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An Elementary Extension of Korn's First Inequality to H(Curl) motivated by Gradient Plasticity with Plastic Spin

by

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A Canonical Extension of Korn's First Inequality to $\mathsf{H}(\mathrm{Curl})$ motivated by Gradient Plasticity with Plastic Spin

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Abstract

We prove a Korn-type inequality in $\overset{\circ}{\mathsf{H}}(\mathrm{Curl};\Omega,\mathbb{R}^{3\times 3})$ for tensor fields P mapping Ω to $\mathbb{R}^{3\times 3}$. More precisely, let $\Omega\subset\mathbb{R}^3$ be a bounded domain with connected Lipschitz boundary $\partial\Omega$. Then, there exists a constant $c>0$ such that

$$c\,\|P\|_{\mathsf{L}^2(\Omega,\mathbb{R}^{3\times 3})}\leq\|\mathrm{sym}\,P\|_{\mathsf{L}^2(\Omega,\mathbb{R}^{3\times 3})}+\|\mathrm{Curl}\,P\|_{\mathsf{L}^2(\Omega,\mathbb{R}^{3\times 3})}\tag{0.1}$$

holds for all tensor fields $P\in\overset{\circ}{\mathsf{H}}(\mathrm{Curl};\Omega,\mathbb{R}^{3\times 3})$, i.e., all $P\in\mathsf{H}(\mathrm{Curl};\Omega,\mathbb{R}^{3\times 3})$ with vanishing tangential trace on $\partial\Omega$. Here, rotation and tangential trace are defined row-wise. For compatible P, i.e., $P=\nabla v$ and thus $\mathrm{Curl}\,P=0$, where $v\in\mathsf{H}^1(\Omega,\mathbb{R}^3)$ are vector fields having components v_n, for which ∇v_n are normal at $\partial\Omega$, the presented estimate (0.1) reduces to a non-standard variant of Korn's first inequality, i.e.,

$$c\,\|\nabla v\|_{\mathsf{L}^2(\Omega,\mathbb{R}^{3\times 3})}\leq\|\mathrm{sym}\,\nabla v\|_{\mathsf{L}^2(\Omega,\mathbb{R}^{3\times 3})}\,.$$

On the other hand, for skew-symmetric P, i.e., $\mathrm{sym}\,P=0$, (0.1) reduces to a non-standard version of Poincaré's estimate. Therefore, since (0.1) admits the classical boundary conditions our result is a common generalization of the two classical estimates, namely Poincaré's resp. Korn's first inequality.

Key Words Korn's inequality, gradient plasticity, theory of Maxwell's equations, Helmholtz decomposition, Poincaré/Friedrichs type estimate

1 Introduction: Infinitesimal Gradient Plasticity

The motivation for our new estimate is a formulation of infinitesimal gradient plasticity [2]. Our model is taken from Neff et al. [9]. Let $\Omega\subset\mathbb{R}^3$ be a bounded domain. The goal is to find the displacement $u:[0,\infty)\times\Omega\mapsto\mathbb{R}^3$ and the possibly non-symmetric plastic distortion tensor $P:[0,\infty)\times\Omega\mapsto\mathbb{R}^{3\times 3}$, such that in $[0,\infty)\times\Omega$

$$\begin{aligned}\mathrm{Div}\,\sigma&=f, &\qquad \sigma&=2\mu\,\mathrm{sym}(\nabla u-P)+\lambda\,\mathrm{tr}(\nabla u-P)\,\mathrm{id},\\ \dot P&\in\Phi(\Sigma), & \Sigma&=\sigma-2\mu\,\mathrm{sym}\,P-\mu L_c^2\,\mathrm{Curl}\,\mathrm{Curl}\,P,\end{aligned}\tag{1.1}$$

hold. The system is completed by the boundary conditions

$$u(t,x)=0, \qquad \nu(x)\times P(t,x)=0 \qquad \forall\,(t,x)\in[0,\infty)\times\partial\Omega$$

and the initial condition $P(0,x)=0$ for all $x\in\Omega$. The underlying thermodynamic potential including the plastic gradients in form of the dislocation density tensor $\operatorname{Curl}P$ is

$$\int_\Omega \mu|\operatorname{sym}(\nabla u-P)|^2+\frac{\lambda}{2}|\operatorname{tr}(\nabla u-P)|^2-f\cdot u+\mu|\operatorname{sym}P|^2+\frac{\mu}{2}L_c^2|\operatorname{Curl}P|^2.$$

Here, μ,λ are the elastic Lamé moduli and σ is the symmetric Cauchy stress tensor. The system is driven by nonzero body forces denoted by f. The exterior normal to the boundary $\partial\Omega$ is denoted by ν and the plastic distortion P is required to satisfy row-wise the homogeneous tangential boundary condition which means that the boundary $\partial\Omega$ is a perfect conductor regarding the plastic distortion. *

Moreover, $\Phi:\mathbb{R}^{3\times3}\mapsto\mathbb{R}^{3\times3}$ is the monotone, multivalued flow-function with $\Phi(0)=0$ and $\Phi(\mathbb{R}^{3\times3}_{\mathrm{sym}})\subset\mathbb{R}^{3\times3}_{\mathrm{sym}}$. In general, Σ is not symmetric even if P is symmetric. Thus, the plastic inhomogeneity is responsible for the plastic spin (the possible non-symmetry of P). The mathematically suitable space for symmetric plastic distortion P is the classical space $\mathsf{H}(\mathrm{curl};\Omega)$ for each row of P [11, 2]. This case appears when choosing $\Phi:\mathbb{R}^{3\times3}\mapsto\mathbb{R}^{3\times3}_{\mathrm{sym}}$.

In the large scale limit $L_c\to0$ we recover a classical elasto-plasticity model with local kinematic hardening and symmetric plastic strain $\varepsilon_p:=\operatorname{sym}P$, since then $\dot P\in\mathbb{R}^{3\times3}_{\mathrm{sym}}$.

Uniqueness of classical solutions for rate-independent and rate-dependent formulations of this model is shown in [9]. The more difficult existence question for the rate-independent model in terms of a weak reformulation is addressed in [9]. First numerical results for a simplified rate-independent irrotational formulation (no plastic spin, i.e., symmetric plastic distortion P) are presented in [11], cf [17]. In [3] the model has been extended to rate-independent isotropic hardening based on the concept of a dissipation function defined in terms of the equivalent plastic strain. From a modeling point of view, it is strongly preferable to again have only the symmetric (rate) part of the plastic distortion P appear in the dissipation potential.

The existence and uniqueness can be settled by recasting the model as a variational inequality, if it is possible to define a bilinear form which is coercive with respect to appropriate spaces. This program has been achieved for other variants of the model in [3]. It had to remain basically open for the above system (1.1). In this case, the appropriate space for the plastic distortion P is the completion $\overset{\circ}{\mathsf{H}}_{\mathrm{sym}}(\mathrm{Curl};\Omega)$ of the linear space

$$\{P\in\mathsf{C}^\infty(\overline{\Omega},\mathbb{R}^{3\times3})\ :\ P_n \text{ normal at } \partial\Omega,\ n=1,2,3\}$$

with respect to the norm $|||\cdot|||$, where P_n are the columns of P^T and

$$|||P|||^2:=\|\operatorname{sym}P\|^2_{\mathsf{L}^2(\Omega)}+\|\operatorname{Curl}P\|^2_{\mathsf{L}^2(\Omega)}.$$

*This homogeneous tangential boundary condition on P is consistent with $\nu\times\nabla u=0$ on $\partial\Omega$ which follows from $u=0$ on $\partial\Omega$.

Despite first appearance, this quadratic form indeed defines a norm as shown in [9]. Thus $\mathring{\mathsf{H}}_{\mathrm{sym}}(\mathrm{Curl};\Omega)$ is a Hilbert-space. However, in this space it is not immediately obvious how to define a linear and bounded tangential trace operator. Since only $\|\operatorname{sym} P\|_{\mathsf{L}^2(\Omega)}$ appears, it is also not clear, how to control the skew-symmetric part of P. Therefore, the crucial embedding

$$\mathring{\mathsf{H}}_{\mathrm{sym}}(\mathrm{Curl};\Omega) \subset \mathsf{L}^2(\Omega)$$

is not clear as well. As a consequence of our main result of this paper we obtain that nevertheless

$$\mathring{\mathsf{H}}_{\mathrm{sym}}(\mathrm{Curl};\Omega) = \mathring{\mathsf{H}}(\mathrm{Curl};\Omega)$$

holds with equivalent norms in case the domain Ω is simply connected and has a Lipschitz boundary. The result of this paper has been announced in [10].

For the proof of our main result (0.1) we combine techniques from electro-magnetic and elastic theory, namely the Helmholtz decomposition, the Maxwell compactness property and Korn's inequality. Their basic variants are well known results which can be found in many books, e.g., in [6] and the literature cited there. More sophisticated and related versions are presented, e.g., in [12, 14, 15, 16, 21] for Maxwell's equations and [1, 8] for Korn's inequality.

This paper is organized as follows. After this motivation we introduce our notation, definitions and provide some background results. In section 3 we give the proof for our main estimates. In the last section 4 we establish a connection to a related result by Garroni et al. [4] for the two-dimensional case.

2 Definitions and Preliminaries

Let Ω be a bounded domain in $\mathbb{R}^3$ with connected Lipschitz continuous boundary $\Gamma := \partial\Omega$.

2.1 Functions and Vector Fields

The usual Lebesgue spaces of square integrable functions, vector or tensor fields on Ω with values in $\mathbb{R}$, $\mathbb{R}^3$ or $\mathbb{R}^{3\times 3}$, respectively, will be denoted by $\mathsf{L}^2(\Omega)$. Moreover, we introduce the standard Sobolev spaces

$$\begin{aligned}
\mathsf{H}(\mathrm{grad};\Omega) &= \{u \in \mathsf{L}^2(\Omega) \,:\, \operatorname{grad} u \in \mathsf{L}^2(\Omega)\}, & \|u\|^2_{\mathsf{H}(\mathrm{Grad};\Omega)} &:= \|u\|^2_{\mathsf{L}^2(\Omega)} + \|\operatorname{grad} u\|^2_{\mathsf{L}^2(\Omega)}\,,\\
\mathsf{H}(\mathrm{curl};\Omega) &= \{v \in \mathsf{L}^2(\Omega) \,:\, \operatorname{curl} v \in \mathsf{L}^2(\Omega)\}, & \|v\|^2_{\mathsf{H}(\mathrm{curl};\Omega)} &:= \|v\|^2_{\mathsf{L}^2(\Omega)} + \|\operatorname{curl} v\|^2_{\mathsf{L}^2(\Omega)}\,,\\
\mathsf{H}(\mathrm{div};\Omega) &= \{v \in \mathsf{L}^2(\Omega) \,:\, \operatorname{div} v \in \mathsf{L}^2(\Omega)\}, & \|v\|^2_{\mathsf{H}(\mathrm{div};\Omega)} &:= \|v\|^2_{\mathsf{L}^2(\Omega)} + \|\operatorname{div} v\|^2_{\mathsf{L}^2(\Omega)}\,.
\end{aligned}$$

$\mathsf{H}(\mathrm{grad};\Omega)$ is often denoted by $\mathsf{H}^1(\Omega)$. Furthermore, we define their closed subspaces $\mathring{\mathsf{H}}(\mathrm{grad};\Omega)$, $\mathring{\mathsf{H}}(\mathrm{curl};\Omega)$ as completition under the respective norms of the scalar resp. vector valued space $\mathring{\mathsf{C}}^\infty(\Omega)$ of compactly supported and smooth test functions resp. vector fields. In the latter Sobolev spaces the usual homogeneous scalar resp. tangential boundary conditions

$$u|_\Gamma = 0, \quad \nu \times v|_\Gamma = 0$$

are generalized, where ν denotes the outer unit normal at Γ. We note in passing that $\nu \times v|_\Gamma = 0$ is equivalent to $\tau \cdot v|_\Gamma = 0$ for all tangential directions τ at Γ, which means that v is normal at Γ. Furthermore, we need the spaces of irrotational or solenoidal vector fields

$$\begin{aligned} \mathsf{H}(\mathrm{curl}_0;\Omega) &:= \{v \in \mathsf{H}(\mathrm{curl};\Omega) \,:\, \mathrm{curl}\, v = 0\}, \\ \overset{\circ}{\mathsf{H}}(\mathrm{curl}_0;\Omega) &:= \{v \in \overset{\circ}{\mathsf{H}}(\mathrm{curl};\Omega) \,:\, \mathrm{curl}\, v = 0\}, \\ \mathsf{H}(\mathrm{div}_0;\Omega) &:= \{v \in \mathsf{H}(\mathrm{div};\Omega) \,:\, \mathrm{div}\, v = 0\}, \end{aligned}$$

where the index 0 indicates vanishing curl or div, respectively. All these spaces are Hilbert spaces. E.g., in classical terms we have $v \in \overset{\circ}{\mathsf{H}}(\mathrm{curl}_0;\Omega)$, if and only if

$$\mathrm{curl}\, v = 0, \quad \nu \times v|_\Gamma = 0.$$

For an introduction of these spaces see [6, p. 11-12, 148] or [5, p. 26]. The most important tool for our analysis is the compact embedding

$$\overset{\circ}{\mathsf{H}}(\mathrm{curl};\Omega) \cap \mathsf{H}(\mathrm{div};\Omega) \hookrightarrow \mathsf{L}^2(\Omega),$$

which is often referred as 'Maxwell compactness property', see [6, p. 158] and [21, 14, 19, 22, 16]. A first immediate consequence is that the space of so called 'harmonic Dirichlet fields'

$$\mathcal{H}(\Omega) := \overset{\circ}{\mathsf{H}}(\mathrm{curl}_0;\Omega) \cap \mathsf{H}(\mathrm{div}_0;\Omega)$$

is finite dimensional. A vector field v belonging to $\mathcal{H}(\Omega)$ means in classical terms that

$$\mathrm{curl}\, v = 0, \quad \mathrm{div}\, v = 0, \quad \nu \times v|_\Gamma = 0.$$

The dimension of $\mathcal{H}(\Omega)$ equals the second Betti number of Ω, see [6, p. 159] and [13, Theorem 1]. Since we assume the boundary Γ to be connected, there are no Dirichlet fields besides zero, i.e.,

$$\mathcal{H}(\Omega) = \{0\}.$$

This condition on the domain Ω resp. its boundary Γ is satisfied e.g. for a ball or a torus.

By a usual indirect argument we achieve another immediate consequence, see [6, p. 158, Theorem 8.9] or [5, Lemma 3.4]:

Lemma 1 (Maxwell Estimate for Vector Fields) *There exists a positive constant c_m, such that for all $v \in \overset{\circ}{\mathsf{H}}(\mathrm{curl};\Omega) \cap \mathsf{H}(\mathrm{div};\Omega)$*

$$\|v\|_{\mathsf{L}^2(\Omega)} \leq c_m \big(\|\mathrm{curl}\, v\|^2_{\mathsf{L}^2(\Omega)} + \|\mathrm{div}\, v\|^2_{\mathsf{L}^2(\Omega)} \big)^{1/2}.$$

By definition of the weak divergence, the projection theorem and Rellich's selection theorem [6, p. 14] we have from [6, p. 148, Theorem 8.3] or [20, Lemma 3.5], [7, Theorem 3.45]

Lemma 2 (Helmholtz Decomposition for Vector Fields) *We have the orthogonal decomposition*

$$\mathsf{L}^2(\Omega) = \mathrm{grad}\, \overset{\circ}{\mathsf{H}}(\mathrm{grad};\Omega) \oplus \mathsf{H}(\mathrm{div}_0;\Omega).$$

2.2 Tensor Fields

We extend our calculus to (3×3)-tensor (matrix) fields. For vector fields v with components in $\mathsf{H}(\mathrm{grad};\Omega)$ and tensor fields P with rows in $\mathsf{H}(\mathrm{curl};\Omega)$ resp. $\mathsf{H}(\mathrm{div};\Omega)$, i.e.,

$$v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}, \quad v_n \in \mathsf{H}(\mathrm{grad};\Omega), \quad P^T = [P_1\, P_2\, P_3], \quad P_n \in \mathsf{H}(\mathrm{curl};\Omega) \text{ resp. } \mathsf{H}(\mathrm{div};\Omega)$$

we define

$$\mathrm{Grad}\, v := \begin{bmatrix} \mathrm{grad}^T v_1 \\ \mathrm{grad}^T v_2 \\ \mathrm{grad}^T v_3 \end{bmatrix} = J_v = \nabla v, \quad \mathrm{Curl}\, P := \begin{bmatrix} \mathrm{curl}^T P_1 \\ \mathrm{curl}^T P_2 \\ \mathrm{curl}^T P_3 \end{bmatrix}, \quad \mathrm{Div}\, P := \begin{bmatrix} \mathrm{div}\, P_1 \\ \mathrm{div}\, P_2 \\ \mathrm{div}\, P_3 \end{bmatrix},$$

where J_v denotes the Jacobian of v and ^{T} the transpose. We note that v and $\mathrm{Div}\, P$ are vector fields, whereas P, $\mathrm{Curl}\, P$ and $\mathrm{Grad}\, v$ are tensor fields. The corresponding Sobolev spaces will be denoted by $\mathsf{H}(\mathrm{Grad};\Omega)$, $\overset{\circ}{\mathsf{H}}(\mathrm{Grad};\Omega)$, $\mathsf{H}(\mathrm{Curl};\Omega)$, $\overset{\circ}{\mathsf{H}}(\mathrm{Curl};\Omega)$, $\mathsf{H}(\mathrm{Curl}_0;\Omega)$, $\overset{\circ}{\mathsf{H}}(\mathrm{Curl}_0;\Omega)$, $\mathsf{H}(\mathrm{Div};\Omega)$, $\mathsf{H}(\mathrm{Div}_0;\Omega)$. As usual, we denote by $\mathrm{sym}\, P := 1/2(P + P^T)$ the symmetric part of a tensor P.

Let us now present our three crucial tools to prove the new estimate. First we have obvious consequences from Lemmas 1 and 2:

Corollary 3 (Maxwell Estimate for Tensor Fields) *For all* $P \in \overset{\circ}{\mathsf{H}}(\mathrm{Curl};\Omega) \cap \mathsf{H}(\mathrm{Div};\Omega)$

$$\|P\|_{\mathsf{L}^2(\Omega)} \le c_m \big(\|\mathrm{Curl}\, P\|^2_{\mathsf{L}^2(\Omega)} + \|\mathrm{Div}\, P\|^2_{\mathsf{L}^2(\Omega)} \big)^{1/2}.$$

Corollary 4 (Helmholtz Decomposition for Tensor Fields) *We have the orthogonal decomposition*

$$\mathsf{L}^2(\Omega) = \mathrm{Grad}\, \overset{\circ}{\mathsf{H}}(\mathrm{Grad};\Omega) \oplus \mathsf{H}(\mathrm{Div}_0;\Omega).$$

The third important tool is Korn's first inequality [6, p. 207] or [18, p. 54]:

Lemma 5 (Korn's First Inequality) *For all* $v \in \overset{\circ}{\mathsf{H}}(\mathrm{Grad};\Omega)$

$$\|\mathrm{Grad}\, v\|_{\mathsf{L}^2(\Omega)} \le \sqrt{2}\, \|\mathrm{sym}\, \mathrm{Grad}\, v\|_{\mathsf{L}^2(\Omega)}.$$

3 Main Results

For tensor fields $P \in \mathsf{H}(\mathrm{Curl};\Omega)$ we define the semi-norm

$$\|\!|P|\!\| := \big(\|\mathrm{sym}\, P\|^2_{\mathsf{L}^2(\Omega)} + \|\mathrm{Curl}\, P\|^2_{\mathsf{L}^2(\Omega)} \big)^{1/2}.$$

Lemma 6 *Let* $\hat{c} := \max\{2, \sqrt{5} c_m\}$. *Then, for all* $P \in \overset{\circ}{\mathsf{H}}(\mathrm{Curl};\Omega)$

$$\|P\|_{\mathsf{L}^2(\Omega)} \le \hat{c}\, \|\!|P|\!\|.$$

Proof Let $P \in \overset{\circ}{\mathsf{H}}(\mathrm{Curl};\Omega)$. According to Corollary 4 we orthogonally decompose

$$P = \operatorname{Grad} v + Q \in \operatorname{Grad} \overset{\circ}{\mathsf{H}}(\mathrm{Grad};\Omega) \oplus \mathsf{H}(\mathrm{Div}_0;\Omega).$$

Then, $\operatorname{Curl} P = \operatorname{Curl} Q$ and we observe $Q \in \overset{\circ}{\mathsf{H}}(\mathrm{Curl};\Omega) \cap \mathsf{H}(\mathrm{Div}_0;\Omega)$ since

$$\operatorname{Grad} \overset{\circ}{\mathsf{H}}(\mathrm{Grad};\Omega) \subset \overset{\circ}{\mathsf{H}}(\mathrm{Curl}_0;\Omega). \tag{3.1}$$

By Corollary 3 we have

$$\|Q\|_{\mathsf{L}^2(\Omega)} \leq c_m \|\operatorname{Curl} P\|_{\mathsf{L}^2(\Omega)}. \tag{3.2}$$

Then, by Lemma 5 and (3.2) we obtain easily

$$\begin{aligned}
\|P\|^2_{\mathsf{L}^2(\Omega)} &= \|\operatorname{Grad} v + Q\|^2_{\mathsf{L}^2(\Omega)} = \|\operatorname{Grad} v\|^2_{\mathsf{L}^2(\Omega)} + \|Q\|^2_{\mathsf{L}^2(\Omega)} \\
&\leq 2\,\|\operatorname{sym}\operatorname{Grad} v\|^2_{\mathsf{L}^2(\Omega)} + \|Q\|^2_{\mathsf{L}^2(\Omega)} = 2\,\|\operatorname{sym}(P-Q)\|^2_{\mathsf{L}^2(\Omega)} + \|Q\|^2_{\mathsf{L}^2(\Omega)} \\
&\leq 4\,\|\operatorname{sym} P\|^2_{\mathsf{L}^2(\Omega)} + 5\,\|Q\|^2_{\mathsf{L}^2(\Omega)} \leq 4\,\|\operatorname{sym} P\|^2_{\mathsf{L}^2(\Omega)} + 5c_m^2\,\|\operatorname{Curl} P\|^2_{\mathsf{L}^2(\Omega)}. \qquad \square
\end{aligned}$$

The immediate consequence is

Theorem 7 *On* $\overset{\circ}{\mathsf{H}}(\mathrm{Curl};\Omega)$ *the norms* $\|\cdot\|_{\mathsf{H}(\mathrm{Curl};\Omega)}$ *and* $|\!|\!|\cdot|\!|\!|$ *are equivalent. In particular,* $|\!|\!|\cdot|\!|\!|$ *is a norm on* $\overset{\circ}{\mathsf{H}}(\mathrm{Curl};\Omega)$ *and*

$$\exists\, c>0 \quad \forall\, P \in \overset{\circ}{\mathsf{H}}(\mathrm{Curl};\Omega) \qquad c\,\|P\|_{\mathsf{H}(\mathrm{Curl};\Omega)} \leq \|\operatorname{sym} P\|_{\mathsf{L}^2(\Omega)} + \|\operatorname{Curl} P\|_{\mathsf{L}^2(\Omega)}.$$

Setting $P := \operatorname{Grad} v$ we obtain by Lemma 6 and (3.1)

Remark 8 (Korn's First Inequality: Tangential-Variant) *For all* $v \in \overset{\circ}{\mathsf{H}}(\mathrm{Grad};\Omega)$

$$\|\operatorname{Grad} v\|_{\mathsf{L}^2(\Omega)} \leq \hat{c}\,\|\operatorname{sym}\operatorname{Grad} v\|_{\mathsf{L}^2(\Omega)}. \tag{3.3}$$

This is Korn's first inequality from Lemma 5 with a larger constant $\hat{c}$. *Since* Γ *is connected, i.e.,* $\mathcal{H}(\Omega) = \{0\}$, *we have* $\operatorname{Grad} \overset{\circ}{\mathsf{H}}(\mathrm{Grad};\Omega) = \overset{\circ}{\mathsf{H}}(\mathrm{Curl}_0;\Omega)$. *Thus,* (3.3) *holds for all* $v \in \mathsf{H}(\mathrm{Grad};\Omega)$ *with* $\operatorname{Grad} v \in \overset{\circ}{\mathsf{H}}(\mathrm{Curl}_0;\Omega)$, *i.e., with* $\operatorname{Grad} v_n$, $n = 1,2,3$, *normal at* Γ, *which then extends Lemma 5 through the (apparently) weaker boundary condition.*

4 Two-Dimensions: a Result of Garroni et al.

Let Ω be a bounded domain in $\mathbb{R}^2$ with connected Lipschitz continuous boundary Γ, which is equivalent (in $\mathbb{R}^2$) to the topological property that Ω is simply connected. For tensor fields $P : \Omega \mapsto \mathbb{R}^{2\times 2}$ we define analogously the Curl-operator by

$$\operatorname{Curl} P = \operatorname{Curl} \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} = \begin{bmatrix} \operatorname{curl}\,[P_{11}\;P_{12}]^T \\ \operatorname{curl}\,[P_{21}\;P_{22}]^T \end{bmatrix} = \begin{bmatrix} \partial_1 P_{12} - \partial_2 P_{11} \\ \partial_1 P_{22} - \partial_2 P_{21} \end{bmatrix},$$

where now curl denotes the two dimensional scalar rotation and Curl P is a vector. With the appropriate changes, Lemma 6 and Theorem 7 hold as well. In particular, there exists a positive constant c, such that

$$c\,\|P\|_{\mathsf{L}^2(\Omega)} \leq \|\operatorname{sym} P\|_{\mathsf{L}^2(\Omega)} + \|\operatorname{Curl} P\|_{\mathsf{L}^2(\Omega)}$$

holds for all $P \in \overset{\circ}{\mathsf{H}}(\mathrm{Curl};\Omega)$.

During the preparation of our paper we got aware that a two-dimensional related result may be inferred from Garroni et al. [4]. Instead of tangential boundary conditions $\nu \times P|_\Gamma = 0$ they impose the normalization condition

$$\int_\Omega \operatorname{skew} P = 0. \tag{4.1}$$

Let us define the total variation measure of the distribution Curl P for $P \in \mathsf{L}^1(\Omega)$ by

$$|\operatorname{Curl} P|_\Omega := \sup_{\substack{v \in \overset{\circ}{\mathsf{C}}{}^1(\Omega) \\ |v|_{\mathsf{L}^\infty(\Omega)} \leq 1}} \langle P, \operatorname{CoGrad} v\rangle_{\mathsf{L}^2(\Omega)}\,, \quad \operatorname{CoGrad} v := \begin{bmatrix} \partial_2 v_1 & -\partial_1 v_1 \\ \partial_2 v_2 & -\partial_1 v_2 \end{bmatrix}.$$

We note

$$\langle P, \operatorname{CoGrad} v\rangle_{\mathsf{L}^2(\Omega)} = \int_\Omega P_{11}\,\partial_2 v_1 - P_{12}\,\partial_1 v_1 + P_{21}\,\partial_2 v_2 - P_{22}\,\partial_1 v_2.$$

Using partial integration, i.e., $\langle P, \operatorname{CoGrad} v\rangle_{\mathsf{L}^2(\Omega)} = \langle \operatorname{Curl} P, v\rangle_{\mathsf{L}^2(\Omega)}$ for $v \in \overset{\circ}{\mathsf{C}}{}^1(\Omega)$, it is easy to see that $|\operatorname{Curl} P|_\Omega = \|\operatorname{Curl} P\|_{\mathsf{L}^1(\Omega)}$ if $\operatorname{Curl} P \in \mathsf{L}^1(\Omega)$. In [4, Theorem 9] it is shown that for Ω having a Lipschitz boundary and a special 'slicing' property, there exists a constant $c > 0$, such that

$$c\,\|P\|_{\mathsf{L}^2(\Omega)} \leq \|\operatorname{sym} P\|_{\mathsf{L}^2(\Omega)} + |\operatorname{Curl} P|_\Omega$$

holds for all $P \in \mathsf{L}^1(\Omega)$ with (4.1). Their proof uses essentially that in $\mathbb{R}^2$ the operators curl and div can be exchanged by the simple transformation, i.e., $\operatorname{curl}\,[v_1, v_2]^T = \operatorname{div}\,[-v_2, v_1]^T$. Thus, such a strong result may not be true in higher space dimensions $N \geq 3$ and it is open whether the normalization condition (4.1) can be exchanged with the more natural tangential boundary conditions.

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