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On Maxwell's and Poincaré's Constants

by

Dirk Pauly

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Dedicated to Sergey Igorevich Repin on the occasion of his 60th birthday

Abstract

We prove that for bounded and convex domains in three dimensions, the Maxwell constants are bounded from below and above by Friedrichs' and Poincaré's constants. In other words, the second Maxwell eigenvalues lie between the square roots of the second Neumann-Laplace and the first Dirichlet-Laplace eigenvalue.
Key Words Maxwell's equations, Maxwell constant, second Maxwell eigenvalue, electro statics, magneto statics, Poincaré's inequality, Friedrichs' inequality, Poincaré's constant, Friedrichs' constant

Contents

1 Introduction

It is well known that, e.g., for bounded Lipschitz domains $\Omega \subset \mathbb{R}^3$, a square integrable vector field v having square integrable divergence $\operatorname{div} v$ and square integrable rotation vector field $\operatorname{rot} v$ as well as vanishing tangential or normal component on the boundary Γ, i.e, $v_{\mathtt{t}}|_\Gamma = 0$ resp. $v_{\mathtt{n}}|_\Gamma = 0$, satisfies the Maxwell estimate

$$\int_\Omega |v|^2 \leq c_{\mathtt{m}}^2 \int_\Omega \big(|\operatorname{rot} v|^2 + |\operatorname{div} v|^2\big), \tag{1.1}$$

if in addition v is perpendicular to the so called Dirichlet or Neumann fields, i.e.,

$$\int_\Omega v \cdot w = 0 \quad \forall\, w \in \mathcal{H}(\Omega),$$

where

$$\mathcal{H}(\Omega) = \begin{cases} \mathcal{H}_{\mathtt{D}}(\Omega) := \{w \in \mathsf{L}^2(\Omega) \,:\, \operatorname{rot} w = 0,\, \operatorname{div} w = 0,\, w_{\mathtt{t}}|_\Gamma = 0\}, & \text{if } v_{\mathtt{t}}|_\Gamma = 0, \\ \mathcal{H}_{\mathtt{N}}(\Omega) := \{w \in \mathsf{L}^2(\Omega) \,:\, \operatorname{rot} w = 0,\, \operatorname{div} w = 0,\, w_{\mathtt{n}}|_\Gamma = 0\}, & \text{if } v_{\mathtt{n}}|_\Gamma = 0 \end{cases}$$

holds. Here, $c_{\mathtt{m}}$ is a positive constant independent of v, which will be called Maxwell constant. See, e.g., [19, 20, 13, 25]. We note that (1.1) is valid in much more general situations modulo some more or less obvious modifications, such as for mixed boundary conditions, in unbounded (like exterior) domains, in domains $\Omega \subset \mathbb{R}^N$, on N-dimensional Riemannian manifolds, for differential forms or in the case of inhomogeneous media. See, e.g.,[10, 15, 17, 20, 21, 22, 25, 26].

So far, to the best of the author's knowledge, general bounds for the Maxwell constants $c_{\mathtt{m}}$ are unknown. On the other hand, at least estimates for $c_{\mathtt{m}}$ from above are very important from the point of view of applications, such as preconditioning or a priori and a posteriori error estimation for numerical methods.

In this contribution we will prove that for bounded and convex domains $\Omega \subset \mathbb{R}^3$

$$c_{\mathtt{p},\circ} \leq c_{\mathtt{m}} \leq c_{\mathtt{p}} \leq \operatorname{diam}(\Omega)/\pi \tag{1.2}$$

holds true, where $0 < c_{\mathtt{p},\circ} < c_{\mathtt{p}}$ are the Poincaré constants, such that for all square integrable functions u having square integrable gradient ∇u

$$\int_\Omega |u|^2 \leq c_{\mathtt{p},\circ}^2 \int_\Omega |\nabla u|^2 \quad \text{resp.} \qquad \int_\Omega |u|^2 \leq c_{\mathtt{p}}^2 \int_\Omega |\nabla u|^2$$

holds, if $u|_\Gamma = 0$ resp. $\int_\Omega u = 0$. While the result (1.2) is already well known in two dimensions, even for general Lipschitz domains $\Omega \subset \mathbb{R}^2$ (except of the last inequality), it is new in three dimensions. We note that the last inequality in (1.2) has been proved in the famous paper of Payne and Weinberger [18], where also the optimality of the estimate was shown. A small mistake in this paper has been corrected later in [3]. We will prove the crucial and from the point of view of applications most interesting inequality $c_{\mathtt{m}} \leq c_{\mathtt{p}}$ also for polyhedral domains in $\mathbb{R}^3$, which might not be convex but still allow the $\mathsf{H}^1(\Omega)$-regularity for solutions of Maxwell's equations. We will give a general result for non-smooth and inhomogeneous, anisotropic media as well, and even a refinement of (1.2). Let us note that our methods are only based on elementary calculations.

2 Preliminaries

Throughout this paper let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain. Many of our results hold true under weaker assumptions on the regularity of the boundary $\Gamma := \partial\Omega$. Essentially we need the compact embeddings (2.3)-(2.5) to hold. We will use the standard Lebesgue spaces $\mathsf{L}^2(\Omega)$ of square integrable functions or vector (or even tensor) fields equipped with the usual $\mathsf{L}^2(\Omega)$-scalar product $\langle\,\cdot\,,\,\cdot\,\rangle_\Omega$ and $\mathsf{L}^2(\Omega)$-norm $|\cdot|_\Omega$. Moreover, we will work with the standard $\mathsf{L}^2(\Omega)$-Sobolev spaces for the gradient $\operatorname{grad} = \nabla$, the rotation rot $= \nabla\times$ and the divergence div $= \nabla\cdot$ denoted by

$$\begin{aligned}
\mathsf{H}^1(\Omega) &:= \mathsf{H}(\mathrm{grad};\Omega), & \mathring{\mathsf{H}}^1(\Omega) &:= \mathring{\mathsf{H}}(\mathrm{grad};\Omega) := \overline{\mathring{\mathsf{C}}^\infty(\Omega)}^{\mathsf{H}^1(\Omega)},\\
\mathsf{D}(\Omega) &:= \mathsf{H}(\mathrm{div};\Omega), & \mathring{\mathsf{D}}(\Omega) &:= \mathring{\mathsf{H}}(\mathrm{div};\Omega) := \overline{\mathring{\mathsf{C}}^\infty(\Omega)}^{\mathsf{D}(\Omega)},\\
\mathsf{R}(\Omega) &:= \mathsf{H}(\mathrm{rot};\Omega), & \mathring{\mathsf{R}}(\Omega) &:= \mathring{\mathsf{H}}(\mathrm{rot};\Omega) := \overline{\mathring{\mathsf{C}}^\infty(\Omega)}^{\mathsf{R}(\Omega)}.
\end{aligned}$$

In the latter three Hilbert spaces the classical homogeneous scalar, normal and tangential boundary traces are generalized, respectively. An index zero at the lower right corner of the latter spaces indicates a vanishing derivative, e.g.,

$$\mathring{\mathsf{R}}_0(\Omega) := \{E \in \mathring{\mathsf{R}}(\Omega) \,:\, \mathrm{rot}\, E = 0\}, \quad \mathsf{D}_0(\Omega) := \{E \in \mathsf{D}(\Omega) \,:\, \mathrm{div}\, E = 0\}.$$

Moreover, we introduce a symmetric, bounded (L^∞) and uniformly positive definite matrix field $\varepsilon : \Omega \to \mathbb{R}^{3\times 3}$ and the spaces of (harmonic) Dirichlet and Neumann fields

$$\mathcal{H}_{\mathtt{D},\varepsilon}(\Omega) := \mathring{\mathsf{R}}_0(\Omega) \cap \varepsilon^{-1}\mathsf{D}_0(\Omega), \quad \mathcal{H}_{\mathtt{N},\varepsilon}(\Omega) := \mathsf{R}_0(\Omega) \cap \varepsilon^{-1}\mathring{\mathsf{D}}_0(\Omega).$$

We will also use the weighted ε-$\mathsf{L}^2(\Omega)$-scalar product $\langle\,\cdot\,,\,\cdot\,\rangle_{\Omega,\varepsilon} := \langle \varepsilon\,\cdot\,,\,\cdot\,\rangle_\Omega$ and the corresponding induced weighted ε-$\mathsf{L}^2(\Omega)$-norm $|\,\cdot\,|_{\Omega,\varepsilon} := \langle\,\cdot\,,\,\cdot\,\rangle^{1/2}_{\Omega,\varepsilon}$. Moreover, $\perp_\varepsilon$ denotes orthogonality with respect to the ε-$\mathsf{L}^2(\Omega)$-scalar product. If we equip $\mathsf{L}^2(\Omega)$ with this weighted scalar product we write $\mathsf{L}^2_\varepsilon(\Omega)$. If ε equals the identity id, we skip it in our notations, e.g., we write $\perp := \perp_{\mathrm{id}}$ and $\mathcal{H}_{\mathtt{D}}(\Omega) := \mathcal{H}_{\mathtt{D},\mathrm{id}}(\Omega)$. By the assumptions on ε we have

$$\exists\, \underline{\varepsilon}, \overline{\varepsilon} > 0 \quad \forall\, E \in \mathsf{L}^2(\Omega) \quad \underline{\varepsilon}^{-2}|E|^2_\Omega \le \langle \varepsilon E, E\rangle_\Omega \le \overline{\varepsilon}^2 |E|^2_\Omega \tag{2.1}$$

and we note $|E|^2_{\Omega,\varepsilon} = \langle \varepsilon E, E\rangle_\Omega = |\varepsilon^{1/2}E|^2_\Omega$ as well as $|\varepsilon E|_\Omega = |\varepsilon^{1/2}E|_{\Omega,\varepsilon}$. Thus, for all $E \in \mathsf{L}^2(\Omega)$

$$\underline{\varepsilon}^{-1}|E|_\Omega \le |E|_{\Omega,\varepsilon} \le \overline{\varepsilon}|E|_\Omega, \qquad \underline{\varepsilon}^{-1}|E|_{\Omega,\varepsilon} \le |\varepsilon E|_\Omega \le \overline{\varepsilon}|E|_{\Omega,\varepsilon}. \tag{2.2}$$

For later purposes let us also define $\hat{\varepsilon} := \max\{\underline{\varepsilon}, \overline{\varepsilon}\}$.

We have the following compact embeddings:

$$\mathring{\mathsf{H}}^1(\Omega) \subset \mathsf{H}^1(\Omega) \hookrightarrow \mathsf{L}^2(\Omega) \qquad \text{(Rellich's selection theorem)} \tag{2.3}$$

$$\mathring{\mathsf{R}}(\Omega) \cap \varepsilon^{-1}\mathsf{D}(\Omega) \hookrightarrow \mathsf{L}^2(\Omega) \qquad \text{(tangential Maxwell compactness property)} \tag{2.4}$$

$$\mathsf{R}(\Omega) \cap \varepsilon^{-1}\mathring{\mathsf{D}}(\Omega) \hookrightarrow \mathsf{L}^2(\Omega) \qquad \text{(normal Maxwell compactness property)} \tag{2.5}$$

It is well known and easy to prove by standard indirect arguments that (2.3) implies the Poincaré estimates

$$\exists\, c_{\mathtt{p},\circ} > 0 \qquad \forall\, u \in \mathring{\mathsf{H}}^1(\Omega) \qquad |u|_\Omega \le c_{\mathtt{p},\circ}|\nabla u|_\Omega, \tag{2.6}$$

$$\exists\, c_{\mathtt{p}} > 0 \qquad \forall\, u \in \mathsf{H}^1(\Omega) \cap \mathbb{R}^\perp \qquad |u|_\Omega \le c_{\mathtt{p}}|\nabla u|_\Omega. \tag{2.7}$$

Furthermore

$$c_{\mathtt{p},\circ}^2 = \frac{1}{\lambda_1} < \frac{1}{\mu_2} = c_{\mathtt{p}}^2$$

holds, where λ_1 is the first Dirichlet and μ_2 the second Neumann eigenvalue of the Laplacian. We even have $0 < \mu_{n+1} < \lambda_n$ for all $n \in \mathbb{N}$, see e.g. [5] and the literature cited there.

Analogously, (2.4) implies $\dim \mathcal{H}_{\mathtt{D},\varepsilon}(\Omega) < \infty^*$, since the unit ball in $\mathcal{H}_{\mathtt{D},\varepsilon}(\Omega)$ is compact, and the tangential Maxwell estimate, i.e., there exists $c_{\mathtt{m},\mathtt{t},\varepsilon} > 0$ such that

$$\forall\, E \in \overset{\circ}{\mathsf{R}}(\Omega) \cap \varepsilon^{-1}\mathsf{D}(\Omega) \qquad |(1-\pi_{\mathtt{D}})E|_{\Omega,\varepsilon} \leq c_{\mathtt{m},\mathtt{t},\varepsilon}\big(|\operatorname{rot} E|_\Omega^2 + |\operatorname{div} \varepsilon E|_\Omega^2\big)^{1/2}, \tag{2.8}$$

where $\pi_{\mathtt{D}} : \mathsf{L}_\varepsilon^2(\Omega) \to \mathcal{H}_{\mathtt{D},\varepsilon}(\Omega)$ denotes the ε-$\mathsf{L}^2(\Omega)$-orthogonal projector onto Dirichlet fields. Similar results hold if one replaces the tangential or electric boundary condition by the normal or magnetic one. More precisely, (2.5) implies $\dim \mathcal{H}_{\mathtt{N},\varepsilon}(\Omega) < \infty$ and the corresponding normal Maxwell estimate, i.e., there exists $c_{\mathtt{m},\mathtt{n},\varepsilon} > 0$ such that

$$\forall\, H \in \mathsf{R}(\Omega) \cap \varepsilon^{-1}\overset{\circ}{\mathsf{D}}(\Omega) \qquad |H - \pi_{\mathtt{N}}H|_{\Omega,\varepsilon} \leq c_{\mathtt{m},\mathtt{n},\varepsilon}\big(|\operatorname{rot} H|_\Omega^2 + |\operatorname{div} \varepsilon H|_\Omega^2\big)^{1/2}, \tag{2.9}$$

where $\pi_{\mathtt{N}} : \mathsf{L}_\varepsilon^2(\Omega) \to \mathcal{H}_{\mathtt{N},\varepsilon}(\Omega)$ denotes the ε-$\mathsf{L}^2(\Omega)$-orthogonal projector onto Neumann fields. We note that $\sqrt{c_{\mathtt{m},\mathtt{t},\varepsilon}^2 + 1}$ can also be seen as the norm of the inverse M^{-1} of the corresponding electro static Maxwell operator

$$\begin{array}{rcccc} M & : & \overset{\circ}{\mathsf{R}}(\Omega) \cap \varepsilon^{-1}\mathsf{D}(\Omega) \cap \mathcal{H}_{\mathtt{D},\varepsilon}(\Omega)^{\perp_\varepsilon} & \longrightarrow & \operatorname{rot} \overset{\circ}{\mathsf{R}}(\Omega) \times \mathsf{L}^2(\Omega) \\ & & E & \longmapsto & (\operatorname{rot} E, \operatorname{div} \varepsilon E) \end{array}.$$

The analogous statement holds for $c_{\mathtt{m},\mathtt{n},\varepsilon}$ as well.

The compact embeddings (2.3)-(2.5) hold for more general bounded domains with weaker regularity of the boundary Γ, such as domains with cone property, restricted cone property or just p-cusp-property. See, e.g., [1, 2, 19, 20, 21, 22, 23, 25, 26, 27, 13]. Note that the Maxwell compactness properties and hence the Maxwell estimates hold for mixed boundary conditions as well, see [10, 7, 9]. The boundedness of the underlying domain Ω is crucial, since one has to work in weighted Sobolev spaces in unbounded (like exterior) domains, see [11, 12, 13, 14, 15, 17, 16, 19, 23].

As always in the theory of Maxwell's equations, we need another crucial tool, the Helmholtz or Weyl decompositions of vector fields into irrotational and solenoidal vector

*$d_{\mathtt{D}} := \dim \mathcal{H}_{\mathtt{D},\varepsilon}(\Omega)$ is finite and independent of ε. In particular, $d_{\mathtt{D}}$ depends just on the topology of Ω. More precisely, $d_{\mathtt{D}} = \beta_2$, the second Betti number of Ω. A similar result holds also for the Neumann fields, i.e., $d_{\mathtt{N}} := \dim \mathcal{H}_{\mathtt{N},\varepsilon}(\Omega) = \beta_1$.

fields. We have

$$\begin{aligned}\mathsf{L}^2_\varepsilon(\Omega) &= \nabla\mathring{\mathsf{H}}^1(\Omega) \oplus_\varepsilon \varepsilon^{-1}\mathsf{D}_0(\Omega)\\ &= \mathring{\mathsf{R}}_0(\Omega) \oplus_\varepsilon \varepsilon^{-1}\operatorname{rot}\mathsf{R}(\Omega)\\ &= \nabla\mathring{\mathsf{H}}^1(\Omega) \oplus_\varepsilon \mathcal{H}_{\mathtt{D},\varepsilon}(\Omega) \oplus_\varepsilon \varepsilon^{-1}\operatorname{rot}\mathsf{R}(\Omega),\\ \mathsf{L}^2_\varepsilon(\Omega) &= \nabla\mathsf{H}^1(\Omega) \oplus_\varepsilon \varepsilon^{-1}\mathring{\mathsf{D}}_0(\Omega)\\ &= \mathsf{R}_0(\Omega) \oplus_\varepsilon \varepsilon^{-1}\operatorname{rot}\mathring{\mathsf{R}}(\Omega)\\ &= \nabla\mathsf{H}^1(\Omega) \oplus_\varepsilon \mathcal{H}_{\mathtt{N},\varepsilon}(\Omega) \oplus_\varepsilon \varepsilon^{-1}\operatorname{rot}\mathring{\mathsf{R}}(\Omega),\end{aligned}$$

where $\oplus_\varepsilon$ denotes the orthogonal sum with respect the latter scalar product, and note

$$\begin{aligned}\nabla\mathring{\mathsf{H}}^1(\Omega) &= \mathring{\mathsf{R}}_0(\Omega) \cap \mathcal{H}_{\mathtt{D},\varepsilon}(\Omega)^{\perp_\varepsilon}, & \varepsilon^{-1}\operatorname{rot}\mathsf{R}(\Omega) &= \varepsilon^{-1}\mathsf{D}_0(\Omega) \cap \mathcal{H}_{\mathtt{D},\varepsilon}(\Omega)^{\perp_\varepsilon},\\ \nabla\mathsf{H}^1(\Omega) &= \mathsf{R}_0(\Omega) \cap \mathcal{H}_{\mathtt{N},\varepsilon}(\Omega)^{\perp_\varepsilon}, & \varepsilon^{-1}\operatorname{rot}\mathring{\mathsf{R}}(\Omega) &= \varepsilon^{-1}\mathring{\mathsf{D}}_0(\Omega) \cap \mathcal{H}_{\mathtt{N},\varepsilon}(\Omega)^{\perp_\varepsilon}.\end{aligned}$$

Moreover, with

$$\begin{aligned}\mathcal{R}(\Omega) &:= \mathsf{R}(\Omega) \cap \operatorname{rot}\mathring{\mathsf{R}}(\Omega) = \mathsf{R}(\Omega) \cap \mathring{\mathsf{D}}_0(\Omega) \cap \mathcal{H}_{\mathtt{N}}(\Omega)^{\perp},\\ \mathring{\mathcal{R}}(\Omega) &:= \mathring{\mathsf{R}}(\Omega) \cap \operatorname{rot}\mathsf{R}(\Omega) = \mathring{\mathsf{R}}(\Omega) \cap \mathsf{D}_0(\Omega) \cap \mathcal{H}_{\mathtt{D}}(\Omega)^{\perp}\end{aligned}$$

we see

$$\operatorname{rot}\mathsf{R}(\Omega) = \operatorname{rot}\mathcal{R}(\Omega), \quad \operatorname{rot}\mathring{\mathsf{R}}(\Omega) = \operatorname{rot}\mathring{\mathcal{R}}(\Omega).$$

Note that all occurring spaces are closed subspaces of $\mathsf{L}^2(\Omega)$, which follows immediately by the estimates (2.6)-(2.9). More details about the Helmholtz decompositions can be found e.g. in [13].

If Ω is even convex[†] we have some simplifications due to the vanishing of Dirichlet and Neumann fields, i.e., $\mathcal{H}_{\mathtt{D},\varepsilon}(\Omega) = \mathcal{H}_{\mathtt{N},\varepsilon}(\Omega) = \{0\}$. Then (2.8) and (2.9) simplify to

$$\forall\, E \in \mathring{\mathsf{R}}(\Omega) \cap \varepsilon^{-1}\mathsf{D}(\Omega) \qquad |E|_{\Omega,\varepsilon} \le c_{\mathtt{m},\mathtt{t},\varepsilon}\big(|\operatorname{rot} E|^2_\Omega + |\operatorname{div}\varepsilon E|^2_\Omega\big)^{1/2}, \tag{2.10}$$

$$\forall\, H \in \mathsf{R}(\Omega) \cap \varepsilon^{-1}\mathring{\mathsf{D}}(\Omega) \qquad |H|_{\Omega,\varepsilon} \le c_{\mathtt{m},\mathtt{n},\varepsilon}\big(|\operatorname{rot} H|^2_\Omega + |\operatorname{div}\varepsilon H|^2_\Omega\big)^{1/2} \tag{2.11}$$

and we have

$$\mathring{\mathsf{R}}_0(\Omega) = \nabla\mathring{\mathsf{H}}^1(\Omega), \quad \mathsf{R}_0(\Omega) = \nabla\mathsf{H}^1(\Omega), \quad \mathsf{D}_0(\Omega) = \operatorname{rot}\mathsf{R}(\Omega), \quad \mathring{\mathsf{D}}_0(\Omega) = \operatorname{rot}\mathring{\mathsf{R}}(\Omega)$$

as well as the simple Helmholtz decompositions

$$\mathsf{L}^2_\varepsilon(\Omega) = \nabla\mathring{\mathsf{H}}^1(\Omega) \oplus_\varepsilon \varepsilon^{-1}\operatorname{rot}\mathsf{R}(\Omega), \quad \mathsf{L}^2_\varepsilon(\Omega) = \nabla\mathsf{H}^1(\Omega) \oplus_\varepsilon \varepsilon^{-1}\operatorname{rot}\mathring{\mathsf{R}}(\Omega). \tag{2.12}$$

The aim of this paper is to give a computable estimate for the two Maxwell constants $c_{\mathtt{m},\mathtt{t},\varepsilon}$ and $c_{\mathtt{m},\mathtt{n},\varepsilon}$.

[†]Note that convex domains are always Lipschitz, see e.g. [8].

3 The Maxwell Estimates

First, we have an estimate for irrotational fields, which is well known.

Lemma 1 *For all* $E \in \nabla\mathring{\mathsf{H}}^1(\Omega) \cap \varepsilon^{-1}\mathsf{D}(\Omega)$ *and all* $H \in \nabla\mathsf{H}^1(\Omega) \cap \varepsilon^{-1}\mathring{\mathsf{D}}(\Omega)$

$$|E|_{\Omega,\varepsilon} \le \underline{\varepsilon} c_{\mathsf{p},\circ} |\operatorname{div} \varepsilon E|_\Omega, \quad |H|_{\Omega,\varepsilon} \le \underline{\varepsilon} c_{\mathsf{p}} |\operatorname{div} \varepsilon H|_\Omega.$$

Proof Pick a scalar potential $\varphi \in \mathring{\mathsf{H}}^1(\Omega)$ with $E = \nabla\varphi$. Then, by (2.6)

$$\begin{aligned} |E|^2_{\Omega,\varepsilon} &= \langle \varepsilon E, \nabla\varphi\rangle_\Omega = -\langle \operatorname{div} \varepsilon E, \varphi\rangle_\Omega \le |\operatorname{div} \varepsilon E|_\Omega |\varphi|_\Omega \le c_{\mathsf{p},\circ} |\operatorname{div} \varepsilon E|_\Omega |\nabla\varphi|_\Omega \\ &= c_{\mathsf{p},\circ} |\operatorname{div} \varepsilon E|_\Omega |E|_\Omega \le \underline{\varepsilon} c_{\mathsf{p},\circ} |\operatorname{div} \varepsilon E|_\Omega |E|_{\Omega,\varepsilon}. \end{aligned}$$

Let $\varphi \in \mathsf{H}^1(\Omega)$ with $H = \nabla\varphi$ and $\varphi \perp \mathbb{R}$. Since $\varepsilon H \in \mathring{\mathsf{D}}(\Omega)$ we obtain as before and by (2.7)

$$\begin{aligned} |H|^2_{\Omega,\varepsilon} &= \langle \varepsilon H, \nabla\varphi\rangle_\Omega = -\langle \operatorname{div} \varepsilon H, \varphi\rangle_\Omega \le |\operatorname{div} \varepsilon H|_\Omega |\varphi|_\Omega \le c_{\mathsf{p}} |\operatorname{div} \varepsilon H|_\Omega |\nabla\varphi|_\Omega \\ &= c_{\mathsf{p}} |\operatorname{div} \varepsilon H|_\Omega |H|_\Omega \le \underline{\varepsilon} c_{\mathsf{p}} |\operatorname{div} \varepsilon H|_\Omega |H|_{\Omega,\varepsilon}, \end{aligned}$$

which finishes the proof. □

Remark 2 *Without any change, Lemma 1 extends to Lipschitz domains* $\Omega \subset \mathbb{R}^N$ *of arbitrary dimension.*

To get similar estimates for solenoidal vector fields we need a crucial lemma from [1, Theorem 2.17], see also [24, 8, 6, 4] for related partial results.

Lemma 3 *Let* Ω *be convex and* $E \in \mathring{\mathsf{R}}(\Omega) \cap \mathsf{D}(\Omega)$ *or* $E \in \mathsf{R}(\Omega) \cap \mathring{\mathsf{D}}(\Omega)$*. Then* $E \in \mathsf{H}^1(\Omega)$ *and*

$$|\nabla E|^2_\Omega \le |\operatorname{rot} E|^2_\Omega + |\operatorname{div} E|^2_\Omega. \tag{3.1}$$

We note that for $E \in \mathring{\mathsf{H}}^1(\Omega)$ it is clear that for any domain $\Omega \subset \mathbb{R}^3$

$$|\nabla E|^2_\Omega = |\operatorname{rot} E|^2_\Omega + |\operatorname{div} E|^2_\Omega \tag{3.2}$$

holds since $-\Delta = \operatorname{rot}\operatorname{rot} - \nabla\operatorname{div}$. This formula is no longer valid if E has just the tangential or normal boundary condition but for convex domains the inequality (3.1) remains true.

Lemma 4 *Let* Ω *be convex. For all vector fields* $E \in \mathring{\mathsf{R}}(\Omega) \cap \varepsilon^{-1}\operatorname{rot}\mathsf{R}(\Omega)$ *and all vector fields* $H \in \mathsf{R}(\Omega) \cap \varepsilon^{-1}\operatorname{rot}\mathring{\mathsf{R}}(\Omega)$

$$|E|_{\Omega,\varepsilon} \le \overline{\varepsilon} c_{\mathsf{p}} |\operatorname{rot} E|_\Omega, \quad |H|_{\Omega,\varepsilon} \le \overline{\varepsilon} c_{\mathsf{p}} |\operatorname{rot} H|_\Omega.$$

Proof Since $\varepsilon E \in \operatorname{rot} \mathsf{R}(\Omega) = \operatorname{rot} \mathcal{R}(\Omega)$ there exists a vector potential field $\Phi \in \mathcal{R}(\Omega)$ with $\operatorname{rot} \Phi = \varepsilon E$ and $\Phi \in \mathsf{H}^1(\Omega)$ by Lemma 3 since $\mathcal{R}(\Omega) = \mathsf{R}(\Omega) \cap \mathring{\mathsf{D}}_0(\Omega)$. Moreover, $\Phi = \operatorname{rot} \Psi$ can be represented by some $\Psi \in \mathring{\mathsf{R}}(\Omega)$. Hence, for any constant vector $a \in \mathbb{R}^3$ we have $\langle \Phi, a \rangle_\Omega = \langle \operatorname{rot} \Psi, a \rangle_\Omega = 0$. Thus, Φ belongs to $\mathsf{H}^1(\Omega) \cap (\mathbb{R}^3)^\perp$. Then, since $E \in \mathring{\mathsf{R}}(\Omega)$ and by Lemma 3 we get

$$\begin{aligned} |E|^2_{\Omega,\varepsilon} &= \langle E, \varepsilon E \rangle_\Omega = \langle E, \operatorname{rot} \Phi \rangle_\Omega = \langle \operatorname{rot} E, \Phi \rangle_\Omega \leq |\operatorname{rot} E|_\Omega |\Phi|_\Omega \leq c_{\mathtt{p}} |\operatorname{rot} E|_\Omega |\nabla \Phi|_\Omega \\ &\leq c_{\mathtt{p}} |\operatorname{rot} E|_\Omega |\operatorname{rot} \Phi|_\Omega = c_{\mathtt{p}} |\operatorname{rot} E|_\Omega |\varepsilon E|_\Omega \leq \overline{\varepsilon} c_{\mathtt{p}} |\operatorname{rot} E|_\Omega |E|_{\Omega,\varepsilon}. \end{aligned}$$

Since $\varepsilon H \in \operatorname{rot} \mathring{\mathsf{R}}(\Omega)$ there exists a vector potential $\Phi \in \mathring{\mathsf{R}}(\Omega)$ with $\operatorname{rot} \Phi = \varepsilon H$. Using the Helmholtz decomposition $\mathsf{L}^2(\Omega) = \mathsf{R}_0(\Omega) \oplus \operatorname{rot} \mathring{\mathsf{R}}(\Omega)$, we decompose

$$\mathsf{R}(\Omega) \ni H = H_0 + H_{\mathrm{rot}} \in \mathsf{R}_0(\Omega) \oplus \mathcal{R}(\Omega).$$

Then, $\operatorname{rot} H_{\mathrm{rot}} = \operatorname{rot} H$ and again by Lemma 3 we see $H_{\mathrm{rot}} \in \mathsf{H}^1(\Omega)$. Let $a \in \mathbb{R}^3$ such that $H_{\mathrm{rot}} - a \in \mathsf{H}^1(\Omega) \cap (\mathbb{R}^3)^\perp$. Since $\Phi \in \mathring{\mathsf{R}}(\Omega)$ and $\langle \operatorname{rot} \Phi, H_0 \rangle_\Omega = 0 = \langle \operatorname{rot} \Phi, a \rangle_\Omega$ as well as by Lemma 3 we obtain

$$\begin{aligned} |H|^2_{\Omega,\varepsilon} &= \langle \varepsilon H, H \rangle_\Omega = \langle \operatorname{rot} \Phi, H \rangle_\Omega = \langle \operatorname{rot} \Phi, H_{\mathrm{rot}} - a \rangle_\Omega \leq |\varepsilon H|_\Omega |H_{\mathrm{rot}} - a|_\Omega \\ &\leq c_{\mathtt{p}} |\varepsilon H|_\Omega |\nabla H_{\mathrm{rot}}|_\Omega \leq \overline{\varepsilon} c_{\mathtt{p}} |H|_{\Omega,\varepsilon} |\operatorname{rot} H_{\mathrm{rot}}|_\Omega = \overline{\varepsilon} c_{\mathtt{p}} |H|_{\Omega,\varepsilon} |\operatorname{rot} H|_\Omega, \end{aligned}$$

completing the proof. □

Remark 5 *It is well known that Lemma 4 holds in two dimensions for any Lipschitz domain $\Omega \subset \mathbb{R}^2$. This follows immediately from Lemma 1 if we take into account that in two dimensions the rotation* rot *is given by the divergence* div *after $90°$-rotation of the vector field to which it is applied. We refer to the appendix for details.*

Theorem 6 *Let Ω be convex. Then, for all vector fields $E \in \mathring{\mathsf{R}}(\Omega) \cap \varepsilon^{-1}\mathsf{D}(\Omega)$ and all vector fields $H \in \mathsf{R}(\Omega) \cap \varepsilon^{-1}\mathring{\mathsf{D}}(\Omega)$*

$$|E|^2_{\Omega,\varepsilon} \leq \underline{\varepsilon}^2 c^2_{\mathtt{p,o}} |\operatorname{div} \varepsilon E|^2_\Omega + \overline{\varepsilon}^2 c^2_{\mathtt{p}} |\operatorname{rot} E|^2_\Omega, \quad |H|^2_{\Omega,\varepsilon} \leq \underline{\varepsilon}^2 c^2_{\mathtt{p}} |\operatorname{div} \varepsilon H|^2_\Omega + \overline{\varepsilon}^2 c^2_{\mathtt{p}} |\operatorname{rot} H|^2_\Omega.$$

Thus, $c_{\mathtt{m,t},\varepsilon} \leq \max\{\underline{\varepsilon} c_{\mathtt{p,o}}, \overline{\varepsilon} c_{\mathtt{p}}\}$ and

$$c_{\mathtt{m,t},\varepsilon}, c_{\mathtt{m,n},\varepsilon} \leq \hat{\varepsilon} c_{\mathtt{p}} \leq \hat{\varepsilon} \operatorname{diam}(\Omega)/\pi.$$

Proof By the Helmholtz decomposition (2.12) we have

$$\mathring{\mathsf{R}}(\Omega) \cap \varepsilon^{-1}\mathsf{D}(\Omega) \ni E = E_\nabla + E_{\mathrm{rot}} \in \nabla \mathring{\mathsf{H}}^1(\Omega) \oplus_\varepsilon \varepsilon^{-1} \operatorname{rot} \mathsf{R}(\Omega)$$

with $E_\nabla \in \nabla \mathring{\mathsf{H}}^1(\Omega) \cap \varepsilon^{-1}\mathsf{D}(\Omega)$ and $E_{\mathrm{rot}} \in \mathring{\mathsf{R}}(\Omega) \cap \varepsilon^{-1} \operatorname{rot} \mathsf{R}(\Omega)$ as well as

$$\operatorname{div} \varepsilon E_\nabla = \operatorname{div} \varepsilon E, \quad \operatorname{rot} E_{\mathrm{rot}} = \operatorname{rot} E.$$

By Lemma 1 and Lemma 4 and orthogonality we obtain

$$|E|^2_{\Omega,\varepsilon} = |E_\nabla|^2_{\Omega,\varepsilon} + |E_{\mathrm{rot}}|^2_{\Omega,\varepsilon} \leq \underline{\varepsilon}^2 c^2_{\mathtt{p},\circ} |\operatorname{div} \varepsilon E|^2_\Omega + \overline{\varepsilon}^2 c^2_{\mathtt{p}} |\operatorname{rot} E|^2_\Omega.$$

Similarly we have

$$\mathsf{R}(\Omega) \cap \varepsilon^{-1} \mathring{\mathsf{D}}(\Omega) \ni H = H_\nabla + H_{\mathrm{rot}} \in \nabla \mathsf{H}^1(\Omega) \oplus_\varepsilon \varepsilon^{-1} \operatorname{rot} \mathring{\mathsf{R}}(\Omega)$$

with $H_\nabla \in \nabla \mathsf{H}^1(\Omega) \cap \varepsilon^{-1} \mathring{\mathsf{D}}(\Omega)$ and $H_{\mathrm{rot}} \in \mathsf{R}(\Omega) \cap \varepsilon^{-1} \operatorname{rot} \mathring{\mathsf{R}}(\Omega)$ as well as

$$\operatorname{div} \varepsilon H_\nabla = \operatorname{div} \varepsilon H, \quad \operatorname{rot} H_{\mathrm{rot}} = \operatorname{rot} H.$$

By Lemma 1 and Lemma 4

$$|H|^2_{\Omega,\varepsilon} = |H_\nabla|^2_{\Omega,\varepsilon} + |H_{\mathrm{rot}}|^2_{\Omega,\varepsilon} \leq \underline{\varepsilon}^2 c^2_{\mathtt{p}} |\operatorname{div} \varepsilon H|^2_\Omega + \overline{\varepsilon}^2 c^2_{\mathtt{p}} |\operatorname{rot} H|^2_\Omega,$$

which finishes the proof. □

Lower bounds can be computed even for general domains Ω:

Theorem 7 *It holds*

$$\frac{c_{\mathtt{p},\circ}}{\underline{\varepsilon}\overline{\varepsilon}^2} \leq c_{\mathtt{m},\mathtt{t},\varepsilon}, \quad \frac{c_{\mathtt{p}}}{\underline{\varepsilon}\overline{\varepsilon}^2} \leq c_{\mathtt{m},\mathtt{n},\varepsilon}.$$

Proof Let λ_1 resp. $\lambda_{1,\varepsilon}$ be the first Dirichlet eigenvalue of the negative Laplacian $-\Delta$ resp. weighted Laplacian $-\operatorname{div} \varepsilon \nabla$, i.e.,

$$\frac{1}{c^2_{\mathtt{p},\circ}} = \lambda_1 = \inf_{0 \neq u \in \mathring{\mathsf{H}}^1(\Omega)} \frac{|\nabla u|^2_\Omega}{|u|^2_\Omega} \geq \frac{1}{\overline{\varepsilon}^2} \inf_{0 \neq u \in \mathring{\mathsf{H}}^1(\Omega)} \frac{|\nabla u|^2_{\Omega,\varepsilon}}{|u|^2_\Omega} = \frac{\lambda_{1,\varepsilon}}{\overline{\varepsilon}^2}.$$

Hence $\lambda_{1,\varepsilon} \leq (\overline{\varepsilon}/c_{\mathtt{p},\circ})^2$. Let $u \in \mathring{\mathsf{H}}^1(\Omega)$ be an eigenfunction to $\lambda_{1,\varepsilon}$. Note that u satisfies

$$\forall \varphi \in \mathring{\mathsf{H}}^1(\Omega) \quad \langle \varepsilon \nabla u, \nabla \varphi \rangle_\Omega = \lambda_{1,\varepsilon} \langle u, \varphi \rangle_\Omega.$$

Then $0 \neq E := \nabla u$ belongs to $\nabla \mathring{\mathsf{H}}^1(\Omega) \cap \varepsilon^{-1} \mathsf{D}(\Omega) = \mathring{\mathsf{R}}_0(\Omega) \cap \varepsilon^{-1} \mathsf{D}(\Omega) \cap \mathcal{H}_{\mathtt{D},\varepsilon}(\Omega)^{\perp_\varepsilon}$ and solves $-\operatorname{div} \varepsilon E = -\operatorname{div} \varepsilon \nabla u = \lambda_{1,\varepsilon} u$. By (2.8) and (2.6) we have

$$|E|_{\Omega,\varepsilon} \leq c_{\mathtt{m},\mathtt{t},\varepsilon} |\operatorname{div} \varepsilon E|_\Omega = c_{\mathtt{m},\mathtt{t},\varepsilon} \lambda_{1,\varepsilon} |u|_\Omega \leq c_{\mathtt{m},\mathtt{t},\varepsilon} \lambda_{1,\varepsilon} c_{\mathtt{p},\circ} |\nabla u|_\Omega \leq \frac{c_{\mathtt{m},\mathtt{t},\varepsilon}}{c_{\mathtt{p},\circ}} \overline{\varepsilon}^2 \underline{\varepsilon} |E|_{\Omega,\varepsilon}$$

yielding $c_{\mathtt{p},\circ} \leq c_{\mathtt{m},\mathtt{t},\varepsilon} \underline{\varepsilon} \overline{\varepsilon}^2$. Now, we follow the same arguments for the Neumann eigenvalues. Let μ_2 resp. $\mu_{2,\varepsilon}$ be the second Neumann eigenvalue of the negative Laplacian $-\Delta$ resp. weighted Laplacian $-\operatorname{div} \varepsilon \nabla$, i.e.,

$$\frac{1}{c^2_{\mathtt{p}}} = \mu_2 = \inf_{0 \neq u \in \mathsf{H}^1(\Omega) \cap \mathbb{R}^\perp} \frac{|\nabla u|^2_\Omega}{|u|^2_\Omega} \geq \frac{1}{\overline{\varepsilon}^2} \inf_{0 \neq u \in \mathsf{H}^1(\Omega) \cap \mathbb{R}^\perp} \frac{|\nabla u|^2_{\Omega,\varepsilon}}{|u|^2_\Omega} = \frac{\mu_{2,\varepsilon}}{\overline{\varepsilon}^2}.$$

Hence $\mu_{2,\varepsilon} \leq (\overline{\varepsilon}/c_{\mathtt{p}})^2$. Let $u \in \mathsf{H}^1(\Omega) \cap \mathbb{R}^\perp$ be an eigenfunction to $\mu_{2,\varepsilon}$. Note that u satisfies

$$\forall \varphi \in \mathsf{H}^1(\Omega) \cap \mathbb{R}^\perp \quad \langle \varepsilon \nabla u, \nabla \varphi \rangle_\Omega = \mu_{2,\varepsilon} \langle u, \varphi \rangle_\Omega$$

and that this relation holds even for all $\varphi \in \mathsf{H}^1(\Omega)$. Then $0 \neq H := \nabla u$ belongs to $\nabla \mathsf{H}^1(\Omega) \cap \varepsilon^{-1} \overset{\circ}{\mathsf{D}}(\Omega) = \mathsf{R}_0(\Omega) \cap \varepsilon^{-1} \overset{\circ}{\mathsf{D}}(\Omega) \cap \mathcal{H}_{\mathtt{N},\varepsilon}(\Omega)^{\perp_\varepsilon}$ and $-\operatorname{div} \varepsilon H = -\operatorname{div} \varepsilon \nabla u = \mu_{2,\varepsilon} u$ holds. By (2.9) and (2.7) we have

$$|H|_{\Omega,\varepsilon} \leq c_{\mathtt{m,n},\varepsilon} |\operatorname{div} \varepsilon H|_\Omega = c_{\mathtt{m,n},\varepsilon} \mu_{2,\varepsilon} |u|_\Omega \leq c_{\mathtt{m,n},\varepsilon} \mu_{2,\varepsilon} c_{\mathtt{p}} |\nabla u|_\Omega \leq \frac{c_{\mathtt{m,n},\varepsilon}}{c_{\mathtt{p}}} \overline{\varepsilon}^2 \underline{\varepsilon} |H|_{\Omega,\varepsilon}$$

yielding $c_{\mathtt{p}} \leq c_{\mathtt{m,n},\varepsilon} \underline{\varepsilon} \overline{\varepsilon}^2$. The proof is complete. □

Remark 8 *The latter proof shows that Theorem 7 extends to any Lipschitz domain $\Omega \subset \mathbb{R}^N$ of arbitrary dimension with the appropriate changes for the rotation operator.*

Combining Theorems 6 and 7 we obtain:

Theorem 9 *Let Ω be convex. Then*

$$\frac{c_{\mathtt{p},\circ}}{\hat{\varepsilon}^3} \leq c_{\mathtt{m,t},\varepsilon} \leq \hat{\varepsilon} c_{\mathtt{p}}, \quad \frac{c_{\mathtt{p},\circ}}{\hat{\varepsilon}^3} < \frac{c_{\mathtt{p}}}{\hat{\varepsilon}^3} \leq c_{\mathtt{m,n},\varepsilon} \leq \hat{\varepsilon} c_{\mathtt{p}}$$

and hence

$$\frac{c_{\mathtt{p},\circ}}{\hat{\varepsilon}^3} \leq c_{\mathtt{m,t},\varepsilon}, c_{\mathtt{m,n},\varepsilon} \leq \hat{\varepsilon} c_{\mathtt{p}} \leq \hat{\varepsilon} \operatorname{diam}(\Omega)/\pi.$$

If additionally $\varepsilon = \mathrm{id}$, then

$$c_{\mathtt{p},\circ} \leq c_{\mathtt{m,t}} \leq c_{\mathtt{m,n}} = c_{\mathtt{p}} \leq \operatorname{diam}(\Omega)/\pi.$$

Remark 10 *Our results extend also to all possibly non-convex polyhedra which allow the $\mathsf{H}^1(\Omega)$-regularity of the Maxwell spaces $\overset{\circ}{\mathsf{R}}(\Omega) \cap \mathsf{D}(\Omega)$ and $\mathsf{R}(\Omega) \cap \overset{\circ}{\mathsf{D}}(\Omega)$ or to domains whose boundaries consist of combinations of convex boundary parts and polygonal parts which allow the $\mathsf{H}^1(\Omega)$-regularity. Is is shown in [4, Theorem 4.1] that (3.1), even (3.2), still holds for all $E \in \mathsf{H}^1(\Omega) \cap \overset{\circ}{\mathsf{R}}(\Omega)$ or $E \in \mathsf{H}^1(\Omega) \cap \overset{\circ}{\mathsf{D}}(\Omega)$ if Ω is a polyhedron*[‡]*. We note that even some non-convex polyhedra admit the $\mathsf{H}^1(\Omega)$-regularity of the Maxwell spaces depending on the angle of the corners, which are not allowed to by too pointy.*

Remark 11

- **(i)** *We conjecture $c_{\mathtt{p},\circ} < c_{\mathtt{m,t}} < c_{\mathtt{m,n}} = c_{\mathtt{p}}$ for convex $\Omega \subset \mathbb{R}^3$.*
- **(ii)** *We note that by Theorem 9 we have given a new proof of the estimate*

$$0 < \mu_2 \leq \lambda_1$$

for convex $\Omega \subset \mathbb{R}^3$. Moreover, the absolute values of the eigenvalues of the different Maxwell operators (tangential or normal boundary condition) lie between $\sqrt{\mu_2}$ and $\sqrt{\lambda_1}$.

[‡]The crucial point is that the unit normal is piecewise constant and hence the curvature is zero.

Finally, we note that in the case $\varepsilon = \mathrm{id}$ we can find some different proofs for the lower bounds in less general settings. For example, if Ω has a connected boundary, then $\mathcal{H}_{\mathtt{D}}(\Omega) = \{0\}$ and hence

$$\begin{aligned}\frac{1}{c_{\mathtt{m},\mathtt{t}}^2} &= \inf_{0\neq E\in\mathring{\mathsf{R}}(\Omega)\cap\mathsf{D}(\Omega)} \frac{|\operatorname{rot} E|_\Omega^2 + |\operatorname{div} E|_\Omega^2}{|E|_\Omega^2}\\ &\leq \inf_{0\neq E\in\mathring{\mathsf{H}}^1(\Omega)} \frac{|\operatorname{rot} E|_\Omega^2 + |\operatorname{div} E|_\Omega^2}{|E|_\Omega^2} = \inf_{0\neq E\in\mathring{\mathsf{H}}^1(\Omega)} \frac{|\nabla E|_\Omega^2}{|E|_\Omega^2} = \frac{1}{c_{\mathtt{p},\circ}^2}\end{aligned}$$

giving $c_{\mathtt{p},\circ} \leq c_{\mathtt{m},\mathtt{t}}$. If Ω is simply connected, then $\mathcal{H}_{\mathtt{N}}(\Omega) = \{0\}$ and hence

$$\begin{aligned}\frac{1}{c_{\mathtt{m},\mathtt{n}}^2} &= \inf_{0\neq H\in\mathsf{R}(\Omega)\cap\mathring{\mathsf{D}}(\Omega)} \frac{|\operatorname{rot} H|_\Omega^2 + |\operatorname{div} H|_\Omega^2}{|H|_\Omega^2}\\ &\leq \inf_{0\neq H\in\mathring{\mathsf{H}}^1(\Omega)} \frac{|\operatorname{rot} H|_\Omega^2 + |\operatorname{div} H|_\Omega^2}{|H|_\Omega^2} = \inf_{0\neq H\in\mathring{\mathsf{H}}^1(\Omega)} \frac{|\nabla H|_\Omega^2}{|H|_\Omega^2} = \frac{1}{c_{\mathtt{p},\circ}^2}\end{aligned}$$

yielding $c_{\mathtt{p},\circ} \leq c_{\mathtt{m},\mathtt{n}}$. Another proof could be like this: Again, we assume that Γ is connected for the tangential case resp. that Ω is simply connected for the normal case. Let $u \in \mathring{\mathsf{H}}^1(\Omega)$ and $\xi \in \mathbb{R}^3$ with $|\xi| = 1$. Then $E := u\xi \in \mathring{\mathsf{H}}^1(\Omega) \subset \mathring{\mathsf{R}}(\Omega) \cap \mathring{\mathsf{D}}(\Omega)$ and since there are no Dirichlet resp. Neumann fields, we get by (2.8) resp. (2.9) and $\operatorname{rot} E = \nabla u \times \xi$, $\operatorname{div} E = \nabla u \cdot \xi$

$$|u|_\Omega^2 = |E|_\Omega^2 \leq c_{\mathtt{m}}^2\big(|\operatorname{rot} E|_\Omega^2 + |\operatorname{div} E|_\Omega^2\big) = c_{\mathtt{m}}^2 |\nabla u|_\Omega^2.$$

Therefore $c_{\mathtt{p},\circ} \leq c_{\mathtt{m}}$, where $c_{\mathtt{m}} = c_{\mathtt{m},\mathtt{t}}$ resp. $c_{\mathtt{m}} = c_{\mathtt{m},\mathtt{n}}$.

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References

[1] C. Amrouche, C. Bernardi, M. Dauge, and V. Girault. Vector potentials in three-dimensional non-smooth domains. *Math. Methods Appl. Sci.*, 21(9):823–864, 1998.

[2] C. Amrouche, P.G. Ciarlet, and P. (Jr.) Ciarlet. Weak vector and scalar potentials. Applications to Poincaré's theorem and Korn's inequality in Sobolev spaces with negative exponents. *Anal. Appl. (Singap.)*, 8(1):1–17, 2010.

[3] M. Bebendorf. A note on the Poincaré inequality for convex domains. *Z. Anal. Anwendungen*, 22(4):751–756, 2003.

[4] M. Costabel. A coercive bilinear form for Maxwell's equations. *J. Math. Anal. Appl.*, 157(2):527–541, 1991.

[5] N. Filonov. On an inequality for the eigenvalues of the Dirichlet and Neumann problems for the Laplace operator. *St. Petersburg Math. J.*, 16(2):413–416, 2005.

[6] V. Girault and P.-A. Raviart. *Finite Element Methods for Navier-Stokes Equations: Theory and Algorithms.* Springer (Series in Computational Mathematics), Heidelberg, 1986.

[7] V. Gol'dshtein, I. Mitrea, and M. Mitrea. Hodge decompositions with mixed boundary conditions and applications to partial differential equations on Lipschitz manifolds. *J. Math. Sci. (N.Y.)*, 172(3):347–400, 2011.

[8] P. Grisvard. *Elliptic Problems in Nonsmooth Domains.* Pitman (Advanced Publishing Program), Boston, 1985.

[9] T. Jakab, I. Mitrea, and M. Mitrea. On the regularity of differential forms satisfying mixed boundary conditions in a class of Lipschitz domains. *Indiana Univ. Math. J.*, 58(5):2043–2071, 2009.

[10] F. Jochmann. A compactness result for vector fields with divergence and curl in $L^q(\Omega)$ involving mixed boundary conditions. *Appl. Anal.*, 66:189–203, 1997.

[11] P. Kuhn and D. Pauly. Regularity results for generalized electro-magnetic problems. *Analysis (Munich)*, 30(3):225–252, 2010.

[12] R. Leis. Zur Theorie elektromagnetischer Schwingungen in anisotropen inhomogenen Medien. *Math. Z.*, 106:213–224, 1968.

[13] R. Leis. *Initial Boundary Value Problems in Mathematical Physics.* Teubner, Stuttgart, 1986.

[14] D. Pauly. Low frequency asymptotics for time-harmonic generalized Maxwell equations in nonsmooth exterior domains. *Adv. Math. Sci. Appl.*, 16(2):591–622, 2006.

[15] D. Pauly. Generalized electro-magneto statics in nonsmooth exterior domains. *Analysis (Munich)*, 27(4):425–464, 2007.

[16] D. Pauly. Complete low frequency asymptotics for time-harmonic generalized Maxwell equations in nonsmooth exterior domains. *Asymptot. Anal.*, 60(3-4):125–184, 2008.

[17] D. Pauly. Hodge-Helmholtz decompositions of weighted Sobolev spaces in irregular exterior domains with inhomogeneous and anisotropic media. *Math. Methods Appl. Sci.*, 31:1509–1543, 2008.

[18] L.E. Payne and H.F. Weinberger. An optimal Poincaré inequality for convex domains. *Arch. Rational Mech. Anal.*, 5:286–292, 1960.

[19] R. Picard. Randwertaufgaben der verallgemeinerten Potentialtheorie. *Math. Methods Appl. Sci.*, 3:218–228, 1981.

[20] R. Picard. On the boundary value problems of electro- and magnetostatics. *Proc. Roy. Soc. Edinburgh Sect. A*, 92:165–174, 1982.

[21] R. Picard. An elementary proof for a compact imbedding result in generalized electromagnetic theory. *Math. Z.*, 187:151–164, 1984.

[22] R. Picard. Some decomposition theorems and their applications to non-linear potential theory and Hodge theory. *Math. Methods Appl. Sci.*, 12:35–53, 1990.

[23] R. Picard, N. Weck, and K.-J. Witsch. Time-harmonic Maxwell equations in the exterior of perfectly conducting, irregular obstacles. *Analysis (Munich)*, 21:231–263, 2001.

[24] J. Saranen. On an inequality of Friedrichs. *Math. Scand.*, 51(2):310–322, 1982.

[25] C. Weber. A local compactness theorem for Maxwell's equations. *Math. Methods Appl. Sci.*, 2:12–25, 1980.

[26] N. Weck. Maxwell's boundary value problems on Riemannian manifolds with nonsmooth boundaries. *J. Math. Anal. Appl.*, 46:410–437, 1974.

[27] K.-J. Witsch. A remark on a compactness result in electromagnetic theory. *Math. Methods Appl. Sci.*, 16:123–129, 1993.

A Appendix: The Maxwell Estimates in Two Dimensions

Finally, we want to note that similar but simpler results hold in two dimensions as well. More precisely, for $N = 2$ the Maxwell constants can be estimated by the Poincaré constants in any bounded Lipschitz domain $\Omega \subset \mathbb{R}^2$. Although this is quite well known, we present the results for convenience and completeness.

As noted before, Lemma 1 holds in any dimension. In two dimensions the rotation rot differs from the divergence div just by a 90°-rotation R given by

$$R := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad R^2 = -\operatorname{id}, \quad R^\top = -R = R^{-1}.$$

The same holds for the co-gradient $\lhd := \operatorname{rot}^*$ (as formal adjoint) and the gradient ∇. More precisely, for smooth functions u and smooth vector fields v we have

$$\operatorname{rot} v = \operatorname{div} Rv = \partial_1 v_2 - \partial_2 v_1, \qquad \lhd u = R\nabla u = \begin{bmatrix} \partial_2 u \\ -\partial_1 u \end{bmatrix},$$

$$\operatorname{div} v = -\operatorname{rot} Rv, \qquad \nabla u = -R \lhd u$$

and thus also $-\Delta u = -\operatorname{div}\nabla u = \operatorname{div} RR\nabla u = \operatorname{rot}\lhd u$. For the vector Laplacian we have $-\Delta v = \lhd\operatorname{rot} - \nabla\operatorname{div}$. Furthermore,

$$v \in \mathsf{R}(\Omega) \quad\Leftrightarrow\quad Rv \in \mathsf{D}(\Omega), \qquad v \in \mathring{\mathsf{R}}(\Omega) \quad\Leftrightarrow\quad Rv \in \mathring{\mathsf{D}}(\Omega).$$

The Helmholtz decompositions read

$$\begin{aligned}
\mathsf{L}^2(\Omega) &= \nabla\mathring{\mathsf{H}}^1(\Omega) \oplus_\varepsilon \varepsilon^{-1}\mathsf{D}_0(\Omega)\\
&= \mathring{\mathsf{R}}_0(\Omega) \oplus_\varepsilon \varepsilon^{-1} \lhd \mathsf{H}^1(\Omega)\\
&= \nabla\mathring{\mathsf{H}}^1(\Omega) \oplus_\varepsilon \mathcal{H}_{\mathtt{D},\varepsilon}(\Omega) \oplus_\varepsilon \varepsilon^{-1} \lhd \mathsf{H}^1(\Omega),\\
\mathsf{L}^2(\Omega) &= \nabla\mathsf{H}^1(\Omega) \oplus_\varepsilon \varepsilon^{-1}\mathring{\mathsf{D}}_0(\Omega)\\
&= \mathsf{R}_0(\Omega) \oplus_\varepsilon \varepsilon^{-1} \lhd \mathring{\mathsf{H}}^1(\Omega)\\
&= \nabla\mathsf{H}^1(\Omega) \oplus_\varepsilon \mathcal{H}_{\mathtt{N},\varepsilon}(\Omega) \oplus_\varepsilon \varepsilon^{-1} \lhd \mathring{\mathsf{H}}^1(\Omega)
\end{aligned}$$

and we note

$$\begin{aligned}
\nabla\mathring{\mathsf{H}}^1(\Omega) &= \mathring{\mathsf{R}}_0(\Omega) \cap \mathcal{H}_{\mathtt{D},\varepsilon}(\Omega)^{\perp_\varepsilon}, &\qquad \varepsilon^{-1} \lhd \mathsf{H}^1(\Omega) &= \varepsilon^{-1}\mathsf{D}_0(\Omega) \cap \mathcal{H}_{\mathtt{D},\varepsilon}(\Omega)^{\perp_\varepsilon},\\
\nabla\mathsf{H}^1(\Omega) &= \mathsf{R}_0(\Omega) \cap \mathcal{H}_{\mathtt{N},\varepsilon}(\Omega)^{\perp_\varepsilon}, &\qquad \varepsilon^{-1} \lhd \mathring{\mathsf{H}}^1(\Omega) &= \varepsilon^{-1}\mathring{\mathsf{D}}_0(\Omega) \cap \mathcal{H}_{\mathtt{N},\varepsilon}(\Omega)^{\perp_\varepsilon}.
\end{aligned}$$

We also need the matrix $\varepsilon_R := -R\varepsilon R$, which fulfills the same estimates as ε, i.e., for all $E \in \mathsf{L}^2(\Omega)$

$$\underline{\varepsilon}^{-2}|E|_\Omega^2 \le \langle \varepsilon_R E, E\rangle_\Omega \le \overline{\varepsilon}^2|E|_\Omega^2,$$

since $\langle \varepsilon_R E, E\rangle_\Omega = \langle \varepsilon RE, RE\rangle_\Omega$ and $|RE|_\Omega = |E|_\Omega$. But then the inverse ε_R^{-1} satisfies for all $E \in \mathsf{L}^2(\Omega)$

$$\overline{\varepsilon}^{-2}|E|_\Omega^2 \le \langle \varepsilon_R^{-1} E, E\rangle_\Omega \le \underline{\varepsilon}^2|E|_\Omega^2,$$

which immediately follows by (2.2), i.e.,

$$\langle \varepsilon_R^{-1}E, E\rangle_\Omega = |\varepsilon_R^{-1/2}E|_\Omega^2 \begin{cases} \le \underline{\varepsilon}^2\langle \varepsilon_R\varepsilon_R^{-1/2}E, \varepsilon_R^{-1/2}E\rangle_\Omega = \underline{\varepsilon}^2|E|_\Omega^2\\ \ge \overline{\varepsilon}^{-2}\langle \varepsilon_R\varepsilon_R^{-1/2}E, \varepsilon_R^{-1/2}E\rangle_\Omega = \overline{\varepsilon}^{-2}|E|_\Omega^2 \end{cases}.$$

Hence, for the inverse matrix $\varepsilon_R^{-1} = -R\varepsilon^{-1}R$ simply $\underline{\varepsilon}$ and $\overline{\varepsilon}$ has to be exchanged. Furthermore, we have $\varepsilon_R^{\pm 1/2} = -R\varepsilon^{\pm 1/2}R$.

For the solenoidal fields we have the following:

Lemma 12 *For all* $E \in \mathring{\mathsf{R}}(\Omega) \cap \varepsilon^{-1} \lhd \mathsf{H}^1(\Omega)$ *and all* $H \in \mathsf{R}(\Omega) \cap \varepsilon^{-1} \lhd \mathring{\mathsf{H}}^1(\Omega)$

$$|E|_{\Omega,\varepsilon} \le \overline{\varepsilon}c_{\mathtt{p}}|\operatorname{rot} E|_\Omega, \quad |H|_{\Omega,\varepsilon} \le \overline{\varepsilon}c_{\mathtt{p},\circ}|\operatorname{rot} H|_\Omega.$$

Proof Since $RE \in \mathring{\mathsf{D}}(\Omega)$ and $R\varepsilon E \in \nabla\mathsf{H}^1(\Omega)$ we have $R\varepsilon E \in \nabla\mathsf{H}^1(\Omega) \cap \varepsilon_R\mathring{\mathsf{D}}(\Omega)$. By Lemma 1 (interchanging $\underline{\varepsilon}$ and $\overline{\varepsilon}$) we get

$$\begin{aligned}
|E|_{\Omega,\varepsilon} &= |\varepsilon^{1/2}E|_\Omega = |R\varepsilon^{-1/2}\varepsilon E|_\Omega = |\varepsilon_R^{-1/2}R\varepsilon E|_\Omega = |R\varepsilon E|_{\Omega,\varepsilon_R^{-1}}\\
&\le \overline{\varepsilon}c_{\mathtt{p}}|\operatorname{div}\varepsilon_R^{-1}R\varepsilon E|_\Omega = \overline{\varepsilon}c_{\mathtt{p}}|\operatorname{rot} E|_\Omega.
\end{aligned}$$

Analogously, as $RH \in \mathsf{D}(\Omega)$ and $R\varepsilon H \in \nabla\mathring{\mathsf{H}}^1(\Omega)$ we have $R\varepsilon H \in \nabla\mathring{\mathsf{H}}^1(\Omega) \cap \varepsilon_R\mathsf{D}(\Omega)$. Again by Lemma 1 (and again interchanging $\underline{\varepsilon}$ and $\overline{\varepsilon}$) we get

$$\begin{aligned}|H|_{\Omega,\varepsilon} = |\varepsilon^{1/2}H|_\Omega = |R\varepsilon^{-1/2}\varepsilon H|_\Omega = |\varepsilon_R^{-1/2}R\varepsilon H|_\Omega = |R\varepsilon H|_{\Omega,\varepsilon_R^{-1}}\\ \leq \overline{\varepsilon}c_{\mathtt{p},\circ}|\operatorname{div}\varepsilon_R^{-1}R\varepsilon H|_\Omega = \overline{\varepsilon}c_{\mathtt{p},\circ}|\operatorname{rot}H|_\Omega,\end{aligned}$$

which completes the proof. □

Finally, the main result is proved as Theorems 6, 7 and 9, but taking into account that there are now possibly Dirichlet and Neumann fields.

Theorem 13 *For all* $E \in \mathring{\mathsf{R}}(\Omega) \cap \varepsilon^{-1}\mathsf{D}(\Omega)$ *and all* $H \in \mathsf{R}(\Omega) \cap \varepsilon^{-1}\mathring{\mathsf{D}}(\Omega)$

$$\begin{aligned}|E - \pi_{\mathtt{D}}E|^2_{\Omega,\varepsilon} \leq \underline{\varepsilon}^2c^2_{\mathtt{p},\circ}|\operatorname{div}\varepsilon E|^2_\Omega + \overline{\varepsilon}^2c^2_{\mathtt{p}}|\operatorname{rot}E|^2_\Omega,\\ |H - \pi_{\mathtt{N}}H|^2_{\Omega,\varepsilon} \leq \underline{\varepsilon}^2c^2_{\mathtt{p}}|\operatorname{div}\varepsilon H|^2_\Omega + \overline{\varepsilon}^2c^2_{\mathtt{p},\circ}|\operatorname{rot}H|^2_\Omega.\end{aligned}$$

Thus

$$\frac{c_{\mathtt{p},\circ}}{\underline{\varepsilon}\overline{\varepsilon}^2} \leq c_{\mathtt{m},\mathtt{t},\varepsilon} \leq \max\{\underline{\varepsilon}c_{\mathtt{p},\circ}, \overline{\varepsilon}c_{\mathtt{p}}\}, \quad \frac{c_{\mathtt{p},\circ}}{\underline{\varepsilon}\overline{\varepsilon}^2} < \frac{c_{\mathtt{p}}}{\underline{\varepsilon}\overline{\varepsilon}^2} \leq c_{\mathtt{m},\mathtt{n},\varepsilon} \leq \max\{\underline{\varepsilon}c_{\mathtt{p}}, \overline{\varepsilon}c_{\mathtt{p},\circ}\}$$

and hence

$$\frac{c_{\mathtt{p},\circ}}{\underline{\varepsilon}\overline{\varepsilon}^2} \leq c_{\mathtt{m},\mathtt{t},\varepsilon}, c_{\mathtt{m},\mathtt{n},\varepsilon} \leq \hat{\varepsilon}c_{\mathtt{p}}.$$

For $\varepsilon = \mathrm{id}$ *it holds*

$$c_{\mathtt{p},\circ} \leq c_{\mathtt{m},\mathtt{t}} \leq c_{\mathtt{m},\mathtt{n}} = c_{\mathtt{p}}$$

and if additionally Ω *is convex we have* $c_{\mathtt{p}} \leq \operatorname{diam}(\Omega)/\pi$.

Proof Using the Helmholtz decomposition we have

$$\mathring{\mathsf{R}}(\Omega) \cap \varepsilon^{-1}\mathsf{D}(\Omega) \cap \mathcal{H}_{\mathtt{D},\varepsilon}(\Omega)^{\perp_\varepsilon} \ni E - \pi_{\mathtt{D}}E = E_\nabla + E_\triangleleft \in \nabla\mathring{\mathsf{H}}^1(\Omega) \oplus_\varepsilon \varepsilon^{-1} \triangleleft \mathsf{H}^1(\Omega)$$

with $E_\nabla \in \nabla\mathring{\mathsf{H}}^1(\Omega) \cap \varepsilon^{-1}\mathsf{D}(\Omega)$ and $E_\triangleleft \in \mathring{\mathsf{R}}(\Omega) \cap \varepsilon^{-1} \triangleleft \mathsf{H}^1(\Omega)$ as well as

$$\operatorname{div}\varepsilon E_\nabla = \operatorname{div}\varepsilon E, \quad \operatorname{rot}E_\triangleleft = \operatorname{rot}E.$$

Thus, by Lemma 1 and Lemma 12 as well as orthogonality we obtain

$$|E - \pi_{\mathtt{D}}E|^2_{\Omega,\varepsilon} = |E_\nabla|^2_{\Omega,\varepsilon} + |E_\triangleleft|^2_{\Omega,\varepsilon} \leq \underline{\varepsilon}^2c^2_{\mathtt{p},\circ}|\operatorname{div}\varepsilon E|^2_\Omega + \overline{\varepsilon}^2c^2_{\mathtt{p}}|\operatorname{rot}E|^2_\Omega.$$

Analogously, we decompose

$$\mathsf{R}(\Omega) \cap \varepsilon^{-1}\mathring{\mathsf{D}}(\Omega) \cap \mathcal{H}_{\mathtt{N},\varepsilon}(\Omega)^{\perp_\varepsilon} \ni H - \pi_{\mathtt{N}}H = H_\nabla + H_\triangleleft \in \nabla\mathsf{H}^1(\Omega) \oplus_\varepsilon \varepsilon^{-1} \triangleleft \mathring{\mathsf{H}}^1(\Omega)$$

with $H_\nabla \in \nabla\mathsf{H}^1(\Omega) \cap \varepsilon^{-1}\mathring{\mathsf{D}}(\Omega)$ and $H_\triangleleft \in \mathsf{R}(\Omega) \cap \varepsilon^{-1} \triangleleft \mathring{\mathsf{H}}^1(\Omega)$ as well as

$$\operatorname{div}\varepsilon H_\nabla = \operatorname{div}\varepsilon H, \quad \operatorname{rot}H_\triangleleft = \operatorname{rot}H.$$

As before, by Lemma 1, Lemma 12 and orthogonality we see

$$|H - \pi_{\mathtt{N}}H|^2_{\Omega,\varepsilon} = |H_\nabla|^2_{\Omega,\varepsilon} + |H_\triangleleft|^2_{\Omega,\varepsilon} \leq \underline{\varepsilon}^2c^2_{\mathtt{p}}|\operatorname{div}\varepsilon H|^2_\Omega + \overline{\varepsilon}^2c^2_{\mathtt{p},\circ}|\operatorname{rot}H|_\Omega,$$

yielding the assertion for the upper bounds. For the lower bounds we refer to Remark 8, which completes the proof. □