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On Korn's First Inequality
for Tangential or Normal Boundary Conditions with Explicit Constants

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ABSTRACT. We will prove that for piecewise C^2 -concave domains in \mathbb{R}^N Korn's first inequality holds for vector fields satisfying homogeneous normal or tangential boundary conditions with explicit Korn constant $\sqrt{2}$.

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1. INTRODUCTION

In [5], Desvillettes and Villani proved a non-standard version of Korn's first inequality

$$(1) \quad |\nabla v|_{L^2(\Omega)} \leq c_{k,n} |\text{sym } \nabla v|_{L^2(\Omega)}$$

on non-axisymmetric sufficiently smooth bounded domains in \mathbb{R}^N for vector fields being tangential at the boundary. Here $c_{k,n} > 0$ denotes the best available constant and the indices k, n refer to 'Korn' and 'homogenous normal boundary condition'. As pointed out in [6], this Korn inequality has an important application in statistical physics, more precisely in the study of relaxation to equilibrium of rarefied gases modeled by Boltzmann's equation.

In the paper at hand, we will show that for piecewise C^2 -domainsⁱ in \mathbb{R}^N with concave or even polyhedral boundary parts (see Definition 2) Korn's first inequality holds for vector fields satisfying (possibly mixed) homogeneous normal or homogenous tangential boundary conditions, see (3) and (4) for a definition of the relevant spaces. In every case the Korn constant can be estimated by $\sqrt{2}$:

$$|\nabla v|_{L^2(\Omega)} \leq \sqrt{2} |\text{sym } \nabla v|_{L^2(\Omega)},$$

see Theorem 8 for a precise statement. The proof of our main theorem consists of a simple combination of two pointwise equalities of the gradient of a vector field, see (5) and (6), and an integration by parts

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ⁱThroughout this contribution, C^2 can always be replaced by $C^{1,1}$.

formula derived e.g. by Grisvard in [7, Theorem 3.1.1.2], see Proposition 3. But before going into details of the proof we shall discuss some disturbing consequences of Theorem 8 seriously questioning at least the physical justification of full normal boundary conditions.

It is well known that Korn's first inequality with full normal boundary condition does not hold if Ω is axisymmetric. We illustrate this fact with a simple example: Let $\Omega \subset \mathbb{R}^3$ be a bounded body of rotation with axis of symmetry $x_1 = x_2 = 0$, e.g., Ω could be a ball, a cylinder or a cone. Then the vector field v defined by $v(x) := (x_2, -x_1, 0)^\top$ belongs to $H^1(\Omega)$ and is tangential to $\partial\Omega$. Hence, $v \in \mathring{H}_n^1(\Omega)$ (for a precise definition of $\mathring{H}_n^1(\Omega)$ see (3)) and

$$\nabla v = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \text{sym } \nabla v = 0, \quad \text{div } v = 0.$$

Thus, Korn's first inequality with full normal boundary condition, see (1), fails for these special domains Ω , i.e., $c_{k,n} = c_{k,n}(\Omega) = \infty$. On the other hand, Theorem 8 applies for every polyhedral approximation Ω_p of Ω and we have

$$\forall v \in \mathring{H}_n^1(\Omega_p) \quad |\nabla v|_{L^2(\Omega_p)} \leq \sqrt{2} |\text{sym } \nabla v|_{L^2(\Omega_p)}.$$

This means the (first) Korn constant can jump from $\sqrt{2}$ to ∞ caused by an arbitrary small deformation of the domain. Many numerical schemes work on polyhedral domains of computation. The Korn constants of all these domains are bounded from above by $\sqrt{2}$. But in many applications the domain of computation is just an approximation of some 'real' domain, whose Korn constant could be much larger.

Furthermore we shall discuss some conjectures on the meaning of Korn's first constant made in [5] and [6]. In [5] Korn's first inequality with normal boundary condition, i.e. (1), is proved for a bounded C^1 -domain $\Omega \subset \mathbb{R}^N$ which is not axisymmetric. An upper bound for the first Korn constant is presented byⁱⁱ

$$(2) \quad c_{k,n}^2 \leq 2N(1 + c_{m,n}^2)(1 + c_k^2)(1 + c_g^{-1}),$$

where c_k denotes the first Korn constant for vector fields in $H^1(\Omega)$ without boundary conditions, $c_{m,n}$ a special Gaffney constant for tangential vector fields in $H^1(\Omega)$ and c_g the so called Grad's number defined by

$$c_g := \frac{1}{|\Omega|} \inf_{|\sigma|=1} \inf_{v_\sigma \in V_{n,\sigma}(\Omega)} |\text{sym } \nabla v_\sigma|_{L^2(\Omega)}^2$$

with the finite dimensional set

$$V_{n,\sigma}(\Omega) := \{v \in \mathring{H}_n^1(\Omega) : \text{div } v = 0 \wedge \text{rot } v = \sigma\}, \quad \sigma \in S^{(N-1)N/2-1}.$$

For a precise definition of and more comments on these constants, see Section 4.

It is now conjectured in [5, pages 607f] and [6, pages 285, 306 and (48)] that the constant $c_{k,n}$ quantifies the deviation of $\Omega \subset \mathbb{R}^N$ from being axisymmetric in the sense that $c_{k,n} = c_{k,n}(\Omega)$ tends to infinity, if Ω is approaching axial symmetry. For such a statement it would be necessary to bound $c_{k,n}$ from below while in [5] only the bound (2) from above is proved. However, Theorem 8 clearly shows that this conjecture becomes false at least if polyhedra are allowed to compete. In [5] and [6, page 609] it is also conjectured that it is Grad's number $c_g = c_g(\Omega)$ steering this blow-up of the Korn constant. It is conjectured that for smooth domains Grad's number tends to zero while the domain Ω is approaching axial symmetry. The following is actually stated in [5, Proposition 5]: $c_g(\Omega) = 0$ if and only if Ω is axisymmetric. Moreover, there is a lower bound on $c_g(\Omega)$ which depends on the shape of Ω . But in order to prove the conjecture it would be necessary to give an upper bound on $c_g(\Omega)$ tending to zero if the domain is approaching axial symmetry. However, this conjecture gets wrong, too, if we allow for polyhedra: In Section 4 we will show that

$$c_g(\Omega_p) = \frac{1}{2}$$

ⁱⁱIn [5] the notations are different. For the constants we have $c_{k,n}^{-2} = K(\Omega)$, $c_k^{-2} = \bar{K}(\Omega)$, $c_{m,n}^2 = C_H(\Omega)$ and $c_g = G(\Omega)$.

holds for every convex bounded polyhedron Ω_ρ . Therefore, for any sequence of bounded and convex polyhedra tending to any axisymmetric domain Grad's number equals $1/2$.

The remaining part of the paper is organized as follows: In Section 2 we give the relevant definitions on the spaces and domains used and establish some equalities and inequalities used in the sequel. In Section 3 we state our main theorem in detail and give the proof. In Section 4 we discuss the constants $c_{k,n}$, c_k , $c_{m,n}$ and c_g and give some further comments on the regularity of the boundary needed in the proof of (1) in [5]. In the last section we provide some more results estimating the gradient of a vector field.

2. PRELIMINARIES

Let Ω be an open subset of \mathbb{R}^N with $2 \leq N \in \mathbb{N}$ and boundary $\Gamma := \partial\Omega$. We introduce the standard scalar valued Lebesgue and Sobolev spaces by $L^2(\Omega)$ and $H^1(\Omega)$, respectively. Moreover, we define $\mathring{H}^1(\Omega)$ as closure in $H^1(\Omega)$ of smooth and compactly supported test functions $\mathring{C}^\infty(\Omega)$. These definitions extend component-wise to vector or matrix fields and we will use the same notations for these spaces throughout the paper. Moreover, we will consistently denote functions by u and vector fields by v . If Ω is Lipschitz, we define the vector valued Sobolev space $\mathring{H}_t^1(\Omega)$ resp. $\mathring{H}_n^1(\Omega)$ as closure in $H^1(\Omega)$ of the set of test vector fields

$$(3) \quad \mathring{C}_t^\infty(\Omega) := \{v \in \mathring{C}^\infty(\overline{\Omega}) : v_t = 0\}, \quad \mathring{C}_n^\infty(\Omega) := \{v \in \mathring{C}^\infty(\overline{\Omega}) : v_n = 0\},$$

respectively, generalizing homogeneous tangential resp. normal boundary conditions. Here, ν denotes the a.e. defined outer unit normal at Γ giving a.e. the tangential resp. normal component

$$v_t := v|_\Gamma - v_n \nu, \quad v_n := \nu \cdot v|_\Gamma$$

of v on Γ . Here, we denote as usual

$$\mathring{C}^\infty(\overline{\Omega}) := \{v|_\Omega : v \in \mathring{C}^\infty(\mathbb{R}^N)\}.$$

For smooth functions or vector fields v in $H^1(\Omega)$ we haveⁱⁱⁱ

$$v \in \mathring{H}^1(\Omega) \Leftrightarrow v|_\Gamma = 0, \quad v \in \mathring{H}_t^1(\Omega) \Leftrightarrow \nu \times v|_\Gamma = 0, \quad v \in \mathring{H}_n^1(\Omega) \Leftrightarrow \nu \cdot v|_\Gamma = 0.$$

If Γ is decomposed into two relatively open subsets Γ_t and $\Gamma_n := \Gamma \setminus \overline{\Gamma_t}$ we define the vector valued H^1 -Sobolev space of mixed boundary conditions $\mathring{H}_{t,n}^1(\Omega)$ as closure in $H^1(\Omega)$ of the set of test vector fields

$$(4) \quad \mathring{C}_{t,n}^\infty(\Omega) := \{v \in \mathring{C}^\infty(\overline{\Omega}) : v_t|_{\Gamma_t} = 0 \wedge v_n|_{\Gamma_n} = 0\},$$

generalizing $\nu \times v|_{\Gamma_t} = 0$ and $\nu \cdot v|_{\Gamma_n} = 0$ for $v \in \mathring{H}_{t,n}^1(\Omega)$, respectively. For matrices $A \in \mathbb{R}^{N \times N}$ we recall the notations

$$\text{sym } A := \frac{1}{2}(A + A^\top), \quad \text{skw } A := \frac{1}{2}(A - A^\top), \quad \text{dev } A := A - \text{id}_A, \quad \text{id}_A := \frac{\text{tr } A}{N} \text{id}$$

with $\text{tr } A := A \cdot \text{id}$ using the pointwise scalar product. By pointwise orthogonality we have

$$|A|^2 = |\text{dev } A|^2 + \frac{1}{N} |\text{tr } A|^2, \quad |A|^2 = |\text{sym } A|^2 + |\text{skw } A|^2, \quad |\text{sym } A|^2 = |\text{dev sym } A|^2 + \frac{1}{N} |\text{tr } A|^2$$

ⁱⁱⁱThe cross-product notation needs an explanation. If we identify vector fields a, b in \mathbb{R}^N with 1-forms α, β we have for $|\beta| = 1$ the identity $\alpha = \beta \wedge * \beta \wedge * \alpha + (-1)^N * \beta \wedge * \beta \wedge \alpha$, where the wedge and Hodge star operations are executed from right to left. Especially in \mathbb{R}^3 we have $*\beta \wedge \alpha \cong b \times a$, $*\beta \wedge * \beta \wedge \alpha \cong b \times b \times a$ and $*\beta \wedge * \alpha \cong b \cdot a$. Hence, $a = (b \cdot a)b - b \times b \times a$. For $b := \nu$ and $a := v|_\Gamma$ we get $v_t = v|_\Gamma - v_n \nu = -\nu \times \nu \times v|_\Gamma$ and we see $v_t = 0$ if and only if $\nu \times v|_\Gamma = 0$. Now, in this spirit the cross product in \mathbb{R}^N for vector fields is generally defined by $b \times a := *\beta \wedge \alpha$, the latter being a $(N-2)$ -form. This yields e.g. $b \times a = b_1 a_2 - b_2 a_1$ in \mathbb{R}^2 or generally $b \times a \in \mathbb{R}^{(N-1)N/2}$ in \mathbb{R}^N .

and hence $|\operatorname{dev} A|, N^{-1/2}|\operatorname{tr} A|, |\operatorname{sym} A|, |\operatorname{skw} A| \leq |A|$. Especially for $A := \nabla v := J_v^\top$, where J_v denotes the Jacobian of $v \in H^1(\Omega)$, we see pointwise a.e.^{iv}

$$|\operatorname{skw} \nabla v|^2 = \frac{1}{2}|\operatorname{rot} v|^2, \quad \operatorname{tr} \nabla v = \operatorname{div} v$$

and

$$(5) \quad |\nabla v|^2 = |\operatorname{dev} \operatorname{sym} \nabla v|^2 + \frac{1}{N}|\operatorname{div} v|^2 + \frac{1}{2}|\operatorname{rot} v|^2.$$

Moreover, we have

$$(6) \quad |\nabla v|^2 = |\operatorname{rot} v|^2 + \langle \nabla v, (\nabla v)^\top \rangle,$$

since

$$2|\operatorname{skw} \nabla v|^2 = \frac{1}{2}|\nabla v - (\nabla v)^\top|^2 = |\nabla v|^2 - \langle \nabla v, (\nabla v)^\top \rangle.$$

The simplest version of Korn's first inequality is the following.

Lemma 1 (Korn's first inequality: \mathring{H}^1 -version). *For all $v \in \mathring{H}^1(\Omega)$*

$$|\nabla v|_{L^2(\Omega)}^2 = 2|\operatorname{dev} \operatorname{sym} \nabla v|_{L^2(\Omega)}^2 + \frac{2-N}{N}|\operatorname{div} v|_{L^2(\Omega)}^2 \leq 2|\operatorname{dev} \operatorname{sym} \nabla v|_{L^2(\Omega)}^2$$

and equality holds if and only if $\operatorname{div} v = 0$ or $N = 2$.

Although the proof is very simple, we present it here, since we will use the underlying idea later.

Proof. For all vector fields $v \in \mathring{C}^\infty(\Omega)$ we have by^v $-\Delta = \operatorname{rot}^* \operatorname{rot} - \nabla \operatorname{div}$ Gaffney's equality

$$(7) \quad |\nabla v|_{L^2(\Omega)}^2 = |\operatorname{rot} v|_{L^2(\Omega)}^2 + |\operatorname{div} v|_{L^2(\Omega)}^2,$$

which extends to all $v \in \mathring{H}^1(\Omega)$ by continuity. Hence, with (5)

$$(8) \quad |\nabla v|_{L^2(\Omega)}^2 = |\operatorname{dev} \operatorname{sym} \nabla v|_{L^2(\Omega)}^2 + \frac{1}{2}|\nabla v|_{L^2(\Omega)}^2 + \frac{2-N}{2N}|\operatorname{div} v|_{L^2(\Omega)}^2$$

and the assertion follows immediately. \square

Recalling that we work with exterior unit normals at the boundaries, we now introduce our admissible domains.

Definition 2. *We call Ω 'piecewise C^2 ', if*

- (i) Γ is strongly Lipschitz, i.e., locally a graph of a Lipschitz function,
- (ii) $\Gamma = \Gamma_0 \cup \Gamma_1$, where Γ_0 has $(N-1)$ -dimensional Lebesgue measure zero, Γ_1 is relatively open in Γ and locally a graph of a C^2 -function.

We call Ω 'piecewise C^2 -convex' resp. 'piecewise C^2 -concave', if Ω is piecewise C^2 and

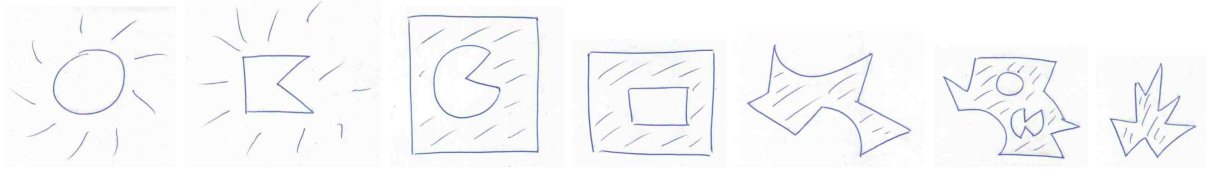
- (iii) *the second fundamental form on Γ_1 induced by $\nabla \nu$ is positive resp. negative semi-definite.*

By assumptions the exterior unit normal ν can be extended into a neighborhood of Γ_1 such that the second fundamental form, i.e., the gradient $\nabla \nu$, and its trace $\operatorname{tr} \nabla \nu = \operatorname{div} \nu = 2H$, where H denotes the mean curvature, are well defined. For precise definitions see e.g. [7, Section 3.1.1].

^{iv}The rot-operator can be defined as follows: We identify smooth vector fields a in \mathbb{R}^N with smooth 1-forms α . Then $\operatorname{rot} a \cong d\alpha$, where d denotes the exterior derivative and $d\alpha$ is a 2-form. For $N = 2$ we obtain the scalar valued rotation $\operatorname{rot} a = \partial_1 a_2 - \partial_2 a_1$ and for $N = 3$ the classical rotation $\operatorname{rot} a$ appears, whereas generally $\operatorname{rot} a(x) \in \mathbb{R}^{(N-1)N/2}$ holds.

^vFor smooth 1-forms in \mathbb{R}^N we have $-\Delta \alpha = d * d * \alpha + (-1)^N * d * d \alpha$. This means for a corresponding smooth vector proxy in \mathbb{R}^N that $-\Delta a = -\nabla \operatorname{div} a + \operatorname{rot}^* \operatorname{rot} a$, where $\operatorname{rot}^* \cong (-1)^N * d *$, the latter mapping 2-forms to 1-forms, denotes the (formal) adjoint of $\operatorname{rot} \cong d$. Hence, rot^* maps smooth vector fields in $\mathbb{R}^{(N-1)N/2}$ to vector fields in \mathbb{R}^N . Especially in \mathbb{R}^3 we have $\operatorname{rot}^* = \operatorname{rot}$ and hence $-\Delta a = -\nabla \operatorname{div} a + \operatorname{rot} \operatorname{rot} a$. In \mathbb{R}^2 it holds $\operatorname{rot}^* = R \nabla$, where R is the 90° -rotation matrix, and hence $-\Delta a = -\nabla \operatorname{div} a + R \nabla \operatorname{rot} a$.

Example 1. *The following domains in \mathbb{R}^2 are piecewise C^2 -concave:*



Our main result is an easy consequence of the pointwise equalities (5) and (6) and the following crucial proposition from Grisvard, [7, Theorem 3.1.1.2].

Proposition 3 (integration by parts). *Let Ω be piecewise C^2 . Then for all $v \in \mathring{C}^\infty(\bar{\Omega})$*

$$|\operatorname{div} v|_{L^2(\Omega)}^2 - \langle \nabla v, (\nabla v)^\top \rangle_{L^2(\Omega)} = \int_{\Gamma_1} (\operatorname{div}_\Gamma(v_n v_t) - 2v_t \cdot \nabla_\Gamma v_n) + \int_{\Gamma_1} (\operatorname{div} \nu |v_n|^2 + ((\nabla \nu) v_t) \cdot v_t)$$

and for $v \in \mathring{C}_{t,n}^\infty(\Omega)$

$$|\operatorname{div} v|_{L^2(\Omega)}^2 - \langle \nabla v, (\nabla v)^\top \rangle_{L^2(\Omega)} = \int_{\Gamma_1} (\operatorname{div} \nu |v_n|^2 + ((\nabla \nu) v_t) \cdot v_t).$$

Here, $\operatorname{div}_\Gamma$ and ∇_Γ are the usual surface differential operators on Γ_1 , which may be identified with the co-derivative $*d*$ on 1-forms and the exterior derivative d on 0-forms on Γ_1 , respectively. Actually in [7] it is assumed that Ω is bounded. But since we assume that v has compact support, the asserted formulas hold for unbounded domains as well.

Remark 4. *We note that in [7] Grisvard uses $-\nabla \nu$ to define the second fundamental form, which implies a negative sign for the curvature term, i.e., the integral*

$$- \int_{\Gamma_1} (\operatorname{div} \nu |v_n|^2 + ((\nabla \nu) v_t) \cdot v_t)$$

appears in [7, Theorem 3.1.1.2]. Moreover, by $\operatorname{div}_\Gamma(v_n v_t) = v_n \operatorname{div}_\Gamma v_t + v_t \cdot \nabla_\Gamma v_n$ on Γ_1 we have

$$\operatorname{div}_\Gamma(v_n v_t) - 2v_t \cdot \nabla_\Gamma v_n = v_n \operatorname{div}_\Gamma v_t - v_t \cdot \nabla_\Gamma v_n.$$

An immediate corollary of Proposition 3 is the following.

Corollary 5 (Gaffney's inequalities). *Let Ω be piecewise C^2 -convex resp. C^2 -concave and $v \in \mathring{H}_{t,n}^1(\Omega)$. Then*

$$|\nabla v|_{L^2(\Omega)}^2 \leq |\operatorname{rot} v|_{L^2(\Omega)}^2 + |\operatorname{div} v|_{L^2(\Omega)}^2 \quad \text{resp.} \quad |\nabla v|_{L^2(\Omega)}^2 \geq |\operatorname{rot} v|_{L^2(\Omega)}^2 + |\operatorname{div} v|_{L^2(\Omega)}^2.$$

If Ω is even a polyhedron, equality holds, i.e.,

$$(9) \quad |\nabla v|_{L^2(\Omega)}^2 = |\operatorname{rot} v|_{L^2(\Omega)}^2 + |\operatorname{div} v|_{L^2(\Omega)}^2.$$

Proof. By continuity it is sufficient to consider $v \in \mathring{C}_{t,n}^\infty(\Omega)$ instead of $v \in \mathring{H}_{t,n}^1(\Omega)$. Using Proposition 3 together with (6) we have

$$(10) \quad |\operatorname{div} v|_{L^2(\Omega)}^2 + |\operatorname{rot} v|_{L^2(\Omega)}^2 = |\nabla v|_{L^2(\Omega)}^2 + \int_{\Gamma_1} (\operatorname{div} \nu |v_n|^2 + ((\nabla \nu) v_t) \cdot v_t).$$

Due to the positive resp. negative semi-definiteness of the second fundamental form, the surface integral is non-negative resp. non-positive resp. vanishes. \square

Remark 6. *For $N = 3$ formula (9) has already been proved in [2, Theorem 4.1].*

Remark 7. By defining the Sobolev spaces with boundary conditions as closures of suitable test vector fields, we avoid discussions about density or approximation arguments and properties. We note that we do not claim

$$\mathring{H}_{t,n}^1(\Omega) = \{v \in H^1(\Omega) : \nu \times v|_{\Gamma_t} = 0 \wedge \nu \cdot v|_{\Gamma_n} = 0\},$$

although this equality seems to be reasonable. On the other hand, it is known, that at least for polyhedra or curved polyhedra^{vi} in \mathbb{R}^2 or \mathbb{R}^3 and either full tangential or full normal boundary condition

$$\mathring{H}_t^1(\Omega) = \{v \in H^1(\Omega) : \nu \times v|_{\Gamma} = 0\}, \quad \mathring{H}_n^1(\Omega) = \{v \in H^1(\Omega) : \nu \cdot v|_{\Gamma} = 0\}$$

hold, see [4, Theorem 2.1, Lemma 2.6 ($J = 1$)] and for the curved case [3, Theorem 2.3] and the corresponding proofs. To the best of the authors knowledge, there are no proofs (yet) for general Lipschitz domains or mixed boundary conditions showing these density properties. We also want to point out that Proposition 3 and formula (10) (and Remark 4) for the special case of $\Omega \subset \mathbb{R}^3$ have been used e.g. in [3, Lemma 2.1, Lemma 2.2] or [1, Lemma 2.11] as well.

3. RESULTS

Theorem 8 (Korn's first inequality: tangential/normal version). *Let $\Omega \subset \mathbb{R}^N$ be piecewise C^2 -concave and $v \in \mathring{H}_{t,n}^1(\Omega)$. Then Korn's first inequality*

$$|\nabla v|_{L^2(\Omega)} \leq \sqrt{2} |\operatorname{dev sym} \nabla v|_{L^2(\Omega)}$$

holds. If Ω is a polyhedron, even

$$|\nabla v|_{L^2(\Omega)}^2 = 2 |\operatorname{dev sym} \nabla v|_{L^2(\Omega)}^2 + \frac{2-N}{N} |\operatorname{div} v|_{L^2(\Omega)}^2 \leq 2 |\operatorname{dev sym} \nabla v|_{L^2(\Omega)}^2$$

is true and equality holds if and only if $\operatorname{div} v = 0$ or $N = 2$.

Proof. We use (5) in combination with Corollary 5 to see

$$|\nabla v|_{L^2(\Omega)}^2 \leq |\operatorname{dev sym} \nabla v|_{L^2(\Omega)}^2 + \frac{1}{2} |\nabla v|_{L^2(\Omega)}^2 + \frac{2-N}{2N} |\operatorname{div} v|_{L^2(\Omega)}^2,$$

which shows the first estimate. If Ω is a polyhedron, we see by Corollary 5 that equality holds in the latter estimate, which proves the other assertions. \square

Remark 9 (unbounded domains). *All our results remain true for slightly weaker Sobolev spaces. In exterior domains, i.e., domains with compact complement, it is common to work in weighted Sobolev spaces like*

$$\begin{aligned} H_{-1}^1(\Omega) &:= \{u \in L_{-1}^2(\Omega) : \nabla u \in L^2(\Omega)\}, \\ L_{-1}^2(\Omega) &:= \{u \in L_{\text{loc}}^2(\Omega) : \rho^{-1}u \in L^2(\Omega)\}, \quad \rho := (1+r^2)^{1/2}, \quad r(x) := |x|. \end{aligned}$$

If $N = 2$ we have to replace $L_{-1}^2(\Omega)$ and $H_{-1}^1(\Omega)$ by $L_{-1,\ln}^2(\Omega)$ and $H_{-1,\ln}^1(\Omega)$, respectively, where u belongs to $L_{-1,\ln}^2(\Omega)$ if $(\ln(e+r)\rho)^{-1}u \in L^2(\Omega)$. The Sobolev spaces generalizing the different boundary conditions are defined as before as closures in $H_{-1}^1(\Omega)$ resp. $H_{-1,\ln}^1(\Omega)$ of respective test functions. For bounded domains, these weighted Sobolev spaces coincide with the standard ones equipped with equivalent scalar products. The reason for working in weighted Sobolev spaces is that the standard Poincaré inequalities do not hold in exterior domains. As proper replacement we have weighted Poincaré inequalities, i.e., for $N \geq 3$

$$\forall u \in H_{-1}^1(\Omega) \quad |u|_{L_{-1}^2(\Omega)} \leq c_p |\nabla u|_{L^2(\Omega)},$$

and we note that the best Poincaré constant for $\mathring{H}_{-1}^1(\Omega)$ satisfies $c_p \leq 2/(N-2)$, see e.g. [9, Poincaré's estimate III, p. 57], [22, Lemma 4.1] or [13, Appendix A.2]. Since our arguments only involve derivatives, it is clear that all our results, mainly Theorem 8 but also the preceding lemmas and corollary, extend easily to the family of Sobolev spaces in $H_{-1}^1(\Omega)$ resp. $H_{-1,\ln}^1(\Omega)$.

^{vi}In our notation, a so called curved polyhedron has got a piecewise C^∞ -boundary.

From the latter remark the following clearly holds true.

Corollary 10 (Korn's first inequality: weighted tangential/normal version). *Theorem 8 extends to all v in $\mathring{H}_{-1,t,n}^1(\Omega)$ resp. $\mathring{H}_{-1,ln,t,n}^1(\Omega)$ if $N = 2$.*

Corollary 11. *Corollary 5 extends to all v in $\mathring{H}_{-1,t,n}^1(\Omega)$ resp. $\mathring{H}_{-1,ln,t,n}^1(\Omega)$ if $N = 2$. Also Lemma 1 extends to all v in $\mathring{H}_{-1}^1(\Omega)$ resp. $\mathring{H}_{-1,ln}^1(\Omega)$ if $N = 2$.*

4. SOME REMARKS ON THE CONSTANTS c_k , $c_{m,n}$ AND c_g

In this section we want to discuss in detail some constants and inequalities used in [5]. In [5] Korn's first inequality with normal boundary condition, i.e. (1), is proved for a bounded C^1 -domain $\Omega \subset \mathbb{R}^N$ which is not axisymmetric. As already mentioned in the introduction an upper bound for the first Korn constant is presented by (2), i.e.,

$$c_{k,n}^2 \leq 2N(1 + c_{m,n}^2)(1 + c_k^2)(1 + c_g^{-1}),$$

which we repeat here for the convenience of the reader. All these constants depend on Ω and we always assume to deal with best possible ones.

4.1. Korn Constant without Boundary Condition c_k . This constant belongs to the standard first Korn inequality without boundary conditions, i.e.,

$$(11) \quad \exists c_k > 0 \quad \forall v \in H^1(\Omega) \quad \exists r_v \in \mathcal{R} \quad |\nabla(v - r_v)|_{L^2(\Omega)} \leq c_k |\text{sym } \nabla v|_{L^2(\Omega)},$$

where \mathcal{R} is the finite dimensional space of rigid motions and r_v the $L^2(\Omega)$ -orthonormal projection onto \mathcal{R} . Especially, (11) holds for any bounded Lipschitz domain $\Omega \subset \mathbb{R}^N$.

4.2. Normal Gaffney Constant $c_{m,n}$. Whereas the literature on c_k is well known, it seems that the knowledge on the normal Gaffney constant $c_{m,n}$ is more restricted to the community dealing with Maxwell's equations as it is explicitly noted in [5]. For this reason we examine it here in more detail. In [5] this constant appears in a special Gaffney inequality for tangential vector fields in $H^1(\Omega)$, i.e., there exists $c_{m,n} > 0$, such that for all $v \in \mathring{H}_n^1(\Omega)$ there exists $n_v \in V_{n,0}(\Omega)$ with

$$|\nabla(v - n_v)|_{L^2(\Omega)} \leq c_{m,n} (|\text{skw } \nabla v|_{L^2(\Omega)}^2 + |\text{tr } \nabla v|_{L^2(\Omega)}^2)^{1/2}$$

or equivalently (with slightly different $c_{m,n}$)

$$(12) \quad |\nabla(v - n_v)|_{L^2(\Omega)} \leq c_{m,n} (|\text{rot } v|_{L^2(\Omega)}^2 + |\text{div } v|_{L^2(\Omega)}^2)^{1/2},$$

where

$$V_{n,0}(\Omega) := \{v \in \mathring{H}_n^1(\Omega) : \text{div } v = 0 \wedge \text{rot } v = 0\}$$

is the finite dimensional^{vii} subspace of $H^1(\Omega)$ -Neumann fields and n_v the $L^2(\Omega)$ -orthonormal projection onto $V_{n,0}(\Omega)$. Inequality (12) can be derived by a Maxwell regularity result, see e.g. [24, 8], stating the following: Let

$$X_n(\Omega) := \{v \in L^2(\Omega) : \text{rot } v \in L^2(\Omega) \wedge \text{div } v \in L^2(\Omega) \wedge \nu \cdot v|_\Gamma = 0\},$$

where the vanishing normal trace has to be understood in the weak sense^{viii}. If $\Omega \subset \mathbb{R}^N$ is a bounded domain and either C^2 or convex, then any vector field v in $X_n(\Omega)$ already belongs to $H^1(\Omega)$, i.e., $v \in \mathring{H}_n^1(\Omega)$. Since $X_n(\Omega)$ together with the norm

$$|v|_{X_n(\Omega)}^2 := |v|_{L^2(\Omega)}^2 + |\text{rot } v|_{L^2(\Omega)}^2 + |\text{div } v|_{L^2(\Omega)}^2$$

^{vii}We remark $V_{n,0}(\Omega) \subset X_{n,0}(\Omega)$ and that even $X_{n,0}(\Omega)$ is finite dimensional.

^{viii}The vanishing normal trace is realized by the closure of test vector fields $\mathring{C}^\infty(\Omega)$ under the graph norm of div viewed as an unbounded operator acting on $L^2(\Omega)$.

is a Hilbert space, we can apply the closed graph theorem to the identity mapping $X_n(\Omega)$ to $H^1(\Omega)$. Therefore there exists $c_{m,n,\text{reg}} > 0$, such that for all $v \in X_n(\Omega) \cap H^1(\Omega) = \mathring{H}_n^1(\Omega)$

$$(13) \quad |\nabla v|_{L^2(\Omega)} \leq c_{m,n,\text{reg}} (|v|_{L^2(\Omega)}^2 + |\text{rot } v|_{L^2(\Omega)}^2 + |\text{div } v|_{L^2(\Omega)}^2)^{1/2}$$

holds. Since the embedding of $X_n(\Omega)$ into $L^2(\Omega)$ is compact even for bounded Lipschitz (or weaker) domains Ω , see [25, 23, 16, 26, 19], we also have the so-called normal Maxwell estimate, i.e., there exists $c_{m,n,\text{est}} > 0$, such that for all $v \in X_n(\Omega)$ there exists $n_v \in X_{n,0}(\Omega)$ with

$$(14) \quad \|v - n_v\|_{L^2(\Omega)} \leq c_{m,n,\text{est}} (|\text{rot } v|_{L^2(\Omega)}^2 + |\text{div } v|_{L^2(\Omega)}^2)^{1/2},$$

where

$$X_{n,0}(\Omega) := \{v \in X_n(\Omega) : \text{div } v = 0 \wedge \text{rot } v = 0\}$$

is the finite dimensional^{ix} subspace of Neumann fields and n_v the $L^2(\Omega)$ -orthonormal projection onto $X_{n,0}(\Omega)$. Now (12) follows immediately by combining (13) and (14) if Ω is bounded and either C^2 or convex with

$$c_{m,n} \leq c_{m,n,\text{reg}} \sqrt{c_{m,n,\text{est}}^2 + 1},$$

since in this case $X_n(\Omega) = \mathring{H}_n^1(\Omega)$ and $X_{n,0}(\Omega) = V_{n,0}(\Omega)$. In the bounded and convex case there are even no Neumann fields, i.e., $X_{n,0}(\Omega) = \{0\}$, and $c_{m,n} \leq 1$ holds, see e.g. [20, 2, 1, 10, 11, 12] for the cases $N = 2$ or $N = 3$, which follows essentially by Corollary 5 and uniform approximation of a convex domain Ω by a sequence of smooth and convex domains. The Neumann fields generally vanish if and only if Ω is simply connected. On the other hand we note that (12) also holds in some non-smooth and non-convex situations as well. For example, by Corollary 14 below we see that (12) is valid if Ω is bounded and piecewise C^2 . Especially, for piecewise C^2 -convex domains we have $c_{m,n} \leq 1$ by Corollary 5. For polyhedra it even holds $c_{m,n} = 1$. We note that in the latter piecewise C^2 -convex case we can choose $n_v = 0$ even if Ω is not simply connected, i.e., even if Neumann fields exist in $X_n(\Omega)$. These possible Neumann fields must vanish by Corollary 5 as soon as they belong to $\mathring{H}_n^1(\Omega)$. Therefore there are domains, e.g. a polyhedron with a reentrant edge, where $\mathring{H}_n^1(\Omega)$ is a closed subspace of $X_n(\Omega)$ in the $X_n(\Omega)$ -topology, but neither $X_n(\Omega) \not\subset \mathring{H}_n^1(\Omega)$ nor $\mathring{H}_n^1(\Omega)$ is dense in $X_n(\Omega)$. To the best knowledge of the authors it is unknown, whether or not (12) holds for general bounded Lipschitz domains or even for general bounded C^1 -domains.

4.3. Grad's Number c_g . Let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain. From the introduction we recall Grad's number

$$c_g = \frac{1}{|\Omega|} \inf_{|\sigma|=1} \inf_{v_\sigma \in V_{n,\sigma}(\Omega)} |\text{sym } \nabla v_\sigma|_{L^2(\Omega)}^2$$

and the finite dimensional set

$$V_{n,\sigma}(\Omega) = \{v \in \mathring{H}_n^1(\Omega) : \text{div } v = 0 \wedge \text{rot } v = \sigma\}, \quad \sigma \in S^{(N-1)N/2-1}.$$

We emphasize that $V_{n,\sigma}(\Omega)$ might be empty, if Ω is not smooth enough, since generally a solution of

$$(15) \quad \text{div } v_\sigma = 0, \text{rot } v_\sigma = \sigma \text{ in } \Omega, \quad v_\sigma \cdot \nu = 0 \text{ on } \Gamma,$$

does not belong to $H^1(\Omega)$. More precisely, (15) admits a solution v_σ

$$v_\sigma \in X_{n,\sigma}(\Omega) := \{v \in X_n(\Omega) : \text{div } v = 0 \wedge \text{rot } v = \sigma\}$$

for any $\sigma \in S^{(N-1)N/2-1}$ since σ belongs to the range of the rotation. This follows by the simple fact that $\sigma = \text{rot } \hat{\sigma} \in \text{rot } H^1(\Omega)$ or equivalently $\Sigma = -\text{skw } \nabla \hat{\Sigma} \in \text{skw } \nabla H^1(\Omega)$ holds with $\hat{\Sigma}(x) := \Sigma x$, where the skew-symmetric matrix $\Sigma \in \mathbb{R}^{N \times N}$ corresponds to $\sigma \in \mathbb{R}^{(N-1)N/2}$ and the vector field $\hat{\Sigma}$ to the vector field $\hat{\sigma}$. An adequate solution theory for these electro-magneto static problems can be found in

^{ix}We note that by the compact embedding $X_n(\Omega) \hookrightarrow L^2(\Omega)$, the unit ball in $X_{n,0}(\Omega)$ is compact.

[14, 15, 21, 17, 18]. In fact, $v_\sigma = \pi \hat{\sigma}$ is the Helmholtz projection π of $\hat{\sigma}$ onto solenoidal vector fields with homogeneous normal boundary condition. Generally, $v_\sigma \notin V_{n,\sigma}(\Omega)$ and thus $V_{n,\sigma}(\Omega) = \emptyset$, i.e., $c_g = +\infty$, is possible even for C^1 -domains^x. On the other hand, if Ω is C^2 or convex, the above mentioned regularity theory for Maxwell's equations shows $v_\sigma \in V_{n,\sigma}(\Omega)$. Moreover, if Ω is convex or simply connected and C^2 , there are even no Neumann fields, which implies in these cases the uniqueness of the solution v_σ and we simply have

$$c_g = \frac{1}{|\Omega|} \inf_{|\sigma|=1} |\text{sym } \nabla v_\sigma|_{L^2(\Omega)}^2.$$

As announced in the introduction we now show that $c_g(\Omega_p) = 1/2$ holds for any bounded and convex polyhedron $\Omega_p \subset \mathbb{R}^N$. For every $\sigma \in S^{(N-1)N/2-1}$ problem (15) has a unique solution $v_\sigma \in \mathring{H}_n^1(\Omega_p)$, i.e. $v_\sigma \in V_{n,\sigma}(\Omega_p)$, (by regularity for static Maxwell's equations in convex domains, see e.g. [20, Theorem 3.1] or [1, Theorem 2.17] for the case $N = 3$) with

$$|\text{rot } v_\sigma|_{L^2(\Omega_p)}^2 = |\sigma|_{L^2(\Omega_p)}^2 = |\Omega_p|.$$

On the other hand, by Corollary 5 and Theorem 8 we also have

$$|\text{rot } v_\sigma|_{L^2(\Omega_p)}^2 = |\nabla v_\sigma|_{L^2(\Omega_p)}^2 = 2|\text{dev sym } \nabla v_\sigma|_{L^2(\Omega_p)}^2 = 2|\text{sym } \nabla v_\sigma|_{L^2(\Omega_p)}^2$$

and hence

$$c_g = \frac{1}{|\Omega_p|} \inf_{|\sigma|=1} |\text{sym } \nabla v_\sigma|_{L^2(\Omega_p)}^2 = \frac{1}{2}.$$

5. SOME MORE ESTIMATES ON THE GRADIENT

In this section we shall combine some more pointwise formulas and estimates on matrices and Jacobians with the integration formula from Proposition 3 in order to get some more equalities and estimates on the norm of gradients.

5.1. Matrices. Let us note a few simple and well known facts about matrices and Jacobians extending the formulas presented in Section 2. The pointwise orthogonal sums

$$A = \text{dev } A \oplus \text{id}_A, \quad A = \text{sym } A \oplus \text{skw } A, \quad \text{sym } A = \text{dev sym } A \oplus \text{id}_A$$

translate to the pointwise equations

$$|A|^2 = |\text{dev } A|^2 + \frac{1}{N} |\text{tr } A|^2, \quad |A|^2 = |\text{sym } A|^2 + |\text{skw } A|^2, \quad |\text{sym } A|^2 = |\text{dev sym } A|^2 + \frac{1}{N} |\text{tr } A|^2$$

and the pointwise estimates

$$|\text{dev } A|, \frac{1}{\sqrt{N}} |\text{tr } A|, |\text{sym } A|, |\text{skw } A| \leq |A|.$$

For $A = \nabla v$ with $v \in H^1(\Omega)$ we see pointwise a.e.

$$(16) \quad |\text{skw } \nabla v|^2 = \frac{1}{2} |\text{rot } v|^2, \quad \text{tr } \nabla v = \text{div } v$$

and

$$|\nabla v|^2 = |\text{sym } \nabla v|^2 + |\text{skw } \nabla v|^2 = |\text{sym } \nabla v|^2 + \frac{1}{2} |\text{rot } v|^2,$$

$$|\nabla v|^2 = |\text{dev sym } \nabla v|^2 + \frac{1}{N} |\text{div } v|^2 + |\text{skw } \nabla v|^2 = |\text{dev sym } \nabla v|^2 + \frac{1}{N} |\text{div } v|^2 + \frac{1}{2} |\text{rot } v|^2.$$

Especially, we see

$$|\text{div } v|^2 + |\text{rot } v|^2 \leq N |\nabla v|^2.$$

^xIn [5, Lemma 4] v_σ is found by solving the Neumann problem, $\Delta \varphi = 0$ in Ω , $\nabla \varphi \cdot \nu = -\hat{\Sigma} \nu$ on Γ , and setting $v_\sigma = \nabla \varphi + \hat{\Sigma}$. But in order to guarantee $v_\sigma \in H^1(\Omega)$ one needs to have $\varphi \in H^2(\Omega)$, which itself is only ensured if Ω is C^2 or convex. Moreover, as pointed out above it seems to be unclear whether [5, (10), (13)], i.e., (12), hold for general C^1 -domains.

Moreover, by

$$2|\text{sym/skw } \nabla v|^2 = \frac{1}{2}|\nabla v \pm (\nabla v)^\top|^2 = |\nabla v|^2 \pm \langle \nabla v, (\nabla v)^\top \rangle$$

we get

$$(17) \quad |\nabla v|^2 = 2|\text{sym } \nabla v|^2 - \langle \nabla v, (\nabla v)^\top \rangle,$$

$$(18) \quad |\nabla v|^2 = 2|\text{skw } \nabla v|^2 + \langle \nabla v, (\nabla v)^\top \rangle = |\text{rot } v|^2 + \langle \nabla v, (\nabla v)^\top \rangle.$$

5.2. Integration by Parts. Defining

$$I_b := \int_{\Gamma_1} (v_n \text{div}_\Gamma v_t - v_t \cdot \nabla_\Gamma v_n), \quad I_c := \int_{\Gamma_1} (\text{div } \nu |v_n|^2 + ((\nabla \nu) v_t) \cdot v_t)$$

Proposition 3 as well as (17) and (18), (16) show:

Lemma 12 (integration by parts). *Let $\Omega \subset \mathbb{R}^N$ be piecewise \mathcal{C}^2 . Then for all $v \in \mathring{\mathcal{C}}^\infty(\overline{\Omega})$*

$$|\nabla v|_{\mathcal{L}^2(\Omega)}^2 - I_b - I_c = 2|\text{sym } \nabla v|_{\mathcal{L}^2(\Omega)}^2 - |\text{div } v|_{\mathcal{L}^2(\Omega)}^2 = 2|\text{dev sym } \nabla v|_{\mathcal{L}^2(\Omega)}^2 + \frac{2-N}{N}|\text{div } v|_{\mathcal{L}^2(\Omega)}^2,$$

$$|\nabla v|_{\mathcal{L}^2(\Omega)}^2 + I_b + I_c = 2|\text{skw } \nabla v|_{\mathcal{L}^2(\Omega)}^2 + |\text{div } v|_{\mathcal{L}^2(\Omega)}^2 = |\text{rot } v|_{\mathcal{L}^2(\Omega)}^2 + |\text{div } v|_{\mathcal{L}^2(\Omega)}^2,$$

which extend by continuity to all $v \in \mathcal{H}^2(\Omega)$. For $v \in \mathring{\mathcal{C}}_{t,n}^\infty(\Omega)$ the integral I_b containing the boundary differential operators vanishes and the formulas (without I_b) extend by continuity to all $v \in \mathring{\mathcal{H}}_{t,n}^1(\Omega)$.

For $N = 2, 3$ these results have already been presented in [3, Lemma 2.2, Theorem 2.3, Remark 2.4].

5.3. Gradient Estimates. By Lemma 12 we get for all $v \in \mathring{\mathcal{H}}_{t,n}^1(\Omega)$

$$I_c = \begin{cases} |\nabla v|_{\mathcal{L}^2(\Omega)}^2 - 2|\text{dev sym } \nabla v|_{\mathcal{L}^2(\Omega)}^2 - \frac{2-N}{N}|\text{div } v|_{\mathcal{L}^2(\Omega)}^2 \\ |\text{rot } v|_{\mathcal{L}^2(\Omega)}^2 + |\text{div } v|_{\mathcal{L}^2(\Omega)}^2 - |\nabla v|_{\mathcal{L}^2(\Omega)}^2 \end{cases}, \quad I_c \leq c \int_{\Gamma_1} |v|^2,$$

where $c > 0$ just depends on the derivatives of ν and Γ_1 . In combination with [7, Theorem 1.5.1.10] we obtain:

Corollary 13. *Let $\Omega \subset \mathbb{R}^N$ be piecewise \mathcal{C}^2 . Then there exists $c > 0$, such that for all $v \in \mathring{\mathcal{H}}_{t,n}^1(\Omega)$ and for all $\epsilon > 0$*

$$(1 - \epsilon)|\nabla v|_{\mathcal{L}^2(\Omega)}^2 \leq 2|\text{dev sym } \nabla v|_{\mathcal{L}^2(\Omega)}^2 + \frac{2-N}{N}|\text{div } v|_{\mathcal{L}^2(\Omega)}^2 + \frac{c}{\epsilon}|v|_{\mathcal{L}^2(\Omega)}^2,$$

$$(1 - \epsilon)|\nabla v|_{\mathcal{L}^2(\Omega)}^2 \leq |\text{rot } v|_{\mathcal{L}^2(\Omega)}^2 + |\text{div } v|_{\mathcal{L}^2(\Omega)}^2 + \frac{c}{\epsilon}|v|_{\mathcal{L}^2(\Omega)}^2.$$

The latter lemma and corollary clearly show that Korn's inequalities and the Maxwell gradient estimate (12) share the same origin. We can also get rid of the $\mathcal{L}^2(\Omega)$ -norm of v on the right hand sides. Let us focus on the second inequality and assume that Ω is a bounded and piecewise \mathcal{C}^2 -domain. We introduce

$$\mathcal{V}_{t,n,0}(\Omega) := \{v \in \mathring{\mathcal{H}}_{t,n}^1(\Omega) : \text{div } v = 0 \wedge \text{rot } v = 0\},$$

which is a finite dimensional (and hence closed) subspace of $\mathcal{L}^2(\Omega)$ since its unit ball is compact by Corollary 13 and Rellich's selection theorem.

Corollary 14 (Gaffney's inequality). *Let $\Omega \subset \mathbb{R}^N$ be a bounded and piecewise \mathcal{C}^2 -domain. Then there exists $c > 0$, such that for all $v \in \mathring{\mathcal{H}}_{t,n}^1(\Omega)$ there exists $n_v \in \mathcal{V}_{t,n,0}(\Omega)$ with*

$$|v - n_v|_{\mathcal{H}^1(\Omega)} \leq c(|\text{rot } v|_{\mathcal{L}^2(\Omega)} + |\text{div } v|_{\mathcal{L}^2(\Omega)})$$

and n_v is the $\mathcal{L}^2(\Omega)$ -orthonormal projection of v onto $\mathcal{V}_{t,n,0}(\Omega)$.

Proof. Since $v - n_v \in \mathring{H}_{t,n}^1(\Omega) \cap V_{t,n,0}(\Omega)^\perp$ as well as $\operatorname{rot}(v - n_v) = \operatorname{rot} v$ and $\operatorname{div}(v - n_v) = \operatorname{div} v$ it is sufficient to show

$$(19) \quad \exists c > 0 \quad \forall v \in \mathring{H}_{t,n}^1(\Omega) \cap V_{t,n,0}(\Omega)^\perp \quad |v|_{H^1(\Omega)} \leq c(|\operatorname{rot} v|_{L^2(\Omega)} + |\operatorname{div} v|_{L^2(\Omega)}).$$

If (19) is wrong, there exists a sequence $(v_n) \subset \mathring{H}_{t,n}^1(\Omega) \cap V_{t,n,0}(\Omega)^\perp$ with

$$|v_n|_{H^1(\Omega)} = 1, \quad |\operatorname{rot} v_n|_{L^2(\Omega)} + |\operatorname{div} v_n|_{L^2(\Omega)} \rightarrow 0.$$

As (v_n) is bounded in $H^1(\Omega)$ there exists a subsequence $(v_{\pi n})$ converging to some v in $L^2(\Omega)$ by Rellich's selection theorem. By Corollary 13, $(v_{\pi n})$ is a Cauchy sequence in $H^1(\Omega)$ and thus

$$v_{\pi n} \rightarrow v \in \mathring{H}_{t,n}^1(\Omega) \cap V_{t,n,0}(\Omega)^\perp \quad \text{in } H^1(\Omega).$$

Since v belongs to $V_{t,n,0}(\Omega)$ as well, we have $v = 0$ in contradiction to $1 = |v_n|_{H^1(\Omega)} \rightarrow |v|_{H^1(\Omega)}$, which proves (19). \square

Remark 15. *As in Remark 9 and Corollaries 10 and 11, there are also versions of Corollary 14 for the case of e.g. a piecewise C^2 exterior domain using polynomially weighted Sobolev spaces.*

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