mathcal{L}_\omega)$ and as $(\mathrm{M} - \omega)\, u_n = f_n$, we obtain $(\mathrm{M} - \omega)\, u = f$. Now estimate (5.3) (along with monotone convergence) shows as before

$$u \in \mathbf{R}_{<-\frac{1}{2}}(\Omega) \quad \text{and} \quad (\Lambda_0 + \sqrt{\varepsilon_0\mu_0}\,\Xi)u \in \mathsf{L}^2_{>-\frac{1}{2}}(\Omega)\,,$$

meaning u is a radiating solution, i. e., $u = \mathcal{L}_\omega f$, which completes the proof. □

Remark 5.2. During the discussion at AANMPDE10 (10th Workshop on Analysis and Advanced Numerical Methods for Partial Differential Equations), M. Waurick and S. Trostorff pointed out that it is sufficient to use weakly convergent subsequences for the construction of the (radiating) solution. This is in fact true (the radiation condition and regularity properties follow from Lemma 4.4 by the boundedness of the sequence and the weak lower semicontinuity of the norms), but it should be noted, that Weck's local selection theorem is still needed to prove (5.2), since here norm convergence is indispensable in order to generate a contradiction. Anyway, we thank both for the vivid discussion and constructive criticism.

Appendix A. Technical tools

Lemma A.1. *Let $\Omega \subset \mathbb{R}^3$ be an arbitrary exterior domain and $s, t, \theta \in \mathbb{R}$ with $t < s$ and $\theta > 0$. Then there exist constants $c, \delta \in (0, \infty)$ such that*

$$\|w\|_{\mathsf{L}^2_t(\Omega)} \le c \cdot \|w\|_{\mathsf{L}^2(\Omega(\delta))} + \theta \cdot \|w\|_{\mathsf{L}^2_s(\Omega)}$$

holds for all $w \in \mathsf{L}^2_s(\Omega)$.

Proof. Let $\mathbb{R}^3 \setminus \Omega \subset \mathrm{U}(r_0)$. For $\tilde{r} \ge r_0$, we obtain

$$\begin{aligned}\|w\|^2_{\mathsf{L}^2_t(\Omega)} &= \|w\|^2_{\mathsf{L}^2_t(\Omega(\tilde{r}))} + \|w\|^2_{\mathsf{L}^2_t(\breve{\mathrm{U}}(\tilde{r}))} \\ &\le \left(1 + \tilde{r}^2\right)^{\max\{0,t\}} \cdot \|w\|^2_{\mathsf{L}^2(\Omega(\tilde{r}))} + \left(1 + \tilde{r}^2\right)^{t-s} \cdot \|w\|^2_{\mathsf{L}^2_s(\breve{\mathrm{U}}(\tilde{r}))} \\ &\le \left(1 + \tilde{r}^2\right)^{\max\{0,t\}} \cdot \|w\|^2_{\mathsf{L}^2(\Omega(\tilde{r}))} + \left(1 + \tilde{r}^2\right)^{t-s} \cdot \|w\|_{\mathsf{L}^2_s(\Omega)}\,.\end{aligned}$$

Since $t < s$, we can choose $\tilde{r}$ such that $\left(1 + \tilde{r}^2\right)^{t-s} \le \theta^2$, which completes the proof. □

Lemma A.2. *For $\tilde{r} > 0$ and $f \in \mathsf{L}^1(\mathbb{R}^n)$, it holds*

$$\liminf_{r\to\infty}\; r \int\limits_{\mathrm{S}(r)} |f|\, d\lambda_s^{n-1} = 0\,.$$

Proof. Otherwise, there exists $\hat{r} > 0$ and $c > 0$ such that

$$\int\limits_{\mathrm{S}(r)} |f|\, d\lambda_s^{n-1} \ge \frac{c}{r} \quad \forall\, r \ge \hat{r}$$

and using Fubini's theorem, we obtain

$$\|f\|^2_{\mathrm{L}^1(\mathbb{R}^n)} \geq \int_{\breve{\mathrm{U}}(\hat{r})} |f|\, d\lambda^n = \int_{\hat{r}}^{\infty} \int_{\mathrm{S}(r)} |f|\, d\lambda_s^{n-1}\, dr \geq c \cdot \int_{\hat{r}}^{\infty} \frac{1}{r}\, dr = \infty\,,$$

a contradiction. □

Lemma A.3. *Let $\Omega \subset \mathbb{R}^3$ be an exterior weak Lipschitz domain with boundary Γ and weak Lipschitz boundary parts Γ_1 and $\Gamma_2 = \Gamma \setminus \overline{\Gamma}_1$. Furthermore, let $\hat{r}, \tilde{r} \in \mathbb{R}_+$ with $\tilde{r} > \hat{r}$ and $\mathbb{R}^3 \setminus \Omega \subset \mathrm{U}(\hat{r})$ as well as $\phi \in \mathsf{C}^0([\hat{r}, \tilde{r}], \mathbb{C})$. If $u \in \mathbf{R}_{t,\Gamma_1}(\Omega) \times \mathbf{R}_{t,\Gamma_2}(\Omega)$ for some $t \in \mathbb{R}$, it holds*

$$\langle\, \Phi \Xi u\,, \Lambda_0 u\,\rangle_{\mathrm{L}^2(\mathrm{G}(\hat{r},\tilde{r}))} = \langle\, \Psi \operatorname{Rot} u\,, \Lambda_0 u\,\rangle_{\mathrm{L}^2(\Omega(\tilde{r}))} + \langle\, \Psi u\,, \operatorname{Rot} \Lambda_0 u\,\rangle_{\mathrm{L}^2(\Omega(\tilde{r}))}\,, \tag{A.1}$$

where $\Phi := \phi \circ r$, $\Psi := \psi \circ r$, and

$$\psi : [0, \tilde{r}] \longrightarrow \mathbb{C},\ \sigma \longmapsto \int_{\max\{\hat{r},\sigma\}}^{\tilde{r}} \phi(\tau)\, d\tau.$$

Proof. As $\mathsf{C}^\infty_{\Gamma_1}(\Omega)$ respectively $\mathsf{C}^\infty_{\Gamma_2}(\Omega)$ is dense in $\mathbf{R}_{t,\Gamma_1}(\Omega)$ respectively $\mathbf{R}_{t,\Gamma_2}(\Omega)$ by definition it is enough to show equation (A.1) for $u = (u_1, u_2) \in \mathsf{C}^\infty_{\Gamma_1}(\Omega) \times \mathsf{C}^\infty_{\Gamma_2}(\Omega) \subset \mathring{\mathsf{C}}^\infty(\mathbb{R}^3)$. Observing that the support of products of u_1 and u_2 is compactly supported in some $\Theta \subset \overline{\Theta} \subset \Omega$, we may choose a cut-off function $\varphi \in \mathring{\mathsf{C}}^\infty(\Omega) \subset \mathring{\mathsf{C}}^\infty(\mathbb{R}^3)$ with $\varphi|_\Theta = 1$ and replace u by $\varphi u =: v =: (E, H)$. Without loss of generality we assume $\mathbb{R}^3 \setminus \Theta \subset \mathrm{U}(\hat{r})$. Using Gauss's divergence theorem we compute

$$\begin{aligned}
\langle\, \Phi \Xi u\,, \Lambda_0 u\,\rangle_{\mathrm{L}^2(\mathrm{G}(\hat{r},\tilde{r}))} &= \int_{\hat{r}}^{\tilde{r}} \phi(r)\, \langle\, \Xi u\,, \Lambda_0 u\,\rangle_{\mathrm{L}^2(\mathrm{S}(r))}\, dr = \int_{\hat{r}}^{\tilde{r}} \phi(r)\, \langle\, \Xi v\,, \Lambda_0 v\,\rangle_{\mathrm{L}^2(\mathrm{S}(r))}\, dr \\
&= \int_{\hat{r}}^{\tilde{r}} \phi(r) (\mu_0 \langle\, \xi \times E\,, H\,\rangle_{\mathrm{L}^2(\mathrm{S}(r))} - \varepsilon_0 \langle\, \xi \times H\,, E\,\rangle_{\mathrm{L}^2(\mathrm{S}(r))}\,)\, dr \\
&= \int_{\hat{r}}^{\tilde{r}} \phi(r) \int_{\mathrm{S}(r)} (\mu_0 \xi \cdot (E \times \overline{H}) - \varepsilon_0 \xi \cdot (H \times \overline{E}))\, d\lambda_s^2\, dr \\
&= \int_{\hat{r}}^{\tilde{r}} \phi(r) \int_{\mathrm{U}(r)} (\mu_0 \operatorname{div}(E \times \overline{H}) - \varepsilon_0 \operatorname{div}(H \times \overline{E}))\, d\lambda^3\, dr\,.
\end{aligned}$$

Note, that

$$\begin{aligned}
&\mu_0 \operatorname{div}(E \times \overline{H}) - \varepsilon_0 \operatorname{div}(H \times \overline{E}) \\
&\quad = \mu_0 (\overline{H} \operatorname{rot} E - E \operatorname{rot} \overline{H}) - \varepsilon_0 (\overline{E} \operatorname{rot} H - H \operatorname{rot} \overline{E})
\end{aligned}$$

$$= \big((\mu_0\overline{H})\,\mathrm{rot}\,E - (\varepsilon_0\overline{E})\,\mathrm{rot}\,H\big) + \big(H\,\mathrm{rot}\,(\varepsilon_0\overline{E}) - E\,\mathrm{rot}\,(\mu_0\overline{H})\big)$$
$$= \overline{\Lambda_0 v}\cdot \mathrm{Rot}\,v + v\cdot \mathrm{Rot}\,\overline{\Lambda_0 v}.$$

Hence, by Fubini's theorem, we see

$$\begin{aligned}
\langle\,\Phi\Xi u\,,\Lambda_0 u\,\rangle_{\mathrm{L}^2(\mathrm{G}(\hat{r},\tilde{r}))} &= \int_{\hat{r}}^{\tilde{r}} \phi(r)\,\big(\langle\,\mathrm{Rot}\,v\,,\Lambda_0 v\,\rangle_{\mathrm{L}^2(\mathrm{U}(r))} + \langle\,v\,,\mathrm{Rot}\,\Lambda_0 v\,\rangle_{\mathrm{L}^2(\mathrm{U}(r))}\big)\,dr\\
&= \int_{\hat{r}}^{\tilde{r}} \phi(r)\int_0^r \big(\langle\,\mathrm{Rot}\,v\,,\Lambda_0 v\,\rangle_{\mathrm{L}^2(\mathrm{S}(\sigma))} + \langle\,v\,,\mathrm{Rot}\,\Lambda_0 v\,\rangle_{\mathrm{L}^2(\mathrm{S}(\sigma))}\big)\,d\sigma\,dr\\
&= \int_0^{\tilde{r}}\int_{\max\{\hat{r},\sigma\}}^{\tilde{r}} \phi(r)\,\big(\langle\,\mathrm{Rot}\,v\,,\Lambda_0 v\,\rangle_{\mathrm{L}^2(\mathrm{S}(\sigma))} + \langle\,v\,,\mathrm{Rot}\,\Lambda_0 v\,\rangle_{\mathrm{L}^2(\mathrm{S}(\sigma))}\big)\,dr\,d\sigma\\
&= \int_0^{\tilde{r}} \psi(\sigma)\,\big(\langle\,\mathrm{Rot}\,v\,,\Lambda_0 v\,\rangle_{\mathrm{L}^2(\mathrm{S}(\sigma))} + \langle\,v\,,\mathrm{Rot}\,\Lambda_0 v\,\rangle_{\mathrm{L}^2(\mathrm{S}(\sigma))}\big)\,d\sigma\\
&= \langle\,\Psi\,\mathrm{Rot}\,v\,,\Lambda_0 v\,\rangle_{\mathrm{L}^2(\mathrm{U}(\tilde{r}))} + \langle\,\Psi v\,,\mathrm{Rot}\,\Lambda_0 v\,\rangle_{\mathrm{L}^2(\mathrm{U}(\tilde{r}))}\\
&= \langle\,\Psi\,\mathrm{Rot}\,v\,,\Lambda_0 v\,\rangle_{\mathrm{L}^2(\Omega(\tilde{r}))} + \langle\,\Psi v\,,\mathrm{Rot}\,\Lambda_0 v\,\rangle_{\mathrm{L}^2(\Omega(\tilde{r}))}\\
&= \langle\,\Psi\,\mathrm{Rot}\,u\,,\Lambda_0 u\,\rangle_{\mathrm{L}^2(\Omega(\tilde{r}))} + \langle\,\Psi u\,,\mathrm{Rot}\,\Lambda_0 u\,\rangle_{\mathrm{L}^2(\Omega(\tilde{r}))}\,,
\end{aligned}$$

where the last line follows by construction of v. □

We end this section with a lemma, which will be needed to prove the polynomial decay and a priori estimate for the Helmholtz equation and can be shown by elementary partial integration.

Lemma A.4. *Let* $w \in \mathbf{H}^2_{\mathrm{loc}}(\mathbb{R}^n)$, $0 \notin \operatorname{supp} w$, $m \in \mathbb{R}$ *and* $\tilde{r} > 0$. *Then with* $\partial_r := \xi\cdot\nabla$*:*

(1) $\mathrm{Re}\int_{\mathrm{U}(\tilde{r})} r^{m+1}\Delta w\partial_r\overline{w}$
$$= \frac{1}{2}\int_{\mathrm{U}(\tilde{r})} r^m\Big((n+m-2)|\nabla w|^2 - 2m|\partial_r w|^2\Big) + \int_{\mathrm{S}(\tilde{r})} r^{m+1}\Big(|\partial_r w|^2 - \frac{1}{2}|\nabla w|^2\Big)$$

(2) $\mathrm{Re}\int_{\mathrm{U}(\tilde{r})} r^m\Delta w\overline{w}$
$$= -\int_{\mathrm{U}(\tilde{r})} r^m\Big(|\nabla w|^2 - \frac{m}{2}\,(n+m-2)\,r^{-2}|w|^2\Big) + \int_{\mathrm{S}(\tilde{r})} r^m\Big(\mathrm{Re}\,(\partial_r w\overline{w}) - \frac{m}{2}\,r^{-1}|w|^2\Big)$$

(3) $\mathrm{Im}\int_{\mathrm{U}(\tilde{r})} r^m\Delta w\overline{w} = -m\int_{\mathrm{U}(\tilde{r})} r^{m-1}\mathrm{Im}\,(\partial_r w\overline{w}) + \frac{1}{2}\int_{\mathrm{S}(\tilde{r})} r^{m+1}|w|^2$

(4) $\mathrm{Re}\int_{\mathrm{U}(\tilde{r})} r^{m+1}w\,\partial_r\,\overline{w} = -\frac{1}{2}\int_{\mathrm{U}(\tilde{r})} r^m\,(n+m)|w|^2 + \frac{1}{2}\int_{\mathrm{S}(\tilde{r})} r^{m+1}|w|^2$

Appendix B. Polynomial decay and a-priori estimate for the Helmholtz equation

In this section we present well-known results for the Helmholtz equation, which we will use to achieve similar results for Maxwell's equations. We start with a regularity result (cf. [27, Lemma 4]) and the polynomial decay (cf. [27, Lemma 5]).

Lemma B.1. *Let $t \in \mathbb{R}$. If $w \in \mathsf{L}^2_t(\mathbb{R}^n)$ and $\Delta w \in \mathsf{L}^2_t(\mathbb{R}^n)$, it holds $w \in \mathbf{H}^2_t(\mathbb{R}^n)$ and*

$$\|w\|_{\mathbf{H}^2_t(\mathbb{R}^n)} \le c \cdot \Big(\|\Delta w\|_{\mathsf{L}^2_t(\mathbb{R}^n)} + \|w\|_{\mathsf{L}^2_t(\mathbb{R}^n)} \Big)$$

with $c \in (0, \infty)$ independent of w and Δw.

Proof. For $t = 0$, we have $w, \Delta w \in \mathsf{L}^2(\mathbb{R}^n)$ and using Fourier transformation, we obtain

$$\begin{aligned} \|\Delta w\|^2_{\mathsf{L}^2(\mathbb{R}^n)} + \|w\|^2_{\mathsf{L}^2(\mathbb{R}^n)} &= \left\| r^2 \mathcal{F}(w) \right\|^2_{\mathsf{L}^2(\mathbb{R}^n)} + \|\mathcal{F}(w)\|^2_{\mathsf{L}^2(\mathbb{R}^n)} \\ &= \int_{\mathbb{R}^n} (r^4 + 1)|\mathcal{F}(w)|^2 \ge \frac{1}{2} \cdot \left\| (1 + r^2)\mathcal{F}(w) \right\|^2_{\mathsf{L}^2(\mathbb{R}^n)}, \end{aligned} \tag{B.1}$$

yielding $w \in \mathbf{H}^2(\mathbb{R}^n)$ and the desired estimate. So let us switch to $t \neq 0$. Then, using a well-known result concerning inner regularity (e. g., [3, Chapter VII, *Section* 3.2, Theorem 1]), we already have $w \in \mathbf{H}^2_{\mathrm{loc}}(\mathbb{R}^n)$. Now let $\tilde{r} > 1$ and define $\eta_{\tilde{r}} \in \mathring{\mathsf{C}}^\infty(\mathbb{R}^n)$ by $\eta_{\tilde{r}}(x) := \rho^t \eta(r(x)/\tilde{r})$. Then $\eta_{\tilde{r}} w \in \mathbf{H}^2(\mathbb{R}^n)$,

$$|\nabla \eta_{\tilde{r}}| \le c \cdot \rho^{t-1} \quad \text{with } \ c = c(t) > 0\,,$$

and

$$\begin{aligned} \langle \nabla(\eta_{\tilde{r}} w), \nabla(\eta_{\tilde{r}} w) \rangle_{\mathsf{L}^2(\mathbb{R}^n)} &= \operatorname{Re} \langle \nabla w, \nabla(\eta^2_{\tilde{r}} w) \rangle_{\mathsf{L}^2(\mathbb{R}^n)} + \left\| (\nabla \eta_{\tilde{r}}) w \right\|^2_{\mathsf{L}^2(\mathbb{R}^n)} \\ &\le c \cdot \Big(\|\eta_{\tilde{r}} \Delta w\|_{\mathsf{L}^2(\mathbb{R}^n)} \, \|\eta_{\tilde{r}} w\|_{\mathsf{L}^2(\mathbb{R}^n)} + \|w\|^2_{\mathsf{L}^2_{t-1}(\mathbb{R}^n)} \Big) \\ &\le c \cdot \Big(\|\Delta w\|^2_{\mathsf{L}^2_t(\mathbb{R}^n)} + \|w\|^2_{\mathsf{L}^2_t(\mathbb{R}^n)} \Big), \end{aligned}$$

with $c = c(n, t) \in (0, \infty)$, hence

$$\|\nabla w\|_{\mathsf{L}^2_t(\mathrm{B}(\tilde{r}))} \le \|\nabla(\eta_{\tilde{r}} w) - (\nabla \eta_{\tilde{r}}) w\|_{\mathsf{L}^2(\mathbb{R}^n)} \le c(n, t) \cdot \Big(\|\Delta w\|_{\mathsf{L}^2_t(\mathbb{R}^n)} + \|w\|_{\mathsf{L}^2_t(\mathbb{R}^n)} \Big).$$

Sending $\tilde{r} \longrightarrow \infty$ (monotone convergence) shows $w \in \mathbf{H}^1_t(\mathbb{R}^n)$ and

$$\|w\|_{\mathbf{H}^1_t(\mathbb{R}^n)} \le c(n, t) \cdot \Big(\|\Delta w\|_{\mathsf{L}^2_t(\mathbb{R}^n)} + \|w\|_{\mathsf{L}^2_t(\mathbb{R}^n)} \Big). \tag{B.2}$$

Moreover,

$$\Delta(\rho^t w) = t\Big(n + (t - 2)\frac{r^2}{1 + r^2}\Big)\rho^{t-2} w + 2r\rho^{t-2}\partial_r w + \rho^t \Delta w\,,$$

such that with (B.2) we obtain

$$\left\|\Delta(\rho^t w)\right\|_{\mathsf{L}^2(\mathbb{R}^n)} \le c \cdot \Big(\|\Delta w\|_{\mathsf{L}^2_t(\mathbb{R}^n)} + \|w\|_{\mathsf{L}^2_t(\mathbb{R}^n)} \Big), \tag{B.3}$$

with $c \in (0, \infty)$ independent of w and Δw. Hence $\Delta(\rho^t w) \in \mathsf{L}^2(\mathbb{R}^n)$ and we may apply the first case. This shows $\rho^t w \in \mathbf{H}^2(\mathbb{R}^n)$ and using (B.1), (B.2) and (B.3), we obtain (uniformly w. r. t. w and Δw)

$$\begin{aligned}
\|w\|_{\mathbf{H}^2_t(\mathbb{R}^n)} &\le c \cdot \Big(\left\|\rho^t w\right\|_{\mathbf{H}^2(\mathbb{R}^n)} + \left\|(\nabla\rho^t)\nabla w\right\|_{\mathsf{L}^2(\mathbb{R}^n)} + \left\|(\nabla\rho^t)w\right\|_{\mathsf{L}^2(\mathbb{R}^n)} + \sum_{|\alpha|=2} \left\|(\partial^\alpha \rho^t)w\right\|_{\mathsf{L}^2(\mathbb{R}^n)} \Big) \\
&\le c \cdot \Big(\left\|\Delta(\rho^t w)\right\|_{\mathsf{L}^2(\mathbb{R}^n)} + \left\|\rho^t w\right\|_{\mathsf{L}^2(\mathbb{R}^n)} + \|\nabla w\|_{\mathsf{L}^2_{t-1}(\mathbb{R}^n)} + \|w\|_{\mathsf{L}^2_{t-1}(\mathbb{R}^n)} \Big) \\
&\le c \cdot \Big(\|\Delta w\|_{\mathsf{L}^2_t(\mathbb{R}^n)} + \|w\|_{\mathsf{L}^2_t(\mathbb{R}^n)} \Big)
\end{aligned}$$

yielding $w \in \mathbf{H}^2_t(\mathbb{R}^n)$ and the required estimate. □

Lemma B.2 (Polynomial decay). *Let $J \Subset \mathbb{R} \setminus (0)$ be some interval, $\gamma \in J$ and $s, t \in \mathbb{R}$ with $t > -1/2$ and $t \le s$. If $w \in \mathsf{L}^2_t(\mathbb{R}^n)$ and $g := (\Delta + \gamma^2)\, w \in \mathsf{L}^2_{s+1}(\mathbb{R}^n)$, it holds*

$$w \in \mathbf{H}^2_s(\mathbb{R}^n) \quad \text{and} \quad \|w\|_{\mathbf{H}^2_s(\mathbb{R}^n)} \le c \cdot \Big(\|g\|_{\mathsf{L}^2_{s+1}(\mathbb{R}^n)} + \|w\|_{\mathsf{L}^2_{s-1}(\mathbb{R}^n)} \Big)$$

with $c = c(n, s, J) \in (0, \infty)$ not depending on γ, g or w.

Proof. The assertion follows directly from Lemma B.1, if we can show

$$w \in \mathsf{L}^2_s(\mathbb{R}^n) \quad \text{with} \quad \|w\|_{\mathsf{L}^2_s(\mathbb{R}^n)} \le c \cdot \Big(\|g\|_{\mathsf{L}^2_{s+1}(\mathbb{R}^n)} + \|w\|_{\mathsf{L}^2_{s-1}(\mathbb{R}^n)} \Big).$$

Therefore, let $v := \check{\chi} w$, where $\check{\chi} \in \mathsf{C}^\infty(\mathbb{R}^n)$ with $\check{\chi} = 1$ on $\check{\mathrm{U}}(1)$ and vanishing in a neighborhood of the origin. By assumption, we already have $w \in \mathbf{H}^2_t(\mathbb{R}^n)$ (cf. Lemma B.1), hence $v \in \mathbf{H}^2_{\mathrm{loc}}(\mathbb{R}^n)$ and we may apply the partial integration rules from Lemma A.4 to

$$\operatorname{Re} \int_{\mathrm{G}(\hat{r},\tilde{r})} (\Delta w + \gamma^2 w)(r^{2t+1}\partial_r \overline{w} + \beta r^{2t}\overline{w}) = \operatorname{Re} \int_{\mathrm{G}(\hat{r},\tilde{r})} (\Delta v + \gamma^2 v)(r^{2t+1}\partial_r \bar{v} + \beta r^{2t}\bar{v}) = \cdots,$$

with $\tilde{r} > \hat{r} \ge 1$ and

$$\beta := \max\big\{(n-1)/2\,, t + (n-1)/2\big\}.$$

After some rearrangements, this leads to

$$\begin{aligned}
&\int_{\mathrm{G}(\hat{r},\tilde{r})} r^{2t}\Big((\beta - (n + 2t - 2)/2)|\nabla w|^2 + ((n + 2t)/2 - \beta)\gamma^2|w|^2\Big) \\
&\quad + 2t \int_{\mathrm{G}(\hat{r},\tilde{r})} r^{2t}|\partial_r w|^2 + \int_{\mathrm{S}(\tilde{r})} \tilde{r}^{2t+1}|\nabla w|^2
\end{aligned}$$

$$
\begin{aligned}
&= -\mathrm{Re} \int\limits_{G(\hat{r},\tilde{r})} (\Delta w + \gamma^2 w)(r^{2t+1}\partial_r \overline{w} + \beta r^{2t}\overline{w}) + t(n+2t-2)\beta \int\limits_{G(\hat{r},\tilde{r})} r^{2t-2}|w|^2 \\
&\quad + \int\limits_{S(\hat{r})} \hat{r}^{2t+1}\Big(\beta t \hat{r}^{-2}|w|^2 - \beta\hat{r}^{-1}\mathrm{Re}\,(\partial_r w\overline{w}) - |\partial_r w|^2\Big) \\
&\quad + \int\limits_{S(\tilde{r})} \tilde{r}^{2t+1}\Big(|\partial_r w|^2 + \beta\tilde{r}^{-1}\mathrm{Re}\,(\partial_r w\overline{w}) - \beta t \tilde{r}^{-2}|w|^2\Big) \\
&\quad + \frac{1}{2}\int\limits_{S(\tilde{r})} \tilde{r}^{2t+1}\Big(|\nabla w|^2 + \gamma^2|w|^2\Big) + \frac{1}{2}\int\limits_{S(\hat{r})} \hat{r}^{2t+1}\Big(|\nabla w|^2 - \gamma^2|w|^2\Big).
\end{aligned} \tag{B.4}
$$

Let us first have a look on the left-hand side. For $t \geq 0$ (i. e., $\beta = t + (n-1)/2$), we skip the second and third integral to obtain

$$
\begin{aligned}
&\int\limits_{G(\hat{r},\tilde{r})} r^{2t}\Big((\beta - (n+2t-2)/2)|\nabla w|^2 + ((n+2t)/2 - \beta)\gamma^2|w|^2\Big) \\
&\qquad + 2t \int\limits_{G(\hat{r},\tilde{r})} r^{2t}|\partial_r w|^2 + \int\limits_{S(\tilde{r})} \tilde{r}^{2t+1}|\nabla w|^2 \\
&\quad \geq \frac{1}{2}\int\limits_{G(\hat{r},\tilde{r})} r^{2t}\Big((2\beta - (n+2t-2))|\nabla w|^2 + ((n+2t) - 2\beta)\gamma^2|w|^2\Big) \\
&\quad = \frac{1}{2}\int\limits_{G(\hat{r},\tilde{r})} r^{2t}\Big(|\nabla w|^2 + \gamma^2|w|^2\Big),
\end{aligned}
$$

while in the case of $t < 0$ (i. e., $\beta = (n-1)/2$) we just skip the third integral and end up with

$$
\begin{aligned}
&\int\limits_{G(\hat{r},\tilde{r})} r^{2t}\Big((\beta - (n+2t-2)/2)|\nabla w|^2 + ((n+2t)/2 - \beta)\gamma^2|w|^2\Big) \\
&\qquad + 2t \int\limits_{G(\hat{r},\tilde{r})} r^{2t}|\partial_r w|^2 + \int\limits_{S(\tilde{r})} \tilde{r}^{2t+1}|\nabla w|^2 \\
&\quad \geq \int\limits_{G(\hat{r},\tilde{r})} r^{2t}\Big((\beta - (n+2t-2)/2 + 2t)|\nabla w|^2 + ((n+2t)/2 - \beta)\gamma^2|w|^2\Big) \\
&\quad = \Big(\frac{1}{2} + t\Big)\int\limits_{G(\hat{r},\tilde{r})} r^{2t}(|\nabla w|^2 + \gamma^2|w|^2),
\end{aligned}
$$

since $|\partial_r w| \leq |\nabla w|$. Thus for arbitrary $t \in \mathbb{R}$ the left-hand side of (B.4) can be estimated from below by

$$
\min\Big\{\frac{1}{2}, \frac{1}{2} + t\Big\} \int\limits_{G(\hat{r},\tilde{r})} r^{2t}\Big(|\nabla w|^2 + \gamma^2|w|^2\Big).
$$

For the right-hand side, we have ($\tilde{r} > 1$)

$$\int_{S(\tilde{r})} \tilde{r}^{2t+1}\Big(|\partial_r w|^2 + \beta\tilde{r}^{-1}\operatorname{Re}(\partial_r w\overline{w}) - \beta t\tilde{r}^{-2}|w|^2\Big)$$

$$\leq \int_{S(\tilde{r})} \tilde{r}^{2t+1}\Big(|\partial_r w|^2 + \beta|\partial_r w\overline{w}| + \beta|t||w|^2\Big) \leq c\cdot\int_{S(\tilde{r})} \tilde{r}^{2t+1}\Big(|\nabla w|^2 + |w|^2\Big),$$

as well as

$$\int_{S(\hat{r})} \hat{r}^{2t+1}\Big(\beta t\hat{r}^{-2}|w|^2 - \beta\hat{r}^{-1}\operatorname{Re}(\partial_r w\overline{w}) - |\partial_r w|^2\Big)$$

$$\leq \int_{S(\tilde{r})} \tilde{r}^{2t+1}\Big(\beta|t|\hat{r}^{-2}|w|^2 + \beta\hat{r}^{-1}|\partial_r w\overline{w}|\Big) \leq c\cdot\int_{S(\tilde{r})} \tilde{r}^{2t+1}\Big(\hat{r}^{-2}|w|^2 + |\nabla w|^2\Big),$$

such that equation (B.4) becomes

$$\min\{\frac{1}{2}, \frac{1}{2}+t\}\int_{G(\hat{r},\tilde{r})} r^{2t}\Big(|\nabla w|^2 + \gamma^2|w|^2\Big)$$

$$\leq \int_{G(\hat{r},\tilde{r})} r^{t+1}|g|\Big(r^t|\nabla w| + \beta r^{t-1}|w|\Big) + \beta\big|t(n+2t-2)\big|\int_{G(\hat{r},\tilde{r})} r^{2t-2}|w|^2$$

$$+ c(n,t)\cdot\left(\int_{S(\hat{r})} \hat{r}^{2t+1}\Big(\hat{r}^{-2}|w|^2 + |\nabla w|^2 - \gamma^2|w|^2\Big) + \int_{S(\tilde{r})} \tilde{r}^{2t+1}\Big(|\nabla w|^2 + |w|^2\Big)\right).$$

By assumption, we have $w \in \mathbf{H}^2_t(\mathbb{R}^n)$, such that according to Lemma A.2 the lower limit for $\tilde{r} \longrightarrow \infty$ of the last boundary integral vanishes. Hence we may replace $G(\hat{r},\tilde{r})$ by $\breve{U}(\hat{r})$ and in addition use Young's inequality to obtain

$$\begin{aligned}
&\left\|r^t\nabla w\right\|^2_{L^2(\breve{U}(\hat{r}))} + \gamma^2\left\|r^t w\right\|^2_{L^2(\breve{U}(\hat{r}))}\\
&\leq c(n,t)\cdot\Big(\left\|r^{t+1}g\right\|^2_{L^2(\breve{U}(\hat{r}))} + \left\|r^{t-1}w\right\|^2_{L^2(\breve{U}(\hat{r}))} + \int_{S(\hat{r})} \hat{r}^{2t+1}\Big(|\nabla w|^2 - \gamma^2|w|^2 + \hat{r}^{-2}|w|^2\Big)\Big) \qquad \text{(B.5)}\\
&\leq c(n,t)\cdot\Big(\|g\|^2_{L^2_{t+1}(\mathbb{R}^n)} + \|w\|^2_{L^2_{t-1}(\mathbb{R}^n)} + \int_{S(\hat{r})} \hat{r}^{2t+1}\Big(|\nabla w|^2 - \gamma^2|w|^2 + \hat{r}^{-2}|w|^2\Big)\Big).
\end{aligned}$$

Now suppose that $s = t$. Then the assertion simply follows by choosing $\hat{r} := 1$ as the trace theorem bounds the surface integral by $\|w\|^2_{\mathbf{H}^2(U(1))}$ and with Lemma B.1

$$\begin{aligned}
\|w\|_{\mathbf{H}^1_t(\mathbb{R}^n)} &\leq c(n,s,J)\cdot\Big(\|g\|_{L^2_{t+1}(\mathbb{R}^n)} + \|w\|_{L^2_{t-1}(\mathbb{R}^n)} + \|w\|_{\mathbf{H}^2(U(2))}\Big)\\
&\leq c(n,s,J)\cdot\Big(\|g\|_{L^2_{t+1}(\mathbb{R}^n)} + \|w\|_{L^2_{t-1}(\mathbb{R}^n)} + \|w\|_{\mathbf{H}^2_{t-1}(\mathbb{R}^n)}\Big)
\end{aligned}$$

$$\leq c(n,s,J)\cdot\Big(\|g\|_{\mathsf{L}^2_{t+1}(\mathbb{R}^n)}+\|w\|_{\mathsf{L}^2_{t-1}(\mathbb{R}^n)}+\|\Delta w\|_{\mathsf{L}^2_{-1}(\mathbb{R}^n)}\Big)$$
$$\leq c(n,s,J)\cdot\Big(\|g\|_{\mathsf{L}^2_{t+1}(\mathbb{R}^n)}+\|w\|_{\mathsf{L}^2_{t-1}(\mathbb{R}^n)}\Big)$$

holds. For the case $w\notin\mathsf{L}^2_s(\mathbb{R}^n)$, let $\hat{s}:=\sup\{m\in\mathbb{R}\,|\,u\in\mathsf{L}^2_m(\mathbb{R}^n)\}$. Then, w. l. o. g.,[4] we may assume

$$\hat{s}-1/2<t<\hat{s}<s\leq t+1/2,$$

hence $\delta:=1-2(s-t)\in(0,1)$. Multiplying (B.5) with $\hat{r}^{-\delta}$ and integrating from 1 to some $\check{r}>1$ leads to:

$$\int_1^{\check{r}}\hat{r}^{-\delta}\int_{\check{\mathrm{U}}(\hat{r})}r^{2t}\Big(|\nabla w|^2+\gamma^2|w|^2\Big)\,d\hat{r}\leq c(n,t)\cdot\Bigg(\int_1^{\check{r}}\hat{r}^{-\delta}\int_{\check{\mathrm{U}}(\hat{r})}r^{2t+2}|g|^2+r^{2t-2}|w|^2\,d\hat{r}$$
$$+\int_{\mathrm{G}(1,\check{r})}r^{2t+1-\delta}\Big(|\nabla w|^2-\gamma^2|w|^2+r^{-2}|w|^2\Big)\Bigg)\tag{B.6}$$

By Fubini's theorem, we have for arbitrary $h\in\mathsf{L}^1(\mathbb{R}^n)$

$$\int_1^{\check{r}}\hat{r}^{-\delta}\int_{\check{\mathrm{U}}(\hat{r})}h\,d\hat{r}=\int_1^{\check{r}}\int_{\hat{r}}^{\infty}\int_{\mathrm{S}(\sigma)}\hat{r}^{-\delta}\,h\,d\sigma\,d\hat{r}=\int_1^{\infty}\int_1^{\min\{\sigma,\check{r}\}}\hat{r}^{-\delta}\int_{\mathrm{S}(\sigma)}h\,d\hat{r}\,d\sigma$$
$$=\int_1^{\infty}(1-\delta)^{-1}\min\{\sigma^{1-\delta}-1,\check{r}^{1-\delta}-1\}\int_{\mathrm{S}(\sigma)}h\,d\sigma$$
$$=\int_1^{\infty}\int_{\mathrm{S}(\sigma)}\underbrace{(1-\delta)^{-1}\min\{r^{1-\delta}-1,\check{r}^{1-\delta}-1\}}_{=:\theta_{\check{r}}}\,h\,d\sigma=\int_{\check{\mathrm{U}}(1)}\theta_{\check{r}}\,h,$$

such that (B.6) becomes (note that $\theta_{\check{r}}\leq(1-\delta)^{-1}\cdot r^{1-\delta}$ and $1-\delta=2(s-t)$)

$$\int_{\check{\mathrm{U}}(1)}\theta_{\check{r}}\,r^{2t}\Big(|\nabla w|^2+\gamma^2|w|^2\Big)\tag{B.7}$$
$$\leq c(n,t)\cdot\Bigg(\int_{\check{\mathrm{U}}(1)}\theta_{\check{r}}\Big(r^{2t+2}|g|^2+r^{2t-2}|w|^2\Big)+\int_{\mathrm{G}(1,\check{r})}r^{2t+1-\delta}\Big(|\nabla w|^2-\gamma^2|w|^2+r^{-2}|w|^2\Big)\Bigg)$$
$$\leq c(n,s)\cdot\Bigg(\|g\|^2_{\mathsf{L}^2_{s+1}(\mathbb{R}^n)}+\|w\|^2_{\mathsf{L}^2_{s-1}(\mathbb{R}^n)}+\int_{\mathrm{G}(1,\check{r})}r^{2s}\Big(|\nabla w|^2-\gamma^2|w|^2\Big)\Bigg).$$

---

4 Otherwise, we replace s and t by $t_k:=t+k/4$ respectively $s_k:=t_{k+1}$, $k=0,1,2,\dots$ and obtain the assertion after finitely many steps of the type $t_k<s_k\leq t_k+1/2$ (cf. Appendix, Section C, Proof of Lemma 4.1).

Finally, look at

$$\operatorname{Re}\int_{G(1,\check{r})} r^{2s} g\overline{w} = \operatorname{Re}\int_{G(1,\check{r})} r^{2s} g\bar{v}\,.$$

Applying Lemma A.4, we obtain (after some rearrangements)

$$\begin{aligned}
&\int_{G(1,\check{r})} r^{2s}\Big(|\nabla w|^2 - \gamma^2|w|^2\Big)\\
&\quad= -\operatorname{Re}\int_{G(1,\check{r})} r^{2s} g\overline{w} + s(n+2s-2)\int_{G(1,\check{r})} r^{2s-2}|w|^2\\
&\qquad+\int_{S(\check{r})}\check{r}^{2s}\Big(\operatorname{Re}(\partial_r w\overline{w}) - s\check{r}^{-1}|w|^2\Big) - \int_{S(1)}\Big(\operatorname{Re}(\partial_r w\overline{w}) - s|w|^2\Big)\\
&\quad\le c(n,s)\cdot\Bigg(\int_{G(1,\check{r})}\Big(r^{2s+2}|g|^2 + r^{2s-2}|w|^2\Big) + \int_{S(1)}\Big(|\nabla w|^2 + |w|^2\Big) + \int_{S(\check{r})}\check{r}^{2s}\Big(|\nabla w|^2 + |w|^2\Big)\Bigg),
\end{aligned}$$

hence (using the trace theorem and Lemma B.1)

$$\begin{aligned}
&\int_{G(1,\check{r})} r^{2s}\Big(|\nabla w|^2 - \gamma^2|w|^2\Big)\\
&\quad\le c(n,s)\cdot\Bigg(\|g\|^2_{\mathsf{L}^2_{s+1}(\mathbb{R}^n)} + \|w\|^2_{\mathsf{L}^2_{s-1}(\mathbb{R}^n)} + \|w\|^2_{\mathsf{H}^2(\mathrm{U}(1))} + \int_{S(\check{r})}\check{r}^{2s}\Big(|\nabla w|^2 + |w|^2\Big)\Bigg) &\text{(B.8)}\\
&\quad\le c(n,s,J)\cdot\Bigg(\|g\|^2_{\mathsf{L}^2_{s+1}(\mathbb{R}^n)} + \|w\|^2_{\mathsf{L}^2_{s-1}(\mathbb{R}^n)} + \int_{S(\check{r})}\check{r}^{2s}\Big(|\nabla w|^2 + |w|^2\Big)\Bigg)
\end{aligned}$$

and inserting (B.8) into (B.7) we end up with

$$\int_{\check{\mathrm{U}}(1)}\theta_{\check{r}}\, r^{2t}(|\nabla w|^2 + \gamma^2|w|^2) \le c(n,s,J)\cdot\Bigg(\|g\|^2_{\mathsf{L}^2_{s+1}(\mathbb{R}^n)} + \|w\|^2_{\mathsf{L}^2_{s-1}(\mathbb{R}^n)} + \int_{S(\check{r})}\check{r}^{2s}\Big(|\nabla w|^2 + |w|^2\Big)\Bigg).$$

Again the lower limit for $\check{r} \longrightarrow \infty$ of the boundary integral vanishes (cf. Lemma A.2 and observe that $w \in \mathsf{H}^2_{s-\frac{1}{2}}(\mathbb{R}^n)$, since $0 < s-t \le 1/2$ by assumption), such that passing to the limit on a suitable subsequence, we obtain

$$\begin{aligned}
\|w\|^2_{\mathsf{L}^2_s(\mathbb{R}^n)} &\le c(n,s,J)\cdot\Bigg(\int_{\check{\mathrm{U}}(1)}(1-\delta)^{-1} r^{2t+1-\delta}\Big(|\nabla w|^2 + \gamma^2|w|^2\Big) + \|w\|^2_{\mathsf{L}^2_s(\mathrm{U}(1))}\Bigg)\\
&\le c(n,s,J)\cdot\Bigg(\lim_{\check{r}\to\infty}\int_{\check{\mathrm{U}}(1)}\theta_{\check{r}} r^{2t}\Big(|\nabla w|^2 + \gamma^2|w|^2\Big) + \|w\|^2_{\mathsf{L}^2_{s-1}(\mathrm{U}(1))}\Bigg)\\
&\le c(n,s,J)\cdot\Big(\|g\|^2_{\mathsf{L}^2_{s+1}(\mathbb{R}^n)} + \|w\|^2_{\mathsf{L}^2_{s-1}(\mathbb{R}^n)}\Big),
\end{aligned}$$

showing $w \in \mathsf{L}^2_s(\mathbb{R}^n)$ and the required estimate. □

Lemma B.3 (A priori estimate). *Let $n \in \mathbb{N}$, $t < -1/2$, $1/2 < s < 1$, and let $J \Subset \mathbb{R}\setminus(0)$ be an interval. Then there exist $c, \delta \in (0,\infty)$, such that for all $\beta \in \mathbb{C}_+$ with $\beta^2 = \nu^2 + i\nu\tau$, $\nu \in J$, $\tau \in (0,1]$ and $g \in \mathsf{L}^2_s(\mathbb{R}^n)$*

$$\begin{aligned}&\left\|(\Delta+\beta)^{-1}g\right\|_{\mathsf{L}^2_t(\mathbb{R}^n)} + \left\|\exp(-i\nu r)(\Delta+\beta)^{-1}g\right\|_{\mathsf{H}^1_{s-2}(\mathbb{R}^n)}\\ &\quad\le c\cdot\left(\|g\|_{\mathsf{L}^2_s(\mathbb{R}^n)} + \left\|(\Delta+\beta)^{-1}g\right\|_{\mathsf{L}^2(\Omega(\delta))}\right)\end{aligned} \tag{B.9}$$

holds.

Ikebe and Saito [6] proved this estimate for the space dimension $n = 3$ and with $t = -s$, which already shows the result also for any $t < -1/2$ as the norms depend monotonic on the parameters s and t. For arbitrary space dimensions, we follow the proof of Vogelsang [21, Satz 4].

Proof. First of all, observe that

$$\Delta : \mathbf{H}^2(\mathbb{R}^n) \subset \mathsf{L}^2(\mathbb{R}^n) \longrightarrow \mathsf{L}^2(\mathbb{R}^n),\ w \longmapsto \Delta w$$

is self-adjoint and, therefore, $w := (\Delta+\beta)^{-1}g \in \mathbf{H}^2(\mathbb{R}^n)$ is well-defined. Moreover, due to the monotone dependence of the norms on the parameters s and t, it is enough to concentrate on the case $t = -s$. With $w_e := \exp(-i\nu r)w$ and $g_e := \exp(-i\nu r)g$, we have $w_e \in \mathbf{H}^2(\Omega)$ and

$$\Delta w_e + i\nu\Big(\tau w_e + \frac{n-1}{r}w_e + 2\partial_r w_e\Big) = g_e\,.$$

Applying Lemma A.4 to

$$\operatorname{Re}\int_{\mathrm{G}(1,\tilde r)} g_e\Big(r^{2s-1}\partial_r\overline{w}_e + \frac{n-1}{2}\,r^{2s-2}\overline{w}_e + \frac{\tau}{2}\,r^{2s-1}\overline{w}_e\Big) = \cdots,$$

with $\tilde r > 1$ and using the same techniques as in the proof of Lemma B.2 we obtain

$$\begin{aligned}&\frac12\int_{\mathrm{G}(1,\tilde r)} r^{2s-2}\Big((4s-4)|\partial_r w_e|^2 - (2s-3)|\nabla w_e|^2\Big) + \frac12\int_{\mathrm{G}(1,\tilde r)} r^{2s-1}\tau|\nabla w_e|^2\\ &= -\operatorname{Re}\int_{\mathrm{G}(1,\tilde r)} g_e\Big(r^{2s-1}\partial_r\overline{w}_e + \frac{n-1}{2}\,r^{2s-2}\overline{w}_e + \frac{\tau}{2}\,r^{2s-1}\overline{w}_e\Big)\\ &\quad+\frac{n-1}{2}(s-1)(n+2s-4)\int_{\mathrm{G}(1,\tilde r)} r^{2s-4}|w_e|^2 + \frac{\tau}{4}(2s-1)(n+2s-3)\int_{\mathrm{G}(1,\tilde r)} r^{2s-3}|w_e|^2\\ &\quad+\frac12\int_{\mathrm{S}(\tilde r)} \tilde r^{2s-1}\Big(2|\partial_r w_e|^2 + \tau\operatorname{Re}(\partial_r w_e\overline{w}_e) - |\nabla w_e|^2 - \frac{(2s-1)}{\tilde r}\,\tau|w_e|^2\Big)\\ &\quad+\frac{(n-1)}{2}\int_{\mathrm{S}(\tilde r)} \tilde r^{2s-2}\Big(\operatorname{Re}(\partial_r w_e\overline{w}_e) - \frac{s-1}{\tilde r}|w_e|^2\Big)\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{2}\int_{\mathrm{S}(1)}\Big(2|\partial_r w_e|^2+\tau\operatorname{Re}\big(\partial_r w_e\overline{w}_e\big)-|\nabla w_e|^2-(2s-1)\,\tau|w_e|^2\Big)\\
&-\frac{(n-1)}{2}\int_{\mathrm{S}(1)}\Big(\operatorname{Re}\big(\partial_r w_e\overline{w}_e\big)-(s-1)|w_e|^2\Big).
\end{aligned}
$$

Since $4s-4<0$ and $|\partial_r w_e|\le|\nabla w_e|$, the left-hand side can be estimated from below

$$
\begin{aligned}
&\frac{1}{2}\int_{\mathrm{G}(1,\tilde r)} r^{2s-2}\Big((4s-4)|\partial_r w_e|^2-(2s-3)|\nabla w_e|^2\Big)+\frac{1}{2}\int_{\mathrm{G}(1,\tilde r)} r^{2s-1}\tau|\nabla w_e|^2\\
&\qquad\ge\frac{1}{2}\int_{\mathrm{G}(1,\tilde r)} r^{2s-2}\Big((4s-4)-(2s-3)\Big)|\nabla w_e|^2=\Big(s-\frac{1}{2}\Big)\int_{\mathrm{G}(1,\tilde r)} r^{2s-2}|\nabla w_e|^2,
\end{aligned}
$$

while for the right-hand side we obtain

$$
\begin{aligned}
&-\operatorname{Re}\int_{\mathrm{G}(1,\tilde r)} g_e\Big(r^{2s-1}\partial_r\overline{w}_e+\frac{n-1}{2}r^{2s-2}\overline{w}_e+\frac{\tau}{2}r^{2s-1}\overline{w}_e\Big)\;+\;\cdots\\
&\le\int_{\mathrm{G}(1,\tilde r)} r^s|g_e|\Big(r^{s-1}|\nabla w_e|+\frac{n-1}{2}r^{s-2}|w_e|+\frac{\tau}{2}r^{s-1}|w_e|\Big)\\
&\quad+c\cdot\Bigg(\int_{\mathrm{G}(1,\tilde r)} r^{2s-4}|w_e|^2+\tau\int_{\mathrm{G}(1,\tilde r)} r^{s-2}|w_e|r^{s-1}|w_e|\\
&\quad+\int_{\mathrm{S}(1)}\Big(|\nabla w_e|^2+|\partial_r w_e\overline{w}_e|+|w_e|^2\Big)+\int_{\mathrm{S}(\tilde r)}\tilde r^{2s-1}\Big(|\nabla w_e|^2+|\partial_r w_e\overline{w}_e|+|w_e|^2\Big)\Bigg),
\end{aligned}
$$

yielding

$$
\begin{aligned}
&\Big(s-\frac{1}{2}\Big)\int_{\mathrm{G}(1,\tilde r)} r^{2s-2}|\nabla w_e|^2\\
&\quad\le\int_{\mathrm{G}(1,\tilde r)} r^s|g_e|\Big(r^{s-1}|\nabla w_e|+\frac{n-1}{2}r^{s-2}|w_e|+\frac{\tau}{2}r^{s-1}|w_e|\Big)+c\cdot\Bigg(\int_{\mathrm{G}(1,\tilde r)} r^{2s-4}|w_e|^2\\
&\quad\quad+\tau\int_{\mathrm{G}(1,\tilde r)} r^{2s-2}|w_e|+\int_{\mathrm{S}(1)}\Big(|\nabla w_e|^2+|w_e|^2\Big)+\int_{\mathrm{S}(\tilde r)}\tilde r^{2s-1}\Big(|\nabla w_e|^2+|w_e|^2\Big)\Bigg).
\end{aligned}
$$

Here, as well as in the sequel, $c\in(0,\infty)$ denotes a generic constant independent of ν, τ, w and g. According to Lemma A.2, the lower limit for $\tilde r\longrightarrow\infty$ of the last boundary integral vanishes. Thus we may omit it and replace $\mathrm{G}(1,\tilde r)$ by $\breve{\mathrm{U}}(1)$, such that using Young's inequality we end up with

$$
\big\|r^{s-1}\,\nabla w_e\big\|^2_{\mathrm{L}^2(\breve{\mathrm{U}}(1))}
$$

$$\leq c \cdot \Big(\big\| r^s g_e \big\|^2_{L^2(\check{U}(1))} + \tau \big\| r^{s-1} w_e \big\|^2_{L^2(\check{U}(1))} + \big\| r^{s-2} w_e \big\|^2_{L^2(\check{U}(1))} + \int_{S(1)} \big(|\nabla w_e|^2 + |w_e|^2 \big) \Big)$$

$$\leq c \cdot \Big(\| g_e \|^2_{L^2(\mathbb{R}^n)} + \tau \| w_e \|^2_{L^2_{s-1}(\mathbb{R}^n)} + \| w_e \|^2_{L^2_{s-2}(\mathbb{R}^n)} + \int_{S(1)} \big(|\nabla w_e|^2 + |w_e|^2 \big) \Big).$$

In addition, the surface integral is bounded by $\|w_e\|^2_{\mathbf{H}^2(U(1))}$ (trace theorem) and Lemma B.1 yields

$$\|w_e\|_{\mathbf{H}^2(U(2))} \leq c \cdot \|w_e\|_{\mathbf{H}^2_{-s}(\mathbb{R}^n)} \leq c \cdot \big(\|g_e\|_{L^2_s(\mathbb{R}^n)} + \|w_e\|_{L^2_{-s}(\mathbb{R}^n)} \big) ,$$

showing

$$\|\nabla w_e\|^2_{L^2_{s-1}(\mathbb{R}^n)} \leq c \cdot \Big(\|g\|^2_{L^2(\mathbb{R}^n)} + \tau \|w_e\|^2_{L^2_{s-1}(\mathbb{R}^n)} + \|w_e\|^2_{L^2_{s-2}(\mathbb{R}^n)} + \|w_e\|^2_{\mathbf{H}^2(U(1))}$$

$$\leq c \cdot \Big(\|g\|^2_{L^2(\mathbb{R}^n)} + \tau \|w\|^2_{L^2_{s-1}(\mathbb{R}^n)} + \|w\|^2_{L^2_{s-2}(\mathbb{R}^n)} + \|w\|^2_{L^2_{-s}(\mathbb{R}^n)} \Big).$$

By the differential equation we see

$$\|g\|_{L^2(\mathbb{R}^n)} \|w\|_{L^2(\mathbb{R}^n)} \geq \big| \mathrm{Im} \langle g , w \rangle_{L^2(\mathbb{R}^n)} \big| = \big| i\nu\tau \langle w , w \rangle_{L^2(\mathbb{R}^n)} \big| = \tau |\nu| \, \|w\|^2_{L^2(\mathbb{R}^n)} ,$$

hence ($-s > s - 2$)

$$\begin{aligned} \|\exp(-i\nu r) w\|_{H^1_{s-2}(\mathbb{R}^n)} &\leq c \cdot \Big(\|w_e\|_{L^2_{s-2}(\mathbb{R}^n)} + \|\nabla w_e\|_{L^2_{s-1}(\mathbb{R}^n)} \Big) \\ &\leq c \cdot \Big(\|g\|^2_{L^2_s(\mathbb{R}^n)} + \tau \|w\|^2_{L^2_{s-1}(\mathbb{R}^n)} + \|w\|^2_{L^2_{-s}(\mathbb{R}^n)} \Big) \qquad \text{(B.10)} \\ &\leq c \cdot \Big(\|g\|_{L^2_s(\mathbb{R}^n)} + \|w\|_{L^2_{-s}(\mathbb{R}^n)} \Big), \end{aligned}$$

and it remains to estimate $\|w\|_{L^2_{-s}(\mathbb{R}^n)}$. For that, we calculate

$$\mathrm{Im} \int_{G(1,\tilde{r})} g_e \overline{w}_e = \mathrm{Im} \int_{G(1,\tilde{r})} \Delta w_e \overline{w}_e + \int_{G(1,\tilde{r})} \nu \Big(\tau w_e + \frac{n-1}{r} w_e \Big) \overline{w}_e + 2\nu \, \mathrm{Re} \int_{G(1,\tilde{r})} \partial_r w_e \overline{w}_e = \cdots ,$$

using Lemma A.4 and obtain

$$\begin{aligned} &\nu \int_{G(1,\tilde{r})} r^{-2s} \Big((2s-1)|w_e|^2 + \tau r |w_e|^2 \Big) \\ &= \mathrm{Im} \int_{G(1,\tilde{r})} r^{1-2s} g_e \overline{w}_e - (2s-1) \int_{G(1,\tilde{r})} r^{-2s} \, \mathrm{Im}\,(\partial_r w_e \overline{w}_e) \\ &\quad + \int_{S(\tilde{r})} r^{1-2s} \Big(\tau |w_e|^2 + \mathrm{Im}\,(\partial_r w_e \overline{w}_e) \Big) - \int_{S(1)} \Big(\tau |w_e|^2 + \mathrm{Im}\,(\partial_r w_e \overline{w}_e) \Big) \\ &\leq \int_{G(1,\tilde{r})} r^s |g_e| r^{1-3s} |w_e| + (2s-1) \int_{G(1,\tilde{r})} r^{s-1} |\partial_r w_e| r^{1-3s} |w_e| \end{aligned}$$

$$+ c \cdot \left(\int\limits_{S(\tilde{r})} r^{1-2s}\Big(|w_e|^2 + |\partial_r w_e \overline{w}_e|\Big) + \int\limits_{S(1)} \Big(|w_e|^2 + |\partial_r w_e \overline{w}_e|\Big) \right)$$

$$\leq \Big(\big\|r^s g_e\big\|_{L^2(G(1,\tilde{r}))} + (2s-1)\,\big\|r^{s-1}\,\nabla w_e\big\|_{L^2(G(1,\tilde{r}))} \Big) \cdot \big\|r^{1-3s} w_e\big\|_{L^2(G(1,\tilde{r}))}$$

$$+ c \cdot \left(\int\limits_{S(1)} \Big(|w_e|^2 + |\nabla w_e|^2\Big) + \int\limits_{S(\tilde{r})} \tilde{r}^{1-2s}\Big(|w_e|^2 + |\nabla w_e|^2\Big) \right).$$

As before, the lower limit for $\tilde{r} \longrightarrow \infty$ of the last boundary integral vanishes (cf. Lemma A.2 and observe that $w_e \in \mathbf{H}^2(\Omega)$, $s > 0$), such that we may omit it and replace $G(1,\tilde{r})$ by $\breve{U}(1)$, yielding (with (B.10))

$$\big\|r^{-s} w_e\big\|^2_{L^2(\breve{U}(1))}$$

$$\leq c \cdot \left(\Big(\big\|r^s g_e\big\|_{L^2(\breve{U}(1))} + \big\|r^{s-1}\nabla w_e\big\|_{L^2(\breve{U}(1))} \Big) \cdot \big\|r^{1-3s} w_e\big\|_{L^2(\breve{U}(1))} + \int\limits_{S(1)} \Big(|w_e|^2 + |\nabla w_e|^2\Big) \right)$$

$$\leq c \cdot \left(\Big(\|g_e\|_{L^2_s(\mathbb{R}^n)} + \|\nabla w_e\|_{L^2_{s-1}(\mathbb{R}^n)} \Big) \cdot \|w_e\|_{L^2_{1-3s}(\mathbb{R}^n)} + \int\limits_{S(1)} \Big(|w_e|^2 + |\nabla w_e|^2\Big) \right).$$

As the surface integral is bounded by $\|w_e\|^2_{\mathbf{H}^2(U(1))}$ (trace theorem) and with (B.10) we obtain

$$\|w_e\|^2_{L^2_{-s}(\mathbb{R}^n)} \leq c \cdot \left(\Big(\|g_e\|_{L^2_s(\mathbb{R}^n)} + \|\nabla w_e\|_{L^2_{s-1}(\mathbb{R}^n)} \Big) \cdot \|w_e\|_{L^2_{1-3s}(\mathbb{R}^n)} + \|w_e\|^2_{\mathbf{H}^2(U(2))} \right)$$

$$\leq c \cdot \left(\Big(\|g_e\|_{L^2_s(\mathbb{R}^n)} + \|w_e\|_{L^2_{-s}(\mathbb{R}^n)} \Big) \cdot \|w_e\|_{L^2_{1-3s}(\mathbb{R}^n)} + \|w_e\|^2_{\mathbf{H}^2(U(2))} \right),$$

hence (Young's inequality)

$$\|w_e\|^2_{L^2_{-s}(\mathbb{R}^n)} \leq c \cdot \Big(\|g_e\|_{L^2_s(\mathbb{R}^n)} + \|w_e\|_{L^2_{1-3s}(\mathbb{R}^n)} + \|w_e\|^2_{\mathbf{H}^2(U(2))} \Big),$$

Finally, using once again Lemma B.1 we arrive at

$$\|w_e\|^2_{L^2_{-s}(\mathbb{R}^n)} \leq c \cdot \Big(\|g_e\|_{L^2_s(\mathbb{R}^n)} + \|w_e\|_{L^2_{1-3s}(\mathbb{R}^n)} \Big),$$

which together with (B.10) and Lemma A.1 implies

$$\|w\|_{L^2_{-s}(\mathbb{R}^n)} + \|\exp(-i\nu r)\, w\|_{H^1_{s-2}(\mathbb{R}^n)} \leq c \cdot \Big(\|g\|_{L^2_s(\mathbb{R}^n)} + \|w\|_{L^2(\Omega(\delta))} \Big) \tag{B.11}$$

with $c, \delta > 0$ independent of ν, τ, w and g. □

Appendix C. Proofs in the case of the time-harmonic Maxwell equations

This section deals with the proofs of the decomposition lemma, the polynomial decay, and the a priori estimate, which we skipped in the main part.

Proof of Lemma 4.1. We start with $u = \eta u + \check{\eta} u$, noting that $\check{\eta} u \in \mathbf{R}_t$. Moreover,

$$\operatorname{Rot} \check{\eta} u = \mathrm{C}_{\operatorname{Rot},\check{\eta}} u + \check{\eta} \operatorname{Rot} u = \mathrm{C}_{\operatorname{Rot},\check{\eta}} u - i\check{\eta}\Lambda f - i\omega\check{\eta}\Lambda u$$

and we have

$$(\operatorname{Rot} + i\omega\Lambda_0)\check{\eta} u = (\mathrm{C}_{\operatorname{Rot},\check{\eta}} - i\omega\check{\eta}\hat{\Lambda})u - i\check{\eta}\Lambda f = f_1 \in \mathsf{L}^2_s,$$

since supp $\nabla\check{\eta}$ is compact and $t + \kappa \geq s$. According to [26, Theorem 4],

$$f_1 = f_{\mathsf{R}} + f_{\mathsf{D}} + f_{\mathcal{S}} \in {}_0\mathsf{R}_s \dotplus {}_0\mathsf{D}_s \dotplus \mathcal{S}_s$$

holds and we obtain

$$i\omega\check{\eta}\Lambda_0 u = f_1 - \operatorname{Rot} \check{\eta} u = f_{\mathsf{D}} - \operatorname{Rot} \check{\eta} u + f_{\mathsf{R}} + f_{\mathcal{S}} .$$

Defining

- $u_1 := -\dfrac{i}{\omega}\Lambda_0^{-1}(f_{\mathsf{R}} + f_{\mathcal{S}}) \in \mathsf{R}_s$;
- $\tilde{u} := \check{\eta} u - u_1 = \dfrac{i}{\omega}\Lambda_0^{-1}(\operatorname{Rot} \check{\eta} u - f_{\mathsf{D}}) \in \mathbf{R}_t \cap {}_0\mathsf{D}_t$,

[8, Lemma 4.2] shows $\tilde{u} \in \mathbf{H}^1_t$ and we have

$$(\operatorname{Rot} + i\omega\Lambda_0)\tilde{u} = \operatorname{Rot}(\check{\eta} u - u_1) + i\omega\Lambda_0\tilde{u} = f_{\mathsf{D}} + \frac{i}{\omega}\,\tilde{\Lambda}_0^{-1}\ \operatorname{Rot} f_{\mathcal{S}} = f_2 \in {}_0\mathsf{D}_s .$$

Next, we solve $(\operatorname{Rot} + 1)u_2 = f_2$. Using Fourier transformation, we look at

$$\hat{u}_2 := (1 + r^2)^{-1}(1 - ir\,\Xi)\,\mathcal{F}(f_2)$$

Since $s > 1/2$ and $f_2 \in \mathsf{L}^2_s$, we obtain $\hat{u} \in \mathsf{L}^2_1$, hence $u_2 := \mathcal{F}^{-1}(\hat{u}_2) \in \mathbf{H}^1$. Moreover, $\mathcal{F}(\mathcal{F}(f_2)) = \mathcal{P}(f_2) \in \mathsf{L}^2_s$ ($\mathcal{P}$: parity operator) yielding $\mathcal{F}(f_2) \in \mathbf{H}^s$ and as product of an $\mathbf{H}^s$-field with bounded C^∞-functions, $\hat{u} \in \mathbf{H}^s$ (cf. [30, Lemma 3.2]), hence $u_2 \in \mathsf{L}^2_s$. In addition a straight forward calculation shows $\mathcal{F}((\operatorname{Rot} + 1)u_2) = \mathcal{F}(f_2)$, which by [8, Lemma 4.2] implies

$$(\operatorname{Rot} + 1)u_2 = f_2 \quad \text{and} \quad u_2 \in \mathbf{H}^1_s \cap {}_0\mathsf{D}_s .$$

Then ($t \leq s$)

$$u_3 := \tilde{u} - u_2 \in \mathbf{H}^1_t \cap {}_0\mathsf{D}_t$$

satisfies

$$(\operatorname{Rot} + i\omega\Lambda_0)u_3 = (\operatorname{Rot} + i\omega\Lambda_0)\tilde{u} - (\operatorname{Rot} + i\omega\Lambda_0)u_2$$

$$= f_2 - (\operatorname{Rot} + 1)u_2 + (1 - i\omega\Lambda_0)u_2 = (1 - i\omega\Lambda_0)u_2 \in \mathsf{H}^1_{\mathsf{s}} \cap {}_0\mathsf{D}_{\mathsf{s}}$$

and using once more [8, Lemma 4.2] we get

$$u_3 \in \mathsf{H}^2_{\mathsf{t}} \cap {}_0\mathsf{D}_{\mathsf{t}} .$$

Finally

$$\begin{aligned}\Delta u_3 &= \operatorname{Rot}(\operatorname{Rot} u_3) = (1 - i\omega\widetilde{\Lambda}_0)\operatorname{Rot} u_2 - i\omega\widetilde{\Lambda}_0 \operatorname{Rot} u_3 \\ &= (1 - i\omega\widetilde{\Lambda}_0)(f_2 - u_2) - i\omega\widetilde{\Lambda}_0((1 - i\omega\Lambda_0)u_2 - i\omega\Lambda_0 u_3) \\ &= (1 - i\omega\widetilde{\Lambda}_0)f_2 - (1 + \omega^2\varepsilon_0\mu_0)u_2 - \omega^2\varepsilon_0\mu_0 u_3\end{aligned}$$

holds, and hence

$$(\Delta + \omega^2\varepsilon_0\mu_0)u_3 = (1 - i\omega\widetilde{\Lambda}_0)f_2 - (1 + \omega^2\varepsilon_0\mu_0)u_2.$$

The asserted estimates follow by straightforward calculations using [8, Lemma 4.2] and the continuity of the projections from $\mathsf{L}^2_{\mathsf{s}}$ into ${}_0\mathsf{R}_{\mathsf{s}}$, ${}_0\mathsf{D}_{\mathsf{s}}$ and $\mathcal{S}_s$. □

Proof of Lemma 4.2. As for $t \geq s - 1$ there is nothing to prove, we concentrate on

$$u \in \mathbf{R}_{\mathsf{t}}(\Omega) \quad \text{with} \quad -1/2 < t < s - 1 .$$

Therefore, assume first that in addition

$$s - \kappa < t \implies t < s < t + \kappa .$$

Then we may apply Lemma 4.1 and decompose the field u in

$$u = \eta u + u_1 + u_2 + u_3 ,$$

with $\eta u + u_1 + u_2 \in \mathbf{R}_{\mathsf{s}}(\Omega)$ and $u_3 \in \mathsf{H}^2_{\mathsf{t}}$ satisfying $(\Delta + \omega^2\varepsilon_0\mu_0)u_3 \in \mathsf{L}^2_{\mathsf{s}}$. Thus the polynomial decay for the Helmholtz equation (cf. Lemma B.2) shows

$$u_3 \in \mathsf{H}^2_{\mathsf{s}-1} \quad \text{and} \quad \|u_3\|_{\mathsf{H}^2_{\mathsf{s}-1}} \leq c \cdot \left(\left\|(\Delta + \omega^2\varepsilon_0\mu_0)u_3\right\|_{\mathsf{L}^2_{\mathsf{s}}} + \|u_3\|_{\mathsf{L}^2_{\mathsf{s}-2}} \right),$$

$c = c(s, J) > 0$, yielding $u = \eta u + u_1 + u_2 + u_3 \in \mathbf{R}_{\mathsf{s}-1}(\Omega)$. Moreover, using the estimates of Lemma 4.1 we obtain uniformly with respect to ω, u, and f

$$\begin{aligned}\|u\|_{\mathbf{R}_{\mathsf{s}-1}(\Omega)} &\leq c \cdot \left(\|f\|_{\mathsf{L}^2_{\mathsf{s}}(\Omega)} + \|u\|_{\mathsf{L}^2_{\mathsf{s}-\kappa}(\Omega)} + \|u_3\|_{\mathsf{L}^2_{\mathsf{s}-1}} \right) \\ &\leq c \cdot \left(\|f\|_{\mathsf{L}^2_{\mathsf{s}}(\Omega)} + \|u\|_{\mathsf{L}^2_{\mathsf{s}-\kappa}(\Omega)} + \left\|(\Delta + \omega^2\varepsilon_0\mu_0)u_3\right\|_{\mathsf{L}^2_{\mathsf{s}}} + \|u_3\|_{\mathsf{L}^2_{\mathsf{s}-2}} \right) \\ &\leq c \cdot \left(\|f\|_{\mathsf{L}^2_{\mathsf{s}}(\Omega)} + \|u\|_{\mathsf{L}^2_{\mathsf{s}-\mathsf{m}}(\Omega)} \right),\end{aligned}$$

where $m := \min\{\kappa, 2\}$ and applying Lemma A.1 we end up with

$$\|u\|_{\mathsf{R}_{s-1}(\Omega)} \leq c \cdot \Big(\|f\|_{\mathsf{L}^2_s(\Omega)} + \|u\|_{\mathsf{L}^2(\Omega(\delta))} \Big),$$

for $c, \delta \in (0, \infty)$ independent of ω, u and f. So let us switch to the case

$$t \leq s - \kappa \quad \Longrightarrow \quad t + \kappa \leq s\,.$$

Here, the idea is to approach s by overlapping intervals to which the first case is applicable. For that, we choose some $\hat{k} \in \mathbb{N}$, such that with $\gamma := (\kappa - 1)/2 > 0$ we have

$$t + \kappa + (\hat{k} - 1) \cdot \gamma \leq s \leq t + \kappa + \hat{k} \cdot \gamma\,,$$

and for $k = 0, 1, \ldots, \hat{k}$ we define

$$t_k := t + k \cdot \gamma \quad \text{as well as} \quad s_k := t_{k+1} + 1 = t_k + (\kappa + 1)/2\,.$$

Then (as $\kappa > 1$)

$$\begin{aligned} t_{k+1} < s_k &= t_{k+1} + 1 = t + \kappa + (k - 1) \cdot \gamma \leq s, \\ t_k < t_{k+1} + 1 = s_k &= t_k + (\kappa + 1)/2 < t_k + \kappa, \end{aligned}$$

such that we can successively apply the first case, ending up with $u \in \mathsf{R}_{s_{\hat{k}}-1}(\Omega)$. If $s = s_{\hat{k}}$, we are done. Otherwise, we choose $t_{\hat{k}+1} := s_{\hat{k}} - 1$ and apply the first case once more, since

$$t_{\hat{k}+1} < s_{\hat{k}} < s \leq t + \kappa + \hat{k} \cdot \gamma = t_{\hat{k}+1} + \kappa\,.$$

Either way, we obtain $u \in \mathsf{R}_{s-1}(\Omega)$ and now the estimate follows as in the first case. □

Proof of Lemma 4.4. Without loss of generality, we may assume $s \in (1/2, 1)$. Then we have $s \in \mathbb{R} \setminus \mathbb{I}$ with $0 < s < \kappa$ and we can apply Lemma 4.1 (with $t = 0$) to decompose $u := \mathcal{L}_\omega f \in \mathsf{R}_{\Gamma_1}(\Omega)$ into

$$u = \eta u + u_1 + u_2 + u_3$$

with $u_3 \in \mathsf{H}^2$ solving

$$(\Delta + \omega^2 \varepsilon_0 \mu_0) u_3 = (1 - i\omega\widetilde{\Lambda}_0) f_2 - (1 + \omega^2 \varepsilon_0 \mu_0) u_2 =: f_3 \in \mathsf{L}^2_s\,,$$

where f_2 is defined as in Lemma 4.1. Moreover, the estimates from Lemma 4.1 along with

$$(\operatorname{Rot} - i\omega\sqrt{\varepsilon_0\mu_0}\,\Xi) u = -i\Lambda f - i\omega(\Lambda_0 + \sqrt{\varepsilon_0\mu_0}\,\Xi) u - i\omega\hat{\Lambda} u$$

yield

$$\begin{aligned}
&\|u\|_{\mathbf{R}_t(\Omega)} + \big\|(\Lambda_0 + \sqrt{\varepsilon_0\mu_0}\,\Xi)u\big\|_{\mathsf{L}^2_{s-1}(\Omega)} \\
&\leq c\cdot\Big(\,\|u\|_{\mathbf{R}_t(\Omega)} + \big\|(\mathrm{Rot} - i\omega\sqrt{\varepsilon_0\mu_0}\,\Xi)u\big\|_{\mathsf{L}^2_{s-1}(\Omega)} + \|f\|_{\mathsf{L}^2_s(\Omega)} + \|u\|_{\mathsf{L}^2_{s-\kappa}(\Omega)}\Big) \qquad \text{(C.1)}\\
&\leq c\cdot\Big(\,\|u_3\|_{\mathsf{L}^2_t} + \big\|(\mathrm{Rot} - i\omega\sqrt{\varepsilon_0\mu_0}\,\Xi)u_3\big\|_{\mathsf{L}^2_{s-1}} + \|f\|_{\mathsf{L}^2_s(\Omega)} + \|u\|_{\mathsf{L}^2_{s-\kappa}(\Omega)}\Big),
\end{aligned}$$

with $c = c(s,t,J) > 0$. Due to the monotonicity of the norms with respect to t and s, we may assume t and s to be close enough to $-1/2$ respectively $1/2$ such that $1 < s - t < \kappa$ holds. Hence, the assertion follows by (C.1) and Lemma A.1, if we can show

$$\|u_3\|_{\mathsf{L}^2_t} + \big\|(\mathrm{Rot} - i\omega\sqrt{\varepsilon_0\mu_0}\,\Xi)u_3\big\|_{\mathsf{L}^2_{s-1}} \leq c\cdot\Big(\,\|f\|_{\mathsf{L}^2_s(\Omega)} + \|u\|_{\mathsf{L}^2_{s-\kappa}(\Omega)}\Big),$$

with $c \in (0,\infty)$ independent of ω, u and f. Therefore, note that the self-adjointness of the Laplacian $\Delta : \mathbf{H}^2 \subset \mathsf{L}^2 \longrightarrow \mathsf{L}^2$ yields $(\Delta + \omega^2\varepsilon_0\mu_0)^{-1} f_3 = u_3$ and applying Lemma B.3 componentwise, we obtain

$$\|u_3\|_{\mathsf{L}^2_t} + \big\|\exp(-i\lambda\sqrt{\varepsilon_0\mu_0}r)u_3\big\|_{\mathsf{H}^1_{s-2}} \leq c\cdot\Big(\,\|f_3\|_{\mathsf{L}^2_s} + \|u_3\|_{\mathsf{L}^2(\Omega(\delta))}\Big).$$

With $\mathrm{Rot}(\exp(-i\lambda\sqrt{\varepsilon_0\mu_0}r)u_3) = \exp(-i\lambda\sqrt{\varepsilon_0\mu_0}r)(\mathrm{Rot} - i\lambda\sqrt{\varepsilon_0\mu_0}\,\Xi)u_3$ this leads to

$$\begin{aligned}
&\|u_3\|_{\mathsf{L}^2_t} + \big\|(\mathrm{Rot} - i\lambda\sqrt{\varepsilon_0\mu_0}\,\Xi)u_3\big\|_{\mathsf{L}^2_{s-1}} \\
&\leq \|u_3\|_{\mathsf{L}^2_t} + \big\|\mathrm{Rot}\,(\exp(-i\lambda\sqrt{\varepsilon_0\mu_0}r)u_3)\big\|_{\mathsf{L}^2_{s-1}} \qquad \text{(C.2)}\\
&\leq \|u_3\|_{\mathsf{L}^2_t} + \big\|\exp(-i\lambda\sqrt{\varepsilon_0\mu_0}r)u_3\big\|_{\mathsf{H}^1_{s-2}} \leq c\cdot\Big(\,\|f_3\|_{\mathsf{L}^2_s} + \|u_3\|_{\mathsf{L}^2(\Omega(\delta))}\Big),
\end{aligned}$$

where $c > 0$ is not depending on ω, u_3 and f_3. But, actually, we would like to estimate $(\mathrm{Rot} - i\omega\sqrt{\varepsilon_0\mu_0}\,\Xi)u_3$. For that, we need some additional arguments, starting with the observation that

$$\omega = |\lambda|(1 + (\sigma/\lambda)^2)^{1/4}\cdot\begin{cases} \exp(i\varphi/2) & \text{for } \lambda > 0 \\ \exp(i(\varphi/2 + \pi)) & \text{for } \lambda < 0 \end{cases}$$
$$\text{with}\quad \varphi := \arctan(\sigma/\lambda) \in \Big(-\frac{\pi}{2}, \frac{\pi}{2}\Big),$$

hence $|\mathrm{Re}\,(\omega)| \geq \sqrt{2}/2\cdot|\lambda|$. Then $|\omega + \lambda| \geq \sqrt{3/2}\cdot|\lambda|$ and we have

$$|\omega - \lambda|^2 = \left|\frac{\omega^2 - \lambda^2}{\omega + \lambda}\right|^2 = \left|\frac{i\sigma\lambda}{\omega + \lambda}\right|^2 \leq \frac{2}{3}\cdot\sigma^2.$$

From this and the resolvent estimate,

$$\|f_3\|_{\mathsf{L}^2} = \big\|(\Delta + \omega^2\varepsilon_0\mu_0)\,u_3\big\|_{\mathsf{L}^2} \geq |\mathrm{Im}\,(\omega^2\varepsilon_0\mu_0)|\cdot\|u_3\|_{\mathsf{L}^2} = \varepsilon_0\mu_0\sigma|\lambda|\cdot\|u_3\|_{\mathsf{L}^2},$$

we obtain ($s > 1/2$)

$$\begin{aligned}\left\|(\mathrm{Rot} - i\omega\sqrt{\varepsilon_0\mu_0}\,\Xi)u_3\right\|_{\mathsf{L}^2_{s-1}} &\leq \left\|(\mathrm{Rot} - i\lambda\sqrt{\varepsilon_0\mu_0}\,\Xi)u_3\right\|_{\mathsf{L}^2_{s-1}} + \left\|(\omega-\lambda)\sqrt{\varepsilon_0\mu_0}\,\Xi u_3\right\|_{\mathsf{L}^2_{s-1}}\\ &\leq \left\|(\mathrm{Rot} - i\lambda\sqrt{\varepsilon_0\mu_0}\,\Xi)u_3\right\|_{\mathsf{L}^2_{s-1}} + c\cdot|\lambda|^{-1}\,\|f_3\|_{\mathsf{L}^2_s}\,,\end{aligned}$$

such that with (C.2) and the estimates from Lemma 4.1 uniformly with respect to ω, u and f

$$\|u_3\|_{\mathsf{L}^2_t} + \left\|(\mathrm{Rot} - i\omega\sqrt{\varepsilon_0\mu_0}\,\Xi)u_3\right\|_{\mathsf{L}^2_{s-1}} \leq c\cdot\Big(\|f\|_{\mathsf{L}^2_s(\Omega)} + \|u\|_{\mathsf{L}^2_{s-\kappa}(\Omega)}\Big). \qquad \square$$

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