SCHRIFTENREIHE DER FAKULTÄT FÜR MATHEMATIK

Hilbert Complexes with Mixed Boundary Conditions: Regular Decompositions, Compact Embeddings, and Functional Analysis ToolBox
Part 2: Elasticity Complex

by

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HILBERT COMPLEXES WITH MIXED BOUNDARY CONDITIONS PART 2: ELASTICITY COMPLEX

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Abstract. We show that the elasticity Hilbert complex with mixed boundary conditions on bounded strong Lipschitz domains is closed and compact. The crucial results are compact embeddings which follow by abstract arguments using functional analysis together with particular regular decompositions. Higher Sobolev order results are proved as well. This paper extends recent results on the de Rham Hilbert complex with mixed boundary conditions from [11] and recent results on the elasticity Hilbert complex with empty or full boundary conditions from [15].

Contents

1. Introduction

In this paper we prove regular decompositions and resulting compact embeddings for the *elasticity complex*

$$\cdots \xrightarrow{\cdots} \mathsf{L}^2(\Omega) \xrightarrow{\operatorname{symGrad}} \mathsf{L}^2_{\mathbb{S}}(\Omega) \xrightarrow{\operatorname{RotRot}_{\mathbb{S}}^\top} \mathsf{L}^2_{\mathbb{S}}(\Omega) \xrightarrow{\operatorname{Div}_{\mathbb{S}}} \mathsf{L}^2(\Omega) \xrightarrow{\cdots} \cdots .$$

This extends the corresponding results from [11] for the de Rham complex

$$\cdots \xrightarrow{\cdots} \mathsf{L}^{q-1,2}(\Omega) \xrightarrow{\mathrm{d}^{q-1}} \mathsf{L}^{q,2}(\Omega) \xrightarrow{\mathrm{d}^{q}} \mathsf{L}^{q+1,2}(\Omega) \xrightarrow{\cdots} \cdots ,$$

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whose 3D-version for vector proxies is given by

$$\cdots \xrightarrow{\cdots} \mathsf{L}^2(\Omega) \xrightarrow{\mathrm{d}^0 \,\widehat{=}\, \mathrm{grad}} \mathsf{L}^2(\Omega) \xrightarrow{\mathrm{d}^1 \,\widehat{=}\, \mathrm{rot}} \mathsf{L}^2(\Omega) \xrightarrow{\mathrm{d}^2 \,\widehat{=}\, \mathrm{div}} \mathsf{L}^2(\Omega) \xrightarrow{\cdots} \cdots .$$

We shall consider mixed boundary conditions on a bounded strong Lipschitz domain $\Omega \subset \mathbb{R}^3$.

Like the de Rham complex, the elasticity complex has the geometric structure of a *Hilbert complex*, i.e.,

$$\cdots \xrightarrow{\cdots} \mathsf{H}_0 \xrightarrow{\mathrm{A}_0} \mathsf{H}_1 \xrightarrow{\mathrm{A}_1} \mathsf{H}_2 \xrightarrow{\cdots} \cdots , \qquad R(\mathrm{A}_0) \subset N(\mathrm{A}_1),$$

where A_0 and A_1 are densely defined and closed (unbounded) linear operators between Hilbert spaces H_ℓ. The corresponding *domain Hilbert complex* is denoted by

$$\cdots \xrightarrow{\cdots} D(\mathrm{A}_0) \xrightarrow{\mathrm{A}_0} D(\mathrm{A}_0) \xrightarrow{\mathrm{A}_1} \mathsf{H}_2 \xrightarrow{\cdots} \cdots .$$

In fact, we show that the assumptions of [11, Lemma 2.22] hold, which provides an elegant, abstract, and short way to prove the crucial compact embeddings

$$D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) \hookrightarrow \mathsf{H}_1 \tag{1}$$

for the elasticity Hilbert complex. In principle, our general technique – compact embeddings by regular decompositions and Rellich's selection theorem – works for all Hilbert complexes known in the literature, see, e.g., [1] for a comprehensive list of such Hilbert complexes.

Roughly speaking a regular decomposition has the form

$$D(\mathrm{A}_1) = \mathsf{H}_1^+ + \mathrm{A}_0\,\mathsf{H}_0^+$$

with regular subspaces $\mathsf{H}_0^+ \subset D(\mathrm{A}_0)$ and $\mathsf{H}_1^+ \subset D(\mathrm{A}_1)$ such that the embeddings $\mathsf{H}_0^+ \hookrightarrow \mathsf{H}_0$ and $\mathsf{H}_1^+ \hookrightarrow \mathsf{H}_1$ are compact. The compactness is typically and simply given by Rellich's selection theorem, which justifies the notion "regular". By applying A_1 any regular decomposition implies regular potentials

$$R(\mathrm{A}_1) = \mathrm{A}_1\,\mathsf{H}_1^+$$

by the complex property. The respective regular potential and decomposition operators

$$\mathcal{P}_{\mathrm{A}_1} : R(\mathrm{A}_1) \to \mathsf{H}_1^+, \qquad \mathcal{Q}_{\mathrm{A}_1}^1 : D(\mathrm{A}_1) \to \mathsf{H}_1^+, \qquad \mathcal{Q}_{\mathrm{A}_1}^0 : D(\mathrm{A}_1) \to \mathsf{H}_0^+$$

are bounded and satisfy $\mathrm{A}_1\,\mathcal{P}_{\mathrm{A}_1} = \mathrm{id}_{R(\mathrm{A}_1)}$ as well as $\mathrm{id}_{D(\mathrm{A}_1)} = \mathcal{Q}_{\mathrm{A}_1}^1 + \mathrm{A}_0\,\mathcal{Q}_{\mathrm{A}_1}^0$.

Note that (1) implies several important results related to the particular Hilbert complex by the so-called FA-ToolBox, such as closed ranges, Friedrichs/Poincaré type estimates, Helmholtz typ decompositions, and comprehensive solution theories, cf. [7, 8, 9, 10] and [13, 14, 15].

For an historical overview on the compact embeddings (1) corresponding to the de Rham complex and Maxwell's equations, i.e., Weck's or Weber-Weck-Picard's selection theorem, see, e.g., the introductions in [2, 6], the original papers [19, 18, 16, 20, 5, 17], and the recent state of the art results for mixed boundary conditions and bounded weak Lipschitz domains in [2, 3, 4]. Compact embeddings (1) corresponding to the biharmonic and the elasticity complex are given in [15] and [13, 14], respectively. Note that in the recent paper [1] similar results have been shown for no or full boundary conditions using an alternative and more algebraic approach, the so-called Bernstein-Gelfand-Gelfand resolution (BGG).

2. Elasticity Complexes I

Throughout this paper, let $\Omega \subset \mathbb{R}^3$ be a *bounded strong Lipschitz domain* with boundary Γ, decomposed into two parts Γ_t and $\Gamma_n := \Gamma \setminus \overline{\Gamma_t}$ with some *relatively open and strong Lipschitz boundary part* $\Gamma_t \subset \Gamma$.

2.1. **Notations and Preliminaries.** We will strongly use the notations and results from our corresponding papers for the elasticity complex [15] and for the de Rham complex [11]. In particular, we recall [11, Section 2, Section 3] including the notion of *extendable domains.*

We utilise the standard Sobolev spaces from [11], e.g., the usual Lebesgue and Sobolev spaces (scalar or tensor valued) $\mathsf{L}^2(\Omega)$ and $\mathsf{H}^k(\Omega)$ with $k \in \mathbb{N}_0$. Boundary conditions are introduced in the *strong sense* as closures of respective test fields, i.e.,

$$\mathsf{H}^k_{\Gamma_t}(\Omega) := \overline{\mathsf{C}^\infty_{\Gamma_t}(\Omega)}^{\mathsf{H}^k(\Omega)},$$

we well as in the *weak sense* by

$$\mathbf{H}^k_{\Gamma_t}(\Omega) := \big\{ u \in \mathsf{H}^k(\Omega) : \langle \partial^\alpha u, \phi \rangle_{\mathsf{L}^2(\Omega)} = (-1)^{|\alpha|} \langle u, \partial^\alpha \phi \rangle_{\mathsf{L}^2(\Omega)} \quad \forall\, \phi \in \mathsf{C}^\infty_{\Gamma_n}(\Omega) \quad \forall\, |\alpha| \leq k \big\}.$$

Lemma 2.1 ([11, Lemma 3.2, Theorem 4.6])**.** $\mathbf{H}^k_{\Gamma_t}(\Omega) = \mathsf{H}^k_{\Gamma_t}(\Omega)$, *i.e., weak and strong boundary conditions coincide for the standard Sobolev spaces with mixed boundary conditions.*

We shall use the abbreviations $\mathsf{H}^k_{\emptyset}(\Omega) = \mathsf{H}^k(\Omega)$ and $\mathsf{H}^0_{\Gamma_t}(\Omega) = \mathsf{L}^2(\Omega)$, where the first notion is actually a density result and incorporated into the notation by purpose.

2.2. **Operators.** Let symGrad, $\operatorname{RotRot}^\top$, and Div (here Grad, Rot, and Div act row-wise as the operators grad, rot, and div from the vector de Rham complex) be realised as densely defined (unbounded) linear operators

$$\begin{aligned}
\mathring{\operatorname{symGrad}}_{\Gamma_t} : D(\mathring{\operatorname{symGrad}}_{\Gamma_t}) \subset \mathsf{L}^2(\Omega) \to \mathsf{L}^2_{\mathbb{S}}(\Omega); &\quad v \mapsto \operatorname{sym}\operatorname{Grad} v = \frac{1}{2}\big(\operatorname{Grad} v + (\operatorname{Grad} v)^\top\big),\\
\mathring{\operatorname{RotRot}}^\top_{\mathbb{S},\Gamma_t} : D(\mathring{\operatorname{RotRot}}^\top_{\mathbb{S},\Gamma_t}) \subset \mathsf{L}^2_{\mathbb{S}}(\Omega) \to \mathsf{L}^2_{\mathbb{S}}(\Omega); &\quad S \mapsto \operatorname{RotRot}^\top S = \operatorname{Rot}\big((\operatorname{Rot} S)^\top\big),\\
\mathring{\operatorname{Div}}_{\mathbb{S},\Gamma_t} : D(\mathring{\operatorname{Div}}_{\mathbb{S},\Gamma_t}) \subset \mathsf{L}^2_{\mathbb{S}}(\Omega) \to \mathsf{L}^2(\Omega); &\quad T \mapsto \operatorname{Div} T
\end{aligned}$$

(S, T, $\operatorname{Grad} v$, $\operatorname{sym}\operatorname{Grad} v$, $\operatorname{Rot} S$, $\operatorname{RotRot}^\top S$ are (3×3)-tensor fields, and v, $\operatorname{Div} T$ are 3-vector fields) with domains of definition

$$D(\mathring{\operatorname{symGrad}}_{\Gamma_t}) := \mathsf{C}^\infty_{\Gamma_t}(\Omega), \qquad D(\mathring{\operatorname{RotRot}}^\top_{\mathbb{S},\Gamma_t}) := \mathsf{C}^\infty_{\mathbb{S},\Gamma_t}(\Omega), \qquad D(\mathring{\operatorname{Div}}_{\mathbb{S},\Gamma_t}) := \mathsf{C}^\infty_{\mathbb{S},\Gamma_t}(\Omega)$$

satisfying the complex properties

$$\mathring{\operatorname{RotRot}}^\top_{\mathbb{S},\Gamma_t} \mathring{\operatorname{symGrad}}_{\Gamma_t} \subset 0, \qquad \mathring{\operatorname{Div}}_{\mathbb{S},\Gamma_t} \mathring{\operatorname{RotRot}}^\top_{\mathbb{S},\Gamma_t} \subset 0.$$

For elementary properties of these operators see, e.g., [15], in particular, we have the collection of formulas presented in Lemma A.1. Here, we introduce the Lebesgue Hilbert space and the test space of symmetric tensor fields

$$\mathsf{L}^2_{\mathbb{S}}(\Omega) := \big\{ S \in \mathsf{L}^2(\Omega) : S^\top = S \big\}, \qquad \mathsf{C}^\infty_{\mathbb{S},\Gamma_t}(\Omega) := \mathsf{C}^\infty_{\Gamma_t}(\Omega) \cap \mathsf{L}^2_{\mathbb{S}}(\Omega),$$

respectively. We get the elasticity complex on smooth tensor fields

$$\cdots \xrightarrow{\ \cdots\ } \mathsf{L}^2(\Omega) \xrightarrow{\mathring{\operatorname{symGrad}}_{\Gamma_t}} \mathsf{L}^2_{\mathbb{S}}(\Omega) \xrightarrow{\mathring{\operatorname{RotRot}}^\top_{\mathbb{S},\Gamma_t}} \mathsf{L}^2_{\mathbb{S}}(\Omega) \xrightarrow{\mathring{\operatorname{Div}}_{\mathbb{S},\Gamma_t}} \mathsf{L}^2(\Omega) \xrightarrow{\ \cdots\ } \cdots.$$

The closures

$$\operatorname{symGrad}_{\Gamma_t} := \overline{\mathring{\operatorname{symGrad}}_{\Gamma_t}}, \qquad \operatorname{RotRot}^\top_{\mathbb{S},\Gamma_t} := \overline{\mathring{\operatorname{RotRot}}^\top_{\mathbb{S},\Gamma_t}}, \qquad \operatorname{Div}_{\mathbb{S},\Gamma_t} := \overline{\mathring{\operatorname{Div}}_{\mathbb{S},\Gamma_t}}$$

and Hilbert space adjoints

$$\operatorname{symGrad}^*_{\Gamma_t} = \mathring{\operatorname{symGrad}}^*_{\Gamma_t}, \qquad (\operatorname{RotRot}^\top_{\mathbb{S},\Gamma_t})^* = (\mathring{\operatorname{RotRot}}^\top_{\mathbb{S},\Gamma_t})^*, \qquad \operatorname{Div}^*_{\mathbb{S},\Gamma_t} = \mathring{\operatorname{Div}}^*_{\mathbb{S},\Gamma_t}$$

are given by the densely defined and closed linear operators

$$\begin{aligned}
\mathrm{A}_0 := \operatorname{symGrad}_{\Gamma_t} : D(\operatorname{symGrad}_{\Gamma_t}) \subset \mathsf{L}^2(\Omega) \to \mathsf{L}^2_{\mathbb{S}}(\Omega); &\quad v \mapsto \operatorname{symGrad} v,\\
\mathrm{A}_1 := \operatorname{RotRot}^\top_{\mathbb{S},\Gamma_t} : D(\operatorname{RotRot}^\top_{\mathbb{S},\Gamma_t}) \subset \mathsf{L}^2_{\mathbb{S}}(\Omega) \to \mathsf{L}^2_{\mathbb{S}}(\Omega); &\quad S \mapsto \operatorname{RotRot}^\top S,\\
\mathrm{A}_2 := \operatorname{Div}_{\mathbb{S},\Gamma_t} : D(\operatorname{Div}_{\mathbb{S},\Gamma_t}) \subset \mathsf{L}^2_{\mathbb{S}}(\Omega) \to \mathsf{L}^2(\Omega); &\quad T \mapsto \operatorname{Div} T,\\
\mathrm{A}_0^* = \operatorname{symGrad}^*_{\Gamma_t} = -\mathbf{Div}_{\mathbb{S},\Gamma_n} : D(\mathbf{Div}_{\mathbb{S},\Gamma_n}) \subset \mathsf{L}^2_{\mathbb{S}}(\Omega) \to \mathsf{L}^2(\Omega); &\quad S \mapsto -\operatorname{Div} S,\\
\mathrm{A}_1^* = (\operatorname{RotRot}^\top_{\mathbb{S},\Gamma_t})^* = \mathbf{RotRot}^\top_{\mathbb{S},\Gamma_n} : D(\mathbf{RotRot}^\top_{\mathbb{S},\Gamma_n}) \subset \mathsf{L}^2_{\mathbb{S}}(\Omega) \to \mathsf{L}^2_{\mathbb{S}}(\Omega); &\quad T \mapsto \operatorname{RotRot}^\top T,
\end{aligned}$$

$$\mathrm{A}_2^* = \mathrm{Div}_{\mathbb{S},\Gamma_t}^* = -\mathbf{symGrad}_{\Gamma_n} : D(\mathbf{symGrad}_{\Gamma_n}) \subset \mathsf{L}^2(\Omega) \to \mathsf{L}^2_{\mathbb{S}}(\Omega); \quad v \mapsto -\operatorname{symGrad} v$$

with domains of definition

$$\begin{aligned} D(\mathrm{A}_0) &= D(\mathrm{symGrad}_{\Gamma_t}) = \mathsf{H}_{\Gamma_t}(\mathrm{symGrad},\Omega), & D(\mathrm{A}_0^*) &= D(\mathbf{Div}_{\mathbb{S},\Gamma_n}) = \mathbf{H}_{\mathbb{S},\Gamma_n}(\mathrm{Div},\Omega),\\ D(\mathrm{A}_1) &= D(\mathrm{RotRot}^\top_{\mathbb{S},\Gamma_t}) = \mathsf{H}_{\mathbb{S},\Gamma_t}(\mathrm{RotRot}^\top,\Omega), & D(\mathrm{A}_1^*) &= D(\mathbf{RotRot}^\top_{\mathbb{S},\Gamma_n}) = \mathbf{H}_{\mathbb{S},\Gamma_n}(\mathrm{RotRot}^\top,\Omega),\\ D(\mathrm{A}_2) &= D(\mathrm{Div}_{\mathbb{S},\Gamma_t}) = \mathsf{H}_{\mathbb{S},\Gamma_t}(\mathrm{Div},\Omega), & D(\mathrm{A}_2^*) &= D(\mathbf{symGrad}_{\Gamma_n}) = \mathbf{H}_{\Gamma_n}(\mathrm{symGrad},\Omega). \end{aligned}$$

We shall introduce the latter Sobolev spaces in the next section.

2.3. **Sobolev Spaces.** Let

$$\begin{aligned} \mathsf{H}(\mathrm{symGrad},\Omega) &:= \big\{v \in \mathsf{L}^2(\Omega) : \operatorname{symGrad} v \in \mathsf{L}^2(\Omega)\big\},\\ \mathsf{H}_{\mathbb{S}}(\mathrm{RotRot}^\top,\Omega) &:= \big\{S \in \mathsf{L}^2_{\mathbb{S}}(\Omega) : \mathrm{RotRot}^\top S \in \mathsf{L}^2(\Omega)\big\},\\ \mathsf{H}_{\mathbb{S}}(\mathrm{Div},\Omega) &:= \big\{T \in \mathsf{L}^2_{\mathbb{S}}(\Omega) : \operatorname{Div} T \in \mathsf{L}^2(\Omega)\big\}. \end{aligned}$$

Note that $M \in \mathsf{H}_{\mathbb{S}}(\mathrm{RotRot}^\top,\Omega)$ implies $\mathrm{RotRot}^\top M \in \mathsf{L}^2_{\mathbb{S}}(\Omega)$, and that we have by Korn's inequality the regularity

$$\mathsf{H}(\mathrm{symGrad},\Omega) = \mathsf{H}^1(\Omega)$$

with equivalent norms. Moreover, we define boundary conditions in the *strong sense* as closures of respective test fields, i.e.,

$$\begin{aligned} \mathsf{H}_{\Gamma_t}(\mathrm{symGrad},\Omega) &:= \overline{\mathsf{C}^\infty_{\Gamma_t}(\Omega)}^{\mathsf{H}(\mathrm{symGrad},\Omega)},\\ \mathsf{H}_{\mathbb{S},\Gamma_t}(\mathrm{RotRot}^\top,\Omega) &:= \overline{\mathsf{C}^\infty_{\mathbb{S},\Gamma_t}(\Omega)}^{\mathsf{H}_{\mathbb{S}}(\mathrm{RotRot}^\top,\Omega)},\\ \mathsf{H}_{\mathbb{S},\Gamma_t}(\mathrm{Div},\Omega) &:= \overline{\mathsf{C}^\infty_{\mathbb{S},\Gamma_t}(\Omega)}^{\mathsf{H}_{\mathbb{S}}(\mathrm{Div},\Omega)}, \end{aligned}$$

and we have $\mathsf{H}_\emptyset(\mathrm{symGrad},\Omega) = \mathsf{H}(\mathrm{symGrad},\Omega) = \mathsf{H}^1(\Omega)$, $\mathsf{H}_{\mathbb{S},\emptyset}(\mathrm{RotRot}^\top,\Omega) = \mathsf{H}_{\mathbb{S}}(\mathrm{RotRot}^\top,\Omega)$, and $\mathsf{H}_{\mathbb{S},\emptyset}(\mathrm{Div},\Omega) = \mathsf{H}_{\mathbb{S}}(\mathrm{Div},\Omega)$, which are density results and incorporated into the notation by purpose. Spaces with vanishing $\mathrm{RotRot}^\top$ and Div are denoted by $\mathsf{H}_{\mathbb{S},\Gamma_t,0}(\mathrm{RotRot}^\top,\Omega)$ and $\mathsf{H}_{\mathbb{S},\Gamma_t,0}(\mathrm{Div},\Omega)$, respectively. Note that, again by Korn's inequality, we have

$$\mathsf{H}_{\Gamma_t}(\mathrm{symGrad},\Omega) = \mathsf{H}^1_{\Gamma_t}(\Omega).$$

We need also the Sobolev spaces with boundary conditions defined in the *weak sense*, i.e.,

$$\begin{aligned} \mathbf{H}_{\Gamma_t}(\mathrm{symGrad},\Omega) &:= \big\{v \in \mathsf{H}(\mathrm{symGrad},\Omega) : \langle \operatorname{symGrad} v, \Phi\rangle_{\mathsf{L}^2(\Omega)} = -\langle v, \operatorname{Div}\Phi\rangle_{\mathsf{L}^2(\Omega)}\\ &\qquad\qquad \forall\, \Phi \in \mathsf{C}^\infty_{\mathbb{S},\Gamma_n}(\Omega)\big\},\\ \mathbf{H}_{\mathbb{S},\Gamma_t}(\mathrm{RotRot}^\top,\Omega) &:= \big\{S \in \mathsf{H}_{\mathbb{S}}(\mathrm{RotRot}^\top,\Omega) : \langle \mathrm{RotRot}^\top S, \Psi\rangle_{\mathsf{L}^2(\Omega)} = \langle S, \mathrm{RotRot}^\top \Psi\rangle_{\mathsf{L}^2(\Omega)}\\ &\qquad\qquad \forall\, \Psi \in \mathsf{C}^\infty_{\mathbb{S},\Gamma_n}(\Omega)\big\},\\ \mathbf{H}_{\mathbb{S},\Gamma_t}(\mathrm{Div},\Omega) &:= \big\{T \in \mathsf{H}_{\mathbb{S}}(\mathrm{Div},\Omega) : \langle \operatorname{Div} T, \phi\rangle_{\mathsf{L}^2(\Omega)} = -\langle T, \operatorname{symGrad}\phi\rangle_{\mathsf{L}^2(\Omega)}\\ &\qquad\qquad \forall\, \phi \in \mathsf{C}^\infty_{\Gamma_n}(\Omega)\big\}. \end{aligned}$$

Note that "*strong* $\subset$ *weak*" holds, i.e., $\mathsf{H}_{\dots}(\cdots,\Omega) \subset \mathbf{H}_{\dots}(\cdots,\Omega)$, e.g.,

$$\mathsf{H}_{\mathbb{S},\Gamma_t}(\mathrm{RotRot}^\top,\Omega) \subset \mathbf{H}_{\mathbb{S},\Gamma_t}(\mathrm{RotRot}^\top,\Omega), \qquad \mathsf{H}_{\mathbb{S},\Gamma_t}(\mathrm{Div},\Omega) \subset \mathbf{H}_{\mathbb{S},\Gamma_t}(\mathrm{Div},\Omega),$$

and that the complex properties hold in both the strong and the weak case, e.g.,

$$\operatorname{symGrad} \mathsf{H}_{\Gamma_t}(\Omega) \subset \mathsf{H}_{\mathbb{S},\Gamma_t,0}(\mathrm{RotRot}^\top,\Omega), \qquad \mathrm{RotRot}^\top \mathbf{H}_{\mathbb{S},\Gamma_t}(\mathrm{RotRot}^\top,\Omega) \subset \mathbf{H}_{\mathbb{S},\Gamma_t,0}(\mathrm{Div},\Omega),$$

which follows immediately by the definitions. In Remark 2.4 below we comment on the question whether "*strong* $=$ *weak*" holds in general.

2.4. **Higher Order Sobolev Spaces.** For $k \in \mathbb{N}_0$ we define higher order Sobolev spaces by

$$\begin{aligned}
\mathsf{H}^k_{\mathbb{S}}(\Omega) &:= \mathsf{H}^k(\Omega) \cap \mathsf{L}^2_{\mathbb{S}}(\Omega),\\
\mathsf{H}^k_{\mathbb{S},\Gamma_t}(\Omega) &:= \overline{\mathsf{C}^{\infty}_{\mathbb{S},\Gamma_t}(\Omega)}^{\mathsf{H}^k(\Omega)} = \mathsf{H}^k_{\Gamma_t}(\Omega) \cap \mathsf{L}^2_{\mathbb{S}}(\Omega),\\
\mathsf{H}^k(\operatorname{symGrad},\Omega) &:= \big\{v \in \mathsf{H}^k(\Omega) : \operatorname{symGrad} v \in \mathsf{H}^k(\Omega)\big\},\\
\mathsf{H}^k_{\Gamma_t}(\operatorname{symGrad},\Omega) &:= \big\{v \in \mathsf{H}^k_{\Gamma_t}(\Omega) \cap \mathsf{H}_{\Gamma_t}(\operatorname{symGrad},\Omega) : \operatorname{symGrad} v \in \mathsf{H}^k_{\Gamma_t}(\Omega)\big\},\\
\mathsf{H}^k_{\mathbb{S}}(\operatorname{RotRot}^\top,\Omega) &:= \big\{S \in \mathsf{H}^k_{\mathbb{S}}(\Omega) : \operatorname{RotRot}^\top S \in \mathsf{H}^k(\Omega)\big\},\\
\mathsf{H}^k_{\mathbb{S},\Gamma_t}(\operatorname{RotRot}^\top,\Omega) &:= \big\{S \in \mathsf{H}^k_{\Gamma_t}(\Omega) \cap \mathsf{H}_{\mathbb{S},\Gamma_t}(\operatorname{RotRot}^\top,\Omega) : \operatorname{RotRot}^\top S \in \mathsf{H}^k_{\Gamma_t}(\Omega)\big\},\\
\mathsf{H}^k_{\mathbb{S}}(\operatorname{Div},\Omega) &:= \big\{T \in \mathsf{H}^k_{\mathbb{S}}(\Omega) : \operatorname{Div} T \in \mathsf{H}^k(\Omega)\big\},\\
\mathsf{H}^k_{\mathbb{S},\Gamma_t}(\operatorname{Div},\Omega) &:= \big\{T \in \mathsf{H}^k_{\Gamma_t}(\Omega) \cap \mathsf{H}_{\mathbb{S},\Gamma_t}(\operatorname{Div},\Omega) : \operatorname{Div} T \in \mathsf{H}^k_{\Gamma_t}(\Omega)\big\}.
\end{aligned}$$

We see $\mathsf{H}^k_{\mathbb{S},\emptyset}(\operatorname{RotRot}^\top,\Omega) = \mathsf{H}^k_{\mathbb{S}}(\operatorname{RotRot}^\top,\Omega)$ and $\mathsf{H}^0_{\mathbb{S},\emptyset}(\operatorname{RotRot}^\top,\Omega) = \mathsf{H}_{\mathbb{S}}(\operatorname{RotRot}^\top,\Omega)$ as well as $\mathsf{H}^0_{\mathbb{S},\Gamma_t}(\operatorname{RotRot}^\top,\Omega) = \mathsf{H}_{\mathbb{S},\Gamma_t}(\operatorname{RotRot}^\top,\Omega)$. Note that for $\Gamma_t \neq \emptyset$ it holds

$$\mathsf{H}^k_{\mathbb{S},\Gamma_t}(\operatorname{RotRot}^\top,\Omega) = \big\{S \in \mathsf{H}^k_{\mathbb{S},\Gamma_t}(\Omega) : \operatorname{RotRot}^\top S \in \mathsf{H}^k_{\Gamma_t}(\Omega)\big\}, \qquad k \geq 2, \tag{2}$$

but for $\Gamma_t \neq \emptyset$ and $k = 0$ and $k = 1$ $\big($as $\mathsf{H}^0_{\mathbb{S},\Gamma_t}(\Omega) = \mathsf{L}^2_{\mathbb{S}}(\Omega)\big)$

$$\begin{aligned}
\mathsf{H}^0_{\mathbb{S},\Gamma_t}(\operatorname{RotRot}^\top,\Omega) &= \mathsf{H}_{\mathbb{S},\Gamma_t}(\operatorname{RotRot}^\top,\Omega)\\
&\subsetneq \big\{S \in \mathsf{H}^0_{\mathbb{S},\Gamma_t}(\Omega) : \operatorname{RotRot}^\top S \in \mathsf{H}^0_{\Gamma_t}(\Omega)\big\} = \mathsf{H}_{\mathbb{S}}(\operatorname{RotRot}^\top,\Omega),\\
\mathsf{H}^1_{\mathbb{S},\Gamma_t}(\operatorname{RotRot}^\top,\Omega) &\subsetneq \big\{S \in \mathsf{H}^1_{\mathbb{S},\Gamma_t}(\Omega) : \operatorname{RotRot}^\top S \in \mathsf{H}^1_{\Gamma_t}(\Omega)\big\},
\end{aligned}$$

respectively. As before, we introduce the kernels

$$\begin{aligned}
\mathsf{H}^k_{\mathbb{S},\Gamma_t,0}(\operatorname{RotRot}^\top,\Omega) &:= \mathsf{H}^k_{\Gamma_t}(\Omega) \cap \mathsf{H}_{\mathbb{S},\Gamma_t,0}(\operatorname{RotRot}^\top,\Omega) = \mathsf{H}^k_{\mathbb{S},\Gamma_t}(\operatorname{RotRot}^\top,\Omega) \cap \mathsf{H}_{\mathbb{S},0}(\operatorname{RotRot}^\top,\Omega)\\
&= \big\{S \in \mathsf{H}^k_{\mathbb{S},\Gamma_t}(\operatorname{RotRot}^\top,\Omega) : \operatorname{RotRot}^\top S = 0\big\}.
\end{aligned}$$

The corresponding remarks and definitions extend to the $\mathsf{H}^k_{\Gamma_t}(\operatorname{symGrad},\Omega)$-spaces and $\mathsf{H}^k_{\mathbb{S},\Gamma_t}(\operatorname{Div},\Omega)$-spaces as well. In particular, we have for $\Gamma_t \neq \emptyset$ and $k \geq 1$

$$\begin{aligned}
\mathsf{H}^k_{\Gamma_t}(\operatorname{symGrad},\Omega) &= \big\{v \in \mathsf{H}^k_{\Gamma_t}(\Omega) : \operatorname{symGrad} v \in \mathsf{H}^k_{\Gamma_t}(\Omega)\big\},\\
\mathsf{H}^k_{\mathbb{S},\Gamma_t}(\operatorname{Div},\Omega) &= \big\{T \in \mathsf{H}^k_{\mathbb{S},\Gamma_t}(\Omega) : \operatorname{Div} T \in \mathsf{H}^k_{\Gamma_t}(\Omega)\big\},
\end{aligned} \tag{3}$$

and

$$\begin{aligned}
\mathsf{H}^0_{\Gamma_t}(\operatorname{symGrad},\Omega) = \mathsf{H}_{\Gamma_t}(\operatorname{symGrad},\Omega) &\subsetneq \big\{v \in \mathsf{H}^0_{\Gamma_t}(\Omega) : \operatorname{symGrad} v \in \mathsf{H}^0_{\Gamma_t}(\Omega)\big\} = \mathsf{H}(\operatorname{symGrad},\Omega),\\
\mathsf{H}^0_{\mathbb{S},\Gamma_t}(\operatorname{Div},\Omega) = \mathsf{H}_{\mathbb{S},\Gamma_t}(\operatorname{Div},\Omega) &\subsetneq \big\{T \in \mathsf{H}^0_{\mathbb{S},\Gamma_t}(\Omega) : \operatorname{Div} T \in \mathsf{H}^0_{\Gamma_t}(\Omega)\big\} = \mathsf{H}_{\mathbb{S}}(\operatorname{Div},\Omega),
\end{aligned}$$

as well as

$$\begin{aligned}
\mathsf{H}^k_{\mathbb{S},\Gamma_t,0}(\operatorname{Div},\Omega) &= \mathsf{H}^k_{\Gamma_t}(\Omega) \cap \mathsf{H}_{\mathbb{S},\Gamma_t,0}(\operatorname{Div},\Omega) = \mathsf{H}^k_{\mathbb{S},\Gamma_t}(\operatorname{Div},\Omega) \cap \mathsf{H}_{\mathbb{S},0}(\operatorname{Div},\Omega)\\
&= \big\{T \in \mathsf{H}^k_{\mathbb{S},\Gamma_t}(\operatorname{Div},\Omega) : \operatorname{Div} T = 0\big\}.
\end{aligned}$$

Analogously, we define the Sobolev spaces $\mathbf{H}^k_{\Gamma_t}(\operatorname{symGrad},\Omega)$, $\mathbf{H}^k_{\mathbb{S},\Gamma_t}(\operatorname{RotRot}^\top,\Omega)$, $\mathbf{H}^k_{\mathbb{S},\Gamma_t}(\operatorname{Div},\Omega)$, and $\mathbf{H}^k_{\mathbb{S},\Gamma_t,0}(\operatorname{RotRot}^\top,\Omega)$, $\mathbf{H}^k_{\mathbb{S},\Gamma_t,0}(\operatorname{Div},\Omega)$ using the respective Sobolev spaces with weak boundary conditions. Note that again "*strong* ⊂ *weak*" holds, i.e., $\mathsf{H}^{\cdots}_{\cdots}(\cdots,\Omega) \subset \mathbf{H}^{\cdots}_{\cdots}(\cdots,\Omega)$, e.g.,

$$\mathsf{H}^k_{\mathbb{S},\Gamma_t}(\operatorname{RotRot}^\top,\Omega) \subset \mathbf{H}^k_{\mathbb{S},\Gamma_t}(\operatorname{RotRot}^\top,\Omega), \qquad \mathsf{H}^k_{\mathbb{S},\Gamma_t}(\operatorname{Div},\Omega) \subset \mathbf{H}^k_{\mathbb{S},\Gamma_t}(\operatorname{Div},\Omega),$$

and that the complex properties hold in both the strong and the weak case, e.g.,

$$\operatorname{symGrad} \mathsf{H}^{k+1}_{\Gamma_t}(\Omega) \subset \mathsf{H}^k_{\mathbb{S},\Gamma_t,0}(\operatorname{RotRot}^\top,\Omega), \qquad \operatorname{RotRot}^\top \mathbf{H}^k_{\mathbb{S},\Gamma_t}(\operatorname{RotRot}^\top,\Omega) \subset \mathbf{H}^k_{\mathbb{S},\Gamma_t,0}(\operatorname{Div},\Omega).$$

Moreover, the corresponding results for (2) and (3) hold for the weak spaces as well.

In the forthcoming sections we shall also investigate whether indeed "*strong* = *weak*" holds. We start with a simple implication from Lemma 2.1.

Corollary 2.2. $\mathbf{H}^k_{\mathbb{S},\Gamma_t}(\Omega) = \mathsf{H}^k_{\mathbb{S},\Gamma_t}(\Omega)$*, i.e., weak and strong boundary conditions coincide for the standard Sobolev spaces of symmetric tensor fields with mixed boundary conditions.*

Lemma 2.1, Corollary 2.2, (2), (3), and Korn's inequality show the following.

Lemma 2.3 (higher order weak and strong partial boundary conditions coincide)**.**

(i) *For $k \geq 0$ it holds* $\mathsf{H}^k_{\Gamma_t}(\operatorname{symGrad}, \Omega) = \mathsf{H}^{k+1}_{\Gamma_t}(\Omega) = \mathbf{H}^{k+1}_{\Gamma_t}(\Omega)$.

(ii) *For $k \geq 1$ it holds*

$$\mathbf{H}^k_{\Gamma_t}(\operatorname{symGrad}, \Omega) = \{v \in \mathsf{H}^k_{\Gamma_t}(\Omega) : \operatorname{symGrad} v \in \mathsf{H}^k_{\Gamma_t}(\Omega)\} = \mathsf{H}^k_{\Gamma_t}(\operatorname{symGrad}, \Omega) = \mathsf{H}^{k+1}_{\Gamma_t}(\Omega),$$
$$\mathbf{H}^k_{\mathbb{S},\Gamma_t}(\operatorname{Div}, \Omega) = \{T \in \mathsf{H}^k_{\mathbb{S},\Gamma_t}(\Omega) : \operatorname{Div} T \in \mathsf{H}^k_{\Gamma_t}(\Omega)\} = \mathsf{H}^k_{\mathbb{S},\Gamma_t}(\operatorname{Div}, \Omega).$$

(iii) *For $k \geq 2$ it holds*

$$\mathbf{H}^k_{\mathbb{S},\Gamma_t}(\operatorname{RotRot}^\top, \Omega) = \{S \in \mathsf{H}^k_{\mathbb{S},\Gamma_t}(\Omega) : \operatorname{RotRot}^\top S \in \mathsf{H}^k_{\Gamma_t}(\Omega)\} = \mathsf{H}^k_{\mathbb{S},\Gamma_t}(\operatorname{RotRot}^\top, \Omega).$$

Remark 2.4 (weak and strong partial boundary conditions coincide)**.** *In* [15] *we could prove the corresponding results "*strong = weak*" for the whole elasticity complex but only with empty or full boundary conditions ($\Gamma_t = \emptyset$ or $\Gamma_t = \Gamma$). Therefore, in these special cases, the adjoints are well-defined on the spaces with strong boundary conditions as well.*

*Lemma 2.3 shows that for higher values of k indeed "*strong = weak*" holds. Thus to show "*strong = weak*" in general we only have to prove that equality holds in the remains cases $k = 0$ and $k = 1$, i.e., we only have to show*

$$\mathbf{H}_{\Gamma_t}(\operatorname{symGrad}, \Omega) \subset \mathsf{H}_{\Gamma_t}(\operatorname{symGrad}, \Omega), \qquad \mathbf{H}_{\mathbb{S},\Gamma_t}(\operatorname{RotRot}^\top, \Omega) \subset \mathsf{H}_{\mathbb{S},\Gamma_t}(\operatorname{RotRot}^\top, \Omega),$$
$$\mathbf{H}_{\mathbb{S},\Gamma_t}(\operatorname{Div}, \Omega) \subset \mathsf{H}^k_{\mathbb{S},\Gamma_t}(\operatorname{Div}, \Omega), \qquad \mathbf{H}^1_{\mathbb{S},\Gamma_t}(\operatorname{RotRot}^\top, \Omega) \subset \mathsf{H}^1_{\mathbb{S},\Gamma_t}(\operatorname{RotRot}^\top, \Omega).$$

The most delicate situation appears due to the second order nature of $\operatorname{RotRot}^\top_{\mathbb{S}}$*. In Corollary 3.11 we shall show using regular decompositions that these results (weak and strong boundary conditions coincide for the elasticity complex for all $k \geq 0$) indeed hold true.*

2.5. **More Sobolev Spaces.** For $k \in \mathbb{N}$ we introduce also slightly less regular higher order Sobolev spaces by

$$\mathsf{H}^{k,k-1}_{\mathbb{S},\Gamma_t}(\operatorname{RotRot}^\top, \Omega) := \{S \in \mathsf{H}^k_{\Gamma_t}(\Omega) \cap \mathsf{H}_{\mathbb{S},\Gamma_t}(\operatorname{RotRot}^\top, \Omega) : \operatorname{RotRot}^\top S \in \mathsf{H}^{k-1}_{\Gamma_t}(\Omega)\},$$
$$\mathbf{H}^{k,k-1}_{\mathbb{S},\Gamma_t}(\operatorname{RotRot}^\top, \Omega) := \{S \in \mathbf{H}^k_{\Gamma_t}(\Omega) \cap \mathbf{H}_{\mathbb{S},\Gamma_t}(\operatorname{RotRot}^\top, \Omega) : \operatorname{RotRot}^\top S \in \mathbf{H}^{k-1}_{\Gamma_t}(\Omega)\},$$

and we extend all conventions of our notations. For the kernels we have

$$\mathsf{H}^{k,k-1}_{\mathbb{S},\Gamma_t,0}(\operatorname{RotRot}^\top, \Omega) = \mathsf{H}^k_{\mathbb{S},\Gamma_t,0}(\operatorname{RotRot}^\top, \Omega), \qquad \mathbf{H}^{k,k-1}_{\mathbb{S},\Gamma_t,0}(\operatorname{RotRot}^\top, \Omega) = \mathbf{H}^k_{\mathbb{S},\Gamma_t,0}(\operatorname{RotRot}^\top, \Omega).$$

Note that, as before, the intersection with $\mathsf{H}_{\mathbb{S},\Gamma_t}(\operatorname{RotRot}^\top, \Omega)$ and $\mathbf{H}_{\mathbb{S},\Gamma_t}(\operatorname{RotRot}^\top, \Omega)$ is only needed if $k = 1$. Again we have "*strong* $\subset$ *weak*", i.e., $\mathsf{H}^{k,k-1}_{\mathbb{S},\Gamma_t}(\operatorname{RotRot}^\top, \Omega) \subset \mathbf{H}^{k,k-1}_{\mathbb{S},\Gamma_t}(\operatorname{RotRot}^\top, \Omega)$, and in both cases (weak and strong) the complex properties hold, e.g.,

$$\operatorname{symGrad} \mathsf{H}^{k+1}_{\Gamma_t}(\Omega) \subset \mathsf{H}^{k,k-1}_{\mathbb{S},\Gamma_t,0}(\operatorname{RotRot}^\top, \Omega), \qquad \operatorname{RotRot}^\top \mathbf{H}^{k,k-1}_{\mathbb{S},\Gamma_t}(\operatorname{RotRot}^\top, \Omega) \subset \mathbf{H}^{k-1}_{\mathbb{S},\Gamma_t,0}(\operatorname{Div}, \Omega).$$

Similar to Lemma 2.3 we have the following.

Lemma 2.5 (higher order weak and strong partial boundary conditions coincide)**.** *For $k \geq 2$*

$$\mathbf{H}^{k,k-1}_{\mathbb{S},\Gamma_t}(\operatorname{RotRot}^\top, \Omega) = \{S \in \mathsf{H}^k_{\mathbb{S},\Gamma_t}(\Omega) : \operatorname{RotRot}^\top S \in \mathsf{H}^{k-1}_{\Gamma_t}(\Omega)\} = \mathsf{H}^{k,k-1}_{\mathbb{S},\Gamma_t}(\operatorname{RotRot}^\top, \Omega).$$

2.6. **Some Elasticity Complexes.** By definition we have densely defined and closed (unbounded) linear operators defining three dual pairs

$$(\operatorname{symGrad}_{\Gamma_t}, \operatorname{symGrad}^*_{\Gamma_t}) = (\operatorname{symGrad}_{\Gamma_t}, -\mathbf{Div}_{\mathbb{S},\Gamma_n}),$$
$$(\operatorname{RotRot}^\top_{\mathbb{S},\Gamma_t}, (\operatorname{RotRot}^\top_{\mathbb{S},\Gamma_t})^*) = (\operatorname{RotRot}^\top_{\mathbb{S},\Gamma_t}, \mathbf{RotRot}^\top_{\mathbb{S},\Gamma_n}),$$
$$(\operatorname{Div}_{\mathbb{S},\Gamma_t}, \operatorname{Div}^*_{\mathbb{S},\Gamma_t}) = (\operatorname{Div}_{\mathbb{S},\Gamma_t}, -\mathbf{symGrad}_{\Gamma_n}).$$

[11, Remark 2.5, Remark 2.6] show the complex properties

$$\operatorname{RotRot}^\top_{\mathbb{S},\Gamma_t} \operatorname{symGrad}_{\Gamma_t} \subset 0, \qquad \operatorname{Div}_{\mathbb{S},\Gamma_t} \operatorname{RotRot}^\top_{\mathbb{S},\Gamma_t} \subset 0,$$

$$-\mathbf{Div}_{\mathbb{S},\Gamma_n}\mathbf{RotRot}^{\top}_{\mathbb{S},\Gamma_n}\subset 0,\qquad -\mathbf{RotRot}^{\top}_{\mathbb{S},\Gamma_n}\mathbf{symGrad}_{\Gamma_n}\subset 0.$$

Hence we get the primal and dual elasticity Hilbert complex

$$\cdots \underset{\cdots}{\overset{\cdots}{\rightleftarrows}} \mathsf{L}^2(\Omega) \underset{-\mathbf{Div}_{\mathbb{S},\Gamma_n}}{\overset{\operatorname{symGrad}_{\Gamma_t}}{\rightleftarrows}} \mathsf{L}^2_{\mathbb{S}}(\Omega) \underset{\mathbf{RotRot}^{\top}_{\mathbb{S},\Gamma_n}}{\overset{\operatorname{RotRot}^{\top}_{\mathbb{S},\Gamma_t}}{\rightleftarrows}} \mathsf{L}^2_{\mathbb{S}}(\Omega) \underset{-\mathbf{symGrad}_{\Gamma_n}}{\overset{\operatorname{Div}_{\mathbb{S},\Gamma_t}}{\rightleftarrows}} \mathsf{L}^2(\Omega) \underset{\cdots}{\overset{\cdots}{\rightleftarrows}} \cdots \tag{4}$$

with the complex properties

$$R(\operatorname{symGrad}_{\Gamma_t})\subset N(\operatorname{RotRot}^{\top}_{\mathbb{S},\Gamma_t}),\qquad R(\mathbf{RotRot}^{\top}_{\mathbb{S},\Gamma_n})\subset N(\mathbf{Div}_{\mathbb{S},\Gamma_n}),$$
$$R(\operatorname{RotRot}^{\top}_{\mathbb{S},\Gamma_t})\subset N(\operatorname{Div}_{\mathbb{S},\Gamma_t}),\qquad R(\mathbf{symGrad}_{\Gamma_n})\subset N(\mathbf{RotRot}^{\top}_{\mathbb{S},\Gamma_n}).$$

The long primal and dual elasticity Hilbert complex, cf. [11, (12)], reads

$$\mathbb{RM}_{\Gamma_t} \underset{\pi_{\mathbb{RM}_{\Gamma_t}}}{\overset{\iota_{\mathbb{RM}_{\Gamma_t}}}{\rightleftarrows}} \mathsf{L}^2(\Omega) \underset{-\mathbf{Div}_{\mathbb{S},\Gamma_n}}{\overset{\operatorname{symGrad}_{\Gamma_t}}{\rightleftarrows}} \mathsf{L}^2_{\mathbb{S}}(\Omega) \underset{\mathbf{RotRot}^{\top}_{\mathbb{S},\Gamma_n}}{\overset{\operatorname{RotRot}^{\top}_{\mathbb{S},\Gamma_t}}{\rightleftarrows}} \mathsf{L}^2_{\mathbb{S}}(\Omega) \underset{-\mathbf{symGrad}_{\Gamma_n}}{\overset{\operatorname{Div}_{\mathbb{S},\Gamma_t}}{\rightleftarrows}} \mathsf{L}^2(\Omega) \underset{\iota_{\mathbb{RM}_{\Gamma_n}}}{\overset{\pi_{\mathbb{RM}_{\Gamma_n}}}{\rightleftarrows}} \mathbb{RM}_{\Gamma_n} \tag{5}$$

with the additional complex properties

$$R(\iota_{\mathbb{RM}_{\Gamma_t}})=N(\operatorname{symGrad}_{\Gamma_t})=\mathbb{RM}_{\Gamma_t},\qquad \overline{R(\mathbf{Div}_{\mathbb{S},\Gamma_n})}=\mathbb{RM}_{\Gamma_t}^{\perp_{\mathsf{L}^2(\Omega)}},$$
$$\overline{R(\operatorname{Div}_{\mathbb{S},\Gamma_t})}=\mathbb{RM}_{\Gamma_n}^{\perp_{\mathsf{L}^2(\Omega)}},\qquad R(\iota_{\mathbb{RM}_{\Gamma_n}})=N(\mathbf{symGrad}_{\Gamma_n})=\mathbb{RM}_{\Gamma_n},$$

where

$$\mathbb{RM}_{\Sigma}=\begin{cases}\{0\} & \text{if } \Sigma\neq\emptyset,\\ \mathbb{RM} & \text{if } \Sigma=\emptyset,\end{cases}\qquad\text{with}\qquad \mathbb{RM}:=\big\{x\mapsto Qx+q : Q\in\mathbb{R}^{3\times 3}\text{ skew},\ q\in\mathbb{R}^3\big\}$$

denoting the global rigid motions in Ω. Note that $\dim\mathbb{RM}=6$.

More generally, in addition to (5), we shall discuss for $k\in\mathbb{N}_0$ the higher Sobolev order (long primal and formally dual) elasticity Hilbert complexes (omitting Ω in the notation)

$$\mathbb{RM}_{\Gamma_t}\xrightarrow{\iota_{\mathbb{RM}_{\Gamma_t}}}\mathsf{H}^k_{\Gamma_t}\xrightarrow{\operatorname{symGrad}^k_{\Gamma_t}}\mathsf{H}^k_{\mathbb{S},\Gamma_t}\xrightarrow{\operatorname{RotRot}^{\top,k}_{\mathbb{S},\Gamma_t}}\mathsf{H}^k_{\mathbb{S},\Gamma_t}\xrightarrow{\operatorname{Div}^k_{\mathbb{S},\Gamma_t}}\mathsf{H}^k_{\Gamma_t}\xrightarrow{\pi_{\mathbb{RM}_{\Gamma_n}}}\mathbb{RM}_{\Gamma_n},$$
$$\mathbb{RM}_{\Gamma_t}\xleftarrow{\pi_{\mathbb{RM}_{\Gamma_t}}}\mathsf{H}^k_{\Gamma_n}\xleftarrow{-\mathbf{Div}^k_{\mathbb{S},\Gamma_n}}\mathsf{H}^k_{\mathbb{S},\Gamma_n}\xleftarrow{\mathbf{RotRot}^{\top,k}_{\mathbb{S},\Gamma_n}}\mathsf{H}^k_{\mathbb{S},\Gamma_n}\xleftarrow{-\mathbf{symGrad}^k_{\Gamma_n}}\mathsf{H}^k_{\Gamma_n}\xleftarrow{\iota_{\mathbb{RM}_{\Gamma_n}}}\mathbb{RM}_{\Gamma_n}$$

with associated domain complexes

$$\mathbb{RM}_{\Gamma_t}\xrightarrow{\iota_{\mathbb{RM}_{\Gamma_t}}}\mathsf{H}^k_{\Gamma_t}(\operatorname{symGrad})\xrightarrow{\operatorname{symGrad}^k_{\Gamma_t}}\mathsf{H}^k_{\mathbb{S},\Gamma_t}(\operatorname{RotRot}^{\top})\xrightarrow{\operatorname{RotRot}^{\top,k}_{\mathbb{S},\Gamma_t}}\mathsf{H}^k_{\mathbb{S},\Gamma_t}(\operatorname{Div})\xrightarrow{\operatorname{Div}^k_{\mathbb{S},\Gamma_t}}\mathsf{H}^k_{\Gamma_t}\xrightarrow{\pi_{\mathbb{RM}_{\Gamma_n}}}\mathbb{RM}_{\Gamma_n},$$
$$\mathbb{RM}_{\Gamma_t}\xleftarrow{\pi_{\mathbb{RM}_{\Gamma_t}}}\mathsf{H}^k_{\Gamma_t}\xleftarrow{-\mathbf{Div}^k_{\mathbb{S},\Gamma_n}}\mathsf{H}^k_{\mathbb{S},\Gamma_n}(\operatorname{Div})\xleftarrow{\mathbf{RotRot}^{\top,k}_{\mathbb{S},\Gamma_n}}\mathsf{H}^k_{\mathbb{S},\Gamma_n}(\operatorname{RotRot}^{\top})\xleftarrow{-\mathbf{symGrad}^k_{\Gamma_n}}\mathsf{H}^k_{\Gamma_n}(\operatorname{symGrad})\xleftarrow{\iota_{\mathbb{RM}_{\Gamma_n}}}\mathbb{RM}_{\Gamma_n}.$$

Additionally, for $k\geq 1$ we will also discuss the following variants of the elasticity complexes

$$\mathbb{RM}_{\Gamma_t}\xrightarrow{\iota_{\mathbb{RM}_{\Gamma_t}}}\mathsf{H}^k_{\Gamma_t}\xrightarrow{\operatorname{symGrad}^k_{\Gamma_t}}\mathsf{H}^k_{\mathbb{S},\Gamma_t}\xrightarrow{\operatorname{RotRot}^{\top,k,k-1}_{\mathbb{S},\Gamma_t}}\mathsf{H}^{k-1}_{\mathbb{S},\Gamma_t}\xrightarrow{\operatorname{Div}^{k-1}_{\mathbb{S},\Gamma_t}}\mathsf{H}^{k-1}_{\Gamma_t}\xrightarrow{\pi_{\mathbb{RM}_{\Gamma_n}}}\mathbb{RM}_{\Gamma_n},$$
$$\mathbb{RM}_{\Gamma_t}\xleftarrow{\pi_{\mathbb{RM}_{\Gamma_t}}}\mathsf{H}^{k-1}_{\Gamma_n}\xleftarrow{-\mathbf{Div}^{k-1}_{\mathbb{S},\Gamma_n}}\mathsf{H}^{k-1}_{\mathbb{S},\Gamma_n}\xleftarrow{\mathbf{RotRot}^{\top,k,k-1}_{\mathbb{S},\Gamma_n}}\mathsf{H}^k_{\mathbb{S},\Gamma_n}\xleftarrow{-\mathbf{symGrad}^k_{\Gamma_n}}\mathsf{H}^k_{\Gamma_n}\xleftarrow{\iota_{\mathbb{RM}_{\Gamma_n}}}\mathbb{RM}_{\Gamma_n}$$

with associated domain complexes

$$\mathbb{RM}_{\Gamma_t}\xrightarrow{\iota_{\mathbb{RM}_{\Gamma_t}}}\mathsf{H}^k_{\Gamma_t}(\operatorname{symGrad})\xrightarrow{\operatorname{symGrad}^k_{\Gamma_t}}\mathsf{H}^{k,k-1}_{\mathbb{S},\Gamma_t}(\operatorname{RotRot}^{\top})\xrightarrow{\operatorname{RotRot}^{\top,k,k-1}_{\mathbb{S},\Gamma_t}}\mathsf{H}^{k-1}_{\mathbb{S},\Gamma_t}(\operatorname{Div})\xrightarrow{\operatorname{Div}^{k-1}_{\mathbb{S},\Gamma_t}}\mathsf{H}^{k-1}_{\Gamma_t}\xrightarrow{\pi_{\mathbb{RM}_{\Gamma_n}}}\mathbb{RM}_{\Gamma_n},$$
$$\mathbb{RM}_{\Gamma_t}\xleftarrow{\pi_{\mathbb{RM}_{\Gamma_t}}}\mathsf{H}^{k-1}_{\Gamma_t}\xleftarrow{-\mathbf{Div}^{k-1}_{\mathbb{S},\Gamma_n}}\mathsf{H}^{k-1}_{\mathbb{S},\Gamma_n}(\operatorname{Div})\xleftarrow{\mathbf{RotRot}^{\top,k,k-1}_{\mathbb{S},\Gamma_n}}\mathsf{H}^{k,k-1}_{\mathbb{S},\Gamma_n}(\operatorname{RotRot}^{\top})\xleftarrow{-\mathbf{symGrad}^k_{\Gamma_n}}\mathsf{H}^k_{\Gamma_n}(\operatorname{symGrad})\xleftarrow{\iota_{\mathbb{RM}_{\Gamma_n}}}\mathbb{RM}_{\Gamma_n}.$$

Here we have introduced the densely defined and closed linear operators

$$\operatorname{symGrad}^k_{\Gamma_t}: D(\operatorname{symGrad}^k_{\Gamma_t})\subset\mathsf{H}^k_{\Gamma_t}(\Omega)\to\mathsf{H}^k_{\mathbb{S},\Gamma_t}(\Omega);\qquad v\mapsto\operatorname{symGrad}v,$$
$$\operatorname{RotRot}^{\top,k}_{\mathbb{S},\Gamma_t}: D(\operatorname{RotRot}^{\top,k}_{\mathbb{S},\Gamma_t})\subset\mathsf{H}^k_{\mathbb{S},\Gamma_t}(\Omega)\to\mathsf{H}^k_{\mathbb{S},\Gamma_t}(\Omega);\qquad S\mapsto\operatorname{RotRot}^{\top}S,$$
$$\operatorname{Div}^k_{\mathbb{S},\Gamma_t}: D(\operatorname{Div}^k_{\mathbb{S},\Gamma_t})\subset\mathsf{H}^k_{\mathbb{S},\Gamma_t}(\Omega)\to\mathsf{H}^k_{\Gamma_t}(\Omega);\qquad T\mapsto\operatorname{Div}T,$$
$$-\mathbf{Div}^k_{\mathbb{S},\Gamma_n}: D(\mathbf{Div}^k_{\mathbb{S},\Gamma_n})\subset\mathsf{H}^k_{\mathbb{S},\Gamma_n}(\Omega)\to\mathsf{H}^k_{\Gamma_n}(\Omega);\qquad S\mapsto-\operatorname{Div}S,$$
$$\mathbf{RotRot}^{\top,k}_{\mathbb{S},\Gamma_n}: D(\mathbf{RotRot}^{\top,k}_{\mathbb{S},\Gamma_n})\subset\mathsf{H}^k_{\mathbb{S},\Gamma_n}(\Omega)\to\mathsf{H}^k_{\mathbb{S},\Gamma_n}(\Omega);\qquad T\mapsto\operatorname{RotRot}^{\top}T,$$

$$-\mathbf{symGrad}^k_{\Gamma_n} : D(\mathbf{symGrad}^k_{\Gamma_n}) \subset \mathsf{H}^k_{\Gamma_n}(\Omega) \to \mathsf{H}^k_{\mathbb{S},\Gamma_n}(\Omega); \qquad v \mapsto -\operatorname{symGrad} v$$

with domains of definition

$$D(\operatorname{symGrad}^k_{\Gamma_t}) = \mathsf{H}^k_{\Gamma_t}(\operatorname{symGrad},\Omega), \qquad D(\mathbf{Div}^k_{\mathbb{S},\Gamma_n}) = \mathbf{H}^k_{\mathbb{S},\Gamma_n}(\operatorname{Div},\Omega),$$
$$D(\operatorname{RotRot}^{\top,k}_{\mathbb{S},\Gamma_t}) = \mathsf{H}^k_{\mathbb{S},\Gamma_t}(\operatorname{RotRot}^\top,\Omega), \qquad D(\mathbf{RotRot}^{\top,k}_{\mathbb{S},\Gamma_n}) = \mathbf{H}^k_{\mathbb{S},\Gamma_n}(\operatorname{RotRot}^\top,\Omega),$$
$$D(\operatorname{Div}^k_{\mathbb{S},\Gamma_t}) = \mathsf{H}^k_{\mathbb{S},\Gamma_t}(\operatorname{Div},\Omega), \qquad D(\mathbf{symGrad}^k_{\Gamma_n}) = \mathbf{H}^k_{\Gamma_n}(\operatorname{symGrad},\Omega),$$

as well as

$$\operatorname{RotRot}^{\top,k,k-1}_{\mathbb{S},\Gamma_t} : D(\operatorname{RotRot}^{\top,k,k-1}_{\mathbb{S},\Gamma_t}) \subset \mathsf{H}^k_{\mathbb{S},\Gamma_t}(\Omega) \to \mathsf{H}^{k-1}_{\mathbb{S},\Gamma_t}(\Omega); \qquad S \mapsto \operatorname{RotRot}^\top S,$$
$$\mathbf{RotRot}^{\top,k,k-1}_{\mathbb{S},\Gamma_n} : D(\mathbf{RotRot}^{\top,k,k-1}_{\mathbb{S},\Gamma_n}) \subset \mathsf{H}^k_{\mathbb{S},\Gamma_n}(\Omega) \to \mathsf{H}^{k-1}_{\mathbb{S},\Gamma_n}(\Omega); \qquad T \mapsto \operatorname{RotRot}^\top T$$

with domains of definition

$$D(\operatorname{RotRot}^{\top,k,k-1}_{\mathbb{S},\Gamma_t}) = \mathsf{H}^{k,k-1}_{\mathbb{S},\Gamma_t}(\operatorname{RotRot}^\top,\Omega), \qquad D(\mathbf{RotRot}^{\top,k,k-1}_{\mathbb{S},\Gamma_n}) = \mathbf{H}^{k,k-1}_{\mathbb{S},\Gamma_n}(\operatorname{RotRot}^\top,\Omega).$$

2.7. **Dirichlet/Neumann Fields.** We also introduce the cohomology space of elastic Dirichlet/Neumann tensor fields (generalised harmonic tensors)

$$\mathcal{H}_{\mathbb{S},\Gamma_t,\Gamma_n,\varepsilon}(\Omega) := N(\operatorname{RotRot}^\top_{\mathbb{S},\Gamma_t}) \cap N(\operatorname{Div}_{\mathbb{S},\Gamma_n}\varepsilon) = \mathsf{H}_{\mathbb{S},\Gamma_t,0}(\operatorname{RotRot}^\top,\Omega) \cap \varepsilon^{-1}\,\mathsf{H}_{\mathbb{S},\Gamma_n,0}(\operatorname{Div},\Omega).$$

Here, $\varepsilon : \mathsf{L}^2_{\mathbb{S}}(\Omega) \to \mathsf{L}^2_{\mathbb{S}}(\Omega)$ is a symmetric and positive topological isomorphism (symmetric and positive bijective bounded linear operator), which introduces a new inner product

$$\langle\,\cdot\,,\,\cdot\,\rangle_{\mathsf{L}^2_{\mathbb{S},\varepsilon}(\Omega)} := \langle\varepsilon\,\cdot\,,\,\cdot\,\rangle_{\mathsf{L}^2_{\mathbb{S}}(\Omega)},$$

where $\mathsf{L}^2_{\mathbb{S},\varepsilon}(\Omega) := \mathsf{L}^2_{\mathbb{S}}(\Omega)$ (as linear space) equipped with the inner product $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathsf{L}^2_{\mathbb{S},\varepsilon}(\Omega)}$. Such *weights* ε shall be called *admissible* and a typical example is given by a symmetric, L^∞-bounded, and uniformly positive definite tensor field $\varepsilon : \Omega \to \mathbb{R}^{(3\times3)\times(3\times3)}$.

3. Elasticity Complexes II

3.1. Regular Potentials and Decompositions I.

3.1.1. *Extendable Domains.*

Theorem 3.1 (regular potential operators for extendable domains)**.** *Let (Ω,Γ_t) be an extendable bounded strong Lipschitz pair and let $k \geq 0$. Then there exist bounded linear regular potential operators*

$$\mathcal{P}^k_{\operatorname{symGrad},\Gamma_t} : \mathbf{H}^k_{\mathbb{S},\Gamma_t,0}(\operatorname{RotRot}^\top,\Omega) \longrightarrow \mathsf{H}^{k+1}_{\Gamma_t}(\Omega) \cap \mathsf{H}^{k+1}(\mathbb{R}^3),$$
$$\mathcal{P}^k_{\operatorname{RotRot}^\top_{\mathbb{S}},\Gamma_t} : \mathbf{H}^k_{\mathbb{S},\Gamma_t,0}(\operatorname{Div},\Omega) \longrightarrow \mathsf{H}^{k+2}_{\mathbb{S},\Gamma_t}(\Omega) \cap \mathsf{H}^{k+2}(\mathbb{R}^3),$$
$$\mathcal{P}^k_{\operatorname{Div}_{\mathbb{S}},\Gamma_t} : \mathsf{H}^k_{\Gamma_t}(\Omega) \cap (\mathbb{RM}_{\Gamma_n})^{\perp_{\mathsf{L}^2(\Omega)}} \longrightarrow \mathsf{H}^{k+1}_{\mathbb{S},\Gamma_t}(\Omega) \cap \mathsf{H}^{k+1}(\mathbb{R}^3).$$

In particular, $\mathcal{P}^{\cdots}_{\cdots}$ are right inverses for symGrad, $\operatorname{RotRot}^\top$*, and* Div*, respectively, i.e.,*

$$\operatorname{symGrad}\mathcal{P}^k_{\operatorname{symGrad},\Gamma_t} = \operatorname{id}_{\mathbf{H}^k_{\mathbb{S},\Gamma_t,0}(\operatorname{RotRot}^\top,\Omega)},$$
$$\operatorname{RotRot}^\top\mathcal{P}^k_{\operatorname{RotRot}^\top_{\mathbb{S}},\Gamma_t} = \operatorname{id}_{\mathbf{H}^k_{\mathbb{S},\Gamma_t,0}(\operatorname{Div},\Omega)},$$
$$\operatorname{Div}\mathcal{P}^k_{\operatorname{Div}_{\mathbb{S}},\Gamma_t} = \operatorname{id}_{\mathsf{H}^k_{\Gamma_t}(\Omega)\cap(\mathbb{RM}_{\Gamma_n})^{\perp_{\mathsf{L}^2(\Omega)}}}.$$

Without loss of generality, $\mathcal{P}^{\cdots}_{\cdots}$ map to tensor fields with a fixed compact support in $\mathbb{R}^3$.

Remark 3.2. *Note that $\mathrm{A}_n\,\mathcal{P}_{\mathrm{A}_n} = \operatorname{id}_{R(\mathrm{A}_n)}$ is a general property of a (bounded regular) potential operator $\mathcal{P}_{\mathrm{A}_n} : R(\mathrm{A}_n) \to \mathsf{H}^+_n$ with $\mathsf{H}^+_n \subset D(\mathrm{A}_n)$.*

Proof of Theorem 3.1. In [15, Theorem 4.2] we have shown the stated results for $\Gamma_t = \Gamma$ and $\Gamma_t = \emptyset$, which is also a crucial ingredient of this proof. Note that in these two special cases always "*strong = weak*" holds as $\mathrm{A}^{**}_n = \overline{\mathrm{A}_n} = \mathrm{A}_n$, and that this argument fails in the remaining cases of mixed boundary conditions. Therefore, let $\emptyset \subsetneq \Gamma_t \subsetneq \Gamma$. Moreover, recall the notion of an extendable domain from [11, Section 3]. In particular, $\widehat{\Omega}$ and the extended domain $\widetilde{\Omega}$ are topologically trivial.

- Let $S \in \mathbf{H}^{k}_{\mathbb{S},\Gamma_t,0}(\operatorname{RotRot}^\top,\Omega)$. By definition, S can be extended through Γ_t by zero to the larger domain $\widetilde{\Omega}$ yielding
$$\widetilde{S} \in \mathbf{H}^{k}_{\mathbb{S},\emptyset,0}(\operatorname{RotRot}^\top,\widetilde{\Omega}) = \mathbf{H}^{k}_{\mathbb{S},0}(\operatorname{RotRot}^\top,\widetilde{\Omega}) = \mathsf{H}^{k}_{\mathbb{S},0}(\operatorname{RotRot}^\top,\widetilde{\Omega}).$$
By [15, Theorem 4.2] there exists $\widetilde{v} \in \mathsf{H}^{k+1}(\mathbb{R}^3)$ such that $\operatorname{symGrad}\widetilde{v} = \widetilde{S}$ in $\widetilde{\Omega}$. Since $\widetilde{S} = 0$ in $\widehat{\Omega}$, $\widetilde{v}$ must be a rigid motion $r \in \mathbb{RM}$ in $\widehat{\Omega}$. Far outside of $\widetilde{\Omega}$ we modify r by a cut-off function such that the resulting vector field $\widetilde{r}$ is compactly supported and $\widetilde{r}|_{\widehat{\Omega}} = r$. Then $v := \widetilde{v} - \widetilde{r} \in \mathsf{H}^{k+1}(\mathbb{R}^3)$ with $v|_{\widehat{\Omega}} = 0$. Hence v belongs to $\mathsf{H}^{k+1}_{\Gamma_t}(\Omega)$ and depends continuously on S. Moreover, v satisfies $\operatorname{symGrad} v = \operatorname{symGrad}\widetilde{v} = \widetilde{S}$ in $\widetilde{\Omega}$, in particular $\operatorname{symGrad} v = S$ in Ω. We put $\mathcal{P}^{k}_{\operatorname{symGrad},\Gamma_t} S := v \in \mathsf{H}^{k+1}_{\Gamma_t}(\Omega)$.
- Let $T \in \mathbf{H}^{k}_{\mathbb{S},\Gamma_t,0}(\operatorname{Div},\Omega)$. By definition, T can be extended through Γ_t by zero to $\widetilde{\Omega}$ giving
$$\widetilde{T} \in \mathbf{H}^{k}_{\mathbb{S},\emptyset,0}(\operatorname{Div},\widetilde{\Omega}) = \mathbf{H}^{k}_{\mathbb{S},0}(\operatorname{Div},\widetilde{\Omega}) = \mathsf{H}^{k}_{\mathbb{S},0}(\operatorname{Div},\widetilde{\Omega}).$$
By [15, Theorem 4.2] there exists $\widetilde{S} \in \mathsf{H}^{k+2}_{\mathbb{S}}(\mathbb{R}^3)$ such that $\operatorname{RotRot}^\top \widetilde{S} = \widetilde{T}$ in $\widetilde{\Omega}$. Since $\widetilde{T} = 0$ in $\widehat{\Omega}$, i.e., $\widetilde{S}|_{\widehat{\Omega}} \in \mathsf{H}^{k+2}_{\mathbb{S},0}(\operatorname{RotRot}^\top,\widehat{\Omega})$, we get again by [15, Theorem 4.2] (or the first part of this proof) $\widetilde{v} \in \mathsf{H}^{k+3}(\mathbb{R}^3)$ such that $\operatorname{symGrad}\widetilde{v} = \widetilde{S}$ in $\widehat{\Omega}$. Then $S := \widetilde{S} - \operatorname{symGrad}\widetilde{v}$ belongs to $\mathsf{H}^{k+2}_{\mathbb{S}}(\mathbb{R}^3)$ and satisfies $S|_{\widehat{\Omega}} = 0$. Thus $S \in \mathsf{H}^{k+2}_{\mathbb{S},\Gamma_t}(\Omega)$ and depends continuously on T. Furthermore, $\operatorname{RotRot}^\top S = \operatorname{RotRot}^\top \widetilde{S} = \widetilde{T}$ in $\widetilde{\Omega}$, in particular $\operatorname{RotRot}^\top S = T$ in Ω. We set $\mathcal{P}^{k}_{\operatorname{RotRot}^\top_{\mathbb{S}},\Gamma_t} T := S \in \mathsf{H}^{k+2}_{\mathbb{S},\Gamma_t}(\Omega)$.
- Let $v \in \mathsf{H}^{k}_{\Gamma_t}(\Omega)$. By definition, v can be extended through Γ_t by zero to $\widetilde{\Omega}$ defining $\widetilde{v} \in \mathsf{H}^{k}(\widetilde{\Omega})$. [15, Theorem 4.2] yields $\widetilde{T} \in \mathsf{H}^{k+1}_{\mathbb{S}}(\mathbb{R}^3)$ such that $\operatorname{Div}\widetilde{T} = \widetilde{v}$ in $\widetilde{\Omega}$. As $\widetilde{v} = 0$ in $\widehat{\Omega}$, i.e., $\widetilde{T}|_{\widehat{\Omega}} \in \mathsf{H}^{k+1}_{\mathbb{S},0}(\operatorname{Div},\widehat{\Omega})$, we get again by [15, Theorem 4.2] (or the second part of this proof) $\widetilde{S} \in \mathsf{H}^{k+3}_{\mathbb{S}}(\mathbb{R}^3)$ such that $\operatorname{RotRot}^\top \widetilde{S} = \widetilde{T}$ holds in $\widehat{\Omega}$. Then $T := \widetilde{T} - \operatorname{RotRot}^\top \widetilde{S}$ belongs to $\mathsf{H}^{k+1}_{\mathbb{S}}(\mathbb{R}^3)$ with $T|_{\widehat{\Omega}} = 0$. Hence T belongs to $\mathsf{H}^{k+1}_{\mathbb{S},\Gamma_t}(\Omega)$ and depends continuously on v. Furthermore, $\operatorname{Div} T = \operatorname{Div}\widetilde{T} = \widetilde{v}$ in $\widetilde{\Omega}$, in particular $\operatorname{Div} T = v$ in Ω. Finally, we define $\mathcal{P}^{k}_{\operatorname{Div}_{\mathbb{S}},\Gamma_t} v := T \in \mathsf{H}^{k+1}_{\mathbb{S},\Gamma_t}(\Omega)$.

The assertion about the compact supports is trivial. □

As a simple consequence of Theorem 3.1 we obtain a few corollaries.

Corollary 3.3 (regular potentials for extendable domains)**.** *Let (Ω,Γ_t) be an extendable bounded strong Lipschitz pair and let $k \geq 0$. Then the regular potentials representations*
$$\begin{aligned}
\mathbf{H}^{k}_{\mathbb{S},\Gamma_t,0}(\operatorname{RotRot}^\top,\Omega) = \mathsf{H}^{k}_{\mathbb{S},\Gamma_t,0}(\operatorname{RotRot}^\top,\Omega) &= \operatorname{symGrad}\mathsf{H}^{k}_{\Gamma_t}(\operatorname{symGrad},\Omega) = \operatorname{symGrad}\mathsf{H}^{k+1}_{\Gamma_t}(\Omega)\\
&= R(\operatorname{symGrad}^{k}_{\Gamma_t}),\\
\mathbf{H}^{k}_{\mathbb{S},\Gamma_t,0}(\operatorname{Div},\Omega) = \mathsf{H}^{k}_{\mathbb{S},\Gamma_t,0}(\operatorname{Div},\Omega) &= \operatorname{RotRot}^\top \mathsf{H}^{k}_{\mathbb{S},\Gamma_t}(\operatorname{RotRot}^\top,\Omega) = \operatorname{RotRot}^\top \mathsf{H}^{k+2}_{\mathbb{S},\Gamma_t}(\Omega)\\
&= \operatorname{RotRot}^\top \mathsf{H}^{k+1,k}_{\mathbb{S},\Gamma_t}(\operatorname{RotRot}^\top,\Omega)\\
&= R(\operatorname{RotRot}^{\top,k}_{\mathbb{S},\Gamma_t}) = R(\operatorname{RotRot}^{\top,k+1,k}_{\mathbb{S},\Gamma_t}),\\
\mathsf{H}^{k}_{\Gamma_t}(\Omega) \cap (\mathbb{RM}_{\Gamma_n})^{\perp_{\mathsf{L}^2(\Omega)}} &= \operatorname{Div}\mathsf{H}^{k}_{\mathbb{S},\Gamma_t}(\operatorname{Div},\Omega) = \operatorname{Div}\mathsf{H}^{k+1}_{\mathbb{S},\Gamma_t}(\Omega)\\
&= R(\operatorname{Div}^{k}_{\mathbb{S},\Gamma_t})
\end{aligned}$$
hold, and the potentials can be chosen such that they depend continuously on the data. In particular, the latter spaces are closed subspaces of $\mathsf{H}^{k}_{\mathbb{S}}(\Omega)$ and $\mathsf{H}^{k}(\Omega)$, respectively.

Proof. By Theorem 3.1 we have
$$\begin{aligned}
\mathbf{H}^{k}_{\mathbb{S},\Gamma_t,0}(\operatorname{Div},\Omega) &= \operatorname{RotRot}^\top \mathcal{P}^{k}_{\operatorname{RotRot}^\top_{\mathbb{S}},\Gamma_t}\mathbf{H}^{k}_{\mathbb{S},\Gamma_t,0}(\operatorname{Div},\Omega) \subset \operatorname{RotRot}^\top \mathsf{H}^{k+2}_{\mathbb{S},\Gamma_t}(\Omega)\\
&\subset \operatorname{RotRot}^\top \mathsf{H}^{k+1,k}_{\mathbb{S},\Gamma_t}(\operatorname{RotRot}^\top,\Omega) \subset \operatorname{RotRot}^\top \mathsf{H}^{k}_{\mathbb{S},\Gamma_t}(\operatorname{RotRot}^\top,\Omega)\\
&\subset \mathsf{H}^{k}_{\mathbb{S},\Gamma_t,0}(\operatorname{Div},\Omega) \subset \mathbf{H}^{k}_{\mathbb{S},\Gamma_t,0}(\operatorname{Div},\Omega).
\end{aligned}$$
The other identities follow analogously. □

Corollary 3.4 (regular decompositions for extendable domains)**.** *Let (Ω,Γ_t) be an extendable bounded strong Lipschitz pair and let $k\geq 0$. Then the bounded regular decompositions*

$$\begin{aligned}\mathbf{H}^k_{\mathbb{S},\Gamma_t}(\operatorname{RotRot}^\top,\Omega)&=\mathsf{H}^{k+2}_{\mathbb{S},\Gamma_t}(\Omega)+\operatorname{symGrad}\mathsf{H}^{k+1}_{\Gamma_t}(\Omega)=R(\mathcal{P}^k_{\operatorname{RotRot}^\top_{\mathbb{S}},\Gamma_t})\dotplus\mathsf{H}^k_{\mathbb{S},\Gamma_t,0}(\operatorname{RotRot}^\top,\Omega)\\&=R(\mathcal{P}^k_{\operatorname{RotRot}^\top_{\mathbb{S}},\Gamma_t})\dotplus\operatorname{symGrad}\mathsf{H}^{k+1}_{\Gamma_t}(\Omega)\\&=R(\mathcal{P}^k_{\operatorname{RotRot}^\top_{\mathbb{S}},\Gamma_t})\dotplus\operatorname{symGrad}R(\mathcal{P}^k_{\operatorname{symGrad},\Gamma_t}),\\\mathbf{H}^k_{\mathbb{S},\Gamma_t}(\operatorname{Div},\Omega)&=\mathsf{H}^{k+1}_{\mathbb{S},\Gamma_t}(\Omega)+\operatorname{RotRot}^\top\mathsf{H}^{k+2}_{\mathbb{S},\Gamma_t}(\Omega)=R(\mathcal{P}^k_{\operatorname{Div}_{\mathbb{S}},\Gamma_t})\dotplus\mathsf{H}^k_{\mathbb{S},\Gamma_t,0}(\operatorname{Div},\Omega)\\&=R(\mathcal{P}^k_{\operatorname{Div}_{\mathbb{S}},\Gamma_t})\dotplus\operatorname{RotRot}^\top\mathsf{H}^{k+2}_{\mathbb{S},\Gamma_t}(\Omega)\\&=R(\mathcal{P}^k_{\operatorname{Div}_{\mathbb{S}},\Gamma_t})\dotplus\operatorname{RotRot}^\top R(\mathcal{P}^k_{\operatorname{RotRot}^\top_{\mathbb{S}},\Gamma_t})\end{aligned}$$

hold with bounded linear regular decomposition operators

$$\begin{aligned}\mathcal{Q}^{k,1}_{\operatorname{RotRot}^\top_{\mathbb{S}},\Gamma_t}&:=\mathcal{P}^k_{\operatorname{RotRot}^\top_{\mathbb{S}},\Gamma_t}\operatorname{RotRot}^\top:\mathbf{H}^k_{\mathbb{S},\Gamma_t}(\operatorname{RotRot}^\top,\Omega)\to\mathsf{H}^{k+2}_{\mathbb{S},\Gamma_t}(\Omega),\\\mathcal{Q}^{k,0}_{\operatorname{RotRot}^\top_{\mathbb{S}},\Gamma_t}&:=\mathcal{P}^k_{\operatorname{symGrad},\Gamma_t}(1-\mathcal{Q}^{k,1}_{\operatorname{RotRot}^\top_{\mathbb{S}},\Gamma_t}):\mathbf{H}^k_{\mathbb{S},\Gamma_t}(\operatorname{RotRot}^\top,\Omega)\to\mathsf{H}^{k+1}_{\Gamma_t}(\Omega),\\\mathcal{Q}^{k,1}_{\operatorname{Div}_{\mathbb{S}},\Gamma_t}&:=\mathcal{P}^k_{\operatorname{Div}_{\mathbb{S}},\Gamma_t}\operatorname{Div}:\mathbf{H}^k_{\mathbb{S},\Gamma_t}(\operatorname{Div},\Omega)\to\mathsf{H}^{k+1}_{\mathbb{S},\Gamma_t}(\Omega),\\\mathcal{Q}^{k,0}_{\operatorname{Div}_{\mathbb{S}},\Gamma_t}&:=\mathcal{P}^k_{\operatorname{RotRot}^\top_{\mathbb{S}},\Gamma_t}(1-\mathcal{Q}^{k,1}_{\operatorname{Div}_{\mathbb{S}},\Gamma_t}):\mathbf{H}^k_{\mathbb{S},\Gamma_t}(\operatorname{Div},\Omega)\to\mathsf{H}^{k+2}_{\mathbb{S},\Gamma_t}(\Omega)\end{aligned}$$

satisfying

$$\begin{aligned}\mathcal{Q}^{k,1}_{\operatorname{RotRot}^\top_{\mathbb{S}},\Gamma_t}+\operatorname{symGrad}\mathcal{Q}^{k,0}_{\operatorname{RotRot}^\top_{\mathbb{S}},\Gamma_t}&=\operatorname{id}_{\mathbf{H}^k_{\mathbb{S},\Gamma_t}(\operatorname{RotRot}^\top,\Omega)},\\\mathcal{Q}^{k,1}_{\operatorname{Div}_{\mathbb{S}},\Gamma_t}+\operatorname{RotRot}^\top\mathcal{Q}^{k,0}_{\operatorname{Div}_{\mathbb{S}},\Gamma_t}&=\operatorname{id}_{\mathbf{H}^k_{\mathbb{S},\Gamma_t}(\operatorname{Div},\Omega)}.\end{aligned}$$

Remark 3.5. *Note that for (bounded linear) potential operators $\mathcal{P}_{\mathrm{A}_{n-1}}$ and $\mathcal{P}_{\mathrm{A}_n}$ the identity*

$$\mathcal{Q}^1_{\mathrm{A}_n}+\mathrm{A}_{n-1}\mathcal{Q}^0_{\mathrm{A}_n}=\operatorname{id}_{D(\mathrm{A}_n)}\quad\text{with}\quad\begin{aligned}\mathcal{Q}^1_{\mathrm{A}_n}&:=\mathcal{P}_{\mathrm{A}_n}\mathrm{A}_n:D(\mathrm{A}_n)\to\mathsf{H}^+_n,\\\mathcal{Q}^0_{\mathrm{A}_n}&:=\mathcal{P}_{\mathrm{A}_{n-1}}(1-\mathcal{Q}^1_{\mathrm{A}_n}):D(\mathrm{A}_n)\to\mathsf{H}^+_{n-1}\end{aligned}$$

is a general structure of a (bounded) regular decomposition. Moreover:

- **(i)** *$R(\mathcal{Q}^1_{\mathrm{A}_n})=R(\mathcal{P}_{\mathrm{A}_n})$ and $R(\mathcal{Q}^0_{\mathrm{A}_n})=R(\mathcal{P}_{\mathrm{A}_{n-1}})$.*
- **(ii)** *$N(\mathrm{A}_n)$ is invariant under $\mathcal{Q}^1_{\mathrm{A}_n}$, as $\mathrm{A}_n=\mathrm{A}_n\mathcal{Q}^1_{\mathrm{A}_n}$ holds by the complex property.*
- **(iii)** *$\mathcal{Q}^1_{\mathrm{A}_n}$ and $\mathrm{A}_{n-1}\mathcal{Q}^0_{\mathrm{A}_n}=1-\mathcal{Q}^1_{\mathrm{A}_n}$ are projections.*
- **(iv)** *There exists $c>0$ such that for all $x\in D(\mathrm{A}_n)$*
$$|\mathcal{Q}^1_{\mathrm{A}_n}x|_{\mathsf{H}^+_n}\leq c|\mathrm{A}_n x|_{\mathsf{H}_{n+1}}.$$
- **(iv')** *In particular, $\mathcal{Q}^1_{\mathrm{A}_n}|_{N(\mathrm{A}_n)}=0$.*

Corollary 3.6 (weak and strong partial boundary conditions coincide for extendable domains)**.** *Let (Ω,Γ_t) be an extendable bounded strong Lipschitz pair and let $k\geq 0$. Then weak and strong boundary conditions coincide, i.e.,*

$$\begin{aligned}\mathbf{H}^k_{\Gamma_t}(\operatorname{symGrad},\Omega)&=\mathsf{H}^k_{\Gamma_t}(\operatorname{symGrad},\Omega)=\mathsf{H}^{k+1}_{\Gamma_t}(\Omega)=\mathbf{H}^{k+1}_{\Gamma_t}(\Omega),\\\mathbf{H}^k_{\mathbb{S},\Gamma_t}(\operatorname{RotRot}^\top,\Omega)&=\mathsf{H}^k_{\mathbb{S},\Gamma_t}(\operatorname{RotRot}^\top,\Omega),\\\mathbf{H}^k_{\mathbb{S},\Gamma_t}(\operatorname{Div},\Omega)&=\mathsf{H}^k_{\mathbb{S},\Gamma_t}(\operatorname{Div},\Omega).\end{aligned}$$

Proof of Corollary 3.4 and Corollary 3.6. Let us pick $S\in\mathbf{H}^k_{\mathbb{S},\Gamma_t}(\operatorname{RotRot}^\top,\Omega)$. By Theorem 3.1 we have $\operatorname{RotRot}^\top S\in\mathbf{H}^k_{\mathbb{S},\Gamma_t,0}(\operatorname{Div},\Omega)$ and $\widehat{S}:=\mathcal{P}^k_{\operatorname{RotRot}^\top_{\mathbb{S}},\Gamma_t}\operatorname{RotRot}^\top S\in\mathsf{H}^{k+2}_{\mathbb{S},\Gamma_t}$. Hence, we obtain $S-\widehat{S}\in\mathbf{H}^k_{\mathbb{S},\Gamma_t,0}(\operatorname{RotRot}^\top,\Omega)$ and Theorem 3.1 shows $v:=\mathcal{P}^k_{\operatorname{symGrad},\Gamma_t}(S-\widehat{S})\in\mathsf{H}^{k+1}_{\Gamma_t}(\Omega)$ and thus

$$S=\widehat{S}+\operatorname{symGrad}v\in\mathsf{H}^{k+2}_{\mathbb{S},\Gamma_t}(\Omega)+\operatorname{symGrad}\mathsf{H}^{k+1}_{\Gamma_t}(\Omega)\subset\mathsf{H}^k_{\mathbb{S},\Gamma_t}(\operatorname{RotRot}^\top,\Omega).$$

For the directness let $S=\mathcal{P}^k_{\operatorname{RotRot}^\top_{\mathbb{S}},\Gamma_t}T\in\mathbf{H}^k_{\mathbb{S},\Gamma_t,0}(\operatorname{RotRot}^\top,\Omega)$ with some $T\in\mathbf{H}^k_{\mathbb{S},\Gamma_t,0}(\operatorname{Div},\Omega)$. Then $0=\operatorname{RotRot}^\top S=T$ and thus $S=0$. The assertions about the corresponding Div-spaces follow analogously. Let $v\in\mathbf{H}^k_{\Gamma_t}(\operatorname{symGrad},\Omega)$. Then $\operatorname{symGrad}v\in\mathbf{H}^k_{\mathbb{S},\Gamma_t,0}(\operatorname{RotRot}^\top,\Omega)$

and Theorem 3.1 yields $\widehat{v} := \mathcal{P}^k_{\operatorname{symGrad},\Gamma_t} \operatorname{symGrad} v \in \mathsf{H}^{k+1}_{\Gamma_t}(\Omega)$. As $\operatorname{symGrad}(v - \widehat{v}) = 0$, we have $v - \widehat{v} =: r \in \mathbb{RM}$, which even vanishes if $\Gamma_t \neq \emptyset$. Hence, $v = \widehat{v} + r \in \mathsf{H}^{k+1}_{\Gamma_t}(\Omega)$. □

By similar arguments we also obtain the following (non-standard) versions of Corollary 3.4 and Corollary 3.6.

Corollary 3.7 (Corollary 3.4 and Corollary 3.6 for non-standard Sobolev spaces). *Let (Ω, Γ_t) be an extendable bounded strong Lipschitz pair and let $k \geq 1$. Then the bounded regular decompositions*

$$\begin{aligned}\mathbf{H}^{k,k-1}_{\mathbb{S},\Gamma_t}(\operatorname{RotRot}^\top, \Omega) &= \mathsf{H}^{k+1}_{\mathbb{S},\Gamma_t}(\Omega) + \operatorname{symGrad} \mathsf{H}^{k+1}_{\Gamma_t}(\Omega) = R(\mathcal{P}^{k-1}_{\operatorname{RotRot}^\top_{\mathbb{S}},\Gamma_t}) \dotplus \mathsf{H}^k_{\mathbb{S},\Gamma_t,0}(\operatorname{RotRot}^\top, \Omega)\\ &= R(\mathcal{P}^{k-1}_{\operatorname{RotRot}^\top_{\mathbb{S}},\Gamma_t}) \dotplus \operatorname{symGrad} \mathsf{H}^{k+1}_{\Gamma_t}(\Omega)\\ &= R(\mathcal{P}^{k-1}_{\operatorname{RotRot}^\top_{\mathbb{S}},\Gamma_t}) \dotplus \operatorname{symGrad} R(\mathcal{P}^k_{\operatorname{symGrad},\Gamma_t}) = \mathsf{H}^{k,k-1}_{\mathbb{S},\Gamma_t}(\operatorname{RotRot}^\top, \Omega)\end{aligned}$$

hold with bounded linear regular decomposition operators

$$\begin{aligned}\mathcal{Q}^{k,k-1,1}_{\operatorname{RotRot}^\top_{\mathbb{S}},\Gamma_t} &:= \mathcal{P}^{k-1}_{\operatorname{RotRot}^\top_{\mathbb{S}},\Gamma_t} \operatorname{RotRot}^\top : \mathbf{H}^{k,k-1}_{\mathbb{S},\Gamma_t}(\operatorname{RotRot}^\top, \Omega) \to \mathsf{H}^{k+1}_{\mathbb{S},\Gamma_t}(\Omega),\\ \mathcal{Q}^{k,k-1,0}_{\operatorname{RotRot}^\top_{\mathbb{S}},\Gamma_t} &:= \mathcal{P}^k_{\operatorname{symGrad},\Gamma_t}(1 - \mathcal{Q}^{k,k-1,1}_{\operatorname{RotRot}^\top_{\mathbb{S}},\Gamma_t}) : \mathbf{H}^{k,k-1}_{\mathbb{S},\Gamma_t}(\operatorname{RotRot}^\top, \Omega) \to \mathsf{H}^{k+1}_{\Gamma_t}(\Omega)\end{aligned}$$

satisfying $\mathcal{Q}^{k,k-1,1}_{\operatorname{RotRot}^\top_{\mathbb{S}},\Gamma_t} + \operatorname{symGrad} \mathcal{Q}^{k,k-1,0}_{\operatorname{RotRot}^\top_{\mathbb{S}},\Gamma_t} = \operatorname{id}_{\mathbf{H}^{k,k-1}_{\mathbb{S},\Gamma_t}(\operatorname{RotRot}^\top,\Omega)}$. In particular, weak and strong boundary conditions coincide also for the non-standard Sobolev spaces.

Recall the Hilbert complexes and cohomology groups from Section 2.6 and Section 2.7.

Theorem 3.8 (closed and exact Hilbert complexes for extendable domains). *Let (Ω, Γ_t) be an extendable bounded strong Lipschitz pair and let $k \geq 0$. The domain complexes of linear elasticity*

$$\mathbb{RM}_{\Gamma_t} \xrightarrow{\iota_{\mathbb{RM}_{\Gamma_t}}} \mathsf{H}^{k+1}_{\Gamma_t} \xrightarrow{\operatorname{symGrad}^k_{\Gamma_t}} \mathsf{H}^k_{\mathbb{S},\Gamma_t}(\operatorname{RotRot}^\top) \xrightarrow{\operatorname{RotRot}^{\top,k}_{\mathbb{S},\Gamma_t}} \mathsf{H}^k_{\mathbb{S},\Gamma_t}(\operatorname{Div}) \xrightarrow{\operatorname{Div}^k_{\mathbb{S},\Gamma_t}} \mathsf{H}^k_{\Gamma_t} \xrightarrow{\pi_{\mathbb{RM}_{\Gamma_n}}} \mathbb{RM}_{\Gamma_n},$$

$$\mathbb{RM}_{\Gamma_t} \xleftarrow{\pi_{\mathbb{RM}_{\Gamma_t}}} \mathsf{H}^k_{\Gamma_t} \xleftarrow{-\operatorname{Div}^k_{\mathbb{S},\Gamma_n}} \mathsf{H}^k_{\mathbb{S},\Gamma_n}(\operatorname{Div}) \xleftarrow{\operatorname{RotRot}^{\top,k}_{\mathbb{S},\Gamma_n}} \mathsf{H}^k_{\mathbb{S},\Gamma_n}(\operatorname{RotRot}^\top) \xleftarrow{-\operatorname{symGrad}^k_{\Gamma_n}} \mathsf{H}^{k+1}_{\Gamma_n} \xleftarrow{\iota_{\mathbb{RM}_{\Gamma_n}}} \mathbb{RM}_{\Gamma_n},$$

and, for $k \geq 1$,

$$\mathbb{RM}_{\Gamma_t} \xrightarrow{\iota_{\mathbb{RM}_{\Gamma_t}}} \mathsf{H}^{k+1}_{\Gamma_t} \xrightarrow{\operatorname{symGrad}^k_{\Gamma_t}} \mathsf{H}^{k,k-1}_{\mathbb{S},\Gamma_t}(\operatorname{RotRot}^\top) \xrightarrow{\operatorname{RotRot}^{\top,k,k-1}_{\mathbb{S},\Gamma_t}} \mathsf{H}^{k-1}_{\mathbb{S},\Gamma_t}(\operatorname{Div}) \xrightarrow{\operatorname{Div}^{k-1}_{\mathbb{S},\Gamma_t}} \mathsf{H}^{k-1}_{\Gamma_t} \xrightarrow{\pi_{\mathbb{RM}_{\Gamma_n}}} \mathbb{RM}_{\Gamma_n},$$

$$\mathbb{RM}_{\Gamma_t} \xleftarrow{\pi_{\mathbb{RM}_{\Gamma_t}}} \mathsf{H}^{k-1}_{\Gamma_t} \xleftarrow{-\operatorname{Div}^{k-1}_{\mathbb{S},\Gamma_n}} \mathsf{H}^{k-1}_{\mathbb{S},\Gamma_n}(\operatorname{Div}) \xleftarrow{\operatorname{RotRot}^{\top,k,k-1}_{\mathbb{S},\Gamma_n}} \mathsf{H}^{k,k-1}_{\mathbb{S},\Gamma_n}(\operatorname{RotRot}^\top) \xleftarrow{-\operatorname{symGrad}^k_{\Gamma_n}} \mathsf{H}^{k+1}_{\Gamma_n} \xleftarrow{\iota_{\mathbb{RM}_{\Gamma_n}}} \mathbb{RM}_{\Gamma_n}$$

are exact and closed Hilbert complexes. In particular, all ranges are closed, all cohomology groups (Dirichlet/Neumann fields) are trivial, and the operators from Theorem 3.1 are associated bounded regular potential operators.

3.1.2. *General Strong Lipschitz Domains.* Similar to [15, Lemma 4.8] we get the following.

Lemma 3.9 (cutting lemma). *Let $\varphi \in \mathsf{C}^\infty(\mathbb{R}^3)$ and let $k \geq 0$.*

- **(i)** *If $T \in \mathbf{H}^k_{\mathbb{S},\Gamma_t}(\operatorname{Div}, \Omega)$, then $\varphi T \in \mathbf{H}^k_{\mathbb{S},\Gamma_t}(\operatorname{Div}, \Omega)$ and $\operatorname{Div}(\varphi T) = \varphi \operatorname{Div} T + T \operatorname{grad} \varphi$ holds.*
- **(ii)** *If $k \geq 1$ and $S \in \mathbf{H}^{k,k-1}_{\mathbb{S},\Gamma_t}(\operatorname{RotRot}^\top, \Omega)$, then $\varphi S \in \mathbf{H}^{k,k-1}_{\mathbb{S},\Gamma_t}(\operatorname{RotRot}^\top, \Omega)$ and*

$$\operatorname{RotRot}^\top(\varphi S) = \varphi \operatorname{RotRot}^\top S + 2 \operatorname{sym}\big((\operatorname{spn} \operatorname{grad} \varphi) \operatorname{Rot} S\big) + \Psi(\operatorname{Grad} \operatorname{grad} \varphi, S)$$

holds with an algebraic operator Ψ. In particular, this holds for $S \in \mathbf{H}^k_{\mathbb{S},\Gamma_t}(\operatorname{RotRot}^\top, \Omega)$.

We proceed by showing regular decompositions for the elasticity complexes extending the results of Corollary 3.4 and Corollary 3.7.

Lemma 3.10 (regular decompositions). *Let $k \geq 0$. Then the bounded regular decompositions*

$$\begin{aligned}\mathbf{H}^k_{\mathbb{S},\Gamma_t}(\operatorname{Div}, \Omega) &= \mathsf{H}^{k+1}_{\mathbb{S},\Gamma_t}(\Omega) + \operatorname{RotRot}^\top \mathsf{H}^{k+2}_{\mathbb{S},\Gamma_t}(\Omega),\\ \mathbf{H}^k_{\mathbb{S},\Gamma_t}(\operatorname{RotRot}^\top, \Omega) &= \mathsf{H}^{k+2}_{\mathbb{S},\Gamma_t}(\Omega) + \operatorname{symGrad} \mathsf{H}^{k+1}_{\Gamma_t}(\Omega)\end{aligned}$$

and, for $k \geq 1$, the non-standard bounded regular decompositions

$$\mathbf{H}^k_{\mathbb{S},\Gamma_t}(\operatorname{RotRot}^\top, \Omega) \subset \mathbf{H}^{k,k-1}_{\mathbb{S},\Gamma_t}(\operatorname{RotRot}^\top, \Omega) = \mathsf{H}^{k+1}_{\mathbb{S},\Gamma_t}(\Omega) + \operatorname{symGrad} \mathsf{H}^{k+1}_{\Gamma_t}(\Omega)$$

hold with bounded linear regular decomposition operators

$$\mathcal{Q}^{k,1}_{\mathrm{RotRot}^\top_{\mathbb{S}},\Gamma_t}:\mathbf{H}^{k}_{\mathbb{S},\Gamma_t}(\mathrm{RotRot}^\top,\Omega)\to\mathsf{H}^{k+2}_{\mathbb{S},\Gamma_t}(\Omega),\qquad \mathcal{Q}^{k,1}_{\mathrm{Div}_{\mathbb{S}},\Gamma_t}:\mathbf{H}^{k}_{\mathbb{S},\Gamma_t}(\mathrm{Div},\Omega)\to\mathsf{H}^{k+1}_{\mathbb{S},\Gamma_t}(\Omega),$$
$$\mathcal{Q}^{k,0}_{\mathrm{RotRot}^\top_{\mathbb{S}},\Gamma_t}:\mathbf{H}^{k}_{\mathbb{S},\Gamma_t}(\mathrm{RotRot}^\top,\Omega)\to\mathsf{H}^{k+1}_{\Gamma_t}(\Omega),\qquad \mathcal{Q}^{k,0}_{\mathrm{Div}_{\mathbb{S}},\Gamma_t}:\mathbf{H}^{k}_{\mathbb{S},\Gamma_t}(\mathrm{Div},\Omega)\to\mathsf{H}^{k+2}_{\mathbb{S},\Gamma_t}(\Omega),$$
$$\mathcal{Q}^{k,k-1,1}_{\mathrm{RotRot}^\top_{\mathbb{S}},\Gamma_t}:\mathbf{H}^{k,k-1}_{\mathbb{S},\Gamma_t}(\mathrm{RotRot}^\top,\Omega)\to\mathsf{H}^{k+1}_{\mathbb{S},\Gamma_t}(\Omega),$$
$$\mathcal{Q}^{k,k-1,0}_{\mathrm{RotRot}^\top_{\mathbb{S}},\Gamma_t}:\mathbf{H}^{k,k-1}_{\mathbb{S},\Gamma_t}(\mathrm{RotRot}^\top,\Omega)\to\mathsf{H}^{k+1}_{\Gamma_t}(\Omega)$$

satisfying

$$\begin{aligned}\mathcal{Q}^{k,1}_{\mathrm{Div}_{\mathbb{S}},\Gamma_t}+\mathrm{RotRot}^\top\,\mathcal{Q}^{k,0}_{\mathrm{Div}_{\mathbb{S}},\Gamma_t}&=\mathrm{id}_{\mathbf{H}^{k}_{\mathbb{S},\Gamma_t}(\mathrm{Div},\Omega)},\\ \mathcal{Q}^{k,1}_{\mathrm{RotRot}^\top_{\mathbb{S}},\Gamma_t}+\mathrm{symGrad}\,\mathcal{Q}^{k,0}_{\mathrm{RotRot}^\top_{\mathbb{S}},\Gamma_t}&=\mathrm{id}_{\mathbf{H}^{k}_{\mathbb{S},\Gamma_t}(\mathrm{RotRot}^\top,\Omega)},\\ \mathcal{Q}^{k,k-1,1}_{\mathrm{RotRot}^\top_{\mathbb{S}},\Gamma_t}+\mathrm{symGrad}\,\mathcal{Q}^{k,k-1,0}_{\mathrm{RotRot}^\top_{\mathbb{S}},\Gamma_t}&=\mathrm{id}_{\mathbf{H}^{k,k-1}_{\mathbb{S},\Gamma_t}(\mathrm{RotRot}^\top,\Omega)},\qquad k\geq 1.\end{aligned}$$

It holds $\mathrm{Div}\,\mathcal{Q}^{k,1}_{\mathrm{Div}_{\mathbb{S}},\Gamma_t}=\mathbf{Div}^{k}_{\mathbb{S},\Gamma_t}$ *and thus* $\mathbf{H}^{k}_{\mathbb{S},\Gamma_t,0}(\mathrm{Div},\Omega)$ *is invariant under* $\mathcal{Q}^{k,1}_{\mathrm{Div}_{\mathbb{S}},\Gamma_t}$. *Analogously,* $\mathrm{RotRot}^\top\,\mathcal{Q}^{k,1}_{\mathrm{RotRot}^\top_{\mathbb{S}},\Gamma_t}=\mathbf{RotRot}^{\top,k}_{\mathbb{S},\Gamma_t}$ *and* $\mathrm{RotRot}^\top\,\mathcal{Q}^{k,k-1,1}_{\mathrm{RotRot}^\top_{\mathbb{S}},\Gamma_t}=\mathbf{RotRot}^{\top,k,k-1}_{\mathbb{S},\Gamma_t}$ *and thus* $\mathbf{H}^{k}_{\mathbb{S},\Gamma_t,0}(\mathrm{RotRot}^\top,\Omega)$ *is invariant under* $\mathcal{Q}^{k,1}_{\mathrm{RotRot}^\top_{\mathbb{S}},\Gamma_t}$ *and* $\mathcal{Q}^{k,k-1,1}_{\mathrm{RotRot}^\top_{\mathbb{S}},\Gamma_t}$*, respectively.*

Corollary 3.11 (weak and strong partial boundary conditions coincide)**.** *Let* $k\geq 0$*. Weak and strong boundary conditions coincide, i.e.,*

$$\begin{aligned}\mathbf{H}^{k}_{\Gamma_t}(\mathrm{symGrad},\Omega)=\mathsf{H}^{k}_{\Gamma_t}(\mathrm{symGrad},\Omega)&=\mathsf{H}^{k+1}_{\Gamma_t}(\Omega)=\mathbf{H}^{k+1}_{\Gamma_t}(\Omega),\\ \mathbf{H}^{k}_{\mathbb{S},\Gamma_t}(\mathrm{Div},\Omega)&=\mathsf{H}^{k}_{\mathbb{S},\Gamma_t}(\mathrm{Div},\Omega),\\ \mathbf{H}^{k}_{\mathbb{S},\Gamma_t}(\mathrm{RotRot}^\top,\Omega)&=\mathsf{H}^{k}_{\mathbb{S},\Gamma_t}(\mathrm{RotRot}^\top,\Omega),\\ \mathbf{H}^{k,k-1}_{\mathbb{S},\Gamma_t}(\mathrm{RotRot}^\top,\Omega)&=\mathsf{H}^{k,k-1}_{\mathbb{S},\Gamma_t}(\mathrm{RotRot}^\top,\Omega),\qquad k\geq 1.\end{aligned}$$

In particular, $\mathbf{symGrad}^{k}_{\Gamma_t}=\mathrm{symGrad}^{k}_{\Gamma_t}$, $\mathbf{RotRot}^{\top,k}_{\mathbb{S},\Gamma_t}=\mathrm{RotRot}^{\top,k}_{\mathbb{S},\Gamma_t}$*, and* $\mathbf{Div}^{k}_{\mathbb{S},\Gamma_t}=\mathrm{Div}^{k}_{\mathbb{S},\Gamma_t}$*, as well as, for* $k\geq 1$, $\mathbf{RotRot}^{\top,k,k-1}_{\mathbb{S},\Gamma_t}=\mathrm{RotRot}^{\top,k,k-1}_{\mathbb{S},\Gamma_t}$.

Proof of Lemma 3.10 and Corollary 3.11. According to [11] and [15], cf. [2, 3, 4], let (U_ℓ,φ_ℓ) be a partition of unity for Ω, i.e.,

$$\Omega=\bigcup_{\ell=-L}^{L}\Omega_\ell,\qquad \Omega_\ell:=\Omega\cap U_\ell,\qquad \varphi_\ell\in\mathsf{C}^\infty_{\partial U_\ell}(U_\ell),$$

and $(\Omega_\ell,\widehat{\Gamma}_{t,\ell})$ are extendable bounded strong Lipschitz pairs. Recall $\Gamma_{t,\ell}:=\Gamma_t\cap U_\ell$ and $\widehat{\Gamma}_{t,\ell}$.

- Let $k\geq 0$ and let $T\in\mathbf{H}^{k}_{\mathbb{S},\Gamma_t}(\mathrm{Div},\Omega)$. Then by definition $T|_{\Omega_\ell}\in\mathbf{H}^{k}_{\mathbb{S},\Gamma_{t,\ell}}(\mathrm{Div},\Omega_\ell)$ and we decompose by Corollary 3.4

$$T|_{\Omega_\ell}=T_{\ell,1}+\mathrm{RotRot}^\top S_{\ell,0}$$

with $T_{\ell,1}:=\mathcal{Q}^{k,1}_{\mathrm{Div}_{\mathbb{S}},\Gamma_{t,\ell}}T|_{\Omega_\ell}\in\mathsf{H}^{k+1}_{\mathbb{S},\Gamma_{t,\ell}}(\Omega_\ell)$ and $S_{\ell,0}:=\mathcal{Q}^{k,0}_{\mathrm{Div}_{\mathbb{S}},\Gamma_{t,\ell}}T|_{\Omega_\ell}\in\mathsf{H}^{k+2}_{\mathbb{S},\Gamma_{t,\ell}}(\Omega_\ell)$. Lemma 3.9 yields

$$\begin{aligned}\varphi_\ell T|_{\Omega_\ell}&=\varphi_\ell T_{\ell,1}+\varphi_\ell\,\mathrm{RotRot}^\top S_{\ell,0}\\ &=\overbrace{\varphi_\ell T_{\ell,1}-2\,\mathrm{sym}\Big((\mathrm{spn}\,\mathrm{grad}\,\varphi_\ell)\,\mathrm{Rot}\,S_{\ell,0}\Big)-\Psi(\mathrm{Grad}\,\mathrm{grad}\,\varphi_\ell,S_{\ell,0})}^{=:T_\ell}\\ &\qquad+\mathrm{RotRot}^\top\underbrace{(\varphi_\ell S_{\ell,0})}_{=:S_\ell}\end{aligned}$$

with $T_\ell \in \mathsf{H}^{k+1}_{\mathbb{S},\widehat{\Gamma}_{t,\ell}}(\Omega_\ell)$ and $S_\ell \in \mathsf{H}^{k+2}_{\mathbb{S},\widehat{\Gamma}_{t,\ell}}(\Omega_\ell)$. Extending T_ℓ and S_ℓ by zero to Ω gives tensor fields $\widetilde{T}_\ell \in \mathsf{H}^{k+1}_{\mathbb{S},\Gamma_t}(\Omega)$ and $\widetilde{S}_\ell \in \mathsf{H}^{k+2}_{\mathbb{S},\Gamma_t}(\Omega)$ as well as

$$\begin{aligned} T = \sum_{\ell=-L}^{L} \varphi_\ell T|_{\Omega_\ell} &= \sum_{\ell=-L}^{L} \widetilde{T}_\ell + \operatorname{RotRot}^\top \sum_{\ell=-L}^{L} \widetilde{S}_\ell \\ &\in \mathsf{H}^{k+1}_{\mathbb{S},\Gamma_t}(\Omega) + \operatorname{RotRot}^\top \mathsf{H}^{k+2}_{\mathbb{S},\Gamma_t}(\Omega) \subset \mathsf{H}^{k}_{\mathbb{S},\Gamma_t}(\operatorname{Div},\Omega). \end{aligned}$$

As all operations have been linear and continuous we set

$$\mathcal{Q}^{k,1}_{\operatorname{Div}_\mathbb{S},\Gamma_t} T := \sum_{\ell=-L}^{L} \widetilde{T}_\ell \in \mathsf{H}^{k+1}_{\mathbb{S},\Gamma_t}(\Omega), \qquad \mathcal{Q}^{k,0}_{\operatorname{Div}_\mathbb{S},\Gamma_t} T := \sum_{\ell=-L}^{L} \widetilde{S}_\ell \in \mathsf{H}^{k+2}_{\mathbb{S},\Gamma_t}(\Omega).$$

- Let $k \geq 1$ and let $S \in \mathbf{H}^{k,k-1}_{\mathbb{S},\Gamma_t}(\operatorname{RotRot}^\top,\Omega)$. Then by definition $S|_{\Omega_\ell} \in \mathbf{H}^{k,k-1}_{\mathbb{S},\Gamma_{t,\ell}}(\operatorname{RotRot}^\top,\Omega_\ell)$ and we decompose by Corollary 3.7

$$S|_{\Omega_\ell} = S_{\ell,1} + \operatorname{symGrad} v_{\ell,0}$$

with $S_{\ell,1} := \mathcal{Q}^{k,k-1,1}_{\operatorname{RotRot}^\top_\mathbb{S},\Gamma_{t,\ell}} S|_{\Omega_\ell} \in \mathsf{H}^{k+1}_{\mathbb{S},\Gamma_{t,\ell}}(\Omega_\ell)$ and $v_{\ell,0} := \mathcal{Q}^{k,k-1,0}_{\operatorname{RotRot}^\top_\mathbb{S},\Gamma_{t,\ell}} S|_{\Omega_\ell} \in \mathsf{H}^{k+1}_{\Gamma_{t,\ell}}(\Omega_\ell)$. Thus

$$\begin{aligned} \varphi_\ell S|_{\Omega_\ell} &= \varphi_\ell S_{\ell,1} + \varphi_\ell \operatorname{symGrad} v_{\ell,0} \\ &= \underbrace{\varphi_\ell S_{\ell,1} - \operatorname{sym}\big(v_{\ell,0}(\operatorname{grad}\varphi_\ell)^\top\big)}_{=:S_\ell} + \operatorname{symGrad}(\underbrace{\varphi_\ell v_{\ell,0}}_{=:v_\ell}) \end{aligned} \tag{6}$$

with $S_\ell \in \mathsf{H}^{k+1}_{\mathbb{S},\widehat{\Gamma}_{t,\ell}}(\Omega_\ell)$ and $v_\ell \in \mathsf{H}^{k+1}_{\widehat{\Gamma}_{t,\ell}}(\Omega_\ell)$. Extending S_ℓ and v_ℓ by zero to Ω gives fields $\widetilde{S}_\ell \in \mathsf{H}^{k+1}_{\mathbb{S},\Gamma_t}(\Omega)$ and $\widetilde{v}_\ell \in \mathsf{H}^{k+1}_{\Gamma_t}(\Omega)$ as well as

$$\begin{aligned} S = \sum_{\ell=-L}^{L} \varphi_\ell S|_{\Omega_\ell} &= \sum_{\ell=-L}^{L} \widetilde{S}_\ell + \operatorname{symGrad} \sum_{\ell=-L}^{L} \widetilde{v}_\ell \\ &\in \mathsf{H}^{k+1}_{\mathbb{S},\Gamma_t}(\Omega) + \operatorname{symGrad} \mathsf{H}^{k+1}_{\Gamma_t}(\Omega) \subset \mathsf{H}^{k,k-1}_{\mathbb{S},\Gamma_t}(\operatorname{RotRot}^\top,\Omega). \end{aligned}$$

As all operations have been linear and continuous we set

$$\mathcal{Q}^{k,k-1,1}_{\operatorname{RotRot}^\top_\mathbb{S},\Gamma_t} S := \sum_{\ell=-L}^{L} \widetilde{S}_\ell \in \mathsf{H}^{k+1}_{\mathbb{S},\Gamma_t}(\Omega), \qquad \mathcal{Q}^{k,k-1,0}_{\operatorname{RotRot}^\top_\mathbb{S},\Gamma_t} S := \sum_{\ell=-L}^{L} \widetilde{v}_\ell \in \mathsf{H}^{k+1}_{\Gamma_t}(\Omega).$$

- Let $k \geq 0$ and let $S \in \mathbf{H}^{k}_{\mathbb{S},\Gamma_t}(\operatorname{RotRot}^\top,\Omega)$. Then by definition $S|_{\Omega_\ell} \in \mathbf{H}^{k}_{\mathbb{S},\Gamma_{t,\ell}}(\operatorname{RotRot}^\top,\Omega_\ell)$ and we decompose by Corollary 3.4

$$S|_{\Omega_\ell} = S_{\ell,1} + \operatorname{symGrad} v_{\ell,0}$$

with $S_{\ell,1} := \mathcal{Q}^{k,1}_{\operatorname{RotRot}^\top_\mathbb{S},\Gamma_{t,\ell}} S|_{\Omega_\ell} \in \mathsf{H}^{k+2}_{\mathbb{S},\Gamma_{t,\ell}}(\Omega_\ell)$ and $v_{\ell,0} := \mathcal{Q}^{k,0}_{\operatorname{RotRot}^\top_\mathbb{S},\Gamma_{t,\ell}} S|_{\Omega_\ell} \in \mathsf{H}^{k+1}_{\Gamma_{t,\ell}}(\Omega_\ell)$. Now we follow the arguments from (6) on. Note that still only $S_\ell \in \mathsf{H}^{k+1}_{\mathbb{S},\widehat{\Gamma}_{t,\ell}}(\Omega_\ell)$ holds, i.e., we have lost one order of regularity for S_ℓ. Nevertheless, we get

$$S \in \mathsf{H}^{k+1}_{\mathbb{S},\Gamma_t}(\Omega) + \operatorname{symGrad} \mathsf{H}^{k+1}_{\Gamma_t}(\Omega),$$

and all operations have been linear and continuous. But this implies by the previous step

$$S \in \mathsf{H}^{k+1,k}_{\mathbb{S},\Gamma_t}(\operatorname{RotRot}^\top,\Omega) + \operatorname{symGrad} \mathsf{H}^{k+1}_{\Gamma_t}(\Omega).$$

Again by the previous step we obtain

$$\begin{aligned} S &\in \mathsf{H}^{k+2}_{\mathbb{S},\Gamma_t}(\Omega) + \operatorname{symGrad} \mathsf{H}^{k+2}_{\Gamma_t}(\Omega) + \operatorname{symGrad} \mathsf{H}^{k+1}_{\Gamma_t}(\Omega) \\ &= \mathsf{H}^{k+2}_{\mathbb{S},\Gamma_t}(\Omega) + \operatorname{symGrad} \mathsf{H}^{k+1}_{\Gamma_t}(\Omega) \subset \mathsf{H}^{k}_{\mathbb{S},\Gamma_t}(\operatorname{RotRot}^\top,\Omega), \end{aligned}$$

and all operations have been linear and continuous.

It remains to prove $\mathbf{H}^k_{\Gamma_t}(\operatorname{symGrad},\Omega) \subset \mathsf{H}^k_{\Gamma_t}(\operatorname{symGrad},\Omega)$. Let $v \in \mathbf{H}^k_{\Gamma_t}(\operatorname{symGrad},\Omega)$. Then we have $\varphi_\ell v \in \mathbf{H}^k_{\widehat{\Gamma}_{t,\ell}}(\operatorname{symGrad},\Omega_\ell) = \mathsf{H}^k_{\widehat{\Gamma}_{t,\ell}}(\operatorname{symGrad},\Omega_\ell) = \mathsf{H}^{k+1}_{\widehat{\Gamma}_{t,\ell}}(\Omega_\ell)$ by Corollary 3.6. Extending $\varphi_\ell v$ by zero to Ω yields vector fields $v_\ell \in \mathsf{H}^{k+1}_{\Gamma_t}(\Omega)$ as well as $v = \sum_\ell \varphi_\ell v = \sum_\ell v_\ell \in \mathsf{H}^{k+1}_{\Gamma_t}(\Omega)$. □

3.2. Mini FA-ToolBox.

3.2.1. *Zero Order Mini FA-ToolBox.* Recall Section 2.7 and let ε, μ be admissible. In Section 2.2 (for $\varepsilon = \mu = \operatorname{id}$) we have seen that the densely defined and closed linear operators

$$\begin{aligned}
\mathrm{A}_0 &= \operatorname{symGrad}_{\Gamma_t} : \mathsf{H}^1_{\Gamma_t}(\Omega) \subset \mathsf{L}^2(\Omega) \to \mathsf{L}^2_{\mathbb{S},\varepsilon}(\Omega),\\
\mathrm{A}_1 &= \mu^{-1}\operatorname{RotRot}^\top_{\mathbb{S},\Gamma_t} : \mathsf{H}_{\mathbb{S},\Gamma_t}(\operatorname{RotRot}^\top,\Omega) \subset \mathsf{L}^2_{\mathbb{S},\varepsilon}(\Omega) \to \mathsf{L}^2_{\mathbb{S},\mu}(\Omega),\\
\mathrm{A}_2 &= \operatorname{Div}_{\mathbb{S},\Gamma_t}\mu : \mu^{-1}\mathsf{H}_{\mathbb{S},\Gamma_t}(\operatorname{Div},\Omega) \subset \mathsf{L}^2_{\mathbb{S},\mu}(\Omega) \to \mathsf{L}^2(\Omega),\\
\mathrm{A}_0^* &= -\operatorname{Div}_{\mathbb{S},\Gamma_n}\varepsilon : \varepsilon^{-1}\mathsf{H}_{\mathbb{S},\Gamma_n}(\operatorname{Div},\Omega) \subset \mathsf{L}^2_{\mathbb{S},\varepsilon}(\Omega) \to \mathsf{L}^2(\Omega),\\
\mathrm{A}_1^* &= \varepsilon^{-1}\operatorname{RotRot}^\top_{\mathbb{S},\Gamma_n} : \mathsf{H}_{\mathbb{S},\Gamma_n}(\operatorname{RotRot}^\top,\Omega) \subset \mathsf{L}^2_{\mathbb{S},\mu}(\Omega) \to \mathsf{L}^2_{\mathbb{S},\varepsilon}(\Omega),\\
\mathrm{A}_2^* &= -\operatorname{symGrad}_{\Gamma_n} : \mathsf{H}^1_{\Gamma_n}(\Omega) \subset \mathsf{L}^2(\Omega) \to \mathsf{L}^2_{\mathbb{S},\mu}(\Omega),
\end{aligned}$$

where we have used Corollary 3.11, build the long primal and dual elasticity Hilbert complex

$$\mathbb{RM}_{\Gamma_t} \underset{\mathrm{A}_{-1}^*=\pi_{\mathbb{RM}_{\Gamma_t}}}{\overset{\mathrm{A}_{-1}=\iota_{\mathbb{RM}_{\Gamma_t}}}{\rightleftarrows}} \mathsf{L}^2(\Omega) \underset{\mathrm{A}_0^*=-\operatorname{Div}_{\mathbb{S},\Gamma_n}\varepsilon}{\overset{\mathrm{A}_0=\operatorname{symGrad}_{\Gamma_t}}{\rightleftarrows}} \mathsf{L}^2_{\mathbb{S},\varepsilon}(\Omega) \underset{\mathrm{A}_1^*=\varepsilon^{-1}\operatorname{RotRot}^\top_{\mathbb{S},\Gamma_n}}{\overset{\mathrm{A}_1=\mu^{-1}\operatorname{RotRot}^\top_{\mathbb{S},\Gamma_t}}{\rightleftarrows}} \mathsf{L}^2_{\mathbb{S},\mu}(\Omega) \underset{\mathrm{A}_2^*=-\operatorname{symGrad}_{\Gamma_n}}{\overset{\mathrm{A}_2=\operatorname{Div}_{\mathbb{S},\Gamma_t}\mu}{\rightleftarrows}} \mathsf{L}^2(\Omega) \underset{\mathrm{A}_3^*=\iota_{\mathbb{RM}_{\Gamma_n}}}{\overset{\mathrm{A}_3=\pi_{\mathbb{RM}_{\Gamma_n}}}{\rightleftarrows}} \mathbb{RM}_{\Gamma_n} \tag{7}$$

cf. (5).

Theorem 3.12 (compact embedding)**.** *The embedding*

$$\mathsf{H}_{\mathbb{S},\Gamma_t}(\operatorname{RotRot}^\top,\Omega) \cap \varepsilon^{-1}\mathsf{H}_{\mathbb{S},\Gamma_n}(\operatorname{Div},\Omega) \hookrightarrow \mathsf{L}^2_{\mathbb{S},\varepsilon}(\Omega)$$

is compact. Moreover, the compactness does not depend on ε.

Proof. Note that this type of compact embedding is independent of ε and μ, cf. [12, Lemma 5.1]. So, let $\varepsilon = \mu = \operatorname{id}$. Lemma 3.10 (for $k = 0$) yields the bounded regular decomposition

$$D(\mathrm{A}_0^*) = \mathsf{H}_{\mathbb{S},\Gamma_n}(\operatorname{Div},\Omega) = \mathsf{H}^1_{\mathbb{S},\Gamma_n}(\Omega) + \operatorname{RotRot}^\top\mathsf{H}^2_{\mathbb{S},\Gamma_n}(\Omega) = \mathsf{H}_1^+ + \mathrm{A}_1^*\,\mathsf{H}_2^+$$

with $\mathsf{H}_1^+ = \mathsf{H}^1_{\mathbb{S},\Gamma_n}(\Omega)$ and $\mathsf{H}_2^+ = \mathsf{H}^2_{\mathbb{S},\Gamma_n}(\Omega)$ and $\mathsf{H}_1 = \mathsf{H}_2 = \mathsf{L}^2_{\mathbb{S}}(\Omega)$. Rellich's selection theorem and [15, Corollary 2.12], cf. [11, Lemma 2.22], yield that $D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) \hookrightarrow \mathsf{H}_1$ is compact. □

Remark 3.13 (compact embedding)**.** *The embeddings*

$$\begin{aligned}
D(\mathrm{A}_0) \cap D(\mathrm{A}_{-1}^*) &= \mathsf{H}^1_{\Gamma_t}(\Omega) \hookrightarrow \mathsf{L}^2(\Omega) = \mathsf{H}_0,\\
D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) &= \mathsf{H}_{\mathbb{S},\Gamma_t}(\operatorname{RotRot}^\top,\Omega) \cap \varepsilon^{-1}\mathsf{H}_{\mathbb{S},\Gamma_n}(\operatorname{Div},\Omega) \hookrightarrow \mathsf{L}^2_{\mathbb{S},\varepsilon}(\Omega) = \mathsf{H}_1,\\
D(\mathrm{A}_2) \cap D(\mathrm{A}_1^*) &= \mu^{-1}\mathsf{H}_{\mathbb{S},\Gamma_t}(\operatorname{Div},\Omega) \cap \mathsf{H}_{\mathbb{S},\Gamma_n}(\operatorname{RotRot}^\top,\Omega) \hookrightarrow \mathsf{L}^2_{\mathbb{S},\mu}(\Omega) = \mathsf{H}_2,\\
D(\mathrm{A}_3) \cap D(\mathrm{A}_2^*) &= \mathsf{H}^1_{\Gamma_n}(\Omega) \hookrightarrow \mathsf{L}^2(\Omega) = \mathsf{H}_3
\end{aligned}$$

are compact, and the compactness does not depend on ε or μ.

Theorem 3.14 (compact elasticity complex)**.** *The long primal and dual elasticity Hilbert complex* (7) *is compact. In particular, the complex is closed.*

Let us recall the reduced operators

$$\begin{aligned}
(\mathrm{A}_0)_\perp &= (\operatorname{symGrad}_{\Gamma_t})_\perp : D\big((\operatorname{symGrad}_{\Gamma_t})_\perp\big) \subset (\mathbb{RM}_{\Gamma_t})^{\perp_{\mathsf{L}^2(\Omega)}} \to R(\operatorname{symGrad}_{\Gamma_t}),\\
(\mathrm{A}_1)_\perp &= (\mu^{-1}\operatorname{RotRot}^\top_{\mathbb{S},\Gamma_t})_\perp : D\big((\mu^{-1}\operatorname{RotRot}^\top_{\mathbb{S},\Gamma_t})_\perp\big) \subset N(\mu^{-1}\operatorname{RotRot}^\top_{\mathbb{S},\Gamma_t})^{\perp_{\mathsf{L}^2_{\mathbb{S},\varepsilon}(\Omega)}} \to R(\mu^{-1}\operatorname{RotRot}^\top_{\mathbb{S},\Gamma_t}),\\
(\mathrm{A}_2)_\perp &= (\operatorname{Div}_{\mathbb{S},\Gamma_t}\mu)_\perp : D\big((\operatorname{Div}_{\mathbb{S},\Gamma_t}\mu)_\perp\big) \subset N(\operatorname{Div}_{\mathbb{S},\Gamma_t}\mu)^{\perp_{\mathsf{L}^2_{\mathbb{S},\mu}(\Omega)}} \to R(\operatorname{Div}_{\mathbb{S},\Gamma_t}\mu),\\
(\mathrm{A}_0^*)_\perp &= -(\operatorname{Div}_{\mathbb{S},\Gamma_n}\varepsilon)_\perp : D\big((\operatorname{Div}_{\mathbb{S},\Gamma_n}\varepsilon)_\perp\big) \subset N(\operatorname{Div}_{\mathbb{S},\Gamma_n}\varepsilon)^{\perp_{\mathsf{L}^2_{\mathbb{S},\varepsilon}(\Omega)}} \to R(\operatorname{Div}_{\mathbb{S},\Gamma_n}\varepsilon),\\
(\mathrm{A}_1^*)_\perp &= (\varepsilon^{-1}\operatorname{RotRot}^\top_{\mathbb{S},\Gamma_n})_\perp : D\big((\varepsilon^{-1}\operatorname{RotRot}^\top_{\mathbb{S},\Gamma_n})_\perp\big) \subset N(\varepsilon^{-1}\operatorname{RotRot}^\top_{\mathbb{S},\Gamma_n})^{\perp_{\mathsf{L}^2_{\mathbb{S},\mu}(\Omega)}} \to R(\varepsilon^{-1}\operatorname{RotRot}^\top_{\mathbb{S},\Gamma_n}),\\
(\mathrm{A}_2^*)_\perp &= (\operatorname{symGrad}_{\Gamma_n})_\perp : D\big((\operatorname{symGrad}_{\Gamma_n})_\perp\big) \subset (\mathbb{RM}_{\Gamma_n})^{\perp_{\mathsf{L}^2(\Omega)}} \to R(\operatorname{symGrad}_{\Gamma_n}),
\end{aligned}$$

with domains of definition

$$D\big((\mathrm{A}_0)_\perp\big) = D(\operatorname{symGrad}_{\Gamma_t}) \cap (\mathbb{RM}_{\Gamma_t})^{\perp_{\mathsf{L}^2(\Omega)}},$$

$$D\big((\mathrm{A}_1)_\perp\big) = D(\mu^{-1}\operatorname{RotRot}^\top_{\mathbb{S},\Gamma_t}) \cap N(\mu^{-1}\operatorname{RotRot}^\top_{\mathbb{S},\Gamma_t})^{\perp_{\mathsf{L}^2_{\mathbb{S},\varepsilon}(\Omega)}} = D(\mu^{-1}\operatorname{RotRot}^\top_{\mathbb{S},\Gamma_t}) \cap R(\varepsilon^{-1}\operatorname{RotRot}^\top_{\mathbb{S},\Gamma_n}),$$

$$D\big((\mathrm{A}_2)_\perp\big) = D(\operatorname{Div}_{\mathbb{S},\Gamma_t}\mu) \cap N(\operatorname{Div}_{\mathbb{S},\Gamma_t}\mu)^{\perp_{\mathsf{L}^2_{\mathbb{S},\mu}(\Omega)}} = D(\operatorname{Div}_{\mathbb{S},\Gamma_t}\mu) \cap R(\operatorname{symGrad}_{\Gamma_n}),$$

$$D\big((\mathrm{A}_0^*)_\perp\big) = D(\operatorname{Div}_{\mathbb{S},\Gamma_n}\varepsilon) \cap N(\operatorname{Div}_{\mathbb{S},\Gamma_n}\varepsilon)^{\perp_{\mathsf{L}^2_{\mathbb{S},\varepsilon}(\Omega)}} = D(\operatorname{Div}_{\mathbb{S},\Gamma_n}\varepsilon) \cap R(\operatorname{symGrad}_{\Gamma_t}),$$

$$D\big((\mathrm{A}_1^*)_\perp\big) = D(\varepsilon^{-1}\operatorname{RotRot}^\top_{\mathbb{S},\Gamma_n}) \cap N(\varepsilon^{-1}\operatorname{RotRot}^\top_{\mathbb{S},\Gamma_n})^{\perp_{\mathsf{L}^2_{\mathbb{S},\mu}(\Omega)}} = D(\varepsilon^{-1}\operatorname{RotRot}^\top_{\mathbb{S},\Gamma_n}) \cap R(\mu^{-1}\operatorname{RotRot}^\top_{\mathbb{S},\Gamma_t}),$$

$$D\big((\mathrm{A}_2^*)_\perp\big) = D(\operatorname{symGrad}_{\Gamma_n}) \cap (\mathbb{RM}_{\Gamma_n})^{\perp_{\mathsf{L}^2(\Omega)}}.$$

Note that $R(\mathrm{A}_n) = R\big((\mathrm{A}_n)_\perp\big)$ and $R(\mathrm{A}_n^*) = R\big((\mathrm{A}_n^*)_\perp\big)$ hold. [11, Lemma 2.9] shows:

Theorem 3.15 (mini FA-ToolBox). *For the zero order elasticity complex it holds:*

- **(i)** *The ranges* $R(\operatorname{symGrad}_{\Gamma_t})$, $R(\mu^{-1}\operatorname{RotRot}^\top_{\mathbb{S},\Gamma_t})$, *and* $R(\operatorname{Div}_{\mathbb{S},\Gamma_t}\mu)$ *are closed.*
- **(i)** *The inverse operators* $(\operatorname{symGrad}_{\Gamma_t})_\perp^{-1}$, $(\mu^{-1}\operatorname{RotRot}^\top_{\mathbb{S},\Gamma_t})_\perp^{-1}$, *and* $(\operatorname{Div}_{\mathbb{S},\Gamma_t}\mu)_\perp^{-1}$ *are compact.*
- **(iii)** *The cohomology group of generalised Dirichlet/Neumann tensor fields* $\mathcal{H}_{\mathbb{S},\Gamma_t,\Gamma_n,\varepsilon}(\Omega)$ *is finite-dimensional. Moreover, the dimension does not depend on* ε.
- **(iv)** *The orthonormal Helmholtz type decompositions*

$$\begin{aligned}\mathsf{L}^2_{\mathbb{S},\varepsilon}(\Omega) &= R(\operatorname{symGrad}_{\Gamma_t}) \oplus_{\mathsf{L}^2_{\mathbb{S},\varepsilon}(\Omega)} N(\operatorname{Div}_{\mathbb{S},\Gamma_n}\varepsilon)\\ &= N(\mu^{-1}\operatorname{RotRot}^\top_{\mathbb{S},\Gamma_t}) \oplus_{\mathsf{L}^2_{\mathbb{S},\varepsilon}(\Omega)} R(\varepsilon^{-1}\operatorname{RotRot}^\top_{\mathbb{S},\Gamma_n})\\ &= R(\operatorname{symGrad}_{\Gamma_t}) \oplus_{\mathsf{L}^2_{\mathbb{S},\varepsilon}(\Omega)} \mathcal{H}_{\mathbb{S},\Gamma_t,\Gamma_n,\varepsilon}(\Omega) \oplus_{\mathsf{L}^2_{\mathbb{S},\varepsilon}(\Omega)} R(\varepsilon^{-1}\operatorname{RotRot}^\top_{\mathbb{S},\Gamma_n})\end{aligned}$$

  hold.
- **(v)** *There exist (optimal)* $c_0, c_1, c_2 > 0$ *such that the Friedrichs/Poincaré type estimates*

$$\begin{aligned}
&\forall\, v \in \mathsf{H}^1_{\Gamma_t}(\Omega) \cap (\mathbb{RM}_{\Gamma_t})^{\perp_{\mathsf{L}^2(\Omega)}} && |v|_{\mathsf{L}^2(\Omega)} \le c_0 |\operatorname{symGrad} v|_{\mathsf{L}^2_{\mathbb{S},\varepsilon}(\Omega)},\\
&\forall\, T \in \varepsilon^{-1}\mathsf{H}_{\mathbb{S},\Gamma_n}(\operatorname{Div},\Omega) \cap R(\operatorname{symGrad}_{\Gamma_t}) && |T|_{\mathsf{L}^2_{\mathbb{S},\varepsilon}(\Omega)} \le c_0 |\operatorname{Div}\varepsilon T|_{\mathsf{L}^2(\Omega)},\\
&\forall\, S \in \mathsf{H}_{\mathbb{S},\Gamma_t}(\operatorname{RotRot}^\top,\Omega) \cap R(\varepsilon^{-1}\operatorname{RotRot}^\top_{\mathbb{S},\Gamma_n}) && |S|_{\mathsf{L}^2_{\mathbb{S},\varepsilon}(\Omega)} \le c_1 |\mu^{-1}\operatorname{RotRot}^\top S|_{\mathsf{L}^2_{\mathbb{S},\mu}(\Omega)},\\
&\forall\, S \in \mathsf{H}_{\mathbb{S},\Gamma_n}(\operatorname{RotRot}^\top,\Omega) \cap R(\mu^{-1}\operatorname{RotRot}^\top_{\mathbb{S},\Gamma_t}) && |S|_{\mathsf{L}^2_{\mathbb{S},\mu}(\Omega)} \le c_1 |\varepsilon^{-1}\operatorname{RotRot}^\top S|_{\mathsf{L}^2_{\mathbb{S},\varepsilon}(\Omega)},\\
&\forall\, T \in \mu^{-1}\mathsf{H}_{\mathbb{S},\Gamma_t}(\operatorname{Div},\Omega) \cap R(\operatorname{symGrad}_{\Gamma_n}) && |T|_{\mathsf{L}^2_{\mathbb{S},\mu}(\Omega)} \le c_2 |\operatorname{Div}\mu T|_{\mathsf{L}^2(\Omega)},\\
&\forall\, v \in \mathsf{H}^1_{\Gamma_n}(\Omega) \cap (\mathbb{RM}_{\Gamma_n})^{\perp_{\mathsf{L}^2(\Omega)}} && |v|_{\mathsf{L}^2(\Omega)} \le c_2 |\operatorname{symGrad} v|_{\mathsf{L}^2_{\mathbb{S},\mu}(\Omega)}
\end{aligned}$$

  hold.
- **(vi)** *For all* $S \in \mathsf{H}_{\mathbb{S},\Gamma_t}(\operatorname{RotRot}^\top,\Omega) \cap \varepsilon^{-1}\mathsf{H}_{\mathbb{S},\Gamma_n}(\operatorname{Div},\Omega) \cap \mathcal{H}_{\mathbb{S},\Gamma_t,\Gamma_n,\varepsilon}(\Omega)^{\perp_{\mathsf{L}^2_{\mathbb{S},\varepsilon}(\Omega)}}$ *it holds*

$$|S|^2_{\mathsf{L}^2_{\mathbb{S},\varepsilon}(\Omega)} \le c_1^2 |\mu^{-1}\operatorname{RotRot}^\top S|^2_{\mathsf{L}^2_{\mathbb{S},\mu}(\Omega)} + c_0^2 |\operatorname{Div}\varepsilon S|^2_{\mathsf{L}^2(\Omega)}.$$

- **(vii)** $\mathcal{H}_{\mathbb{S},\Gamma_t,\Gamma_n,\varepsilon}(\Omega) = \{0\}$, *if* (Ω, Γ_t) *is extendable.*

3.2.2. *Higher Order Mini FA-ToolBox.* For simplicity, let $\varepsilon = \mu = \mathrm{id}$. From Section 2.6 we recall the densely defined and closed higher Sobolev order operators

$$\begin{aligned}
\operatorname{symGrad}^k_{\Gamma_t} &: \mathsf{H}^{k+1}_{\Gamma_t}(\Omega) \subset \mathsf{H}^k_{\Gamma_t}(\Omega) \to \mathsf{H}^k_{\mathbb{S},\Gamma_t}(\Omega),\\
\operatorname{RotRot}^{\top,k}_{\mathbb{S},\Gamma_t} &: \mathsf{H}^k_{\mathbb{S},\Gamma_t}(\operatorname{RotRot}^\top,\Omega) \subset \mathsf{H}^k_{\mathbb{S},\Gamma_t}(\Omega) \to \mathsf{H}^k_{\mathbb{S},\Gamma_t}(\Omega),\\
\operatorname{RotRot}^{\top,k,k-1}_{\mathbb{S},\Gamma_t} &: \mathsf{H}^{k,k-1}_{\mathbb{S},\Gamma_t}(\operatorname{RotRot}^\top,\Omega) \subset \mathsf{H}^k_{\mathbb{S},\Gamma_t}(\Omega) \to \mathsf{H}^{k-1}_{\mathbb{S},\Gamma_t}(\Omega), \qquad k \ge 1,\\
\operatorname{Div}^k_{\mathbb{S},\Gamma_t} &: \mathsf{H}^k_{\mathbb{S},\Gamma_t}(\operatorname{Div},\Omega) \subset \mathsf{H}^k_{\mathbb{S},\Gamma_t}(\Omega) \to \mathsf{H}^k_{\Gamma_t}(\Omega),
\end{aligned} \tag{8}$$

building the long elasticity Hilbert complexes

$$\mathbb{RM}_{\Gamma_t} \xrightarrow{\iota_{\mathbb{RM}_{\Gamma_t}}} \mathsf{H}^k_{\Gamma_t}(\Omega) \xrightarrow{\operatorname{symGrad}^k_{\Gamma_t}} \mathsf{H}^k_{\mathbb{S},\Gamma_t}(\Omega) \xrightarrow{\operatorname{RotRot}^{\top,k}_{\mathbb{S},\Gamma_t}} \mathsf{H}^k_{\mathbb{S},\Gamma_t}(\Omega) \xrightarrow{\operatorname{Div}^k_{\mathbb{S},\Gamma_t}} \mathsf{H}^k_{\Gamma_t}(\Omega) \xrightarrow{\pi_{\mathbb{RM}_{\Gamma_n}}} \mathbb{RM}_{\Gamma_n}, \quad k \ge 0, \tag{9}$$

$$\mathbb{RM}_{\Gamma_t} \xrightarrow{\iota_{\mathbb{RM}_{\Gamma_t}}} \mathsf{H}^k_{\Gamma_t}(\Omega) \xrightarrow{\operatorname{symGrad}^k_{\Gamma_t}} \mathsf{H}^k_{\mathbb{S},\Gamma_t}(\Omega) \xrightarrow{\operatorname{RotRot}^{\top,k,k-1}_{\mathbb{S},\Gamma_t}} \mathsf{H}^{k-1}_{\mathbb{S},\Gamma_t}(\Omega) \xrightarrow{\operatorname{Div}^{k-1}_{\mathbb{S},\Gamma_t}} \mathsf{H}^{k-1}_{\Gamma_t}(\Omega) \xrightarrow{\pi_{\mathbb{RM}_{\Gamma_n}}} \mathbb{RM}_{\Gamma_n}, \quad k \geq 1. \tag{10}$$

We start with regular representations implied by Lemma 3.10 and Corollary 3.11.

Theorem 3.16 (regular representations and closed ranges)**.** *Let $k \geq 0$. Then the regular potential representations*

$$\begin{aligned}
R(\operatorname{symGrad}^k_{\Gamma_t}) &= \operatorname{symGrad} \mathsf{H}^k_{\Gamma_t}(\operatorname{symGrad}, \Omega) = \operatorname{symGrad} \mathsf{H}^{k+1}_{\Gamma_t}(\Omega) \\
&= \mathsf{H}^k_{\mathbb{S},\Gamma_t}(\Omega) \cap R(\operatorname{symGrad}_{\Gamma_t}) \\
&= \mathsf{H}^k_{\mathbb{S},\Gamma_t}(\Omega) \cap \mathsf{H}_{\mathbb{S},\Gamma_t,0}(\operatorname{RotRot}^\top, \Omega) \cap \mathcal{H}_{\mathbb{S},\Gamma_t,\Gamma_n,\varepsilon}(\Omega)^{\perp_{\mathsf{L}^2_{\mathbb{S},\varepsilon}(\Omega)}} \\
&= \mathsf{H}^k_{\mathbb{S},\Gamma_t,0}(\operatorname{RotRot}^\top, \Omega) \cap \mathcal{H}_{\mathbb{S},\Gamma_t,\Gamma_n,\varepsilon}(\Omega)^{\perp_{\mathsf{L}^2_{\mathbb{S},\varepsilon}(\Omega)}}, \\
R(\operatorname{RotRot}^{\top,k+1,k}_{\mathbb{S},\Gamma_t}) = R(\operatorname{RotRot}^{\top,k}_{\mathbb{S},\Gamma_t}) &= \operatorname{RotRot}^\top \mathsf{H}^k_{\mathbb{S},\Gamma_t}(\operatorname{RotRot}^\top, \Omega) = \operatorname{RotRot}^\top \mathsf{H}^{k+2}_{\mathbb{S},\Gamma_t}(\Omega) \\
&= \operatorname{RotRot}^\top \mathsf{H}^{k+1,k}_{\mathbb{S},\Gamma_t}(\operatorname{RotRot}^\top, \Omega) \\
&= \mathsf{H}^k_{\mathbb{S},\Gamma_t}(\Omega) \cap R(\operatorname{RotRot}^\top_{\mathbb{S},\Gamma_t}) \\
&= \mathsf{H}^k_{\mathbb{S},\Gamma_t}(\Omega) \cap \mathsf{H}_{\mathbb{S},\Gamma_t,0}(\operatorname{Div}, \Omega) \cap \mathcal{H}_{\mathbb{S},\Gamma_n,\Gamma_t,\varepsilon}(\Omega)^{\perp_{\mathsf{L}^2_{\mathbb{S}}(\Omega)}}, \\
&= \mathsf{H}^k_{\mathbb{S},\Gamma_t,0}(\operatorname{Div}, \Omega) \cap \mathcal{H}_{\mathbb{S},\Gamma_n,\Gamma_t,\varepsilon}(\Omega)^{\perp_{\mathsf{L}^2_{\mathbb{S}}(\Omega)}}, \\
R(\operatorname{Div}^k_{\mathbb{S},\Gamma_t}) &= \operatorname{Div} \mathsf{H}^k_{\mathbb{S},\Gamma_t}(\operatorname{Div}, \Omega) = \operatorname{Div} \mathsf{H}^{k+1}_{\mathbb{S},\Gamma_t}(\Omega) \\
&= \mathsf{H}^k_{\Gamma_t}(\Omega) \cap R(\operatorname{Div}_{\mathbb{S},\Gamma_t}) = \mathsf{H}^k_{\Gamma_t}(\Omega) \cap (\mathbb{RM}_{\Gamma_n})^{\perp_{\mathsf{L}^2(\Omega)}}
\end{aligned}$$

hold. In particular, the latter spaces are closed subspaces of $\mathsf{H}^k_{\mathbb{S}}(\Omega)$ and $\mathsf{H}^k(\Omega)$, respectively, and all ranges of the higher Sobolev order operators in (8) *are closed. Moroever, the long elasticity Hilbert complexes* (9) *and* (10) *are closed.*

Note that in Theorem 3.16 we claim nothing about bounded regular potential operators, leaving the question of bounded potentials to the next sections, cf. Theorem 3.24.

Proof of Theorem 3.16. We only show the representations for $R(\operatorname{RotRot}^{\top,k}_{\mathbb{S},\Gamma_t})$. The other follow analogously, but simpler. By Lemma 3.10 and Corollary 3.11 we have

$$\begin{aligned}
\operatorname{RotRot}^\top \mathsf{H}^{k+2}_{\mathbb{S},\Gamma_t}(\Omega) &\subset \operatorname{RotRot}^\top \mathsf{H}^{k+1,k}_{\mathbb{S},\Gamma_t}(\operatorname{RotRot}^\top, \Omega) = R(\operatorname{RotRot}^{\top,k+1,k}_{\mathbb{S},\Gamma_t}) \\
&\subset \operatorname{RotRot}^\top \mathsf{H}^k_{\mathbb{S},\Gamma_t}(\operatorname{RotRot}^\top, \Omega) = R(\operatorname{RotRot}^{\top,k}_{\mathbb{S},\Gamma_t}) = \operatorname{RotRot}^\top \mathsf{H}^{k+2}_{\mathbb{S},\Gamma_t}(\Omega).
\end{aligned}$$

In particular,

$$R(\operatorname{RotRot}^{\top,k}_{\mathbb{S},\Gamma_t}) = \operatorname{RotRot}^\top \mathsf{H}^k_{\mathbb{S},\Gamma_t}(\operatorname{RotRot}^\top, \Omega) = \operatorname{RotRot}^\top \mathsf{H}^{k+2}_{\mathbb{S},\Gamma_t}(\Omega). \tag{11}$$

Moreover,

$$\begin{aligned}
R(\operatorname{RotRot}^{\top,k}_{\mathbb{S},\Gamma_t}) &\subset \mathsf{H}^k_{\mathbb{S},\Gamma_t,0}(\operatorname{Div}, \Omega) \cap \mathcal{H}_{\mathbb{S},\Gamma_n,\Gamma_t,\varepsilon}(\Omega)^{\perp_{\mathsf{L}^2_{\mathbb{S}}(\Omega)}} \\
&= \mathsf{H}^k_{\mathbb{S},\Gamma_t}(\Omega) \cap \mathsf{H}_{\mathbb{S},\Gamma_t,0}(\operatorname{Div}, \Omega) \cap \mathcal{H}_{\mathbb{S},\Gamma_n,\Gamma_t,\varepsilon}(\Omega)^{\perp_{\mathsf{L}^2_{\mathbb{S}}(\Omega)}} = \mathsf{H}^k_{\mathbb{S},\Gamma_t}(\Omega) \cap R(\operatorname{RotRot}^\top_{\mathbb{S},\Gamma_t}),
\end{aligned}$$

since by Theorem 3.15 (iv)

$$R(\operatorname{RotRot}^\top_{\mathbb{S},\Gamma_t}) = \mathsf{H}_{\mathbb{S},\Gamma_t,0}(\operatorname{Div}, \Omega) \cap \mathcal{H}_{\mathbb{S},\Gamma_n,\Gamma_t,\varepsilon}(\Omega)^{\perp_{\mathsf{L}^2_{\mathbb{S}}(\Omega)}}. \tag{12}$$

Thus it remains to show

$$\mathsf{H}^k_{\mathbb{S},\Gamma_t,0}(\operatorname{Div}, \Omega) \cap \mathcal{H}_{\mathbb{S},\Gamma_n,\Gamma_t,\varepsilon}(\Omega)^{\perp_{\mathsf{L}^2_{\mathbb{S}}(\Omega)}} \subset \operatorname{RotRot}^\top \mathsf{H}^k_{\mathbb{S},\Gamma_t}(\operatorname{RotRot}^\top, \Omega), \qquad k \geq 1.$$

For this, let $k \geq 1$ and $T \in \mathsf{H}^k_{\mathbb{S},\Gamma_t,0}(\operatorname{Div}, \Omega) \cap \mathcal{H}_{\mathbb{S},\Gamma_n,\Gamma_t,\varepsilon}(\Omega)^{\perp_{\mathsf{L}^2_{\mathbb{S}}(\Omega)}}$. By (12) and (11) we have

$$T \in R(\operatorname{RotRot}^\top_{\mathbb{S},\Gamma_t}) = \operatorname{RotRot}^\top \mathsf{H}^2_{\mathbb{S},\Gamma_t}(\Omega)$$

and hence there is $S_1 \in \mathsf{H}^2_{\mathbb{S},\Gamma_t}(\Omega)$ such that $\operatorname{RotRot}^\top S_1 = T$. We see $S_1 \in \mathsf{H}^2_{\mathbb{S},\Gamma_t}(\operatorname{RotRot}^\top, \Omega)$ resp. $S_1 \in \mathsf{H}^1_{\mathbb{S},\Gamma_t}(\operatorname{RotRot}^\top, \Omega)$ if $k = 1$. Hence we are done for $k = 1$ and $k = 2$. For $k \geq 2$ we have $T \in \operatorname{RotRot}^\top \mathsf{H}^2_{\mathbb{S},\Gamma_t}(\operatorname{RotRot}^\top,\Omega) = \operatorname{RotRot}^\top \mathsf{H}^4_{\mathbb{S},\Gamma_t}(\Omega)$ by (11). Thus there is $S_2 \in \mathsf{H}^4_{\mathbb{S},\Gamma_t}(\Omega)$ such that $\operatorname{RotRot}^\top S_2 = T$. Then $S_2 \in \mathsf{H}^4_{\mathbb{S},\Gamma_t}(\operatorname{RotRot}^\top,\Omega)$ resp. $S_2 \in \mathsf{H}^3_{\mathbb{S},\Gamma_t}(\operatorname{RotRot}^\top,\Omega)$ if $k=3$, and we are done for $k=3$ and $k=4$. After finitely many steps, we observe that T belongs to $\operatorname{RotRot}^\top \mathsf{H}^k_{\mathbb{S},\Gamma_t}(\operatorname{RotRot}^\top,\Omega)$, finishing the proof. □

The reduced operators corresponding to (8) are

$$(\operatorname{symGrad}^k_{\Gamma_t})_\perp : D\big((\operatorname{symGrad}^k_{\Gamma_t})_\perp\big) \subset (\mathbb{RM}_{\Gamma_t})^{\perp_{\mathsf{H}^k_{\Gamma_t}(\Omega)}} \to R(\operatorname{symGrad}^k_{\Gamma_t}),$$
$$(\operatorname{RotRot}^{\top,k}_{\mathbb{S},\Gamma_t})_\perp : D\big((\operatorname{RotRot}^{\top,k}_{\mathbb{S},\Gamma_t})_\perp\big) \subset N(\operatorname{RotRot}^{\top,k}_{\mathbb{S},\Gamma_t})^{\perp_{\mathsf{H}^k_{\mathbb{S},\Gamma_t}(\Omega)}} \to R(\operatorname{RotRot}^{\top,k}_{\mathbb{S},\Gamma_t}),$$
$$(\operatorname{RotRot}^{\top,k,k-1}_{\mathbb{S},\Gamma_t})_\perp : D\big((\operatorname{RotRot}^{\top,k,k-1}_{\mathbb{S},\Gamma_t})_\perp\big) \subset N(\operatorname{RotRot}^{\top,k}_{\mathbb{S},\Gamma_t})^{\perp_{\mathsf{H}^k_{\mathbb{S},\Gamma_t}(\Omega)}} \to R(\operatorname{RotRot}^{\top,k-1}_{\mathbb{S},\Gamma_t}), \quad k \geq 1,$$
$$(\operatorname{Div}^k_{\mathbb{S},\Gamma_t})_\perp : D\big((\operatorname{Div}^k_{\mathbb{S},\Gamma_t})_\perp\big) \subset N(\operatorname{Div}^k_{\mathbb{S},\Gamma_t})^{\perp_{\mathsf{H}^k_{\mathbb{S},\Gamma_t}(\Omega)}} \to R(\operatorname{Div}^k_{\mathbb{S},\Gamma_t})$$

with domains of definition

$$D\big((\operatorname{symGrad}^k_{\Gamma_t})_\perp\big) = D(\operatorname{symGrad}^k_{\Gamma_t}) \cap (\mathbb{RM}_{\Gamma_t})^{\perp_{\mathsf{H}^k_{\Gamma_t}(\Omega)}},$$
$$D\big((\operatorname{RotRot}^{\top,k}_{\mathbb{S},\Gamma_t})_\perp\big) = D(\operatorname{RotRot}^{\top,k}_{\mathbb{S},\Gamma_t}) \cap N(\operatorname{RotRot}^{\top,k}_{\mathbb{S},\Gamma_t})^{\perp_{\mathsf{H}^k_{\mathbb{S},\Gamma_t}(\Omega)}},$$
$$D\big((\operatorname{RotRot}^{\top,k,k-1}_{\mathbb{S},\Gamma_t})_\perp\big) = D(\operatorname{RotRot}^{\top,k,k-1}_{\mathbb{S},\Gamma_t}) \cap N(\operatorname{RotRot}^{\top,k}_{\mathbb{S},\Gamma_t})^{\perp_{\mathsf{H}^k_{\mathbb{S},\Gamma_t}(\Omega)}}, \quad k \geq 1,$$
$$D\big((\operatorname{Div}^k_{\mathbb{S},\Gamma_t})_\perp\big) = D(\operatorname{Div}^k_{\mathbb{S},\Gamma_t}) \cap N(\operatorname{Div}^k_{\mathbb{S},\Gamma_t})^{\perp_{\mathsf{H}^k_{\mathbb{S},\Gamma_t}(\Omega)}}.$$

[11, Lemma 2.1] and Theorem 3.16 yield:

Theorem 3.17 (closed ranges and bounded inverse operators)**.** *Let $k \geq 0$. Then:*

(i) $R(\operatorname{symGrad}^k_{\Gamma_t}) = R\big((\operatorname{symGrad}^k_{\Gamma_t})_\perp\big)$ *are closed and, equivalently, the inverse operator*

$$(\operatorname{symGrad}^k_{\Gamma_t})_\perp^{-1} : R(\operatorname{symGrad}^k_{\Gamma_t}) \to D\big((\operatorname{symGrad}^k_{\Gamma_t})_\perp\big)$$
$$\textit{resp.}\qquad (\operatorname{symGrad}^k_{\Gamma_t})_\perp^{-1} : R(\operatorname{symGrad}^k_{\Gamma_t}) \to D(\operatorname{symGrad}^k_{\Gamma_t})$$

is bounded. Equivalently, there is $c > 0$ such that for all $v \in D\big((\operatorname{symGrad}^k_{\Gamma_t})_\perp\big)$

$$|v|_{\mathsf{H}^k(\Omega)} \leq c|\operatorname{symGrad} v|_{\mathsf{H}^k_{\mathbb{S}}(\Omega)}.$$

(ii) $R\big((\operatorname{RotRot}^{\top,k}_{\mathbb{S},\Gamma_t})_\perp\big) = R(\operatorname{RotRot}^{\top,k}_{\mathbb{S},\Gamma_t}) = R(\operatorname{RotRot}^{\top,k+1,k}_{\mathbb{S},\Gamma_t}) = R\big((\operatorname{RotRot}^{\top,k+1,k}_{\mathbb{S},\Gamma_t})_\perp\big)$ *are closed and, equivalently, the inverse operators*

$$(\operatorname{RotRot}^{\top,k}_{\mathbb{S},\Gamma_t})_\perp^{-1} : R(\operatorname{RotRot}^{\top,k}_{\mathbb{S},\Gamma_t}) \to D\big((\operatorname{RotRot}^{\top,k}_{\mathbb{S},\Gamma_t})_\perp\big)$$
$$\textit{resp.}\qquad (\operatorname{RotRot}^{\top,k}_{\mathbb{S},\Gamma_t})_\perp^{-1} : R(\operatorname{RotRot}^{\top,k}_{\mathbb{S},\Gamma_t}) \to D(\operatorname{RotRot}^{\top,k}_{\mathbb{S},\Gamma_t}),$$
$$(\operatorname{RotRot}^{\top,k+1,k}_{\mathbb{S},\Gamma_t})_\perp^{-1} : R(\operatorname{RotRot}^{\top,k}_{\mathbb{S},\Gamma_t}) \to D\big((\operatorname{RotRot}^{\top,k+1,k}_{\mathbb{S},\Gamma_t})_\perp\big)$$
$$\textit{resp.}\qquad (\operatorname{RotRot}^{\top,k+1,k}_{\mathbb{S},\Gamma_t})_\perp^{-1} : R(\operatorname{RotRot}^{\top,k}_{\mathbb{S},\Gamma_t}) \to D(\operatorname{RotRot}^{\top,k+1,k}_{\mathbb{S},\Gamma_t})$$

are bounded. Equivalently, there is $c > 0$ such that for all $S \in D\big((\operatorname{RotRot}^{\top,k}_{\mathbb{S},\Gamma_t})_\perp\big)$ resp. $S \in D\big((\operatorname{RotRot}^{\top,k+1,k}_{\mathbb{S},\Gamma_t})_\perp\big)$

$$|S|_{\mathsf{H}^k_{\mathbb{S}}(\Omega)} \leq c|\operatorname{RotRot}^\top S|_{\mathsf{H}^k_{\mathbb{S}}(\Omega)} \quad \textit{resp.} \quad |S|_{\mathsf{H}^{k+1}_{\mathbb{S}}(\Omega)} \leq c|\operatorname{RotRot}^\top S|_{\mathsf{H}^k_{\mathbb{S}}(\Omega)}.$$

(iii) $R(\operatorname{Div}^k_{\mathbb{S},\Gamma_t}) = R\big((\operatorname{Div}^k_{\mathbb{S},\Gamma_t})_\perp\big)$ *are closed and, equivalently, the inverse operator*

$$(\operatorname{Div}^k_{\mathbb{S},\Gamma_t})_\perp^{-1} : R(\operatorname{Div}^k_{\mathbb{S},\Gamma_t}) \to D\big((\operatorname{Div}^k_{\mathbb{S},\Gamma_t})_\perp\big)$$
$$\textit{resp.}\qquad (\operatorname{Div}^k_{\mathbb{S},\Gamma_t})_\perp^{-1} : R(\operatorname{Div}^k_{\mathbb{S},\Gamma_t}) \to D(\operatorname{Div}^k_{\mathbb{S},\Gamma_t})$$

is bounded. Equivalently, there is $c > 0$ such that for all $T \in D\big((\operatorname{Div}^k_{\mathbb{S},\Gamma_t})_\perp\big)$

$$|T|_{\mathsf{H}^k_{\mathbb{S}}(\Omega)} \leq c|\operatorname{Div} T|_{\mathsf{H}^k(\Omega)}.$$

Lemma 3.18 (Schwarz' lemma)**.** *Let $0 \leq |\alpha| \leq k$.*

(i) *For* $S \in \mathsf{H}^{k}_{\mathbb{S},\Gamma_t}(\operatorname{RotRot}^\top,\Omega)$ *resp.* $S \in \mathsf{H}^{k+1,k}_{\mathbb{S},\Gamma_t}(\operatorname{RotRot}^\top,\Omega)$ *it holds* $\partial^\alpha S \in \mathsf{H}_{\mathbb{S},\Gamma_t}(\operatorname{RotRot}^\top,\Omega)$ *resp.* $\partial^\alpha S \in \mathsf{H}^{1,0}_{\mathbb{S},\Gamma_t}(\operatorname{RotRot}^\top,\Omega)$ *and* $\operatorname{RotRot}^\top \partial^\alpha S = \partial^\alpha \operatorname{RotRot}^\top S$.

(ii) *For* $T \in \mathsf{H}^{k}_{\mathbb{S},\Gamma_t}(\operatorname{Div},\Omega)$ *it holds* $\partial^\alpha T \in \mathsf{H}_{\mathbb{S},\Gamma_t}(\operatorname{Div},\Omega)$ *and* $\operatorname{Div}\partial^\alpha T = \partial^\alpha \operatorname{Div} T$.

Proof. Let $S \in \mathsf{H}^{k}_{\mathbb{S},\Gamma_t}(\operatorname{RotRot}^\top,\Omega)$. For $\Phi \in \mathsf{C}^\infty_{\Gamma_n}(\Omega)$ we have

$$\begin{aligned}\langle \partial^\alpha S, \operatorname{RotRot}^\top \Phi\rangle_{\mathsf{L}^2_{\mathbb{S}}(\Omega)} &= (-1)^{|\alpha|}\langle S, \operatorname{RotRot}^\top \partial^\alpha \Phi\rangle_{\mathsf{L}^2_{\mathbb{S}}(\Omega)}\\ &= (-1)^{|\alpha|}\langle \operatorname{RotRot}^\top S, \partial^\alpha \Phi\rangle_{\mathsf{L}^2_{\mathbb{S}}(\Omega)} = \langle \partial^\alpha \operatorname{RotRot}^\top S, \Phi\rangle_{\mathsf{L}^2_{\mathbb{S}}(\Omega)}\end{aligned}$$

as $S \in \mathsf{H}^{k}_{\mathbb{S},\Gamma_t}(\Omega) \cap \mathsf{H}_{\mathbb{S},\Gamma_t}(\operatorname{RotRot}^\top,\Omega)$ and $\operatorname{RotRot}^\top S \in \mathsf{H}^{k}_{\mathbb{S},\Gamma_t}(\Omega)$. Hence

$$\partial^\alpha S \in \mathbf{H}_{\mathbb{S},\Gamma_t}(\operatorname{RotRot}^\top,\Omega) = \mathsf{H}_{\mathbb{S},\Gamma_t}(\operatorname{RotRot}^\top,\Omega)$$

by Corollary 3.11 and $\operatorname{RotRot}^\top \partial^\alpha S = \partial^\alpha \operatorname{RotRot}^\top S$. The other assertions follow analogously. □

Theorem 3.19 (compact embedding)**.** *Let* $k \geq 0$. *Then the embedding*

$$\mathsf{H}^{k}_{\mathbb{S},\Gamma_t}(\operatorname{RotRot}^\top,\Omega) \cap \mathsf{H}^{k}_{\mathbb{S},\Gamma_n}(\operatorname{Div},\Omega) \hookrightarrow \mathsf{H}^{k}_{\mathbb{S},\Gamma}(\Omega)$$

is compact.

Proof. We follow in close lines the proof of [15, Theorem 4.11], cf. [11, Theorem 4.16], using induction. The case $k = 0$ is given by Theorem 3.12. Let $k \geq 1$ and let (S_ℓ) be a bounded sequence in $\mathsf{H}^{k}_{\mathbb{S},\Gamma_t}(\operatorname{RotRot}^\top,\Omega) \cap \mathsf{H}^{k}_{\mathbb{S},\Gamma_n}(\operatorname{Div},\Omega)$. Note that

$$\mathsf{H}^{k}_{\mathbb{S},\Gamma_t}(\operatorname{RotRot}^\top,\Omega) \cap \mathsf{H}^{k}_{\mathbb{S},\Gamma_n}(\operatorname{Div},\Omega) \subset \mathsf{H}^{k}_{\mathbb{S},\Gamma_t}(\Omega) \cap \mathsf{H}^{k}_{\mathbb{S},\Gamma_n}(\Omega) = \mathsf{H}^{k}_{\mathbb{S},\Gamma}(\Omega).$$

By assumption and w.l.o.g. we have that (S_ℓ) is a Cauchy sequence in $\mathsf{H}^{k-1}_{\mathbb{S},\Gamma}(\Omega)$. Moreover, for all $|\alpha| = k$ we have $\partial^\alpha S_\ell \in \mathsf{H}_{\mathbb{S},\Gamma_t}(\operatorname{RotRot}^\top,\Omega) \cap \mathsf{H}_{\mathbb{S},\Gamma_n}(\operatorname{Div},\Omega)$ with $\operatorname{RotRot}^\top \partial^\alpha S_\ell = \partial^\alpha \operatorname{RotRot}^\top S_\ell$ and $\operatorname{Div}\partial^\alpha S_\ell = \partial^\alpha \operatorname{Div} S_\ell$ by Lemma 3.18. Hence $(\partial^\alpha S_\ell)$ is a bounded sequence in the zero order space $\mathsf{H}_{\mathbb{S},\Gamma_t}(\operatorname{RotRot}^\top,\Omega) \cap \mathsf{H}_{\mathbb{S},\Gamma_n}(\operatorname{Div},\Omega)$. Thus, w.l.o.g. $(\partial^\alpha S_\ell)$ is a Cauchy sequence in $\mathsf{L}^2_{\mathbb{S}}(\Omega)$ by Theorem 3.12. Finally, (S_ℓ) is a Cauchy sequence in $\mathsf{H}^{k}_{\mathbb{S},\Gamma}(\Omega)$, finishing the proof. □

Remark 3.20 (compact embedding)**.** *For* $k \geq 1$, *cf.* [15, Remark 4.12]*, there is another and slightly more general proof using a variant of* [11, Lemma 2.22].

For this, let (S_ℓ) *be a bounded sequence in* $\mathsf{H}^{k}_{\mathbb{S},\Gamma_t}(\operatorname{RotRot}^\top,\Omega) \cap \mathsf{H}^{k}_{\mathbb{S},\Gamma_n}(\operatorname{Div},\Omega)$. *In particular,* (S_ℓ) *is bounded in* $\mathsf{H}^{k,k-1}_{\mathbb{S},\Gamma_t}(\operatorname{RotRot}^\top,\Omega) \cap \mathsf{H}^{k}_{\mathbb{S},\Gamma_n}(\operatorname{Div},\Omega)$. *According to Lemma 3.10 we decompose* $S_\ell = T_\ell + \operatorname{symGrad} v_\ell$ *with* $T_\ell \in \mathsf{H}^{k+1}_{\mathbb{S},\Gamma_t}(\Omega)$ *and* $v_\ell \in \mathsf{H}^{k+1}_{\Gamma_t}(\Omega)$. *By the boundedness of the regular decomposition operators,* (T_ℓ) *and* (v_ℓ) *are bounded in* $\mathsf{H}^{k+1}_{\mathbb{S},\Gamma_t}(\Omega)$ *and* $\mathsf{H}^{k+1}_{\Gamma_t}(\Omega)$*, respectively. W.l.o.g.* (T_ℓ) *and* (v_ℓ) *converge in* $\mathsf{H}^{k}_{\mathbb{S},\Gamma_t}(\Omega)$ *and* $\mathsf{H}^{k}_{\Gamma_t}(\Omega)$*, respectively. For all* $0 \leq |\alpha| \leq k$ *Lemma 3.18 yields* $(\partial^\alpha S_\ell) \subset \mathsf{H}_{\mathbb{S},\Gamma_n}(\operatorname{Div},\Omega)$ *and* $\operatorname{Div}\partial^\alpha T = \partial^\alpha \operatorname{Div} T$. *With* $S_{\ell,l} := S_\ell - S_l$, $T_{\ell,l} := T_\ell - T_l$, *and* $v_{\ell,l} := v_\ell - v_l$ *we get*

$$\begin{aligned}|S_{\ell,l}|^2_{\mathsf{H}^k_{\mathbb{S}}(\Omega)} &= \langle S_{\ell,l}, T_{\ell,l}\rangle_{\mathsf{H}^k_{\mathbb{S}}(\Omega)} + \langle S_{\ell,l}, \operatorname{symGrad} v_{\ell,l}\rangle_{\mathsf{H}^k_{\mathbb{S}}(\Omega)}\\ &= \langle S_{\ell,l}, T_{\ell,l}\rangle_{\mathsf{H}^k_{\mathbb{S}}(\Omega)} - \langle \operatorname{Div} S_{\ell,l}, v_{\ell,l}\rangle_{\mathsf{H}^k(\Omega)} \leq c\big(|T_{\ell,l}|_{\mathsf{H}^k_{\mathbb{S}}(\Omega)} + |v_{\ell,l}|_{\mathsf{H}^k(\Omega)}\big) \to 0.\end{aligned}$$

The latter remark shows immediately:

Theorem 3.21 (compact embedding)**.** *Let* $k \geq 1$. *Then the embedding*

$$\mathsf{H}^{k,k-1}_{\mathbb{S},\Gamma_t}(\operatorname{RotRot}^\top,\Omega) \cap \mathsf{H}^{k}_{\mathbb{S},\Gamma_n}(\operatorname{Div},\Omega) \hookrightarrow \mathsf{H}^{k}_{\mathbb{S},\Gamma}(\Omega)$$

is compact.

Theorem 3.22 (Friedrichs/Poincaré type estimate)**.** *There exists* $\widehat{c}_k > 0$ *such that for all* S *in* $\mathsf{H}^{k}_{\mathbb{S},\Gamma_t}(\operatorname{RotRot}^\top,\Omega) \cap \mathsf{H}^{k}_{\mathbb{S},\Gamma_n}(\operatorname{Div},\Omega) \cap \mathcal{H}_{\mathbb{S},\Gamma_t,\Gamma_n,\mathrm{id}}(\Omega)^{\perp_{\mathsf{L}^2_{\mathbb{S}}(\Omega)}}$

$$|S|_{\mathsf{H}^k_{\mathbb{S}}(\Omega)} \leq \widehat{c}_k\big(|\operatorname{RotRot}^\top S|_{\mathsf{H}^k_{\mathbb{S}}(\Omega)} + |\operatorname{Div} S|_{\mathsf{H}^k(\Omega)}\big).$$

The condition $\mathcal{H}_{\mathbb{S},\Gamma_t,\Gamma_n,\mathrm{id}}(\Omega)^{\perp_{\mathsf{L}^2_{\mathbb{S}}(\Omega)}}$ *can be replaced by the weaker conditions* $\mathcal{H}^k_{\mathbb{S},\Gamma_t,\Gamma_n,\mathrm{id}}(\Omega)^{\perp_{\mathsf{L}^2_{\mathbb{S}}(\Omega)}}$ *or* $\mathcal{H}^k_{\mathbb{S},\Gamma_t,\Gamma_n,\mathrm{id}}(\Omega)^{\perp_{\mathsf{H}^k_{\mathbb{S}}(\Omega)}}$. *In particular, it holds*

$$\forall\, S \in \mathsf{H}^k_{\mathbb{S},\Gamma_t}(\operatorname{RotRot}^\top,\Omega) \cap R(\operatorname{RotRot}^{\top,k}_{\mathbb{S},\Gamma_n}) \qquad |S|_{\mathsf{H}^k_{\mathbb{S}}(\Omega)} \le \widehat{c}_k |\operatorname{RotRot}^\top S|_{\mathsf{H}^k_{\mathbb{S}}(\Omega)},$$
$$\forall\, S \in \mathsf{H}^k_{\mathbb{S},\Gamma_n}(\operatorname{Div},\Omega) \cap R(\operatorname{symGrad}^k_{\Gamma_t}) \qquad |S|_{\mathsf{H}^k_{\mathbb{S}}(\Omega)} \le \widehat{c}_k |\operatorname{Div} S|_{\mathsf{H}^k(\Omega)}$$

with

$$R(\operatorname{RotRot}^{\top,k+1,k}_{\mathbb{S},\Gamma_n}) = R(\operatorname{RotRot}^{\top,k}_{\mathbb{S},\Gamma_n}) = \mathsf{H}^k_{\mathbb{S},\Gamma_n,0}(\operatorname{Div},\Omega) \cap \mathcal{H}_{\mathbb{S},\Gamma_t,\Gamma_n,\mathrm{id}}(\Omega)^{\perp_{\mathsf{L}^2_{\mathbb{S}}(\Omega)}},$$
$$R(\operatorname{symGrad}^k_{\Gamma_t}) = \mathsf{H}^k_{\mathbb{S},\Gamma_t,0}(\operatorname{RotRot}^\top,\Omega) \cap \mathcal{H}_{\mathbb{S},\Gamma_t,\Gamma_n,\mathrm{id}}(\Omega)^{\perp_{\mathsf{L}^2_{\mathbb{S}}(\Omega)}}.$$

Analogously, for $k \ge 1$ *there exists* $\widehat{c}_{k,k-1} > 0$ *such that*

$$|S|_{\mathsf{H}^k_{\mathbb{S}}(\Omega)} \le \widehat{c}_{k,k-1}\big(|\operatorname{RotRot}^\top S|_{\mathsf{H}^{k-1}_{\mathbb{S}}(\Omega)} + |\operatorname{Div} S|_{\mathsf{H}^k(\Omega)}\big)$$

for all S *in* $\mathsf{H}^{k,k-1}_{\mathbb{S},\Gamma_t}(\operatorname{RotRot}^\top,\Omega) \cap \mathsf{H}^k_{\mathbb{S},\Gamma_n}(\operatorname{Div},\Omega) \cap \mathcal{H}_{\mathbb{S},\Gamma_t,\Gamma_n,\mathrm{id}}(\Omega)^{\perp_{\mathsf{L}^2_{\mathbb{S}}(\Omega)}}$. *Moreover,*

$$\forall\, S \in \mathsf{H}^{k,k-1}_{\mathbb{S},\Gamma_t}(\operatorname{RotRot}^\top,\Omega) \cap R(\operatorname{RotRot}^{\top,k}_{\mathbb{S},\Gamma_n}) \qquad |S|_{\mathsf{H}^k_{\mathbb{S}}(\Omega)} \le \widehat{c}_{k,k-1} |\operatorname{RotRot}^\top S|_{\mathsf{H}^{k-1}_{\mathbb{S}}(\Omega)}.$$

Proof. We follow the proof of [11, Theorem 4.17]. To show the first estimate, we use a standard strategy and assume the contrary. Then there is a sequence

$$(S_\ell) \subset \mathsf{H}^k_{\mathbb{S},\Gamma_t}(\operatorname{RotRot}^\top,\Omega) \cap \mathsf{H}^k_{\mathbb{S},\Gamma_n}(\operatorname{Div},\Omega) \cap \mathcal{H}_{\mathbb{S},\Gamma_t,\Gamma_n,\mathrm{id}}(\Omega)^{\perp_{\mathsf{L}^2_{\mathbb{S}}(\Omega)}}$$

with $|S_\ell|_{\mathsf{H}^k_{\mathbb{S}}(\Omega)} = 1$ and $|\operatorname{RotRot}^\top S_\ell|_{\mathsf{H}^k_{\mathbb{S}}(\Omega)} + |\operatorname{Div} S_\ell|_{\mathsf{H}^k(\Omega)} \to 0$. Hence we may assume that S_ℓ converges weakly to some S in $\mathsf{H}^k_{\mathbb{S}}(\Omega) \cap \mathcal{H}_{\mathbb{S},\Gamma_t,\Gamma_n,\mathrm{id}}(\Omega) \cap \mathcal{H}_{\mathbb{S},\Gamma_t,\Gamma_n,\mathrm{id}}(\Omega)^{\perp_{\mathsf{L}^2_{\mathbb{S}}(\Omega)}}$. Thus $S = 0$. By Theorem 3.19 (S_ℓ) converges strongly to 0 in $\mathsf{H}^k_{\mathbb{S}}(\Omega)$, in contradiction to $|S_\ell|_{\mathsf{H}^k(\Omega)} = 1$. The other estimates follow analogously resp. with Theorem 3.16 by restriction. □

Remark 3.23 (Friedrichs/Poincaré/Korn type estimate). *Similar to Theorem 3.22 and by Rellich's selection theorem there exists* $c > 0$ *such that for all* $v \in \mathsf{H}^{k+1}_{\Gamma_t}(\Omega) \cap (\mathbb{RM}_{\Gamma_t})^{\perp_{\mathsf{L}^2(\Omega)}}$

$$|v|_{\mathsf{H}^k(\Omega)} \le c |\operatorname{symGrad} v|_{\mathsf{H}^k_{\mathbb{S}}(\Omega)}.$$

As in Theorem 3.17, $(\mathbb{RM}_{\Gamma_t})^{\perp_{\mathsf{L}^2(\Omega)}}$ *can be replaced by* $(\mathbb{RM}_{\Gamma_t})^{\perp_{\mathsf{H}^k_{\Gamma_t}(\Omega)}}$.

3.3. Regular Potentials and Decompositions II.

Let $k \ge 0$. According to Theorem 3.17 the inverses of the reduced operators

$$(\operatorname{symGrad}^k_{\Gamma_t})^{-1}_\perp : R(\operatorname{symGrad}^k_{\Gamma_t}) \to D(\operatorname{symGrad}^k_{\Gamma_t}) = \mathsf{H}^{k+1}_{\Gamma_t}(\Omega),$$
$$(\operatorname{RotRot}^{\top,k}_{\mathbb{S},\Gamma_t})^{-1}_\perp : R(\operatorname{RotRot}^{\top,k}_{\mathbb{S},\Gamma_t}) \to D(\operatorname{RotRot}^{\top,k}_{\mathbb{S},\Gamma_t}) = \mathsf{H}^k_{\mathbb{S},\Gamma_t}(\operatorname{RotRot}^\top,\Omega),$$
$$(\operatorname{RotRot}^{\top,k+1,k}_{\mathbb{S},\Gamma_t})^{-1}_\perp : R(\operatorname{RotRot}^{\top,k}_{\mathbb{S},\Gamma_t}) \to D(\operatorname{RotRot}^{\top,k+1,k}_{\mathbb{S},\Gamma_t}) = \mathsf{H}^{k+1,k}_{\mathbb{S},\Gamma_t}(\operatorname{RotRot}^\top,\Omega),$$
$$(\operatorname{Div}^k_{\mathbb{S},\Gamma_t})^{-1}_\perp : R(\operatorname{Div}^k_{\mathbb{S},\Gamma_t}) \to D(\operatorname{Div}^k_{\mathbb{S},\Gamma_t}) = \mathsf{H}^k_{\mathbb{S},\Gamma_t}(\operatorname{Div},\Omega)$$

are bounded and we recall the bounded linear regular decomposition operators

$$\mathcal{Q}^{k,1}_{\operatorname{Div}_{\mathbb{S}},\Gamma_t} : \mathsf{H}^k_{\mathbb{S},\Gamma_t}(\operatorname{Div},\Omega) \to \mathsf{H}^{k+1}_{\mathbb{S},\Gamma_t}(\Omega), \qquad \mathcal{Q}^{k,0}_{\operatorname{Div}_{\mathbb{S}},\Gamma_t} : \mathsf{H}^k_{\mathbb{S},\Gamma_t}(\operatorname{Div},\Omega) \to \mathsf{H}^{k+2}_{\mathbb{S},\Gamma_t}(\Omega),$$
$$\mathcal{Q}^{k,1}_{\operatorname{RotRot}^\top_{\mathbb{S}},\Gamma_t} : \mathsf{H}^k_{\mathbb{S},\Gamma_t}(\operatorname{RotRot}^\top,\Omega) \to \mathsf{H}^{k+2}_{\mathbb{S},\Gamma_t}(\Omega), \qquad \mathcal{Q}^{k,0}_{\operatorname{RotRot}^\top_{\mathbb{S}},\Gamma_t} : \mathsf{H}^k_{\mathbb{S},\Gamma_t}(\operatorname{RotRot}^\top,\Omega) \to \mathsf{H}^{k+1}_{\Gamma_t}(\Omega),$$
$$\mathcal{Q}^{k+1,k,1}_{\operatorname{RotRot}^\top_{\mathbb{S}},\Gamma_t} : \mathsf{H}^{k+1,k}_{\mathbb{S},\Gamma_t}(\operatorname{RotRot}^\top,\Omega) \to \mathsf{H}^{k+2}_{\mathbb{S},\Gamma_t}(\Omega), \qquad \mathcal{Q}^{k+1,k,0}_{\operatorname{RotRot}^\top_{\mathbb{S}},\Gamma_t} : \mathsf{H}^{k+1,k}_{\mathbb{S},\Gamma_t}(\operatorname{RotRot}^\top,\Omega) \to \mathsf{H}^{k+2}_{\Gamma_t}(\Omega)$$

from Lemma 3.10. Similar to [11, Theorem 4.18, Theorem 5.2], cf. [11, Lemma 2.22, Theorem 2.23], we obtain the following sequence of results:

Theorem 3.24 (bounded regular potentials from bounded regular decompositions). *For* $k \ge 0$ *there exist bounded linear regular potential operators*

$$\mathcal{P}^k_{\operatorname{symGrad},\Gamma_t} := (\operatorname{symGrad}^k_{\Gamma_t})^{-1}_\perp : \mathsf{H}^k_{\mathbb{S},\Gamma_t,0}(\operatorname{RotRot}^\top,\Omega) \cap \mathcal{H}_{\mathbb{S},\Gamma_t,\Gamma_n,\varepsilon}(\Omega)^{\perp_{\mathsf{L}^2_{\mathbb{S},\varepsilon}(\Omega)}} \to \mathsf{H}^{k+1}_{\Gamma_t}(\Omega),$$

$$\mathcal{P}^{k}_{\mathrm{RotRot}^\top_{\mathbb{S}},\Gamma_t} := \mathcal{Q}^{k,1}_{\mathrm{RotRot}^\top_{\mathbb{S}},\Gamma_t}(\mathrm{RotRot}^{\top,k}_{\mathbb{S},\Gamma_t})^{-1}_{\perp} : \mathsf{H}^k_{\mathbb{S},\Gamma_t,0}(\mathrm{Div},\Omega)\cap\mathcal{H}_{\mathbb{S},\Gamma_n,\Gamma_t,\varepsilon}(\Omega)^{\perp_{\mathsf{L}^2_{\mathbb{S}}(\Omega)}} \to \mathsf{H}^{k+2}_{\mathbb{S},\Gamma_t}(\Omega),$$

$$\mathcal{P}^{k+1,k}_{\mathrm{RotRot}^\top_{\mathbb{S}},\Gamma_t} := \mathcal{Q}^{k+1,k,1}_{\mathrm{RotRot}^\top_{\mathbb{S}},\Gamma_t}(\mathrm{RotRot}^{\top,k+1,k}_{\mathbb{S},\Gamma_t})^{-1}_{\perp} : \mathsf{H}^k_{\mathbb{S},\Gamma_t,0}(\mathrm{Div},\Omega)\cap\mathcal{H}_{\mathbb{S},\Gamma_n,\Gamma_t,\varepsilon}(\Omega)^{\perp_{\mathsf{L}^2_{\mathbb{S}}(\Omega)}} \to \mathsf{H}^{k+2}_{\mathbb{S},\Gamma_t}(\Omega),$$

$$\mathcal{P}^{k}_{\mathrm{Div}_{\mathbb{S}},\Gamma_t} := \mathcal{Q}^{k,1}_{\mathrm{Div}_{\mathbb{S}},\Gamma_t}(\mathrm{Div}^{k}_{\mathbb{S},\Gamma_t})^{-1}_{\perp} : \mathsf{H}^k_{\Gamma_t}(\Omega)\cap(\mathbb{RM}_{\Gamma_n})^{\perp_{\mathsf{L}^2(\Omega)}} \to \mathsf{H}^{k+1}_{\mathbb{S},\Gamma_t}(\Omega),$$

such that

$$\mathrm{symGrad}\,\mathcal{P}^{k}_{\mathrm{symGrad},\Gamma_t} = \mathrm{id}\,|_{\mathsf{H}^k_{\mathbb{S},\Gamma_t,0}(\mathrm{RotRot}^\top,\Omega)\cap\mathcal{H}_{\mathbb{S},\Gamma_t,\Gamma_n,\varepsilon}(\Omega)^{\perp_{\mathsf{L}^2_{\mathbb{S},\varepsilon}(\Omega)}}},$$

$$\mathrm{RotRot}^\top\,\mathcal{P}^{k+1,k}_{\mathrm{RotRot}^\top_{\mathbb{S}},\Gamma_t} = \mathrm{RotRot}^\top\,\mathcal{P}^{k}_{\mathrm{RotRot}^\top_{\mathbb{S}},\Gamma_t} = \mathrm{id}\,|_{\mathsf{H}^k_{\mathbb{S},\Gamma_t,0}(\mathrm{Div},\Omega)\cap\mathcal{H}_{\mathbb{S},\Gamma_n,\Gamma_t,\varepsilon}(\Omega)^{\perp_{\mathsf{L}^2_{\mathbb{S}}(\Omega)}}},$$

$$\mathrm{Div}\,\mathcal{P}^{k}_{\mathrm{Div}_{\mathbb{S}},\Gamma_t} = \mathrm{id}\,|_{\mathsf{H}^k_{\Gamma_t}(\Omega)\cap(\mathbb{RM}_{\Gamma_n})^{\perp_{\mathsf{L}^2(\Omega)}}}.$$

In particular, all potentials in Theorem 3.16 can be chosen such that they depend continuously on the data. $\mathcal{P}^{k}_{\mathrm{symGrad},\Gamma_t}$*,* $\mathcal{P}^{k}_{\mathrm{RotRot}^\top_{\mathbb{S}},\Gamma_t}$*,* $\mathcal{P}^{k+1,k}_{\mathrm{RotRot}^\top_{\mathbb{S}},\Gamma_t}$*, and* $\mathcal{P}^{k}_{\mathrm{Div}_{\mathbb{S}},\Gamma_t}$ *are right inverses of* symGrad, $\mathrm{RotRot}^\top$*, and* Div*, respectively.*

Theorem 3.25 (bounded regular decompositions from bounded regular potentials)**.** *For* $k \geq 0$ *the bounded regular decompositions*

$$\begin{aligned}
\mathsf{H}^k_{\mathbb{S},\Gamma_t}(\mathrm{Div},\Omega) &= \mathsf{H}^{k+1}_{\mathbb{S},\Gamma_t}(\Omega) + \mathsf{H}^k_{\mathbb{S},\Gamma_t,0}(\mathrm{Div},\Omega) = \mathsf{H}^{k+1}_{\mathbb{S},\Gamma_t}(\Omega) + \mathrm{RotRot}^\top\,\mathsf{H}^{k+2}_{\mathbb{S},\Gamma_t}(\Omega)\\
&= R(\widetilde{\mathcal{Q}}^{k,1}_{\mathrm{Div}_{\mathbb{S}},\Gamma_t}) \dotplus \mathsf{H}^k_{\mathbb{S},\Gamma_t,0}(\mathrm{Div},\Omega)\\
&= R(\widetilde{\mathcal{Q}}^{k,1}_{\mathrm{Div}_{\mathbb{S}},\Gamma_t}) \dotplus R(\widetilde{\mathcal{N}}^{k}_{\mathrm{Div}_{\mathbb{S}},\Gamma_t}),\\
\mathsf{H}^k_{\mathbb{S},\Gamma_t}(\mathrm{RotRot}^\top,\Omega) &= \mathsf{H}^{k+2}_{\mathbb{S},\Gamma_t}(\Omega) + \mathsf{H}^k_{\mathbb{S},\Gamma_t,0}(\mathrm{RotRot}^\top,\Omega) = \mathsf{H}^{k+2}_{\mathbb{S},\Gamma_t}(\Omega) + \mathrm{symGrad}\,\mathsf{H}^{k+1}_{\Gamma_t}(\Omega)\\
&= R(\widetilde{\mathcal{Q}}^{k,1}_{\mathrm{RotRot}^\top_{\mathbb{S}},\Gamma_t}) \dotplus \mathsf{H}^k_{\mathbb{S},\Gamma_t,0}(\mathrm{RotRot}^\top,\Omega)\\
&= R(\widetilde{\mathcal{Q}}^{k,1}_{\mathrm{RotRot}^\top_{\mathbb{S}},\Gamma_t}) \dotplus R(\widetilde{\mathcal{N}}^{k}_{\mathrm{RotRot}^\top_{\mathbb{S}},\Gamma_t}),\\
\mathsf{H}^{k+1,k}_{\mathbb{S},\Gamma_t}(\mathrm{RotRot}^\top,\Omega) &= \mathsf{H}^{k+2}_{\mathbb{S},\Gamma_t}(\Omega) + \mathsf{H}^{k+1}_{\mathbb{S},\Gamma_t,0}(\mathrm{RotRot}^\top,\Omega) = \mathsf{H}^{k+2}_{\mathbb{S},\Gamma_t}(\Omega) + \mathrm{symGrad}\,\mathsf{H}^{k+2}_{\Gamma_t}(\Omega)\\
&= R(\widetilde{\mathcal{Q}}^{k+1,k,1}_{\mathrm{RotRot}^\top_{\mathbb{S}},\Gamma_t}) \dotplus \mathsf{H}^{k+1}_{\mathbb{S},\Gamma_t,0}(\mathrm{RotRot}^\top,\Omega)\\
&= R(\widetilde{\mathcal{Q}}^{k+1,k,1}_{\mathrm{RotRot}^\top_{\mathbb{S}},\Gamma_t}) \dotplus R(\widetilde{\mathcal{N}}^{k+1,k}_{\mathrm{RotRot}^\top_{\mathbb{S}},\Gamma_t})
\end{aligned}$$

hold with bounded linear regular decomposition operators

$$\begin{aligned}
\widetilde{\mathcal{Q}}^{k,1}_{\mathrm{Div}_{\mathbb{S}},\Gamma_t} &:= \mathcal{P}^{k}_{\mathrm{Div}_{\mathbb{S}},\Gamma_t}\,\mathrm{Div}^{k}_{\mathbb{S},\Gamma_t} : \mathsf{H}^k_{\mathbb{S},\Gamma_t}(\mathrm{Div},\Omega) \to \mathsf{H}^{k+1}_{\mathbb{S},\Gamma_t}(\Omega),\\
\widetilde{\mathcal{Q}}^{k,1}_{\mathrm{RotRot}^\top_{\mathbb{S}},\Gamma_t} &:= \mathcal{P}^{k}_{\mathrm{RotRot}^\top_{\mathbb{S}},\Gamma_t}\,\mathrm{RotRot}^{\top,k}_{\mathbb{S},\Gamma_t} : \mathsf{H}^k_{\mathbb{S},\Gamma_t}(\mathrm{RotRot}^\top,\Omega) \to \mathsf{H}^{k+2}_{\mathbb{S},\Gamma_t}(\Omega),\\
\widetilde{\mathcal{Q}}^{k+1,k,1}_{\mathrm{RotRot}^\top_{\mathbb{S}},\Gamma_t} &:= \mathcal{P}^{k+1,k}_{\mathrm{RotRot}^\top_{\mathbb{S}},\Gamma_t}\,\mathrm{RotRot}^{\top,k+1,k}_{\mathbb{S},\Gamma_t} : \mathsf{H}^{k+1,k}_{\mathbb{S},\Gamma_t}(\mathrm{RotRot}^\top,\Omega) \to \mathsf{H}^{k+2}_{\mathbb{S},\Gamma_t}(\Omega),\\
\widetilde{\mathcal{N}}^{k}_{\mathrm{Div}_{\mathbb{S}},\Gamma_t} &: \mathsf{H}^k_{\mathbb{S},\Gamma_t}(\mathrm{Div},\Omega) \to \mathsf{H}^k_{\mathbb{S},\Gamma_t,0}(\mathrm{Div},\Omega),\\
\widetilde{\mathcal{N}}^{k}_{\mathrm{RotRot}^\top_{\mathbb{S}},\Gamma_t} &: \mathsf{H}^k_{\mathbb{S},\Gamma_t}(\mathrm{RotRot}^\top,\Omega) \to \mathsf{H}^k_{\mathbb{S},\Gamma_t,0}(\mathrm{RotRot}^\top,\Omega),\\
\widetilde{\mathcal{N}}^{k+1,k}_{\mathrm{RotRot}^\top_{\mathbb{S}},\Gamma_t} &: \mathsf{H}^{k+1,k}_{\mathbb{S},\Gamma_t}(\mathrm{RotRot}^\top,\Omega) \to \mathsf{H}^{k+1}_{\mathbb{S},\Gamma_t,0}(\mathrm{RotRot}^\top,\Omega)
\end{aligned}$$

satisfying

$$\begin{aligned}
\mathrm{id}_{\mathsf{H}^k_{\mathbb{S},\Gamma_t}(\mathrm{Div},\Omega)} &= \widetilde{\mathcal{Q}}^{k,1}_{\mathrm{Div}_{\mathbb{S}},\Gamma_t} + \widetilde{\mathcal{N}}^{k}_{\mathrm{Div}_{\mathbb{S}},\Gamma_t},\\
\mathrm{id}_{\mathsf{H}^k_{\mathbb{S},\Gamma_t}(\mathrm{RotRot}^\top,\Omega)} &= \widetilde{\mathcal{Q}}^{k,1}_{\mathrm{RotRot}^\top_{\mathbb{S}},\Gamma_t} + \widetilde{\mathcal{N}}^{k}_{\mathrm{RotRot}^\top_{\mathbb{S}},\Gamma_t},\\
\mathrm{id}_{\mathsf{H}^{k+1,k}_{\mathbb{S},\Gamma_t}(\mathrm{RotRot}^\top,\Omega)} &= \widetilde{\mathcal{Q}}^{k+1,k,1}_{\mathrm{RotRot}^\top_{\mathbb{S}},\Gamma_t} + \widetilde{\mathcal{N}}^{k+1,k}_{\mathrm{RotRot}^\top_{\mathbb{S}},\Gamma_t}.
\end{aligned}$$

Corollary 3.26 (bounded regular kernel decompositions)**.** *For* $k \geq 0$ *the bounded regular kernel decompositions*

$$\mathsf{H}^k_{\mathbb{S},\Gamma_t,0}(\mathrm{Div},\Omega) = \mathsf{H}^{k+1}_{\mathbb{S},\Gamma_t,0}(\mathrm{Div},\Omega) + \mathrm{RotRot}^\top\,\mathsf{H}^{k+2}_{\mathbb{S},\Gamma_t}(\Omega),$$

$$\mathsf{H}^{k}_{\mathbb{S},\Gamma_t,0}(\operatorname{RotRot}^\top,\Omega) = \mathsf{H}^{k+2}_{\mathbb{S},\Gamma_t,0}(\operatorname{RotRot}^\top,\Omega) + \operatorname{symGrad}\mathsf{H}^{k+1}_{\Gamma_t}(\Omega)$$

hold.

Remark 3.27 (bounded regular decompositions from bounded regular potentials)**.** *It holds*

$$\operatorname{Div}\widetilde{\mathcal{Q}}^{k,1}_{\operatorname{Div}_{\mathbb{S}},\Gamma_t} = \operatorname{Div}\mathcal{Q}^{k,1}_{\operatorname{Div}_{\mathbb{S}},\Gamma_t} = \operatorname{Div}^{k}_{\mathbb{S},\Gamma_t}$$

and hence $\mathsf{H}^{k}_{\mathbb{S},\Gamma_t,0}(\operatorname{Div},\Omega)$ *is invariant under* $\mathcal{Q}^{k,1}_{\operatorname{Div}_{\mathbb{S}},\Gamma_t}$ *and* $\widetilde{\mathcal{Q}}^{k,1}_{\operatorname{Div}_{\mathbb{S}},\Gamma_t}$*. Analogously,*

$$\begin{aligned}\operatorname{RotRot}^\top\widetilde{\mathcal{Q}}^{k,1}_{\operatorname{RotRot}^\top_{\mathbb{S}},\Gamma_t} &= \operatorname{RotRot}^\top\mathcal{Q}^{k,1}_{\operatorname{RotRot}^\top_{\mathbb{S}},\Gamma_t} = \operatorname{RotRot}^{\top,k}_{\mathbb{S},\Gamma_t},\\ \operatorname{RotRot}^\top\widetilde{\mathcal{Q}}^{k+1,k,1}_{\operatorname{RotRot}^\top_{\mathbb{S}},\Gamma_t} &= \operatorname{RotRot}^\top\mathcal{Q}^{k+1,k,1}_{\operatorname{RotRot}^\top_{\mathbb{S}},\Gamma_t} = \operatorname{RotRot}^{\top,k+1,k}_{\mathbb{S},\Gamma_t}.\end{aligned}$$

Thus $\mathsf{H}^{k}_{\mathbb{S},\Gamma_t,0}(\operatorname{RotRot}^\top,\Omega)$ *and* $\mathsf{H}^{k+1}_{\mathbb{S},\Gamma_t,0}(\operatorname{RotRot}^\top,\Omega)$ *are invariant under* $\mathcal{Q}^{k,1}_{\operatorname{RotRot}^\top_{\mathbb{S}},\Gamma_t}$ *and* $\mathcal{Q}^{k+1,k,1}_{\operatorname{RotRot}^\top_{\mathbb{S}},\Gamma_t}$*, respectively. Moreover,*

$$R(\widetilde{\mathcal{Q}}^{k,1}_{\operatorname{Div}_{\mathbb{S}},\Gamma_t}) = R(\mathcal{P}^{k}_{\operatorname{Div}_{\mathbb{S}},\Gamma_t}), \qquad \widetilde{\mathcal{Q}}^{k,1}_{\operatorname{Div}_{\mathbb{S}},\Gamma_t} = \mathcal{Q}^{k,1}_{\operatorname{Div}_{\mathbb{S}},\Gamma_t}(\operatorname{Div}^{k}_{\mathbb{S},\Gamma_t})^{-1}_{\perp}\operatorname{Div}^{k}_{\mathbb{S},\Gamma_t}.$$

Therefore, we have $\widetilde{\mathcal{Q}}^{k,1}_{\operatorname{Div}_{\mathbb{S}},\Gamma_t}|_{D((\operatorname{Div}^{k}_{\mathbb{S},\Gamma_t})_\perp)} = \mathcal{Q}^{k,1}_{\operatorname{Div}_{\mathbb{S}},\Gamma_t}|_{D((\operatorname{Div}^{k}_{\mathbb{S},\Gamma_t})_\perp)}$ *and thus* $\widetilde{\mathcal{Q}}^{k,1}_{\operatorname{Div}_{\mathbb{S}},\Gamma_t}$ *may differ from* $\mathcal{Q}^{k,1}_{\operatorname{Div}_{\mathbb{S}},\Gamma_t}$ *only on* $N(\operatorname{Div}^{k}_{\mathbb{S},\Gamma_t}) = \mathsf{H}^{k}_{\mathbb{S},\Gamma_t,0}(\operatorname{Div},\Omega)$*. Analogously,*

$$\begin{aligned}R(\widetilde{\mathcal{Q}}^{k,1}_{\operatorname{RotRot}^\top_{\mathbb{S}},\Gamma_t}) &= R(\mathcal{P}^{k}_{\operatorname{RotRot}^\top_{\mathbb{S}},\Gamma_t}), & \widetilde{\mathcal{Q}}^{k,1}_{\operatorname{RotRot}^\top_{\mathbb{S}},\Gamma_t} &= \mathcal{Q}^{k,1}_{\operatorname{RotRot}^\top_{\mathbb{S}},\Gamma_t}(\operatorname{RotRot}^{\top,k}_{\mathbb{S},\Gamma_t})^{-1}_{\perp}\operatorname{RotRot}^{\top,k}_{\mathbb{S},\Gamma_t},\\ R(\widetilde{\mathcal{Q}}^{k+1,k,1}_{\operatorname{RotRot}^\top_{\mathbb{S}},\Gamma_t}) &= R(\mathcal{P}^{k+1,k}_{\operatorname{RotRot}^\top_{\mathbb{S}},\Gamma_t}), & \widetilde{\mathcal{Q}}^{k+1,k,1}_{\operatorname{RotRot}^\top_{\mathbb{S}},\Gamma_t} &= \mathcal{Q}^{k+1,k,1}_{\operatorname{RotRot}^\top_{\mathbb{S}},\Gamma_t}(\operatorname{RotRot}^{\top,k+1,k}_{\mathbb{S},\Gamma_t})^{-1}_{\perp}\operatorname{RotRot}^{\top,k+1,k}_{\mathbb{S},\Gamma_t}.\end{aligned}$$

Hence

$$\begin{aligned}\widetilde{\mathcal{Q}}^{k,1}_{\operatorname{RotRot}^\top_{\mathbb{S}},\Gamma_t}|_{D((\operatorname{RotRot}^{\top,k}_{\mathbb{S},\Gamma_t})_\perp)} &= \mathcal{Q}^{k,1}_{\operatorname{RotRot}^\top_{\mathbb{S}},\Gamma_t}|_{D((\operatorname{RotRot}^{\top,k}_{\mathbb{S},\Gamma_t})_\perp)},\\ \widetilde{\mathcal{Q}}^{k+1,k,1}_{\operatorname{RotRot}^\top_{\mathbb{S}},\Gamma_t}|_{D((\operatorname{RotRot}^{\top,k+1,k}_{\mathbb{S},\Gamma_t})_\perp)} &= \mathcal{Q}^{k+1,k,1}_{\operatorname{RotRot}^\top_{\mathbb{S}},\Gamma_t}|_{D((\operatorname{RotRot}^{\top,k+1,k}_{\mathbb{S},\Gamma_t})_\perp)}\end{aligned}$$

and thus $\widetilde{\mathcal{Q}}^{k,1}_{\operatorname{RotRot}^\top_{\mathbb{S}},\Gamma_t}$ *and* $\widetilde{\mathcal{Q}}^{k+1,k,1}_{\operatorname{RotRot}^\top_{\mathbb{S}},\Gamma_t}$ *may differ from* $\mathcal{Q}^{k,1}_{\operatorname{RotRot}^\top_{\mathbb{S}},\Gamma_t}$ *and* $\mathcal{Q}^{k+1,k,1}_{\operatorname{RotRot}^\top_{\mathbb{S}},\Gamma_t}$ *only on the kernels* $N(\operatorname{RotRot}^{\top,k}_{\mathbb{S},\Gamma_t}) = \mathsf{H}^{k}_{\mathbb{S},\Gamma_t,0}(\operatorname{RotRot}^\top,\Omega)$ *and* $N(\operatorname{RotRot}^{\top,k+1,k}_{\mathbb{S},\Gamma_t}) = \mathsf{H}^{k+1}_{\mathbb{S},\Gamma_t,0}(\operatorname{RotRot}^\top,\Omega)$*, respectively.*

Remark 3.28 (projections)**.** *Recall Theorem 3.25, e.g., for* $\operatorname{RotRot}^{\top,k}_{\mathbb{S},\Gamma_t}$

$$\mathsf{H}^{k}_{\mathbb{S},\Gamma_t}(\operatorname{RotRot}^\top,\Omega) = R(\widetilde{\mathcal{Q}}^{k,1}_{\operatorname{RotRot}^\top_{\mathbb{S}},\Gamma_t}) \dotplus R(\widetilde{\mathcal{N}}^{k}_{\operatorname{RotRot}^\top_{\mathbb{S}},\Gamma_t}).$$

- **(i)** $\widetilde{\mathcal{Q}}^{k,1}_{\operatorname{RotRot}^\top_{\mathbb{S}},\Gamma_t}$, $\widetilde{\mathcal{N}}^{k}_{\operatorname{RotRot}^\top_{\mathbb{S}},\Gamma_t} = 1 - \widetilde{\mathcal{Q}}^{k,1}_{\operatorname{RotRot}^\top_{\mathbb{S}},\Gamma_t}$ *are projections.*
- **(i')** $\widetilde{\mathcal{Q}}^{k,1}_{\operatorname{RotRot}^\top_{\mathbb{S}},\Gamma_t}\widetilde{\mathcal{N}}^{k}_{\operatorname{RotRot}^\top_{\mathbb{S}},\Gamma_t} = \widetilde{\mathcal{N}}^{k}_{\operatorname{RotRot}^\top_{\mathbb{S}},\Gamma_t}\widetilde{\mathcal{Q}}^{k,1}_{\operatorname{RotRot}^\top_{\mathbb{S}},\Gamma_t} = 0$.
- **(ii)** *For* $I_\pm := \widetilde{\mathcal{Q}}^{k,1}_{\operatorname{RotRot}^\top_{\mathbb{S}},\Gamma_t} \pm \widetilde{\mathcal{N}}^{k}_{\operatorname{RotRot}^\top_{\mathbb{S}},\Gamma_t}$ *it holds* $I_+ = I_-^2 = \operatorname{id}_{\mathsf{H}^{k}_{\mathbb{S},\Gamma_t}(\operatorname{RotRot}^\top,\Omega)}$*. Therefore,* I_+*,* I_-^2*, as well as* $I_- = 2\widetilde{\mathcal{Q}}^{k,1}_{\operatorname{RotRot}^\top_{\mathbb{S}},\Gamma_t} - \operatorname{id}_{\mathsf{H}^{k}_{\mathbb{S},\Gamma_t}(\operatorname{RotRot}^\top,\Omega)}$ *are topological isomorphisms on* $\mathsf{H}^{k}_{\mathbb{S},\Gamma_t}(\operatorname{RotRot}^\top,\Omega)$.
- **(iii)** *There exists* $c > 0$ *such that for all* $S \in \mathsf{H}^{k}_{\mathbb{S},\Gamma_t}(\operatorname{RotRot}^\top,\Omega)$

$$\begin{aligned}c|\widetilde{\mathcal{Q}}^{k,1}_{\operatorname{RotRot}^\top_{\mathbb{S}},\Gamma_t}S|_{\mathsf{H}^{k+2}_{\mathbb{S}}(\Omega)} &\leq |\operatorname{RotRot}^\top S|_{\mathsf{H}^{k}_{\mathbb{S}}(\Omega)} \leq |S|_{\mathsf{H}^{k}_{\mathbb{S}}(\operatorname{RotRot}^\top,\Omega)},\\ |\widetilde{\mathcal{N}}^{k}_{\operatorname{RotRot}^\top_{\mathbb{S}},\Gamma_t}S|_{\mathsf{H}^{k}_{\mathbb{S}}(\Omega)} &\leq |S|_{\mathsf{H}^{k}_{\mathbb{S}}(\Omega)} + |\widetilde{\mathcal{Q}}^{k,1}_{\operatorname{RotRot}^\top_{\mathbb{S}},\Gamma_t}S|_{\mathsf{H}^{k}_{\mathbb{S}}(\Omega)}.\end{aligned}$$

- **(iii')** *For* $S \in \mathsf{H}^{k}_{\mathbb{S},\Gamma_t,0}(\operatorname{RotRot}^\top,\Omega)$ *we have* $\widetilde{\mathcal{Q}}^{k,1}_{\operatorname{RotRot}^\top_{\mathbb{S}},\Gamma_t}S = 0$ *and* $\widetilde{\mathcal{N}}^{k}_{\operatorname{RotRot}^\top_{\mathbb{S}},\Gamma_t}S = S$*. In particular,* $\widetilde{\mathcal{N}}^{k}_{\operatorname{RotRot}^\top_{\mathbb{S}},\Gamma_t}$ *is onto.*

Similar results to (i)-(iii') hold for $\operatorname{Div}^k_{\mathbb{S},\Gamma_t}$ *and* $\operatorname{RotRot}^{\top,k+1,k}_{\mathbb{S},\Gamma_t}$ *as well. In particular,* $\widetilde{\mathcal{Q}}^{k,1}_{\operatorname{Div}_{\mathbb{S}},\Gamma_t}$, $\widetilde{\mathcal{N}}^{k}_{\operatorname{Div}_{\mathbb{S}},\Gamma_t}$, *and* $\widetilde{\mathcal{Q}}^{k+1,k,1}_{\operatorname{RotRot}^\top_{\mathbb{S}},\Gamma_t}$, $\widetilde{\mathcal{N}}^{k+1,k}_{\operatorname{RotRot}^\top_{\mathbb{S}},\Gamma_t}$ *are projections and there exists* $c>0$ *such that for all* $T\in\mathsf{H}^k_{\mathbb{S},\Gamma_t}(\operatorname{Div},\Omega)$ *and all* $S\in\mathsf{H}^{k+1,k}_{\mathbb{S},\Gamma_t}(\operatorname{RotRot}^\top,\Omega)$

$$|\widetilde{\mathcal{Q}}^{k,1}_{\operatorname{Div}_{\mathbb{S}},\Gamma_t}T|_{\mathsf{H}^{k+1}_{\mathbb{S}}(\Omega)}\leq c|\operatorname{Div}T|_{\mathsf{H}^k(\Omega)},\qquad |\widetilde{\mathcal{Q}}^{k+1,k,1}_{\operatorname{RotRot}^\top_{\mathbb{S}},\Gamma_t}S|_{\mathsf{H}^{k+2}_{\mathbb{S}}(\Omega)}\leq c|\operatorname{RotRot}^\top S|_{\mathsf{H}^k_{\mathbb{S}}(\Omega)}.$$

Corollary 3.26 shows:

Corollary 3.29 (bounded regular higher order kernel decompositions)**.** *For* $k,\ell\geq 0$ *the bounded regular kernel decompositions*

$$N(\operatorname{Div}^k_{\mathbb{S},\Gamma_t})=\mathsf{H}^k_{\mathbb{S},\Gamma_t,0}(\operatorname{Div},\Omega)=\mathsf{H}^\ell_{\mathbb{S},\Gamma_t,0}(\operatorname{Div},\Omega)+\operatorname{RotRot}^\top\mathsf{H}^{k+2}_{\mathbb{S},\Gamma_t}(\Omega),$$
$$N(\operatorname{RotRot}^{\top,k}_{\mathbb{S},\Gamma_t})=\mathsf{H}^k_{\mathbb{S},\Gamma_t,0}(\operatorname{RotRot}^\top,\Omega)=\mathsf{H}^\ell_{\mathbb{S},\Gamma_t,0}(\operatorname{RotRot}^\top,\Omega)+\operatorname{symGrad}\mathsf{H}^{k+1}_{\Gamma_t}(\Omega)$$

hold. In particular, for $k=0$ *and all* $\ell\geq 0$

$$N(\operatorname{Div}_{\mathbb{S},\Gamma_t})=\mathsf{H}_{\mathbb{S},\Gamma_t,0}(\operatorname{Div},\Omega)=\mathsf{H}^\ell_{\mathbb{S},\Gamma_t,0}(\operatorname{Div},\Omega)+\operatorname{RotRot}^\top\mathsf{H}^{2}_{\mathbb{S},\Gamma_t}(\Omega),$$
$$N(\operatorname{RotRot}^{\top}_{\mathbb{S},\Gamma_t})=\mathsf{H}_{\mathbb{S},\Gamma_t,0}(\operatorname{RotRot}^\top,\Omega)=\mathsf{H}^\ell_{\mathbb{S},\Gamma_t,0}(\operatorname{RotRot}^\top,\Omega)+\operatorname{symGrad}\mathsf{H}^{1}_{\Gamma_t}(\Omega).$$

3.4. **Dirichlet/Neumann Fields.** From Theorem 3.15 (iv) we recall the orthonormal Helmholtz type decompositions (for $\mu=1$)

$$\begin{aligned}\mathsf{L}^2_{\mathbb{S},\varepsilon}(\Omega)&=R(\operatorname{symGrad}_{\Gamma_t})\oplus_{\mathsf{L}^2_{\mathbb{S},\varepsilon}(\Omega)}N(\operatorname{Div}_{\mathbb{S},\Gamma_n}\varepsilon)\\&=N(\operatorname{RotRot}^\top_{\mathbb{S},\Gamma_t})\oplus_{\mathsf{L}^2_{\mathbb{S},\varepsilon}(\Omega)}R(\varepsilon^{-1}\operatorname{RotRot}^\top_{\mathbb{S},\Gamma_n})\\&=R(\operatorname{symGrad}_{\Gamma_t})\oplus_{\mathsf{L}^2_{\mathbb{S},\varepsilon}(\Omega)}\mathcal{H}_{\mathbb{S},\Gamma_t,\Gamma_n,\varepsilon}(\Omega)\oplus_{\mathsf{L}^2_{\mathbb{S},\varepsilon}(\Omega)}R(\varepsilon^{-1}\operatorname{RotRot}^\top_{\mathbb{S},\Gamma_n}),\\N(\operatorname{RotRot}^\top_{\mathbb{S},\Gamma_t})&=R(\operatorname{symGrad}_{\Gamma_t})\oplus_{\mathsf{L}^2_{\mathbb{S},\varepsilon}(\Omega)}\mathcal{H}_{\mathbb{S},\Gamma_t,\Gamma_n,\varepsilon}(\Omega),\\N(\operatorname{Div}_{\mathbb{S},\Gamma_n}\varepsilon)&=\mathcal{H}_{\mathbb{S},\Gamma_t,\Gamma_n,\varepsilon}(\Omega)\oplus_{\mathsf{L}^2_{\mathbb{S},\varepsilon}(\Omega)}R(\varepsilon^{-1}\operatorname{RotRot}^\top_{\mathbb{S},\Gamma_n}).\end{aligned}\tag{13}$$

Let us denote the $\mathsf{L}^2_{\mathbb{S},\varepsilon}(\Omega)$-orthonormal projector onto $N(\operatorname{Div}_{\mathbb{S},\Gamma_n}\varepsilon)$ and $N(\operatorname{RotRot}^\top_{\mathbb{S},\Gamma_t})$ by

$$\pi_{\operatorname{Div}}:\mathsf{L}^2_{\mathbb{S},\varepsilon}(\Omega)\to N(\operatorname{Div}_{\mathbb{S},\Gamma_n}\varepsilon),\qquad \pi_{\operatorname{RotRot}^\top}:\mathsf{L}^2_{\mathbb{S},\varepsilon}(\Omega)\to N(\operatorname{RotRot}^\top_{\mathbb{S},\Gamma_t}),$$

respectively. Then

$$\pi_{\operatorname{Div}}|_{N(\operatorname{RotRot}^\top_{\mathbb{S},\Gamma_t})}:N(\operatorname{RotRot}^\top_{\mathbb{S},\Gamma_t})\to\mathcal{H}_{\mathbb{S},\Gamma_t,\Gamma_n,\varepsilon}(\Omega),$$
$$\pi_{\operatorname{RotRot}^\top}|_{N(\operatorname{Div}_{\mathbb{S},\Gamma_n}\varepsilon)}:N(\operatorname{Div}_{\mathbb{S},\Gamma_n}\varepsilon)\to\mathcal{H}_{\mathbb{S},\Gamma_t,\Gamma_n,\varepsilon}(\Omega)$$

are onto. Moreover,

$$\pi_{\operatorname{Div}}|_{R(\operatorname{symGrad}_{\Gamma_t})}=0,\qquad \pi_{\operatorname{RotRot}^\top}|_{R(\varepsilon^{-1}\operatorname{RotRot}^\top_{\mathbb{S},\Gamma_n})}=0,$$
$$\pi_{\operatorname{Div}}|_{\mathcal{H}_{\mathbb{S},\Gamma_t,\Gamma_n,\varepsilon}(\Omega)}=\operatorname{id}_{\mathcal{H}_{\mathbb{S},\Gamma_t,\Gamma_n,\varepsilon}(\Omega)},\qquad \pi_{\operatorname{RotRot}^\top}|_{\mathcal{H}_{\mathbb{S},\Gamma_t,\Gamma_n,\varepsilon}(\Omega)}=\operatorname{id}_{\mathcal{H}_{\mathbb{S},\Gamma_t,\Gamma_n,\varepsilon}(\Omega)}.$$

Therefore, by Corollary 3.29 and for all $\ell\geq 0$

$$\mathcal{H}_{\mathbb{S},\Gamma_t,\Gamma_n,\varepsilon}(\Omega)=\pi_{\operatorname{Div}}N(\operatorname{RotRot}^\top_{\mathbb{S},\Gamma_t})=\pi_{\operatorname{Div}}\mathsf{H}^\ell_{\mathbb{S},\Gamma_t,0}(\operatorname{RotRot}^\top,\Omega),$$
$$\mathcal{H}_{\mathbb{S},\Gamma_t,\Gamma_n,\varepsilon}(\Omega)=\pi_{\operatorname{RotRot}^\top}N(\operatorname{Div}_{\mathbb{S},\Gamma_n}\varepsilon)=\pi_{\operatorname{RotRot}^\top}\varepsilon^{-1}\mathsf{H}^\ell_{\mathbb{S},\Gamma_n,0}(\operatorname{Div},\Omega),$$

where we have used $N(\operatorname{Div}_{\mathbb{S},\Gamma_n}\varepsilon)=\varepsilon^{-1}\mathsf{H}_{\mathbb{S},\Gamma_n,0}(\operatorname{Div},\Omega)$. Hence with

$$\mathsf{H}^\infty_{\mathbb{S},\Gamma_t,0}(\operatorname{RotRot}^\top,\Omega):=\bigcap_{k\geq0}\mathsf{H}^k_{\mathbb{S},\Gamma_t,0}(\operatorname{RotRot}^\top,\Omega),\qquad \mathsf{H}^\infty_{\mathbb{S},\Gamma_n,0}(\operatorname{Div},\Omega):=\bigcap_{k\geq0}\mathsf{H}^k_{\mathbb{S},\Gamma_n,0}(\operatorname{Div},\Omega)$$

we have the following result:

Theorem 3.30 (smooth pre-bases of Dirichlet/Neumann fields)**.** *Let* $d_{\Omega,\Gamma_t}:=\dim\mathcal{H}_{\mathbb{S},\Gamma_t,\Gamma_n,\varepsilon}(\Omega)$. *Then*

$$\pi_{\operatorname{Div}}\mathsf{H}^\infty_{\mathbb{S},\Gamma_t,0}(\operatorname{RotRot}^\top,\Omega)=\mathcal{H}_{\mathbb{S},\Gamma_t,\Gamma_n,\varepsilon}(\Omega)=\pi_{\operatorname{RotRot}^\top}\varepsilon^{-1}\mathsf{H}^\infty_{\mathbb{S},\Gamma_n,0}(\operatorname{Div},\Omega).$$

Moreover, there exists a smooth $\operatorname{RotRot}^\top$-pre-basis *and a smooth* Div-pre-basis *of* $\mathcal{H}_{\mathbb{S},\Gamma_t,\Gamma_n,\varepsilon}(\Omega)$, *i.e., there are linear independent smooth fields*

$$\mathcal{B}^{\operatorname{RotRot}^\top}_{\mathbb{S},\Gamma_t}(\Omega) := \{B^{\operatorname{RotRot}^\top}_{\mathbb{S},\Gamma_t,\ell}\}_{\ell=1}^{d_{\Omega,\Gamma_t}} \subset \mathsf{H}^\infty_{\mathbb{S},\Gamma_t,0}(\operatorname{RotRot}^\top,\Omega),$$
$$\mathcal{B}^{\operatorname{Div}}_{\mathbb{S},\Gamma_n}(\Omega) := \{B^{\operatorname{Div}}_{\mathbb{S},\Gamma_n,\ell}\}_{\ell=1}^{d_{\Omega,\Gamma_t}} \subset \mathsf{H}^\infty_{\mathbb{S},\Gamma_n,0}(\operatorname{Div},\Omega),$$

such that $\pi_{\operatorname{Div}}\mathcal{B}^{\operatorname{RotRot}^\top}_{\mathbb{S},\Gamma_t}(\Omega)$ *and* $\pi_{\operatorname{RotRot}^\top}\varepsilon^{-1}\mathcal{B}^{\operatorname{Div}}_{\mathbb{S},\Gamma_n}(\Omega)$ *are both bases of* $\mathcal{H}_{\mathbb{S},\Gamma_t,\Gamma_n,\varepsilon}(\Omega)$. *In particular,*

$$\operatorname{Lin}\pi_{\operatorname{Div}}\mathcal{B}^{\operatorname{RotRot}^\top}_{\mathbb{S},\Gamma_t}(\Omega) = \mathcal{H}_{\mathbb{S},\Gamma_t,\Gamma_n,\varepsilon}(\Omega) = \operatorname{Lin}\pi_{\operatorname{RotRot}^\top}\varepsilon^{-1}\mathcal{B}^{\operatorname{Div}}_{\mathbb{S},\Gamma_n}(\Omega).$$

Note that $(1-\pi_{\operatorname{Div}})$ and $(1-\pi_{\operatorname{RotRot}^\top})$ are the $\mathsf{L}^2_{\mathbb{S},\varepsilon}(\Omega)$-orthonormal projectors onto the ranges $R(\operatorname{symGrad}_{\Gamma_t})$ and $R(\varepsilon^{-1}\operatorname{RotRot}^\top_{\mathbb{S},\Gamma_n})$, respectively, i.e.,

$$(1-\pi_{\operatorname{Div}}) : \mathsf{L}^2_{\mathbb{S},\varepsilon}(\Omega) \to R(\operatorname{symGrad}_{\Gamma_t}), \qquad (1-\pi_{\operatorname{RotRot}^\top}) : \mathsf{L}^2_{\mathbb{S},\varepsilon}(\Omega) \to R(\varepsilon^{-1}\operatorname{RotRot}^\top_{\mathbb{S},\Gamma_n}).$$

By (13), Theorem 3.16, and Theorem 3.30 we have, e.g.,

$$\begin{aligned}
\mathsf{H}_{\mathbb{S},\Gamma_t,0}(\operatorname{RotRot}^\top,\Omega) &= R(\operatorname{symGrad}_{\Gamma_t}) \oplus_{\mathsf{L}^2_{\mathbb{S},\varepsilon}(\Omega)} \mathcal{H}_{\mathbb{S},\Gamma_t,\Gamma_n,\varepsilon}(\Omega)\\
&= R(\operatorname{symGrad}_{\Gamma_t}) \oplus_{\mathsf{L}^2_{\mathbb{S},\varepsilon}(\Omega)} \operatorname{Lin}\pi_{\operatorname{Div}}\mathcal{B}^{\operatorname{RotRot}^\top}_{\mathbb{S},\Gamma_t}(\Omega)\\
&= R(\operatorname{symGrad}_{\Gamma_t}) + (\pi_{\operatorname{Div}}-1)\operatorname{Lin}\mathcal{B}^{\operatorname{RotRot}^\top}_{\mathbb{S},\Gamma_t}(\Omega) + \operatorname{Lin}\mathcal{B}^{\operatorname{RotRot}^\top}_{\mathbb{S},\Gamma_t}(\Omega)\\
&= R(\operatorname{symGrad}_{\Gamma_t}) + \operatorname{Lin}\mathcal{B}^{\operatorname{RotRot}^\top}_{\mathbb{S},\Gamma_t}(\Omega),\\
\mathsf{H}^k_{\mathbb{S},\Gamma_t,0}(\operatorname{RotRot}^\top,\Omega) &= R(\operatorname{symGrad}_{\Gamma_t}) \cap \mathsf{H}^k_{\mathbb{S},\Gamma_t,0}(\operatorname{RotRot}^\top,\Omega) + \operatorname{Lin}\mathcal{B}^{\operatorname{RotRot}^\top}_{\mathbb{S},\Gamma_t}(\Omega),\\
&= R(\operatorname{symGrad}^k_{\Gamma_t}) + \operatorname{Lin}\mathcal{B}^{\operatorname{RotRot}^\top}_{\mathbb{S},\Gamma_t}(\Omega).
\end{aligned} \tag{14}$$

Similarly, we obtain a decomposition of $\mathsf{H}^k_{\mathbb{S},\Gamma_n,0}(\operatorname{Div},\Omega)$ using $\mathcal{B}^{\operatorname{Div}}_{\mathbb{S},\Gamma_n}(\Omega)$. We conclude:

Theorem 3.31 (bounded regular direct decompositions)**.** *Let* $k \geq 0$. *Then the bounded regular direct decompositions*

$$\begin{aligned}
\mathsf{H}^k_{\mathbb{S},\Gamma_t}(\operatorname{RotRot}^\top,\Omega) &= R(\widetilde{\mathcal{Q}}^{k,1}_{\operatorname{RotRot}^\top_{\mathbb{S}},\Gamma_t}) \dotplus \mathsf{H}^k_{\mathbb{S},\Gamma_t,0}(\operatorname{RotRot}^\top,\Omega),\\
\mathsf{H}^{k+1,k}_{\mathbb{S},\Gamma_t}(\operatorname{RotRot}^\top,\Omega) &= R(\widetilde{\mathcal{Q}}^{k+1,k,1}_{\operatorname{RotRot}^\top_{\mathbb{S}},\Gamma_t}) \dotplus \mathsf{H}^{k+1}_{\mathbb{S},\Gamma_t,0}(\operatorname{RotRot}^\top,\Omega),\\
\mathsf{H}^k_{\mathbb{S},\Gamma_t,0}(\operatorname{RotRot}^\top,\Omega) &= \operatorname{symGrad}\mathsf{H}^{k+1}_{\Gamma_t}(\Omega) \dotplus \operatorname{Lin}\mathcal{B}^{\operatorname{RotRot}^\top}_{\mathbb{S},\Gamma_t}(\Omega),\\
\mathsf{H}^k_{\mathbb{S},\Gamma_n}(\operatorname{Div},\Omega) &= R(\widetilde{\mathcal{Q}}^{k,1}_{\operatorname{Div}_{\mathbb{S}},\Gamma_n}) \dotplus \mathsf{H}^k_{\mathbb{S},\Gamma_n,0}(\operatorname{Div},\Omega),\\
\mathsf{H}^k_{\mathbb{S},\Gamma_n,0}(\operatorname{Div},\Omega) &= \operatorname{RotRot}^\top\mathsf{H}^{k+2}_{\mathbb{S},\Gamma_n}(\Omega) \dotplus \operatorname{Lin}\mathcal{B}^{\operatorname{Div}}_{\mathbb{S},\Gamma_n}(\Omega)
\end{aligned}$$

hold. Note that $R(\widetilde{\mathcal{Q}}^{k,1}_{\operatorname{RotRot}^\top_{\mathbb{S}},\Gamma_t}), R(\widetilde{\mathcal{Q}}^{k+1,k,1}_{\operatorname{RotRot}^\top_{\mathbb{S}},\Gamma_t}) \subset \mathsf{H}^{k+2}_{\mathbb{S},\Gamma_t}(\Omega)$ *and* $R(\widetilde{\mathcal{Q}}^{k,1}_{\operatorname{Div}_{\mathbb{S}},\Gamma_n}) \subset \mathsf{H}^{k+1}_{\mathbb{S},\Gamma_n}(\Omega)$.

Remark 3.32 (bounded regular direct decompositions)**.** *In particular, for* $k=0$

$$\begin{aligned}
\mathsf{H}_{\mathbb{S},\Gamma_t}(\operatorname{RotRot}^\top,\Omega) &= R(\widetilde{\mathcal{Q}}^{0,1}_{\operatorname{RotRot}^\top_{\mathbb{S}},\Gamma_t}) \dotplus \mathsf{H}_{\mathbb{S},\Gamma_t,0}(\operatorname{RotRot}^\top,\Omega),\\
\mathsf{H}_{\mathbb{S},\Gamma_t,0}(\operatorname{RotRot}^\top,\Omega) &= \operatorname{symGrad}\mathsf{H}^1_{\Gamma_t}(\Omega) \dotplus \operatorname{Lin}\mathcal{B}^{\operatorname{RotRot}^\top}_{\mathbb{S},\Gamma_t}(\Omega)\\
&= \operatorname{symGrad}\mathsf{H}^1_{\Gamma_t}(\Omega) \oplus_{\mathsf{L}^2_{\mathbb{S},\varepsilon}(\Omega)} \mathcal{H}_{\mathbb{S},\Gamma_t,\Gamma_n,\varepsilon}(\Omega),\\
\mathsf{H}_{\mathbb{S},\Gamma_n}(\operatorname{Div},\Omega) &= R(\widetilde{\mathcal{Q}}^{0,1}_{\operatorname{Div}_{\mathbb{S}},\Gamma_n}) \dotplus \mathsf{H}_{\mathbb{S},\Gamma_n,0}(\operatorname{Div},\Omega),\\
\varepsilon^{-1}\mathsf{H}_{\mathbb{S},\Gamma_n,0}(\operatorname{Div},\Omega) &= \varepsilon^{-1}\operatorname{RotRot}^\top\mathsf{H}^2_{\mathbb{S},\Gamma_n}(\Omega) \dotplus \varepsilon^{-1}\operatorname{Lin}\mathcal{B}^{\operatorname{Div}}_{\mathbb{S},\Gamma_n}(\Omega)\\
&= \varepsilon^{-1}\operatorname{RotRot}^\top\mathsf{H}^2_{\mathbb{S},\Gamma_n}(\Omega) \oplus_{\mathsf{L}^2_{\mathbb{S},\varepsilon}(\Omega)} \mathcal{H}_{\mathbb{S},\Gamma_t,\Gamma_n,\varepsilon}(\Omega)
\end{aligned}$$

and

$$\begin{aligned}
\mathsf{L}^2_{\mathbb{S},\varepsilon}(\Omega) &= \mathsf{H}_{\mathbb{S},\Gamma_t,0}(\operatorname{RotRot}^\top,\Omega) \oplus_{\mathsf{L}^2_{\mathbb{S},\varepsilon}(\Omega)} \varepsilon^{-1}\operatorname{RotRot}^\top\mathsf{H}^2_{\mathbb{S},\Gamma_n}(\Omega)\\
&= \operatorname{symGrad}\mathsf{H}^1_{\Gamma_t}(\Omega) \oplus_{\mathsf{L}^2_{\mathbb{S},\varepsilon}(\Omega)} \varepsilon^{-1}\mathsf{H}_{\mathbb{S},\Gamma_n,0}(\operatorname{Div},\Omega).
\end{aligned}$$

Proof of Theorem 3.31. Theorem 3.25 and (14) show

$$\mathsf{H}^{k}_{\mathbb{S},\Gamma_t}(\operatorname{RotRot}^\top,\Omega) = R(\widetilde{\mathcal{Q}}^{k,1}_{\operatorname{RotRot}^\top_{\mathbb{S}},\Gamma_t}) \dotplus \mathsf{H}^{k}_{\mathbb{S},\Gamma_t,0}(\operatorname{RotRot}^\top,\Omega),$$
$$\mathsf{H}^{k+1,k}_{\mathbb{S},\Gamma_t}(\operatorname{RotRot}^\top,\Omega) = R(\widetilde{\mathcal{Q}}^{k+1,k,1}_{\operatorname{RotRot}^\top_{\mathbb{S}},\Gamma_t}) \dotplus \mathsf{H}^{k+1}_{\mathbb{S},\Gamma_t,0}(\operatorname{RotRot}^\top,\Omega),$$
$$\mathsf{H}^{k}_{\mathbb{S},\Gamma_t,0}(\operatorname{RotRot}^\top,\Omega) = \operatorname{symGrad}\mathsf{H}^{k+1}_{\Gamma_t}(\Omega) + \operatorname{Lin}\mathcal{B}^{\operatorname{RotRot}^\top}_{\mathbb{S},\Gamma_t}(\Omega).$$

To prove the directness, let

$$\sum_{\ell=1}^{d_{\Omega,\Gamma_t}} \lambda_\ell B^{\operatorname{RotRot}^\top}_{\mathbb{S},\Gamma_t,\ell} \in \operatorname{symGrad}\mathsf{H}^{k+1}_{\Gamma_t}(\Omega) \cap \mathcal{B}^{\operatorname{RotRot}^\top}_{\mathbb{S},\Gamma_t}(\Omega).$$

Then $0 = \sum_\ell \lambda_\ell \pi_{\operatorname{Div}} B^{\operatorname{RotRot}^\top}_{\mathbb{S},\Gamma_t,\ell} \in \operatorname{Lin}\pi_{\operatorname{Div}} B^{\operatorname{RotRot}^\top}_{\mathbb{S},\Gamma_t,\ell}$ and hence $\lambda_\ell = 0$ for all ℓ as $\pi_{\operatorname{Div}} B^{\operatorname{RotRot}^\top}_{\mathbb{S},\Gamma_t,\ell}$ is a basis of $\mathcal{H}_{\mathbb{S},\Gamma_t,\Gamma_n,\varepsilon}(\Omega)$ by Theorem 3.30. Concerning the boundedness of the decompositions, let

$$\mathsf{H}^{k}_{\mathbb{S},\Gamma_t,0}(\operatorname{RotRot}^\top,\Omega) \ni S = \operatorname{symGrad} v + B, \qquad v \in \mathsf{H}^{k+1}_{\Gamma_t}(\Omega), \quad B \in \operatorname{Lin}\mathcal{B}^{\operatorname{RotRot}^\top}_{\mathbb{S},\Gamma_t}(\Omega).$$

By Theorem 3.24 $\operatorname{symGrad} v \in R(\operatorname{symGrad}^k_{\Gamma_t})$ and $u := \mathcal{P}^k_{\operatorname{symGrad},\Gamma_t}\operatorname{symGrad} v \in \mathsf{H}^{k+1}_{\Gamma_t}(\Omega)$ solves $\operatorname{symGrad} u = \operatorname{symGrad} v$ with $|u|_{\mathsf{H}^{k+1}(\Omega)} \le c|\operatorname{symGrad} v|_{\mathsf{H}^k_{\mathbb{S}}(\Omega)}$. Therefore,

$$|u|_{\mathsf{H}^{k+1}(\Omega)} + |B|_{\mathsf{H}^k_{\mathbb{S}}(\Omega)} \le c\big(|\operatorname{symGrad} v|_{\mathsf{H}^k_{\mathbb{S}}(\Omega)} + |B|_{\mathsf{H}^k_{\mathbb{S}}(\Omega)}\big) \le c\big(|S|_{\mathsf{H}^k_{\mathbb{S}}(\Omega)} + |B|_{\mathsf{H}^k_{\mathbb{S}}(\Omega)}\big).$$

Note that the mapping

$$I_{\pi,\operatorname{Div}} : \operatorname{Lin}\mathcal{B}^{\operatorname{RotRot}^\top}_{\mathbb{S},\Gamma_t}(\Omega) \to \operatorname{Lin}\pi_{\operatorname{Div}}\mathcal{B}^{\operatorname{RotRot}^\top}_{\mathbb{S},\Gamma_t}(\Omega) = \mathcal{H}_{\mathbb{S},\Gamma_t,\Gamma_n,\varepsilon}(\Omega); \quad B^{\operatorname{RotRot}^\top}_{\mathbb{S},\Gamma_t,\ell} \mapsto \pi_{\operatorname{Div}} B^{\operatorname{RotRot}^\top}_{\mathbb{S},\Gamma_t,\ell}$$

is a topological isomorphism (between finite dimensional spaces and with arbitrary norms). Thus

$$|B|_{\mathsf{H}^k_{\mathbb{S}}(\Omega)} \le c|B|_{\mathsf{L}^2_{\mathbb{S}}(\Omega)} \le c|\pi_{\operatorname{Div}} B|_{\mathsf{L}^2_{\mathbb{S}}(\Omega)} = c|\pi_{\operatorname{Div}} S|_{\mathsf{L}^2_{\mathbb{S}}(\Omega)} \le c|S|_{\mathsf{L}^2_{\mathbb{S}}(\Omega)} \le c|S|_{\mathsf{H}^k_{\mathbb{S}}(\Omega)}.$$

Finally, we see $S = \operatorname{symGrad} u + B \in \operatorname{symGrad}\mathsf{H}^{k+1}_{\Gamma_t}(\Omega) \dotplus \operatorname{Lin}\mathcal{B}^{\operatorname{RotRot}^\top}_{\mathbb{S},\Gamma_t}(\Omega)$ and

$$|u|_{\mathsf{H}^{k+1}(\Omega)} + |B|_{\mathsf{H}^k_{\mathbb{S}}(\Omega)} \le c|S|_{\mathsf{H}^k_{\mathbb{S}}(\Omega)}.$$

The other assertions for Div follow analogously. □

Remark 3.33 (bounded regular direct decompositions)**.** *By Theorem 3.31 we have, e.g.,*

$$\begin{aligned}\mathsf{H}^{k}_{\mathbb{S},\Gamma_t}(\operatorname{RotRot}^\top,\Omega) &= R(\widetilde{\mathcal{Q}}^{k,1}_{\operatorname{RotRot}^\top_{\mathbb{S}},\Gamma_t}) \dotplus \operatorname{Lin}\mathcal{B}^{\operatorname{RotRot}^\top}_{\mathbb{S},\Gamma_t}(\Omega) \dotplus \operatorname{symGrad}\mathsf{H}^{k+1}_{\Gamma_t}(\Omega)\\ &= \mathsf{H}^{k+2}_{\mathbb{S},\Gamma_t}(\Omega) + \operatorname{symGrad}\mathsf{H}^{k+1}_{\Gamma_t}(\Omega)\end{aligned}$$

with bounded linear regular direct decomposition operators

$$\widehat{\mathcal{Q}}^{k,1}_{\operatorname{RotRot}^\top_{\mathbb{S}},\Gamma_t} : \mathsf{H}^{k}_{\mathbb{S},\Gamma_t}(\operatorname{RotRot}^\top,\Omega) \to R(\widetilde{\mathcal{Q}}^{k,1}_{\operatorname{RotRot}^\top_{\mathbb{S}},\Gamma_t}) \subset \mathsf{H}^{k+2}_{\mathbb{S},\Gamma_t}(\Omega),$$
$$\widehat{\mathcal{Q}}^{k,\infty}_{\operatorname{RotRot}^\top_{\mathbb{S}},\Gamma_t} : \mathsf{H}^{k}_{\mathbb{S},\Gamma_t}(\operatorname{RotRot}^\top,\Omega) \to \operatorname{Lin}\mathcal{B}^{\operatorname{RotRot}^\top}_{\mathbb{S},\Gamma_t}(\Omega) \subset \mathsf{H}^{\infty}_{\mathbb{S},\Gamma_t,0}(\operatorname{RotRot}^\top,\Omega) \subset \mathsf{H}^{k+2}_{\mathbb{S},\Gamma_t}(\Omega),$$
$$\widehat{\mathcal{Q}}^{k,0}_{\operatorname{RotRot}^\top_{\mathbb{S}},\Gamma_t} : \mathsf{H}^{k}_{\mathbb{S},\Gamma_t}(\operatorname{RotRot}^\top,\Omega) \to \mathsf{H}^{k+1}_{\Gamma_t}(\Omega)$$

satisfying $\widehat{\mathcal{Q}}^{k,1}_{\operatorname{RotRot}^\top_{\mathbb{S}},\Gamma_t} + \widehat{\mathcal{Q}}^{k,\infty}_{\operatorname{RotRot}^\top_{\mathbb{S}},\Gamma_t} + \operatorname{symGrad}\widehat{\mathcal{Q}}^{k,0}_{\operatorname{RotRot}^\top_{\mathbb{S}},\Gamma_t} = \operatorname{id}_{\mathsf{H}^k_{\mathbb{S},\Gamma_t}(\operatorname{RotRot}^\top,\Omega)}$.

A closer inspection of the latter proof allows for a more precise description of these bounded decomposition operators. For this, let $S \in \mathsf{H}^{k}_{\mathbb{S},\Gamma_t}(\operatorname{RotRot}^\top,\Omega)$*. According to Theorem 3.25 and Remark 3.28 we decompose*

$$S = S_R + S_N \in R(\widetilde{\mathcal{Q}}^{k,1}_{\operatorname{RotRot}^\top_{\mathbb{S}},\Gamma_t}) \dotplus R(\widetilde{\mathcal{N}}^{k}_{\operatorname{RotRot}^\top_{\mathbb{S}},\Gamma_t})$$

with $R(\widetilde{\mathcal{N}}^{k}_{\operatorname{RotRot}^\top_{\mathbb{S}},\Gamma_t}) = \mathsf{H}^{k}_{\mathbb{S},\Gamma_t,0}(\operatorname{RotRot}^\top,\Omega) = N(\operatorname{RotRot}^{\top,k}_{\mathbb{S},\Gamma_t})$ *as well as* $S_R = \widetilde{\mathcal{Q}}^{k,1}_{\operatorname{RotRot}^\top_{\mathbb{S}},\Gamma_t} S$ *and* $S_N = \widetilde{\mathcal{N}}^{k}_{\operatorname{RotRot}^\top_{\mathbb{S}},\Gamma_t} S$*. By Theorem 3.31 we further decompose*

$$\mathsf{H}^{k}_{\mathbb{S},\Gamma_t,0}(\operatorname{RotRot}^\top,\Omega) \ni S_N = \operatorname{symGrad} u + B \in \operatorname{symGrad}\mathsf{H}^{k+1}_{\Gamma_t}(\Omega) \dotplus \operatorname{Lin}\mathcal{B}^{\operatorname{RotRot}^\top}_{\mathbb{S},\Gamma_t}(\Omega).$$

Then $\pi_{\mathrm{Div}}S_N = \pi_{\mathrm{Div}}B \in \mathcal{H}_{\mathbb{S},\Gamma_t,\Gamma_n,\varepsilon}(\Omega)$ *and thus* $B = I^{-1}_{\pi,\mathrm{Div}}\pi_{\mathrm{Div}}S_N \in \operatorname{Lin}\mathcal{B}^{\mathrm{RotRot}^\top}_{\mathbb{S},\Gamma_t}(\Omega)$. *Therefore,* $u = \mathcal{P}^k_{\mathrm{symGrad},\Gamma_t}\operatorname{symGrad}u = \mathcal{P}^k_{\mathrm{symGrad},\Gamma_t}(S_N - B) = \mathcal{P}^k_{\mathrm{symGrad},\Gamma_t}(1 - I^{-1}_{\pi,\mathrm{Div}}\pi_{\mathrm{Div}})S_N$. *Finally we see*

$$\begin{aligned}
\widehat{\mathcal{Q}}^{k,1}_{\mathrm{RotRot}^\top_{\mathbb{S}},\Gamma_t} &= \widetilde{\mathcal{Q}}^{k,1}_{\mathrm{RotRot}^\top_{\mathbb{S}},\Gamma_t} = \mathcal{P}^k_{\mathrm{RotRot}^\top_{\mathbb{S}},\Gamma_t}\operatorname{RotRot}^{\top,k}_{\mathbb{S},\Gamma_t} = \mathcal{Q}^{k,1}_{\mathrm{RotRot}^\top_{\mathbb{S}},\Gamma_t}(\operatorname{RotRot}^{\top,k}_{\mathbb{S},\Gamma_t})^{-1}_\perp\operatorname{RotRot}^{\top,k}_{\mathbb{S},\Gamma_t},\\
\widehat{\mathcal{Q}}^{k,\infty}_{\mathrm{RotRot}^\top_{\mathbb{S}},\Gamma_t} &= I^{-1}_{\pi,\mathrm{Div}}\pi_{\mathrm{Div}}\widetilde{\mathcal{N}}^k_{\mathrm{RotRot}^\top_{\mathbb{S}},\Gamma_t},\\
\widehat{\mathcal{Q}}^{k,0}_{\mathrm{RotRot}^\top_{\mathbb{S}},\Gamma_t} &= \mathcal{P}^k_{\mathrm{symGrad},\Gamma_t}(1 - I^{-1}_{\pi,\mathrm{Div}}\pi_{\mathrm{Div}})\widetilde{\mathcal{N}}^k_{\mathrm{RotRot}^\top_{\mathbb{S}},\Gamma_t}
\end{aligned}$$

with $\widetilde{\mathcal{N}}^k_{\mathrm{RotRot}^\top_{\mathbb{S}},\Gamma_t} = 1 - \widetilde{\mathcal{Q}}^{k,1}_{\mathrm{RotRot}^\top_{\mathbb{S}},\Gamma_t}$. *Analogously, we have*

$$\begin{aligned}
\mathsf{H}^{k+1,k}_{\mathbb{S},\Gamma_t}(\operatorname{RotRot}^\top,\Omega) &= R(\widetilde{\mathcal{Q}}^{k+1,k,1}_{\mathrm{RotRot}^\top_{\mathbb{S}},\Gamma_t}) \dotplus \operatorname{Lin}\mathcal{B}^{\mathrm{RotRot}^\top}_{\mathbb{S},\Gamma_t}(\Omega) \dotplus \operatorname{symGrad}\mathsf{H}^{k+2}_{\Gamma_t}(\Omega)\\
&= \mathsf{H}^{k+2}_{\mathbb{S},\Gamma_t}(\Omega) + \operatorname{symGrad}\mathsf{H}^{k+2}_{\Gamma_t}(\Omega),\\
\mathsf{H}^k_{\mathbb{S},\Gamma_n}(\operatorname{Div},\Omega) &= R(\widetilde{\mathcal{Q}}^{k,1}_{\mathrm{Div}_{\mathbb{S}},\Gamma_n}) \dotplus \operatorname{Lin}\mathcal{B}^{\mathrm{Div}}_{\mathbb{S},\Gamma_n}(\Omega) \dotplus \operatorname{RotRot}^\top\mathsf{H}^{k+2}_{\mathbb{S},\Gamma_n}(\Omega)\\
&= \mathsf{H}^{k+1}_{\mathbb{S},\Gamma_n}(\Omega) + \operatorname{RotRot}^\top\mathsf{H}^{k+2}_{\mathbb{S},\Gamma_n}(\Omega)
\end{aligned}$$

with bounded linear regular direct decomposition operators

$$\begin{aligned}
\widehat{\mathcal{Q}}^{k+1,k,1}_{\mathrm{RotRot}^\top_{\mathbb{S}},\Gamma_t} &: \mathsf{H}^{k+1,k}_{\mathbb{S},\Gamma_t}(\operatorname{RotRot}^\top,\Omega) \to R(\widetilde{\mathcal{Q}}^{k+1,k,1}_{\mathrm{RotRot}^\top_{\mathbb{S}},\Gamma_t}) \subset \mathsf{H}^{k+2}_{\mathbb{S},\Gamma_t}(\Omega),\\
\widehat{\mathcal{Q}}^{k+1,k,\infty}_{\mathrm{RotRot}^\top_{\mathbb{S}},\Gamma_t} &: \mathsf{H}^{k+1,k}_{\mathbb{S},\Gamma_t}(\operatorname{RotRot}^\top,\Omega) \to \operatorname{Lin}\mathcal{B}^{\mathrm{RotRot}^\top}_{\mathbb{S},\Gamma_t}(\Omega) \subset \mathsf{H}^\infty_{\mathbb{S},\Gamma_t,0}(\operatorname{RotRot}^\top,\Omega) \subset \mathsf{H}^{k+2}_{\mathbb{S},\Gamma_t}(\Omega),\\
\widehat{\mathcal{Q}}^{k+1,k,0}_{\mathrm{RotRot}^\top_{\mathbb{S}},\Gamma_t} &: \mathsf{H}^{k+1,k}_{\mathbb{S},\Gamma_t}(\operatorname{RotRot}^\top,\Omega) \to \mathsf{H}^{k+2}_{\Gamma_t}(\Omega),\\
\widehat{\mathcal{Q}}^{k,1}_{\mathrm{Div}_{\mathbb{S}},\Gamma_n} &: \mathsf{H}^k_{\mathbb{S},\Gamma_n}(\operatorname{Div},\Omega) \to R(\widetilde{\mathcal{Q}}^{k,1}_{\mathrm{Div}_{\mathbb{S}},\Gamma_n}) \subset \mathsf{H}^{k+1}_{\mathbb{S},\Gamma_n}(\Omega),\\
\widehat{\mathcal{Q}}^{k,\infty}_{\mathrm{Div}_{\mathbb{S}},\Gamma_n} &: \mathsf{H}^k_{\mathbb{S},\Gamma_n}(\operatorname{Div},\Omega) \to \operatorname{Lin}\mathcal{B}^{\mathrm{Div}}_{\mathbb{S},\Gamma_n}(\Omega) \subset \mathsf{H}^\infty_{\mathbb{S},\Gamma_n,0}(\operatorname{Div},\Omega) \subset \mathsf{H}^{k+1}_{\mathbb{S},\Gamma_n}(\Omega),\\
\widehat{\mathcal{Q}}^{k,0}_{\mathrm{Div}_{\mathbb{S}},\Gamma_n} &: \mathsf{H}^k_{\mathbb{S},\Gamma_n}(\operatorname{Div},\Omega) \to \mathsf{H}^{k+2}_{\mathbb{S},\Gamma_n}(\Omega)
\end{aligned}$$

satisfying

$$\begin{aligned}
\widehat{\mathcal{Q}}^{k+1,k,1}_{\mathrm{RotRot}^\top_{\mathbb{S}},\Gamma_t} + \widehat{\mathcal{Q}}^{k+1,k,\infty}_{\mathrm{RotRot}^\top_{\mathbb{S}},\Gamma_t} + \operatorname{symGrad}\widehat{\mathcal{Q}}^{k+1,k,0}_{\mathrm{RotRot}^\top_{\mathbb{S}},\Gamma_t} &= \mathrm{id}_{\mathsf{H}^{k+1,k}_{\mathbb{S},\Gamma_t}(\operatorname{RotRot}^\top,\Omega)},\\
\widehat{\mathcal{Q}}^{k,1}_{\mathrm{Div}_{\mathbb{S}},\Gamma_n} + \widehat{\mathcal{Q}}^{k,\infty}_{\mathrm{Div}_{\mathbb{S}},\Gamma_n} + \operatorname{RotRot}^\top\widehat{\mathcal{Q}}^{k,0}_{\mathrm{Div}_{\mathbb{S}},\Gamma_n} &= \mathrm{id}_{\mathsf{H}^k_{\mathbb{S},\Gamma_n}(\operatorname{Div},\Omega)}
\end{aligned}$$

and

$$\begin{aligned}
\widehat{\mathcal{Q}}^{k+1,k,1}_{\mathrm{RotRot}^\top_{\mathbb{S}},\Gamma_t} &= \widetilde{\mathcal{Q}}^{k+1,k,1}_{\mathrm{RotRot}^\top_{\mathbb{S}},\Gamma_t} = \mathcal{P}^{k+1,k}_{\mathrm{RotRot}^\top_{\mathbb{S}},\Gamma_t}\operatorname{RotRot}^{\top,k+1,k}_{\mathbb{S},\Gamma_t},\\
\widehat{\mathcal{Q}}^{k+1,k,\infty}_{\mathrm{RotRot}^\top_{\mathbb{S}},\Gamma_t} &= I^{-1}_{\pi,\mathrm{Div}}\pi_{\mathrm{Div}}\widetilde{\mathcal{N}}^{k+1,k}_{\mathrm{RotRot}^\top_{\mathbb{S}},\Gamma_t},\\
\widehat{\mathcal{Q}}^{k+1,k,0}_{\mathrm{RotRot}^\top_{\mathbb{S}},\Gamma_t} &= \mathcal{P}^{k+1}_{\mathrm{symGrad},\Gamma_t}(1 - I^{-1}_{\pi,\mathrm{Div}}\pi_{\mathrm{Div}})\widetilde{\mathcal{N}}^{k+1,k}_{\mathrm{RotRot}^\top_{\mathbb{S}},\Gamma_t},\\
\widehat{\mathcal{Q}}^{k,1}_{\mathrm{Div}_{\mathbb{S}},\Gamma_n} &= \widetilde{\mathcal{Q}}^{k,1}_{\mathrm{Div}_{\mathbb{S}},\Gamma_n} = \mathcal{P}^k_{\mathrm{Div}_{\mathbb{S}},\Gamma_n}\operatorname{Div}^k_{\mathbb{S},\Gamma_n},\\
\widehat{\mathcal{Q}}^{k,\infty}_{\mathrm{Div}_{\mathbb{S}},\Gamma_n} &= I^{-1}_{\pi,\mathrm{RotRot}^\top}\pi_{\mathrm{RotRot}^\top}\widetilde{\mathcal{N}}^k_{\mathrm{Div}_{\mathbb{S}},\Gamma_n},\\
\widehat{\mathcal{Q}}^{k,0}_{\mathrm{Div}_{\mathbb{S}},\Gamma_n} &= \mathcal{P}^k_{\mathrm{RotRot}^\top_{\mathbb{S}},\Gamma_n}(1 - I^{-1}_{\pi,\mathrm{RotRot}^\top}\pi_{\mathrm{RotRot}^\top})\widetilde{\mathcal{N}}^k_{\mathrm{Div}_{\mathbb{S}},\Gamma_n}
\end{aligned}$$

with

$$\begin{aligned}
\widetilde{\mathcal{N}}^{k+1,k}_{\mathrm{RotRot}^\top_{\mathbb{S}},\Gamma_t} &= 1 - \widetilde{\mathcal{Q}}^{k+1,k,1}_{\mathrm{RotRot}^\top_{\mathbb{S}},\Gamma_t}, & \widetilde{\mathcal{N}}^k_{\mathrm{Div}_{\mathbb{S}},\Gamma_n} &= 1 - \widehat{\mathcal{Q}}^{k,1}_{\mathrm{Div}_{\mathbb{S}},\Gamma_n},\\
\mathcal{P}^{k+1,k}_{\mathrm{RotRot}^\top_{\mathbb{S}},\Gamma_t} &= \mathcal{Q}^{k+1,k,1}_{\mathrm{RotRot}^\top_{\mathbb{S}},\Gamma_t}(\operatorname{RotRot}^{\top,k+1,k}_{\mathbb{S},\Gamma_t})^{-1}_\perp, & \mathcal{P}^k_{\mathrm{Div}_{\mathbb{S}},\Gamma_n} &= \mathcal{Q}^{k,1}_{\mathrm{Div}_{\mathbb{S}},\Gamma_n}(\operatorname{Div}^k_{\mathbb{S},\Gamma_n})^{-1}_\perp,
\end{aligned}$$

and

$$I_{\pi,\mathrm{RotRot}^\top} : \operatorname{Lin}\mathcal{B}^{\mathrm{Div}}_{\mathbb{S},\Gamma_n}(\Omega) \to \operatorname{Lin}\pi_{\mathrm{RotRot}^\top}\mathcal{B}^{\mathrm{Div}}_{\mathbb{S},\Gamma_n}(\Omega) = \mathcal{H}_{\mathbb{S},\Gamma_n,\Gamma_n,\varepsilon}(\Omega); \quad B^{\mathrm{Div}}_{\mathbb{S},\Gamma_n,\ell} \mapsto \pi_{\mathrm{RotRot}^\top}B^{\mathrm{Div}}_{\mathbb{S},\Gamma_n,\ell}.$$

Noting

$$R(\varepsilon^{-1}\operatorname{RotRot}^\top_{\mathbb{S},\Gamma_n}) \perp_{\mathsf{L}^2_{\mathbb{S},\varepsilon}(\Omega)} \mathcal{B}^{\mathrm{RotRot}^\top}_{\mathbb{S},\Gamma_t}(\Omega), \qquad R(\operatorname{symGrad}_{\Gamma_t}) \perp_{\mathsf{L}^2_{\mathbb{S}}(\Omega)} \mathcal{B}^{\mathrm{Div}}_{\mathbb{S},\Gamma_n}(\Omega) \tag{15}$$

we see:

Theorem 3.34 (alternative Dirichlet/Neumann projections). *It holds*

$$\begin{aligned}\mathcal{H}_{\mathbb{S},\Gamma_t,\Gamma_n,\varepsilon}(\Omega)\cap\mathcal{B}^{\mathrm{RotRot}^\top}_{\mathbb{S},\Gamma_t}(\Omega)^{\perp_{\mathsf{L}^2_{\mathbb{S},\varepsilon}(\Omega)}}&=\{0\},\\ N(\operatorname{Div}_{\mathbb{S},\Gamma_n}\varepsilon)\cap\mathcal{B}^{\mathrm{RotRot}^\top}_{\mathbb{S},\Gamma_t}(\Omega)^{\perp_{\mathsf{L}^2_{\mathbb{S},\varepsilon}(\Omega)}}&=R(\varepsilon^{-1}\operatorname{RotRot}^\top_{\mathbb{S},\Gamma_n}),\\ \mathcal{H}_{\mathbb{S},\Gamma_t,\Gamma_n,\varepsilon}(\Omega)\cap\mathcal{B}^{\mathrm{Div}}_{\mathbb{S},\Gamma_n}(\Omega)^{\perp_{\mathsf{L}^2_{\mathbb{S}}(\Omega)}}&=\{0\},\\ N(\operatorname{RotRot}^\top_{\mathbb{S},\Gamma_t})\cap\mathcal{B}^{\mathrm{Div}}_{\mathbb{S},\Gamma_n}(\Omega)^{\perp_{\mathsf{L}^2_{\mathbb{S}}(\Omega)}}&=R(\operatorname{symGrad}_{\Gamma_t}).\end{aligned}$$

Moreover, for all $k\geq 0$

$$\begin{aligned}N(\operatorname{Div}^k_{\mathbb{S},\Gamma_n}\varepsilon)\cap\mathcal{B}^{\mathrm{RotRot}^\top}_{\mathbb{S},\Gamma_t}(\Omega)^{\perp_{\mathsf{L}^2_{\mathbb{S},\varepsilon}(\Omega)}}&=R(\varepsilon^{-1}\operatorname{RotRot}^{\top,k}_{\mathbb{S},\Gamma_n})=\varepsilon^{-1}\operatorname{RotRot}^\top\mathsf{H}^{k+2}_{\mathbb{S},\Gamma_n}(\Omega),\\ N(\operatorname{RotRot}^{\top,k}_{\mathbb{S},\Gamma_t})\cap\mathcal{B}^{\mathrm{Div}}_{\mathbb{S},\Gamma_n}(\Omega)^{\perp_{\mathsf{L}^2_{\mathbb{S}}(\Omega)}}&=R(\operatorname{symGrad}^k_{\Gamma_t})=\operatorname{symGrad}\mathsf{H}^{k+1}_{\Gamma_t}(\Omega).\end{aligned}$$

Proof. For $k=0$ and $S\in\mathcal{H}_{\mathbb{S},\Gamma_t,\Gamma_n,\varepsilon}(\Omega)\cap\mathcal{B}^{\mathrm{RotRot}^\top}_{\mathbb{S},\Gamma_t}(\Omega)^{\perp_{\mathsf{L}^2_{\mathbb{S},\varepsilon}(\Omega)}}$ we have

$$0=\langle S,B^{\mathrm{RotRot}^\top}_{\mathbb{S},\Gamma_t,\ell}\rangle_{\mathsf{L}^2_{\mathbb{S},\varepsilon}(\Omega)}=\langle\pi_{\mathrm{Div}}S,B^{\mathrm{RotRot}^\top}_{\mathbb{S},\Gamma_t,\ell}\rangle_{\mathsf{L}^2_{\mathbb{S},\varepsilon}(\Omega)}=\langle S,\pi_{\mathrm{Div}}B^{\mathrm{RotRot}^\top}_{\mathbb{S},\Gamma_t,\ell}\rangle_{\mathsf{L}^2_{\mathbb{S},\varepsilon}(\Omega)}$$

and hence $S=0$ by Theorem 3.30. Analogously, we see for $S\in\mathcal{H}_{\mathbb{S},\Gamma_t,\Gamma_n,\varepsilon}(\Omega)\cap\mathcal{B}^{\mathrm{Div}}_{\mathbb{S},\Gamma_n}(\Omega)^{\perp_{\mathsf{L}^2_{\mathbb{S}}(\Omega)}}$

$$0=\langle S,B^{\mathrm{Div}}_{\mathbb{S},\Gamma_n,\ell}\rangle_{\mathsf{L}^q(\Omega)}=\langle\pi_{\mathrm{RotRot}^\top}S,\varepsilon^{-1}B^{\mathrm{Div}}_{\mathbb{S},\Gamma_n,\ell}\rangle_{\mathsf{L}^2_{\mathbb{S},\varepsilon}(\Omega)}=\langle S,\pi_{\mathrm{RotRot}^\top}\varepsilon^{-1}B^{\mathrm{Div}}_{\mathbb{S},\Gamma_n,\ell}\rangle_{\mathsf{L}^2_{\mathbb{S},\varepsilon}(\Omega)}$$

and thus $S=0$ again by Theorem 3.30. According to (13) we can decompose

$$\begin{aligned}N(\operatorname{Div}_{\mathbb{S},\Gamma_n}\varepsilon)&=R(\varepsilon^{-1}\operatorname{RotRot}^\top_{\mathbb{S},\Gamma_n})\oplus_{\mathsf{L}^2_{\mathbb{S},\varepsilon}(\Omega)}\mathcal{H}_{\mathbb{S},\Gamma_t,\Gamma_n,\varepsilon}(\Omega),\\ N(\operatorname{RotRot}^\top_{\mathbb{S},\Gamma_t})&=R(\operatorname{symGrad}_{\Gamma_t})\oplus_{\mathsf{L}^2_{\mathbb{S},\varepsilon}(\Omega)}\mathcal{H}_{\mathbb{S},\Gamma_t,\Gamma_n,\varepsilon}(\Omega),\end{aligned}$$

which shows by (15) the other two assertions. Let $k\geq 0$. The case $k=0$ and Theorem 3.16 show

$$\begin{aligned}N(\operatorname{RotRot}^{\top,k}_{\mathbb{S},\Gamma_t})\cap\mathcal{B}^{\mathrm{Div}}_{\mathbb{S},\Gamma_n}(\Omega)^{\perp_{\mathsf{L}^2_{\mathbb{S}}(\Omega)}}&=\mathsf{H}^k_{\mathbb{S},\Gamma_t}(\Omega)\cap N(\operatorname{RotRot}^\top_{\mathbb{S},\Gamma_t})\cap\mathcal{B}^{\mathrm{Div}}_{\mathbb{S},\Gamma_n}(\Omega)^{\perp_{\mathsf{L}^2_{\mathbb{S}}(\Omega)}}\\ &=\mathsf{H}^k_{\mathbb{S},\Gamma_t}(\Omega)\cap R(\operatorname{symGrad}_{\Gamma_t})\\ &=R(\operatorname{symGrad}^k_{\Gamma_t})=\operatorname{symGrad}\mathsf{H}^{k+1}_{\Gamma_t}(\Omega).\end{aligned}$$

Analogously,

$$\begin{aligned}N(\operatorname{Div}^k_{\mathbb{S},\Gamma_n}\varepsilon)\cap\mathcal{B}^{\mathrm{RotRot}^\top}_{\mathbb{S},\Gamma_t}(\Omega)^{\perp_{\mathsf{L}^2_{\mathbb{S},\varepsilon}(\Omega)}}&=\varepsilon^{-1}\mathsf{H}^k_{\mathbb{S},\Gamma_n}(\Omega)\cap N(\operatorname{Div}_{\mathbb{S},\Gamma_n}\varepsilon)\cap\mathcal{B}^{\mathrm{RotRot}^\top}_{\mathbb{S},\Gamma_t}(\Omega)^{\perp_{\mathsf{L}^2_{\mathbb{S},\varepsilon}(\Omega)}}\\ &=\varepsilon^{-1}\mathsf{H}^k_{\mathbb{S},\Gamma_n}(\Omega)\cap R(\varepsilon^{-1}\operatorname{RotRot}^\top_{\mathbb{S},\Gamma_n})\\ &=R(\varepsilon^{-1}\operatorname{RotRot}^{\top,k}_{\mathbb{S},\Gamma_n})=\varepsilon^{-1}\operatorname{RotRot}^\top\mathsf{H}^{k+2}_{\mathbb{S},\Gamma_n}(\Omega),\end{aligned}$$

completing the proof. □

Theorem 3.31 implies:

Theorem 3.35 (cohomology groups). *It holds*

$$\frac{N(\operatorname{RotRot}^{\top,k}_{\mathbb{S},\Gamma_t})}{R(\operatorname{symGrad}^k_{\Gamma_t})}\cong\operatorname{Lin}\mathcal{B}^{\mathrm{RotRot}^\top}_{\mathbb{S},\Gamma_t}(\Omega)\cong\mathcal{H}_{\mathbb{S},\Gamma_t,\Gamma_n,\varepsilon}(\Omega)\cong\operatorname{Lin}\mathcal{B}^{\mathrm{Div}}_{\mathbb{S},\Gamma_n}(\Omega)\cong\frac{N(\operatorname{Div}^k_{\mathbb{S},\Gamma_n})}{R(\operatorname{RotRot}^{\top,k}_{\mathbb{S},\Gamma_n})}.$$

In particular, the dimensions of the cohomology groups (Dirichlet/Neumann fields) are independent of k and ε and it holds

$$d_{\Omega,\Gamma_t}=\dim\big(N(\operatorname{RotRot}^{\top,k}_{\mathbb{S},\Gamma_t})/R(\operatorname{symGrad}^k_{\Gamma_t})\big)=\dim\big(N(\operatorname{Div}^k_{\mathbb{S},\Gamma_n})/R(\operatorname{RotRot}^{\top,k}_{\mathbb{S},\Gamma_n})\big).$$

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Appendix A. Elementary Formulas

From [13, 14, 15] and [12] we have the following collection of formulas related to the elasticity and the biharmonic complex.

Lemma A.1 ([12, Lemma 12.10])**.** *Let u, v, w, and S belong to $\mathsf{C}^\infty(\mathbb{R}^3)$.*

- $(\operatorname{spn} v)\, w = v \times w = -(\operatorname{spn} w)\, v$ *and* $(\operatorname{spn} v)(\operatorname{spn}^{-1} S) = -Sv$, *if* $\operatorname{sym} S = 0$
- $\operatorname{sym}\operatorname{spn} v = 0$ *and* $\operatorname{dev}(u \operatorname{id}) = 0$
- $\operatorname{tr}\operatorname{Grad} v = \operatorname{div} v$ *and* $2\operatorname{skw}\operatorname{Grad} v = \operatorname{spn}\operatorname{rot} v$
- $\operatorname{Div}(u \operatorname{id}) = \operatorname{grad} u$ *and* $\operatorname{Rot}(u \operatorname{id}) = -\operatorname{spn}\operatorname{grad} u$,
  in particular, $\operatorname{rot}\operatorname{Div}(u \operatorname{id}) = 0$ *and* $\operatorname{rot}\operatorname{spn}^{-1}\operatorname{Rot}(u \operatorname{id}) = 0$
  and $\operatorname{sym}\operatorname{Rot}(u \operatorname{id}) = 0$
- $\operatorname{Div}\operatorname{spn} v = -\operatorname{rot} v$ *and* $\operatorname{Div}\operatorname{skw} S = -\operatorname{rot}\operatorname{spn}^{-1}\operatorname{skw} S$,
  in particular, $\operatorname{div}\operatorname{Div}\operatorname{skw} S = 0$
- $\operatorname{Rot}\operatorname{spn} v = (\operatorname{div} v)\operatorname{id} - (\operatorname{Grad} v)^\top$
  and $\operatorname{Rot}\operatorname{skw} S = (\operatorname{div}\operatorname{spn}^{-1}\operatorname{skw} S)\operatorname{id} - (\operatorname{Grad}\operatorname{spn}^{-1}\operatorname{skw} S)^\top$
- $\operatorname{dev}\operatorname{Rot}\operatorname{spn} v = -(\operatorname{dev}\operatorname{Grad} v)^\top$
- $-2\operatorname{Rot}\operatorname{sym}\operatorname{Grad} v = 2\operatorname{Rot}\operatorname{skw}\operatorname{Grad} v = -(\operatorname{Grad}\operatorname{rot} v)^\top$

- $2\operatorname{spn}^{-1}\operatorname{skw}\operatorname{Rot}S = \operatorname{Div}S^\top - \operatorname{grad}\operatorname{tr}S = \operatorname{Div}\big(S - (\operatorname{tr}S)\operatorname{id}\big)^\top$,
  in particular, $\operatorname{rot}\operatorname{Div}S^\top = 2\operatorname{rot}\operatorname{spn}^{-1}\operatorname{skw}\operatorname{Rot}S$
  and $2\operatorname{skw}\operatorname{Rot}S = \operatorname{spn}\operatorname{Div}S^\top$, *if* $\operatorname{tr}S = 0$
- $\operatorname{tr}\operatorname{Rot}S = 2\operatorname{div}\operatorname{spn}^{-1}\operatorname{skw}S$, *in particular,* $\operatorname{tr}\operatorname{Rot}S = 0$, *if* $\operatorname{skw}S = 0$,
  and $\operatorname{tr}\operatorname{Rot}\operatorname{sym}S = 0$ *and* $\operatorname{tr}\operatorname{Rot}\operatorname{skw}S = \operatorname{tr}\operatorname{Rot}S$
- $2(\operatorname{Grad}\operatorname{spn}^{-1}\operatorname{skw}S)^\top = (\operatorname{tr}\operatorname{Rot}\operatorname{skw}S)\operatorname{id} - 2\operatorname{Rot}\operatorname{skw}S$
- $3\operatorname{Div}(\operatorname{dev}\operatorname{Grad}v)^\top = 2\operatorname{grad}\operatorname{div}v$
- $2\operatorname{Rot}\operatorname{sym}\operatorname{Grad}v = -2\operatorname{Rot}\operatorname{skw}\operatorname{Grad}v = -\operatorname{Rot}\operatorname{spn}\operatorname{rot}v = (\operatorname{Grad}\operatorname{rot}v)^\top$
- $2\operatorname{Div}\operatorname{sym}\operatorname{Rot}S = -2\operatorname{Div}\operatorname{skw}\operatorname{Rot}S = \operatorname{rot}\operatorname{Div}S^\top$
- $\operatorname{Rot}(\operatorname{Rot}\operatorname{sym}S)^\top = \operatorname{sym}\operatorname{Rot}(\operatorname{Rot}S)^\top$
- $\operatorname{Rot}(\operatorname{Rot}\operatorname{skw}S)^\top = \operatorname{skw}\operatorname{Rot}(\operatorname{Rot}S)^\top$

All formulas extend to distributions as well.

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