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3 Weck's selection theorem: The Maxwell compactness property for bounded weak Lipschitz domains with mixed boundary conditions in arbitrary dimensions

Abstract: It is proved that the space of differential forms with weak exterior and co-derivative, is compactly embedded into the space of square integrable differential forms. Mixed boundary conditions on weak Lipschitz domains are considered. Furthermore, canonical applications such as Maxwell estimates, Helmholtz decompositions and a static solution theory are proved. As a side product and crucial tool for our proofs, we show the existence of regular potentials and regular decompositions as well.

Keywords: Maxwell compactness property, weak Lipschitz domain, Maxwell estimate, Helmholtz decomposition, electro-magneto statics, mixed boundary conditions, vector potentials

MSC 2010: 35A23, 35Q61

1 Introduction

The aim of this contribution is to prove a compact embedding, so called "Weck's selection theorem" or (generalized) Maxwell compactness property [28, 29, 24], of differential q-forms with weak exterior and co-derivative into the space of square integrable q-forms subject to mixed boundary conditions on bounded weak Lipschitz domains $\Omega \subset \mathbb{R}^N$, i. e.,

$$\mathring{\mathsf{D}}^q_{\Gamma_\tau}(\Omega) \cap \varepsilon^{-1}\mathring{\Delta}^q_{\Gamma_\nu}(\Omega) \hookrightarrow \mathsf{L}^{2,q}(\Omega)$$

is compact. The main result is given by Theorem 4.8. Here, $N \geq 2$ and $0 \leq q \leq N$ are natural numbers, the dimension of the domain Ω and the rank of the differential

Note: In memoriam of our dear friend and mentor, Karl-Josef (Charlie) Witsch (1948–2017).
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forms, respectively. This generalises the results from [1], where bounded weak Lipschitz domains in the classical setting of $\mathbb{R}^3$ were considered. In fact, the results from [1] can be recovered by setting $N = 3$ and $q = 1$ or $q = 2$.

Similar results for strong Lipschitz domains in three dimensions can be found in [11, 8]. For a historical overview of the mathematical treatment of Weck's selection theorem (Maxwell compactness property), see [1, 13, 25] and the literature cited therein. In particular, let us mention the important papers [28, 27, 24, 3, 32, 11, 25]. We emphasise that in [32] Witsch was able to go even beyond Lipschitz regularity (p-cusps). In [30], Weck applied Witsch's ideas to the theory of elasticity.

The central role of compact embeddings of this type can, for example, be seen in connection with Hilbert space complexes, where the compact embeddings immediately provide closed ranges, solution theories by continuous inverses, Friedrichs/Poincaré-type estimates, and access to Hodge–Helmholtz-type decompositions, Fredholm theory, div–curl-type lemmas, and a posteriori error estimation; see [21, 20, 22]. In exterior domains, where local versions of the compact embeddings hold, one obtains radiation solutions (scattering theory) with the help of Eidus' limiting absorption principle [5–7]; see [14–16, 18, 17, 19]. We elaborate on some of these applications in our Section 5.

Finally, we note that by the same arguments as in [24] our results extend to Riemannian manifolds.

2 Notation, preliminaries and outline of the proof

Let $\Omega \subset \mathbb{R}^N$ be a bounded weak Lipschitz domain. For a precise definition of weak Lipschitz domains, see Definitions 2.3 and 2.5. In short, Ω is an N-dimensional $\mathsf{C}^{0,1}$-submanifold of $\mathbb{R}^N$ with boundary, i. e., a manifold with Lipschitz atlas. Let $\Gamma := \partial\Omega$, which is itself an $(N-1)$-dimensional Lipschitz-manifold without boundary, consist of two relatively open subsets Γ_τ and Γ_ν such that $\overline{\Gamma}_\tau \cup \overline{\Gamma}_\nu = \Gamma$ and $\Gamma_\tau \cap \Gamma_\nu = \emptyset$. The separating set $\overline{\Gamma}_\tau \cap \overline{\Gamma}_\nu$ (interface) will be assumed to be a, not necessarily connected, $(N-2)$-dimensional Lipschitz-submanifold of Γ. We shall call (Ω, Γ_τ) a weak Lipschitz pair.

We will be working in the framework of alternating differential forms; see, for example, [10]. The vector space $\mathring{\mathsf{C}}^{\infty,q}(\Omega)$ is defined as the subset of $\mathsf{C}^{\infty,q}(\Omega)$, the set of smooth alternating differential forms of rank q, having compact support in Ω. Together with the inner product,

$$\langle E, H\rangle_{\mathsf{L}^{2,q}(\Omega)} := \int_\Omega E \wedge \star H$$

it is an inner product space.[1] We may then define $\mathsf{L}^{2,q}(\Omega)$ as the completion of $\mathring{\mathsf{C}}^{\infty,q}(\Omega)$ with respect to the corresponding norm. $\mathsf{L}^{2,q}(\Omega)$ can be identified with those q-forms having L^2-coefficients with respect to any coordinate system. Using the weak version of Stokes' theorem,

$$\langle \mathrm{d}\, E, H\rangle_{\mathsf{L}^{2,q+1}(\Omega)} = -\langle E, \delta\, H\rangle_{\mathsf{L}^{2,q}(\Omega)}, \qquad E \in \mathring{\mathsf{C}}^{\infty,q}(\Omega),\ H \in \mathring{\mathsf{C}}^{\infty,q+1}(\Omega), \tag{1}$$

weak versions of the exterior derivative and co-derivative can be defined. Here, d is the exterior derivative, $\delta = (-1)^{N(q-1)} \star \mathrm{d} \star$ the co-derivative and $\star$ the Hodge-star-operator on Ω. We thus introduce the Sobolev (Hilbert) spaces (equipped with their natural graph norms)

$$\mathsf{D}^q(\Omega) := \{E \in \mathsf{L}^{2,q}(\Omega) : \mathrm{d}\, E \in \mathsf{L}^{2,q+1}(\Omega)\}, \quad \Delta^q(\Omega) := \{E \in \mathsf{L}^{2,q}(\Omega) : \delta\, E \in \mathsf{L}^{2,q-1}(\Omega)\}$$

in the distributional sense. It holds

$$\star \mathsf{D}^q(\Omega) = \Delta^{N-q}(\Omega), \qquad \star \Delta^q(\Omega) = \mathsf{D}^{N-q}(\Omega).$$

We further define the test forms

$$\mathring{\mathsf{C}}^{\infty,q}_{\Gamma_\tau}(\Omega) := \{\varphi|_\Omega \ : \ \varphi \in \mathring{\mathsf{C}}^{\infty,q}(\mathbb{R}^N),\ \operatorname{dist}(\operatorname{supp}\varphi, \Gamma_\tau) > 0\}$$

and note that $\mathring{\mathsf{C}}^{\infty,q}_{\emptyset}(\Omega) = \mathsf{C}^{\infty,q}(\overline{\Omega})$. We now take care of boundary conditions. First, we introduce strong boundary conditions as closures of test forms by

$$\mathring{\mathsf{D}}^q_{\Gamma_\tau}(\Omega) := \overline{\mathring{\mathsf{C}}^{\infty,q}_{\Gamma_\tau}(\Omega)}^{\mathsf{D}^q(\Omega)}, \quad \mathring{\Delta}^q_{\Gamma_\nu}(\Omega) := \overline{\mathring{\mathsf{C}}^{\infty,q}_{\Gamma_\nu}(\Omega)}^{\Delta^q(\Omega)}. \tag{2}$$

For the full boundary case $\Gamma_\tau = \Gamma$ (resp., $\Gamma_\nu = \Gamma$), we set

$$\mathring{\mathsf{D}}^q(\Omega) := \mathring{\mathsf{D}}^q_{\Gamma_\tau}(\Omega), \quad \mathring{\Delta}^q(\Omega) := \mathring{\Delta}^q_{\Gamma_\nu}(\Omega).$$

Furthermore, we define weak boundary conditions in the spaces

$$\begin{aligned} \mathring{\mathbf{D}}^q_{\Gamma_\tau}(\Omega) &:= \{E \in \mathsf{D}^q(\Omega) \ : \ \langle E, \delta\,\varphi\rangle_{\mathsf{L}^{2,q}(\Omega)} = -\langle \mathrm{d}\, E, \varphi\rangle_{\mathsf{L}^{2,q+1}(\Omega)} \text{ for all } \varphi \in \mathring{\mathsf{C}}^{\infty,q+1}_{\Gamma_\nu}(\Omega)\}, \\ \mathring{\boldsymbol{\Delta}}^q_{\Gamma_\nu}(\Omega) &:= \{H \in \Delta^q(\Omega) \ : \ \langle H, \mathrm{d}\,\varphi\rangle_{\mathsf{L}^{2,q}(\Omega)} = -\langle \delta\, H, \varphi\rangle_{\mathsf{L}^{2,q-1}(\Omega)} \text{ for all } \varphi \in \mathring{\mathsf{C}}^{\infty,q-1}_{\Gamma_\tau}(\Omega)\}, \end{aligned} \tag{3}$$

and again for $\Gamma_\tau = \Gamma$ (resp., $\Gamma_\nu = \Gamma$), we set

$$\mathring{\mathbf{D}}^q(\Omega) := \mathring{\mathbf{D}}^q_{\Gamma_\tau}(\Omega), \quad \mathring{\boldsymbol{\Delta}}^q(\Omega) := \mathring{\boldsymbol{\Delta}}^q_{\Gamma_\nu}(\Omega).$$

We note that in Definitions (1) and (2), the smooth test forms can by mollification be replaced by their respective Lipschitz continuous counterparts, e. g., $\mathring{\mathsf{C}}^{\infty,q}_{\Gamma_\tau}(\Omega)$ can be

1 For simplicity, we work in a real Hilbert space setting.

replaced by $\mathring{\mathsf{C}}^{0,1,q}_{\Gamma_\tau}(\Omega)$. Similarly, in Definition (3) the smooth test forms can by completion be replaced by their respective closures, i. e., $\mathring{\mathsf{C}}^{\infty,q+1}_{\Gamma_\nu}(\Omega)$ and $\mathring{\mathsf{C}}^{\infty,q-1}_{\Gamma_\tau}(\Omega)$ can be replaced by $\mathring{\Delta}^{q+1}_{\Gamma_\nu}(\Omega)$ and $\mathring{\mathsf{D}}^{q-1}_{\Gamma_\tau}(\Omega)$, respectively. In (2) and (3), homogeneous tangential and normal traces on Γ_τ, respectively Γ_ν, are generalised. Clearly,

$$\mathring{\mathsf{D}}^q_{\Gamma_\tau}(\Omega) \subset \mathring{\mathbf{D}}^q_{\Gamma_\tau}(\Omega), \quad \mathring{\Delta}^q_{\Gamma_\nu}(\Omega) \subset \mathring{\mathbf{\Delta}}^q_{\Gamma_\nu}(\Omega)$$

and it will later be shown that in fact equality holds under our regularity assumptions on the boundary. In case of full boundary conditions, the equality even holds without any assumptions on the regularity of the boundary, as can be seen by a short functional analytic argument (see [1]) but which is unavailable for the mixed boundary case.

We define the closed subspaces

$$\mathsf{D}^q_0(\Omega) := \{E \in \mathsf{D}^q(\Omega) \,:\, \mathrm{d}\,E = 0\}, \quad \Delta^q_0(\Omega) := \{E \in \Delta^q(\Omega) \,:\, \delta\,E = 0\}$$

as well as $\mathring{\mathsf{D}}^q_{\Gamma_\tau,0}(\Omega) := \mathring{\mathsf{D}}^q_{\Gamma_\tau}(\Omega) \cap \mathsf{D}^q_0(\Omega)$ and $\mathring{\Delta}^q_{\Gamma_\nu,0}(\Omega) := \mathring{\Delta}^q_{\Gamma_\nu}(\Omega) \cap \Delta^q_0(\Omega)$. Analogously, for the weak spaces,

$$\mathring{\mathbf{D}}^q_{\Gamma_\tau,0}(\Omega) := \mathring{\mathbf{D}}^q_{\Gamma_\tau}(\Omega) \cap \mathsf{D}^q_0(\Omega), \quad \mathring{\mathbf{\Delta}}^q_{\Gamma_\nu,0}(\Omega) := \mathring{\mathbf{\Delta}}^q_{\Gamma_\nu}(\Omega) \cap \Delta^q_0(\Omega).$$

In addition to the latter canonical Sobolev spaces, we will also need the classical Sobolev spaces for the Euclidean components of q-forms. Note that Ω, together with the global identity chart, is a N-dimensional Riemannian manifold. In particular, q-forms $E \in \mathsf{L}^{2,q}(\Omega)$ can be represented globally in Cartesian coordinates by their components E_I, i. e., $E = \sum_I E_I \mathrm{d}x^I$. Here, we use the ordered multi-index notation $\mathrm{d}x^I = \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_q}$ for $I = (i_1, \ldots, i_q) \in \{1, \ldots, N\}^q$. The inner product for $E, H \in \mathsf{L}^{2,q}(\Omega)$ is given by

$$\langle E, H\rangle_{\mathsf{L}^{2,q}(\Omega)} = \int_\Omega E \wedge \star H = \sum_I \int_\Omega E_I H_I = \sum_I \langle E_I, H_I\rangle_{\mathsf{L}^2(\Omega)} = \langle \vec{E}, \vec{H}\rangle_{\mathsf{L}^2(\Omega)},$$

where we introduce the vector proxy notation:

$$\vec{E} = [E_I]_I \in \mathsf{L}^2(\Omega; \mathbb{R}^{N_q}), \quad N_q := \binom{N}{q}.$$

For $k \in \mathbb{N}$, we can now define the Sobolev space $\mathsf{H}^{k,q}(\Omega)$ as the subset of $\mathsf{L}^{2,q}(\Omega)$ having each component E_I in $\mathsf{H}^k(\Omega)$. In these cases, we have for $|\alpha| \le k$,

$$\partial^\alpha E = \sum_I \partial^\alpha E_I \mathrm{d}x^I \quad \text{and} \quad \langle E, H\rangle_{\mathsf{H}^{k,q}(\Omega)} := \sum_{0 \le |\alpha| \le k} \langle \partial^\alpha E, \partial^\alpha H\rangle_{\mathsf{L}^{2,q}(\Omega)}$$

and we use the vector proxy notation also for the gradient, i. e.,

$$\nabla \vec{E} = [\partial_n E_I]_{n,I} = [\ldots \nabla E_I \ldots]_I \in \mathsf{L}^2(\Omega; \mathbb{R}^{N \times N_q}).$$

In particular, for $E, H \in \mathsf{H}^{1,q}(\Omega)$,

$$\begin{aligned}\langle E, H\rangle_{\mathsf{H}^{1,q}(\Omega)} &= \langle E, H\rangle_{\mathsf{L}^{2,q}(\Omega)} + \sum_{n=1}^{N}\langle \partial_n E, \partial_n H\rangle_{\mathsf{L}^{2,q}(\Omega)} = \sum_I \Big(\int_\Omega E_I H_I + \sum_n \int_\Omega \partial_n E_I \partial_n H_I\Big)\\ &= \sum_I \big(\langle E_I, H_I\rangle_{\mathsf{L}^2(\Omega)} + \langle \nabla E_I, \nabla H_I\rangle_{\mathsf{L}^2(\Omega)}\big)\\ &= \langle \vec{E}, \vec{H}\rangle_{\mathsf{L}^2(\Omega)} + \langle \nabla\vec{E}, \nabla\vec{H}\rangle_{\mathsf{L}^2(\Omega)} = \langle \vec{E}, \vec{H}\rangle_{\mathsf{H}^1(\Omega)}.\end{aligned}$$

Boundary conditions for $\mathsf{H}^{1,q}(\Omega)$-forms can again be defined strongly and weakly, i. e., by closure

$$\mathring{\mathsf{H}}^{1,q}_{\Gamma_\tau}(\Omega) := \overline{\mathring{\mathsf{C}}^{\infty,q}_{\Gamma_\tau}(\Omega)}^{\mathsf{H}^{1,q}(\Omega)}$$

and by integration by parts

$$\mathring{\mathbf{H}}^{1,q}_{\Gamma_\tau}(\Omega) := \{E \in \mathsf{H}^{1,q}(\Omega) \,:\, \langle E_I, \partial_n \phi\rangle_{\mathsf{L}^2(\Omega)} = -\langle \partial_n E_I, \phi\rangle_{\mathsf{L}^2(\Omega)} \text{ for all } n, I \text{ and all } \phi \in \mathring{\mathsf{C}}^{\infty}_{\Gamma_\nu}(\Omega)\},$$

respectively. Let us also introduce the following Sobolev type spaces:

$$\begin{aligned}\mathsf{D}^{k,q}(\Omega) &:= \{E \in \mathsf{H}^{k,q}(\Omega) \,:\, \mathrm{d}\,E \in \mathsf{H}^{k,q+1}(\Omega)\},\\ \Delta^{k,q}(\Omega) &:= \{E \in \mathsf{H}^{k,q}(\Omega) \,:\, \delta\,E \in \mathsf{H}^{k,q-1}(\Omega)\}.\end{aligned}$$

Remark 2.1. We emphasise that by switching Γ_τ and Γ_ν we can define the respective boundary conditions on the other part of the boundary as well. Moreover, all definitions of our spaces extend literally to any open subset $\Omega \subset \mathbb{R}^N$ and any relatively open complementary boundary pairs Γ_τ and Γ_ν.

Finally, we introduce our transformations ε.

Definition 2.2. A transformation $\varepsilon : \mathsf{L}^{2,q}(\Omega) \to \mathsf{L}^{2,q}(\Omega)$ will be called admissible, if ε is bounded, symmetric, and uniformly positive definite. More precisely, ε is a self-adjoint operator on $\mathsf{L}^{2,q}(\Omega)$ and there exists $\underline{\varepsilon}, \overline{\varepsilon} > 0$ such that for all $E \in \mathsf{L}^{2,q}(\Omega)$

$$\underline{\varepsilon}|\varepsilon E|_{\mathsf{L}^{2,q}(\Omega)} \le |E|_{\mathsf{L}^{2,q}(\Omega)} \le \overline{\varepsilon}\sqrt{\langle \varepsilon E, E\rangle_{\mathsf{L}^{2,q}(\Omega)}}.$$

2.1 Lipschitz domains

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with boundary $\Gamma := \partial\Omega$. We introduce the setting we will be working in. Define (cf. Figure 3.2)

$$\begin{aligned}I := (-1,1), \quad B := I^N \subset \mathbb{R}^N, \quad B_\pm &:= \{x \in B \,:\, \pm x_N > 0\}, \quad B_0 := \{x \in B \,:\, x_N = 0\},\\ B_{0,\pm} &:= \{x \in B_0 \,:\, \pm x_1 > 0\}, \quad B_{0,0} := \{x \in B_0 \,:\, x_1 = 0\}.\end{aligned}$$

Definition 2.3 (Weak Lipschitz domain). Ω is called weak Lipschitz, if the boundary Γ is a Lipschitz submanifold of the manifold $\overline{\Omega}$, i. e., there exist a finite open covering $U_1, \dots, U_K \subset \mathbb{R}^N$ of Γ and vector fields $\phi_k : U_k \to B$, such that for $k = 1, \dots, K$
(i) $\phi_k \in \mathsf{C}^{0,1}(U_k, B)$ is bijective and $\psi_k := \phi_k^{-1} \in \mathsf{C}^{0,1}(B, U_k)$;
(ii) $\phi_k(U_k \cap \Omega) = B_-$

hold.

Remark 2.4. For $k = 1, \dots, K$, we have $\phi_k(U_k \setminus \overline{\Omega}) = B_+$ and $\phi_k(U_k \cap \Gamma) = B_0$.

Definition 2.5 (Weak Lipschitz domain and weak Lipschitz interface). Let Ω be weak Lipschitz. A relatively open subset Γ_τ of Γ is called weak Lipschitz, if Γ_τ is a Lipschitz submanifold of Γ, i. e., there are an open covering $U_1, \dots, U_K \subset \mathbb{R}^N$ of Γ and vector fields $\phi_k := U_k \to B$, such that for $k = 1, \dots, K$ and in addition to (i), (ii) in Definition 2.3 one of
(iii) $U_k \cap \Gamma_\tau = \emptyset$;
(iii′) $U_k \cap \Gamma_\tau = U_k \cap \Gamma \;\Rightarrow\; \phi_k(U_k \cap \Gamma_\tau) = B_0$;
(iii″) $\emptyset \neq U_k \cap \Gamma_\tau \neq U_k \cap \Gamma \;\Rightarrow\; \phi_k(U_k \cap \Gamma_\tau) = B_{0,-}$

holds. We define $\Gamma_\nu := \Gamma \setminus \overline{\Gamma}_\tau$ to be the relatively open complement of Γ_τ.

Definition 2.6 (Weak Lipschitz pair). A pair (Ω, Γ_τ) conforming to Definitions 2.3 and 2.5 will be called weak Lipschitz.

Remark 2.7. If (Ω, Γ_τ) is weak Lipschitz, so is (Ω, Γ_ν). Moreover, for the cases (iii), (iii′) and (iii″) in Definition 2.5 we further have
(iii) $U_k \cap \Gamma_\tau = \emptyset \;\Rightarrow\; U_k \cap \Gamma_\nu = U_k \cap \Gamma \;\Rightarrow\; \phi_k(U_k \cap \Gamma_\nu) = B_0$;
(iii′) $U_k \cap \Gamma_\tau = U_k \cap \Gamma \;\Rightarrow\; U_k \cap \Gamma_\nu = \emptyset$;
(iii″) $\emptyset \neq U_k \cap \Gamma_\tau \neq U_k \cap \Gamma \;\Rightarrow\; \emptyset \neq U_k \cap \Gamma_\nu \neq U_k \cap \Gamma \;\Rightarrow\; \phi_k(U_k \cap \Gamma_\nu) = B_{0,+}$ and $\phi_k(U_k \cap \overline{\Gamma}_\tau \cap \overline{\Gamma}_\nu) = B_{0,0}$.

In the literature, the notion of a Lipschitz domain $\Omega \subset \mathbb{R}^N$ is often used for a strong Lipschitz domain. For this, let us define for $x \in \mathbb{R}^N$,

$$x' := (x_1, x_2, \dots, x_{N-1}), \quad x'' := (x_2, \dots, x_{N-1}).$$

Definition 2.8 (Strong Lipschitz domain). Ω is called strong Lipschitz, if there are an open covering $U_1, \dots, U_K \subset \mathbb{R}^N$ of Γ, rigid body motions $R_k = A_k + a_k$, $A_k \in \mathbb{R}^{N \times N}$ orthogonal, $a_k \in \mathbb{R}^N$ and $\xi_k \in \mathsf{C}^{0,1}(I^{N-1}, I)$, such that for $k = 1, \dots, K$
(i) $R_k(U_k \cap \Omega) = \{x \in B : x_N < \xi_k(x')\}$.

Remark 2.9. For $k = 1, \dots, K$, we have

$$R_k(U_k \setminus \overline{\Omega}) = \{x \in B : x_N > \xi_k(x')\}, \quad R_k(U_k \cap \Gamma) = \{x \in B : x_N = \xi_k(x')\}.$$

Definition 2.10 (Strong Lipschitz domain and strong Lipschitz interface). Let Ω be strong Lipschitz. A relatively open subset Γ_τ of Γ is called strong Lipschitz, if there exist an open covering $U_1, \dots, U_K \subset \mathbb{R}^N$ of Γ, rigid body motions R_k, and $\xi_k \in \mathsf{C}^{0,1}(I^{N-1}, I)$, $\zeta_k \in \mathsf{C}^{0,1}(I^{N-2}, I)$, such that for $k = 1, \dots, K$ and in addition to (i) in Definition 2.8 one of

(ii) $U_k \cap \Gamma_\tau = \emptyset$;

(ii′) $U_k \cap \Gamma_\tau = U_k \cap \Gamma \;\Rightarrow\; R_k(U_k \cap \Gamma_\tau) = \{x \in B : x_N = \xi_k(x')\}$;

(ii″) $\emptyset \neq U_k \cap \Gamma_\tau \neq U_k \cap \Gamma \;\Rightarrow\; R_k(U_k \cap \Gamma_\tau) = \{x \in B : x_N = \xi_k(x'),\; x_1 < \zeta_k(x'')\}$

holds. We define $\Gamma_\nu := \Gamma \setminus \overline{\Gamma}_\tau$ to be the relatively open complement of Γ_τ.

Definition 2.11 (Strong Lipschitz pair). A pair (Ω, Γ_τ) conforming to Definitions 2.8 and 2.10 will be called strong Lipschitz.

Remark 2.12. If (Ω, Γ_τ) is strong Lipschitz, so is (Ω, Γ_ν). Moreover, for the cases (ii), (ii′) and (ii″) in Definition 2.10 we further have

(ii) $U_k \cap \Gamma_\tau = \emptyset \;\Rightarrow\; U_k \cap \Gamma_\nu = U_k \cap \Gamma \;\Rightarrow\; R_k(U_k \cap \Gamma_\nu) = \{x \in B : x_N = \xi_k(x')\}$;

(ii′) $U_k \cap \Gamma_\tau = U_k \cap \Gamma \;\Rightarrow\; U_k \cap \Gamma_\nu = \emptyset$;

(ii″) $\emptyset \neq U_k \cap \Gamma_\tau \neq U_k \cap \Gamma \;\Rightarrow\; \emptyset \neq U_k \cap \Gamma_\nu \neq U_k \cap \Gamma \;\Rightarrow$

$$R_k(U_k \cap \Gamma_\nu) = \{x \in B : x_N = \xi_k(x'),\; x_1 > \zeta_k(x'')\},$$
$$R_k(U_k \cap \overline{\Gamma}_\tau \cap \overline{\Gamma}_\nu) = \{x \in B : x_N = \xi_k(x'),\; x_1 = \zeta_k(x'')\}.$$

Remark 2.13. The following holds:

(i) Ω strong Lipschitz $\Rightarrow$ Ω weak Lipschitz

(ii) (Ω, Γ_τ) strong Lipschitz pair $\Rightarrow$ (Ω, Γ_τ) weak Lipschitz pair

For a proof just define $\phi_k := \varphi_k \circ R_k$ with $\varphi_k : U_k \to B$ given by

$$\varphi_k(x) := \begin{bmatrix} x_1 - \zeta_k(x'') \\ x'' \\ x_N - \xi_k(x') \end{bmatrix}.$$

Note that the contrary does not hold as the implicit function theorem is not available for Lipschitz maps.

For later purposes, we introduce special notation for the half-cube domain

$$\Xi := B_-, \quad \gamma := \partial \Xi \tag{4}$$

and its relatively open boundary parts γ_τ and $\gamma_\nu := \gamma \setminus \overline{\gamma_\tau}$. We will only consider the cases

$$\gamma_\nu = \emptyset, \quad \gamma_\nu = B_0, \quad \gamma_\nu = B_{0,+} \tag{5}$$

and we note that Ξ and $\gamma, \gamma_\tau, \gamma_\nu$ are strong Lipschitz, see Figure 3.1.

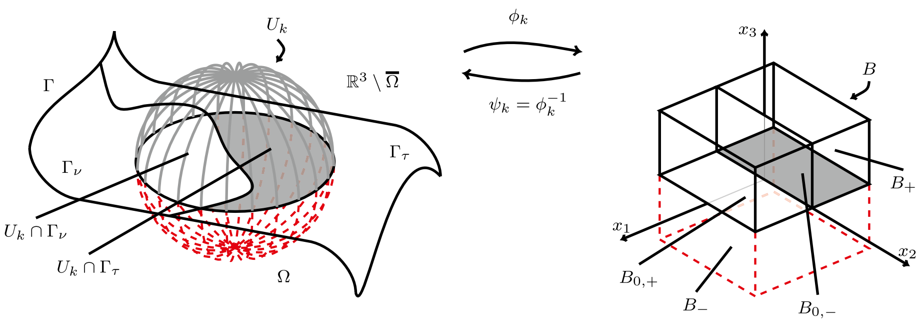

Figure 3.1: Mappings ϕ_k and ψ_k between a ball U_k and the cube B.

2.2 Outline of the proof

Let (Ω, Γ_τ) be a weak Lipschitz pair for a bounded domain $\Omega \subset \mathbb{R}^N$.

- As a first step, we observe $\mathring{\mathsf{H}}^{1,q}_{\Gamma_\tau}(\Omega) = \mathring{\mathbf{H}}^{1,q}_{\Gamma_\tau}(\Omega)$, i. e., for the $\mathsf{H}^{1,q}(\Omega)$-spaces the strong and weak definitions of the boundary conditions coincide; see Lemma 2.14.
- In the second and essential step, we construct various regular $\mathsf{H}^{1,q}$-potentials on simple domains, mainly for the half-cube Ξ from (4) with the special boundary constellations (5), i. e.,

$$\mathring{\mathbf{D}}^q_{\Gamma_\nu,0}(\Xi) = \mathring{\mathsf{D}}^q_{\gamma_\nu,0}(\Xi) = \mathrm{d}\,\mathring{\mathsf{H}}^{1,q-1}_{\gamma_\nu}(\Xi), \quad \mathring{\mathbf{\Delta}}^q_{\Gamma_\nu,0}(\Xi) = \mathring{\Delta}^q_{\Gamma_\nu,0}(\Xi) = \delta\,\mathring{\mathsf{H}}^{1,q+1}_{\gamma_\nu}(\Xi);$$

  see Section 3. Potentials of this type are called regular potentials.
- In the third step, Section 3.3, it is shown that the strong and weak definitions of the boundary conditions coincide on the half-cube Ξ from (4) with the special boundary constellation (5), i. e.,

$$\mathring{\mathbf{D}}^q_{\Gamma_\nu}(\Xi) = \mathring{\mathsf{D}}^q_{\Gamma_\nu}(\Xi), \quad \mathring{\mathbf{\Delta}}^q_{\Gamma_\nu}(\Xi) = \mathring{\Delta}^q_{\Gamma_\nu}(\Xi). \tag{6}$$

- The fourth step proves the compact embedding on the half-cube Ξ from (4) with the special boundary constellations (5), i. e.,

$$\mathring{\mathsf{D}}^q_{\Gamma_\tau}(\Xi) \cap \varepsilon^{-1}\mathring{\Delta}^q_{\Gamma_\nu}(\Xi) \hookrightarrow \mathsf{L}^{2,q}(\Xi) \tag{7}$$

  is compact; see Section 4.1.
- In the fifth step, Theorem 4.7, (6) is established for weak Lipschitz domains, i. e.,

$$\mathring{\mathbf{D}}^q_{\Gamma_\tau}(\Omega) = \mathring{\mathsf{D}}^q_{\Gamma_\tau}(\Omega), \quad \mathring{\mathbf{\Delta}}^q_{\Gamma_\nu}(\Omega) = \mathring{\Delta}^q_{\Gamma_\nu}(\Omega).$$

- In the last step, we finally prove the compact embedding (7) for weak Lipschitz pairs, i. e.,

$$\mathring{\mathsf{D}}^q_{\Gamma_\tau}(\Omega) \cap \varepsilon^{-1}\mathring{\Delta}^q_{\Gamma_\nu}(\Omega) \hookrightarrow \mathsf{L}^{2,q}(\Omega)$$

  is compact; see our main result Theorem 4.8.

2.3 Some important results

Within our proofs, we need a few important technical lemmas. First, the strong and weak definitions of the boundary conditions coincide for $\mathsf{H}^{1,q}(\Omega)$-forms, which is a density result for $\mathsf{H}^{1,q}(\Omega)$-forms. This is an immediate consequence of the corresponding scalar result, whose proof can be found in [11, Lemma 2, Lemma 3] and with a simplified proof in [1, Lemma 3.1].

Lemma 2.14 (Weak and strong boundary conditions coincide for $\mathsf{H}^{1,q}(\Omega)$). *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and let (Ω, Γ_τ) be a weak Lipschitz pair as well as*

$$\mathring{\underset{\sim}{\mathsf{H}}}{}^{1,q}_{\Gamma_\tau}(\Omega) := \{u \in \mathsf{H}^{1,q}(\Omega) \,:\, u|_{\Gamma_\tau} = 0\}$$

in the sense of traces. Then $\mathring{\mathbf{H}}^{1,q}_{\Gamma_\tau}(\Omega) = \mathring{\underset{\sim}{\mathsf{H}}}{}^{1,q}_{\Gamma_\tau}(\Omega) = \mathring{\mathsf{H}}^{1,q}_{\Gamma_\tau}(\Omega)$.

Another crucial tool in our arguments is a universal extension operator for the Sobolev spaces $\mathsf{D}^{k,q}(\Omega)$ and $\Delta^{k,q}(\Omega)$ given in [9], which is based on the universal extension operator for standard Sobolev spaces $\mathsf{H}^k(\Omega)$ introduced by Stein in [26]. "Universality" in this context means that the operator, which is given by a single formula, is able to extend all orders of Sobolev spaces simultaneously. More precisely, the following theorem, which is taken from [9, Theorem 3.6], holds.

Lemma 2.15 (Stein's extension operator). *Let $\Omega \subset \mathbb{R}^N$ be a bounded strong Lipschitz domain. Then for $k \in \mathbb{N}_0$ and $0 \le q \le N$ there exists a (universal) linear and continuous extension operator*

$$\mathcal{E} \,:\, \mathsf{D}^{k,q}(\Omega) \to \mathsf{D}^{k,q}(\mathbb{R}^N).$$

More precisely, $\mathcal{E}$ satisfies $\mathcal{E}E = E$ a. e. in Ω and there exists $c > 0$ such that for all $E \in \mathsf{D}^{k,q}(\Omega)$

$$|\mathcal{E}E|_{\mathsf{D}^{k,q}(\mathbb{R}^N)} \le c|E|_{\mathsf{D}^{k,q}(\Omega)}.$$

Furthermore, $\mathcal{E}$ can be chosen such that $\mathcal{E}E$ has a fixed compact support in $\mathbb{R}^N$ for all $E \in \mathsf{D}^{k,q}(\Omega)$.

Our third lemma summarises well-known and fundamental results for the theory of Maxwell's equations from [23, 24]. For this, we denote orthogonality and the orthogonal sum in $\mathsf{L}^{2,q}(\Omega)$ by $\perp$ and $\oplus$, respectively, and introduce the harmonic Dirichlet and Neumann forms

$$\mathcal{H}^q_D(\Omega) := \mathring{\mathsf{D}}^q_0(\Omega) \cap \Delta^q_0(\Omega), \qquad \mathcal{H}^q_N(\Omega) := \mathsf{D}^q_0(\Omega) \cap \mathring{\Delta}^q_0(\Omega),$$

respectively.

Lemma 2.16 (Picard's generalisation of Weck's selection theorem, Helmholtz decompositions and Maxwell estimates). *Let $\Omega \subset \mathbb{R}^N$ be a bounded weak Lipschitz domain. Then the embeddings*

$$\mathring{\mathsf{D}}^q(\Omega) \cap \Delta^q(\Omega) \hookrightarrow \mathsf{L}^{2,q}(\Omega), \qquad \mathsf{D}^q(\Omega) \cap \mathring{\Delta}^q(\Omega) \hookrightarrow \mathsf{L}^{2,q}(\Omega)$$

are compact and $\mathcal{H}^q_D(\Omega)$, $\mathcal{H}^q_N(\Omega)$ are finite-dimensional. Moreover, the Helmholtz decompositions

$$\begin{aligned} \mathsf{L}^{2,q}(\Omega) &= \mathrm{d}\,\mathring{\mathsf{D}}^{q-1}(\Omega) \oplus \Delta^q_0(\Omega) & \mathsf{L}^{2,q}(\Omega) &= \mathrm{d}\,\mathsf{D}^{q-1}(\Omega) \oplus \mathring{\Delta}^q_0(\Omega) \\ &= \mathring{\mathsf{D}}^q_0(\Omega) \oplus \delta\,\Delta^{q+1}(\Omega) & &= \mathsf{D}^q_0(\Omega) \oplus \delta\,\mathring{\Delta}^{q+1}(\Omega) \\ &= \mathrm{d}\,\mathring{\mathsf{D}}^{q-1}(\Omega) \oplus \mathcal{H}^q_D(\Omega) \oplus \delta\,\Delta^{q+1}(\Omega), & &= \mathrm{d}\,\mathsf{D}^{q-1}(\Omega) \oplus \mathcal{H}^q_N(\Omega) \oplus \delta\,\mathring{\Delta}^{q+1}(\Omega) \end{aligned}$$

are valid. In particular, all ranges are closed subspaces of $\mathsf{L}^{2,q}(\Omega)$ and

$$\begin{aligned} \mathrm{d}\,\mathring{\mathsf{D}}^{q-1}(\Omega) &= \mathring{\mathsf{D}}^q_0(\Omega) \cap \mathcal{H}^q_D(\Omega)^\perp, & \mathrm{d}\,\mathsf{D}^{q-1}(\Omega) &= \mathsf{D}^q_0(\Omega) \cap \mathcal{H}^q_N(\Omega)^\perp, \\ \delta\,\Delta^{q+1}(\Omega) &= \Delta^q_0(\Omega) \cap \mathcal{H}^q_D(\Omega)^\perp, & \delta\,\mathring{\Delta}^{q+1}(\Omega) &= \mathring{\Delta}^q_0(\Omega) \cap \mathcal{H}^q_N(\Omega)^\perp. \end{aligned}$$

Furthermore, there exists $c > 0$ such that

$$c|E|_{\mathsf{L}^{2,q}(\Omega)} \le |\mathrm{d}\,E|_{\mathsf{L}^{2,q+1}(\Omega)} + |\delta\,E|_{\mathsf{L}^{2,q-1}(\Omega)}$$

holds for all $E \in \mathring{\mathsf{D}}^q(\Omega) \cap \Delta^q(\Omega) \cap \mathcal{H}^q_D(\Omega)^\perp$ and all $E \in \mathsf{D}^q(\Omega) \cap \mathring{\Delta}^q(\Omega) \cap \mathcal{H}^q_N(\Omega)^\perp$, i. e., the Maxwell (or Friedrichs–Poincaré-type) estimates are valid.

Corollary 2.17 (Refined Helmholtz decompositions). *Let $\Omega \subset \mathbb{R}^N$ be a bounded weak Lipschitz domain. Then*

$$\begin{aligned} \mathring{\mathsf{D}}^q(\Omega) &= \mathring{\mathsf{D}}^q_0(\Omega) \oplus \big(\mathring{\mathsf{D}}^q(\Omega) \cap \delta\,\Delta^{q+1}(\Omega)\big), & \mathrm{d}\,\mathring{\mathsf{D}}^q(\Omega) &= \mathrm{d}\big(\mathring{\mathsf{D}}^q(\Omega) \cap \delta\,\Delta^{q+1}(\Omega)\big), \\ \mathsf{D}^q(\Omega) &= \mathsf{D}^q_0(\Omega) \oplus \big(\mathsf{D}^q(\Omega) \cap \delta\,\mathring{\Delta}^{q+1}(\Omega)\big), & \mathrm{d}\,\mathsf{D}^q(\Omega) &= \mathrm{d}\big(\mathsf{D}^q(\Omega) \cap \delta\,\mathring{\Delta}^{q+1}(\Omega)\big), \\ \Delta^q(\Omega) &= \big(\mathrm{d}\,\mathring{\mathsf{D}}^{q-1}(\Omega) \cap \Delta^q(\Omega)\big) \oplus \Delta^q_0(\Omega), & \delta\,\Delta^q(\Omega) &= \delta\big(\mathrm{d}\,\mathring{\mathsf{D}}^{q-1}(\Omega) \cap \Delta^q(\Omega)\big), \\ \mathring{\Delta}^q(\Omega) &= \big(\mathrm{d}\,\mathsf{D}^{q-1}(\Omega) \cap \mathring{\Delta}^q(\Omega)\big) \oplus \mathring{\Delta}^q_0(\Omega), & \delta\,\mathring{\Delta}^q(\Omega) &= \delta\big(\mathrm{d}\,\mathsf{D}^{q-1}(\Omega) \cap \mathring{\Delta}^q(\Omega)\big). \end{aligned}$$

Let $\pi_{q,\Omega} : \mathsf{L}^{2,q}(\Omega) \to \delta\,\mathring{\Delta}^{q+1}(\Omega)$ be the orthonormal Helmholtz projector onto $\delta\,\mathring{\Delta}^{q+1}(\Omega)$. By the latter corollary, $\pi_{q,\Omega}$ maps $\mathsf{D}^q(\Omega)$ to

$$\mathsf{D}^q(\Omega) \cap \delta\,\mathring{\Delta}^{q+1}(\Omega) = \mathsf{D}^q(\Omega) \cap \mathring{\Delta}^q_0(\Omega) \cap \mathcal{H}^q_N(\Omega)^\perp.$$

Corollary 2.18 (Maxwell estimate for d and Neumann boundary condition). *Assume $\Omega \subset \mathbb{R}^N$ to be a bounded weak Lipschitz domain. Then for all $E \in \mathsf{D}^q(\Omega)$ it holds $\pi_{q,\Omega}E \in \mathsf{D}^q(\Omega) \cap \delta\,\mathring{\Delta}^{q+1}(\Omega)$ and $\mathrm{d}\,\pi_{q,\Omega}E = \mathrm{d}\,E$ as well as*

$$c|\pi_{q,\Omega}E|_{\mathsf{L}^{2,q}(\Omega)} \le |\mathrm{d}\,E|_{\mathsf{L}^{2,q+1}(\Omega)},$$

with c from Lemma 2.16.

If $\Omega = \mathbb{R}^N$, a similar theory holds true utilising polynomially weighted Sobolev spaces; see [23] for details. Let $\pi_{q,\mathbb{R}^N} : \mathsf{L}^{2,q}(\mathbb{R}^N) \to \Delta_0^q(\mathbb{R}^N)$ be the orthonormal Helmholtz projector onto $\Delta_0^q(\mathbb{R}^N)$.

Lemma 2.19 (Helmholtz decompositions and Maxwell estimate for d in the whole space). *It holds $\mathcal{H}_N^q(\mathbb{R}^N) = \mathcal{H}_D^q(\mathbb{R}^N) = \{0\}$ and*

$$\mathsf{L}^{2,q}(\mathbb{R}^N) = \mathsf{D}_0^q(\mathbb{R}^N) \oplus \Delta_0^q(\mathbb{R}^N), \quad \mathsf{D}^q(\mathbb{R}^N) = \mathsf{D}_0^q(\mathbb{R}^N) \oplus (\mathsf{D}^q(\mathbb{R}^N) \cap \Delta_0^q(\mathbb{R}^N)).$$

Moreover, for all $E \in \mathsf{D}^q(\mathbb{R}^N)$ it holds $\pi_{q,\mathbb{R}^N} E \in \mathsf{D}^q(\mathbb{R}^N) \cap \Delta_0^q(\mathbb{R}^N)$ and $\mathrm{d}\,\pi_{q,\mathbb{R}^N} E = \mathrm{d}\,E$ as well as

$$|\pi_{q,\mathbb{R}^N} E|_{\mathsf{D}^q(\mathbb{R}^N)} \leq |E|_{\mathsf{D}^q(\mathbb{R}^N)}.$$

Regularity in the whole space (see, e. g., [12, (4.7) or Lemma 4.2(i)]) shows the following result.

Lemma 2.20 (Regularity in the whole space). $\mathsf{D}^q(\mathbb{R}^N) \cap \Delta^q(\mathbb{R}^N) = \mathsf{H}^{1,q}(\mathbb{R}^N)$ *with equal norms. More precisely, $E \in \mathsf{D}^q(\mathbb{R}^N) \cap \Delta^q(\mathbb{R}^N)$ if and only if $E \in \mathsf{H}^{1,q}(\mathbb{R}^N)$ and*

$$|E|^2_{\mathsf{H}^{1,q}(\mathbb{R}^N)} = |E|^2_{\mathsf{L}^{2,q}(\mathbb{R}^N)} + |\mathrm{d}\,E|^2_{\mathsf{L}^{2,q+1}(\mathbb{R}^N)} + |\delta\,E|^2_{\mathsf{L}^{2,q-1}(\mathbb{R}^N)}.$$

3 Regular potentials

As one of our main steps (step 4), in Section 4.1 the compact embedding is proved on the half-cube $\Xi \subset \mathbb{R}^N$. This will be achieved (in step 2) by constructing regular $\mathsf{H}^1(\Xi)$-potentials for d-free and δ-free $\mathsf{L}^{2,q}(\Xi)$-forms, which will then enable us to use Rellich's selection theorem. This section is devoted to the construction and existence of these regular potentials, i. e., to step 2.

3.1 Regular potentials without boundary conditions

Let us recall

$$\mathrm{d}\,\mathsf{D}^{q-1}(\Omega) = \mathsf{D}_0^q(\Omega) \cap \mathcal{H}_N^q(\Omega)^\perp, \quad \delta\,\Delta^{q+1}(\Omega) = \Delta_0^q(\Omega) \cap \mathcal{H}_D^q(\Omega)^\perp$$

from Lemma 2.16. The next two lemmas ensure the existence of $\mathsf{H}^{1,q}(\Omega)$-potentials without boundary conditions for strong Lipschitz domains.

Lemma 3.1 (Regular potential for d without boundary condition). *Let $\Omega \subset \mathbb{R}^N$ be a bounded strong Lipschitz domain. Then there exists a continuous linear operator*

$$\mathcal{T}_\mathrm{d} : \mathsf{D}_0^q(\Omega) \cap \mathcal{H}_N^q(\Omega)^\perp \to \mathsf{H}^{1,q-1}(\mathbb{R}^N) \cap \Delta_0^{q-1}(\mathbb{R}^N)$$

such that for all $E \in \mathsf{D}_0^q(\Omega) \cap \mathcal{H}_N^q(\Omega)^\perp$

$$\mathrm{d}\,\mathcal{T}_{\mathrm{d}} E = E \quad \textit{in } \Omega.$$

Especially,

$$\mathsf{D}_0^q(\Omega) \cap \mathcal{H}_N^q(\Omega)^\perp = \mathrm{d}\,\mathsf{H}^{1,q-1}(\Omega) = \mathrm{d}\big(\mathsf{H}^{1,q-1}(\Omega) \cap \Delta_0^{q-1}(\Omega)\big)$$

and the regular potential depends continuously on the data. Particularly, these are closed subspaces of $\mathsf{L}^{2,q}(\Omega)$ *and* $\mathcal{T}_{\mathrm{d}}$ *is a right inverse to* d. *By a simple cut-off technique* $\mathcal{T}_{\mathrm{d}}$ *may be modified to*

$$\mathcal{T}_{\mathrm{d}} : \mathsf{D}_0^q(\Omega) \cap \mathcal{H}_N^q(\Omega)^\perp \to \mathsf{H}^{1,q-1}(\mathbb{R}^N)$$

such that $\mathcal{T}_{\mathrm{d}} E$ *has a fixed compact support in* $\mathbb{R}^N$ *for all* $E \in \mathsf{D}_0^q(\Omega) \cap \mathcal{H}_N^q(\Omega)^\perp$.

Proof. Suppose $E \in \mathsf{D}_0^q(\Omega) \cap \mathcal{H}_N^q(\Omega)^\perp$. By Lemma 2.16, there exists $H \in \mathsf{D}^{q-1}(\Omega)$ with $\mathrm{d}\,H = E$ in Ω. Applying Corollary 2.18, we get $\pi_{q-1,\Omega} H \in \mathsf{D}^{q-1}(\Omega) \cap \delta\,\mathring{\Delta}^q(\Omega)$ with $\mathrm{d}\,\pi_{q-1,\Omega} H = \mathrm{d}\,H = E$ and

$$|\pi_{q-1,\Omega} H|_{\mathsf{D}^{q-1}(\Omega)} \le c|E|_{\mathsf{L}^{2,q}(\Omega)}.$$

Note that $\pi_{q-1,\Omega} H$ is uniquely determined. By the Stein extension operator $\mathcal{E} : \mathsf{D}^{0,q-1}(\Omega) \to \mathsf{D}^{0,q-1}(\mathbb{R}^N)$ from Lemma 2.15, we have $\mathcal{E}\pi_{q-1,\Omega} H \in \mathsf{D}^{0,q-1}(\mathbb{R}^N)$ with compact support. Projecting again, now with Lemma 2.19 onto $\Delta_0^{q-1}(\mathbb{R}^N)$, we obtain $\pi_{q-1,\mathbb{R}^N} \mathcal{E}\pi_{q-1,\Omega} H \in \mathsf{D}^{q-1}(\mathbb{R}^N) \cap \Delta_0^{q-1}(\mathbb{R}^N)$ (again uniquely determined) with $\mathrm{d}\,\pi_{q-1,\mathbb{R}^N} \mathcal{E}\pi_{q-1,\Omega} H = \mathrm{d}\,\mathcal{E}\pi_{q-1,\Omega} H$ and

$$|\pi_{q-1,\mathbb{R}^N} \mathcal{E}\pi_{q-1,\Omega} H|_{\mathsf{D}^{q-1}(\mathbb{R}^N)} \le |\mathcal{E}\pi_{q-1,\Omega} H|_{\mathsf{D}^{q-1}(\mathbb{R}^N)} \le c|\pi_{q-1,\Omega} H|_{\mathsf{D}^{q-1}(\Omega)}.$$

Lemma 2.20 shows $\pi_{q-1,\mathbb{R}^N} \mathcal{E}\pi_{q-1,\Omega} H \in \mathsf{H}^{1,q-1}(\mathbb{R}^N) \cap \Delta_0^{q-1}(\mathbb{R}^N)$ with

$$|\pi_{q-1,\mathbb{R}^N} \mathcal{E}\pi_{q-1,\Omega} H|_{\mathsf{H}^{1,q-1}(\mathbb{R}^N)} = |\pi_{q-1,\mathbb{R}^N} \mathcal{E}\pi_{q-1,\Omega} H|_{\mathsf{D}^{q-1}(\mathbb{R}^N)}.$$

Finally, $\mathcal{T}_{\mathrm{d}} E := \pi_{q-1,\mathbb{R}^N} \mathcal{E}\pi_{q-1,\Omega} H \in \mathsf{H}^{1,q-1}(\mathbb{R}^N) \cap \Delta_0^{q-1}(\mathbb{R}^N)$ meets our needs as

$$|\mathcal{T}_{\mathrm{d}} E|_{\mathsf{H}^{1,q-1}(\mathbb{R}^N)} \le c|E|_{\mathsf{L}^{2,q}(\Omega)}$$

and $\mathrm{d}\,\mathcal{T}_{\mathrm{d}} E = \mathrm{d}\,\pi_{q-1,\mathbb{R}^N} \mathcal{E}\pi_{q-1,\Omega} H = \mathrm{d}\,\mathcal{E}\pi_{q-1,\Omega} H = \mathrm{d}\,\pi_{q-1,\Omega} H = \mathrm{d}\,H = E$ in Ω. □

By Hodge-$\star$-duality, we get a corresponding result for the δ-operator.

Lemma 3.2 (Regular potential for δ without boundary condition). *Let* $\Omega \subset \mathbb{R}^N$ *be a bounded strong Lipschitz domain. Then there exists a continuous linear operator,*

$$\mathcal{T}_\delta : \Delta_0^q(\Omega) \cap \mathcal{H}_D^q(\Omega)^\perp \to \mathsf{H}^{1,q+1}(\mathbb{R}^N) \cap \mathsf{D}_0^{q+1}(\mathbb{R}^N),$$

such that for all $E \in \Delta_0^q(\Omega) \cap \mathcal{H}_D^q(\Omega)^\perp$

$$\delta\,\mathcal{T}_\delta E = E \quad \text{in } \Omega.$$

Especially,

$$\Delta_0^q(\Omega) \cap \mathcal{H}_D^q(\Omega)^\perp = \delta\,\mathsf{H}^{1,q+1}(\Omega) = \delta\big(\mathsf{H}^{1,q+1}(\Omega) \cap \mathsf{D}_0^{q+1}(\Omega)\big)$$

and the regular potential depends continuously on the data. In particular, these are closed subspaces of $\mathsf{L}^{2,q}(\Omega)$ *and* $\mathcal{T}_\delta$ *is a right inverse to* δ. *By a simple cut-off technique* $\mathcal{T}_\delta$ *may be modified to*

$$\mathcal{T}_\delta : \Delta_0^q(\Omega) \cap \mathcal{H}_D^q(\Omega)^\perp \to \mathsf{H}^{1,q+1}(\mathbb{R}^N)$$

such that $\mathcal{T}_\delta E$ *has a fixed compact support in* $\mathbb{R}^N$ *for all* $E \in \Delta_0^q(\Omega) \cap \mathcal{H}_D^q(\Omega)^\perp$.

3.2 Regular potentials with boundary conditions for the half-cube

Now we start constructing $\mathsf{H}^{1,q}(\Xi)$-potentials on Ξ with boundary conditions. Let us recall our special setting on the half-cube

$$\Xi = B_- \quad \text{and} \quad \gamma_\nu = \emptyset, \quad \gamma_\nu = B_0 \quad \text{or} \quad \gamma_\nu = B_{0,+}.$$

Furthermore (cf. Figure 3.2), we extend Ξ over γ_ν by

$$\tilde{\Xi} = \operatorname{int}(\overline{\Xi} \cup \overline{\hat{\Xi}}), \quad \hat{\Xi} := \begin{cases} \{x \in B \,:\, x_N > 0\} = B_+, & \text{if } \gamma_\nu = B_0, \\ \{x \in B \,:\, x_N, x_1 > 0\} = \{x \in B_+ \,:\, x_1 > 0\} =: B_{+,+}, & \text{if } \gamma_\nu = B_{0,+}. \end{cases}$$

Lemma 3.3 (Regular potential for d with partial boundary condition on the half-cube). *There exists a continuous linear operator*

$$\mathcal{S}_{\mathrm{d}} : \mathring{\mathbf{D}}_{\gamma_\nu,0}^q(\Xi) \to \mathsf{H}^{1,q-1}(\mathbb{R}^N) \cap \mathring{\mathsf{H}}_{\gamma_\nu}^{1,q-1}(\Xi),$$

such that for all $H \in \mathring{\mathbf{D}}_{\gamma_\nu,0}^q(\Xi)$

$$\mathrm{d}\,\mathcal{S}_{\mathrm{d}} H = H \quad \text{in } \Xi.$$

Especially,

$$\mathring{\mathbf{D}}_{\gamma_\nu,0}^q(\Xi) = \mathring{\mathsf{D}}_{\gamma_\nu,0}^q(\Xi) = \mathrm{d}\,\mathring{\mathsf{H}}_{\gamma_\nu}^{1,q-1}(\Xi) = \mathrm{d}\,\mathring{\mathsf{D}}_{\gamma_\nu}^{q-1}(\Xi) = \mathrm{d}\,\mathring{\mathbf{D}}_{\gamma_\nu}^{q-1}(\Xi)$$

and the regular $\mathring{\mathsf{H}}_{\gamma_\nu}^{1,q-1}(\Xi)$*-potential depends continuously on the data. In particular, these spaces are closed subspaces of* $\mathsf{L}^{2,q}(\Xi)$ *and* $\mathcal{S}_{\mathrm{d}}$ *is a right inverse to* d. *Without loss of generality,* $\mathcal{S}_{\mathrm{d}}$ *maps to forms with a fixed compact support in* $\mathbb{R}^N$.

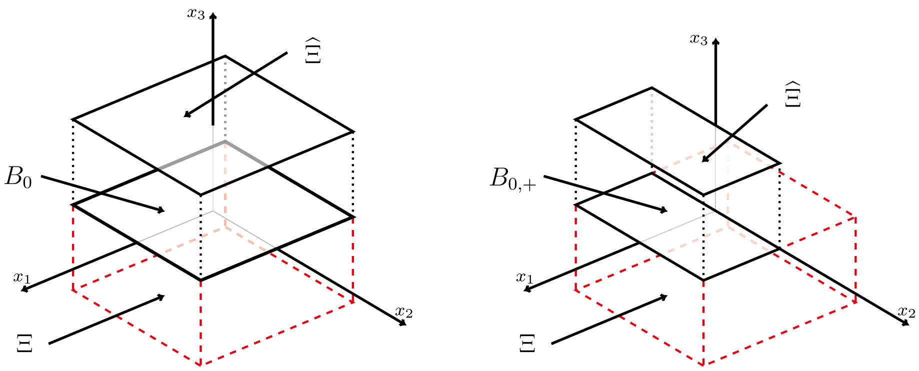

Figure 3.2: The half-cube $\Xi = B_-$, extended by $\widehat{\Xi}$ to the polygonal domain $\widetilde{\Xi}$, and the rectangles $\gamma_\nu = B_0$ and $\gamma_\nu = B_{0,+}$.

Proof. The case $\gamma_\nu = \emptyset$ is done in Lemma 3.1. Hence let $\gamma_\nu = B_0$ or $\gamma_\nu = B_{0,+}$. Suppose $H \in \mathring{\mathsf{D}}^q_{\gamma_\nu,0}(\Xi)$ and define $\widetilde{H} \in \mathsf{L}^{2,q}(\widetilde{\Xi})$ as extension of H by zero to $\widehat{\Xi}$ by

$$\widetilde{H} := \begin{cases} H & \text{in } \Xi, \\ 0 & \text{in } \widehat{\Xi}. \end{cases} \tag{8}$$

By definition of $\mathring{\mathsf{D}}^q_{\gamma_\nu,0}(\Xi)$ (definition of the weak boundary condition), it follows $\mathrm{d}\,\widetilde{H} = 0$ in $\widetilde{\Xi}$, i. e., $\widetilde{H} \in \mathsf{D}^q_0(\widetilde{\Xi})$. Because $\widetilde{\Xi}$ is strong Lipschitz and topologically trivial, especially $\mathcal{H}^q_N(\widetilde{\Xi}) = \{0\}$, Lemma 3.1 yields a regular potential $E = \mathcal{T}_\mathrm{d}\widetilde{H} \in \mathsf{H}^{1,q-1}(\mathbb{R}^N) \cap \mathsf{D}^{q-1}_0(\mathbb{R}^N)$ with $\mathrm{d}\,E = \widetilde{H}$ in $\widetilde{\Xi}$ and

$$|E|_{\mathsf{H}^{1,q-1}(\mathbb{R}^N)} \le c|\widetilde{H}|_{\mathsf{L}^{2,q}(\widetilde{\Xi})} \le c|H|_{\mathsf{L}^{2,q}(\Xi)}.$$

In particular, $E \in \mathsf{H}^{1,q-1}(\widehat{\Xi})$ and $\mathrm{d}\,E = 0$ in $\widehat{\Xi}$, i. e., $E \in \mathsf{H}^{1,q-1}(\widehat{\Xi}) \cap \mathsf{D}^{q-1}_0(\widehat{\Xi})$. Using Lemma 3.1 again, this time in $\widehat{\Xi}$, we obtain $F = \mathcal{T}_\mathrm{d}E \in \mathsf{H}^{1,q-2}(\mathbb{R}^N) \subset \mathsf{H}^{1,q-2}(\widehat{\Xi})$ with $\mathrm{d}\,F = E$ in $\widehat{\Xi}$ and

$$|F|_{\mathsf{H}^{1,q-2}(\mathbb{R}^N)} \le c|E|_{\mathsf{L}^{2,q}(\widehat{\Xi})}.$$

Since $E \in \mathsf{H}^{1,q-1}(\widehat{\Xi})$, we have $F \in \mathsf{D}^{1,q-2}(\widehat{\Xi})$. Let $\mathcal{E} : \mathsf{D}^{1,q-2}(\widehat{\Xi}) \to \mathsf{D}^{1,q-2}(\mathbb{R}^N)$ be the Stein extension operator from Lemma 2.15. Then

$$\begin{array}{rcccc} \mathcal{S}_\mathrm{d} & : & \mathring{\mathsf{D}}^q_{\gamma_\nu,0}(\Xi) & \longrightarrow & \mathsf{H}^{1,q-1}(\mathbb{R}^N) \\ & & H & \longmapsto & E - \mathrm{d}(\mathcal{E}F) \end{array}$$

is linear and continuous as

$$\begin{aligned} |\mathcal{S}_\mathrm{d}H|_{\mathsf{H}^{1,q-1}(\mathbb{R}^N)} &\le |E|_{\mathsf{H}^{1,q-1}(\mathbb{R}^N)} + |\mathcal{E}F|_{\mathsf{D}^{1,q-2}(\mathbb{R}^N)} \\ &\le |E|_{\mathsf{H}^{1,q-1}(\mathbb{R}^N)} + |F|_{\mathsf{D}^{1,q-2}(\widehat{\Xi})} \le |E|_{\mathsf{H}^{1,q-1}(\mathbb{R}^N)} \le c|H|_{\mathsf{L}^{2,q}(\Xi)}. \end{aligned}$$

Since $\mathcal{S}_{\mathrm{d}}H = 0$ in $\widehat{\Xi}$, we have $\mathcal{S}_{\mathrm{d}}H|_{\gamma_\nu} = 0$, which means $\mathcal{S}_{\mathrm{d}}H \in \mathring{\mathsf{H}}^{1,q-1}_{\underset{\sim}{\gamma_\nu}}(\Xi)$. Therefore, by Lemma 2.14 we see $\mathcal{S}_{\mathrm{d}}H \in \mathring{\mathsf{H}}^{1,q-1}_{\gamma_\nu}(\Xi) \subset \mathring{\mathsf{D}}^{q-1}_{\gamma_\nu}(\Xi) \subset \mathring{\mathbf{D}}^{q-1}_{\gamma_\nu}(\Xi)$. Moreover, $\mathrm{d}(\mathcal{S}_{\mathrm{d}}H) = \mathrm{d}\,E = \widetilde{H}$ in $\widetilde{\Xi}$, especially $\mathrm{d}(\mathcal{S}_{\mathrm{d}}H) = H$ in Ξ. Finally, we note

$$\mathrm{d}\,\mathring{\mathsf{H}}^{1,q-1}_{\gamma_\nu}(\Xi) \subset \mathrm{d}\,\mathring{\mathsf{D}}^{q-1}_{\gamma_\nu}(\Xi) \subset \mathring{\mathsf{D}}^{q}_{\gamma_\nu,0}(\Xi),\ \mathrm{d}\,\mathring{\mathbf{D}}^{q-1}_{\gamma_\nu}(\Xi) \subset \mathring{\mathbf{D}}^{q}_{\gamma_\nu,0}(\Xi) \subset \mathrm{d}\,\mathring{\mathsf{H}}^{1,q-1}_{\gamma_\nu}(\Xi),$$

completing the proof. □

Again by Hodge-$\star$-duality, we obtain the following.

Lemma 3.4 (Regular potential for δ with partial boundary condition on the half-cube). *There exists a continuous linear operator*

$$\mathcal{S}_\delta : \mathring{\mathbf{\Delta}}^{q}_{\gamma_\nu,0}(\Xi) \to \mathsf{H}^{1,q+1}(\mathbb{R}^N) \cap \mathring{\mathsf{H}}^{1,q+1}_{\gamma_\nu}(\Xi),$$

such that for all $H \in \mathring{\mathbf{\Delta}}^{q}_{\gamma_\nu,0}(\Xi)$

$$\delta\, \mathcal{S}_\delta H = H \quad \textit{in } \Xi.$$

Especially

$$\mathring{\mathbf{\Delta}}^{q}_{\gamma_\nu,0}(\Xi) = \mathring{\Delta}^{q}_{\gamma_\nu,0}(\Xi) = \delta\, \mathring{\mathsf{H}}^{1,q+1}_{\gamma_\nu}(\Xi) = \delta\, \mathring{\Delta}^{q+1}_{\gamma_\nu}(\Xi) = \delta\, \mathring{\mathbf{\Delta}}^{q+1}_{\gamma_\nu}(\Xi)$$

and the regular $\mathring{\mathsf{H}}^{1,q+1}_{\gamma_\nu}(\Xi)$*-potential depends continuously on the data. In particular, these spaces are closed subspaces of* $\mathsf{L}^{2,q}(\Xi)$ *and* $\mathcal{S}_\delta$ *is a right inverse to* δ. *Without loss of generality,* $\mathcal{S}_\delta$ *maps to forms with a fixed compact support in* $\mathbb{R}^N$.

3.3 Weak and strong boundary conditions coincide for the half-cube

Now the two main density results immediately follow. We note that this has already been proved for the $\mathsf{H}^{1,q}(\Omega)$-spaces in Lemma 2.14, i. e., $\mathring{\mathbf{H}}^{1,q}_{\Gamma_\tau}(\Omega) = \mathring{\mathsf{H}}^{1,q}_{\Gamma_\tau}(\Omega)$.

Lemma 3.5 (Weak and strong boundary conditions coincide for the half-cube).

$$\mathring{\mathbf{D}}^{q}_{\gamma_\nu}(\Xi) = \mathring{\mathsf{D}}^{q}_{\gamma_\nu}(\Xi) \quad \textit{and} \quad \mathring{\mathbf{\Delta}}^{q}_{\gamma_\nu}(\Xi) = \mathring{\Delta}^{q}_{\gamma_\nu}(\Xi).$$

Proof. Suppose $E \in \mathring{\mathbf{D}}^{q}_{\gamma_\nu}(\Xi)$, and therefore $\mathrm{d}\,E \in \mathring{\mathbf{D}}^{q+1}_{\gamma_\nu,0}(\Xi)$. By Lemma 3.3, there exists $H = \mathcal{S}_{\mathrm{d}}\,\mathrm{d}\,E \in \mathring{\mathsf{H}}^{1,q}_{\gamma_\nu}(\Xi)$ with $\mathrm{d}\,H = \mathrm{d}\,E$. By Lemma 3.3, we get $E - H \in \mathring{\mathbf{D}}^{q}_{\gamma_\nu,0}(\Xi) = \mathring{\mathsf{D}}^{q}_{\gamma_\nu,0}(\Xi)$, and hence $E \in \mathring{\mathsf{D}}^{q}_{\gamma_\nu}(\Xi)$. □

4 Weck's selection theorem

4.1 The compact embedding for the half-cube

First, we show the main result on the half-cube $\Xi = B_{-}$ with the special boundary patches

$$\gamma_\nu = \emptyset, \quad \gamma_\nu = B_0 \quad \text{or} \quad \gamma_\nu = B_{0,+}$$

from the latter section. To this end, let ε be an admissible transformation on $\mathsf{L}^{2,q}(\Xi)$ and let us consider the densely defined and closed (unbounded) linear operator

$$\mathrm{d}_\tau^{q-1} : \mathring{\mathsf{D}}_{\gamma_\tau}^{q-1}(\Xi) \subset \mathsf{L}^{2,q-1}(\Xi) \to \mathsf{L}_\varepsilon^{2,q}(\Xi); \quad E \mapsto \mathrm{d}\,E$$

together with its (Hilbert space) adjoint

$$-\,\delta_\nu^q\,\varepsilon := \left(\mathrm{d}_\tau^{q-1}\right)^* : \varepsilon^{-1}\mathring{\boldsymbol{\Delta}}_{\gamma_\nu}^q(\Xi) \subset \mathsf{L}_\varepsilon^{2,q}(\Xi) \to \mathsf{L}^{2,q-1}(\Xi); \quad H \mapsto -\,\delta\,\varepsilon H.$$

Note that by Lemma 3.5 we have $\mathring{\boldsymbol{\Delta}}_{\gamma_\nu}^q(\Xi) = \mathring{\Delta}_{\gamma_\nu}^q(\Xi)$. Here, $\mathsf{L}_\varepsilon^{2,q}(\Xi)$ denotes $\mathsf{L}^{2,q}(\Xi)$ equipped with the inner product $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathsf{L}_\varepsilon^{2,q}(\Xi)} := \langle\varepsilon\,\cdot\,,\,\cdot\,\rangle_{\mathsf{L}^{2,q}(\Xi)}$. Let $\oplus_\varepsilon$ denote the orthogonal sum with respect to the $\mathsf{L}_\varepsilon^{2,q}$-scalar product. The projection theorem yields immediately.

Lemma 4.1 (Regular Helmholtz decompositions for the half-cube). *The Helmholtz decompositions*

$$\mathsf{L}_\varepsilon^{2,q}(\Xi) = \mathring{\mathsf{D}}_{\gamma_\tau,0}^q(\Xi) \oplus_\varepsilon \varepsilon^{-1}\mathring{\Delta}_{\gamma_\nu,0}^q(\Xi), \qquad \mathring{\mathsf{D}}_{\gamma_\tau,0}^q(\Xi) = \mathrm{d}\,\mathring{\mathsf{H}}_{\gamma_\tau}^{1,q-1}(\Xi), \qquad \mathring{\Delta}_{\gamma_\nu,0}^q(\Xi) = \delta\,\mathring{\mathsf{H}}_{\gamma_\nu}^{1,q+1}(\Xi)$$

hold. Moreover, the refined Helmholtz decompositions

$$\mathring{\mathsf{D}}_{\gamma_\tau}^q(\Xi) = \mathrm{d}\,\mathring{\mathsf{H}}_{\gamma_\tau}^{1,q-1}(\Xi) \oplus_\varepsilon \left(\mathring{\mathsf{D}}_{\gamma_\tau}^q(\Xi) \cap \varepsilon^{-1}\,\delta\,\mathring{\mathsf{H}}_{\gamma_\nu}^{1,q+1}(\Xi)\right),$$
$$\varepsilon^{-1}\mathring{\Delta}_{\gamma_\nu}^q(\Xi) = \left(\mathrm{d}\,\mathring{\mathsf{H}}_{\gamma_\tau}^{1,q-1}(\Xi) \cap \varepsilon^{-1}\mathring{\Delta}_{\gamma_\nu}^q(\Xi)\right) \oplus_\varepsilon \varepsilon^{-1}\,\delta\,\mathring{\mathsf{H}}_{\gamma_\nu}^{1,q+1}(\Xi),$$
$$\mathring{\mathsf{D}}_{\gamma_\tau}^q(\Xi) \cap \varepsilon^{-1}\mathring{\Delta}_{\gamma_\nu}^q(\Xi) = \left(\mathrm{d}\,\mathring{\mathsf{H}}_{\gamma_\tau}^{1,q-1}(\Xi) \cap \varepsilon^{-1}\mathring{\Delta}_{\gamma_\nu}^q(\Xi)\right) \oplus_\varepsilon \left(\mathring{\mathsf{D}}_{\gamma_\tau}^q(\Xi) \cap \varepsilon^{-1}\,\delta\,\mathring{\mathsf{H}}_{\gamma_\nu}^{1,q+1}(\Xi)\right)$$

are valid, and the respective regular potentials, given by the operators $\mathcal{S}_{\mathrm{d}}$ and $\mathcal{S}_\delta$ from Lemma 3.3 and Lemma 3.4, respectively, depend continuously on the data.

Proof. The projection theorem yields $\mathsf{L}_\varepsilon^{2,q}(\Xi) = \overline{\mathrm{d}\,\mathring{\mathsf{D}}_{\gamma_\tau}^{q-1}(\Xi)} \oplus_\varepsilon \varepsilon^{-1}\mathring{\boldsymbol{\Delta}}_{\gamma_\nu,0}^q(\Xi)$. Furthermore,

$$\overline{\mathrm{d}\,\mathring{\mathsf{D}}_{\gamma_\tau}^{q-1}(\Xi)} = \mathrm{d}\,\mathring{\mathsf{D}}_{\gamma_\tau}^{q-1}(\Xi) = \mathrm{d}\,\mathring{\mathsf{H}}_{\gamma_\tau}^{1,q-1}(\Xi) = \mathring{\mathsf{D}}_{\gamma_\tau,0}^q(\Xi)$$

by Lemma 3.3 and

$$\mathring{\boldsymbol{\Delta}}_{\gamma_\nu,0}^q(\Xi) = \mathring{\Delta}_{\gamma_\nu,0}^q(\Xi) = \delta\,\mathring{\mathsf{H}}_{\gamma_\nu}^{1,q+1}(\Xi)$$

by Lemma 3.4. The other assertions follow immediately. □

Lemma 4.2 (Weck's selection theorem for the half-cube). *The embedding*

$$\mathring{\mathsf{D}}^{q}_{\gamma_\tau}(\Xi) \cap \varepsilon^{-1}\mathring{\Delta}^{q}_{\gamma_\nu}(\Xi) \hookrightarrow \mathsf{L}^{2,q}_{\varepsilon}(\Xi)$$

is compact.

Proof. Let $(H_n)_{n\in\mathbb{N}}$ be a bounded sequence in $\mathring{\mathsf{D}}^{q}_{\gamma_\tau}(\Xi) \cap \varepsilon^{-1}\mathring{\Delta}^{q}_{\gamma_\nu}(\Xi)$. By Lemma 4.1, we can decompose

$$H_n = H_n^{\mathrm{d}} + H_n^{\delta} = \mathrm{d}\,E_n^{\mathrm{d}} + \varepsilon^{-1}\,\delta\,E_n^{\delta} \in \big(\mathrm{d}\,\mathring{\mathsf{H}}^{1,q-1}_{\gamma_\tau}(\Xi) \cap \varepsilon^{-1}\mathring{\Delta}^{q}_{\gamma_\nu}(\Xi)\big) \oplus_{\varepsilon} \big(\mathring{\mathsf{D}}^{q}_{\gamma_\tau}(\Xi) \cap \varepsilon^{-1}\,\delta\,\mathring{\mathsf{H}}^{1,q+1}_{\gamma_\nu}(\Xi)\big),$$

with $E_n^{\mathrm{d}} = \mathcal{S}_{\mathrm{d}} H_n^{\mathrm{d}}$ and $E_n^{\delta} = \mathcal{S}_{\delta} H_n^{\delta}$. Then $\mathrm{d}\,H_n^{\delta} = \mathrm{d}\,H_n$ and $\delta\,\varepsilon H_n^{\mathrm{d}} = \delta\,\varepsilon H_n$ as well as

$$|E_n^{\mathrm{d}}|_{\mathsf{H}^{1,q-1}(\Xi)} \le c\,|H_n^{\mathrm{d}}|_{\mathsf{L}^{2,q}(\Xi)} \le c\,|H_n|_{\mathsf{L}^{2,q}_{\varepsilon}(\Xi)},$$
$$|E_n^{\delta}|_{\mathsf{H}^{1,q+1}(\Xi)} \le c\,|H_n^{\delta}|_{\mathsf{L}^{2,q}(\Xi)} \le c\,|H_n|_{\mathsf{L}^{2,q}_{\varepsilon}(\Xi)}.$$

By Rellich's selection theorem and without loss of generality, (E_n^{d}) and (E_n^{δ}) converge in $\mathsf{L}^{2,q-1}(\Xi)$ and $\mathsf{L}^{2,q+1}(\Xi)$, respectively. Moreover,

$$\begin{aligned}
|H_n^{\mathrm{d}} - H_m^{\mathrm{d}}|^2_{\mathsf{L}^{2,q}_{\varepsilon}(\Xi)} &= \langle H_n^{\mathrm{d}} - H_m^{\mathrm{d}}, \mathrm{d}(E_n^{\mathrm{d}} - E_m^{\mathrm{d}})\rangle_{\mathsf{L}^{2,q}_{\varepsilon}(\Xi)} \\
&= -\langle \delta\,\varepsilon(H_n^{\mathrm{d}} - H_m^{\mathrm{d}}), E_n^{\mathrm{d}} - E_m^{\mathrm{d}}\rangle_{\mathsf{L}^{2,q-1}(\Xi)} \le c\,|E_n^{\mathrm{d}} - E_m^{\mathrm{d}}|_{\mathsf{L}^{2,q-1}(\Xi)}, \\
|H_n^{\delta} - H_m^{\delta}|^2_{\mathsf{L}^{2,q}_{\varepsilon}(\Xi)} &= \langle H_n^{\delta} - H_m^{\delta}, \varepsilon^{-1}\,\delta(E_n^{\delta} - E_m^{\delta})\rangle_{\mathsf{L}^{2,q}_{\varepsilon}(\Xi)} \\
&= -\langle \mathrm{d}(H_n^{\delta} - H_m^{\delta}), E_n^{\delta} - E_m^{\delta}\rangle_{\mathsf{L}^{2,q+1}(\Xi)} \le c\,|E_n^{\delta} - E_m^{\delta}|_{\mathsf{L}^{2,q+1}(\Xi)}.
\end{aligned}$$

Thus (H_n^{d}) and (H_n^{δ}) converge in $\mathsf{L}^{2,q}_{\varepsilon}(\Xi)$ and altogether (H_n) converges in $\mathsf{L}^{2,q}_{\varepsilon}(\Xi)$ as well. □

Remark 4.3. The use of Helmholtz decompositions and regular potentials in the proof of Lemma 4.2 demonstrates the main idea behind an elegant proof of a compact embedding. This general idea carries over to proofs of compact embeddings related to other kinds of Hilbert complexes as well, arising, e. g., in elasticity, general relativity or biharmonic problems; see, for example, [22].

4.2 The compact embedding for weak Lipschitz domains

The aim of this section is to transfer Lemma 4.2 to arbitrary weak Lipschitz pairs (Ω, Γ_τ). To this end, we will employ a technical lemma, whose proof is sketched in [24, Section 3] and [31, Remark 2]. We give a detailed proof in the Appendix. Let us consider the following situation: Let $\Theta, \widetilde{\Theta}$ be two bounded domains in $\mathbb{R}^N$ with boundaries $\Upsilon := \partial\,\Theta$, $\widetilde{\Upsilon} := \partial\,\widetilde{\Theta}$ and let $\Upsilon_0 \subset \Upsilon$ be relatively open. Moreover, let

$$\phi : \Theta \to \widetilde{\Theta}, \qquad \psi := \phi^{-1} : \widetilde{\Theta} \to \Theta$$

be Lipschitz diffeomorphisms, this is, $\phi \in \mathsf{C}^{0,1}(\Theta, \widetilde{\Theta})$ and $\psi = \phi^{-1} \in \mathsf{C}^{0,1}(\widetilde{\Theta}, \Theta)$. Then $\widetilde{\Theta} = \phi(\Theta)$, $\widetilde{\Upsilon} = \phi(\Upsilon)$ and we define $\widetilde{\Upsilon}_0 := \phi(\Upsilon_0)$.

Lemma 4.4 (Pull-back lemma for Lipschitz transformations). *Let $E \in \mathring{\mathbf{D}}^q_{\Upsilon_0}(\Theta)$, respectively, $E \in \mathring{\mathsf{D}}^q_{\Upsilon_0}(\Theta)$ and $H \in \varepsilon^{-1}\mathring{\mathbf{\Delta}}^q_{\Upsilon_0}(\Theta)$, respectively, $H \in \varepsilon^{-1}\mathring{\Delta}^q_{\Upsilon_0}(\Theta)$ for an admissible transformation ε on $\mathsf{L}^{2,q}(\Theta)$. Then*

$$\psi^* E \in \mathring{\mathbf{D}}^q_{\widetilde{\Upsilon}_0}(\widetilde{\Theta}),\ \textit{resp.},\ \mathring{\mathsf{D}}^q_{\widetilde{\Upsilon}_0}(\widetilde{\Theta}) \qquad \textit{and} \qquad \mathrm{d}\,\psi^* E = \psi^*\,\mathrm{d}\,E,$$

$$\psi^* H \in \mu^{-1}\mathring{\mathbf{\Delta}}^q_{\widetilde{\Upsilon}_0}(\widetilde{\Theta}),\ \textit{resp.},\ \mu^{-1}\mathring{\Delta}^q_{\widetilde{\Upsilon}_0}(\widetilde{\Theta}) \quad \textit{and} \quad \delta\,\mu\psi^* H = \pm \star \mathrm{d}\,\psi^* \star \varepsilon H = \pm \star \psi^* \star \delta\,\varepsilon H,$$

where $\mu := (-1)^{qN-1} \star \psi^* \star \varepsilon\phi^*$ is an admissible transformation. Moreover, there exists $c > 0$, independent of E and H, such that

$$|\psi^* E|_{\mathsf{D}^q(\widetilde{\Theta})} \leq c|E|_{\mathsf{D}^q(\Theta)}, \qquad |\psi^* H|_{\mu^{-1}\Delta^q(\widetilde{\Theta})} \leq c|H|_{\varepsilon^{-1}\Delta^q(\Theta)}.$$

Let (Ω, Γ_τ) be a bounded weak Lipschitz pair as introduced in Definitions 2.3 and 2.5. We adjust Lemma 4.4 to our situation: Let $U_1, \dots, U_K$ be an open covering of Γ according to Definitions 2.3 and 2.5 and set $U_0 := \Omega$. Therefore, $U_0, \dots, U_K$ is an open covering of $\overline{\Omega}$. Moreover, let $\chi_k \in \mathring{\mathsf{C}}^\infty(U_k)$, $k \in \{0, \dots, K\}$, be a partition of unity subordinate to the open covering $U_0, \dots, U_K$. Now suppose $k \in \{1, \dots, K\}$. We define

$$\begin{aligned}
&\Omega_k := U_k \cap \Omega, && \Gamma_k := U_k \cap \Gamma, && \Gamma_{\tau,k} := U_k \cap \Gamma_\tau, && \Gamma_{\nu,k} := U_k \cap \Gamma_\nu,\\
&\widehat{\Gamma}_k := \partial\Omega_k, && \Sigma_k := \widehat{\Gamma}_k \setminus \Gamma, && \widehat{\Gamma}_{\tau,k} := \operatorname{int}(\Gamma_{\tau,k} \cup \overline{\Sigma}_k), && \widehat{\Gamma}_{\nu,k} := \operatorname{int}(\Gamma_{\nu,k} \cup \overline{\Sigma}_k),\\
& && \sigma := \gamma \setminus \overline{B}_0, && \widehat{\gamma}_\tau := \operatorname{int}(\gamma_\tau \cup \overline{\sigma}), && \widehat{\gamma}_\nu := \operatorname{int}(\gamma_\nu \cup \overline{\sigma}).
\end{aligned}$$

Lemma 4.4 will from now on be used with

$$\Theta := \Omega_k, \quad \widetilde{\Theta} := \Xi, \qquad \phi := \phi_k : \Omega_k \to \Xi, \quad \psi := \psi_k : \Xi \to \Omega_k$$

and with one of the following cases:

$$\Upsilon_0 := \Gamma_{\tau,k}, \quad \Upsilon_0 := \widehat{\Gamma}_{\tau,k}, \quad \Upsilon_0 := \Gamma_{\nu,k}, \quad \Upsilon_0 := \widehat{\Gamma}_{\nu,k}.$$

Then $\Upsilon = \widehat{\Gamma}_k$ and $\widetilde{\Upsilon} = \phi_k(\widehat{\Gamma}_k) = \gamma$ as well as (depending on the respective case)

$$\begin{aligned}
&\widetilde{\Upsilon}_0 = \phi_k(\Gamma_{\tau,k}) = \gamma_\tau, && \widetilde{\Upsilon}_0 = \phi_k(\widehat{\Gamma}_{\tau,k}) = \widehat{\gamma}_\tau, && \gamma_\tau \in \{\emptyset, B_0, B_{0,-}\}, && \gamma_\nu = \gamma \setminus \overline{\gamma}_\tau,\\
&\widetilde{\Upsilon}_0 = \phi_k(\Gamma_{\nu,k}) = \gamma_\nu, && \widetilde{\Upsilon}_0 = \phi_k(\widehat{\Gamma}_{\nu,k}) = \widehat{\gamma}_\nu, && \gamma_\nu \in \{\emptyset, B_0, B_{0,+}\}, && \gamma_\tau = \gamma \setminus \overline{\gamma}_\nu.
\end{aligned}$$

Remark 4.5. Lemmas 3.3, 3.4, 3.5, 4.1, 4.2 hold for $\gamma_\nu = B_{0,-}$ without any (substantial) modification as well.

It is straightforward to show the following.

Lemma 4.6 (Localisation). *Let (Ω, Γ_τ) be a bounded weak Lipschitz pair and let k be in $\{1,\dots,K\}$. Then for $E \in \mathring{\mathbf{D}}^q_{\Gamma_\tau}(\Omega)$, respectively, $E \in \mathring{\mathsf{D}}^q_{\Gamma_\tau}(\Omega)$ and $H \in \mathring{\mathbf{\Delta}}^q_{\Gamma_\nu}(\Omega)$, respectively, $H \in \mathring{\Delta}^q_{\Gamma_\nu}(\Omega)$ it holds*

$$E \in \mathring{\mathbf{D}}^q_{\Gamma_{\tau,k}}(\Omega_k), \qquad \chi_k E \in \mathring{\mathbf{D}}^q_{\widehat{\Gamma}_{\tau,k}}(\Omega_k), \qquad H \in \mathring{\mathbf{\Delta}}^q_{\Gamma_{\nu,k}}(\Omega_k), \qquad \chi_k H \in \mathring{\mathbf{\Delta}}^q_{\widehat{\Gamma}_{\nu,k}}(\Omega_k),$$
$$E \in \mathring{\mathsf{D}}^q_{\Gamma_{\tau,k}}(\Omega_k), \qquad \chi_k E \in \mathring{\mathsf{D}}^q_{\widehat{\Gamma}_{\tau,k}}(\Omega_k), \qquad H \in \mathring{\Delta}^q_{\Gamma_{\nu,k}}(\Omega_k), \qquad \chi_k H \in \mathring{\Delta}^q_{\widehat{\Gamma}_{\nu,k}}(\Omega_k).$$

Theorem 4.7 (Weak and strong boundary conditions coincide). *Let the pair (Ω, Γ_τ) be bounded and weak Lipschitz. Then $\mathring{\mathbf{D}}^q_{\Gamma_\tau}(\Omega) = \mathring{\mathsf{D}}^q_{\Gamma_\tau}(\Omega)$ and $\mathring{\mathbf{\Delta}}^q_{\Gamma_\nu}(\Omega) = \mathring{\Delta}^q_{\Gamma_\nu}(\Omega)$.*

Proof. Suppose $E \in \mathring{\mathbf{D}}^q_{\Gamma_\tau}(\Omega)$. Then we see $\chi_0 E \in \mathring{\mathsf{D}}^q(\Omega) \subset \mathring{\mathsf{D}}^q_{\Gamma_\tau}(\Omega)$ by mollification. Let $k \in \{1,\dots,K\}$. Then $\chi_k E \in \mathring{\mathbf{D}}^q_{\widehat{\Gamma}_{\tau,k}}(\Omega_k)$ by Lemma 4.6. Lemma 4.4, Lemma 3.5 (with $\gamma_\nu := \gamma_\tau$) and Remark 4.5 yield

$$\psi_k^*(\chi_k E) \in \mathring{\mathbf{D}}^q_{\widehat{\gamma}_\tau}(\Xi) = \mathring{\mathsf{D}}^q_{\widehat{\gamma}_\tau}(\Xi), \qquad \widehat{\gamma}_\tau = \phi_k(\widehat{\Gamma}_{\tau,k}), \qquad \gamma_\tau \in \{\emptyset, B_0, B_{0,-}\}.$$

Then $\chi_k E = \chi_k \phi_k^* \psi_k^* E \in \mathring{\mathsf{D}}^q_{\widehat{\Gamma}_{\tau,k}}(\Omega_k) \subset \mathring{\mathsf{D}}^q_{\Gamma_\tau}(\Omega)$ follows by Lemma 4.4. Therefore, we obtain $E = \sum_k \chi_k E \in \mathring{\mathsf{D}}^q_{\Gamma_\tau}(\Omega)$. $\mathring{\mathbf{\Delta}}^q_{\Gamma_\nu}(\Omega) = \mathring{\Delta}^q_{\Gamma_\nu}(\Omega)$ follows analogously or by Hodge-$\star$-duality. □

Now the compact embedding for bounded weak Lipschitz pairs (Ω, Γ_τ) can be proved.

Theorem 4.8 (Weck's selection theorem). *Let (Ω, Γ_τ) be a bounded weak Lipschitz pair and let ε be an admissible transformation on $\mathsf{L}^{2,q}(\Omega)$. Then the embedding*

$$\mathring{\mathsf{D}}^q_{\Gamma_\tau}(\Omega) \cap \varepsilon^{-1} \mathring{\Delta}^q_{\Gamma_\nu}(\Omega) \hookrightarrow \mathsf{L}^{2,q}_\varepsilon(\Omega)$$

is compact.

Proof. Suppose (E_n) is a bounded sequence in $\mathring{\mathsf{D}}^q_{\Gamma_\tau}(\Omega) \cap \varepsilon^{-1} \mathring{\Delta}^q_{\Gamma_\nu}(\Omega)$. Then by mollification

$$E_{0,n} := \chi_0 E_n \in \mathring{\mathsf{D}}^q(\Omega) \cap \varepsilon^{-1} \mathring{\Delta}^q(\Omega)$$

and $E_{0,n}$ even has compact support in Ω. By classical results (see [28, 29, 24]), $(E_{0,n})$ contains a subsequence, which is again denoted by $(E_{0,n})$, converging in $\mathsf{L}^{2,q}_\varepsilon(\Omega)$. Let $k \in \{1,\dots,K\}$. By Lemma 4.6,

$$E_{k,n} := \chi_k E_n \in \mathring{\mathsf{D}}^q_{\widehat{\Gamma}_{\tau,k}}(\Omega_k), \qquad \varepsilon E_{k,n} \in \mathring{\Delta}^q_{\widehat{\Gamma}_{\nu,k}}(\Omega_k),$$

and the sequence $(E_{k,n})$ is bounded in $\mathring{\mathsf{D}}^q_{\widehat{\Gamma}_{\tau,k}}(\Omega_k) \cap \varepsilon^{-1} \mathring{\Delta}^q_{\widehat{\Gamma}_{\nu,k}}(\Omega_k)$ by the product rule. By Lemma 4.4, we have $\psi_k^* E_{k,n} \in \mathring{\mathsf{D}}^q_{\widehat{\gamma}_\tau}(\Xi)$ and

$$|\psi_k^* E_{k,n}|_{\mathsf{D}^q(\Xi)} \le c |E_{k,n}|_{\mathsf{D}^q(\Omega_k)},$$

showing that $(\psi_k^* E_{k,n})$ is bounded in $\mathring{\mathsf{D}}^q_{\widehat{\gamma}_\tau}(\Xi)$. Analogously, $(\psi_k^* E_{k,n}) \subset \mu_k^{-1}\mathring{\Delta}^q_{\widehat{\gamma}_\nu}(\Xi)$ is bounded in $\mu_k^{-1}\mathring{\Delta}^q_{\widehat{\gamma}_\nu}(\Xi)$ with the admissible transformation $\mu_k := (-1)^{qN-1} \star \psi_k^* \star \varepsilon\phi_k^*$. Thus $(\psi_k^* E_{k,n})$ is bounded in

$$\mathring{\mathsf{D}}^q_{\widehat{\gamma}_\tau}(\Xi) \cap \mu_k^{-1}\mathring{\Delta}^q_{\widehat{\gamma}_\nu}(\Xi) \subset \mathring{\mathsf{D}}^q_{\widehat{\gamma}_\tau}(\Xi) \cap \mu_k^{-1}\mathring{\Delta}^q_{\widehat{\gamma}_\nu}(\Xi), \quad \gamma_\nu \in \{\emptyset, B_0, B_{0,+}\}, \quad \widehat{\gamma}_\tau = \gamma \setminus \overline{\gamma}_\nu.$$

Thus, by Lemma 4.2 and without loss of generality, $(\psi_k^* E_{k,n})$ is a Cauchy sequence in $\mathsf{L}^{2,q}(\Xi)$. Now

$$E_{k,n} = \phi_k^* \psi_k^* E_{k,n} \in \mathsf{L}^{2,q}(\Omega_k)$$

and Lemma 4.4 yields

$$|E_{k,n} - E_{k,m}|_{\mathsf{L}^{2,q}(\Omega_k)} \le c|\psi_k^* E_{k,n} - \psi_k^* E_{k,m}|_{\mathsf{L}^{2,q}(\Xi)}.$$

Hence $(E_{k,n})$ is a Cauchy sequence in $\mathsf{L}^{2,q}(\Omega_k)$ and so in $\mathsf{L}^{2,q}_\varepsilon(\Omega)$ for their extensions by zero to Ω. Finally, extracting convergent subsequences for $k = 1, \dots, K$, we see that

$$(E_n) = \left(\sum_{k=0}^{K} \chi_k E_n\right) = \left(\sum_{k=0}^{K} E_{k,n}\right)$$

is a Cauchy sequence in $\mathsf{L}^{2,q}_\varepsilon(\Omega)$. □

Remark 4.9 (Independence of the transformation). By standard techniques, it can be shown that Weck's selection theorem is independent of the transformation ε, i. e., the compactness of the embedding in Theorem 4.8 does not depend on ε. For details, see [2].

5 Applications

From now on, let $\Omega \subset \mathbb{R}^N$ be a bounded domain and let (Ω, Γ_τ) be a weak Lipschitz pair as well as $\varepsilon : \mathsf{L}^{2,q}(\Omega) \to \mathsf{L}^{2,q}(\Omega)$ be admissible. Then by Theorem 4.8, the embedding

$$\mathring{\mathsf{D}}^q_{\Gamma_\tau}(\Omega) \cap \varepsilon^{-1}\mathring{\Delta}^q_{\Gamma_\nu}(\Omega) \hookrightarrow \mathsf{L}^{2,q}(\Omega) \tag{9}$$

is compact. The results of this section immediately follow in the framework of a general functional analytic toolbox; see [21, 20, 22]. For details, see also the proofs in [1] for the classical case of vector analysis.

5.1 The Maxwell estimate

A first consequence of (9) is that the space of so-called "harmonic" Dirichlet–Neumann forms

$$\mathcal{H}^q_\varepsilon(\Omega) := \mathring{\mathsf{D}}^q_{\Gamma_\tau,0}(\Omega) \cap \varepsilon^{-1}\mathring{\Delta}^q_{\Gamma_\nu,0}(\Omega)$$

is finite-dimensional, as the unit ball in $\mathcal{H}_\varepsilon^q(\Omega)$ is compact by (9). Using the Helmholtz projections of Theorem 5.2, we see that the dimension of $\mathcal{H}_\varepsilon^q(\Omega)$ does not depend on ε, in particular $\dim \mathcal{H}_\varepsilon^q(\Omega) = \mathcal{H}^q(\Omega)$. By a standard indirect argument, (9) immediately implies the so-called Maxwell estimate.

Theorem 5.1 (Maxwell estimate). *There exists a positive constant c_m, such that for all $E \in \mathring{\mathsf{D}}_{\Gamma_\tau}^q(\Omega) \cap \varepsilon^{-1}\mathring{\Delta}_{\Gamma_\nu}^q(\Omega) \cap \mathcal{H}_\varepsilon^q(\Omega)^{\perp_\varepsilon}$*

$$|E|_{\mathsf{L}_\varepsilon^{2,q}(\Omega)} \leq c_\mathsf{m} \left(|\operatorname{d} E|_{\mathsf{L}^{2,q+1}(\Omega)}^2 + |\operatorname{\delta} \varepsilon E|_{\mathsf{L}^{2,q-1}(\Omega)}^2\right)^{1/2}.$$

Here, we denote by $\perp_\varepsilon$ orthogonality with respect to the $\mathsf{L}_\varepsilon^{2,q}(\Omega)$-inner product.

5.2 Helmholtz decompositions

Applying the projection theorem to the densely defined and closed (unbounded) linear operators,

$$\operatorname{d}_\tau^{q-1} : \mathring{\mathsf{D}}_{\Gamma_\tau}^{q-1}(\Omega) \subset \mathsf{L}^{2,q-1}(\Omega) \to \mathsf{L}_\varepsilon^{2,q}(\Omega); \quad E \mapsto \operatorname{d} E$$

with (Hilbert space) adjoint (see Theorem 4.7)

$$-\operatorname{\delta}_\nu^q \varepsilon := (\operatorname{d}_\tau^{q-1})^* : \varepsilon^{-1}\mathring{\Delta}_{\Gamma_\nu}^q(\Omega) \subset \mathsf{L}_\varepsilon^{2,q}(\Omega) \to \mathsf{L}^{2,q-1}(\Omega); \quad H \mapsto -\operatorname{\delta} \varepsilon H$$

and

$$-\varepsilon^{-1} \operatorname{\delta}_\nu^{q+1} : \varepsilon^{-1}\mathring{\Delta}_{\Gamma_\nu}^{q+1}(\Omega) \subset \mathsf{L}^{2,q+1}(\Omega) \to \mathsf{L}_\varepsilon^{2,q}(\Omega); \quad H \mapsto -\varepsilon^{-1} \operatorname{\delta} H$$

with adjoint (see Theorem 4.7)

$$\operatorname{d}_\tau^q := (-\varepsilon^{-1} \operatorname{\delta}_\nu^{q+1})^* : \mathring{\mathsf{D}}_{\Gamma_\tau}^q(\Omega) \subset \mathsf{L}_\varepsilon^{2,q}(\Omega) \to \mathsf{L}^{2,q+1}(\Omega); \quad E \mapsto \operatorname{d} E$$

we obtain the Helmholtz decompositions

$$\mathsf{L}_\varepsilon^{2,q}(\Omega) = \overline{\operatorname{d} \mathring{\mathsf{D}}_{\Gamma_\tau}^{q-1}(\Omega)} \oplus_\varepsilon \varepsilon^{-1}\mathring{\Delta}_{\Gamma_\nu,0}^q(\Omega), \tag{10}$$

$$\mathsf{L}_\varepsilon^{2,q}(\Omega) = \mathring{\mathsf{D}}_{\Gamma_\tau,0}^q(\Omega) \oplus_\varepsilon \overline{\varepsilon^{-1} \operatorname{\delta} \mathring{\Delta}_{\Gamma_\nu}^{q+1}(\Omega)}. \tag{11}$$

Therefore, $\mathring{\mathsf{D}}_{\Gamma_\tau,0}^q(\Omega) = \overline{\operatorname{d} \mathring{\mathsf{D}}_{\Gamma_\tau}^{q-1}(\Omega)} \oplus_\varepsilon \mathcal{H}_\varepsilon^q(\Omega)$ and, altogether, we get the refined Helmholtz decomposition

$$\mathsf{L}_\varepsilon^{2,q}(\Omega) = \overline{\operatorname{d} \mathring{\mathsf{D}}_{\Gamma_\tau}^{q-1}(\Omega)} \oplus_\varepsilon \mathcal{H}_\varepsilon^q(\Omega) \oplus_\varepsilon \overline{\varepsilon^{-1} \operatorname{\delta} \mathring{\Delta}_{\Gamma_\nu}^{q+1}(\Omega)}. \tag{12}$$

Theorem 5.2 (Helmholtz decompositions). *The orthonormal decompositions*

$$\mathsf{L}_\varepsilon^{2,q}(\Omega) = \operatorname{d} \mathring{\mathsf{D}}_{\Gamma_\tau}^{q-1}(\Omega) \oplus_\varepsilon \varepsilon^{-1}\mathring{\Delta}_{\Gamma_\nu,0}^q(\Omega)$$

$$
\begin{aligned}
&= \mathring{\mathsf{D}}^q_{\Gamma_\tau,0}(\Omega) \oplus_\varepsilon \varepsilon^{-1}\,\delta\,\mathring{\Delta}^{q+1}_{\Gamma_\nu}(\Omega)\\
&= \mathrm{d}\,\mathring{\mathsf{D}}^{q-1}_{\Gamma_\tau}(\Omega) \oplus_\varepsilon \mathcal{H}^q_\varepsilon(\Omega) \oplus_\varepsilon \varepsilon^{-1}\,\delta\,\mathring{\Delta}^{q+1}_{\Gamma_\nu}(\Omega)
\end{aligned}
$$

hold. Furthermore,

$$
\begin{aligned}
\mathrm{d}\,\mathring{\mathsf{D}}^q_{\Gamma_\tau}(\Omega) &= \mathrm{d}\big(\mathring{\mathsf{D}}^q_{\Gamma_\tau}(\Omega) \cap \varepsilon^{-1}\,\delta\,\mathring{\Delta}^{q+1}_{\Gamma_\nu}(\Omega)\big) = \mathrm{d}\big(\mathring{\mathsf{D}}^q_{\Gamma_\tau}(\Omega) \cap \varepsilon^{-1}\mathring{\Delta}^q_{\Gamma_\nu,0}(\Omega) \cap \mathcal{H}^q_\varepsilon(\Omega)^{\perp_\varepsilon}\big),\\
\delta\,\mathring{\Delta}^q_{\Gamma_\nu}(\Omega) &= \delta\big(\mathring{\Delta}^q_{\Gamma_\nu}(\Omega) \cap \varepsilon\,\mathrm{d}\,\mathring{\mathsf{D}}^{q-1}_{\Gamma_\tau}(\Omega)\big) = \delta\big(\mathring{\Delta}^q_{\Gamma_\nu}(\Omega) \cap \varepsilon\big(\mathring{\mathsf{D}}^q_{\Gamma_\tau,0}(\Omega) \cap \mathcal{H}^q_\varepsilon(\Omega)^{\perp_\varepsilon}\big)\big)
\end{aligned}
$$

and

$$
\begin{aligned}
\mathrm{d}\,\mathring{\mathsf{D}}^{q-1}_{\Gamma_\tau}(\Omega) &= \mathring{\mathsf{D}}^q_{\Gamma_\tau,0}(\Omega) \cap \mathcal{H}^q_\varepsilon(\Omega)^{\perp_\varepsilon}, & \delta\,\mathring{\Delta}^{q+1}_{\Gamma_\nu}(\Omega) &= \mathring{\Delta}^q_{\Gamma_\nu,0}(\Omega) \cap \mathcal{H}^q_\varepsilon(\Omega)^{\perp},\\
\mathring{\mathsf{D}}^q_{\Gamma_\tau,0}(\Omega) &= \mathrm{d}\,\mathring{\mathsf{D}}^{q-1}_{\Gamma_\tau}(\Omega) \oplus_\varepsilon \mathcal{H}^q_\varepsilon(\Omega), & \mathring{\Delta}^q_{\Gamma_\nu,0}(\Omega) &= \delta\,\mathring{\Delta}^{q+1}_{\Gamma_\nu}(\Omega) \oplus_{\varepsilon^{-1}} \varepsilon\,\mathcal{H}^q_\varepsilon(\Omega).
\end{aligned}
$$

The ranges $\mathrm{d}\,\mathring{\mathsf{D}}^{q-1}_{\Gamma_\tau}(\Omega)$ *and* $\delta\,\mathring{\Delta}^{q+1}_{\Gamma_\nu}(\Omega)$ *are closed subspaces of* $\mathsf{L}^{2,q}_\varepsilon(\Omega)$. *Moreover, the* d-, *respectively,* δ-*potentials are uniquely determined in* $\mathring{\mathsf{D}}^q_{\Gamma_\tau}(\Omega) \cap \varepsilon^{-1}\mathring{\Delta}^q_{\Gamma_\nu,0}(\Omega) \cap \mathcal{H}^q_\varepsilon(\Omega)^{\perp_\varepsilon}$ *and* $\mathring{\Delta}^q_{\Gamma_\nu}(\Omega) \cap \varepsilon(\mathring{\mathsf{D}}^q_{\Gamma_\tau,0}(\Omega) \cap \mathcal{H}^q_\varepsilon(\Omega)^{\perp_\varepsilon})$, *respectively, and depend continuously on their respective images.*

Proof. For $\varepsilon = \mathrm{id}$, (10) and (11) yield

$$
\begin{aligned}
\mathring{\Delta}^q_{\Gamma_\nu}(\Omega) &= \big(\overline{\mathrm{d}\,\mathring{\mathsf{D}}^{q-1}_{\Gamma_\tau}(\Omega)} \cap \mathring{\Delta}^q_{\Gamma_\nu}(\Omega)\big) \oplus \mathring{\Delta}^q_{\Gamma_\nu,0}(\Omega),\\
\mathring{\mathsf{D}}^q_{\Gamma_\tau}(\Omega) &= \mathring{\mathsf{D}}^q_{\Gamma_\tau,0}(\Omega) \oplus \big(\mathring{\mathsf{D}}^q_{\Gamma_\tau}(\Omega) \cap \overline{\delta\,\mathring{\Delta}^{q+1}_{\Gamma_\nu}(\Omega)}\big)
\end{aligned}
$$

and thus with (10), (11) and (12)

$$
\begin{aligned}
\delta\,\mathring{\Delta}^q_{\Gamma_\nu}(\Omega) &= \delta\big(\mathring{\Delta}^q_{\Gamma_\nu}(\Omega) \cap \overline{\mathrm{d}\,\mathring{\mathsf{D}}^{q-1}_{\Gamma_\tau}(\Omega)}\big) = \delta\big(\mathring{\mathsf{D}}^q_{\Gamma_\tau,0}(\Omega) \cap \mathring{\Delta}^q_{\Gamma_\nu}(\Omega) \cap \mathcal{H}^q(\Omega)^{\perp}\big),\\
\mathrm{d}\,\mathring{\mathsf{D}}^q_{\Gamma_\tau}(\Omega) &= \mathrm{d}\big(\mathring{\mathsf{D}}^q_{\Gamma_\tau}(\Omega) \cap \overline{\delta\,\mathring{\Delta}^{q+1}_{\Gamma_\nu}(\Omega)}\big) = \mathrm{d}\big(\mathring{\mathsf{D}}^q_{\Gamma_\tau}(\Omega) \cap \mathring{\Delta}^q_{\Gamma_\nu,0}(\Omega) \cap \mathcal{H}^q(\Omega)^{\perp}\big).
\end{aligned}
$$

Now Theorem 5.1 implies the closedness of the ranges and the continuity of the potentials. The other assertions follow immediately. □

Corollary 5.3 (Refined Helmholtz decompositions). *It holds*

$$
\begin{aligned}
\mathring{\mathsf{D}}^q_{\Gamma_\tau}(\Omega) &= \mathrm{d}\,\mathring{\mathsf{D}}^{q-1}_{\Gamma_\tau}(\Omega) \oplus_\varepsilon \big(\mathring{\mathsf{D}}^q_{\Gamma_\tau}(\Omega) \cap \varepsilon^{-1}\mathring{\Delta}^q_{\Gamma_\nu,0}(\Omega)\big)\\
&= \mathring{\mathsf{D}}^q_{\Gamma_\tau,0}(\Omega) \oplus_\varepsilon \big(\mathring{\mathsf{D}}^q_{\Gamma_\tau}(\Omega) \cap \varepsilon^{-1}\,\delta\,\mathring{\Delta}^{q+1}_{\Gamma_\nu}(\Omega)\big)\\
&= \mathrm{d}\,\mathring{\mathsf{D}}^{q-1}_{\Gamma_\tau}(\Omega) \oplus_\varepsilon \mathcal{H}^q_\varepsilon(\Omega) \oplus_\varepsilon \big(\mathring{\mathsf{D}}^q_{\Gamma_\tau}(\Omega) \cap \varepsilon^{-1}\,\delta\,\mathring{\Delta}^{q+1}_{\Gamma_\nu}(\Omega)\big),\\
\varepsilon^{-1}\mathring{\Delta}^q_{\Gamma_\nu}(\Omega) &= \big(\mathrm{d}\,\mathring{\mathsf{D}}^{q-1}_{\Gamma_\tau}(\Omega) \cap \varepsilon^{-1}\mathring{\Delta}^q_{\Gamma_\nu}(\Omega)\big) \oplus_\varepsilon \varepsilon^{-1}\mathring{\Delta}^q_{\Gamma_\nu,0}(\Omega)\\
&= \big(\mathring{\mathsf{D}}^q_{\Gamma_\tau,0}(\Omega) \cap \varepsilon^{-1}\mathring{\Delta}^q_{\Gamma_\nu}(\Omega)\big) \oplus_\varepsilon \varepsilon^{-1}\,\delta\,\mathring{\Delta}^{q+1}_{\Gamma_\nu}(\Omega)\\
&= \big(\mathrm{d}\,\mathring{\mathsf{D}}^{q-1}_{\Gamma_\tau}(\Omega) \cap \varepsilon^{-1}\mathring{\Delta}^q_{\Gamma_\nu}(\Omega)\big) \oplus_\varepsilon \mathcal{H}^q_\varepsilon(\Omega) \oplus_\varepsilon \varepsilon^{-1}\,\delta\,\mathring{\Delta}^{q+1}_{\Gamma_\nu}(\Omega).
\end{aligned}
$$

5.3 Static solution theory

As a further application, we turn to the boundary value problem of generalized electro and magnetostatics with mixed boundary values: Let $F \in \mathsf{L}^{2,q+1}(\Omega)$, $G \in \mathsf{L}^{2,q-1}(\Omega)$, E_τ, $E_\nu \in \mathsf{L}^{2,q}_\varepsilon(\Omega)$ and let ε be admissible. The problem is to find $E \in \mathsf{D}^q(\Omega) \cap \varepsilon^{-1}\Delta^q(\Omega)$ such that

$$\begin{aligned} \mathrm{d}\, E &= F, \\ \delta\, \varepsilon E &= G, \\ E - E_\tau &\in \mathring{\mathsf{D}}^q_{\Gamma_\tau}(\Omega), \\ \varepsilon(E - E_\nu) &\in \mathring{\Delta}^q_{\Gamma_\nu}(\Omega). \end{aligned} \tag{13}$$

For uniqueness, we require the additional conditions

$$\langle \varepsilon E, D_\ell \rangle_{\mathsf{L}^{2,q}_\varepsilon(\Omega)} = \alpha_\ell \in \mathbb{R}, \quad \ell = 1, \ldots, d, \tag{14}$$

where d is the dimension and $\{D_\ell\}$ an ε-orthonormal basis of $\mathcal{H}^q_\varepsilon(\Omega)$. The boundary values on Γ_τ and Γ_ν, respectively, are realised by the given volume forms E_τ and E_ν, respectively.

Theorem 5.4 (Static solution theory). (13) *admits a solution, if and only if*

$$E_\tau \in \mathsf{D}^q(\Omega), \quad E_\nu \in \varepsilon^{-1}\Delta^q(\Omega),$$

and

$$F - \mathrm{d}\, E_\tau \perp \mathring{\Delta}^{q+1}_{\Gamma_\nu,0}(\Omega), \quad G - \delta\, \varepsilon E_\nu \perp \mathring{\mathsf{D}}^{q-1}_{\Gamma_\tau,0}(\Omega). \tag{15}$$

The solution $E \in \mathsf{D}^q(\Omega) \cap \varepsilon^{-1}\Delta^q(\Omega)$ can be chosen in a way such that condition (14) *with $\alpha \in \mathbb{R}^d$ is fulfilled, which then uniquely determines the solution. Furthermore, the solution depends linearly and continuously on the data.*

Note that (15) is equivalent to

$$F - \mathrm{d}\, E_\tau \in \mathrm{d}\, \mathring{\mathsf{D}}^q_{\Gamma_\tau}(\Omega), \qquad G - \delta\, \varepsilon E_\nu \in \delta\, \mathring{\Delta}^q_{\Gamma_\nu}(\Omega).$$

For homogeneous boundary data, i. e., $E_\tau = E_\nu = 0$, the latter theorem immediately follows from a functional analytic toolbox (see [21, 20, 22]), which even states a sharper result: The linear static Maxwell-operator

$$\begin{array}{rcccc} M & : & \mathring{\mathsf{D}}^q_{\Gamma_\tau}(\Omega) \cap \varepsilon^{-1}\mathring{\Delta}^q_{\Gamma_\nu}(\Omega) & \longrightarrow & \mathrm{d}\, \mathring{\mathsf{D}}^q_{\Gamma_\tau}(\Omega) \times \delta\, \mathring{\Delta}^q_{\Gamma_\nu}(\Omega) \times \mathbb{R}^d \\ & & E & \longmapsto & (\mathrm{d}\, E, \delta\, \varepsilon E, (\langle \varepsilon E, D_\ell \rangle_{\mathsf{L}^{2,q}_\varepsilon(\Omega)})_{\ell=1}^d) \end{array}$$

is a topological isomorphism. Its inverse M^{-1} maps not only continuously onto its domain of definition $\mathring{\mathsf{D}}^q_{\Gamma_\tau}(\Omega) \cap \varepsilon^{-1}\mathring{\Delta}^q_{\Gamma_\nu}(\Omega)$, but also compactly into $\mathsf{L}^{2,q}_\varepsilon(\Omega)$ by (9). For homogeneous kernel data, i. e., for

$$\begin{array}{rcccc} M_0 & : & \mathring{\mathsf{D}}^q_{\Gamma_\tau}(\Omega) \cap \varepsilon^{-1}\mathring{\Delta}^q_{\Gamma_\nu}(\Omega) \cap \mathcal{H}^q_\varepsilon(\Omega)^{\perp_\varepsilon} & \longrightarrow & \mathrm{d}\, \mathring{\mathsf{D}}^q_{\Gamma_\tau}(\Omega) \times \delta\, \mathring{\Delta}^q_{\Gamma_\nu}(\Omega) \\ & & E & \longmapsto & (\mathrm{d}\, E, \delta\, \varepsilon E) \end{array},$$

we have $|M_0^{-1}| \le (c_{\mathrm{m}}^2 + 1)^{1/2}$. For details and a proof of Theorem 5.4 in the classical setting of vector analysis, see [1].

5.4 General regular potentials and decompositions

A closer inspection of the proof of Lemma 3.3 shows that Lemma 3.3 and Lemma 3.4 hold for more general situations. Using the partition of unity from Section 4.2 and the concept of extendable strong Lipschitz pairs, we can even generalise Lemma 3.3 and Lemma 3.4 to general strong Lipschitz pairs. Note that by Theorem 5.2

$$\mathrm{d}\,\mathring{\mathsf{D}}^q_{\Gamma_\tau}(\Omega) = \mathring{\mathsf{D}}^{q+1}_{\Gamma_\tau,0}(\Omega) \cap \mathcal{H}^{q+1}_\varepsilon(\Omega)^{\perp_\varepsilon}, \qquad \mathring{\mathsf{D}}^{q+1}_{\Gamma_\tau,0}(\Omega) = \mathrm{d}\,\mathring{\mathsf{D}}^q_{\Gamma_\tau}(\Omega) \oplus_\varepsilon \mathcal{H}^{q+1}_\varepsilon(\Omega). \tag{16}$$

Theorem 5.5 (Regular potentials and decompositions for strong Lipschitz domains). *Let $\Omega \subset \mathbb{R}^N$ and let (Ω, Γ_τ) be a bounded strong Lipschitz pair.*

(i) *There exists a continuous linear operator*

$$\mathcal{S}^q_{\mathrm{d}} : \mathrm{d}\,\mathring{\mathsf{D}}^{q-1}_{\Gamma_\tau}(\Omega) \to \mathring{\mathsf{H}}^{1,q-1}_{\Gamma_\tau}(\Omega),$$

such that $\mathrm{d}\,\mathcal{S}^q_{\mathrm{d}} = \mathrm{id}\,|_{\mathrm{d}\,\mathring{\mathsf{D}}^{q-1}_{\Gamma_\tau}(\Omega)}$. *Especially,*

$$\mathrm{d}\,\mathring{\mathsf{D}}^{q-1}_{\Gamma_\tau}(\Omega) = \mathrm{d}\,\mathring{\mathsf{H}}^{1,q-1}_{\Gamma_\tau}(\Omega)$$

and the regular $\mathring{\mathsf{H}}^{1,q-1}_{\Gamma_\tau}(\Omega)$*-potential depends continuously on the data. In particular, these spaces are closed subspaces of* $\mathsf{L}^{2,q}(\Omega)$ *and* $\mathcal{S}^q_{\mathrm{d}}$ *is a right inverse to* d.

(ii) *The regular decompositions*

$$\begin{aligned}\mathring{\mathsf{D}}^q_{\Gamma_\tau}(\Omega) &= \mathring{\mathsf{H}}^{1,q}_{\Gamma_\tau}(\Omega) + \mathrm{d}\,\mathring{\mathsf{H}}^{1,q-1}_{\Gamma_\tau}(\Omega)\\ &= \mathcal{S}^{q+1}_{\mathrm{d}}\,\mathrm{d}\,\mathring{\mathsf{D}}^q_{\Gamma_\tau}(\Omega) \dotplus \mathring{\mathsf{D}}^q_{\Gamma_\tau,0}(\Omega),\end{aligned} \qquad \begin{aligned}\mathring{\mathsf{D}}^q_{\Gamma_\tau,0}(\Omega) &= \mathrm{d}\,\mathring{\mathsf{H}}^{1,q-1}_{\Gamma_\tau}(\Omega) + \big(\mathring{\mathsf{H}}^{1,q}_{\Gamma_\tau}(\Omega) \cap \mathring{\mathsf{D}}^q_{\Gamma_\tau,0}(\Omega)\big)\\ &= \mathrm{d}\,\mathring{\mathsf{H}}^{1,q-1}_{\Gamma_\tau}(\Omega) \oplus \mathcal{H}^q(\Omega)\\ &= \mathrm{d}\,\mathring{\mathsf{H}}^{1,q-1}_{\Gamma_\tau}(\Omega) \oplus_\varepsilon \mathcal{H}^q_\varepsilon(\Omega)\end{aligned}$$

hold with linear and continuous regular decomposition, respectively, potential operators, which can be defined explicitly by the orthonormal Helmholtz projectors and the operators $\mathcal{S}^q_{\mathrm{d}}$. *Note that* $\mathcal{H}^q(\Omega)$ *is a subspace of smooth forms, i. e., it holds* $\mathcal{H}^q(\Omega) = \mathring{\mathsf{D}}^q_{\Gamma_\tau,0}(\Omega) \cap \mathring{\Delta}^q_{\Gamma_\nu,0}(\Omega) \cap \mathsf{C}^{\infty,q}(\Omega)$.

Hodge-$\star$-duality yields the corresponding results for the co-derivative δ.

For details, see [2]. In the case of no or full boundary conditions, related results on regular potentials and regular decompositions are presented in [4].

Appendix A. Proof of Lemma 4.4 (pull-back lemma for Lipschitz transformations)

We start out by proving the assertions for the exterior derivative.

A.1 Without boundary conditions

Let $E = \sum_I E_I \,\mathrm{d}\,x^I \in \mathsf{D}^q(\Theta)$. We have to show $\psi^* E \in \mathsf{D}^q(\widetilde{\Theta})$ with $\mathrm{d}\,\psi^* E = \psi^* \,\mathrm{d}\, E$.

(i) Let us first consider $\Phi = \sum_I \Phi_I \,\mathrm{d}\,x^I \in \mathsf{C}^{0,1,q}(\Theta)$, i. e., $\Phi_I \in \mathsf{C}^{0,1}(\Theta)$ for all I. In the following, we denote by $\widetilde{\cdot}$ the composition with ψ. We have

$$\mathrm{d}\,\psi_j = \sum_i \partial_i \psi_j \,\mathrm{d}\,x^i, \qquad \psi^* \Phi = \sum_I \widetilde{\Phi}_I \psi^* \,\mathrm{d}\,x^I = \sum_I \widetilde{\Phi}_I (\mathrm{d}\,\psi_{i_1}) \wedge \cdots \wedge (\mathrm{d}\,\psi_{i_q}),$$
$$\mathrm{d}\,\Phi = \sum_{I,j} \partial_j \Phi_I (\mathrm{d}\,x_j) \wedge (\mathrm{d}\,x^I).$$

By Rademacher's theorem, $\widetilde{\Phi}_I = \Phi_I \circ \psi$ and ψ_j belong to $\mathsf{C}^{0,1}(\widetilde{\Theta}) \subset \mathsf{H}^1(\widetilde{\Theta})$ and the chain rule holds, i. e., $\partial_i \widetilde{\Phi}_I = \sum_j \widetilde{\partial_j \Phi_I} \partial_i \psi_j$. As $\psi_j \in \mathsf{H}^1(\widetilde{\Theta})$ we get $\mathrm{d}\,\psi_j \in \mathsf{D}^1_0(\widetilde{\Theta})$ by

$$\langle \mathrm{d}\,\psi_j, \delta\,\varphi \rangle_{\mathsf{L}^{2,1}(\widetilde{\Theta})} = -\langle \psi_j, \delta\,\delta\,\varphi \rangle_{\mathsf{L}^{2,0}(\widetilde{\Theta})} = 0$$

for all $\varphi \in \mathring{\mathsf{C}}^{\infty,2}(\widetilde{\Theta})$. Thus by definition, we see

$$\begin{aligned}
\mathrm{d}\,\psi^* \Phi &= \sum_I (\mathrm{d}\,\widetilde{\Phi}_I) \wedge (\mathrm{d}\,\psi_{i_1}) \wedge \cdots \wedge (\mathrm{d}\,\psi_{i_q}) = \sum_{I,i} \partial_i \widetilde{\Phi}_I (\mathrm{d}\,x^i) \wedge (\mathrm{d}\,\psi_{i_1}) \wedge \cdots \wedge (\mathrm{d}\,\psi_{i_q}) \\
&= \sum_{I,i,j} \widetilde{\partial_j \Phi_I} \partial_i \psi_j (\mathrm{d}\,x^i) \wedge (\mathrm{d}\,\psi_{i_1}) \wedge \cdots \wedge (\mathrm{d}\,\psi_{i_q}) \\
&= \sum_{I,j} \widetilde{\partial_j \Phi_I} (\mathrm{d}\,\psi_j) \wedge (\mathrm{d}\,\psi_{i_1}) \wedge \cdots \wedge (\mathrm{d}\,\psi_{i_q}).
\end{aligned}$$

On the other hand, it holds

$$\psi^* \,\mathrm{d}\,\Phi = \sum_{I,j} \widetilde{\partial_j \Phi_I} (\psi^* \,\mathrm{d}\,x_j) \wedge (\psi^* \,\mathrm{d}\,x^I) = \sum_{I,j} \widetilde{\partial_j \Phi_I} (\mathrm{d}\,\psi_j) \wedge (\mathrm{d}\,\psi_{i_1}) \wedge \cdots \wedge (\mathrm{d}\,\psi_{i_q}).$$

Therefore, $\psi^* \Phi \in \mathsf{D}^q(\widetilde{\Theta})$ and $\mathrm{d}\,\psi^* \Phi = \psi^* \,\mathrm{d}\,\Phi$.

(ii) For general $E \in \mathsf{D}^q(\Theta)$, we pick $\Phi \in \mathring{\mathsf{C}}^{\infty,q+1}(\widetilde{\Theta})$. Note $\operatorname{supp}\Phi \subset\subset \widetilde{\Theta} = \phi(\Theta)$. Replacing ψ by ϕ in (i) we have $\phi^* \star \Phi \in \mathsf{D}^{N-q-1}(\Theta)$ with $\mathrm{d}\,\phi^* \star \Phi = \phi^* \,\mathrm{d} \star\Phi$ and, since $\phi^* \star \Phi = \sum_I \widetilde{(\star\Phi)_I} \phi^* \,\mathrm{d}\,x^I$ holds, $\operatorname{supp}\phi^* \star \Phi \subset\subset \Theta$. By standard mollification, we obtain a sequence $(\Psi_n) \subset \mathring{\mathsf{C}}^{\infty,N-q-1}(\Theta)$ with $\Psi_n \to \phi^* \star \Phi$ in $\mathsf{D}^{N-q-1}(\Theta)$. Furthermore, $\star\Psi_n \in \mathring{\mathsf{C}}^{\infty,q+1}(\Theta)$. Then

$$\langle \psi^* E, \delta\,\Phi \rangle_{\mathsf{L}^{2,q}(\widetilde{\Theta})} = \int_{\widetilde{\Theta}} \psi^* E \wedge \star\,\delta\,\Phi = \pm \int_{\widetilde{\Theta}} \psi^* E \wedge \psi^* \phi^* \,\mathrm{d} \star\Phi = \pm \int_{\widetilde{\Theta}} \psi^* (E \wedge \phi^* \,\mathrm{d} \star\Phi)$$

$$
\begin{aligned}
&= \pm \int_{\Theta} E \wedge \phi^* \, \mathrm{d} \star \Phi = \pm \int_{\Theta} E \wedge \mathrm{d} \phi^* \star \Phi \leftarrow \pm \int_{\Theta} E \wedge \mathrm{d} \Psi_n \\
&= \pm \int_{\Theta} E \wedge \star \star \mathrm{d} \star \star \Psi_n = \pm \langle E, \delta \star \Psi_n \rangle_{\mathsf{L}^{2,q}(\Theta)} \\
&= \pm \langle \mathrm{d} E, \star \Psi_n \rangle_{\mathsf{L}^{2,q+1}(\Theta)} \to \pm \big\langle \mathrm{d} E, \star \phi^* \star \Phi \big\rangle_{\mathsf{L}^{2,q+1}(\Theta)} = \pm \int_{\Theta} \mathrm{d} E \wedge \phi^* \star \Phi \\
&= \pm \int_{\widetilde{\Theta}} \psi^* (\mathrm{d} E \wedge \phi^* \star \Phi) = \pm \int_{\widetilde{\Theta}} (\psi^* \, \mathrm{d} E) \wedge \star \Phi = - \big\langle \psi^* \, \mathrm{d} E, \Phi \big\rangle_{\mathsf{L}^{2,q+1}(\widetilde{\Theta})}
\end{aligned}
$$

and hence $\psi^* E \in \mathsf{D}^q(\widetilde{\Theta})$ with $\mathrm{d}\, \psi^* E = \psi^* \, \mathrm{d} E$.

(iii) Let $E \in \mathsf{D}^q(\Theta)$. By (ii), we know $\psi^* E \in \mathsf{D}^q(\widetilde{\Theta})$ with $\mathrm{d}\, \psi^* E = \psi^* \, \mathrm{d} E$. Hence

$$
\begin{aligned}
|\psi^* E|^2_{\mathsf{L}^{2,q}(\widetilde{\Theta})} &= \int_{\widetilde{\Theta}} \psi^* E \wedge \star \psi^* E = \int_{\Theta} \phi^* \psi^* E \wedge \phi^* \star \psi^* E \\
&= \pm \int_{\Theta} E \wedge \star (\star \phi^* \star \psi^*) E \leq c |E|^2_{\mathsf{L}^{2,q}(\Theta)}
\end{aligned}
$$

and

$$
|\mathrm{d}\, \psi^* E|_{\mathsf{L}^{2,q+1}(\widetilde{\Theta})} = |\psi^* \, \mathrm{d} E|_{\mathsf{L}^{2,q+1}(\widetilde{\Theta})} \leq c |\mathrm{d} E|_{\mathsf{L}^{2,q+1}(\Theta)}.
$$

A.2 With strong boundary condition

Let $E \in \mathring{\mathsf{D}}^q_{\Upsilon_0}(\Theta)$ and $(E_n) \subset \mathring{\mathsf{C}}^{\infty,q}_{\Upsilon_0}(\Theta)$ with $E_n \to E$ in $\mathsf{D}^q(\Theta)$. By Appendix A.1(ii), we know $\psi^* E_n, \psi^* E \in \mathsf{D}^q(\widetilde{\Theta})$ with $\mathrm{d}\, \psi^* E_n = \psi^* \, \mathrm{d} E_n$ as well as $\mathrm{d}\, \psi^* E = \psi^* \, \mathrm{d} E$. Furthermore, $\psi^* E_n$ has compact support away from $\widetilde{\Upsilon}_0$. Using standard mollification, we obtain $\psi^* E_n \in \mathring{\mathsf{D}}^q_{\widetilde{\Upsilon}_0}(\widetilde{\Theta})$. Moreover, by A.1(iii), $\psi^* E_n \to \psi^* E$ in $\mathsf{D}^q(\widetilde{\Theta})$. Therefore, $\psi^* E \in \mathring{\mathsf{D}}^q_{\widetilde{\Upsilon}_0}(\widetilde{\Theta})$ with $\mathrm{d}\, \psi^* E = \psi^* \, \mathrm{d} E$.

A.3 With weak boundary condition

Let $E \in \mathring{\mathbf{D}}^q_{\Upsilon_0}(\Theta)$ and $\Phi \in \mathring{\mathsf{C}}^{\infty,q+1}_{\widetilde{\Upsilon}_1}(\widetilde{\Theta})$, where $\Upsilon_1 = \Upsilon \setminus \overline{\Upsilon}_0$. By Appendix A.1(ii), we again know $\psi^* E \in \mathsf{D}^q(\widetilde{\Theta})$ with $\mathrm{d}\, \psi^* E = \psi^* \, \mathrm{d} E$. Moreover, by Appendix A.2, we have $\phi^* \star \Phi \in \mathring{\mathsf{D}}^{N-q-1}_{\Upsilon_1}(\Theta)$, and hence $\star \phi^* \star \Phi \in \mathring{\Delta}^{q+1}_{\Upsilon_1}(\Theta)$. We repeat the calculation from Appendix A.1(ii) to arrive at

$$
\begin{aligned}
\big\langle \psi^* E, \delta \Phi \big\rangle_{\mathsf{L}^{2,q}(\widetilde{\Theta})} &= \int_{\widetilde{\Theta}} \psi^* E \wedge \star \delta \Phi = \pm \big\langle E, \star \phi^* \, \mathrm{d} \star \Phi \big\rangle_{\mathsf{L}^{2,q}(\Theta)} \\
&= \pm \big\langle E, \star \mathrm{d} \phi^* \star \Phi \big\rangle_{\mathsf{L}^{2,q}(\Theta)} = \pm \big\langle E, \delta \star \phi^* \star \Phi \big\rangle_{\mathsf{L}^{2,q}(\Theta)}
\end{aligned}
$$

$$= \pm\langle \mathrm{d}\,E, \star\phi^* \star \Phi\rangle_{\mathsf{L}^{2,q+1}(\Theta)} = -\langle \psi^*\,\mathrm{d}\,E, \Phi\rangle_{\mathsf{L}^{2,q+1}(\widetilde{\Theta})} = -\langle \mathrm{d}\,\psi^* E, \Phi\rangle_{\mathsf{L}^{2,q+1}(\widetilde{\Theta})}$$

and, therefore, $\psi^* E \in \mathring{\mathsf{D}}^q_{\widetilde{\Upsilon}_0}(\widetilde{\Theta})$.

A.4 Assertions for the co-derivative

It holds by Appendix A.1(ii),

$$\varepsilon H \in \Delta^q(\Theta) \;\Leftrightarrow\; \star\varepsilon H \in \mathsf{D}^{N-q}(\Theta) \;\Leftrightarrow\; \psi^* \star \varepsilon\phi^*\psi^* H \in \mathsf{D}^{N-q}(\widetilde{\Theta}) \;\Leftrightarrow\; \mu\psi^* H \in \Delta^q(\widetilde{\Theta}).$$

Moreover, using Appendix A.1(iii) μ is admissible since for all $H \in \mathsf{L}^{2,q}(\widetilde{\Theta})$,

$$\begin{aligned}
\langle \mu H, H\rangle_{\mathsf{L}^{2,q}(\widetilde{\Theta})} &= \pm\langle \star\psi^* \star \varepsilon\phi^* H, H\rangle_{\mathsf{L}^{2,q}(\widetilde{\Theta})} = \pm\langle \psi^* \star \varepsilon\phi^* H, \star H\rangle_{\mathsf{L}^{2,N-q}(\widetilde{\Theta})} \\
&= \pm\int_{\widetilde{\Theta}} \psi^* \star \varepsilon\phi^* H \wedge H = \pm\int_{\Theta} \star\varepsilon\phi^* H \wedge \star\star\phi^* H \\
&= \pm\langle \varepsilon\phi^* H, \phi^* H\rangle_{\mathsf{L}^{2,q}(\Theta)} \geq c|\phi^* H|^2_{\mathsf{L}^{2,q}(\Theta)} \geq c|H|^2_{\mathsf{L}^{2,q}(\widetilde{\Theta})}.
\end{aligned}$$

Furthermore,

$$\delta\,\mu\psi^* H = \pm\star\mathrm{d}\,\psi^* \star \varepsilon H = \pm\star\psi^* \star \delta\,\varepsilon H.$$

The remaining assertions now follow by Appendix A.1–A.3 and Hodge-$\star$-duality.

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