Part 2: Numerics

Discrete Differential Forms

Finite Elements

3D de Rahm Complex

$$
\begin{array}{ccccccccc}
0 & \to & C^\infty(X) & \xrightarrow{\text{grad}} & C^\infty(X) & \xrightarrow{\text{curl}} & C^\infty(X) & \xrightarrow{\text{div}} & C^\infty(X) \to 0 \\
 & & \| & & \| & & \| & & \| \\
0 & \to & \mathcal{D}^0(X) & \xrightarrow[\delta]{d} & \mathcal{D}^1(X) & \xrightarrow[\delta]{d} & \mathcal{D}^2(X) & \xrightarrow[\delta]{d} & \mathcal{D}^3(X) \xrightarrow{d} 0 \\
 & & \downarrow * & & \downarrow * & & \downarrow * & & \downarrow * \\
0 & \xleftarrow{\delta} & \mathcal{D}^3(X) & \xleftarrow{d} & \mathcal{D}^2(X) & \xleftarrow{d} & \mathcal{D}^1(X) & \xleftarrow{d} & \mathcal{D}^0(X) \leftarrow 0 \\
 & & \| & & \| & & \| & & \| \\
0 & \leftarrow & C^\infty(X) & \xleftarrow{\text{div}} & C^\infty(X) & \xleftarrow{-\text{curl}} & C^\infty(X) & \xleftarrow{\text{grad}} & C^\infty(X) \leftarrow 0
\end{array}
$$

2D

$$
\begin{array}{ccccccccc}
0 & \to & C^\infty(X) & \xrightarrow{\text{grad}} & C^\infty(X) & \xrightarrow{\text{curl}} & C^\infty(X) & \to & 0 \\
 & & & & & & & & \\
0 & \to & \mathcal{D}^0(X) & \xrightarrow[\delta]{d} & \mathcal{D}^1(X) & \xrightarrow[\delta]{d} & \mathcal{D}^2(X) & \xrightarrow{d} & 0 \\
 & & \downarrow * & & \downarrow * & & \downarrow * & & \\
0 & \leftarrow & \mathcal{D}^2(X) & \leftarrow & \mathcal{D}^1(X) & \leftarrow & \mathcal{D}^0(X) & \leftarrow & 0 \\
 & & \| & & \| & & \| & & \\
0 & \leftarrow & C^\infty(X) & \xleftarrow{\text{div}} & C^\infty(X) & \xleftarrow{\text{grad}} & C^\infty(X) & \leftarrow & 0
\end{array}
$$

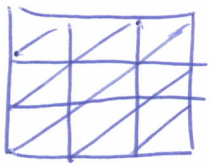
$$\text{curl}(a,b) = \left(\frac{\partial b}{\partial x}, -\frac{\partial a}{\partial y}\right)$$

(!)$_p$

Braess : Finite Elements

Classical Def FE is a triple (T, Π, Σ)

- T is a polyedr
- Π is a subspace of the Hilbert space
- Σ is a set of linear independent functionals over Π. Every $p \in \Pi$ is uniquely determined by the functionals from Σ (unisolvence)



- Decompose into triangles
- Basis functions for Π
- Conditions for conformity (subspace of Hilbert space)

Let us consider a conforming FE method, i.e.

$$V_h \subset V$$

Find u $\forall v$: $a(u, v) = F(v)$ (*) a bilinear, F linear

instead solve

Find u_h $\forall v_h$: $a(u_h, v_h) = F(v_h)$ (**)

$u_h \in V_h$ $v_h \in V_h$

Let us suppose we have a solution for (**) then

$$\|u - u_h\|_a = \inf_{v_h \in V_h} \|u - v_h\|_a$$ Best approximation

(< min)$_p$

$$\|a\|_a^2 = a(u,u)$$

$$a(u, v_h) = F(v_h) \quad \forall v_h \in V_h$$

$$- \; a(u_h, v_h) = F(v_h)$$

$$a(u - u_h, v_h) = 0$$ Galerkin-orthogonality

$= e_h$

$$\|u-u_h\|_a^2 = a(u-u_h, u-u_h)$$

$$= a(u-u_h, u-u_h-v_h)$$

$$\leq \|u-u_h\|_a \, \|u-u_h-v_h\|_a \quad \text{CS}$$

$$\Rightarrow \|u-u_h\|_a^2 \leq \|u-u_h-v_h\|_a$$

$$\Rightarrow \|u-u_h\|_a \leq \inf_{w_h \in V_h} \|u-w_h\|_a \quad \text{Best approximation}$$

$\Rightarrow$ error is small if V_h is rich

The richness of V_h is measured by interpolation error estimates

$$\|u - I_h u\|_a \leq C(h, u, f)$$

$I_h : V \to V_h$

To compute such error estimates for I_h the so called Bramble-Hilbert-lemma is often used:

$\Omega \subset \mathbb{R}^2$ lipschitz and $L : H^k(\Omega) \to Y$ (lin., bounded)$_p$

$P_{k-1} \subset \ker L$

$$\Rightarrow \|Lu\|_Y \leq \|L\|_Y \cdot |u|_{H^k(\Omega)} \quad (\leq \|L\| \, |u|_{H^k(\Omega)})_p$$

$\uparrow$ semi-norm

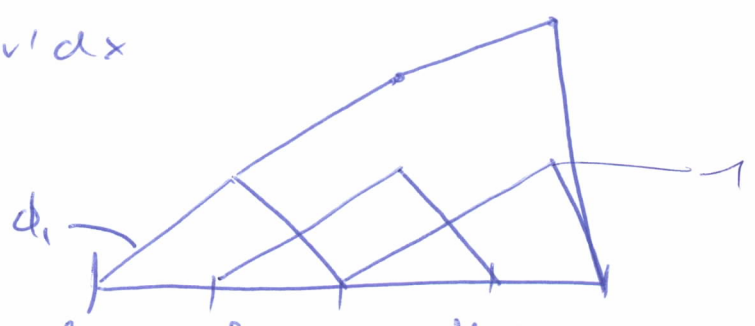
Simplest case: $v \in C_0^\infty([0,1]) = \mathring{C}^\infty([0,1])$, $V \in C^\infty([0,1])$, $v(0) = v(1) = 0$

1D: $u'' = 1$, $u(0) = u(1) = 0$, $[0,1]$

$$\int_0^1 1 \cdot v = \int_0^1 u'' v \, dx = -\int_0^1 u' v' dx + u'v\Big|_0^1 = -\int_0^1 u'v' dx$$

$$a(u,v) = -\int_0^1 u'v' dx$$

$$F(v) = \int_0^1 1 \cdot v \, dx$$



V_h = space of piecewise linear functions that are continuous (affine linear)$_p$

= ~~space~~ span of hat functions

$V_h = \operatorname{span}\{\phi_1, \dots, \phi_n\}$

$$v_h = \sum_{i=1}^{n} v_i \phi_i \qquad v = (v_1, v_2, \dots, v_n)$$

Putting them into eq gives

$$\sum_{i,j} v_i u_j \, a(\phi_i, \phi_j) = \sum_i v_i \, f(\phi_i) \qquad \forall\, v_h \hat{=} (v_1, \dots, v_n)$$

$$(\Rightarrow \langle k u, v \rangle = \langle f, v \rangle) \qquad (\nwarrow F)_p$$

Choose v as $e_1, \dots, e_n$

$$k u = f$$

$k = (a(\phi_i, \phi_j))_{i,j}$ Stiffness matrix

$f = (f(\phi_i))_{i,1}$ Load vector

$$a(u_h, v_h) = f(v_h) \quad \forall\, v_h \iff k u = f$$

Exercise: $k = ?$, $f = ?$

Conformity Conditions H^1, $H(\mathrm{div})$, $H(\mathrm{curl})$

Patching Conditions

$H^1(\Omega)$

Let $\Omega \subset \mathbb{R}^d$ and τ a triangulation.

Denote FE by $T \in \tau$.

Then for $k \geq 1$. A function $v: \bar{\Omega} \to \mathbb{R}$ which is piecewise $C^k(\Omega)$ is in $H^1 \iff v \in C^{k-1}(\Omega)$.

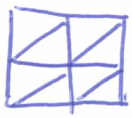
"$\Leftarrow$": (Important direction)

$$\int_\Omega v \, \partial_i \varphi \, dx = \sum_{T_j} \int_{T_j} v \, \partial_i \varphi \, dx \qquad \text{"Green's formula"}$$

$$= \sum_{T_j} - \int_{T_j} \partial_i v \, \varphi \, dx + \int_{\partial T_j} \varphi \, v \, \nu_i \, ds$$

$$\overset{!}{=} \int_\Omega - \partial_i v \, \varphi \, dx$$

where div is the well defined weak derivative of v
(and it is defined piecewise)
For this to hold the sum over the secondary term
should vanish; This is the case since (in our case)
since v is continuous and φ has compact support in Ω.



(Traces!)$_p$ $(H^{\frac{1}{2}})_p$

$H(\mathrm{div})$-conformity

$V_h \subset H(\mathrm{div}, \Omega)$:

Green's formula

$$\int_\Omega v \cdot \mathrm{grad}\,\varphi \, dx = -\int_\Omega \mathrm{div}\, v \, \varphi \, dx + \int_{\partial\Omega} \varphi \cdot v \cdot \nu \, ds$$

Let v be our FE function and φ with compact support

$$\int v \cdot \mathrm{grad}\,\varphi \, dx = \sum_{T_j} \int_{T_j} v \cdot \mathrm{grad}\,\varphi \, dx$$

$$= \sum_{T_j} \left(-\int_{T_j} \mathrm{div}\, v \, \varphi \, dx + \int_{\partial T_j} \varphi \; v \cdot \nu \, ds \right)$$

(Traces! $H^{-\frac{1}{2}}$)$_p$

$$\overset{!}{=} -\int_\Omega \mathrm{div}\, v \, \varphi \, dx$$

Here the second sum vanishes (for φ with compact support)
if the integral of the <u>normal component</u> of v
is identical from both sides of an edge (face).
This condition is especially fulfilled if the normal
component is continuous across edges.

$\Rightarrow$ div v is well defined.

$$\int_\Omega \langle v, \operatorname{curl} w\rangle \, dx = \int_\Omega \langle \operatorname{curl} v, w\rangle \, dx - \int_{\partial\Omega} \langle v \times \nu, w\rangle \, ds$$

$H(\operatorname{curl})$-conformity

$$\int \langle v, \operatorname{curl} w\rangle \, dx = \int_{T_j} \sum_{T_j} \langle v, \operatorname{curl} w\rangle \, dx \qquad (\text{Traces! } H^{-\frac{1}{2}})_p$$

$$= \sum_{T_j} \left(\int_{T_j} \langle \operatorname{curl} v, w\rangle \, dx - \int_{\partial T_j} \langle v \times \nu, w\rangle \, dx \right)$$

$$\overset{!}{=} \sum_{T_j} \int_{T_j} \langle \operatorname{curl} v, w\rangle \, dx$$

For this ($\overset{!}{=}$) to hold it suffices that $v \times \nu$ is identical from both sides of an edge (face).

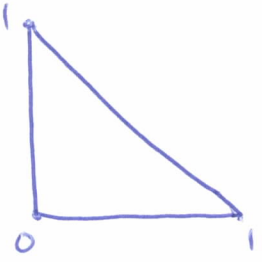
$\Rightarrow$ $\operatorname{curl} v$ is well defined; $v \in H(\operatorname{curl})$

Implementing Conforming H^1-elements

$$a(u,v) = \int_\Omega (\operatorname{grad} u)^T \operatorname{grad} v \, dx \qquad , \Omega = (0,1)^2$$

$$F(v) = \int_\Omega f v \, dx$$

Ref



basis $\varphi_1, \varphi_2, \varphi_3$

where $\varphi_i(p_j) = \delta_{ij}$, φ_i linear (affine linear)$_p$

$\varphi = (\varphi_1, \varphi_2, \varphi_3)^T$

basis functions and

$u_h = u_T^T \varphi$

$$K_T = \int_T (\operatorname{grad} u_h)^T \operatorname{grad} v_h \, dx$$

$$= \int_T u_T^T D\varphi (D\varphi)^T v_T \, dx$$

$$= \underset{(\cdot)_p}{u_T^T} \int_{T_{ref}} D\hat{\varphi} B^{-1} B^{-T} (D\hat{\varphi})^T |\det B| \, dx \, \underset{(\cdot)_p}{v_T}$$

$$\left(\partial_i u_h = u_{T_j} \partial_i \varphi_j = (\varphi'^T u_T)_i \Rightarrow \operatorname{grad} u_h = \varphi'^T u_T = D\varphi^T u_T \right)_p$$

$$f_T = \underset{(\cdot)_p}{v_T} \int_{T_{ref}} \hat{\varphi} \, f(Bx+d) \, |\det B| \, dx$$

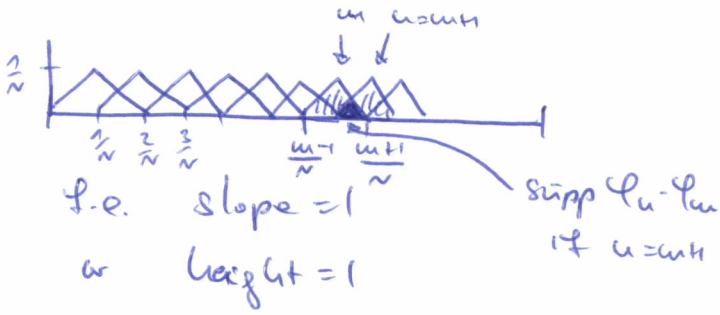
For a triangle in 2D with corners a_1, a_2, a_3

$$B = (a_2 - a_1, a_3 - a_1)$$

Exercise 1

$u'' = 1, \quad u(0) = u(1) = 0 \qquad [0,1] \qquad t_n = \frac{n}{N}$

$$\varphi_1(t) = \begin{cases} t & , t \in [0, \frac{1}{N}] \\ -t + \frac{2}{N} & , t \in [\frac{1}{N}, \frac{2}{N}] \end{cases}$$



$$\varphi_n(t) = \varphi_1(t - \tfrac{n-1}{N}) \qquad \operatorname{supp}\varphi_n = [\tfrac{n-1}{N}, \tfrac{n+1}{N}]$$

$$= \begin{cases} t - \frac{n-1}{N} & , t \in [\frac{n-1}{N}, \frac{n}{N}) \\ -t + \frac{2}{N} + \frac{n-1}{N} & , t \in [\frac{n}{N}, \frac{n+1}{N}] \end{cases}$$

$$= -t + \frac{n+1}{N}$$

$\Rightarrow \quad \varphi_n \quad , \quad n = 1, \dots, N-1$

$$a(u,v) = -\int_0^1 u'v'\,dx \qquad , \qquad F(v) = \int_0^1 v\,dx$$

$$a(\varphi_i, \varphi_j) = -\int_0^1 \varphi_i'(t)\,\varphi_j'(t)\,dt = 0 \quad , \text{if} \quad i-j \notin \{0, \pm 1\}$$

$$\varphi_n'(t) = \begin{cases} 1 & , t \in [\frac{n-1}{N}, \frac{n}{N}) \\ -1 & , t \in [\frac{n}{N}, \frac{n+1}{N}] \end{cases}$$

$$\Rightarrow \quad a(\varphi_n, \varphi_m) = \begin{cases} -\int_{\frac{m-1}{N}}^{\frac{m+1}{N}} 1\,dt = -\frac{2}{N} & , n = m \\ -\int_{\frac{m}{N}}^{\frac{m+1}{N}} \varphi_n'\varphi_m'\,dt & , n = m+1 \\ = -\int_{\frac{m}{N}}^{\frac{m+1}{N}} 1 \cdot (-1)\,dt = \frac{1}{N} & \\ \frac{1}{N} \quad (\text{sym.}) & , n = m-1 \end{cases}$$

$$\Rightarrow \quad k = \frac{1}{N} \begin{bmatrix} -2 & 1 & & 0 \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ 0 & & 1 & -2 \end{bmatrix}$$

$$f(\varphi_u) = \int_0^1 \varphi_u(t)\,dt = |\Delta| = \frac{1}{N^2}$$

$$\Rightarrow \quad f = \frac{1}{N^2} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$$

$$\Rightarrow \quad k \in \mathbb{R}^{(N-1)\times(N-1)}, \quad f \in \mathbb{R}^{N-1}$$

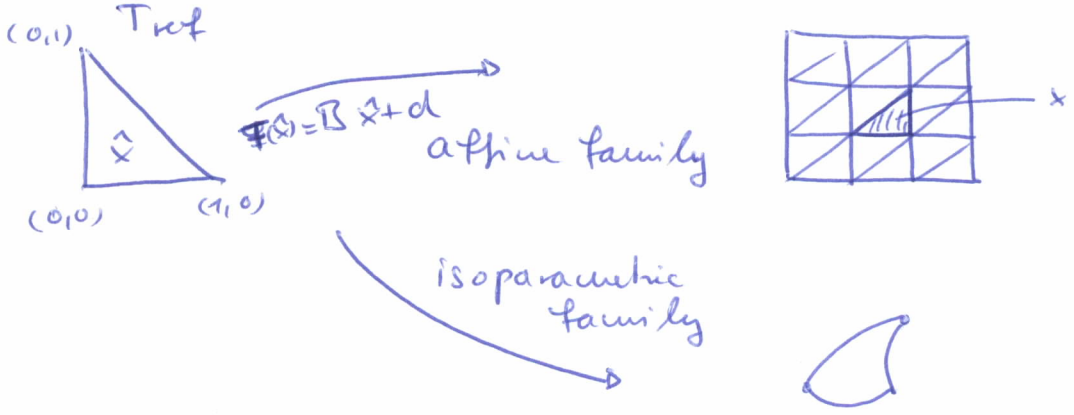
$$ku = f \quad \Leftrightarrow \quad \tilde{k}u = \tilde{f}$$

$$\tilde{k} = \begin{bmatrix} -2 & 1 & & 0 \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ 0 & & 1 & -2 \end{bmatrix}, \quad \tilde{f} = \frac{1}{N} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$$

Then

$$u_h = \sum_{u=1}^{N} u_u \varphi_u(t), \quad u_h(t_u) = u_u \underbrace{\varphi_u(t_u)}_{=\frac{1}{N}} = \frac{1}{N} u_u$$

put the vector $u := \begin{bmatrix} u_h(t_0) \\ \vdots \\ u_h(t_N) \end{bmatrix} = \frac{1}{N} \begin{bmatrix} 0 \\ u_1 \\ \vdots \\ u_{N-1} \\ 0 \end{bmatrix}$



$$\hat{\varphi}(\hat{x}) = \varphi(x)$$

Consider one element T

$\varphi_1, \dots, \varphi_m$ basis functions on T with local numbering

$$\varphi = \begin{bmatrix} \varphi_1 \\ \vdots \\ \varphi_m \end{bmatrix}, \quad D\varphi(x) = D(\hat{\varphi} \circ F^{-1})(x) = D\hat{\varphi}(\hat{x}) \underbrace{DF^{-1}(x)}_{= B^{-1}}$$

$$\Rightarrow \quad k_T = \int_T D\varphi (D\varphi)^T, \qquad k = (a(\phi_i, \phi_j))_{i,j}$$

$$= \int_{T_{ref}} D\hat{\varphi}\, B^{-1} B (D\hat{\varphi})^T |\det B|\, d\hat{x}$$

$$f_T = \int_{T_{ref}} \hat{\varphi}\, \varphi(F(x)) \, |\det B| \, d\hat{x}$$

$$k = \sum R_T^T k_T R_T$$

Recall $k = (a(\phi_i, \phi_j))_{i,j}$

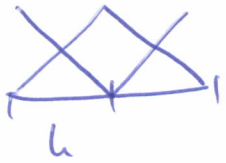
$k_T = (a(\phi_l, \phi_k))_{l,k}$ (local numbering)

Then let $r = \tau(T, l)$ be the translation from local to global numbering. Then

$$r_{rl}^{(T)} = 1 \quad \text{if} \quad r = \tau(T, l)$$

$$r_{rl} = 0 \quad \text{otherwise}$$

$$h = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} h_T \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} h_T \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$



$$h_T = \frac{1}{h} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\frac{1}{h} \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix} u = h \begin{pmatrix} 1/2 \\ 1 \\ 1/2 \end{pmatrix}$$

$$\mathcal{B} = (a_2 - a_1, a_3 - a_1)$$

Raviart-Thomas of Arbitrary Order

($H(div)$ - elements)

Let $P_k(\mathbb{R})$ be the space of polynomials of degree $\leq k$, $\tilde{P}_k(\mathbb{R})$ with $= k$.

The construction of RT-space follows the following philosophy

- choose basis functions from P_{k+1}
- ensure conformity condition for $H(div)$

This will achieved by

- demanding that $\hat{\varphi}$ restricted to a face/edge is of degree $\leq k$
- introducing enough degrees of freedom to ensure continuity of normal component

$$RT_k(\Omega) = P_k(\Omega)^d + x\,\tilde{P}_k(\Omega)$$

we can write $RT_k(\Omega)$ in 3D

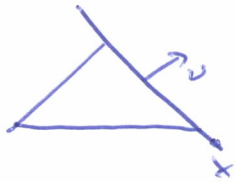
$$RT_k(\Omega) = P_k(\Omega)^d + \{ q \in \tilde{P}_{k+1}(\Omega)^d : \underbrace{x \cdot q = 0}_{x \perp\!\!\!\perp q} \}$$

conformity?

$$\langle v, p + xq \rangle = \langle v, p \rangle + \langle v, x \rangle q$$

$p \in P_k(\Omega)^d$, $P_k(\Omega)$

$= \langle v, p \rangle + \text{constant} \cdot q \in P_k(\Omega)$



$\langle v, x \rangle$ constant

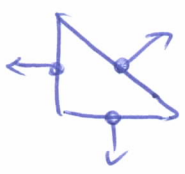
If we now $(2d)$ $(k+1)$ dof on each face/edge then the polynomials on the face/edge is uniquely determined (from both sides and therefore the normal component is continuous.)

The dimension of RT_k in 2D: $(k+1)(k+3)$

3D: $\frac{1}{2}(k+1)(k+2)(k+4)$

2D: $k=0$: $p = \begin{pmatrix} a \\ b \end{pmatrix} + c \begin{pmatrix} x \\ y \end{pmatrix}$, dof a, b, c #=3

on the refrence triangle the normal components are



$$\left\langle \begin{pmatrix} a \\ b \end{pmatrix} + c \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right\rangle = -a$$

$$\left\langle \begin{pmatrix} a \\ b \end{pmatrix} + c \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right\rangle = -b$$

$$\left\langle \begin{pmatrix} a \\ b \end{pmatrix} + c \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\rangle = a + b + c$$

We will now use the normal components at edge midpoints as dof.

We have defined new dof by functionals $\langle q, v \rangle$ thus for we can write dof

$$\xi \mapsto \int_e \langle \xi(x), v(x) \rangle \, ds$$

(use midpoint quadrature to get)

3D : $k=0$

$$p = \begin{pmatrix} a \\ b \\ c \end{pmatrix} + d \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

dof 4

$k=1$

$$p = \begin{pmatrix} a_1 + a_2 x + a_3 y + a_4 z \\ b_1 + b_2 x + b_3 y + b_4 z \\ c_1 + c_2 x + c_3 y + c_4 z \end{pmatrix} + (a_5 + b_5 y + c_5 z) \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

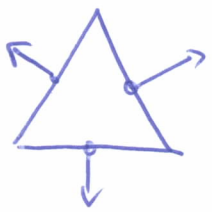
dof = 15

2D : $k=1$

$$p = \begin{pmatrix} a + bx + cy \\ d + ex + fy \end{pmatrix} + (gx + h) \begin{pmatrix} x \\ y \end{pmatrix}$$

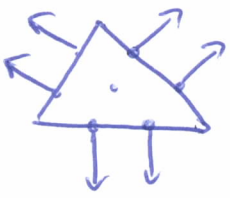
dof = 8

2D $k=0$

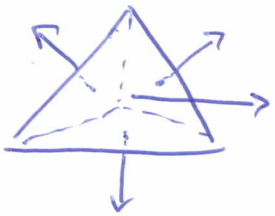


dof 3

$k=1$

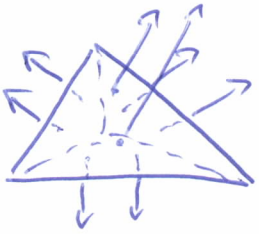


3D $k=0$



dof 4

$k=1$



Nédélec-Elements (H(curl)-elements)

$$N_k(\Omega) = P_k(\Omega)^d + \{ q \in \tilde{P}_{k+1}(\Omega)^d ; \langle x, q \rangle = 0 \}$$

Dual $\quad \varphi \mapsto \int_e \langle \varphi(s), t(s) \rangle ds$

Lowest order N_0

$$p = \begin{pmatrix} a \\ b \end{pmatrix} + c \begin{pmatrix} y \\ -x \end{pmatrix}$$

N_1

$$p = \begin{bmatrix} a + cx + dy \\ b + fy + gy \end{bmatrix} + \begin{pmatrix} iy^2 + jxy \\ jy^2 - ixy \end{pmatrix}$$

3D 0st order $\quad P_0(\Omega) + p$

with $\quad p = \begin{pmatrix} a_1 x_1 + b_1 x_2 + c_1 x_3 \\ a_2 x_1 + b_2 x_2 + c_2 x_3 \\ a_3 x_1 + b_3 x_2 + c_3 x_3 \end{pmatrix}$

$\langle x, p \rangle = 0$ gives

$$a_1 x_1^2 + b_2 x_2^2 + c_2 x_3^2 + (b_1 + a_2) x_1 x_2 + (c_1 + a_3) x_1 x_3 + (c_2 + b_3) x_2 x_3 = 0$$

$$\Rightarrow \quad a_1 = b_2 = c_2 = 0, \; b_1 = -a_2, \; c_1 = -a_3, \; c_2 = -b_3$$

$$\Rightarrow \quad p = \begin{pmatrix} b x_2 + c x_3 \\ -b_1 x_1 + d x_2 \\ -c x_1 - d x_2 \end{pmatrix} = \begin{pmatrix} \pm a \\ \pm c \\ \pm b \end{pmatrix} \times x$$

FEEC

Polynomials Differential Forms

$\mathcal{P}_k(\Omega)$ polyn $\leq k$ with $d = \dim \Omega$ variables

$$\dim \mathcal{P}_k(\Omega) = \binom{k+d}{k} = \frac{(k+d)!}{k!\,d!}$$

$$p_k(x) = \sum_{i_1+\cdots+i_d \leq k} a_{i_1\cdots i_d}\, x_1^{i_1} \cdots x_d^{i_d}, \quad a_{i_1\cdots i_d} \in \mathbb{R}$$

$$\tilde{p}_k(x) = \sum_{i_1+\cdots+i_d = k} a_{i_1} \cdots a_{i_d}\, x_1^{i_1} \cdots x_d^{i_d}$$

Definition Define $\mathcal{P}_k^q(\Omega)$ polynomials q-forms of degree $\leq k$ as $\omega \in \Lambda^q(\Omega)$ where

$$\omega = \sum_I p_I(x)\, dx^I$$

where $p_I \in \mathcal{P}_k(\Omega)$,

i.e. the coefficient functions of the forms are polynomials of degree $\leq k$. Unlike $\tilde{\mathcal{P}}_k^q(\Omega)$ for the space of polynomial q-forms of degree $= k$, i.e.

$$p_I \in \tilde{\mathcal{P}}_k(\Omega)$$

Definition We call a FE space defined by piecewise polynomial q-forms on a triangulation τ from $\mathcal{P}_k^q(T)$ conform if

$$\omega \in \mathcal{D}^q(\Omega)$$

or in other words

$$\|\omega\| + \|d\omega\| < \infty$$

this is equivalent to the classical setting where $v_h \in L_2(\Omega)$, $\begin{pmatrix} \text{grad} \\ \text{curl} \\ \text{div} \end{pmatrix} v_h \in L_2$.

3D:
$$0 \to P_k^0(\Omega) \xrightarrow{d} P_{k-1}^1(\Omega) \xrightarrow{d} P_{k-2}^2(\Omega) \xrightarrow{d} P_{k-3}^3(\Omega) \xrightarrow{d} 0$$

2D: $0 \to P_k^0(\Omega) \xrightarrow{d} P_{k-1}^1(\Omega) \xrightarrow{d} P_{k-2}^2(\Omega) \xrightarrow{d} 0$

" polynomial de Rham complex"

This is also valid for $\tilde{P}_k^3(\Omega)$.

Exercise 2 $\quad \Omega = (0,1)^2 \quad u, v \in \mathring{H}^1(\Omega)$

$$-\Delta u = 1$$

~~$a(u,v) =$~~

$$\langle -\Delta u, v \rangle_\Omega = -\int_\Omega \Delta u \, v \, d\lambda = \int_\Omega \langle \nabla u, \nabla v \rangle - \int_{\partial\Omega} \partial_\nu u \cdot v \, d\sigma$$

$$= \langle f, v \rangle_\Omega$$

$$\Rightarrow \quad a(u,v) = \langle \nabla u, \nabla v \rangle_\Omega = \langle f, v \rangle = F(v) \qquad \forall v$$

$$u = \sum u_u \varphi_u \quad , \quad v = \sum v_u \varphi_u$$

$$\Rightarrow \quad a(u,v) = u_u v_m \langle \nabla \varphi_u, \nabla \varphi_m \rangle_\Omega$$

$$= u_u \underbrace{a(\varphi_u, \varphi_m)}_{= k} v_m$$

$$= \langle \vec{u}, k\vec{v} \rangle = \vec{u}^T k \vec{v} \overset{k \text{ sym.}}{=} \langle k\vec{u}, \vec{v} \rangle$$

$$F(v) = v_u \underbrace{\langle f, \varphi_u \rangle_\Omega}_{= f_u} = \langle \vec{f}, \vec{v} \rangle$$

$\forall \vec{v}$

$$\Rightarrow \quad k\vec{u} = \vec{f}$$

$$k_{um} = \int_\Omega \langle \nabla \varphi_u, \nabla \varphi_m \rangle d\lambda = \sum_T \int_T \langle \nabla \varphi_u, \nabla \varphi_m \rangle d\lambda$$

$$= \sum_T \int_{T_{ref} = F^{-1}(T)} \langle B_T^{-1} \nabla \hat{\varphi}_u(\hat{x}), B_T^{-1} \nabla \hat{\varphi}_m(\hat{x}) \rangle \underbrace{|\det F'(\hat{x})|}_{= B_T} d\hat{x}$$

$$F(\hat{x}) = B_T \hat{x} + d_T = x, \quad dx = \underbrace{|\det F'(\hat{x})|}_{= |\det B_T|} d\hat{x} \quad , \quad F^{-1}(x) = B_T^{-1}(x - d_T) = \hat{x}$$

$$\partial_i \varphi_u(x) = \partial_i (\varphi_u \circ F \circ F^{-1})(x) \qquad (F^{-1})'(x) = B_T^{-1}$$

$$= \partial_j (\varphi_u \circ F)(\hat{x}) \, \partial_i F_j^{-1}(x)$$

$$= \partial_j \hat{\varphi}_u(\hat{x}) \, \partial_i F_j^{-1}(x) = ((F^{-1})'(x) \nabla \hat{\varphi}_u(\hat{x}))_i$$

$$\Rightarrow \quad \nabla \varphi_u(x) = (F^{-1})'(x) \nabla \hat{\varphi}_u(\hat{x})$$

$$= B_T^{-1} \nabla \hat{\varphi}_u(\hat{x})$$

$$\Rightarrow k_{ij} = \sum_T \int_{T_{ref}} (\nabla \hat{\varphi}_i(\hat{x}))^T B_T^{-T} B_T^{-1} \nabla \hat{\varphi}_j(\hat{x}) |\det B| d\hat{x} \quad (18)$$

$$\varphi = \begin{bmatrix} \varphi_1 \\ \vdots \\ \varphi_N \end{bmatrix} \Rightarrow \varphi' = \begin{bmatrix} (\nabla \varphi_1)^T \\ \vdots \\ (\nabla \varphi_N)^T \end{bmatrix}, \quad \varphi'^T = [\nabla \varphi_1, \dots, \nabla \varphi_N]$$

$$\Rightarrow k = \sum_T \int_{T_{ref}} \hat{\varphi}'^T B_T^{-T} B_T^{-1} \hat{\varphi}' |\det B_T| d\hat{x}$$

? transposition error ?

$$f_i = \int_\Omega f \varphi_i \, dx = \sum_T \int_T f \varphi_i \, dx$$

$$= \sum_T \int_{T_{ref}} f \circ F \cdot \varphi_i \circ F \, |\det B_T| \, d\hat{x}$$

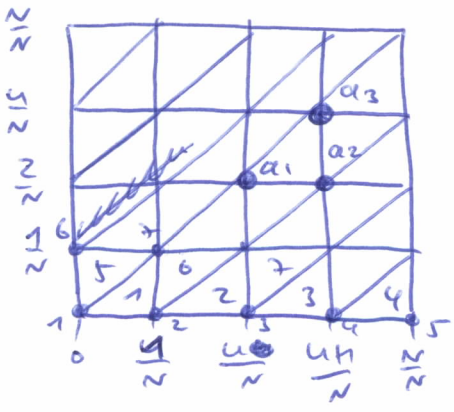
$$\Rightarrow \vec{f} = \sum_T \int_{T_{ref}} f \circ F \cdot \hat{\varphi} \, |\det B_T| \, d\hat{x}$$

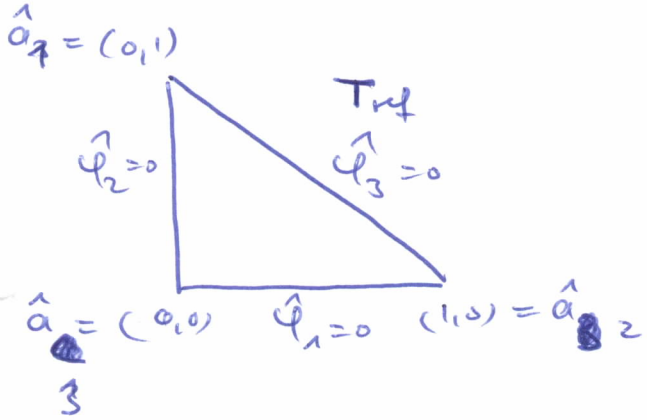
$$\Rightarrow k = \sum_T \int_T \varphi' \varphi'^T \, dx$$

$$\Gamma \quad \varphi' = (\varphi \circ F \circ F^{-1})'$$

$$= \hat{\varphi}' (F^{-1})' = \hat{\varphi}' B_T^{-1}$$

$$= \sum_T \int_{T_{ref}} \hat{\varphi}' B_T^{-1} B_T^{-T} \hat{\varphi}'^T |\det B_T| \, d\hat{x}$$





$$\hat{\varphi}(\hat{x}) = \alpha + \beta \hat{x}_1 + \gamma \hat{x}_2$$

$$= \alpha \cdot \underbrace{1}_{\hat{\varphi}_1(\hat{x})} + \beta \underbrace{\hat{x}_1}_{\hat{\varphi}_2(\hat{x})} + \gamma \underbrace{\hat{x}_2}_{\hat{\varphi}_3(\hat{x})}$$

$$\sim$$

$$\hat{\varphi}_1(\hat{x}) := \hat{x}_1$$
$$\hat{\varphi}_2(\hat{x}) := \hat{x}_2$$
$$\hat{\varphi}_3(\hat{x}) := 1 - \hat{x}_1 - \hat{x}_2$$

$$\hat{\varphi}(\hat{x}) = \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ 1 - \hat{x}_1 - \hat{x}_2 \end{bmatrix}, \quad \hat{\varphi}'(\hat{x}) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{bmatrix}$$

$$k = \sum_T \int_{T_{ref}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{bmatrix} B_T^{-1} B_T^{-T} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix} |\det B_T| \, d\lambda$$

$$\vec{\varphi} = \sum_{T}^{\varphi=1} \int_{T_{ref}} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ 1 - \hat{x}_1 - \hat{x}_2 \end{bmatrix} |\det B_T| \, d\lambda$$

$$B_T \hat{x} + d_T = x$$

$$\Rightarrow B_T(\hat{a}_2) + d_T = a_2$$

$$= d_T$$

$$B_T(\hat{a}_1) + d_T = a_1$$

$$= B_T\begin{pmatrix}0\\1\end{pmatrix} + d_T$$

$$B_T(\hat{a}_3) + d_T = a_3$$

$$= B_T\begin{pmatrix}1\\0\end{pmatrix} + d_T$$

$$\Rightarrow d_T = a_2$$

$$\Rightarrow B_T\begin{pmatrix}0\\1\end{pmatrix} = a_1 - a_2 \quad , \quad B_T\begin{pmatrix}1\\0\end{pmatrix} = a_3 - a_2$$

$$\Rightarrow B_T = B_T\, id = B_T\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = [a_3 - a_2 \,,\, a_1 - a_2]$$

$$\Rightarrow F(\hat{x}) = B_T \hat{x} + d_T = a_2 + [a_3 - a_2 \,,\, a_1 - a_2]\,\hat{x}$$

$$\Rightarrow k = \sum_T \int_{T_{ref}} \begin{bmatrix}1 & 0\\ 0 & 1\\ -1 & -1\end{bmatrix} [a_{3T} - a_{2T} \,,\, a_{1T} - a_{2T}]^{-1}$$

$$\begin{bmatrix} a_{T3}^T - a_{T2}^T \\ a_{T1}^T - a_{T2}^T \end{bmatrix}^{-1} \begin{bmatrix}1 & 0 & -1\\ 0 & 1 & -1\end{bmatrix}$$

$$\cdot \left|\det [a_{T3} - a_{T2} \,,\, a_{T1} - a_{T2}]\right| d\hat{x}$$

$$= \sum_T \underbrace{|T_{ref}|}_{= \frac{1}{2}}$$

1D:

$\hat{\varphi}_1(\hat{x}) = \hat{x}$

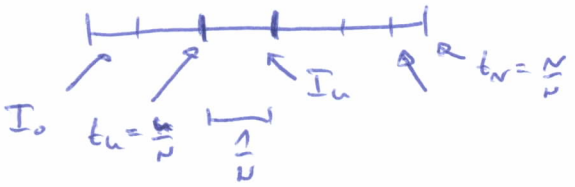
$\hat{\varphi}_2(\hat{x}) = 1 - \hat{x}$

$\hat{\varphi} = \begin{bmatrix} \hat{\varphi}_1 \\ \hat{\varphi}_2 \end{bmatrix}$, $\hat{x} = x$

$\hat{\varphi}' = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$F(\hat{x}) = B\hat{x} + d = \frac{1}{N}\hat{x} + \frac{u}{N}$

$F'(\hat{x}) = \frac{1}{N}$

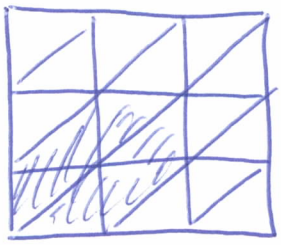


$x = F(\hat{x})$

$\Rightarrow K_{ij} = \int_I \varphi'_i \varphi'_j \, dx = \sum_{u=0}^{N-1} \int_{I_u} \varphi'_i \varphi'_j \, dx$

$= \sum_{u=0}^{N-1}$

~~Das kann~~ This can not be possible. $\varphi_u \notin H^1(I)$ or $H^1_0(I)$ ↑ continuation by 0 to I



$$k = \sum_T R_T^T k_T R_T$$

$$= (a(\phi_i, \phi_j))_{i,j}$$

$$a(\phi_i, \phi_j) = \int_\Omega \nabla\phi_i \nabla\phi_j \, dx$$

$$= \int_{\text{supp}\,\phi_i \cap \text{supp}\,\phi_j} \nabla\phi_i \nabla\phi_j \, dx = \sum_{T \in \text{supp}\,\phi_i \cap \text{supp}\,\phi_j} \int \nabla\phi_i \nabla\phi_j \, dx$$

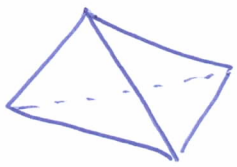
$$\omega = \sum_I p_I \, dx_I$$

$H^0 \quad (\phi_1, \phi_2, \phi_3) \to \phi_1 \in A^0(T)$

$H(\text{div}) \quad (\phi_1, \phi_2, \phi_3) \to \phi_1 \, dx_2 \wedge dx_3 + \phi_2 \, dx_1 \wedge x_3 + \phi_3 \, dx_1 \wedge dx_2$

For a polynomial differential form ω we choose the values

$$k = \int_\varphi \omega \wedge \zeta_{i,m}$$



3D

$\Delta 3$ volume

$\Delta 2$ set of faces

$\Delta 1$ set of edges

$\Delta 0$ points

where $\varphi \in \Delta_m(T)$, $0 \le m \le d$

$\zeta \in P^{m-q}_{k-(m-q)}(\varphi)$

Lowest order N in 2D $H(\mathrm{curl})$

$k=0$

$$p = \begin{pmatrix} a \\ b \end{pmatrix} + c \begin{pmatrix} y \\ -x \end{pmatrix}$$

$$\omega = (a + cy)\,dx_1 + (b + cx)\,dx_2$$

$m = 0, 1, 2$; $q = 1$

for $m=2$ $\quad \xi \in \mathcal{P}^{2-1}_{0-2+1} = \mathcal{P}^1_{-1} := \{0\}$

for $m=1$ $\quad \xi \in \mathcal{P}^{1-1}_{0+1-1} = \mathcal{P}^0_0 =$ constants

for $m=0$ $\quad \xi \in \mathcal{P}^{0-1}_{0-0+1} = \mathcal{P}^{-1}_1 = \{0\}$

Thus for N_0 $(k=0)$

$$\int_e \underbrace{\omega \wedge 1}_{= \omega \cdot 1 = \omega}$$

e an edge in 2D

Conformity condition using Diff. Forms

$$\langle E, H \rangle_\Omega = \int_\Omega E \wedge *H \quad \left(= \int_\Omega \langle E, H \rangle \overbrace{*1}^{=\hat{\omega}} = \int_\Omega \langle E, H \rangle \, dx \right)$$

Product rule

$$d(E \wedge *H) = dE \wedge *H + E \wedge * \delta H$$

and

$$d(\omega \wedge \xi) = d\omega \wedge \xi + (-1)^q \omega \wedge d\xi$$

$\Downarrow$

$$\langle dE, H \rangle_\Omega + \langle E, \delta H \rangle_\Omega = \int_\Omega dE \wedge *H + E \wedge * \delta H$$

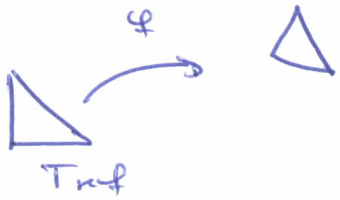
$$= \int_\Omega d(E \wedge *H) = \int_\Omega \mathrm{tr}^*(E \wedge *H)$$

$$= \int_{\partial\Omega} \mathrm{tr}^* E \wedge \mathrm{tr}^* *H = \int_{\partial\Omega} \langle \gamma_t E, \gamma_n H \rangle \overset{!}{=} 0$$

H^1 read dE as the gradient and $\gamma_t E = E$;
H is the test function with compact support
$H(div, \Omega)$ choose E as $(d-1)$-form and H as test function with compact support
$(d=3)$ $H(curl, \Omega)$ choose E as $\underbrace{(d-2)}_{=1}$-form and H as test function with compact support.

We can then obtain conformity for a piecewise smooth form ω (piecewise C^∞, piecewise polynomial) if the correct trace is continuous over $f \in \Delta_{d-1}$.

Polynomial Space



Affine Invariance of P_k^q
φ affine function

$$\varphi^*(P_k^q(\Omega_1)) \subset P_k^q(\Omega_2) \quad (=)$$

Affine Families

- $v(x) = \hat{p}(\underbrace{F^{-1}(x)}_{\text{affine mapping}}) \qquad \hat{p} \in \Pi_{ref}$
- the functionals $l(v)$ are defined on T_{ref}

Argyris HCT (C^1-elements; are not affine families)

How to find affine invariant polynomial spaces other than P_k^q ?

$$\varphi^* d P_{k+1}^{q-1} = d\varphi^* P_{k+1}^{q-1} \subset d P_{k+1}^{q-1}$$

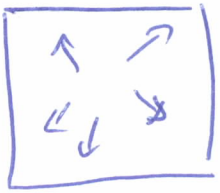
$\Rightarrow$ $d P_{k+1}^{q-1}$ is affine invariant

there exists an operator

$$d(\pi(\omega)) = \omega \quad , \text{if } d\omega = 0$$

potential operator

There is an explicit formula for computing $\pi(\omega)$



$d\omega = 0 \iff$ vector field u is conservative, curl free, irrotational, is a gradient field

compute potential $\nabla p = u$ $\xleftarrow[\text{shaped}]{\text{star}}$

$$f^* \pi(\tilde{P}^{q+1}_{k-1}) = \pi f^* \tilde{P}^{q+1}_{k-1} \subset \pi \tilde{P}^{q+1}_{k-1}$$

only for <u>linear</u> f

$\pi(\tilde{P}^{q+1}_{k-1})$ is not affine invariant

$$P^q_k \subsetneq P^q_{k+1,-} = P^q_k \oplus \pi_0 \tilde{P}^{q+1}_k \quad \text{is affine invariant}$$

$$\subsetneq P^q_{k+1}$$

$\tilde{P}^{q+1}_k$ restricted to closed forms $dp = 0$

$$? \quad {}_0\tilde{P}^{q+1}_k = d\,\tilde{P}^q_{k+1} \qquad (?)_p$$

RT $\quad P_k(T) + \{P_k(T)^d : p \times x = 0\}$

N $\quad$ " $\quad \langle p, x \rangle = 0$

2D: $P^0_{k,-}$ Lagrange elements H^1 degree $\le k$

$P^1_{k,-}$ RT of order $k-1$ degree k

$P^2_{k,-}$ (discontinuous ...)

3D: $P^0_{k,-}$ Lagrange elements degree $\le k$

$P^1_{k,-}$ N $H(\mathrm{curl})$ order $k-1$ degree k

$P^2_{k,-}$ RT $H(\mathrm{div})$ N-$H(\mathrm{div})$ order $k-1$

$P^3_{k,-}$ (discontinuous ...)

Implementing Raviart-Thomas RT

Consider a mixed system $u=0$ on $\partial\Omega$

$$\int_\Omega q\,p\,dx - \int_\Omega \mathrm{div}\,q\cdot u\,dx = 0$$

$$\int_\Omega v\,\mathrm{div}\,p\,dx = -\int_\Omega \varphi\, v\,dx$$

Find $(p,u) \in H(\mathrm{div}) \times L_2(\Omega)$

Choose basis ψ_i

for $p, q \in RT_0$,

ϕ_i for $u, v \in P_0$

(piecewise constant)

$$u = \sum_{i=1}^{m} u_i \phi_i$$

$$v = \sum_{i=1}^{m} v_i \phi_i$$

$$p = \sum_{i=1}^{n} p_i \psi_i$$

$$q = \sum_{i=1}^{n} q_i \psi_i$$

$$\Gamma \quad \int_\Omega (p - \nabla u)\, q\,dx = 0$$

$$\int_\Omega (\varphi + \mathrm{div}\,p)\, v\,dx = 0$$

$$\Rightarrow \quad \nabla u = p$$

$$\mathrm{div}\,p = -\varphi$$

$$\Rightarrow -\Delta u = \varphi + f \quad \Big| \quad (!)_p$$

$$u|_{\partial\Omega} = 0$$

$$\sum_{i,j=1}^{n} q_i p_j \int_\Omega \varphi_i \varphi_j \, dx - \sum_{i=1}^{n} \sum_{k=1}^{m} q_i u_k \int_\Omega \operatorname{div} \varphi_i \, \phi_k \, dx = 0$$

$$\sum_{i=1}^{n} \sum_{k=1}^{m} v_k p_i \int_\Omega \phi_k \operatorname{div} \varphi_i \, dx$$

$$= -\sum_{k=1}^{m} v_k \int_\Omega f \phi_k \, dx$$

$$(q v) \begin{pmatrix} B & C^T \\ C & 0 \end{pmatrix} \begin{pmatrix} p \\ u \end{pmatrix} = (q v) \begin{pmatrix} 0 \\ -f \end{pmatrix} \qquad \forall q, v$$

$$\begin{pmatrix} B & C^T \\ C & 0 \end{pmatrix} \begin{pmatrix} p \\ u \end{pmatrix} = \begin{pmatrix} 0 \\ -f \end{pmatrix}$$

$$B = \left(\int_\Omega \varphi_i \varphi_j \, dx \right)_{i,j = 1, \dots, n}$$

$$C = \left(\int_\Omega \phi_k \operatorname{div} \varphi_i \, dx \right)_{\substack{k = 1, \dots, m \\ i = 1, \dots, n}}$$

this is 1 on every element

Exercise 3 — $\hat{\varphi}_i$ continued by 0 to all of $\mathbb{R}$!

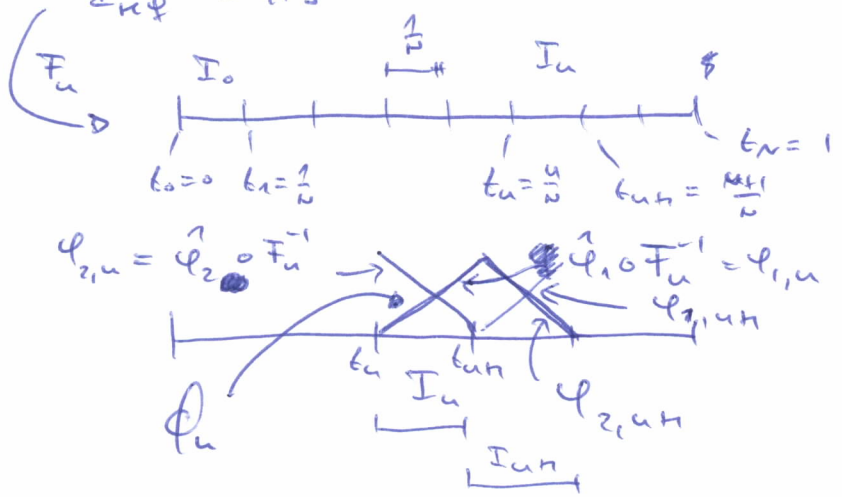
1D:

$\hat{\varphi}_1(x) = x$

$\hat{\varphi}_2(x) = 1 - x$

$\hat{\varphi} = \begin{bmatrix} \hat{\varphi}_1 \\ \hat{\varphi}_2 \end{bmatrix}$, $\hat{\varphi}' = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$I_{ref} = [0,1]$



$F_u(\hat{x}) = B\hat{x} + d$

$= \frac{1}{N}\hat{x} + \frac{u}{N} = x$

$F_u'(\hat{x}) = \frac{1}{N}$

$F_u^{-1}(x) = B^{-1}(x - d)$

$= N\left(x - \frac{u}{N}\right) = Nx - u$

$(F_u^{-1})'(x) = N$

basis functions: $\tilde{\phi}_u(x) = \alpha \varphi_{1,u}(x) + \beta \varphi_{2,u+1}(x)$

$= \alpha(Nx - u) + \beta(1 - Nx + u)$

$= \begin{cases} \alpha(Nx - u), & x \in I_u \\ \beta(1 - Nx + u + 1), & x \in I_{u+1} \end{cases}$

$\lim_{x \to t_{u+1}} \phi_u(x) = \alpha$

$\stackrel{!}{=} \beta = \lim_{x \to t_{u+1}} \phi_u(x)$

$\Rightarrow \alpha = \beta \Rightarrow \tilde{\phi}_u(x) = \alpha \varphi_{1,u}(x) + \alpha \varphi_{2,u+1}(x)$

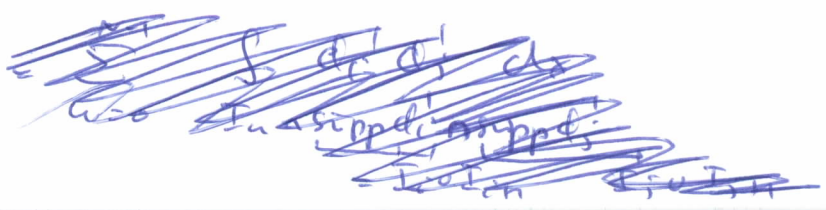
$\alpha = 1 \Rightarrow \phi_u(x) = \begin{cases} Nx - u, & x \in I_u \\ 2 + u - Nx, & x \in I_{u+1} \end{cases}$

~~Then:~~

$a(\phi_i, \phi_j) = \int_\Omega \phi_i'^2(x)\,dx = \sum_{u=0}^{N-1} \int_{I_u} \phi_u'^2(x+1)\,dx$

Then

$a(\phi_i, \phi_j) = \int_\Omega \phi_i' \phi_j'\,dx = \sum_{u=0}^{N-1} \int_{I_u} \phi_i' \cdot \phi_j'\,dx$



$i = j$:

$$a(\phi_i, \phi_i) = \sum_{u=0}^{N-1} \int_{I_u} \phi_i' \phi_i' \, dx = \int_{I_i} \phi_i'^2 \, dx + \int_{I_{i+1}} \phi_i'^2 \, dx$$

$$= \int_{I_i} \varphi_{1,i}'^2 \circ dx + \int_{I_{i+1}} \varphi_{2,i+1}'^2 \circ dx$$

$$= \int_{I_i} (\hat{\varphi}_1' \circ F_i^{-1}(x) \cdot (F_i^{-1})'(x))^2 \, dx$$

$x = F_i(\hat{x})$

$$+ \int_{I_{i+1}} (\hat{\varphi}_2' \circ F_{i+1}^{-1}(x) \cdot (F_{i+1}^{-1})'(x))^2 \, dx$$

$x = F_{i+1}(\hat{x})$

$$= N^2 \int_{I_{ref}} (\hat{\varphi}_1')^2 \frac{d\hat{x}}{N} + N^2 \int_{I_{ref}} (\hat{\varphi}_2')^2 \frac{d\hat{x}}{N}$$

$= 1$

$$= N + N = 2N$$

$i = j+1$,

$$a(\phi_i, \phi_j) = \sum_{u=0}^{N-1} \int_{I_u} \phi_i' \phi_j' \, dx = \int_{I_{j+1}} \phi_{j+1}' \phi_j' \, dx$$

$$= \int_{I_{j+1}} \varphi_{1,j+1}'(x) \, \varphi_{2,j+1}'(x) \, dx$$

$= N$

$$= \int_{I_{j+1}} \hat{\varphi}_1' \circ F_{j+1}^{-1}(x) \cdot (F_{j+1}^{-1})'(x) \cdot \hat{\varphi}_2' \circ F_{j+1}^{-1} \cdot \underbrace{(F_{j+1}^{-1})'(x)}_{=N} \, dx$$

$$= N^2 \int_{I_{ref}} \underbrace{\hat{\varphi}_1'}_{1} \underbrace{\hat{\varphi}_2'}_{-1} \frac{d\hat{x}}{N}$$

$$= -N$$

$$k = N \begin{bmatrix} 2 & -1 & & & 0 \\ -1 & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & -1 \\ 0 & & & -1 & 2 \end{bmatrix}$$

Mixed Methods

The Laplace $\Delta u = \operatorname{div} \operatorname{grad} u = -f$

$$\Rightarrow \quad \operatorname{grad} u = \sigma$$
$$\operatorname{div} \sigma = -f$$

$(\sigma, u) \in L_2(\Omega)^d \times H^1(\Omega)$, $\forall (\tau, v) \in \dots$

$$(\sigma, \tau)_\Omega - (\operatorname{grad} u, \tau)_\Omega = 0$$

$$(\sigma, \operatorname{grad} v)_\Omega = (f, v)_\Omega$$

Find (u, λ) such that for all (v, μ)

$$a(u, v) + b(\lambda, v) = (f, v) \qquad (*)$$
$$b(u, \mu) = (g, \mu)$$

$$A u + B \lambda = f$$
$$B u = g$$

For the solution of saddle point problems the so called inf-sup condition is essential.

$$\inf_{\mu \in \Pi} \sup_{v \in X} \frac{b(v, \mu)}{\|v\| \, \|\mu\|} > \beta > 0 \qquad (**)$$

"Babuška - Brezzi - condition"

The saddle point problem (*) is uniquely solvable if a is elliptic on the kernel of B, i.e.

$$a(v,v) \geq \alpha \|v\|^2 \qquad \forall v \in V = \ker B = \{ v : b(v,\mu) = 0 \ \forall \mu \}$$

and (**) is satisfied.

$$\text{inf sup} \iff \sup_{v \in V} \frac{b(v,\mu)}{\|v\|} \geq \beta \|\mu\| \qquad \forall \mu \in \Pi$$

$$b(\mu_1, v) = b(\mu_2, v) \qquad \forall v \in V$$

$$0 = b(\mu_1 - \mu_2, v) \geq \beta \|\mu\|$$

$$\Rightarrow \mu_1 = \mu_2 \quad \text{thus} \quad b(\mu, v) = B^T \mu \text{ is injective (?)}$$

If inf sup condition is not fulfilled this will result in instability of numerical method, "oscillations", "checkerboard modes"

Stable discrete mixed methods have to fulfill a discrete inf sup-condition for V_h, M_h, with a parameter β independent of h

$$\inf_{\mu \in \Pi_h} \sup_{v \in V_h} \frac{b(v,\mu)}{\|v\| \|\mu\|} \geq \beta > 0 \qquad \beta \text{ independent of } h$$

For injectivity of B^T the condition

$$\inf_{\mu \in \Pi_h} \sup_{v \in V_h} \frac{b(v,\mu)}{\|v\| \|\mu\|} > 0$$

would be sufficient. But then the operator $B^T B$ may degenerate for $h \to 0$. (constant β becomes smaller and smaller)

It can be shown that the inf sup condition is equivalent the existence of a decomposition of

$X = V \oplus V^\perp$. $\oplus = \oplus_a$?

For $u \in X$ there exists an orthogonal decomposition

$$u = v + w \qquad (\star)$$

such that $\|w\|_X \leq \beta^{-1} \|Bu\|_\Pi$ $(\leq)_p$

$$V = \{ v \in X : b(v,\mu) = 0 \ \forall \mu \in \Pi \}$$

Constructing mixed FE spaces from Differential Forms calculus automatically gives a decomposition of the form $(\star)$. The condition $(\star)$ has still to be shown. This decomposition relies on exact sequences of the discrete de Rham complex.

$$\to P_{k+1}^{q-1}(\Omega) \xrightarrow{d_{k+1}} P_k^q(\Omega) \xrightarrow{d_{k+1}} P_{k-1}^{q+1}(\Omega)$$
$$\xleftarrow{\delta_k} \qquad\qquad \xleftarrow{\delta_{k+1}}$$

If Ω star shape then P_k^q can be decomposed into an orthogonal sum $P_k^q(\Omega) = V \oplus V^\perp$

where $V = \text{range}(d_{k-1}) = \ker(d_{k+1})$

$$V^\perp = \ker(\delta_k) = \text{range}(\delta_{k+1})$$

"discrete Hodge decomposition"

For this an interpolation operator I_k satisfying

$$d I_{k+1} u = I_k \, du$$

is necessary. Such regularity

$h_T = \text{diam } T$

δ_T = radius of a sphere contained in

shape regular $\quad c_1 \delta_T \geq h_T$

with constant uniform for the whole triangulation (34)

quasi uniform: $c_1 \rho_T \geq c_2 h_T \geq h_{max}$

h_{max} is diam T_{max} the element with largest diameter.

condition $\quad c \operatorname{vol} T \leq h_T^d \leq \tilde{c} \operatorname{vol} T \qquad (*)$

If you have $(*)$ for continuous differential forms you can define an operator I_h

$$\omega \in \mathbb{P}^q_{h,-} \qquad \int_f I_h \omega \wedge v = \int_f \omega \wedge v \qquad \forall v \in \mathbb{P}^{m-q}_{h-m+q}$$

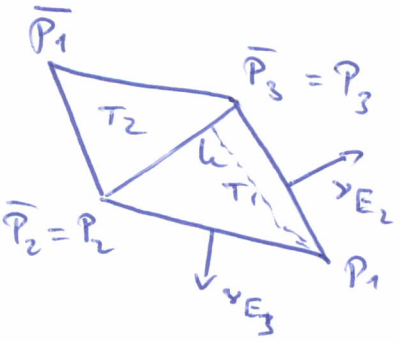
for faces $f \in \Delta_m(\tau)$ with an estimate

$$\|\omega - I_h \omega\|_{L^{2,q}} \leq c h^t |\omega|_{H^{t,q}}, \quad \frac{d-q}{2} < t < h+1$$

$$\begin{tikzcd} D^q(X) \arrow[r, "d"] \arrow[d, "I_h"'] & D^{q+1}(X) \arrow[d, "I_h"] \\ \mathbb{P}^q_{h,-} \arrow[r, "d"] & \mathbb{P}^{q+1}_{h-1,-} \end{tikzcd}$$

The diagram commutes.

Clément-like interpolation for non-continuous differential forms.



$$\varphi_E|_{T_1} = \frac{|E|}{2|T_1|}(x - P_1)$$

$$\varphi_E|_{T_2} = -\frac{|E|}{2|T_2|}(x - \bar{P}_1)$$

$$\varphi_E \cdot \nu_{E_{R(!)_p}} = \begin{cases} 1 & \text{on } E \\ 0 & \text{on other edges} \end{cases}$$

$$|T| = \frac{|E| \cdot G}{2}$$

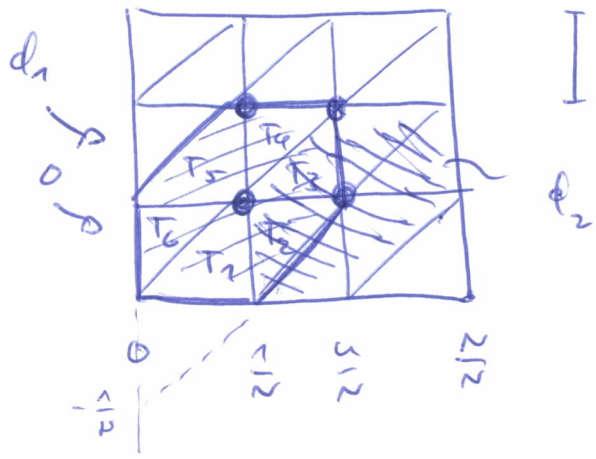
φ_E lin. independent

φ_E explicit basis for RT_0 in 2D

$$\text{div}\,\varphi_E|_{T_{1/2}} = \pm \begin{cases} |E|/|T_{1/2}| & \text{on } T_{1/2} \\ 0 \end{cases}$$

Exercise 4

$$-\Delta u = f \qquad \Omega = (0,1)^2$$
$$u|_{\partial\Omega} = 0$$



$\Rightarrow (N+1)^2$ points

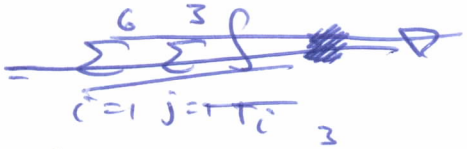
$\Rightarrow (N-1)^2$ basis functions

$6T \times 3$ dof $= 18$ dof

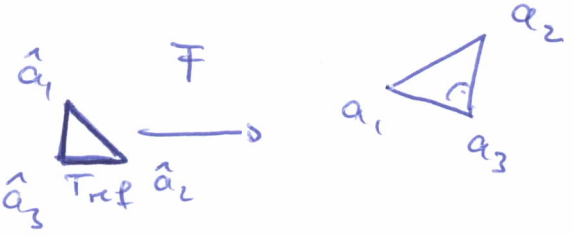
$\in H^1(\Omega) \Rightarrow$ 1 dof

$$a(u,v) = u_n v_m \int_\Omega \nabla\phi_n \nabla\phi_m \, dx$$
$$= a(\phi_n, \phi_m)$$

$$a(\phi_n, \phi_n) = \int_\Omega |\nabla\phi_n|^2 dx = \sum_{i=1}^{6} \int_{T_i} |\nabla\phi_n|^2 dx$$



$$\phi_n|_{T_i} = \sum_{s=1}^{3} \varphi_{s,n} = \varphi_{j,n}$$
$$= \hat{\varphi}_j \circ F$$



$$F(\hat{x}) = B\hat{x} + d$$
$$F(\hat{a}_3) = d \overset{!}{=} a_3$$
$$F(\hat{a}_2) = B\begin{pmatrix}1\\0\end{pmatrix} + d \overset{!}{=} a_2$$
$$F(\hat{a}_1) = B\begin{pmatrix}0\\1\end{pmatrix} + d \overset{!}{=} a_1$$

$$\Rightarrow B = B\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = (a_2 - a_3 \quad a_1 - a_3)$$
$$\Rightarrow F(\hat{x}) = (a_2 - a_3 \quad a_1 - a_3)\,\hat{x} + a_3$$
$$\Rightarrow F'(\hat{x}) = B = (a_2 - a_3 \quad a_1 - a_3)$$
$$F^{-1}(x) = B^{-1}(x - a_3)$$