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On the Maxwell Inequalities for Bounded and Convex Domains

by

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Abstract

For a bounded and convex domain in three dimensions we show that the Maxwell constants are bounded from below and above by Friedrichs' and Poincaré's constants.

Key Words Maxwell's equations, Maxwell constant, second Maxwell eigenvalue, electro statics, magneto statics, Poincaré's inequality, Friedrichs' inequality, Poincaré's constant, Friedrichs' constant

Contents

1 Introduction

Let $\Omega \subset \mathbb{R}^3$ be a bounded and convex domain. It is well known that, e.g., by Rellich's selection theorem using standard indirect arguments, the Poincaré* inequalities

$$\exists\, c_{\mathrm{p},\circ} > 0 \qquad \forall\, u \in \overset{\circ}{\mathsf{H}}{}^1 \qquad |u| \le c_{\mathrm{p},\circ} |\nabla u|, \tag{1.1}$$

$$\exists\, c_{\mathrm{p}} > 0 \qquad \forall\, u \in \mathsf{H}^1 \cap \mathbb{R}^{\perp} \qquad |u| \le c_{\mathrm{p}} |\nabla u| \tag{1.2}$$

hold. Here, $c_{\mathrm{p},\circ}$ and c_{p} are the Poincaré constants, which satisfy

$$0 < c_{\mathrm{p},\circ} = 1/\sqrt{\lambda_1} < 1/\sqrt{\mu_2} = c_{\mathrm{p}},$$

where λ_1 is the first Dirichlet and μ_2 the second Neumann eigenvalue of the Laplacian. By $\langle\,\cdot\,,\,\cdot\,\rangle$ and $|\cdot|$ we denote the standard inner product and induced norm in L^2 and we will write the usual L^2-Sobolev spaces as H^1 and $\overset{\circ}{\mathsf{H}}{}^1$, the latter is defined as the closure in H^1 of smooth and compactly supported test functions. All spaces and norms are defined on

*The estimate (1.1) is often called Friedrichs'/Steklov inequality as well.

Ω. Moreover, we introduce the standard Sobolev spaces for the rotation and divergence by R and D. As before, we will denote the closures of test vector fields in the respective graph norms by $\overset{\circ}{\mathsf{R}}$ and $\overset{\circ}{\mathsf{D}}$. An index zero at the lower right corner of the latter spaces indicates a vanishing derivative, e.g.,

$$\mathsf{R}_0 := \{E \in \mathsf{R} \, : \, \operatorname{rot} E = 0\}, \quad \overset{\circ}{\mathsf{D}}_0 := \{E \in \overset{\circ}{\mathsf{D}} \, : \, \operatorname{div} E = 0\}.$$

As Ω is convex, it is especially simply connected and has got a connected boundary. Hence, the Neumann and Dirichlet fields of Ω vanish, i.e., $\mathsf{R}_0 \cap \overset{\circ}{\mathsf{D}}_0 = \overset{\circ}{\mathsf{R}}_0 \cap \mathsf{R}_0 = \{0\}$. By the Maxwell compactness properties, i.e., the compactness of the two embeddings

$$\overset{\circ}{\mathsf{R}} \cap \mathsf{D} \hookrightarrow \mathsf{L}^2, \quad \mathsf{R} \cap \overset{\circ}{\mathsf{D}} \hookrightarrow \mathsf{L}^2,$$

(and again by a standard indirect argument) the Maxwell inequalities

$$\exists\, c_{\mathtt{m,t}} > 0 \qquad \forall\, E \in \overset{\circ}{\mathsf{R}} \cap \mathsf{D} \qquad |E| \leq c_{\mathtt{m,t}} \big(|\operatorname{rot} E|^2 + |\operatorname{div} E|^2 \big)^{1/2}, \tag{1.3}$$

$$\exists\, c_{\mathtt{m,n}} > 0 \qquad \forall\, H \in \mathsf{R} \cap \overset{\circ}{\mathsf{D}} \qquad |H| \leq c_{\mathtt{m,n}} \big(|\operatorname{rot} H|^2 + |\operatorname{div} H|^2 \big)^{1/2} \tag{1.4}$$

hold. To the best of the author's knowledge, general bounds for the Maxwell constants $c_{\mathtt{m,t}}$ and $c_{\mathtt{m,n}}$ are missing. On the other hand, at least estimates for $c_{\mathtt{m,t}}$ and $c_{\mathtt{m,n}}$ from above are very important from the point of view of applications, such as preconditioning or a priori and a posteriori error estimation for numerical methods.

In the paper at hand we will prove that

$$c_{\mathtt{p},\circ} \leq c_{\mathtt{m,t}} \leq c_{\mathtt{m,n}} = c_{\mathtt{p}} \leq \operatorname{diam}(\Omega)/\pi \tag{1.5}$$

holds true. We note that (1.5) is already well known in two dimensions, even for general Lipschitz domains $\Omega \subset \mathbb{R}^2$ (except of the last inequality), but new in three dimensions. Furthermore, the last inequality in (1.5) has been proved in the famous paper of Payne and Weinberger [9], where also the optimality of the estimate was shown. This paper contains a small mistake, which has been corrected in [2].

2 Results and Proofs

We start with an inequality for irrotational fields.

Lemma 1 *For all* $E \in \nabla \overset{\circ}{\mathsf{H}}{}^1 \cap \mathsf{D}$ *and all* $H \in \nabla \mathsf{H}^1 \cap \overset{\circ}{\mathsf{D}}$

$$|E| \leq c_{\mathtt{p},\circ} |\operatorname{div} E|, \quad |H| \leq c_{\mathtt{p}} |\operatorname{div} H|.$$

Proof Let $\varphi \in \overset{\circ}{\mathsf{H}}{}^1$ with $E = \nabla\varphi$. By (1.1) we get

$$|E|^2 = \langle E, \nabla\varphi \rangle = -\langle \operatorname{div} E, \varphi \rangle \leq |\operatorname{div} E| |\varphi| \leq c_{\mathtt{p},\circ} |\operatorname{div} E| |\nabla\varphi| = c_{\mathtt{p},\circ} |\operatorname{div} E| |E|.$$

Let $\varphi \in \mathsf{H}^1$ with $H = \nabla\varphi$ and $\varphi \bot \mathbb{R}$. Since $H \in \overset{\circ}{\mathsf{D}}$ and by (1.2) we obtain

$$|H|^2 = \langle H, \nabla\varphi \rangle = -\langle \operatorname{div} H, \varphi \rangle \leq |\operatorname{div} H||\varphi| \leq c_{\mathsf{p}}|\operatorname{div} H||\nabla\varphi| = c_{\mathsf{p}}|\operatorname{div} H||H|,$$

completing the proof. □

Remark 2 *Clearly, Lemma 1 extends to arbitrary Lipschitz domains $\Omega \subset \mathbb{R}^N$, $N \in \mathbb{N}$.*

As usual in the theory of Maxwell's equations, we need another crucial tool, the Helmholtz decompositions of vector fields into irrotational and solenoidal vector fields. For convex domains, these decompositions are very simple. We have

$$\mathsf{L}^2 = \nabla\overset{\circ}{\mathsf{H}}{}^1 \oplus \operatorname{rot} \mathsf{R}, \quad \mathsf{L}^2 = \nabla\mathsf{H}^1 \oplus \operatorname{rot} \overset{\circ}{\mathsf{R}}, \tag{2.1}$$

where $\oplus$ denotes the orthogonal sum in L^2. We note

$$\overset{\circ}{\mathsf{R}}_0 = \nabla\overset{\circ}{\mathsf{H}}{}^1, \quad \mathsf{R}_0 = \nabla\mathsf{H}^1, \quad \mathsf{D}_0 = \operatorname{rot} \mathsf{R}, \quad \overset{\circ}{\mathsf{D}}_0 = \operatorname{rot} \overset{\circ}{\mathsf{R}}.$$

Moreover, with

$$\overset{\circ}{\mathcal{R}} := \overset{\circ}{\mathsf{R}} \cap \operatorname{rot} \mathsf{R}, \quad \mathcal{R} := \mathsf{R} \cap \operatorname{rot} \overset{\circ}{\mathsf{R}}$$

we have

$$\overset{\circ}{\mathsf{R}} = \nabla\overset{\circ}{\mathsf{H}}{}^1 \oplus \overset{\circ}{\mathcal{R}}, \quad \mathsf{R} = \nabla\mathsf{H}^1 \oplus \mathcal{R} \tag{2.2}$$

and see

$$\operatorname{rot} \overset{\circ}{\mathsf{R}} = \operatorname{rot} \overset{\circ}{\mathcal{R}}, \quad \operatorname{rot} \mathsf{R} = \operatorname{rot} \mathcal{R}.$$

We note that all occurring spaces of range-type are closed subspaces of L^2, which follows immediately by the estimates (1.1)-(1.4). More details about the Helmholtz decompositions can be found e.g. in [6].

To get similar inequalities for solenoidal vector fields as in Lemma 1 we need a crucial lemma from [1, Theorem 2.17], see also [10, 5, 4, 3] for related partial results.

Lemma 3 *Let E belong to $\overset{\circ}{\mathsf{R}} \cap \mathsf{D}$ or $\mathsf{R} \cap \overset{\circ}{\mathsf{D}}$. Then $E \in \mathsf{H}^1$ and*

$$|\nabla E|^2 \leq |\operatorname{rot} E|^2 + |\operatorname{div} E|^2. \tag{2.3}$$

We emphasize that for $E \in \overset{\circ}{\mathsf{H}}{}^1$ and any domain $\Omega \subset \mathbb{R}^3$

$$|\nabla E|^2 = |\operatorname{rot} E|^2 + |\operatorname{div} E|^2 \tag{2.4}$$

holds since $-\Delta = \operatorname{rot}\operatorname{rot} - \nabla \operatorname{div}$. This formula is no longer valid if E has just the tangential or normal boundary condition but for convex domains the inequality (2.3) remains true.

Lemma 4 *For all vector fields E in $\mathring{\mathsf{R}} \cap \operatorname{rot} \mathsf{R}$ or $\mathsf{R} \cap \operatorname{rot} \mathring{\mathsf{R}}$*

$$|E| \le c_{\mathtt{p}} |\operatorname{rot} E|.$$

Proof Let $E \in \operatorname{rot} \mathsf{R} = \operatorname{rot} \mathcal{R}$ and $\Phi \in \mathcal{R}$ with $\operatorname{rot} \Phi = E$. Then $\Phi \in \mathsf{H}^1$ by Lemma 3 since $\mathcal{R} = \mathsf{R} \cap \mathring{\mathsf{D}}_0$. Moreover, $\Phi = \operatorname{rot} \Psi$ can be represented by some $\Psi \in \mathring{\mathsf{R}}$. Hence, for any constant vector $a \in \mathbb{R}^3$ we have $\langle \Phi, a \rangle = 0$. Thus, Φ belongs to $\mathsf{H}^1 \cap (\mathbb{R}^3)^\perp$. Then, since $E \in \mathring{\mathsf{R}}$ and by Lemma 3 we get

$$|E|^2 = \langle E, \operatorname{rot} \Phi \rangle = \langle \operatorname{rot} E, \Phi \rangle \le |\operatorname{rot} E||\Phi| \le c_{\mathtt{p}} |\operatorname{rot} E||\nabla \Phi| \le c_{\mathtt{p}} |\operatorname{rot} E|| \underbrace{\operatorname{rot} \Phi}_{=E} |.$$

If $E \in \operatorname{rot} \mathring{\mathsf{R}}$ there exists $\Phi \in \mathring{\mathsf{R}}$ with $\operatorname{rot} \Phi = E$. Using (2.2) we decompose

$$E = E_0 + E_{\mathrm{rot}} \in \mathsf{R}_0 \oplus \mathcal{R}.$$

Then, $\operatorname{rot} E_{\mathrm{rot}} = \operatorname{rot} E$ and again by Lemma 3 we see $E_{\mathrm{rot}} \in \mathsf{H}^1$. Let $a \in \mathbb{R}^3$ such that $E_{\mathrm{rot}} - a \in \mathsf{H}^1 \cap (\mathbb{R}^3)^\perp$. Since $\Phi \in \mathring{\mathsf{R}}$, $\langle \operatorname{rot} \Phi, H_0 \rangle$ and $\langle \operatorname{rot} \Phi, a \rangle$ vanish. By Lemma 3

$$|E|^2 = \langle \operatorname{rot} \Phi, E \rangle = \langle \underbrace{\operatorname{rot} \Phi}_{=E}, E_{\mathrm{rot}} - a \rangle \le |E||E_{\mathrm{rot}} - a| \le c_{\mathtt{p}} |E||\nabla E_{\mathrm{rot}}| \le c_{\mathtt{p}} |E|| \underbrace{\operatorname{rot} E_{\mathrm{rot}}}_{=\operatorname{rot} E} |$$

holds, which completes the proof. □

Remark 5 *It is well known that Lemma 4 holds in two dimensions for any Lipschitz domain $\Omega \subset \mathbb{R}^2$. This follows immediately from Lemma 1 if we take into account that in two dimensions the rotation* rot *is given by the divergence* div *after $90°$-rotation of the vector field to which it is applied.*

Theorem 6 *For all vector fields $E \in \mathring{\mathsf{R}} \cap \mathsf{D}$ and $H \in \mathsf{R} \cap \mathring{\mathsf{D}}$*

$$|E|^2 \le c_{\mathtt{p},\circ}^2 |\operatorname{div} E|^2 + c_{\mathtt{p}}^2 |\operatorname{rot} E|^2, \quad |H|^2 \le c_{\mathtt{p}}^2 |\operatorname{div} H|^2 + c_{\mathtt{p}}^2 |\operatorname{rot} H|^2$$

hold, i.e., $c_{\mathtt{m,t}}, c_{\mathtt{m,n}} \le c_{\mathtt{p}}$. Moreover, $c_{\mathtt{p},\circ} \le c_{\mathtt{m,t}} \le c_{\mathtt{m,n}} = c_{\mathtt{p}} \le \operatorname{diam}(\Omega)/\pi$.

Proof By the Helmholtz decomposition (2.1) we have

$$\mathring{\mathsf{R}} \cap \mathsf{D} \ni E = E_\nabla + E_{\mathrm{rot}} \in \nabla \mathring{\mathsf{H}}^1 \oplus \operatorname{rot} \mathsf{R}$$

with $E_\nabla \in \nabla \mathring{\mathsf{H}}^1 \cap \mathsf{D}$ and $E_{\mathrm{rot}} \in \mathring{\mathsf{R}} \cap \operatorname{rot} \mathsf{R}$ as well as $\operatorname{div} E_\nabla = \operatorname{div} E$ and $\operatorname{rot} E_{\mathrm{rot}} = \operatorname{rot} E$. By Lemma 1 and Lemma 4 and orthogonality we obtain

$$|E|^2 = |E_\nabla|^2 + |E_{\mathrm{rot}}|^2 \le c_{\mathtt{p},\circ}^2 |\operatorname{div} E|^2 + c_{\mathtt{p}}^2 |\operatorname{rot} E|^2.$$

Similarly we have

$$\mathsf{R} \cap \mathring{\mathsf{D}} \ni H = H_\nabla + H_{\mathrm{rot}} \in \nabla \mathsf{H}^1 \oplus \operatorname{rot} \mathring{\mathsf{R}}$$

with $H_\nabla \in \nabla\mathsf{H}^1 \cap \mathring{\mathsf{D}}$ and $H_{\mathrm{rot}} \in \mathsf{R} \cap \operatorname{rot} \mathring{\mathsf{R}}$ as well as $\operatorname{div} H_\nabla = \operatorname{div} H$ and $\operatorname{rot} H_{\mathrm{rot}} = \operatorname{rot} H$. As before,

$$|H|^2 = |H_\nabla|^2 + |H_{\mathrm{rot}}|^2 \leq c_{\mathtt{p}}^2 |\operatorname{div} H|^2 + c_{\mathtt{p}}^2 |\operatorname{rot} H|^2.$$

This shows the upper bounds. For the lower bounds, let λ_1 be the first Dirichlet eigenvalue of the negative Laplacian $-\Delta$, i.e.,

$$\frac{1}{c_{\mathtt{p},\circ}^2} = \lambda_1 = \inf_{0 \neq u \in \mathring{\mathsf{H}}^1} \frac{|\nabla u|^2}{|u|^2},$$

and let $u \in \mathring{\mathsf{H}}^1$ be an eigenfunction to λ_1. Note that u satisfies

$$\forall\, \varphi \in \mathring{\mathsf{H}}^1 \quad \langle \nabla u, \nabla \varphi \rangle = \lambda_1 \langle u, \varphi \rangle.$$

Then $0 \neq E := \nabla u \in \nabla \mathring{\mathsf{H}}^1 \cap \mathsf{D} = \mathring{\mathsf{R}}_0 \cap \mathsf{D}$ and $-\operatorname{div} E = -\operatorname{div} \nabla u = \lambda_1 u$. By (1.3) and (1.1) we have

$$|E| \leq c_{\mathtt{m,t}} |\operatorname{div} E| = c_{\mathtt{m,t}} \lambda_1 |u| \leq c_{\mathtt{m,t}} \lambda_1 c_{\mathtt{p},\circ} |\nabla u| = \frac{c_{\mathtt{m,t}}}{c_{\mathtt{p},\circ}} |E|,$$

yielding $c_{\mathtt{p},\circ} \leq c_{\mathtt{m,t}}$. Now, let μ_2 be the second Neumann eigenvalue of the negative Laplacian $-\Delta$, i.e.,

$$\frac{1}{c_{\mathtt{p}}^2} = \mu_2 = \inf_{0 \neq u \in \mathsf{H}^1 \cap \mathbb{R}^\perp} \frac{|\nabla u|^2}{|u|^2},$$

and let $u \in \mathsf{H}^1 \cap \mathbb{R}^\perp$ be an eigenfunction to μ_2. Note that u satisfies

$$\forall\, \varphi \in \mathsf{H}^1 \cap \mathbb{R}^\perp \quad \langle \nabla u, \nabla \varphi \rangle = \mu_2 \langle u, \varphi \rangle$$

and that this relation holds even for all $\varphi \in \mathsf{H}^1$. Then $0 \neq H := \nabla u \in \nabla \mathsf{H}^1 \cap \mathring{\mathsf{D}} = \mathsf{R}_0 \cap \mathring{\mathsf{D}}$ and satisfies $-\operatorname{div} H = -\operatorname{div} \nabla u = \mu_2 u$. By (1.4) and (1.2) we have

$$|H| \leq c_{\mathtt{m,n}} |\operatorname{div} H| = c_{\mathtt{m,n}} \mu_2 |u| \leq c_{\mathtt{m,n}} \mu_2 c_{\mathtt{p}} |\nabla u| = \frac{c_{\mathtt{m,n}}}{c_{\mathtt{p}}} |H|,$$

yielding $c_{\mathtt{p}} \leq c_{\mathtt{m,n}}$ and completing the proof. □

Remark 7

(i) *It is unclear but most probable that* $c_{\mathtt{p},\circ} < c_{\mathtt{m,t}} < c_{\mathtt{m,n}} = c_{\mathtt{p}}$ *holds. In forthcoming publications [7, 8] we will show more and sharper estimates on the Maxwell constants, showing additional and sharp relations between the Maxwell and the Poincaré/Friedrichs/Steklov constants.*

(ii) *Our results extend also to all polyhedra which allow the* H^1*-regularity of the Maxwell spaces* $\overset{\circ}{\mathsf{R}} \cap \mathsf{D}$ *and* $\mathsf{R} \cap \overset{\circ}{\mathsf{D}}$ *or to domains whose boundaries consist of combinations of convex boundary parts and polygonal parts which allow the* H^1*-regularity. Is is shown in [3, Theorem 4.1] that* (2.3)*, even* (2.4)*, still holds for all* $E \in \mathsf{H}^1 \cap \overset{\circ}{\mathsf{R}}$ *or* $E \in \mathsf{H}^1 \cap \overset{\circ}{\mathsf{D}}$ *if* Ω *is a polyhedron*†*. We note that even some non-convex polyhedra admit the* H^1*-regularity of the Maxwell spaces depending on the angle of the corners, which are not allowed to by too pointy.*

(iii) *Looking at the proof, the lower bounds* $c_{\mathtt{p},\circ} \leq c_{\mathtt{m},\mathtt{t}}$ *and* $c_{\mathtt{p}} \leq c_{\mathtt{m},\mathtt{n}}$ *remain true in more general situations, i.e., for bounded Lipschitz*‡ *domains* $\Omega \subset \mathbb{R}^3$*.*

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†The crucial point is that the unit normal is piecewise constant and hence the curvature is zero.

‡The Lipschitz assumption can also be weakened. It is sufficient that Ω admits the Maxwell compactness properties.

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