SCHRIFTENREIHE DER FAKULTÄT FÜR MATHEMATIK

On Korn's First Inequality for Mixed Tangential and Normal Boundary Conditions on Bounded Lipschitz-Domains in $\mathbb{R}^N$

by
Sebastian Bauer & Dirk Pauly

SM-UDE-791 2016

Received: January 14, 2016

On Korn's First Inequality for Mixed Tangential and Normal Boundary Conditions on Bounded Lipschitz-Domains in $\mathbb{R}^N$

SEBASTIAN BAUER AND DIRK PAULY

Abstract. We prove that for bounded Lipschitz domains in $\mathbb{R}^N$ Korn's first inequality holds for vector fields satisfying homogeneous mixed normal and tangential boundary conditions.

Contents

1. Introduction

Recently, motivated by [3, 4] and inspired by the ideas and techniques presented in [10, 12, 11] for estimating the Maxwell constants, we have shown in [2] that Korn's first inequality, i.e.,

$$|\nabla v|_{\mathsf{L}^2(\Omega)} \leq c_{\mathsf{k}} |\operatorname{sym} \nabla v|_{\mathsf{L}^2(\Omega)}, \tag{1}$$

holds with $c_{\mathsf{k}} = \sqrt{2}$ for all vector fields v satisfying (possibly mixed) homogeneous normal or homogenous tangential boundary conditions and for all piecewise $\mathsf{C}^{1,1}$-domains $\Omega \subset \mathbb{R}^N$, $N \geq 2$, with concave boundary parts. In this contribution, we extend (1) to any bounded (strong) Lipschitz domain $\Omega \subset \mathbb{R}^N$, $N \geq 2$. As pointed out in [4], this Korn inequality has an important application in statistical physics, more precisely in the study of relaxation to equilibrium of rarefied gases modeled by Boltzmann's equation.

2. Preliminaries

We will utilize the notations from [2]. Throughout this paper and unless otherwise explicitly stated, let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a bounded domain with Lipschitz boundary $\Gamma := \partial\,\Omega$, i.e., locally Γ can be represented as a graph of a Lipschitz function. As in [2], we introduce the standard scalar valued Lebesgue and Sobolev spaces by $\mathsf{L}^2(\Omega)$ and $\mathsf{H}^1(\Omega)$, respectively. These definitions extend componentwise to vector or matrix, or more general tensor fields and we will use the same notations for these spaces. Moreover, we will consistently denote functions by u and vector fields by v. We define the vector valued H^1-Sobolev space $\mathring{\mathsf{H}}^1_{\mathsf{t}}(\Omega)$ resp. $\mathring{\mathsf{H}}^1_{\mathsf{n}}(\Omega)$ as closure in $\mathsf{H}^1(\Omega)$ of the set of test vector fields

$$\mathring{\mathsf{C}}^\infty_{\mathsf{t}}(\Omega) := \big\{ v|_\Omega : v \in \mathring{\mathsf{C}}^\infty(\mathbb{R}^N) \wedge v_{\mathsf{t}} = 0 \big\}, \qquad \mathring{\mathsf{C}}^\infty_{\mathsf{n}}(\Omega) := \big\{ v|_\Omega : v \in \mathring{\mathsf{C}}^\infty(\mathbb{R}^N) \wedge v_{\mathsf{n}} = 0 \big\}, \tag{2}$$

Date: January 14, 2016.
1991 Mathematics Subject Classification. 49J40 / 82C40 / 76P05.
Key words and phrases. Korn inequality, tangential and normal boundary conditions, Boltzmann equation.

respectively, generalizing homogeneous tangential resp. normal boundary conditions. Here, ν denotes the a.e. defined outer unit normal at Γ giving a.e. the tangential resp. normal component

$$v_{\mathsf{n}} := \nu \cdot v|_{\Gamma}, \quad v_{\mathsf{t}} := v|_{\Gamma} - v_{\mathsf{n}}\nu$$

of v on Γ. We assume additionally that Γ is decomposed into two relatively open subsets Γ_{t} and $\Gamma_{\mathsf{n}} := \Gamma \setminus \overline{\Gamma_{\mathsf{t}}}$ and introduce the vector valued H^1-Sobolev space of mixed boundary conditions $\mathring{\mathsf{H}}^1_{\mathsf{t},\mathsf{n}}(\Omega)$ as closure in $\mathsf{H}^1(\Omega)$ of the set of test vector fields

$$\mathring{\mathsf{C}}^{\infty}_{\mathsf{t},\mathsf{n}}(\Omega) := \big\{ v|_{\Omega} \, : \, v \in \mathring{\mathsf{C}}^{\infty}(\mathbb{R}^N) \, \wedge \, v_{\mathsf{t}}|_{\Gamma_{\mathsf{t}}} = 0 \, \wedge \, v_{\mathsf{n}}|_{\Gamma_{\mathsf{n}}} = 0 \big\}. \tag{3}$$

3. Korn's Second Inequality

It is well known that Korn's second inequality can easily be proved by a simple H^{-1}-argument. Let us illustrate a simple and short proof: In the sense of distributions we have e.g. for all vector fields $v \in \mathsf{L}^2(\Omega)$ that the components of $\nabla\nabla v$[i] consist only of components of $\nabla \operatorname{sym} \nabla v$, i.e.,

$$\forall\, i,j,k = 1,\dots,N \qquad \partial_i \partial_j v_k = \partial_i \operatorname{sym}_{j,k} \nabla v + \partial_j \operatorname{sym}_{i,k} \nabla v - \partial_k \operatorname{sym}_{i,j} \nabla v, \tag{4}$$

where $\operatorname{sym}_{j,k} T := (\operatorname{sym} T)_{j,k}$. By [7, Corollary 2.1] we obtain (for scalar functions) the Poincaré estimate

$$\exists\, c > 0 \quad \forall\, u \in \mathsf{L}^2(\Omega) \quad \frac{1}{c}|u - \pi_{\mathbb{R}} u|_{\mathsf{L}^2(\Omega)} \leq |\nabla u|_{\mathsf{H}^{-1}(\Omega)} \leq c|u|_{\mathsf{L}^2(\Omega)}, \quad \pi_{\mathbb{R}} u = \fint_{\Omega} u := \frac{1}{|\Omega|}\int_{\Omega} u, \tag{5}$$

where the original result for Lipschitz boundaries is due to Nečas from the 1960s, see [5] for the case of a smooth domain. Here $\pi_{\mathbb{R}} u$ denotes the $\mathsf{L}^2(\Omega)$-orthogonal projection of u onto $\mathbb{R}$.

Remark 1. *The best constant for the first inequality in* (5)*, i.e.,*

$$\exists\, c > 0 \qquad \forall\, u \in \mathsf{L}^2_0(\Omega) := \{u \in \mathsf{L}^2(\Omega) \, : \, \langle u, 1 \rangle_{\mathsf{L}^2(\Omega)} = 0\} \qquad |u|_{\mathsf{L}^2(\Omega)} \leq c|\nabla u|_{\mathsf{H}^{-1}(\Omega)},$$

is the inverse of the well known inf-sup- or LBB-constant

$$c_{\mathsf{LBB}} := \inf_{0 \neq u \in \mathsf{L}^2_0(\Omega)} \sup_{v \in \mathring{\mathsf{H}}^1(\Omega)} \frac{\langle u, \operatorname{div} v \rangle_{\mathsf{L}^2(\Omega)}}{|u|_{\mathsf{L}^2(\Omega)} |\nabla v|_{\mathsf{L}^2(\Omega)}} = \inf_{0 \neq u \in \mathsf{L}^2_0(\Omega)} \frac{|\nabla u|_{\mathsf{H}^{-1}(\Omega)}}{|u|_{\mathsf{L}^2(\Omega)}}.$$

We note that the LBB-constant can be bounded from below by the inverse of the continuity constant c_A of the H^1-potential operator $A : \mathsf{L}^2_0(\Omega) \to \mathring{\mathsf{H}}^1(\Omega)$ with $\operatorname{div} Au = u$, i.e.,

$$\forall\, u \in \mathsf{L}^2_0(\Omega) \qquad |\nabla Au|_{\mathsf{L}^2(\Omega)} \leq c_A |u|_{\mathsf{L}^2(\Omega)}.$$

This follows directly by setting $v := Au$ and

$$c_{\mathsf{LBB}} \geq \inf_{0 \neq u \in \mathsf{L}^2_0(\Omega)} \frac{|u|^2_{\mathsf{L}^2(\Omega)}}{|u|_{\mathsf{L}^2(\Omega)} |\nabla Au|_{\mathsf{L}^2(\Omega)}} \geq \frac{1}{c_A}.$$

Alternatively, for $0 \neq u \in \mathsf{L}^2_0(\Omega)$ we have $\nabla Au \neq 0$ and

$$|u|^2_{\mathsf{L}^2(\Omega)} = \frac{\langle u, \operatorname{div} Au \rangle_{\mathsf{L}^2(\Omega)}}{|\nabla Au|_{\mathsf{L}^2(\Omega)}} |\nabla Au|_{\mathsf{L}^2(\Omega)} \leq c_A |u|_{\mathsf{L}^2(\Omega)} |\nabla u|_{\mathsf{H}^{-1}(\Omega)} \quad \Rightarrow \quad |u|_{\mathsf{L}^2(\Omega)} \leq c_A |\nabla u|_{\mathsf{H}^{-1}(\Omega)},$$

which also shows $1/c_A \leq c_{\mathsf{LBB}}$.

Let $v \in \mathsf{H}^1(\Omega)$. Combining (4) and (5) we get with a generic constant $c > 0$

$$|\nabla v - G_v|_{\mathsf{L}^2(\Omega)} \leq c|\nabla\nabla v|_{\mathsf{H}^{-1}(\Omega)} \leq c|\nabla \operatorname{sym} \nabla v|_{\mathsf{H}^{-1}(\Omega)} \leq c|\operatorname{sym} \nabla v|_{\mathsf{L}^2(\Omega)}, \qquad G_v := \fint_{\Omega} \nabla v, \tag{6}$$

[i] We denote by ∇v the transpose of the Jacobian of v and by $\nabla\nabla v$ the tensor of second derivatives of v.

where $G_v = \pi_{\mathbb{R}^{N\times N}} \nabla v$ is the $\mathsf{L}^2(\Omega)$-orthogonal projection of ∇v onto $\mathbb{R}^{N\times N}$. By Gauß' theorem and [8, Theorem 1.5.1.10] we have for all $\epsilon > 0$

$$|G_v|_{\mathsf{L}^2(\Omega)} \leq \epsilon |\nabla v|_{\mathsf{L}^2(\Omega)} + \frac{c}{\epsilon}|v|_{\mathsf{L}^2(\Omega)},$$

which together with (6) immediately yields:

Theorem 2 (Korn's second inequality). *There exists $c > 0$ such that for all $v \in \mathsf{H}^1(\Omega)$*

$$|\nabla v|_{\mathsf{L}^2(\Omega)} \leq c\big(|\operatorname{sym}\nabla v|_{\mathsf{L}^2(\Omega)} + |v|_{\mathsf{L}^2(\Omega)}\big).$$

By standard mollification we see that the restrictions of $\overset{\circ}{\mathsf{C}}{}^\infty(\mathbb{R}^N)$-vector fields to Ω are dense in

$$\mathsf{S}(\Omega) := \{v \in \mathsf{L}^2(\Omega) \,:\, \operatorname{sym}\nabla v \in \mathsf{L}^2(\Omega)\},$$

even if Ω just has the segment property. Especially $\mathsf{H}^1(\Omega)$ is dense in $\mathsf{S}(\Omega)$. This shows immediately:

Theorem 3 (H^1-regularity). *It holds $\mathsf{S}(\Omega) = \mathsf{H}^1(\Omega)$.*

Proof. Let $v \in \mathsf{S}(\Omega)$. By density, there exists a sequence $(v_n) \subset \mathsf{H}^1(\Omega)$ converging to v in $\mathsf{S}(\Omega)$. By Theorem 2 (v_n) is a Cauchy sequence in $\mathsf{H}^1(\Omega)$ converging to v, yielding $v \in \mathsf{H}^1(\Omega)$. □

Remark 4. *The latter arguments show, that any domain allowing for Poincaré's (or Nečas') estimate* (5) *fulfills the Korn type inequality* (6)*. If for the domain additionally Gauß theorem holds, Korn's second inequality Theorem 6 holds. In these domains we have also the H^1-regularity Theorem 3, provided that the segment property holds.*

To apply standard solution theories for linear elasticity, such as Fredholm's alternative for bounded domains or Eidus' limiting absorption principle [6] for exterior domains, it is most important to ensure for bounded domains the compact embedding

$$\mathsf{S}(\Omega) \hookrightarrow \mathsf{L}^2(\Omega). \tag{7}$$

As long as Korn's second inequality, i.e., the continuous embedding $\mathsf{S}(\Omega) \hookrightarrow \mathsf{H}^1(\Omega)$, holds true, this follows easily by Rellich's selection theorem, i.e., the compact embedding $\mathsf{H}^1(\Omega) \hookrightarrow \mathsf{L}^2(\Omega)$. As shown in [13], there are bounded irregular domains, more precisely bounded domains with the p-cusp property, see [14, Definition 3] or [13, Definition 2], with $1 < p < 2$, for which Korn's second inequality fails and so the embedding $\mathsf{S}(\Omega) \subset \mathsf{H}^1(\Omega)$ by the closed graph theorem[ii], but the important compact embedding (7) remains valid.

We emphasize that by [13, Theorem 2] the compact embedding (7) holds for bounded domains having the p-cusp property with $1 \leq p < 2$[iii], and that (7) implies immediately a Poincaré type inequality for elasticity by a standard indirect argument. For this we define

$$\mathsf{S}_0(\Omega) := \{v \in \mathsf{S}(\Omega) \,:\, \operatorname{sym}\nabla v = 0\} = \{v \in \mathsf{L}^2(\Omega) \,:\, \operatorname{sym}\nabla v = 0\}.$$

It is well known that even for any domain Ω

$$\mathsf{S}_0(\Omega) = \mathcal{R}$$

holds, where $\mathcal{R} := \{Sx + a \,:\, S \in \mathfrak{so} \wedge a \in \mathbb{R}^N\}$ is the space rigid motions and $\mathfrak{so} = \mathfrak{so}(N)$ the vector space of constant skew-symmetric matrices. This follows easily for $v \in \mathsf{S}_0(\Omega)$ by approximating Ω by smooth domains Ω_n, in each of which $v_n := v|_{\Omega_n}$ equals to the same rigid motion $r \in \mathcal{R}$.

[ii] The identity mapping $\mathrm{id}_{\mathsf{S}} : \mathsf{S}(\Omega) \to \mathsf{H}^1(\Omega)$ is continuous, if and only if id_{S} is closed, if and only if $\mathsf{S}(\Omega) \subset \mathsf{H}^1(\Omega)$.

[iii] For $p = 1$ the 1-cusp property equals the strict cone property, which itself holds for strong Lipschitz domains.

Theorem 5 (Poincaré inequality for elasticity)**.** *Let* Ω *be bounded and possess the* p*-cusp property with some* $1 \leq p < 2$*. There exists* $c > 0$ *such that for all* $v \in \mathsf{S}(\Omega) \cap \mathcal{R}^{\perp}$

$$|v|_{\mathsf{L}^2(\Omega)} \leq c|\operatorname{sym}\nabla v|_{\mathsf{L}^2(\Omega)}.$$

Equivalently, for all $v \in \mathsf{S}(\Omega)$

$$|v - r_v|_{\mathsf{L}^2(\Omega)} \leq c|\operatorname{sym}\nabla v|_{\mathsf{L}^2(\Omega)}, \qquad r_v := \pi_{\mathcal{R}} v.$$

Here and throughout the paper, we denote orthogonality in $\mathsf{L}^2(\Omega)$ by $\perp$. Moreover, $\pi_{\mathcal{R}} v$ denotes the $\mathsf{L}^2(\Omega)$-orthogonal projection of v onto the rigid motions $\mathcal{R}$.

Proof. If the assertion was wrong, there exists a sequence $(v_n) \subset \mathsf{S}(\Omega) \cap \mathcal{R}^{\perp}$ with $|v_n|_{\mathsf{L}^2(\Omega)} = 1$ and $|\operatorname{sym}\nabla v_n|_{\mathsf{L}^2(\Omega)} \to 0$. By (7) we can assume without loss of generality that (v_n) converges in $\mathsf{L}^2(\Omega)$ to some $v \in \mathsf{L}^2(\Omega)$. But then $v \in \mathsf{S}_0(\Omega) \cap \mathcal{R}^{\perp} = \{0\}$, in contradiction to $1 = |v_n|_{\mathsf{L}^2(\Omega)} \to |v|_{\mathsf{L}^2(\Omega)} = 0$. $\square$

For bounded domains with the p-cusp property with some $1 \leq p < 2$ and by Lemma 5 the variational static linear elasticity problem, for $f \in \mathsf{L}^2(\Omega)$ find $v \in \mathsf{S}(\Omega) \cap \mathcal{R}^{\perp}$ such that

$$\forall\, \varphi \in \mathsf{S}(\Omega) \cap \mathcal{R}^{\perp} \qquad \langle \operatorname{sym}\nabla v, \operatorname{sym}\nabla \varphi\rangle_{\mathsf{L}^2(\Omega)} = \langle f, \varphi\rangle_{\mathsf{L}^2(\Omega)},$$

is uniquely solvable with continuous resp. compact inverse $\mathsf{L}^2(\Omega) \to \mathsf{S}(\Omega)$ resp. $\mathsf{L}^2(\Omega) \to \mathsf{L}^2(\Omega)$, which shows that Fredholm's alternative holds for the corresponding reduced operators.

4. Korn's First Inequality

By Rellich's selection theorem, Theorem 2 and an indirect argument we can easily prove:

Theorem 6 (Korn's first inequality without boundary conditions)**.** *There exists* $c > 0$ *such that for all* $v \in \mathsf{H}^1(\Omega)$ *with* $\nabla v \perp \mathfrak{so}$

$$|\nabla v|_{\mathsf{L}^2(\Omega)} \leq c|\operatorname{sym}\nabla v|_{\mathsf{L}^2(\Omega)}. \tag{8}$$

Equivalently for all $v \in \mathsf{H}^1(\Omega)$

$$|\nabla v - S_v|_{\mathsf{L}^2(\Omega)} \leq c|\operatorname{sym}\nabla v|_{\mathsf{L}^2(\Omega)}, \qquad S_v := \operatorname{skw} \fint_{\Omega} \nabla v.$$

Here, $S_v = \pi_{\mathfrak{so}} \nabla v$ is the $\mathsf{L}^2(\Omega)$-orthogonal projection of ∇v onto $\mathfrak{so}$.

Proof. The equivalence is clear by the orthogonal projection.[iv] If (8) was wrong, there exists a sequence $(v_n) \subset \mathsf{H}^1(\Omega)$ with $\nabla v_n \perp \mathfrak{so}$ and $|\nabla v_n|_{\mathsf{L}^2(\Omega)} = 1$ and $|\operatorname{sym}\nabla v_n|_{\mathsf{L}^2(\Omega)} \to 0$. Without loss of generality we can assume $v_n \perp \mathbb{R}^N$. By Poincare's inequality (v_n) is bounded in $\mathsf{H}^1(\Omega)$. Thus, by Rellich's selection theorem we can assume without loss of generality that (v_n) converges in $\mathsf{L}^2(\Omega)$ to some $v \in \mathsf{L}^2(\Omega)$. By Theorem 2 (v_n) is a Cauchy sequence in $\mathsf{H}^1(\Omega)$. Therefore (v_n) converges in $\mathsf{H}^1(\Omega)$ to $v \in \mathsf{H}^1(\Omega) \cap (\mathbb{R}^N)^{\perp}$ with $\operatorname{sym}\nabla v = 0$ and $\nabla v \perp \mathfrak{so}$. But then ∇v is even constant and belongs to $\mathfrak{so}$. Hence $\nabla v = 0$[v] in contradiction to $1 = |\nabla v_n|_{\mathsf{L}^2(\Omega)} \to |\nabla v|_{\mathsf{L}^2(\Omega)} = 0$. $\square$

Using Poincare's inequality we immediately obtain:

[iv] We can also compute it by hand: For $v \in \mathsf{H}^1(\Omega)$ with $\nabla v \perp \mathfrak{so}$ we see

$$|S_v|^2 = \frac{1}{|\Omega|}\langle \operatorname{skw}\int_{\Omega} \nabla v, S_v\rangle = \frac{1}{|\Omega|}\langle \nabla v, S_v\rangle_{\mathsf{L}^2(\Omega)} = 0$$

since $S_v \in \mathfrak{so}$. For $v \in \mathsf{H}^1(\Omega)$ and $T \in \mathfrak{so}$ we have

$$\langle \nabla v - S_v, T\rangle_{\mathsf{L}^2(\Omega)} = \int_{\Omega}\langle \operatorname{skw}\nabla v, T\rangle - \langle S_v, T\rangle_{\mathsf{L}^2(\Omega)} = \langle \int_{\Omega} \operatorname{skw}\nabla v, T\rangle - |\Omega|\langle S_v, T\rangle = 0,$$

implying $v + s_v \in \mathsf{H}^1(\Omega)$ with $\nabla(v + s_v) = (\nabla v - S_v) \perp \mathfrak{so}$ and $\operatorname{sym}\nabla(v + s_v) = \operatorname{sym}(\nabla v - S_v) = \operatorname{sym}\nabla v$, where $s_v(x) := S_v x$.

[v] We note that even $v \in \mathbb{R}^N$ holds and thus $v = 0$.

Corollary 7 (Korn's first inequality without boundary conditions)**.** *There exists* $c > 0$ *such that for all* $v \in \mathsf{H}^1(\Omega) \cap (\mathbb{R}^N)^\perp$ *with* $\nabla v \perp \mathfrak{so}$

$$|v|_{\mathsf{H}^1(\Omega)} \leq c|\operatorname{sym}\nabla v|_{\mathsf{L}^2(\Omega)}.$$

In order to prove Korn's first inequality in $\overset{\circ}{\mathsf{H}}{}^1_{\mathsf{t,n}}(\Omega)$ we need a kind of Poincaré type estimate on this space first. It should be noted that in general mixed boundary conditions are not sufficent to rule out a kernel of the gradient operator. We consider, for example, the cube $\Omega = (0, 1)^3 \subset \mathbb{R}^3$ with Γ_{t} beeing the union of top and bottom together with the constant vector field $R(x) = (0, 0, 1)^t$. Then clearly $R \in \overset{\circ}{\mathsf{H}}{}^1_{\mathsf{t,n}}(\Omega)$. On this account, such constant vector fields have to be excluded by hand.

Lemma 8 (Poincaré inequality with tangential or normal boundary conditions)**.** *There exists* $c > 0$ *such that*

$$|v|_{\mathsf{L}^2(\Omega)} \leq c|\nabla v|_{\mathsf{L}^2(\Omega)} \quad \textit{for all} \quad v \in \overset{\circ}{\mathsf{H}}{}^1_{\mathsf{t,n}}(\Omega) \cap \big(\overset{\circ}{\mathsf{H}}{}^1_{\mathsf{t,n}}(\Omega) \cap \mathbb{R}^N\big)^\perp.$$

Proof. If the assertion was wrong, there exists some sequence $(v_n) \subset \overset{\circ}{\mathsf{H}}{}^1_{\mathsf{t,n}}(\Omega) \cap \big(\overset{\circ}{\mathsf{H}}{}^1_{\mathsf{t,n}}(\Omega) \cap \mathbb{R}^N\big)^\perp$ with $|v_n|_{\mathsf{L}^2(\Omega)} = 1$ and $|\nabla v_n|_{\mathsf{L}^2(\Omega)} \to 0$. Thus, by Rellich's selection theorem we can assume without loss of generality that (v_n) converges in $\mathsf{L}^2(\Omega)$ to some $v \in \mathsf{L}^2(\Omega)$. Hence, (v_n) is a Cauchy sequence in $\mathsf{H}^1(\Omega)$ and converges in $\mathsf{H}^1(\Omega)$ to $v \in \overset{\circ}{\mathsf{H}}{}^1_{\mathsf{t,n}}(\Omega) \cap \big(\overset{\circ}{\mathsf{H}}{}^1_{\mathsf{t,n}}(\Omega) \cap \mathbb{R}^N\big)^\perp$ with $\nabla v = 0$. Therefore, v is a constant in $\overset{\circ}{\mathsf{H}}{}^1_{\mathsf{t,n}}(\Omega) \cap \mathbb{R}^N$ and must vanish in contradiction to $1 = |v_n|_{\mathsf{L}^2(\Omega)} \to |v|_{\mathsf{L}^2(\Omega)} = 0$. □

As an easy consequence we get

Corollary 9. $\nabla \overset{\circ}{\mathsf{H}}{}^1_{\mathsf{t,n}}(\Omega)$ *is a closed subspace of* $\mathsf{L}^2(\Omega)$.

Proof. Let $(v_n) \subset \overset{\circ}{\mathsf{H}}{}^1_{\mathsf{t,n}}(\Omega)$ such that $\nabla v_n \to G \in \mathsf{L}^2(\Omega)$ in $\mathsf{L}^2(\Omega)$. Without loss of generality we can assume $(v_n) \subset \overset{\circ}{\mathsf{H}}{}^1_{\mathsf{t,n}}(\Omega) \cap \big(\overset{\circ}{\mathsf{H}}{}^1_{\mathsf{t,n}}(\Omega) \cap \mathbb{R}^N\big)^\perp$, otherwise we replace v_n by

$$\tilde{v}_n := v_n - \pi_{\overset{\circ}{\mathsf{H}}{}^1_{\mathsf{t,n}}(\Omega)\cap\mathbb{R}^N} v_n \in \overset{\circ}{\mathsf{H}}{}^1_{\mathsf{t,n}}(\Omega) \cap \big(\overset{\circ}{\mathsf{H}}{}^1_{\mathsf{t,n}}(\Omega) \cap \mathbb{R}^N\big)^\perp,$$

where $\pi_{\overset{\circ}{\mathsf{H}}{}^1_{\mathsf{t,n}}(\Omega)\cap\mathbb{R}^N}$ is the orthogonal projektor onto $\overset{\circ}{\mathsf{H}}{}^1_{\mathsf{t,n}}(\Omega) \cap \mathbb{R}^N$. Because of Lemma 8 (v_n) is a Cauchy sequence in $\overset{\circ}{\mathsf{H}}{}^1_{\mathsf{t,n}}(\Omega)$, which converges in $\mathsf{H}^1(\Omega)$ to $v \in \overset{\circ}{\mathsf{H}}{}^1_{\mathsf{t,n}}(\Omega)$. Hence, $G \leftarrow \nabla v_n \to \nabla v \in \nabla \overset{\circ}{\mathsf{H}}{}^1_{\mathsf{t,n}}(\Omega)$. □

To exclude the kernel of the $\operatorname{sym}\nabla$-operator on $\overset{\circ}{\mathsf{H}}{}^1_{\mathsf{t,n}}(\Omega)$, we define

$$\mathcal{K} := \{\nabla v \,:\, v \in \overset{\circ}{\mathsf{H}}{}^1_{\mathsf{t,n}}(\Omega) \wedge \operatorname{sym}\nabla v = 0\} = \nabla\big(\mathcal{R} \cap \overset{\circ}{\mathsf{H}}{}^1_{\mathsf{t,n}}(\Omega)\big) = \mathfrak{so} \cap \nabla \overset{\circ}{\mathsf{H}}{}^1_{\mathsf{t,n}}(\Omega).$$

Theorem 10 (Korn's first inequality with tangential or normal boundary conditions)**.** *There exists* $c > 0$ *such that for all* $v \in \overset{\circ}{\mathsf{H}}{}^1_{\mathsf{t,n}}(\Omega)$ *with* $\nabla v \perp \mathcal{K}$

$$|\nabla v|_{\mathsf{L}^2(\Omega)} \leq c|\operatorname{sym}\nabla v|_{\mathsf{L}^2(\Omega)}. \tag{9}$$

Equivalently, for all $v \in \overset{\circ}{\mathsf{H}}{}^1_{\mathsf{t,n}}(\Omega)$

$$|\nabla v - \pi_{\mathcal{K}}\nabla v|_{\mathsf{L}^2(\Omega)} \leq c|\operatorname{sym}\nabla v|_{\mathsf{L}^2(\Omega)}.$$

Here, $\pi_{\mathcal{K}}\nabla v$ denotes the $\mathsf{L}^2(\Omega)$-orthogonal projection of ∇v onto $\mathcal{K}$. For a proof we follow in close lines the one of Theorem 6, but present it again in some detail for the convenience of the reader.

Proof. Equivalence is again clear by the orthogonal projection. If (9) was wrong, there exists a sequence $(v_n) \subset \overset{\circ}{\mathsf{H}}{}^1_{\mathsf{t},\mathsf{n}}(\Omega)$ with $\nabla v_n \perp \mathcal{K}$ and $|\nabla v_n|_{\mathsf{L}^2(\Omega)} = 1$ and $|\operatorname{sym} \nabla v_n|_{\mathsf{L}^2(\Omega)} \to 0$. Without loss of generality we can assume $(v_n) \subset \overset{\circ}{\mathsf{H}}{}^1_{\mathsf{t},\mathsf{n}}(\Omega) \cap \big(\overset{\circ}{\mathsf{H}}{}^1_{\mathsf{t},\mathsf{n}}(\Omega) \cap \mathbb{R}^N\big)^{\perp}$. By Lemma 8 (v_n) is bounded in $\mathsf{H}^1(\Omega)$, and thus, using Rellich's selection theorem, we can assume without loss of generality that (v_n) converges in $\mathsf{L}^2(\Omega)$ to some $v \in \mathsf{L}^2(\Omega)$. By Theorem 2 (v_n) is a Cauchy sequence in $\mathsf{H}^1(\Omega)$. Therefore, (v_n) converges in $\mathsf{H}^1(\Omega)$ to $v \in \overset{\circ}{\mathsf{H}}{}^1_{\mathsf{t},\mathsf{n}}(\Omega)$ with $\operatorname{sym} \nabla v = 0$ and $\nabla v \perp \mathcal{K}$. But then, ∇v is even a constant in $\mathfrak{so}$, i.e., $\nabla v \in \mathcal{K}$, in contradiction to $1 = |\nabla v_n|_{\mathsf{L}^2(\Omega)} \to |\nabla v|_{\mathsf{L}^2(\Omega)} = 0$. □

5. Discussing the Set $\mathcal{K}$

In this chapter we shall discuss which combinations of domains Ω and boundary parts Γ_{t} allow for a non-constant rigid motion $R \in \overset{\circ}{\mathsf{H}}{}^1_{\mathsf{t},\mathsf{n}}(\Omega) \cap \mathcal{R}$, i.e. $\mathcal{K} \neq \{0\}$. We start with the case $\Gamma_{\mathsf{t}} = \Gamma$.

Theorem 11. *If $\Gamma_{\mathsf{t}} = \Gamma$, then $\mathcal{K} = \{0\}$ and there is a constant $c > 0$ such that for all $v \in \overset{\circ}{\mathsf{H}}{}^1_{\mathsf{t}}(\Omega)$*

$$|\nabla v|_{\mathsf{L}^2(\Omega)} \leq c |\operatorname{sym} \nabla v|_{\mathsf{L}^2}.$$

Proof. We give a proof by contradiction. Assume $R \in \mathcal{R} \cap \overset{\circ}{\mathsf{H}}{}^1_{\mathsf{t}}(\Omega)$ and $R \neq 0$. Let us define the null space $\mathcal{N}_R := \{x \in \mathbb{R}^N : R(x) = 0\}$. Then $\mathcal{N}_R$ is an empty set or an affine plane in $\mathbb{R}^N$ with dimension $d_{\mathcal{N}_R} \leq N-2$. We recall that ν is the outer normal at Γ defined a.e. on Γ w.r.t. $(N-1)$-dimensional Lebesgue-measure. Since R is normal on Γ, we conclude for almost all $x \in \Gamma \setminus \mathcal{N}_R$

$$\nu(x) = \pm \frac{R(x)}{|R(x)|}. \tag{10}$$

Because Ω is locally on one side of the boundary Γ, the unit normal ν can not change sign in (10) in every connected component of $\Gamma \setminus \mathcal{N}_R$. But since $d_{\mathcal{N}_R} \leq N-2$, it follows that $\Gamma \setminus \mathcal{N}_R$ is connected, and

$$\nu(x) = \frac{R(x)}{|R(x)|} \quad \text{for almost all} \quad x \in \Gamma \setminus \mathcal{N}_R, \quad \text{or} \quad \nu(x) = -\frac{R(x)}{|R(x)|} \quad \text{for almost all} \quad x \in \Gamma \setminus \mathcal{N}_R. \tag{11}$$

Note that $\Gamma \cap \mathcal{N}_R$ has measure zero. From this, it follows that in (11) we can replace $\Gamma \setminus \mathcal{N}_R$ by Γ. Let's say $\nu(x) = \frac{R(x)}{|R(x)|}$. With Gauss Theorem we conclude the contradiction

$$0 = \int_\Omega \operatorname{div} R = \int_\Gamma \nu \cdot R = \int_\Gamma |R| > 0.$$

□

Next we turn to the normal boundary condition, i.e. $\Gamma_{\mathsf{t}} = \emptyset$. In [3] it is proved that for smooth bounded domains $\Omega \subset \mathbb{R}^N$ Korn's first inequality holds for all $v \in \overset{\circ}{\mathsf{H}}{}^1_{\mathsf{n}}(\Omega)$, i.e. $\mathcal{K} = \{0\}$, if and only if Ω is not axisymmetric. Furthermore an explicit upper bound on the constant is given.[vi] In that contribution and here axisymmetry is defined as follows.

Definition 12. *Ω is called axisymmetric if there is a non-trivial rigid motion $R \in \mathcal{R}$ tangential to the boundary Γ of Ω, i.e. $R \in \overset{\circ}{\mathsf{H}}{}^1_{\mathsf{n}}(\Omega)$.*

In a more elementary approach in $\mathbb{R}^3$ a domain is called axisymmetric w.r.t. to an axis a if it is a body of rotation around this axis. In order to show that in $\mathbb{R}^3$ both concepts coincide for bounded Lipschitz domains, we make use of the invariance of a Lipschitz boundary under the flow of a tangential vector field.

[vi] In [3] a C^1-boundary is assumed, but it seems that for the proof of [3, Lemma 4] actually a C^2-boundary is needed in order to guaranty H^1-regularity of $\nabla \phi$, where ϕ is the solution of [3, (14)].

Proposition 13. *Let $\Omega \subset \mathbb{R}^N$ be a (not necessarily bounded) domain with a (strong) Lipschitz boundary $\Gamma := \partial\,\Omega$ and $R : \mathbb{R}^N \longrightarrow \mathbb{R}^N$ a locally Lipschitz continuous vector field that is tangential on Γ a.e. w.r.t. the $(N-1)$-dimensional Lebesgue measure on Γ. Let $p \in \Gamma$ and $t \mapsto \gamma(t)$ the maximal solution of the ordinary differential equation*

$$\dot{\gamma} = R(\gamma)\,, \quad \gamma(0) = p\,. \tag{12}$$

existing on the interval I_p. Then for all $t \in I_p$

$$\gamma(t) \in \Gamma\,. \tag{13}$$

This proposition is a variant of Nagumo's invariance theorem, see [1, Theorem 2, p. 180], c.f. also [9], where the tangential condition on R is defined in terms of a so called 'Bouligand contigent cone'. As we need this statement for a Lipschitz boundary we give a self-contained proof in the Appendix.

The next lemma states that for bounded domains in $\mathbb{R}^3$ both definitions of axisymmetry coincide. An elementary proof is provided in the appendix.

Lemma 14. *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain.*

(i) *Assume $\sigma, b \in \mathbb{R}^3$, $|\sigma| = 1$ and let $g = \{\lambda\sigma + b \,:\, \lambda \in \mathbb{R}\}$. Assume that Ω is axisymmetric w.r.t. the axis g. Then the vector field R with $R(x) := \sigma \wedge (x - b)$ is a rigid-motion which is tangential on Γ, i.e. $R \in \mathcal{R} \cap \overset{\circ}{\mathsf{H}}{}^1_{\mathsf{n}}(\Omega)$.*

(ii) *Let $R \in \mathcal{R} \cap \overset{\circ}{\mathsf{H}}{}^1_{\mathsf{n}}(\Omega)$, $R(x) = \omega\sigma \wedge x + b$ for all $x \in \mathbb{R}^3$ with $\sigma, b \in \mathbb{R}^3$, $|\sigma| = 1$ and $\omega \in \mathbb{R}$. Then $\omega \neq 0$, $\langle b, \sigma\rangle = 0$, and Ω is axisymmetric w.r.t. the axis $g = \left\{\lambda\sigma + \frac{1}{w}\sigma \wedge b \,:\, \lambda \in \mathbb{R}\right\}$.*

Remark 15. *There are rigid motions tangential to the boundary of some unbounded domains in $\mathbb{R}^3$, which do not exibit any axis of symmetry. Consider, for example, a domain built from a plane square which simultanously is lifted along and rotated around the axis perpendicular to it, e.g.*

$$\Omega = \left\{ \big(x_1\cos(t) - x_2\sin(t), x_1\sin(t) + x_2\cos(t), t\big) \,:\, |x_1| + |x_2| < 1, t \in \mathbb{R} \right\}\,.$$

Then $R(x) = (-x_2, x_1, 1)^t$ is tangential to Γ.

Using Definition 12, Korn's first inequality for normal boundary conditions is more or less obvious.

Theorem 16. *Let $\Gamma_{\mathsf{t}} = \emptyset$. Then Korn's first inequality holds for all $v \in \overset{\circ}{\mathsf{H}}{}^1_{\mathsf{n}}(\Omega)$, if and only if $\mathcal{K} = \{0\}$, if and only if Ω is not axisymmetric.*

Proof. The first 'if and only if' is just the assertion of Theorem 10. For the second 'if and only if' according to the definition of 'axisymmetry' the only remaining issue is to prove that there is no constant vector field tangential to a bounded Lipschitz domain (in that case we would have a non-trivial rigid-motion, which gives no contribution to $\mathcal{K}$). Assume that a constant vector $a \in \mathbb{R}^N$ tangential to Γ, i.e. $a \in \overset{\circ}{\mathsf{H}}{}^1_{\mathsf{n}}(\Omega)$, and let $\hat{x} \in \Gamma$. According to Proposition 13 the unbounded curve $t \mapsto \hat{x} + ta$ would remain in Γ. This contradicts the boundedness of Ω. $\square$

Remark 17. *The latter proof shows that a bounded domian is axissymmetric if and only if there is a non-constant rigid motion tangential to the boundary.*

For mixed boundary conditions there are domains of rather special type with $\mathcal{K} \neq \{0\}$. Consider, for example, a half cylinder

$$\Omega = \left\{x \in \mathbb{R}^3 \,:\, x_1 > 0, \quad x_1^2 + x_2^2 < 1, \quad 0 < x_3 < 1\right\},$$

or more generally, consider the domain

$$\Omega = \left\{(r\cos\phi, r\sin\phi, x_3)^t \,:\, \phi_1 < \phi < \phi_2, \quad 0 < x_3 < 1, \quad 0 < r < h(x_3)\right\}$$

with $\Gamma_{\mathsf{t}} = \Gamma \cap \left\{(r\cos(\phi_{1/2}), r\sin(\phi_{1/2}), x_3)^t \,:\, 0 \le r, \quad 0 < x_3 < 1\right\}$ for some positive Lipschitz function $h : \mathbb{R} \to \mathbb{R}$ and some $-\pi < \phi_1 < \phi_2 < \pi$. Define $R(x) = (-x_2, x_1, 0)^t$. Then, R is a rigid motion and $R \in \overset{\circ}{\mathsf{H}}{}^1_{\mathsf{t,n}}(\Omega)$. In the next theorem we will show that in $\mathbb{R}^3$ all bounded domains Ω with $\mathcal{K} \neq \{0\}$ are compositions of subdomains of this kind.

Theorem 18. *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain and $\Gamma_{\mathsf{t}} \neq \emptyset$ as well as $\Gamma_{\mathsf{t}} \neq \Gamma$. Assume that there is a non-constant rigid motion $R \in \mathcal{R} \cap \overset{\circ}{\mathsf{H}}{}^1_{\mathsf{t},\mathsf{n}}(\Omega)$, $R(x) = w\sigma \wedge x + b$ for all $x \in \mathbb{R}^3$ with $w \in \mathbb{R}$ and $|\sigma| = 1$. Define $g_R \subset \mathbb{R}^3$ by $g_R := \big\{\lambda\sigma + \frac{1}{w}\sigma \wedge b \,:\, \lambda \in \mathbb{R}\big\}$. Then $\langle\sigma, b\rangle = 0$, Γ_{t} is a subset of a union of affine planes, where each of these planes contains g_R. Every connected component of Γ_{n} is a subset of a surface which is axisymmetric w.r.t. g_R.*

By this theorem the cube already mentioned, $\Omega = (0,\,1)^3 \subset \mathbb{R}^3$, Γ_{t} beeing the union of top and bottom, has a trivial kernel $\mathcal{K}$, which means Korn's first inequality holds on $\overset{\circ}{\mathsf{H}}{}^1_{\mathsf{t},\mathsf{n}}(\Omega)$, while Poincaré's inequality only holds on $\overset{\circ}{\mathsf{H}}{}^1_{\mathsf{t},\mathsf{n}}(\Omega) \cap ((0,0,1)^t)^{\perp}$.

Proof. First we note that the scalar-product $\langle\sigma, b\rangle$ is independent of the choosen cartesian coordinates, i.e. if we chose another positively oriented euclidian coordinate system (y_1, y_2, y_3) and represent the vectorfield R by means of the y-coordinates, then there exist vectors $\sigma_y, b_y \in \mathbb{R}^3$ with $|\sigma_y| = 1$ and $R(y) = w\sigma_y \wedge y + b_y$ for all $y \in \mathbb{R}^3$. Furthermore $\langle\sigma_y, b_y\rangle = \langle\sigma, b\rangle$. In the same way the representation of the axis g_R associated to R is independent of the cartesian coordiantes chosen; in y-coordinates we have $g_R = \big\{\lambda\sigma_y + \frac{1}{w}\sigma_y \wedge b_y \,:\, \lambda \in \mathbb{R}\big\}$.

Suppose $R \in \mathcal{R} \cap \overset{\circ}{\mathsf{H}}{}^1_{\mathsf{t},\mathsf{n}}(\Omega)$ and R non-constant. We fix some $p \in \Gamma_{\mathsf{t}}$ together with a neighborhood $U \subset \mathbb{R}^3$ of p, an open subset $V \subset \mathbb{R}^2$, euclidian coordinates $(x_1, x_2, x_3) = (x', x_3)$ and a Lipschitz map $h : V \subset \mathbb{R}^2 \to \mathbb{R}$, such that for all $x \in U$ we have $x = (x', x^3) \in \Gamma_{\mathsf{t}}$ iff $x^3 = h(x')$. Since R is normal and by Rademacher's theorem, we have

$$R(x', h(x')) = f(x')\,(\nabla_{x'} h(x'), -1)^t \tag{14}$$

with some function $f : V \subset \mathbb{R}^2 \to \mathbb{R}$ a.e. in V.

In x-coordinates R can be represented by $R(x) = w\sigma \wedge x + b$ with some $b, \sigma \in \mathbb{R}^3$, $|\sigma| = 1$ and $w \in \mathbb{R}$, $w \neq 0$. From (14) we conclude

$$b_1 + w\sigma_2 h(x') - w\sigma_3 x_2 = f(x')\,\partial_1\, h(x'), \tag{15}$$

$$b_2 + w\sigma_3 x_1 - w\sigma_1 h(x') = f(x')\,\partial_2\, h(x'), \tag{16}$$

$$b_3 + w\sigma_1 x_2 - w\sigma_2 x_1 = -f(x'). \tag{17}$$

We differentiate (in the sense of distributions) (15) w.r.t. x_2 and (16) w.r.t. x_1, compute the difference as well as the sum of the resulting equations, and conclude using (17)

$$\sigma_3 = \sigma_1\,\partial_1\, h + \sigma_2\,\partial_2\, h\,, \tag{18}$$

$$0 = f\,\partial_1\,\partial_2\, h\,. \tag{19}$$

Differerentiating (15) w.r.t. x_1 and (16) w.r.t. x_2 yields

$$f\,\partial_1^2\, h = f\,\partial_2^2\, h = 0\,. \tag{20}$$

Now we multiply (15) by σ_1, (16) by σ_2, equate the resulting equations for $\sigma_1\sigma_2 h$, use (17,18), and obtain

$$0 = \langle b, \sigma\rangle\,. \tag{21}$$

From (19, 20) we conclude that $\nabla_{x'} h$ is constant on connected components of $V \cap \{f \neq 0\}$. Therefore, h is an affine function on each part and continuous on the whole of V. Note that $\{f = 0\}$ is a subset of the line $\mathcal{N}_{\sigma,b} := \big\{x' \in \mathbb{R}^2 \,:\, b_3 + w\sigma_1 x_2 - w\sigma_2 x_1 = 0\big\}$. Now we extend the affine function from one connected component of $V \cap \{f \neq 0\}$ to $\mathbb{R}^2$ and call the resulting affine function $\tilde{h}$. Because of (18) the plane $\mathcal{E}_{\tilde{h}} := \Big\{\Big(x', \tilde{h}(x')\Big) \,:\, x' \in \mathbb{R}^2\Big\}$ is collinear to g_R. Recalling $\langle\sigma, b\rangle = 0$, it is straightforward to check that g_R is the affine null space of R. Now we use this fact together with the collinearity of $\mathcal{E}_{\tilde{h}}$ and g_R in order to prove $g_R \subset \mathcal{E}_{\tilde{h}}$. It is sufficent to show that $\mathcal{E}_{\tilde{h}} \cap \{R = 0\}$ is not void. But in view of (17) and (14) this is obvious.

Now let $p \in \Gamma_{\mathsf{n}}$. Since $\langle\sigma, b\rangle = 0$, the solutions γ of $\dot{\gamma} = R(\gamma)$ are circles, contained in planes perpendicular to g_R and with centers on g_R (see also the computations in the proof of Lemma 14).

Hence, applying Proposition 13, every connected component is a subset of some hypersurface being axisymmetric w.r.t. g_R. □

6. Appendix

Proof of Proposition 13. Clearly, it is sufficient to prove the invariance locally. Since Γ is Lipschitz, after rotation there is a neighborhood $U = V \times I$ of p with $V \subset \mathbb{R}^{N-1}$, $I \subset \mathbb{R}$, orthornormal coordinates $(x^1, \dots, x^N) = (x', x^N) \in V \times I$, a point $x'_0 \in V$ and a Lipschitz continuous function $h : V \longrightarrow I$ such that $p = (x'_0, h(x'_0))$, and for all $x \in U$ we have $x \in \Gamma$ iff $x^N = h(x')$. By Rademacher's theorem h is differentiable a.e. with respect to the $N-1$-dimensional Lebegue-measure on V, and $\nabla_{x'} h \in L^\infty(V)$. Furthermore, the set of the $N-1$ vectors

$$t_1(x') = (1, 0, \dots, 0, \partial_1 h(x'))^t, \dots, t_{N-1}(x') = (0, \dots, 0, 1, \partial_{N-1} h(x'))^t$$

gives a basis of the tangential space of Γ in the point $(x', h(x'))$ for almost all $x' \in V$. Therefore, on $\Gamma \cap U$ we have two representations of the vector field R, one representation in the coordinate vectors of $x^1, \dots, x^N$ holding on the whole of U,

$$R(x) = R_U(x) = \left(R_U^1(x), \dots, R_U^N(x)\right)^t,$$

and the functions R_U^i, $i = 1, \dots N$ are Lipschitz continuous functions on U. On the other hand, for almost all $x' \in V$

$$R(x', h(x')) = R_V^1(x') t_1(x') + \dots + R_V^{N-1}(x') t_{N-1}(x').$$

We define $R_V = (R_V^1, \dots, R_V^{N-1})^t$. Comparison yields a.e. on V

$$R_U^i(x', h(x')) = R_V^i(x') \quad \text{for all} \quad i = 1, \dots, N-1\,. \tag{22}$$

Hence, R_V is Lipschitz continuous on V. Furthermore,

$$R_U^N(x', h(x')) = R_V^1(x')\, \partial_1 h(x') + \ldots + R_V^{N-1}(x')\, \partial_{N-1} h(x') = R_V(x') \cdot \nabla_{x'} h(x') \tag{23}$$

for almost all $x' \in V$. Since h is Lipschitz on V and R_U^N is Lipschitz on U, $R_V \cdot \nabla_{x'} h$ is also Lipschitz on V. Now we define the flow of R_V: For $x' \in V$ we define $\psi(\,\cdot\,, x')$ as the solution of the ordinary differential equation

$$\dot\psi(t, x') = R_V\left(\psi(t, x')\right)\,, \quad \psi(0, x') = x'\,. \tag{24}$$

Since R_V is Lipschitz on V, we can restrict the flow such that for some $\varepsilon > 0$ and some neighborhood $\bar V \subset V$ of x'_0 the solution ψ is Lipschitz continuous on $(-\varepsilon, \varepsilon) \times \bar V$. Next we lift up this flow to Γ and define

$$\gamma_V(t) := \left(\psi(t, x'_0), h\left(\psi(t, x'_0)\right)\right)^t.$$

By definition $\gamma_V(0) = p$ and $\gamma_V(t) \in \Gamma$ for all $t \in (-\varepsilon, \varepsilon)$.

In the next step we have to prove that γ_V is also a solution of (12) on $(-\varepsilon, \varepsilon)$. With regard to (22) it only remains to prove that the mapping $t \mapsto h(\psi(t, x'_0))$ is classically differentiable with derivative $\partial_t\left(h(\psi(t, x'_0)\right) = R_U^N\left(\psi(t, x'_0), h(\psi(t, x'_0))\right)$. We denote the l-dimensional Lebesgue measure by $\mathcal{L}^l$. For all $t \in (-\varepsilon, \varepsilon)$ it holds that $\psi(t, \cdot\,)$ is a bi-Lipschitz homeomorphism with inverse Lipschitz transformation $\psi(t, \cdot\,)^{-1} = \psi(-t, \cdot\,)$. Therefore, if $\mathcal{L}^{N-1}(\psi(t, \cdot\,)(\tilde V)) = 0$ for some set $\tilde V \subset \bar V$, then also $\mathcal{L}^{N-1}(\tilde V) = 0$, because $\tilde V = \psi(-t, \cdot\,)\left(\psi(t, \cdot\,)(\tilde V)\right)$. Fix a measureable set $V_0 \subset V$ such that $\mathcal{L}^{N-1}(V_0) = 0$ and h is classically differentiable for every $x' \in V \setminus V_0$. Let us define

$$W_0 := \left\{(t, x) \in (-\varepsilon, \varepsilon) \times \bar V \,:\, \psi(t, x) \in V_0\right\}.$$

Then W_0 is measureable and using Tonelli' and Fubini's theorems and the transformation formula we obtain

$$\mathcal{L}^N(W_0) = \int_{(-\varepsilon, \varepsilon) \times \bar V} \mathbf{1}_{W_0} \le c \int_{(-\varepsilon, \varepsilon)} \int_{V_0} 1 = 0.$$

Therefore, and since ψ is differentiable w.r.t. t everywhere, we have using (23)

$$\partial_t\, h(\psi(t, x')) = \nabla h(\psi(t, x')) \cdot \partial_t\, \psi(t, x') = R_U^N\left(\psi(t, x'), h(\psi(t, x'))\right) \tag{25}$$

for almost all $(t,x') \in (-\varepsilon,\,\varepsilon)\times\bar V$; consequently this formula holds in the distributional sense.

Because $h\circ\psi$ is continuous and its distributional derivative w.r.t. t is continuous, too, it is also differentiable w.r.t. t in the classical sense. This can be seen as follows: We define

$$v(t,x') := h\left(\psi(0,x')\right) + \int_0^t R^N\big(\psi(\tau,x'),h(\psi(\tau,x'))\big)\,\mathrm{d}\tau\,.$$

Hence, the vector field v is classically differentiable w.r.t. t and $\partial_t\,v(t,x') = R_U^N\big(\psi(t,x'),h(\psi(t,x'))\big)$ holds for all $(t,x')\in(-\varepsilon,\,\varepsilon)\times\bar V$. Furthermore,

$$\int_{(-\varepsilon,\,\varepsilon)\times\bar V}(v-h\circ\psi)\,\partial_t\,\phi = 0 \quad\text{for all}\quad \phi\in\overset{\circ}{\mathsf{C}}{}^\infty\big((-\varepsilon,\,\varepsilon)\times\bar V\big)\,.$$

This yields

$$h\circ\psi(t,x') = v(t,x') + w(x').$$

Since for all $x'\in\bar V$ we have

$$h\circ\psi(0,x') = v(0,x'),$$

we finally conclude $w=0$ on $\bar V$ and hence $v = h\circ\psi$. □

Proof of Lemma 14. For (i) we choose $\sigma_1,\sigma_2\in\mathbb{R}^3$ such that the set $\{\sigma_1,\sigma_2,\sigma\}$ gives an positively oriented orthornormal base of $\mathbb{R}^3$. Let $x\in\Gamma$ and define $r := \operatorname{dist}(g,x)$. Since Ω is axisymmetric w.r.t. g, for all $t\in\mathbb{R}$

$$\gamma(t) := \langle x,\sigma\rangle\sigma + \left(\langle b,\sigma_1\rangle + r\cos(t)\right)\sigma_1 + \left(\langle b,\sigma_2\rangle + r\sin(t)\right)\sigma_2\in\Gamma\,.$$

Therefore, $\dot\gamma\,(t)$ is a tangential vector to Γ located in x. On the other hand

$$\begin{aligned}
R\,(x) &= \sigma\wedge(x-b)\\
&= \sigma_2\langle x-b,\sigma_1\rangle - \sigma_1\langle x-b,\sigma_2\rangle\\
&= \sigma_2\langle(\langle b,\sigma_1\rangle + r\cos(t))\,\sigma_1 - b,\sigma_1\rangle - \sigma_1\langle(\langle b,\sigma_2\rangle + r\sin(t))\,\sigma_2 - b,\sigma_2\rangle\\
&= \sigma_2 r\cos(t) - \sigma_1 r\sin(t)\\
&= \dot\gamma\,(t)\ ,
\end{aligned}$$

which yields $R\in\overset{\circ}{\mathsf{H}}{}^1_{\mathsf{n}}(\Omega)\cap\mathcal{R}$.

No we turn to the proof of (ii). If $\omega=0$ then $x(t) = x_0 + tb$ remains in Γ for all t if $x_0\in\Gamma$ (Proposition 13) and Ω would be unbounded. Therefore, we have $\omega\neq 0$. Choose again $\sigma_1,\sigma_2\in\mathbb{R}^3$ such that the set $\{\sigma_1,\sigma_2,\sigma\}$ gives an orthornormal base of $\mathbb{R}^3$ with positive orientation.

The solution of the ordinary differential equation system

$$\begin{aligned}
\dot s_1 &= -\omega s_2 + \langle b,\sigma_1\rangle, &\qquad s_1(0) &= \langle\hat x,\sigma_1\rangle,\\
\dot s_2 &= \omega s_1 + \langle b,\sigma_2\rangle, & s_2(0) &= \langle\hat x,\sigma_2\rangle,\\
\dot s_3 &= \langle b,\sigma\rangle, & s_3(0) &= \langle\hat x,\sigma\rangle,
\end{aligned}$$

is given by

$$\begin{aligned}
s_1(t) &= c_1\cos(\omega t) - c_2\sin(\omega t) - \frac{1}{w}\langle b,\sigma_2\rangle,\\
s_2(t) &= c_1\sin(\omega t) + c_2\cos(\omega t) + \frac{1}{w}\langle b,\sigma_1\rangle,\\
s_3(t) &= \langle\hat x,\sigma\rangle + t\langle b,\sigma\rangle,
\end{aligned}$$

where c_1 and c_2 are uniquely definded by the initial conditons on s_1 and s_2. Then

$$x(t) := s_1(t)\sigma_1 + s_2(t)\sigma_2 + s_3(t)\sigma$$

is the unique solution of

$$\dot x = R(x),\quad\text{with initial conditon}\quad x(0) = \hat x\,.$$

Due to Proposition 13 and since $R \in \overset{\circ}{\mathsf{H}}{}^1_{\mathsf{n}}(\Omega)$, we have $x(t) \in \Gamma$ for all $t \in \mathbb{R}$. Because Ω is bounded, we conclude $\langle b, \sigma \rangle = 0$. Therefore, the trajectory $t \mapsto x(t)$ is a circle lying in a plane perpendicular to σ with center $-\frac{1}{w}\langle b, \sigma_2\rangle\sigma_1 + \frac{1}{w}\langle b, \sigma_1\rangle\sigma_2 + \langle \hat{x}, \sigma\rangle\sigma$. Consequently, Ω is axisymmetric w.r.t. to g. $\square$

REFERENCES

[1] Jean-Pierre Aubin and Arrigo Cellina. *Differential inclusions*, volume 264 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1984. Set-valued maps and viability theory.

[2] S. Bauer and D. Pauly. On Korn's first inequality for tangential or normal boundary conditions with explicit constants, 2015.

[3] L. Desvillettes and C. Villani. On a variant of Korn's inequality arising in statistical mechanics. A tribute to J. L. Lions. *ESAIM Control Optim. Calc. Var.*, 8:603–619, 2002.

[4] L. Desvillettes and C. Villani. On the trend to global equilibrium for spatially inhomogeneous kinetic systems: the Boltzmann equation. *Invent. Math.*, 159(2):245–316, 2005.

[5] G. Duvaut and J.-L. Lions. *Les Inéquations en Mécanique et en Physique*. Dunod, Paris, 1972.

[6] D.M. Eidus. On the principle of limiting absorption. *Mat. Sb. (N.S.)*, 57(99):13–44, 1962.

[7] V. Girault and P.-A. Raviart. *Finite Element Methods for Navier-Stokes Equations: Theory and Algorithms*. Springer (Series in Computational Mathematics), Heidelberg, 1986.

[8] P. Grisvard. *Elliptic Problems in Nonsmooth Domains*. Pitman (Advanced Publishing Program), Boston, 1985.

[9] Mitio Nagumo. Über die Lage der Integralkurven gewöhnlicher Differentialgleichungen. *Proc. Phys.-Math. Soc. Japan (3)*, 24:551–559, 1942.

[10] D. Pauly. On constants in Maxwell inequalities for bounded and convex domains. *Zapiski POMI*, 435:46-54, 2014*, & J. Math. Sci. (N.Y.)*, 2014.

[11] D. Pauly. On the Maxwell constants in 3D. *Math. Methods Appl. Sci.*, 2014.

[12] D. Pauly. On Maxwell's and Poincaré's constants. *Discrete Contin. Dyn. Syst. Ser. S*, 8(3):607–618, 2015.

[13] N. Weck. Local compactness for linear elasticity in irregular domains. *Math. Methods Appl. Sci.*, 17:107–113, 1994.

[14] K.-J. Witsch. A remark on a compactness result in electromagnetic theory. *Math. Methods Appl. Sci.*, 16:123–129, 1993.

FAKULTÄT FÜR MATHEMATIK, UNIVERSITÄT DUISBURG-ESSEN, CAMPUS ESSEN, GERMANY
E-mail address, Sebastian Bauer: `sebastian.bauer.seuberlich@uni-due.de`
E-mail address, Dirk Pauly: `dirk.pauly@uni-due.de`