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On an Extension of Korn's First Inequality to Incompatible Tensor Fields on Domains of Arbitrary Dimensions

by

Patrizio Neff, Dirk Pauly and Karl-Josef Witsch

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Abstract

For a bounded domain Ω in $\mathbb{R}^N$ with Lipschitz boundary $\Gamma = \partial\Omega$ and a relatively open and non-empty subset Γ_t of Γ, we prove the existence of a positive constant c such that inequality

$$c\,\|T\|_{\mathsf{L}^2(\Omega,\mathbb{R}^{N\times N})} \leq \|\operatorname{sym} T\|_{\mathsf{L}^2(\Omega,\mathbb{R}^{N\times N})} + \|\operatorname{Curl} T\|_{\mathsf{L}^2(\Omega,\mathbb{R}^{N\times N(N-1)/2})} \tag{0.1}$$

holds for all tensor fields $T \in \overset{\circ}{\mathsf{H}}(\operatorname{Curl};\Gamma_t,\Omega,\mathbb{R}^{N\times N})$, this is, for all square-integrable tensor fields $T:\Omega\to\mathbb{R}^{N\times N}$ having a row-wise square-integrable rotation tensor field $\operatorname{Curl} T:\Omega\to\mathbb{R}^{N\times N(N-1)/2}$ and vanishing row-wise tangential trace on Γ_t.

For compatible tensor fields $T = \nabla v$ with $v \in \mathsf{H}^1(\Omega,\mathbb{R}^N)$ having vanishing tangential Neumann trace on Γ_t the inequality (0.1) reduces to a non-standard variant of Korn's first inequality since $\operatorname{Curl} T = 0$, while for skew-symmetric tensor fields T Poincaré's inequality is recovered.

If $\Gamma_t = \emptyset$, our estimate (0.1) still holds at least for simply connected Ω and for all tensor fields $T \in \mathsf{H}(\operatorname{Curl};\Omega,\mathbb{R}^{N\times N})$ which are $\mathsf{L}^2(\Omega,\mathbb{R}^{N\times N})$-perpendicular to $\mathfrak{so}(N)$, i.e., to all skew-symmetric constant tensors.

Key Words Korn's inequality, Poincaré's inequality, Maxwell's equations, Helmholtz' decomposition, gradient plasticity, incompatible tensor fields, differential forms, mixed boundary conditions

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1 Introduction and Main Results

We extend the Korn-type inequalities from [16] presented earlier in less general settings in [13, 12, 15, 14] to the N-dimensional case. For this, let $N \in \mathbb{N}$ and Ω be a bounded domain in $\mathbb{R}^N$ as well as Γ_t be an open subset of its boundary $\Gamma := \partial\Omega$. Our main result reads:

Theorem 1 (Main Theorem) *Let the pair (Ω, Γ_t) be admissible*[*]. *There exist constants $0 < c_1 \le c_2$ such that the following estimates hold:*

(i) *If $\Gamma_t \neq \emptyset$, then the inequality*

$$\|T\|_{\mathsf{L}^2(\Omega)} \le c_1 \big(\|\operatorname{sym} T\|_{\mathsf{L}^2(\Omega)} + \|\operatorname{Curl} T\|_{\mathsf{L}^2(\Omega)} \big)$$

holds for all tensor fields $T \in \overset{\circ}{\mathsf{H}}(\mathrm{Curl}; \Gamma_t, \Omega)$. In other words, on $\overset{\circ}{\mathsf{H}}(\mathrm{Curl}; \Gamma_t, \Omega)$ the right hand side defines a norm equivalent to the standard norm in $\mathsf{H}(\mathrm{Curl}; \Omega)$.

(ii) *If $\Gamma_t = \emptyset$, then for all tensor fields $T \in \mathsf{H}(\mathrm{Curl}; \Omega)$*[†] *there exists a piece-wise constant skew-symmetric tensor field A, such that*

$$\|T - A\|_{\mathsf{L}^2(\Omega)} \le c_2 \big(\|\operatorname{sym} T\|_{\mathsf{L}^2(\Omega)} + \|\operatorname{Curl} T\|_{\mathsf{L}^2(\Omega)} \big).$$

Note that, in general $A \notin \mathsf{H}(\mathrm{Curl}; \Omega)$.

(ii') *If $\Gamma_t = \emptyset$ and Ω is additionally simply connected, then for all tensor fields T in $\mathsf{H}(\mathrm{Curl}; \Omega)$ there exists a uniquely determined constant skew-symmetric tensor field $A = A_T \in \mathfrak{so}(N)$*[‡], *such that*

$$\|T - A_T\|_{\mathsf{L}^2(\Omega)} \le c_1 \big(\|\operatorname{sym} T\|_{\mathsf{L}^2(\Omega)} + \|\operatorname{Curl} T\|_{\mathsf{L}^2(\Omega)} \big).$$

Since $A_T \in \mathsf{H}(\mathrm{Curl}_0; \Omega)$ one can easily estimate $\|T - A_T\|_{\mathsf{H}(\mathrm{Curl};\Omega)}$ as well. Moreover, $T - A_T \in \mathsf{H}(\mathrm{Curl}; \Omega) \cap \mathfrak{so}(N)^\perp$ and $A_T = 0$ if and only if $T \perp \mathfrak{so}(N)$. Thus, the inequality in (i) holds for all $T \in \mathsf{H}(\mathrm{Curl}; \Omega) \cap \mathfrak{so}(N)^\perp$ as well. Therefore, also on $\mathsf{H}(\mathrm{Curl}; \Omega) \cap \mathfrak{so}(N)^\perp$ the right hand side defines a norm equivalent to the standard norm in $\mathsf{H}(\mathrm{Curl}; \Omega)$.

[*] The precise meaning of 'admissible' will given in Definition 25.

[†] If $\Gamma_t = \emptyset$, then $\overset{\circ}{\mathsf{H}}(\mathrm{Curl}; \Gamma_t, \Omega) = \mathsf{H}(\mathrm{Curl}; \Omega)$.

[‡] $\mathfrak{so}(N)$ denotes the set of all constant skew-symmetric tensors, i.e., $(N \times N)$-matrices.

Remark 2

- **(i)** *Here, the differential operator* Curl *denotes the row-wise application of the standard* curl *in* $\mathbb{R}^N$ *and a tensor field* T *belongs to the Hilbert Sobolev-type space* $\overset{\circ}{\mathsf{H}}(\mathrm{Curl};\Gamma_t,\Omega)$ *if* T *and its distributional* $\mathrm{Curl}\,T$ *belong both to the standard Lebesgue spaces* $\mathsf{L}^2(\Omega)$ *and the row-wise weak tangential trace of* T *vanishes at the boundary part* Γ_t. *Exact definitions of all spaces and operators used will be given in section 2.*
- **(ii)** *In (ii') the special constant skew-symmetric tensor field* A_T *is given explicitly by* $A_T = \pi_{\mathfrak{so}(N)}T \in \mathfrak{so}(N)$, *where* $\pi_{\mathfrak{so}(N)} : \mathsf{L}^2(\Omega) \to \mathfrak{so}(N)$ *denotes the* $\mathsf{L}^2(\Omega)$*-orthogonal projection onto* $\mathfrak{so}(N)$ *and can be represented by*

$$\pi_{\mathfrak{so}(N)}T = \operatorname{skew} \fint_\Omega T\,d\lambda \in \mathfrak{so}(N).$$

Furthermore, A_T *can also be computed by*

$$A_T = A_R := \pi_{\mathfrak{so}(N)}R = \operatorname{skew} \fint_\Omega R\,d\lambda \in \mathfrak{so}(N),$$

where R *denotes the Helmholtz projection of* T *onto* $\mathsf{H}(\mathrm{Curl}_0;\Omega)$ *according to Corollary 18.*

- **(iii)** *The constants* c_1 *and* c_2 *are given by* (3.3) *and* (3.4) *and these depend in a simply algebraic way only on the constants* $c_{\mathtt{k}}, c_{\mathtt{m}}$ *in Korn's first and the Maxwell inequality.*

For the proof of Theorem 1 we follow in close lines the proofs from [16]. Therefore, again we need to combine three crucial tools, namely

- a Maxwell estimate, Corollary 17;
- a Helmholtz decomposition, Corollary 18;
- a generalized version of Korn's first inequality, Lemma 29.

Our assumptions on the domain Ω and the part of the boundary Γ_t, i.e., on the pair (Ω,Γ_t), are precisely made for this three major tools to hold. We will present these assumptions in section 2 and a pair (Ω,Γ_t) satisfying those will be called admissible.

Theorem 1 can be looked at as a common generalization and formulation of two well known and very important classical inequalities, namely **Korn's first** and **Poincaré's inequality**. This is, taking irrotational tensor fields T, i.e., $\mathrm{Curl}\,T = 0$, then a non-standard version of Korn's first inequality

$$c\,\|T - A_T\|_{\mathsf{L}^2(\Omega)} \le \|\operatorname{sym} T\|_{\mathsf{L}^2(\Omega)}$$

holds for all $T \in \overset{\circ}{\mathsf{H}}(\mathrm{Curl}_0;\Gamma_t,\Omega)$, where $A_T = 0$ if $\Gamma_t \neq \emptyset$. Another, less general choice, is $T = \nabla v$ yielding

$$c\,\|\nabla v - A_{\nabla v}\|_{\mathsf{L}^2(\Omega)} \le \|\operatorname{sym}\nabla v\|_{\mathsf{L}^2(\Omega)}$$

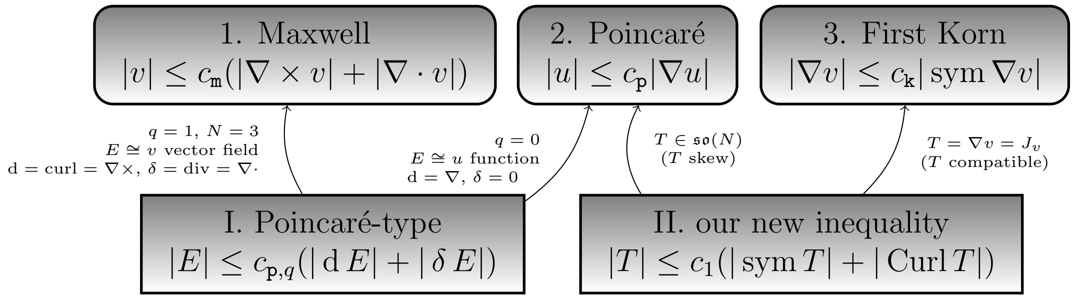

Figure 1: The three fundamental inequalities are implied by two. For the constants we have $c_{\mathtt{p}} = c_{\mathtt{p},0}$, $c_{\mathtt{m}} = c_{\mathtt{p},1}$ and $c_{\mathtt{k}}, c_{\mathtt{p}} \leq c_1$.

with some vector field v belonging to $\overset{\circ}{\mathsf{H}}{}^1(\Gamma_t;\Omega)$ or just to $\mathsf{H}^1(\Omega)$ with ∇v_n, $n = 1,\dots,N$, normal at Γ_t. Note that

$$\nabla\overset{\circ}{\mathsf{H}}{}^1(\Gamma_t;\Omega),\ \nabla\{v \in \mathsf{H}^1(\Omega)\ :\ \nabla v_n \text{ normal at } \Gamma_t\,\forall\, n = 1,\dots,N\} \subset \overset{\circ}{\mathsf{H}}(\operatorname{Curl}_0;\Gamma_t,\Omega).$$

On the other hand, taking a skew-symmetric tensor field T, i.e., $\operatorname{sym} T = 0$, then Poincaré's inequality in disguise

$$c\,\|T - A_T\|_{\mathsf{L}^2(\Omega)} \leq \|\operatorname{Curl} T\|_{\mathsf{L}^2(\Omega)}$$

appears, where again $A_T = 0$ if $\Gamma_t \neq \emptyset$. We note that since T can be identified with a vector field v and the Curl is as good as the gradient ∇ on v we have

$$c\,\|v - c_v\|_{\mathsf{L}^2(\Omega)} \leq \|\nabla v\|_{\mathsf{L}^2(\Omega)}$$

with $c_v \in \mathbb{R}^N$ and $c_v = 0$ if $\Gamma_t \neq \emptyset$. These connections between Korn's first and Poincaré's inequalities and also to the Maxwell inequalities and the more general Poincaré-type inequalities are illustrated in Figure 1.

2 Definitions and Preliminaries

As mentioned before, let generally $N \in \mathbb{N}$ and Ω be a bounded domain in $\mathbb{R}^N$ as well as Γ_t be an open subset of the boundary $\Gamma = \partial\Omega$. We will use the notations from our earlier papers [14] and [13, 12, 15, 16].

2.1 Differential Forms

In particular, we denote the Lebesgue spaces of square-integrable q-forms[§] by $\mathsf{L}^{2,q}(\Omega)$. Moreover, we have the standard Sobolev-type spaces for the exterior derivative d and

[§] alternating differential forms of rank $q \in \{0,\dots,N\}$

co-derivative $\delta := (-1)^{(q-1)N} * \mathrm{d} *$ (acting on q-forms)

$$\mathsf{D}^q(\Omega) := \{E \in \mathsf{L}^{2,q}(\Omega) \,:\, \mathrm{d}\,E \in \mathsf{L}^{2,q+1}(\Omega)\},$$
$$\Delta^q(\Omega) := \{H \in \mathsf{L}^{2,q}(\Omega) \,:\, \delta\,H \in \mathsf{L}^{2,q-1}(\Omega)\},$$

where as usual $*$ denotes Hodge's star isomorphism. $\mathring{\mathsf{C}}^{\infty,q}(\Omega)$ is the space of smooth and compactly supported q-forms on Ω, often called test space. Due to the more complex geometry and topology of the domain Ω and its boundary parts Γ, Γ_t we need some more test spaces

$$\mathring{\mathsf{C}}^{\infty,q}(\Gamma_t, \Omega) := \{E \in \mathsf{C}^{\infty,q}(\overline{\Omega}) \,:\, \mathrm{dist}(\mathrm{supp}\,E, \Gamma_t) > 0\},$$
$$\mathsf{C}^{\infty,q}(\overline{\Omega}) := \{E|_\Omega \,:\, E \in \mathring{\mathsf{C}}^{\infty,q}(\mathbb{R}^N)\}.$$

Then, we define

$$\mathring{\mathsf{D}}^q(\Gamma_t, \Omega) := \overline{\mathring{\mathsf{C}}^{\infty,q}(\Gamma_t, \Omega)}$$

taking the closure in $\mathsf{D}^q(\Omega)$ and note that a q-form in $\mathring{\mathsf{D}}^q(\Gamma_t, \Omega)$ has generalized vanishing tangential trace¶ on Γ_t. If $\Gamma_t = \Gamma$ we can identify $\mathring{\mathsf{C}}^{\infty,q}(\Gamma_t, \Omega)$ with $\mathring{\mathsf{C}}^{\infty,q}(\Omega)$ and write

$$\mathring{\mathsf{D}}^q(\Gamma_t, \Omega) = \overline{\mathring{\mathsf{C}}^{\infty,q}(\Gamma_t, \Omega)} = \overline{\mathring{\mathsf{C}}^{\infty,q}(\Omega)} =: \mathring{\mathsf{D}}^q(\Omega).$$

If $\Gamma_t = \emptyset$ we have $\mathring{\mathsf{C}}^{\infty,q}(\Gamma_t, \Omega) = \mathsf{C}^{\infty,q}(\overline{\Omega})$ and thus

$$\mathring{\mathsf{D}}^q(\Gamma_t, \Omega) = \overline{\mathring{\mathsf{C}}^{\infty,q}(\Gamma_t, \Omega)} = \overline{\mathsf{C}^{\infty,q}(\overline{\Omega})} \subset \mathsf{D}^q(\Omega).$$

Equality in the last relation means the density result $\overline{\mathsf{C}^{\infty,q}(\overline{\Omega})} = \mathsf{D}^q(\Omega)$, which holds, e.g., if Ω has the segment property‖. The latter is valid, e.g., for domains with Lipschitz boundary. An index 0 at the lower right corner indicates vanishing derivatives, e.g.,

$$\mathring{\mathsf{D}}^q_0(\Gamma_t, \Omega) := \{E \in \mathring{\mathsf{D}}^q(\Gamma_t, \Omega) \,:\, \mathrm{d}\,E = 0\}.$$

Analogously, we introduce the corresponding Sobolev-type spaces for the co-derivative δ which are usually assigned to the boundary complement $\Gamma_n := \Gamma \setminus \overline{\Gamma_t}$ of Γ_t. We have, e.g.,

$$\Delta^q_0(\Omega) = \{H \in \Delta^q(\Omega) \,:\, \delta\,H = 0\}, \quad \mathring{\Delta}^q(\Gamma_n, \Omega), \quad \mathring{\Delta}^q_0(\Gamma_n, \Omega),$$

where in the latter spaces a vanishing normal trace on Γ_n is generalized. Moreover, we define the spaces of so called 'harmonic Dirichlet-Neumann forms'

$$\mathcal{H}^q(\Omega) := \mathring{\mathsf{D}}^q_0(\Gamma_t, \Omega) \cap \mathring{\Delta}^q_0(\Gamma_n, \Omega). \tag{2.1}$$

¶This can be seen easily by Stokes' theorem.

‖See, e.g., [1, 33, 7].

We note that in classical terms a harmonic Dirichlet-Neumann q-form E satisfies

$$\mathrm{d}\,E = 0, \quad \delta\,E = 0, \quad \iota^* E|_{\Gamma_t} = 0, \quad \iota^* * E|_{\Gamma_n} = 0,$$

where ι^* denotes the pullback of the canonical embedding $\iota : \Gamma \hookrightarrow \overline{\Omega}$ and the restrictions to Γ_t and Γ_n should be understood as pullbacks as well. Equipped with their natural graph norms all these spaces are Hilbert spaces.

Now, we can begin to introduce our regularity assumptions on the boundary Γ and the interface $\gamma := \overline{\Gamma_t} \cap \overline{\Gamma_n}$. We start with the following:

Definition 3 *The pair (Ω, Γ_t) has the 'Maxwell compactness property'* (MCP)*, if for all q the embeddings*

$$\overset{\circ}{\mathsf{D}}{}^q(\Gamma_t, \Omega) \cap \overset{\circ}{\Delta}{}^q(\Gamma_n, \Omega) \hookrightarrow \mathsf{L}^2(\Omega)$$

are compact.

Remark 4

(i) *There exists a substantial amount of literature and different proofs for the* MCP*. See for example the papers and books of Costabel, Kuhn, Leis, Pauly, Picard, Saranen, Weber, Weck, Witsch [3, 8, 9, 10, 11, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 28, 29, 30, 31, 32]. All these papers are concerned with the special cases $\Gamma_t = \Gamma$ resp. $\Gamma_t = \emptyset$. For the case $N = 3$, $q = 1$, i.e., $\Omega \subset \mathbb{R}^3$, we refer to [3, 9, 10, 11, 22, 24, 26, 28, 29, 30, 32], whereas for the general case, i.e., $\Omega \subset \mathbb{R}^N$ or even Ω a Riemannian manifold, we correspond to [8, 17, 18, 19, 20, 21, 23, 25, 31]. We note that even weaker regularity of Γ than Lipschitz is sufficient for the* MCP *to hold. The first proof of the* MCP *for non-smooth domains and even for smooth Riemannian manifolds with non-smooth boundaries (cone property) was given in 1974 by Weck in [31]. To the best of our knowledge, the strongest result so far can be found in [26]. See also our discussion in [16]. An interesting proof has been given by Costabel in [3]. He made the detour of showing more fractional Sobolev regularity for the vector fields. More precisely, he was able to prove that for Lipschitz domains $\Omega \subset \mathbb{R}^3$ and $q = 1$ the embedding*

$$\overset{\circ}{\mathsf{D}}{}^q(\Omega) \cap \Delta^q(\Omega) \hookrightarrow \mathsf{H}^{1/2}(\Omega) \tag{2.2}$$

is continuous. Then, for all $0 \leq k < 1/2$ the embeddings

$$\overset{\circ}{\mathsf{D}}{}^q(\Omega) \cap \Delta^q(\Omega) \hookrightarrow \mathsf{H}^k(\Omega)$$

are compact, especially for $k = 0$, where $\mathsf{H}^k(\Omega) = \mathsf{L}^2(\Omega)$ holds.

(ii) *For the general case $\emptyset \subset \Gamma_t \subset \Gamma$ with possibly $\emptyset \subsetneq \Gamma_t \subsetneq \Gamma$, Jochmann gave a proof for the* MCP *in [6], where he considered the special case of a bounded domain $\Omega \subset \mathbb{R}^3$. He can admit Ω to be Lipschitz and γ to be a Lipschitz interface. Generalizing the ideas of Weck in [31], Kuhn showed in his dissertation [7] that the* MCP *holds for* *smooth domains $\Omega \subset \mathbb{R}^N$ or even for smooth Riemannian manifolds Ω with smooth boundary and admissible interface γ. See also our discussion in [16].*

A result, which meets our needs, has been proved quite recently by M. Mitrea and his collaborators. More precisely, we will use results by Gol'dshtein and Mitrea (I. & M.) from [4]. In the language of this paper we assume Ω to be a weakly Lipschitz domain, this is, Ω is a Lipschitz manifold with boundary, see [4, Definition 3.6], and $\Gamma_t \subset \Gamma$ to be an admissible patch (yielding γ to be an admissible path), i.e., Γ_t is a Lipschitz submanifold with boundary, see [4, Definition 3.7]. Roughly speaking, Ω and Γ_t are defined by Lipschitz functions. Here, the main point in proving the MCP, *i.e., [4, Proposition 4.4, (4.21)], is that then Ω is locally Lipschitz diffeomorphic to a 'creased domain' in $\mathbb{R}^N$, first introduced by Brown in [2]. See [4, Section 3.6] for more details and to find the statement 'Informally speaking, the pieces in which the boundary is partitioned are admissible patches which meet at an angle $< \pi$. In particular, creased domains are inherently non-smooth'. Whereas in [4] everything is defined in the more general framework of manifolds, in [5] the* MCP *is proved by Jakab and Mitrea (I. & M.) for creased domains $\Omega \subset \mathbb{R}^N$. By the Lipschitz diffeomorphisms, the* MCP *holds then for general manifolds/domains Ω as well. In [5] the authors follow and generalize the idea* (2.2) *of Costabel from [3]. Particularly, in [5, (1.2), Theorem 1.1, (1.9)] the following regularity result has been proved: For all q the embeddings*

$$\mathring{\mathsf{D}}^q(\Gamma_t,\Omega) \cap \mathring{\Delta}^q(\Gamma_n,\Omega) \hookrightarrow \mathsf{H}^{1/2}(\Omega)$$

are continuous. Therefore, as before, for all q and for all $0 \le k < 1/2$ the embeddings

$$\mathring{\mathsf{D}}^q(\Gamma_t,\Omega) \cap \mathring{\Delta}^q(\Gamma_n,\Omega) \hookrightarrow \mathsf{H}^{k}(\Omega)$$

are compact, giving the MCP *for $k = 0$.*

By [4, Proposition 4.4, (4.21)] and the latter remark we have:

Theorem 5 *Let Ω be a weakly Lipschitz domain and Γ_t be an admissible patch, i.e., let Ω be a (weakly) Lipschitz domain and Γ_t be an Lipschitz patch of Γ. Then the pair (Ω, Γ_t) has the* MCP.

Corollary 6 *Let the pair (Ω, Γ_t) have the* MCP. *Then, for all q the spaces of harmonic Dirichlet-Neumann forms $\mathcal{H}^q(\Omega)$ are finite dimensional.*

Proof The MCP implies immediately that the unit ball in $\mathcal{H}^q(\Omega)$ is compact. □

For details about the particular dimensions see [22] or [4]. We note that the dimensions of $\mathcal{H}^q(\Omega)$ depend only on topological properties of the pair (Ω, Γ_t).

Lemma 7 (Poincaré-type Estimate for Differential Forms) *Let the pair (Ω, Γ_t) have the* MCP. *Then, for all q there exist positive constants $c_{\mathsf{p},q}$, such that*

$$\|E\|_{\mathsf{L}^{2,q}(\Omega)} \le c_{\mathsf{p},q}\big(\,\|\mathrm{d}\,E\|^2_{\mathsf{L}^{2,q+1}(\Omega)} + \|\delta\,E\|^2_{\mathsf{L}^{2,q-1}(\Omega)}\,\big)^{1/2}$$

holds for all $E \in \overset{\circ}{\mathsf{D}}{}^q(\Gamma_t,\Omega) \cap \overset{\circ}{\Delta}{}^q(\Gamma_n,\Omega) \cap \mathcal{H}^q(\Omega)^\perp$. *Moreover,*

$$\|(\mathrm{id}-\pi_q)E\|_{\mathsf{L}^{2,q}(\Omega)} \leq c_{\mathsf{p},q}\big(\,\|\mathrm{d}\,E\|^2_{\mathsf{L}^{2,q+1}(\Omega)} + \|\delta\,E\|^2_{\mathsf{L}^{2,q-1}(\Omega)}\,\big)^{1/2}$$

holds for all $E \in \overset{\circ}{\mathsf{D}}{}^q(\Gamma_t,\Omega) \cap \overset{\circ}{\Delta}{}^q(\Gamma_n,\Omega)$*, where* $\pi_q : \mathsf{L}^{2,q}(\Omega) \to \mathcal{H}^q(\Omega)$ *denotes the* $\mathsf{L}^{2,q}(\Omega)$*-orthogonal projection onto the Dirichlet-Neumann forms* $\mathcal{H}^q(\Omega)$*.*

Here and throughout the paper, $\perp$ denotes orthogonality in $\mathsf{L}^{2,q}(\Omega)$.

Proof A standard indirect argument utilizing the MCP yields the desired estimates. $\square$

By Stokes' theorem and approximation always

$$\overset{\circ}{\mathsf{D}}{}^q_0(\Gamma_t,\Omega) \subset (\delta\,\overset{\circ}{\Delta}{}^{q+1}(\Gamma_n,\Omega))^\perp, \quad \overset{\circ}{\Delta}{}^q_0(\Gamma_n,\Omega) \subset (\mathrm{d}\,\overset{\circ}{\mathsf{D}}{}^{q-1}(\Gamma_t,\Omega))^\perp$$

hold. Equality in the latter relations is not clear and needs another assumption on the pair (Ω,Γ_t).

Definition 8 *The pair* (Ω,Γ_t) *has the 'Maxwell approximation property'* (MAP)*, if for all* q

$$\overset{\circ}{\mathsf{D}}{}^q_0(\Gamma_t,\Omega) = (\delta\,\overset{\circ}{\Delta}{}^{q+1}(\Gamma_n,\Omega))^\perp, \quad \overset{\circ}{\Delta}{}^q_0(\Gamma_n,\Omega) = (\mathrm{d}\,\overset{\circ}{\mathsf{D}}{}^{q-1}(\Gamma_t,\Omega))^\perp.$$

Remark 9 *By* $*$*-duality the pair* (Ω,Γ_t) *has the* MAP*, if and only if the pair* (Ω,Γ_n) *has the* MAP*, i.e., if and only if for all* q

$$\overset{\circ}{\mathsf{D}}{}^q_0(\Gamma_n,\Omega) = (\delta\,\overset{\circ}{\Delta}{}^{q+1}(\Gamma_t,\Omega))^\perp, \quad \overset{\circ}{\Delta}{}^q_0(\Gamma_t,\Omega) = (\mathrm{d}\,\overset{\circ}{\mathsf{D}}{}^{q-1}(\Gamma_n,\Omega))^\perp.$$

Remark 10 *If* $\Gamma_t = \Gamma$ *or* $\Gamma_t = \emptyset$*, the* MAP *is simply given by the projection theorem in Hilbert spaces and by the definitions of the respective Sobolev spaces. For the general case* $\emptyset \subset \Gamma_t \subset \Gamma$ *with possibly* $\emptyset \subsetneq \Gamma_t \subsetneq \Gamma$*, Jochmann proved the* MAP *in [6] considering the special case of a bounded domain* $\Omega \subset \mathbb{R}^3$*. As in Remark 4 he needs* Ω *to be Lipschitz and* γ *to be a Lipschitz interface. Kuhn showed the* MAP *in [7] for smooth domains* $\Omega \subset \mathbb{R}^N$ *or even for smooth Riemannian manifolds* Ω *with smooth boundary and admissible interface* γ*. Again, a sufficient result for us has been given recently by Gol'dshtein and Mitrea (I. & M.) in [4, Theorem 4.3, (4.16)]. Like in Remark 4, for this* Ω *has to be a weakly Lipschitz domain and* $\Gamma_t \subset \Gamma$ *to be an admissible patch.*

By [4, Theorem 4.3, (4.16)] and the latter remark we have:

Theorem 11 *Let* Ω *be a weakly Lipschitz domain and* Γ_t *be an admissible patch, i.e., let* Ω *be a (weakly) Lipschitz domain and* Γ_t *be an Lipschitz patch of* Γ*. Then the pair* (Ω,Γ_t) *has the* MAP*.*

Lemma 12 (Hodge-Helmholtz Decomposition for Differential Forms) *Let the pair* (Ω,Γ_t) *have the* MAP*. Then, the orthogonal decompositions*

$$\begin{aligned}\mathsf{L}^{2,q}(\Omega)&=\overline{\mathrm{d}\,\mathring{\mathsf{D}}^{q-1}(\Gamma_t,\Omega)}\oplus\mathring{\Delta}^q_0(\Gamma_n,\Omega)\\&=\mathring{\mathsf{D}}^q_0(\Gamma_t,\Omega)\oplus\overline{\delta\,\mathring{\Delta}^{q+1}(\Gamma_n,\Omega)}\\&=\overline{\mathrm{d}\,\mathring{\mathsf{D}}^{q-1}(\Gamma_t,\Omega)}\oplus\mathcal{H}^q(\Omega)\oplus\overline{\delta\,\mathring{\Delta}^{q+1}(\Gamma_n,\Omega)}\end{aligned}$$

hold. If the pair (Ω,Γ_t) *has additionally the* MCP*, then*

$$\begin{aligned}\mathrm{d}\,\mathring{\mathsf{D}}^{q-1}(\Gamma_t,\Omega)&=\mathrm{d}\left(\mathring{\mathsf{D}}^{q-1}(\Gamma_t,\Omega)\cap\delta\,\mathring{\Delta}^q(\Gamma_n,\Omega)\right)=\mathring{\mathsf{D}}^q_0(\Gamma_t,\Omega)\cap\mathcal{H}^q(\Omega)^\perp,\\\delta\,\mathring{\Delta}^{q+1}(\Gamma_n,\Omega)&=\delta\left(\mathring{\Delta}^{q+1}(\Gamma_n,\Omega)\cap\mathrm{d}\,\mathring{\mathsf{D}}^q(\Gamma_t,\Omega)\right)=\mathring{\Delta}^q_0(\Gamma_n,\Omega)\cap\mathcal{H}^q(\Omega)^\perp\end{aligned}$$

and these are closed subspaces of $\mathsf{L}^{2,q}(\Omega)$*. Moreover, then the orthogonal decompositions*

$$\begin{aligned}\mathsf{L}^{2,q}(\Omega)&=\mathrm{d}\,\mathring{\mathsf{D}}^{q-1}(\Gamma_t,\Omega)\oplus\mathring{\Delta}^q_0(\Gamma_n,\Omega)\\&=\mathring{\mathsf{D}}^q_0(\Gamma_t,\Omega)\oplus\delta\,\mathring{\Delta}^{q+1}(\Gamma_n,\Omega)\\&=\mathrm{d}\,\mathring{\mathsf{D}}^{q-1}(\Gamma_t,\Omega)\oplus\mathcal{H}^q(\Omega)\oplus\delta\,\mathring{\Delta}^{q+1}(\Gamma_n,\Omega)\end{aligned}$$

hold.

Here, $\oplus$ denotes the $\mathsf{L}^{2,q}(\Omega)$-orthogonal sum and all closures are taken in $\mathsf{L}^{2,q}(\Omega)$.

Proof By the projection theorem in Hilbert space and the MAP we obtain immediately the two $\mathsf{L}^{2,q}(\Omega)$-orthogonal decompositions

$$\overline{\mathrm{d}\,\mathring{\mathsf{D}}^{q-1}(\Gamma_t,\Omega)}\oplus\mathring{\Delta}^q_0(\Gamma_n,\Omega)=\mathsf{L}^{2,q}(\Omega)=\mathring{\mathsf{D}}^q_0(\Gamma_t,\Omega)\oplus\overline{\delta\,\mathring{\Delta}^{q+1}(\Gamma_n,\Omega)},$$

where the closures are taken in $\mathsf{L}^{2,q}(\Omega)$. Since

$$\mathrm{d}\,\mathring{\mathsf{D}}^{q-1}(\Gamma_t,\Omega)\subset\mathring{\mathsf{D}}^q_0(\Gamma_t,\Omega),\quad\delta\,\mathring{\Delta}^{q+1}(\Gamma_n,\Omega)\subset\mathring{\Delta}^q_0(\Gamma_n,\Omega)$$

and applying the latter decompositions separately to $\mathring{\Delta}^q_0(\Gamma_n,\Omega)$ or $\mathring{\mathsf{D}}^q_0(\Gamma_t,\Omega)$ we get a refined decomposition, namely

$$\mathsf{L}^{2,q}(\Omega)=\overline{\mathrm{d}\,\mathring{\mathsf{D}}^{q-1}(\Gamma_t,\Omega)}\oplus\mathcal{H}^q(\Omega)\oplus\overline{\delta\,\mathring{\Delta}^{q+1}(\Gamma_n,\Omega)}.$$

Applying this decomposition to $\mathring{\mathsf{D}}^{q-1}(\Gamma_t,\Omega)$ and $\mathring{\Delta}^{q+1}(\Gamma_n,\Omega)$ yields also

$$\begin{aligned}\mathrm{d}\,\mathring{\mathsf{D}}^{q-1}(\Gamma_t,\Omega)&=\mathrm{d}\left(\mathring{\mathsf{D}}^{q-1}(\Gamma_t,\Omega)\cap\overline{\delta\,\mathring{\Delta}^q(\Gamma_n,\Omega)}\right),\\\delta\,\mathring{\Delta}^{q+1}(\Gamma_n,\Omega)&=\delta\left(\mathring{\Delta}^{q+1}(\Gamma_n,\Omega)\cap\overline{\mathrm{d}\,\mathring{\mathsf{D}}^q(\Gamma_t,\Omega)}\right).\end{aligned}$$

Now, Lemma 7 shows that $\mathrm{d}\,\mathring{\mathsf{D}}^{q-1}(\Gamma_t,\Omega)$ and $\delta\,\mathring{\Delta}^{q+1}(\Gamma_n,\Omega)$ are even closed subspaces of $\mathsf{L}^{2,q}(\Omega)$. Hence, we obtain the asserted Hodge-Helmholtz decompositions of $\mathsf{L}^{2,q}(\Omega)$. $\square$

q	0	1	2	3
d	grad	curl	div	0
δ	0	div	$-$ curl	grad
$\overset{\circ}{\mathsf{D}}{}^q(\Gamma_t,\Omega)$	$\overset{\circ}{\mathsf{H}}(\mathrm{grad};\Gamma_t,\Omega)$	$\overset{\circ}{\mathsf{H}}(\mathrm{curl};\Gamma_t,\Omega)$	$\overset{\circ}{\mathsf{H}}(\mathrm{div};\Gamma_t,\Omega)$	$\mathsf{L}^2(\Omega)$
$\overset{\circ}{\Delta}{}^q(\Gamma_n,\Omega)$	$\mathsf{L}^2(\Omega)$	$\overset{\circ}{\mathsf{H}}(\mathrm{div};\Gamma_n,\Omega)$	$\overset{\circ}{\mathsf{H}}(\mathrm{curl};\Gamma_n,\Omega)$	$\overset{\circ}{\mathsf{H}}(\mathrm{grad};\Gamma_n,\Omega)$
$\iota^*_{\Gamma_t}E$	$E\vert_{\Gamma_t}$	$\nu\times E\vert_{\Gamma_t}$	$\nu\cdot E\vert_{\Gamma_t}$	0
$\circledast\iota^*_{\Gamma_n}*E$	0	$\nu\cdot E\vert_{\Gamma_n}$	$-\nu\times(\nu\times E)\vert_{\Gamma_n}$	$E\vert_{\Gamma_n}$

Figure 2: identification table for q-forms and vector proxies in $\mathbb{R}^3$

2.2 Functions and Vector Fields

We turn to the special case $q=1$, the case of vector fields, and use the notations and identifications from [14] and [12, 15, 16]. Especially, $\mathsf{L}^{2,q}(\Omega)$ can be identified with the usual Lebesgue spaces of square integrable functions or vector fields on Ω with values in $\mathbb{R}^n$, $n:=n_{N,q}:=\binom{N}{q}$, and will be denoted by $\mathsf{L}^2(\Omega):=\mathsf{L}^2(\Omega,\mathbb{R}^n)$. We have the standard Sobolev spaces

$$\begin{aligned}\mathsf{H}(\mathrm{grad};\Omega)&:=\{u\in\mathsf{L}^2(\Omega,\mathbb{R})\ :\ \mathrm{grad}\,u\in\mathsf{L}^2(\Omega,\mathbb{R}^N)\},\\ \mathsf{H}(\mathrm{div};\Omega)&:=\{v\in\mathsf{L}^2(\Omega,\mathbb{R}^N)\ :\ \mathrm{div}\,v\in\mathsf{L}^2(\Omega,\mathbb{R})\},\\ \mathsf{H}(\mathrm{curl};\Omega)&:=\{v\in\mathsf{L}^2(\Omega,\mathbb{R}^N)\ :\ \mathrm{curl}\,v\in\mathsf{L}^2(\Omega,\mathbb{R}^{N(N-1)/2})\}\end{aligned}$$

and by natural isomorphic identification

$$\mathsf{D}^0(\Omega)\cong\mathsf{H}(\mathrm{grad};\Omega),\quad \Delta^1(\Omega)\cong\mathsf{H}(\mathrm{div};\Omega),\quad \mathsf{D}^1(\Omega)\cong\mathsf{H}(\mathrm{curl};\Omega).$$

Generally $\mathsf{D}^q(\Omega)\cong\Delta^{N-q}(\Omega)$ holds by Hodge star duality. For $v\in\mathsf{C}^\infty(\Omega)$ and $N=3,4$

$$\mathrm{curl}\,v=\begin{bmatrix}\partial_2 v_3-\partial_3 v_2\\ \partial_3 v_1-\partial_1 v_3\\ \partial_1 v_2-\partial_2 v_1\end{bmatrix}\in\mathbb{R}^3,\quad \mathrm{curl}\,v=\begin{bmatrix}\partial_1 v_2-\partial_2 v_1\\ \partial_1 v_3-\partial_3 v_1\\ \partial_1 v_4-\partial_4 v_1\\ \partial_2 v_3-\partial_3 v_2\\ \partial_2 v_4-\partial_4 v_2\\ \partial_3 v_4-\partial_4 v_3\end{bmatrix}\in\mathbb{R}^6$$

hold, whereas $\mathrm{curl}\,v=\partial_1 v_2-\partial_2 v_1\in\mathbb{R}$ or $\mathrm{curl}\,v\in\mathbb{R}^{10}$ for $N=2$ or $N=5$, respectively.

Moreover, we have the closed subspaces

$$\overset{\circ}{\mathsf{H}}(\mathrm{grad};\Gamma_t,\Omega),\quad \overset{\circ}{\mathsf{H}}(\mathrm{curl};\Gamma_t,\Omega),\quad \overset{\circ}{\mathsf{H}}(\mathrm{div};\Gamma_n,\Omega),$$

in which the homogeneous scalar, tangential and normal boundary conditions

$$u|_{\Gamma_t}=0,\quad \nu\times v|_{\Gamma_t}=0,\quad \nu\cdot v|_{\Gamma_n}=0$$

are generalized, as reincarnations of $\mathring{\mathsf{D}}^0(\Gamma_t,\Omega)$, $\mathring{\mathsf{D}}^1(\Gamma_t,\Omega)$ and $\mathring{\Delta}^1(\Gamma_n,\Omega)$, respectively. Here ν denotes the outer unit normal at Γ. If $\Gamma_t=\Gamma$ (and $\Gamma_n=\emptyset$) resp. $\Gamma_t=\emptyset$ (and $\Gamma_n=\Gamma$) we obtain the usual Sobolev spaces

$$\mathring{\mathsf{H}}(\text{grad};\Omega),\quad \mathring{\mathsf{H}}(\text{curl};\Omega),\quad \mathsf{H}(\text{div};\Omega)$$

resp.

$$\mathsf{H}(\text{grad};\Omega),\quad \mathsf{H}(\text{curl};\Omega),\quad \mathring{\mathsf{H}}(\text{div};\Omega).$$

We note that $\mathsf{H}(\text{grad};\Omega)$ and $\mathring{\mathsf{H}}(\text{grad};\Omega)$ coincide with the usual standard Sobolev spaces $\mathsf{H}^1(\Omega)$ and $\mathring{\mathsf{H}}^1(\Omega)$, respectively. As before, the index 0, now attached to the symbols curl or div, indicates vanishing curl or div, e.g.,

$$\mathring{\mathsf{H}}(\text{curl}_0;\Gamma_t,\Omega)=\{v\in\mathring{\mathsf{H}}(\text{curl};\Gamma_t,\Omega)\,:\,\text{curl}\,v=0\},$$
$$\mathsf{H}(\text{div}_0;\Omega)=\{v\in\mathsf{H}(\text{div};\Omega)\,:\,\text{div}\,v=0\}.$$

Finally, we denote the 'harmonic Dirichlet-Neumann fields' by

$$\mathcal{H}^1(\Omega)\cong\mathcal{H}(\Omega):=\mathring{\mathsf{H}}(\text{curl}_0;\Gamma_t,\Omega)\cap\mathring{\mathsf{H}}(\text{div}_0;\Gamma_n,\Omega).$$

Assuming the MCP for the pair (Ω,Γ_t), then $\mathcal{H}(\Omega)$ is finite dimensional by Corollary 6 and we have the two (out of four) compact embeddings

$$\mathring{\mathsf{H}}(\text{grad};\Gamma_t,\Omega)\hookrightarrow\mathsf{L}^2(\Omega),\tag{2.3}$$
$$\mathring{\mathsf{H}}(\text{curl};\Gamma_t,\Omega)\cap\mathring{\mathsf{H}}(\text{div};\Gamma_n,\Omega)\hookrightarrow\mathsf{L}^2(\Omega),\tag{2.4}$$

i.e., Rellich's selection theorem ($q=0$) and the vectorial Maxwell's compactness property ($q=1$). Moreover, by Lemma 7 we get the following Poincaré and Maxwell estimates:

Corollary 13 (Poincaré Estimate for Functions) *Let the pair (Ω,Γ_t) have the* MCP *and $c_{\mathsf{p}}:=c_{\mathsf{p},0}$. Then*

$$\|u\|_{\mathsf{L}^2(\Omega)}\le c_{\mathsf{p}}\,\|\text{grad}\,u\|_{\mathsf{L}^2(\Omega)}$$

holds for all $u\in\mathring{\mathsf{H}}(\text{grad};\Gamma_t,\Omega)$ if $\Gamma_t\neq\emptyset$ and for all $u\in\mathsf{H}(\text{grad};\Omega)\cap\mathbb{R}^\perp$ if $\Gamma_t=\emptyset$. Moreover, for all $u\in\mathsf{H}(\text{grad};\Omega)$

$$\|(\text{id}-\pi_0)u\|_{\mathsf{L}^2(\Omega)}\le c_{\mathsf{p}}\,\|\text{grad}\,u\|_{\mathsf{L}^2(\Omega)}$$

holds, where $\pi_0:\mathsf{L}^2(\Omega)\to\mathbb{R}$ denotes the $\mathsf{L}^2(\Omega)$-orthogonal projection onto the constants.

We note that if $\Gamma_t\neq\emptyset$ we have $\mathcal{H}^0(\Omega)=\{0\}$. Furthermore, $\mathcal{H}^0(\Omega)=\mathbb{R}$ and $\mathring{\mathsf{H}}(\text{grad};\Gamma_t,\Omega)=\mathsf{H}(\text{grad};\Omega)$ hold if $\Gamma_t=\emptyset$.

Corollary 14 (Maxwell Estimate for Vector Fields) *Let the pair* (Ω, Γ_t) *have the* MCP *and* $c_{\mathtt{m}} := c_{\mathtt{p},1}$. *Then*

$$\|v\|_{\mathsf{L}^2(\Omega)} \leq c_{\mathtt{m}}\big(\|\operatorname{curl} v\|^2_{\mathsf{L}^2(\Omega)} + \|\operatorname{div} v\|^2_{\mathsf{L}^2(\Omega)} \big)^{1/2}$$

holds for all $v \in \overset{\circ}{\mathsf{H}}(\operatorname{curl}; \Gamma_t, \Omega) \cap \overset{\circ}{\mathsf{H}}(\operatorname{div}; \Gamma_n, \Omega) \cap \mathcal{H}(\Omega)^\perp$ *as well as*

$$\|(\operatorname{id} - \pi_1)v\|_{\mathsf{L}^2(\Omega)} \leq c_{\mathtt{m}}\big(\|\operatorname{curl} v\|^2_{\mathsf{L}^2(\Omega)} + \|\operatorname{div} v\|^2_{\mathsf{L}^2(\Omega)} \big)^{1/2}$$

holds for all $v \in \overset{\circ}{\mathsf{H}}(\operatorname{curl}; \Gamma_t, \Omega) \cap \overset{\circ}{\mathsf{H}}(\operatorname{div}; \Gamma_n, \Omega)$, *where again* $\pi_1 : \mathsf{L}^2(\Omega) \to \mathcal{H}(\Omega)$ *denotes the* $\mathsf{L}^2(\Omega)$*-orthogonal projection onto the Dirichlet-Neumann fields* $\mathcal{H}(\Omega)$.

Lemma 12 yields:

Corollary 15 (Helmholtz Decompositions for Vector Fields) *Let the pair* (Ω, Γ_t) *have the* MCP *and the* MAP. *Then, the orthogonal decompositions*

$$\begin{aligned} \mathsf{L}^2(\Omega) &= \operatorname{grad} \overset{\circ}{\mathsf{H}}(\operatorname{grad}; \Gamma_t, \Omega) \oplus \overset{\circ}{\mathsf{H}}(\operatorname{div}_0; \Gamma_n, \Omega) \\ &= \overset{\circ}{\mathsf{H}}(\operatorname{curl}_0; \Gamma_t, \Omega) \oplus \big(\overset{\circ}{\mathsf{H}}(\operatorname{div}_0; \Gamma_n, \Omega) \cap \mathcal{H}(\Omega)^\perp\big) \end{aligned}$$

hold.

2.3 Tensor Fields

Next, we extend our calculus to tensor fields, i.e., matrix fields. For vector fields v with components in $\mathsf{H}(\operatorname{grad}; \Omega)$ and tensor fields T with rows in $\mathsf{H}(\operatorname{curl}; \Omega)$ resp. $\mathsf{H}(\operatorname{div}; \Omega)$, i.e.,

$$v = \begin{bmatrix} v_1 \\ \vdots \\ v_N \end{bmatrix}, \quad v_n \in \mathsf{H}(\operatorname{grad}; \Omega), \quad T = \begin{bmatrix} T_1^\top \\ \vdots \\ T_N^\top \end{bmatrix}, \quad T_n \in \mathsf{H}(\operatorname{curl}; \Omega) \text{ resp. } \mathsf{H}(\operatorname{div}; \Omega)$$

for $n = 1, \ldots, N$ we define (in the weak sense)

$$\operatorname{Grad} v := \begin{bmatrix} \operatorname{grad}^\top v_1 \\ \vdots \\ \operatorname{grad}^\top v_N \end{bmatrix} = J_v, \quad \operatorname{Curl} T := \begin{bmatrix} \operatorname{curl}^\top T_1 \\ \vdots \\ \operatorname{curl}^\top T_N \end{bmatrix}, \quad \operatorname{Div} T := \begin{bmatrix} \operatorname{div} T_1 \\ \vdots \\ \operatorname{div} T_N \end{bmatrix},$$

where J_v** denotes the Jacobian of v and $^\top$ the transpose. We note that v and $\operatorname{Div} T$ are N-vector fields, T and $\operatorname{Grad} v$ are $(N \times N)$-tensor fields, whereas $\operatorname{Curl} T$ is a $(N \times N(N-1)/2)$-tensor field. The corresponding Sobolev spaces will be denoted by

$$\mathsf{H}(\operatorname{Grad}; \Omega), \quad \mathsf{H}(\operatorname{Curl}; \Omega), \quad \mathsf{H}(\operatorname{Curl}_0; \Omega), \quad \mathsf{H}(\operatorname{Div}; \Omega), \quad \mathsf{H}(\operatorname{Div}_0; \Omega)$$

and

$$\overset{\circ}{\mathsf{H}}(\operatorname{Grad}; \Gamma_t, \Omega), \quad \overset{\circ}{\mathsf{H}}(\operatorname{Curl}; \Gamma_t, \Omega), \quad \overset{\circ}{\mathsf{H}}(\operatorname{Curl}_0; \Gamma_t, \Omega), \quad \overset{\circ}{\mathsf{H}}(\operatorname{Div}; \Gamma_n, \Omega), \quad \overset{\circ}{\mathsf{H}}(\operatorname{Div}_0; \Gamma_n, \Omega),$$

again with the usual notations if $\Gamma_t \in \{\emptyset, \Gamma\}$.

From Corollaries 13, 14 and 15 we obtain immediately:

**Sometimes, the Jacobian J_v is also denoted by ∇v.

Corollary 16 (Poincaré Estimate for Vector Fields) *Let the pair* (Ω, Γ_t) *have the* MCP*. Then*

$$\|v\|_{\mathsf{L}^2(\Omega)} \leq c_{\mathtt{p}} \, \|\operatorname{Grad} v\|_{\mathsf{L}^2(\Omega)}$$

holds for all $v \in \overset{\circ}{\mathsf{H}}(\operatorname{Grad}; \Gamma_t, \Omega)$ *if* $\Gamma_t \neq \emptyset$ *and for all* $v \in \mathsf{H}(\operatorname{Grad}; \Omega) \cap (\mathbb{R}^N)^\perp$ *if* $\Gamma_t = \emptyset$*. Moreover, for all* $v \in \mathsf{H}(\operatorname{Grad}; \Omega)$

$$\big\|(\operatorname{id} - \pi_0^N) v\big\|_{\mathsf{L}^2(\Omega)} \leq c_{\mathtt{p}} \, \|\operatorname{Grad} v\|_{\mathsf{L}^2(\Omega)}$$

holds, where $\pi_0^N : \mathsf{L}^2(\Omega) \to \mathbb{R}^N$ *denotes the* $\mathsf{L}^2(\Omega)$*-orthogonal projection onto* $\mathbb{R}^N$*.*

Corollary 17 (Maxwell Estimate for Tensor Fields) *Let the pair* (Ω, Γ_t) *have the* MCP*. Then*

$$\|T\|_{\mathsf{L}^2(\Omega)} \leq c_{\mathtt{m}} \big(\|\operatorname{Curl} T\|^2_{\mathsf{L}^2(\Omega)} + \|\operatorname{Div} T\|^2_{\mathsf{L}^2(\Omega)} \big)^{1/2}$$

holds for all $T \in \overset{\circ}{\mathsf{H}}(\operatorname{Curl}; \Gamma_t, \Omega) \cap \overset{\circ}{\mathsf{H}}(\operatorname{Div}; \Gamma_n, \Omega) \cap (\mathcal{H}(\Omega)^N)^\perp$ *as well as*

$$\big\|(\operatorname{id} - \pi_1^N) T\big\|_{\mathsf{L}^2(\Omega)} \leq c_{\mathtt{m}} \big(\|\operatorname{Curl} T\|^2_{\mathsf{L}^2(\Omega)} + \|\operatorname{Div} T\|^2_{\mathsf{L}^2(\Omega)} \big)^{1/2}$$

holds for all $T \in \overset{\circ}{\mathsf{H}}(\operatorname{Curl}; \Gamma_t, \Omega) \cap \overset{\circ}{\mathsf{H}}(\operatorname{Div}; \Gamma_n, \Omega)$*, where* $\pi_1^N : \mathsf{L}^2(\Omega) \to \mathcal{H}(\Omega)^N$ *denotes the* $\mathsf{L}^2(\Omega)$*-orthogonal projection onto the (N-times)-Dirichlet-Neumann fields* $\mathcal{H}(\Omega)^N$*.*

Corollary 18 (Helmholtz Decompositions for Tensor Fields) *Let the pair* (Ω, Γ_t) *have the* MCP *and the* MAP*. Then, the orthogonal decompositions*

$$\begin{aligned} \mathsf{L}^2(\Omega) &= \operatorname{Grad} \overset{\circ}{\mathsf{H}}(\operatorname{Grad}; \Gamma_t, \Omega) \oplus \overset{\circ}{\mathsf{H}}(\operatorname{Div}_0; \Gamma_n, \Omega) \\ &= \overset{\circ}{\mathsf{H}}(\operatorname{Curl}_0; \Gamma_t, \Omega) \oplus \big(\overset{\circ}{\mathsf{H}}(\operatorname{Div}_0; \Gamma_n, \Omega) \cap (\mathcal{H}(\Omega)^N)^\perp \big) \end{aligned}$$

hold.

We also need Korn's First Inequality.

Definition 19 (Korn's Second Inequality) *The domain* Ω *has the 'Korn property'* (KP)*, if*

(i) *Korn's second inequality holds, this is, there exists a constant* $c > 0$*, such that for all vector fields* $v \in \mathsf{H}(\operatorname{Grad}; \Omega)$

$$c \, \|\operatorname{Grad} v\|_{\mathsf{L}^2(\Omega)} \leq \|v\|_{\mathsf{L}^2(\Omega)} + \|\operatorname{sym} \operatorname{Grad} v\|_{\mathsf{L}^2(\Omega)} \, ,$$

(ii) *and Rellich's selection theorem holds for* $\mathsf{H}(\operatorname{grad}; \Omega)$*, this is, the natural embedding* $\mathsf{H}(\operatorname{grad}; \Omega) \hookrightarrow \mathsf{L}^2(\Omega)$ *is compact.*

Here, we introduce the symmetric and skew-symmetric parts

$$\operatorname{sym} T := \frac{1}{2}(T + T^\top), \quad \operatorname{skew} T := T - \operatorname{sym} T = \frac{1}{2}(T - T^\top)$$

of a tensor field $T = \operatorname{skew} T + \operatorname{sym} T$[††].

Remark 20 *There exists a rich amount of literature for the* KP*, which we do not intend to list here. We refer to our overview on Korn's inequalities in [16].*

Theorem 21 *Korn's second inequality holds for domains Ω having the strict cone property. For domains Ω with the segment property, Rellich's selection theorem for $\mathsf{H}(\operatorname{grad};\Omega)$ is valid. Thus, e.g., Lipschitz domains Ω possess the* KP.

Proof Book of Leis [11]. □

By a standard indirect argument we immediately obtain:

Corollary 22 (Korn's First Inequality: Standard Version) *Let Ω have the* KP*. Then, there exists a constant $c_{\mathsf{k},s} > 0$ such that the following holds:*

(i) *If $\Gamma_t \neq \emptyset$ then*

$$(1 + c_{\mathsf{p}}^2)^{-1/2} \|v\|_{\mathsf{H}^1(\Omega)} \leq \|\operatorname{Grad} v\|_{\mathsf{L}^2(\Omega)} \leq c_{\mathsf{k},s} \|\operatorname{sym} \operatorname{Grad} v\|_{\mathsf{L}^2(\Omega)} \tag{2.5}$$

holds for all vector fields $v \in \overset{\circ}{\mathsf{H}}(\operatorname{Grad};\Gamma_t,\Omega)$.

(ii) *If $\Gamma_t = \emptyset$, then the inequalities (2.5) hold for all vector fields $v \in \mathsf{H}(\operatorname{Grad};\Omega)$ with $\operatorname{Grad} v \perp \mathfrak{so}(N)$ and $v \perp \mathbb{R}^N$. Moreover, the second inequality of (2.5) holds for all vector fields $v \in \mathsf{H}(\operatorname{Grad};\Omega)$ with $\operatorname{Grad} v \perp \mathfrak{so}(N)$. For all $v \in \mathsf{H}(\operatorname{Grad};\Omega)$*

$$(1 + c_{\mathsf{p}}^2)^{-1/2} \|v - r_v\|_{\mathsf{H}^1(\Omega)} \leq \|\operatorname{Grad} v - A_{\operatorname{Grad} v}\|_{\mathsf{L}^2(\Omega)} \leq c_{\mathsf{k},s} \|\operatorname{sym} \operatorname{Grad} v\|_{\mathsf{L}^2(\Omega)} \tag{2.6}$$

holds, where the ridgid motion r_v and the skew-symmetric tensor $A_{\operatorname{Grad} v} = \operatorname{Grad} r_v$ are given by $r_v(x) := A_{\operatorname{Grad} v} x + b_v$ and

$$A_{\operatorname{Grad} v} := \operatorname{skew} \oint_\Omega \operatorname{Grad} v \, d\lambda \in \mathfrak{so}(N), \quad b_v := \oint_\Omega v \, d\lambda - A_{\operatorname{Grad} v} \oint_\Omega x \, d\lambda_x \in \mathbb{R}^N.$$

We note $v - r_v \perp \mathbb{R}^N$ and $\operatorname{Grad}(v - r_v) = \operatorname{Grad} v - A_{\operatorname{Grad} v} \perp \mathfrak{so}(N)$.

Here, we generally define

$$\oint_\Omega u \, d\lambda := \lambda(\Omega)^{-1} \int_\Omega u \, d\lambda, \quad \lambda \text{ Lebesgue's measure.}$$

We note that $A_{\operatorname{Grad} v} = \pi_{\mathfrak{so}(N)} \operatorname{Grad} v \in \mathfrak{so}(N)$, where $\pi_{\mathfrak{so}(N)} : \mathsf{L}^2(\Omega) \to \mathfrak{so}(N)$ denotes the $\mathsf{L}^2(\Omega)$-orthogonal projection onto the constant skew-symmetric tensor fields $\mathfrak{so}(N)$. Moreover, we have generally for square integrable $(N \times N)$-tensor fields T

$$\pi_{\mathfrak{so}(N)} T := A_T := \operatorname{skew} \oint_\Omega T \, d\lambda \in \mathfrak{so}(N). \tag{2.7}$$

[††]Note that $\operatorname{sym} T$ and $\operatorname{skew} T$ are point-wise orthogonal with respect to the standard inner product in $\mathbb{R}^{N \times N}$.

2.4 Sliceable and Admissible Domains

The essential tools to prove our main result Theorem 1 are

- the Maxwell estimate for tensor fields (Corollary 17),
- the Helmholtz decomposition for tensor fields (Corollary 18),
- and a generalized version of Korn's first inequality (Corollary 22).

For the first two tools the pair (Ω, Γ_t) needs to have the MCP and the MAP. The third tool will be provided in Lemma 29 and needs at least the KP. As already pointed out, these three properties hold, e.g., for Lipschitz domains Ω and admissible boundary patches Γ_t. Moreover, we will make use of the fact that any irrotational vector field is already a gradient if the underlying domain is simply connected. For this, we present a trick, the concept of sliceable domains, which we have used already in [16].

Definition 23 *The pair (Ω, Γ_t) is called 'sliceable', if there exist $J \in \mathbb{N}$ and $\Omega_j \subset \Omega$, $j = 1, \ldots, J$, such that $\Omega \setminus (\Omega_1 \cup \ldots \cup \Omega_J)$ has zero Lebesgue-measure and for $j = 1, \ldots, J$*

(i) *Ω_j are open, disjoint and simply connected subdomains of Ω having the* KP*,*

(ii) *$\Gamma_{t,j} := \mathrm{int}_{\mathsf{rel}}(\overline{\Omega_j} \cap \Gamma_t) \neq \emptyset$, if $\Gamma_t \neq \emptyset$.*

Here, $\mathrm{int}_{\mathsf{rel}}$ denotes the interior with respect to the topology on Γ.

Remark 24 *From a practical point of view, <u>all</u> domains considered in applications are sliceable, but it is unclear whether every Lipschitz pair (Ω, Γ_t) is already sliceable.*

Now, we can introduce our general assumptions on the domain and its boundary parts.

Definition 25 *The pair (Ω, Γ_t) is called 'admissible', if*

- *the pair (Ω, Γ_t) possesses the* MCP*and the* MAP*,*
- *and the pair (Ω, Γ_t) is sliceable.*

Remark 26 *In particular, the pair (Ω, Γ_t) is admissible if*

- *Ω has a Lipschitz boundary Γ,*
- *Γ_t is a Lipschitz patch,*
- *(Ω, Γ_t) is sliceable.*

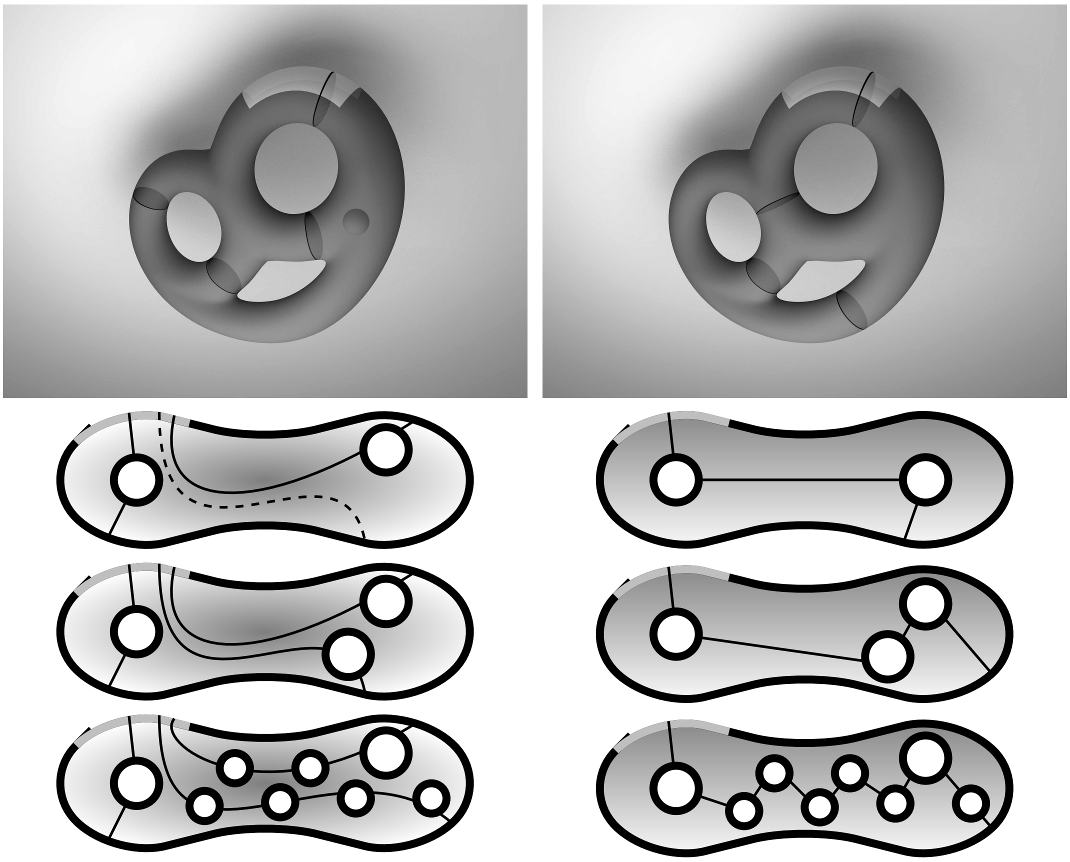

Figure 3: Some ways to 'cut' sliceable domains Ω in $\mathbb{R}^3$ and $\mathbb{R}^2$ into two $(J = 2)$ or more $(J = 3, 4)$ 'pieces'. The boundary part Γ_t is colored in light gray. Roughly speaking, a domain is sliceable if it can be cut into finitely many simply connected Lipschitz pieces Ω_j, i.e., any closed curve inside some piece Ω_j is homotop to a point, this is, one has to cut all 'handles'. In three and higher dimensions, holes inside Ω are permitted, but this is forbidden in the two-dimensional case. Note that, in these examples it is always possible to slice Ω into two $(J = 2)$ pieces.

3 Proofs

Let the pair (Ω, Γ_t) be admissible. On our way to prove our main result we follow in close lines the arguments of [16, section 3]. First we prove a non-standard version of Korn's first inequality Corollary 22, which will be presented as Lemma 29. Then, we prove our main result. Although, all subsequent proofs are very similar to the ones given in [16, Lemmas 8, 9, 12, Theorem 14], we will repeat them here for the convenience of the reader.

Lemma 27 *Let* $\Gamma_t \neq \emptyset$ *and* $u \in \mathsf{H}(\mathrm{grad};\Omega)$ *with* $\mathrm{grad}\, u \in \overset{\circ}{\mathsf{H}}(\mathrm{curl}_0;\Gamma_t,\Omega)$*. Then, u is constant on any connected component of* Γ_t*.*

Proof Let $x \in \Gamma_t$ and $B_{2r} := B_{2r}(x)$ be the open ball of radius $2r > 0$ around x such that B_{2r} is covered by a Lipschitz-chart domain and $\Gamma \cap B_{2r} \subset \Gamma_t$. Moreover, we pick a cut-off function $\varphi \in \overset{\circ}{\mathsf{C}}^\infty(B_{2r})$ with $\varphi|_{B_r} = 1$. Then, $\varphi\, \mathrm{grad}\, u \in \overset{\circ}{\mathsf{H}}(\mathrm{curl};\Omega \cap B_{2r})$. Thus, the extension by zero v of $\varphi\, \mathrm{grad}\, u$ to B_{2r} belongs to $\mathsf{H}(\mathrm{curl};B_{2r})$. Hence, $v|_{B_r} \in \mathsf{H}(\mathrm{curl}_0;B_r)$. Since B_r is simply connected, there exists a $\tilde{u} \in \mathsf{H}(\mathrm{grad};B_r)$ with $\mathrm{grad}\,\tilde{u} = v$ in B_r. In $B_r \setminus \overline{\Omega}$ we have $v = 0$. Therefore, $\tilde{u}|_{B_r \setminus \overline{\Omega}} = \tilde{c}$ with some $\tilde{c} \in \mathbb{R}$. Moreover, $\mathrm{grad}\, u = v = \mathrm{grad}\,\tilde{u}$ holds in $B_r \cap \Omega$, which yields $u = \tilde{u} + c$ in $B_r \cap \Omega$ with some $c \in \mathbb{R}$. Finally, $u|_{B_r \cap \Gamma_t} = \tilde{c} + c$ is constant. Therefore, u is locally constant and hence the assertion follows. □

Lemma 28 (Korn's First Inequality: Tangential Version) *Let* $\Gamma_t \neq \emptyset$*. Then, there exists a constant* $c_{\mathsf{k},t} \geq c_{\mathsf{k},s}$*, such that*

$$\|\mathrm{Grad}\, v\|_{\mathsf{L}^2(\Omega)} \leq c_{\mathsf{k},t}\, \|\mathrm{sym}\,\mathrm{Grad}\, v\|_{\mathsf{L}^2(\Omega)}$$

holds for all $v \in \mathsf{H}(\mathrm{Grad};\Omega)$ *with* $\mathrm{Grad}\, v \in \overset{\circ}{\mathsf{H}}(\mathrm{Curl}_0;\Gamma_t,\Omega)$.

In classical terms, $\mathrm{Grad}\, v \in \overset{\circ}{\mathsf{H}}(\mathrm{Curl}_0;\Gamma_t,\Omega)$ means that $\mathrm{grad}\, v_n = \nabla v_n$, $n = 1, \ldots, N$, are normal at Γ_t.

Proof We pick a relatively open connected component $\tilde{\Gamma} \neq \emptyset$ of Γ_t. Then, there exists a constant vector $c_v \in \mathbb{R}^3$ such that $v - c_v$ belongs to $\overset{\circ}{\mathsf{H}}(\mathrm{Grad};\tilde{\Gamma},\Omega)$ by Lemma 27 applied to each component of v. Corollary 22 (i) (with $\Gamma_t = \tilde{\Gamma}$ and a possibly larger $c_{\mathsf{k},t}$) completes the proof. □

Now, we extend Korn's first inequality from gradient to merely irrotational tensor fields.

Lemma 29 (Korn's First Inequality: Irrotational Version) *There exists* $c_{\mathsf{k}} \geq c_{\mathsf{k},t} > 0$*, such that the following inequalities hold:*

(i) *If* $\Gamma_t \neq \emptyset$*, then for all tensor fields* $T \in \overset{\circ}{\mathsf{H}}(\mathrm{Curl}_0;\Gamma_t,\Omega)$

$$\|T\|_{\mathsf{L}^2(\Omega)} \leq c_{\mathsf{k}}\, \|\mathrm{sym}\, T\|_{\mathsf{L}^2(\Omega)}\,. \tag{3.1}$$

(ii) *If $\Gamma_t = \emptyset$, then for all tensor fields $T \in \mathsf{H}(\operatorname{Curl}_0; \Omega)$ there exists a piece-wise constant skew-symmetric tensor field A such that*

$$\|T - A\|_{\mathsf{L}^2(\Omega)} \leq c_{\mathtt{k}} \|\operatorname{sym} T\|_{\mathsf{L}^2(\Omega)} .$$

(ii') *If $\Gamma_t = \emptyset$ and Ω is additionally simply connected, then (ii) holds with the uniquely determined constant skew-symmetric tensor field $A := A_T = \pi_{\mathfrak{so}(N)} T$ given by* (2.7)*. Moreover, $T - A_T \in \mathsf{H}(\operatorname{Curl}_0; \Omega) \cap \mathfrak{so}(N)^{\perp}$ and $A_T = 0$ if and only if $T \perp \mathfrak{so}(N)$. Thus,* (3.1) *holds for all $T \in \mathsf{H}(\operatorname{Curl}_0; \Omega) \cap \mathfrak{so}(N)^{\perp}$ as well.*

Again we note that in classical terms a tensor $T \in \overset{\circ}{\mathsf{H}}(\operatorname{Curl}_0; \Gamma_t, \Omega)$ is irrotational and the vector field $T\tau|_{\Gamma_t}$ vanishes for all tangential vector fields τ at Γ. Moreover, the sliceability of (Ω, Γ_t) is precisely needed for Lemma 29 to hold.

Proof We start with proving (i). Let $\Gamma_t \neq \emptyset$ and $T \in \overset{\circ}{\mathsf{H}}(\operatorname{Curl}_0; \Gamma_t, \Omega)$. We choose a sequence $(T^{\ell}) \subset \overset{\circ}{\mathsf{C}}{}^{\infty}(\Gamma_t; \Omega)$ converging to T in $\mathsf{H}(\operatorname{Curl}; \Omega)$. According to Definition 23 we decompose Ω into $\Omega_1 \cup \ldots \cup \Omega_J$ and pick some $1 \leq j \leq J$. Then, the restriction $T_j := T|_{\Omega_j}$ belongs to $\mathsf{H}(\operatorname{Curl}_0; \Omega_j)$ and $(T^{\ell}|_{\overline{\Omega_j}}) \subset \overset{\circ}{\mathsf{C}}{}^{\infty}(\Gamma_{t,j}; \Omega)$ converges to T_j in $\mathsf{H}(\operatorname{Curl}; \Omega_j)$. Thus, $T_j \in \overset{\circ}{\mathsf{H}}(\operatorname{Curl}_0; \Gamma_{t,j}, \Omega_j)$. Since Ω_j is simply connected, there exists a potential vector field $v_j \in \mathsf{H}(\operatorname{Grad}; \Omega_j)$ with $\operatorname{Grad} v_j = T_j$ and Lemma 28 yields

$$\|T_j\|_{\mathsf{L}^2(\Omega_j)} \leq c_{\mathtt{k},t,j} \|\operatorname{sym} T_j\|_{\mathsf{L}^2(\Omega_j)} , \qquad c_{\mathtt{k},t,j} > 0.$$

This can be done for each j. Summing up, we obtain

$$\|T\|_{\mathsf{L}^2(\Omega)} \leq c_{\mathtt{k}} \|\operatorname{sym} T\|_{\mathsf{L}^2(\Omega)} , \qquad c_{\mathtt{k}} := \max_{j=1,\ldots,J} c_{\mathtt{k},t,j},$$

proving (i). Now, we assume $\Gamma_t = \emptyset$. To show (ii), let $T \in \mathsf{H}(\operatorname{Curl}_0; \Omega)$ and, as before, let Ω be decomposed into $\Omega_1 \cup \ldots \cup \Omega_J$ by Definition 23. Again, since every Ω_j is simply connected and $T_j \in \mathsf{H}(\operatorname{Curl}_0; \Omega_j)$, there exist vector fields $v_j \in \mathsf{H}(\operatorname{Grad}; \Omega_j)$ with $\operatorname{Grad} v_j =: T_j = T$ in Ω_j. By Korn's first inequality, Corollary (22) (ii), there exist positive $c_{\mathtt{k},s,j}$ and $A_{T_j} \in \mathfrak{so}(N)$ with

$$\left\|T_j - A_{T_j}\right\|_{\mathsf{L}^2(\Omega_j)} \leq c_{\mathtt{k},s,j} \|\operatorname{sym} T_j\|_{\mathsf{L}^2(\Omega_j)} , \quad A_{T_j} = \operatorname{skew} \oint_{\Omega_j} T_j \, d\lambda = \operatorname{skew} \oint_{\Omega_j} T \, d\lambda.$$

We define the piece-wise constant skew-symmetric tensor field A a.e. by $A|_{\Omega_j} := A_{T_j}$ and set $c_{\mathtt{k}} := \max_{j=1,\ldots,J} c_{\mathtt{k},s,j}$. Summing up, gives (ii). We have also proved the first assertion of (ii'), since we do not have to slice if Ω is simply connected. The remaining assertion of (ii') are trivial, since $\pi_{\mathfrak{so}(N)} : \mathsf{L}^2(\Omega) \to \mathfrak{so}(N)$ is a $\mathsf{L}^2(\Omega)$-orthogonal projector. We note that this can be seen also by direct calculations: To show that $T - A_T$ belongs to $\mathsf{H}(\operatorname{Curl}_0; \Omega) \cap \mathfrak{so}(N)^{\perp}$ we note $A_T \in \mathsf{H}(\operatorname{Curl}_0; \Omega)$ and compute for all $A \in \mathfrak{so}(N)$

$$\langle A_T, A \rangle_{\mathsf{L}^2(\Omega)} = \langle \int_{\Omega} T \, d\lambda, A \rangle_{\mathbb{R}^{N \times N}} = \int_{\Omega} \langle T, A \rangle_{\mathbb{R}^{N \times N}} \, d\lambda = \langle T, A \rangle_{\mathsf{L}^2(\Omega)} .$$

Hence, $A_T = 0$ implies $T \perp \mathfrak{so}(N)$. On the other hand, setting $A := A_T$ shows that $T \perp \mathfrak{so}(N)$ also implies $A_T = 0$. □

We are ready to prove our main theorem.

Proof of Theorem 1 Let $\Gamma_t \neq \emptyset$ and $T \in \overset{\circ}{\mathsf{H}}(\mathrm{Curl};\Gamma_t,\Omega)$. By Corollary 18 we have

$$T = R + S \in \overset{\circ}{\mathsf{H}}(\mathrm{Curl}_0;\Gamma_t,\Omega) \oplus \big(\overset{\circ}{\mathsf{H}}(\mathrm{Div}_0;\Gamma_n,\Omega) \cap (\mathcal{H}(\Omega)^N)^\perp\big).$$

Moreover, by Corollary 17 we obtain

$$\|S\|_{\mathsf{L}^2(\Omega)} \leq c_{\mathtt{m}} \|\mathrm{Curl}\, T\|_{\mathsf{L}^2(\Omega)} \tag{3.2}$$

since $\mathrm{Curl}\, S = \mathrm{Curl}\, T$ and $S \in \overset{\circ}{\mathsf{H}}(\mathrm{Curl};\Gamma_t,\Omega) \cap \overset{\circ}{\mathsf{H}}(\mathrm{Div}_0;\Gamma_n,\Omega) \cap (\mathcal{H}(\Omega)^N)^\perp$. Then, by orthogonality, Lemma 29 (i) for R and (3.2)

$$\begin{aligned}\|T\|^2_{\mathsf{L}^2(\Omega)} = \|R\|^2_{\mathsf{L}^2(\Omega)} + \|S\|^2_{\mathsf{L}^2(\Omega)} &\leq c_{\mathtt{k}}^2 \|\mathrm{sym}\, R\|^2_{\mathsf{L}^2(\Omega)} + \|S\|^2_{\mathsf{L}^2(\Omega)}\\ &\leq 2c_{\mathtt{k}}^2 \|\mathrm{sym}\, T\|^2_{\mathsf{L}^2(\Omega)} + (1+2c_{\mathtt{k}}^2)\, \|S\|^2_{\mathsf{L}^2(\Omega)}\\ &\leq c_1^2\big(\|\mathrm{sym}\, T\|^2_{\mathsf{L}^2(\Omega)} + \|\mathrm{Curl}\, T\|^2_{\mathsf{L}^2(\Omega)}\big)\end{aligned}$$

with

$$c_1 := \max\{\sqrt{2}c_{\mathtt{k}}, c_{\mathtt{m}}\sqrt{1+2c_{\mathtt{k}}^2}\} \tag{3.3}$$

follows, which proves (i). Now, let $\Gamma_t = \emptyset$ and $T \in \mathsf{H}(\mathrm{Curl};\Omega)$. First, we show (ii'). We follow in close lines the first part of the proof. For the convenience of the reader, we repeat the previous arguments in this special case. According to Corollary 18 we orthogonally decompose

$$T = R + S \in \mathsf{H}(\mathrm{Curl}_0;\Omega) \oplus \big(\overset{\circ}{\mathsf{H}}(\mathrm{Div}_0;\Omega) \cap (\mathcal{H}(\Omega)^N)^\perp\big).$$

Then, $\mathrm{Curl}\, S = \mathrm{Curl}\, T$ and $S \in \mathsf{H}(\mathrm{Curl};\Omega) \cap \overset{\circ}{\mathsf{H}}(\mathrm{Div}_0;\Omega) \cap (\mathcal{H}(\Omega)^N)^\perp$. Again, by Corollary 17 we have (3.2). Note that

$$A_R = \pi_{\mathfrak{so}(N)} R = \mathrm{skew} \fint_\Omega R\, d\lambda \in \mathfrak{so}(N) \subset \mathsf{H}(\mathrm{Curl}_0;\Omega).$$

As before, by orthogonality, Lemma 29 (ii') applied to R and (3.2)

$$\begin{aligned}\|T - A_R\|^2_{\mathsf{L}^2(\Omega)} = \|R - A_R\|^2_{\mathsf{L}^2(\Omega)} + \|S\|^2_{\mathsf{L}^2(\Omega)} &\leq c_{\mathtt{k}}^2 \|\mathrm{sym}\, R\|^2_{\mathsf{L}^2(\Omega)} + \|S\|^2_{\mathsf{L}^2(\Omega)}\\ &\leq 2c_{\mathtt{k}}^2 \|\mathrm{sym}\, T\|^2_{\mathsf{L}^2(\Omega)} + (1+2c_{\mathtt{k}}^2)\, \|S\|^2_{\mathsf{L}^2(\Omega)}\\ &\leq c_1^2\big(\|\mathrm{sym}\, T\|^2_{\mathsf{L}^2(\Omega)} + \|\mathrm{Curl}\, T\|^2_{\mathsf{L}^2(\Omega)}\big).\end{aligned}$$

For $S = \mathrm{Curl}^* X$ with $X \in \overset{\circ}{\mathsf{H}}(\mathrm{Curl}^*;\Omega) = \overset{\circ}{\mathsf{H}}(\mathrm{Div}_0;\Omega) \cap (\mathcal{H}(\Omega)^N)^\perp$, where $\mathrm{Curl}^* \cong -\delta_2$ denotes the formal adjoint of $\mathrm{Curl} \cong \mathrm{d}_1$, and all $A \in \mathfrak{so}(N)$ we have

$$\langle A_S, A\rangle_{\mathsf{L}^2(\Omega)} = \langle \int_\Omega S\, d\lambda, A\rangle_{\mathbb{R}^{N\times N}} = \langle \mathrm{Curl}^* X, A\rangle_{\mathsf{L}^2(\Omega)} = \langle X, \mathrm{Curl}\, A\rangle_{\mathsf{L}^2(\Omega)} = 0,$$

which shows $A_S = 0$ by setting $A := A_S$. Hence $A_T = A_R$. The proof of (ii') is complete, since all other remaining assertions are trivial. Finally, to show (ii), we follow the proof of (ii') up to the point, where A_R was introduced. Now, by Lemma 29 (ii) for R we get a piece-wise constant skew-symmetric tensor $A := A_R$. We note that in general A does not belong to $\mathsf{H}(\mathrm{Curl};\Omega)$ anymore. Hence, we loose the $\mathsf{L}^2(\Omega)$-orthogonality $R - A \perp S$. But again, by Lemma 29 (ii) and (3.2)

$$\begin{aligned}\|T - A\|_{\mathsf{L}^2(\Omega)} &\leq \|R - A\|_{\mathsf{L}^2(\Omega)} + \|S\|_{\mathsf{L}^2(\Omega)} \leq c_{\mathtt{k}} \|\mathrm{sym}\, R\|_{\mathsf{L}^2(\Omega)} + \|S\|_{\mathsf{L}^2(\Omega)}\\ &\leq c_{\mathtt{k}} \|\mathrm{sym}\, T\|_{\mathsf{L}^2(\Omega)} + (1 + c_{\mathtt{k}}) \|S\|_{\mathsf{L}^2(\Omega)}\\ &\leq c_{\mathtt{k}} \|\mathrm{sym}\, T\|_{\mathsf{L}^2(\Omega)} + (1 + c_{\mathtt{k}}) c_{\mathtt{m}} \|\mathrm{Curl}\, T\|_{\mathsf{L}^2(\Omega)}\\ &\leq c_2 \big(\|\mathrm{sym}\, T\|^2_{\mathsf{L}^2(\Omega)} + \|\mathrm{Curl}\, T\|^2_{\mathsf{L}^2(\Omega)} \big)^{1/2}\end{aligned}$$

with

$$c_2 := \sqrt{2} \max\{c_{\mathtt{k}}, c_{\mathtt{m}}(1 + c_{\mathtt{k}})\}, \tag{3.4}$$

which proves (ii). □

4 One Additional Result

As in [16, sections 3.4] we can prove a generalization for media with structural changes. To apply the main result from [27], let $\mu \in \mathsf{C}^0(\overline{\Omega})$ be a $(N \times N)$-matrix field satisfying $\det \mu \geq \hat{\mu} > 0$.

Corollary 30 *Let $\Gamma_t \neq \emptyset$ and let the pair (Ω, Γ_t) be admissible. Then there exists $c > 0$ such that*

$$c \|T\|_{\mathsf{L}^2(\Omega)} \leq \|\mathrm{sym}(\mu T)\|_{\mathsf{L}^2(\Omega)} + \|\mathrm{Curl}\, T\|_{\mathsf{L}^2(\Omega)}$$

holds for all tensor fields $T \in \mathring{\mathsf{H}}(\mathrm{Curl};\Gamma_t,\Omega)$. In other words, on $\mathring{\mathsf{H}}(\mathrm{Curl};\Gamma_t,\Omega)$ the right hand side defines a norm equivalent to the standard norm in $\mathsf{H}(\mathrm{Curl};\Omega)$.

A Construction of Hodge-Helmholtz Projections

We want to point out how to compute the projections in the Hodge-Helmholtz decompositions in Lemma 12. Recalling from Lemma 12 the orthogonal decompositions

$$\begin{aligned}\mathsf{L}^{2,q}(\Omega) &= \mathrm{d}\, \mathring{\mathsf{D}}^{q-1}(\Gamma_t, \Omega) \oplus \mathring{\Delta}^q_0(\Gamma_n, \Omega)\\ &= \mathring{\mathsf{D}}^q_0(\Gamma_t, \Omega) \oplus \delta\, \mathring{\Delta}^{q+1}(\Gamma_n, \Omega)\\ &= \mathrm{d}\, \mathring{\mathsf{D}}^{q-1}(\Gamma_t, \Omega) \oplus \mathcal{H}^q(\Omega) \oplus \delta\, \mathring{\Delta}^{q+1}(\Gamma_n, \Omega)\end{aligned}$$

we denote the corresponding $\mathsf{L}^{2,q}(\Omega)$-orthogonal projections by π_{d}, π_δ and $\pi_{\mathcal{H}}$. Then, we have $\pi_{\mathcal{H}} = \mathrm{id} - \pi_{\mathrm{d}} - \pi_\delta$ and

$$\begin{aligned}
&\pi_{\mathrm{d}}\,\mathsf{L}^{2,q}(\Omega) = \mathrm{d}\,\mathring{\mathsf{D}}^{q-1}(\Gamma_t,\Omega) = \mathrm{d}\,\mathsf{X}^{q-1}(\Omega), && \mathsf{X}^{q-1}(\Omega) := \mathring{\mathsf{D}}^{q-1}(\Gamma_t,\Omega) \cap \delta\,\mathring{\Delta}^{q}(\Gamma_n,\Omega),\\
&\pi_\delta\,\mathsf{L}^{2,q}(\Omega) = \delta\,\mathring{\Delta}^{q+1}(\Gamma_n,\Omega) = \delta\,\mathsf{Y}^{q+1}(\Omega), && \mathsf{Y}^{q+1}(\Omega) := \mathring{\Delta}^{q+1}(\Gamma_n,\Omega) \cap \mathrm{d}\,\mathring{\mathsf{D}}^{q}(\Gamma_t,\Omega),\\
&\pi_{\mathcal{H}}\,\mathsf{L}^{2,q}(\Omega) = \mathcal{H}^q(\Omega).
\end{aligned}$$

By Poincaré's estimate, i.e., Lemma 7, we have

$$\forall\, E \in \mathsf{X}^{q-1}(\Omega) \qquad \|E\|_{\mathsf{L}^{2,q-1}(\Omega)} \le c_{\mathsf{p},q-1}\,\|\mathrm{d}\,E\|_{\mathsf{L}^{2,q}(\Omega)}\,, \tag{A.1}$$

$$\forall\, H \in \mathsf{Y}^{q+1}(\Omega) \qquad \|H\|_{\mathsf{L}^{2,q+1}(\Omega)} \le c_{\mathsf{p},q+1}\,\|\delta\,H\|_{\mathsf{L}^{2,q}(\Omega)}\,. \tag{A.2}$$

Hence, the bilinear forms

$$(\tilde{E},E) \mapsto \left\langle \mathrm{d}\,\tilde{E}, \mathrm{d}\,E \right\rangle_{\mathsf{L}^{2,q}(\Omega)}, \qquad (\tilde{H},H) \mapsto \left\langle \delta\,\tilde{H}, \delta\,H \right\rangle_{\mathsf{L}^{2,q}(\Omega)}$$

are continuous and coercive over $\mathsf{X}^{q-1}(\Omega)$ and $\mathsf{Y}^{q+1}(\Omega)$, respectively. Moreover, for any $F \in \mathsf{L}^{2,q}(\Omega)$ the linear functionals

$$E \mapsto \langle F, \mathrm{d}\,E\rangle_{\mathsf{L}^{2,q}(\Omega)}\,, \qquad H \mapsto \langle F, \delta\,H\rangle_{\mathsf{L}^{2,q}(\Omega)}$$

are continuous over $\mathsf{X}^{q-1}(\Omega)$ respectively $\mathsf{Y}^{q+1}(\Omega)$. Thus, by Lax-Milgram's theorem we get unique solutions $E_{\mathrm{d}} \in \mathsf{X}^{q-1}(\Omega)$ and $H_\delta \in \mathsf{Y}^{q+1}(\Omega)$ of the two variational problems

$$\langle \mathrm{d}\,E_{\mathrm{d}}, \mathrm{d}\,E\rangle_{\mathsf{L}^{2,q}(\Omega)} = \langle F, \mathrm{d}\,E\rangle_{\mathsf{L}^{2,q}(\Omega)} \qquad \forall\, E \in \mathsf{X}^{q-1}(\Omega), \tag{A.3}$$

$$\langle \delta\,H_\delta, \delta\,H\rangle_{\mathsf{L}^{2,q}(\Omega)} = \langle F, \delta\,H\rangle_{\mathsf{L}^{2,q}(\Omega)} \qquad \forall\, H \in \mathsf{Y}^{q+1}(\Omega) \tag{A.4}$$

and the corresponding solution operators, mapping F to E_{d} and H_δ, respectively, are continuous. In fact, we have as usual

$$\|\mathrm{d}\,E_{\mathrm{d}}\|_{\mathsf{L}^{2,q}(\Omega)} \le \|F\|_{\mathsf{L}^{2,q}(\Omega)}\,, \quad \|\delta\,H_\delta\|_{\mathsf{L}^{2,q}(\Omega)} \le \|F\|_{\mathsf{L}^{2,q}(\Omega)}\,,$$

respectively, and therefore together with (A.1) and (A.2)

$$\begin{aligned}
\|E_{\mathrm{d}}\|_{\mathsf{X}^{q-1}(\Omega)} &= \|E_{\mathrm{d}}\|_{\mathsf{D}^{q-1}(\Omega)} \le \sqrt{1 + c_{\mathsf{p},q-1}^2}\,\|F\|_{\mathsf{L}^{2,q}(\Omega)}\,,\\
\|H_\delta\|_{\mathsf{Y}^{q+1}(\Omega)} &= \|H_\delta\|_{\Delta^{q+1}(\Omega)} \le \sqrt{1 + c_{\mathsf{p},q+1}^2}\,\|F\|_{\mathsf{L}^{2,q}(\Omega)}\,.
\end{aligned}$$

Since $\mathrm{d}\,\mathring{\mathsf{D}}^{q-1}(\Gamma_t,\Omega) = \mathrm{d}\,\mathsf{X}^{q-1}(\Omega)$ and $\delta\,\mathring{\Delta}^{q+1}(\Gamma_n,\Omega) = \delta\,\mathsf{Y}^{q+1}(\Omega)$ we see that (A.3) and (A.4) hold also for $E \in \mathring{\mathsf{D}}^{q-1}(\Gamma_t,\Omega)$ and $H \in \mathring{\Delta}^{q+1}(\Gamma_n,\Omega)$, respectively, and that

$$\begin{aligned}
F - \mathrm{d}\,E_{\mathrm{d}} &\in \big(\mathrm{d}\,\mathsf{X}^{q-1}(\Omega)\big)^\perp = \big(\mathrm{d}\,\mathring{\mathsf{D}}^{q-1}(\Gamma_t,\Omega)\big)^\perp = \mathring{\Delta}^q_0(\Gamma_n,\Omega),\\
F - \delta\,H_\delta &\in \big(\delta\,\mathsf{Y}^{q+1}(\Omega)\big)^\perp = \big(\delta\,\mathring{\Delta}^{q+1}(\Gamma_n,\Omega)\big)^\perp = \mathring{\mathsf{D}}^q_0(\Gamma_t,\Omega).
\end{aligned}$$

Hence, we have found our projections since

$$\pi_{\mathrm{d}} F := \mathrm{d}\, E_{\mathrm{d}} \in \mathrm{d}\, \mathsf{X}^{q-1}(\Omega) \subset \overset{\circ}{\mathsf{D}}{}^{q}_{0}(\Gamma_t, \Omega),$$
$$\pi_{\delta} F := \delta\, H_{\delta} \in \delta\, \mathsf{Y}^{q+1}(\Omega) \subset \overset{\circ}{\Delta}{}^{q}_{0}(\Gamma_n, \Omega)$$

and

$$\pi_{\mathcal{H}} F := F - \mathrm{d}\, E_{\mathrm{d}} - \delta\, H_{\delta} \in \overset{\circ}{\mathsf{D}}{}^{q}_{0}(\Gamma_t, \Omega) \cap \overset{\circ}{\Delta}{}^{q}_{0}(\Gamma_n, \Omega) = \mathcal{H}^q(\Omega).$$

Explicit formulas for the dimensions of $\mathcal{H}^q(\Omega)$ or explicit constructions of bases of $\mathcal{H}^q(\Omega)$ depending on the topology of the pair (Ω, Γ_t) can be found, e.g., in [22] for the case $\Gamma_t = \Gamma$ or $\Gamma_t = \emptyset$, or in [4] for the general case.

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