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A Global div-curl-Lemma for Mixed Boundary Conditions in Weak Lipschitz Domains and a Corresponding Generalized A_0^*-A_1-Lemma in Hilbert Spaces

by

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A Global div-curl-Lemma for Mixed Boundary Conditions in Weak Lipschitz Domains and a Corresponding Generalized A_0^*-A_1-Lemma in Hilbert Spaces

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Abstract. We prove global and local versions of the so called div-curl-lemma, also known as compensated compactness, for mixed boundary conditions as well as bounded weak Lipschitz domains in 3D and weak Lipschitz interfaces. We will generalize our results using an abstract Hilbert space setting, which shows corresponding results to hold in arbitrary dimensions as well as for various differential operators. The crucial tools are Hilbert complexes and related compact embeddings.

Contents

1. Introduction

We will prove a global version of the so called div-curl-lemma used for compensated compactness, stating that under certain (mixed tangential and normal) boundary conditions and (very weak) regularity assumptions on a domain $\Omega \subset \mathbb{R}^3$ the following holds:

Let $\Omega \subset \mathbb{R}^3$ be a bounded weak Lipschitz domain with boundary Γ and boundary parts Γ_{t} and Γ_{n}. Let (E_n) and (H_n) be two sequences bounded in $\mathsf{L}^2(\Omega)$, such that $(\operatorname{curl} E_n)$ and $(\operatorname{div} H_n)$ are also bounded in $\mathsf{L}^2(\Omega)$ and $\nu \times E_n = 0$ on Γ_{t} and $\nu \cdot H_n = 0$ on Γ_{n}. Then there exist subsequences, again denoted by (E_n) and (H_n), such that (E_n), $(\operatorname{curl} E_n)$ and (H_n), $(\operatorname{div} H_n)$ converge weakly to E, $\operatorname{curl} E$ resp. H, $\operatorname{div} H$ in $\mathsf{L}^2(\Omega)$ and the inner products converge as well, i.e.,

$$\int_\Omega E_n \cdot H_n \to \int_\Omega E \cdot H.$$

A local version (distributional like convergence for arbitrary domains and no boundary conditions needed) of this div-curl-lemma is then immediately obtained.

Let $\Omega \subset \mathbb{R}^3$ be an open set. Let (E_n) and (H_n) be two sequences bounded in $\mathsf{L}^2(\Omega)$, such that $(\operatorname{curl} E_n)$ and $(\operatorname{div} H_n)$ are also bounded in $\mathsf{L}^2(\Omega)$. Then there exist subsequences, again denoted by (E_n) and (H_n),

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such that (E_n), $(\operatorname{curl} E_n)$ *and* (H_n), $(\operatorname{div} H_n)$ *converge weakly to* E, $\operatorname{curl} E$ *resp.* H, $\operatorname{div} H$ *in* $\mathsf{L}^2(\Omega)$ *and the inner products converge in the distributional sense as well, i.e., for all* $\varphi \in \mathring{\mathsf{C}}^\infty(\Omega)$ *it holds*

$$\int_\Omega (E_n \cdot H_n)\varphi \to \int_\Omega (E \cdot H)\varphi.$$

For details, see Theorem 3.1 and Corollary 3.2. We will also show a generalization to a natural Hilbert space setting in Theorem 4.7. In Section 5 we apply this result to some more differential operators in 3D and ND, appearing, e.g., in generalized electro-magnetics, for the biharmonic equation, in general relativity, for gravitational waves, and in the theory of linear elasticity and plasticity.

The div-curl-lemma, or compensated compactness, see the original papers by Murat [13] and Tartar [23] or [7, 22], and its variants and extensions have plenty of important applications. For an extensive discussion and a historical overview of the div-curl-lemma see [24]. More recent discussions can be found, e.g., in [5, 25] and in the nice preprint [26]. The div-curl-lemma is widely used in the theory of homogenization of (nonlinear) partial differential equations, see, e.g., [22]. Moreover, it is crucial in establishing compactness and regularity results for nonlinear partial differential equations such as harmonic maps, see, e.g., [9, 8, 19]. Numerical applications can be found, e.g., in [2]. It is further a crucial tool in the homogenization of stochastic partial differential equations, especially with certain random coefficients, see, e.g., the survey [1] and the literature cited therein, e.g., [10].

Let us also mention that the div-curl-lemma is particularly useful to treat homogenization of problems arising in plasticity, see, e.g., a recent preprint on this topic [21], for which the preprint [20] provides the important key div-curl-lemma. As in [20, 21] $\mathsf{H}^1(\Omega)$-potentials are used, these contributions are restricted to smooth, e.g., C^2 or convex, domains and to full boundary conditions. On the other hand, using Weck's selection theorem (2.1) it is easily possible to extend these results even to bounded weak Lipschitz domains of arbitrary topology and to the case of mixed boundary conditions.

Generally, for problems related to Maxwell's equations the detour over $\mathsf{H}^1(\Omega)$ instead of using Weck's selection theorem seems to be the wrong way to deal with such equations. Most of the arguments simply fail, and if not, the results are usually limited to smooth domains and trivial topologies. Mixed boundary conditions cannot be treated properly. Since the early 1970's, see the original paper by Weck [28] for Weck's selection theorem, it is well known, that the $\mathsf{H}^1(\Omega)$-detour is often not helpful and does not lead to satisfying results.

2. Definitions and Preliminaries

Let $\Omega \subset \mathbb{R}^3$ be a bounded weak Lipschitz domain, see [3, Definition 2.3] for details, with boundary $\Gamma := \partial\,\Omega$, which is divided into two relatively open weak Lipschitz subsets Γ_{t} and $\Gamma_{\mathsf{n}} := \Gamma \setminus \overline{\Gamma_{\mathsf{t}}}$ (its complement), see [3, Definition 2.5] for details. Note that strong Lipschitz (graph of Lipschitz functions) implies weak Lipschitz (Lipschitz manifolds) for the boundary as well as the interface. Throughout this paper we shall assume the latter regularity on Ω and Γ_{t}.

Recently, in [3], Weck's selection theorem, also known as the Maxwell compactness property, has been shown to hold for such bounded weak Lipschitz domains and mixed boundary conditions. More precisely, the embedding

$$\mathring{\mathsf{R}}_{\Gamma_{\mathsf{t}}}(\Omega) \cap \mathring{\mathsf{D}}_{\Gamma_{\mathsf{n}}}(\Omega) \hookrightarrow\!\!\!\!\rightarrow \mathsf{L}^2(\Omega) \tag{2.1}$$

is compact, see [3, Theorem 4.7]. A short historical overview of Weck's selection theorem is given in the introduction of [3], see also the original paper [28] and [18, 27, 6, 29, 11, 12] for simpler proofs and generalizations.

Here the usual Lebesgue and Sobolev spaces are denoted by $\mathsf{L}^2(\Omega)$ and $\mathsf{H}^1(\Omega)$ as well as

$$\mathsf{R}(\Omega) := \big\{E \in \mathsf{L}^2(\Omega) \,:\, \operatorname{rot} E \in \mathsf{L}^2(\Omega)\big\}, \quad \mathsf{D}(\Omega) := \big\{E \in \mathsf{L}^2(\Omega) \,:\, \operatorname{div} E \in \mathsf{L}^2(\Omega)\big\},$$

where we prefer to write rot instead of curl. $\mathsf{R}(\Omega)$ and $\mathsf{D}(\Omega)$ are also written as $H(\operatorname{rot}, \Omega)$ or $H(\operatorname{curl}, \Omega)$ resp. $H(\operatorname{div}, \Omega)$ in the literature. With the help of test functions and test vector fields

$$\mathring{\mathsf{C}}^\infty_{\Gamma_{\mathsf{t}}}(\Omega) := \big\{\varphi|_\Omega \,:\, \varphi \in \mathring{\mathsf{C}}^\infty(\mathbb{R}^3),\ \operatorname{dist}(\operatorname{supp}\varphi, \Gamma_{\mathsf{t}}) > 0\big\}$$

we define the closed subspaces

$$\mathring{\mathsf{H}}^1_{\Gamma_t}(\Omega) := \overline{\mathring{\mathsf{C}}^\infty_{\Gamma_t}(\Omega)}^{\mathsf{H}^1(\Omega)}, \quad \mathring{\mathsf{R}}_{\Gamma_t}(\Omega) := \overline{\mathring{\mathsf{C}}^\infty_{\Gamma_t}(\Omega)}^{\mathsf{R}(\Omega)}, \quad \mathring{\mathsf{D}}_{\Gamma_n}(\Omega) := \overline{\mathring{\mathsf{C}}^\infty_{\Gamma_n}(\Omega)}^{\mathsf{D}(\Omega)} \tag{2.2}$$

as closures of test functions respectively vector fields. In (2.2) homogeneous scalar, tangential and normal traces on Γ_t resp. Γ_n are generalized, respectively. To avoid case studies when using the Poincaré estimate, we also define

$$\mathring{\mathsf{H}}^1_\emptyset(\Omega) := \mathsf{H}^1(\Omega) \cap \mathbb{R}^{\perp_{\mathsf{L}^2(\Omega)}} = \big\{ u \in \mathsf{H}^1(\Omega) \,:\, \int_\Omega u = 0 \big\}.$$

Let us emphasize that our assumptions also allow for Rellich's selection theorem, i.e., the embedding

$$\mathring{\mathsf{H}}^1_{\Gamma_t}(\Omega) \hookrightarrow\!\!\!\!\rightarrow \mathsf{L}^2(\Omega) \tag{2.3}$$

is compact, see, e.g., [3, Theorem 4.8]. By density we have the two rules of partial integration

$$\forall\, u \in \mathring{\mathsf{H}}^1_{\Gamma_t}(\Omega) \qquad \forall\, H \in \mathring{\mathsf{D}}_{\Gamma_n}(\Omega) \qquad \langle \nabla u, H \rangle_{\mathsf{L}^2(\Omega)} = -\langle u, \operatorname{div} H \rangle_{\mathsf{L}^2(\Omega)}, \tag{2.4}$$

$$\forall\, E \in \mathring{\mathsf{R}}_{\Gamma_t}(\Omega) \qquad \forall\, H \in \mathring{\mathsf{R}}_{\Gamma_n}(\Omega) \qquad \langle \operatorname{rot} E, H \rangle_{\mathsf{L}^2(\Omega)} = \langle E, \operatorname{rot} H \rangle_{\mathsf{L}^2(\Omega)}. \tag{2.5}$$

We emphasize that, besides Weck's selection theorem, the resulting Maxwell estimates (Friedrichs/Poincaré type estimates), Helmholtz decompositions, closed ranges, continuous and compact inverse operators, and an adequate electro-magneto static solution theory for bounded weak Lipschitz domains and mixed boundary conditions, another important result has been shown in [3]. It holds

$$\begin{aligned}
\mathring{\mathsf{H}}^1_{\Gamma_t}(\Omega) &= \big\{ u \in \mathsf{H}^1(\Omega) \,:\, \langle \nabla u, \Phi \rangle_{\mathsf{L}^2(\Omega)} = -\langle u, \operatorname{div} \Phi \rangle_{\mathsf{L}^2(\Omega)} \text{ for all } \Phi \in \mathring{\mathsf{C}}^\infty_{\Gamma_n}(\Omega) \big\}, \\
\mathring{\mathsf{R}}_{\Gamma_t}(\Omega) &= \big\{ E \in \mathsf{R}(\Omega) \,:\, \langle \operatorname{rot} E, \Phi \rangle_{\mathsf{L}^2(\Omega)} = \langle E, \operatorname{rot} \Phi \rangle_{\mathsf{L}^2(\Omega)} \text{ for all } \Phi \in \mathring{\mathsf{C}}^\infty_{\Gamma_n}(\Omega) \big\}, \\
\mathring{\mathsf{D}}_{\Gamma_n}(\Omega) &= \big\{ H \in \mathsf{D}(\Omega) \,:\, \langle \operatorname{div} H, \varphi \rangle_{\mathsf{L}^2(\Omega)} = -\langle H, \nabla \varphi \rangle_{\mathsf{L}^2(\Omega)} \text{ for all } \varphi \in \mathring{\mathsf{C}}^\infty_{\Gamma_t}(\Omega) \big\},
\end{aligned} \tag{2.6}$$

i.e., strong and weak definitions of boundary conditions coincide, see [3, Theorem 4.5]. Furthermore, we define the closed subspaces of irrotational resp. solenoidal vector fields

$$\mathsf{R}_0(\Omega) := \big\{ E \in \mathsf{R}(\Omega) \,:\, \operatorname{rot} E = 0 \big\}, \quad \mathsf{D}_0(\Omega) := \big\{ E \in \mathsf{D}(\Omega) \,:\, \operatorname{div} E = 0 \big\}$$

as well as

$$\mathring{\mathsf{R}}_{\Gamma_t,0}(\Omega) := \mathring{\mathsf{R}}_{\Gamma_t}(\Omega) \cap \mathsf{R}_0(\Omega), \quad \mathring{\mathsf{D}}_{\Gamma_n,0}(\Omega) := \mathring{\mathsf{D}}_{\Gamma_n}(\Omega) \cap \mathsf{D}_0(\Omega).$$

A direct consequence of (2.1) is the compactness of the unit ball in

$$\mathcal{H}(\Omega) := \mathring{\mathsf{R}}_{\Gamma_t,0}(\Omega) \cap \mathring{\mathsf{D}}_{\Gamma_n,0}(\Omega),$$

the space of so called Dirichlet-Neumann fields. Hence $\mathcal{H}(\Omega)$ is finite dimensional. Another immediate consequence of Weck's selection theorem (2.1), using a standard indirect argument, is the so called Maxwell estimate, i.e.,

$$\exists\, c_{\mathsf{m}} > 0 \quad \forall\, E \in \mathring{\mathsf{R}}_{\Gamma_t}(\Omega) \cap \mathring{\mathsf{D}}_{\Gamma_n}(\Omega) \cap \mathcal{H}(\Omega)^{\perp_{\mathsf{L}^2(\Omega)}} \qquad |E|_{\mathsf{L}^2(\Omega)} \leq c_{\mathsf{m}} \big(|\operatorname{rot} E|_{\mathsf{L}^2(\Omega)} + |\operatorname{div} E|_{\mathsf{L}^2(\Omega)} \big) \tag{2.7}$$

or, equivalently,

$$\forall\, E \in \mathring{\mathsf{R}}_{\Gamma_t}(\Omega) \cap \mathring{\mathsf{D}}_{\Gamma_n}(\Omega) \qquad |E - \pi E|_{\mathsf{L}^2(\Omega)} \leq c_{\mathsf{m}} \big(|\operatorname{rot} E|_{\mathsf{L}^2(\Omega)} + |\operatorname{div} E|_{\mathsf{L}^2(\Omega)} \big), \tag{2.8}$$

see [3, Theorem 5.1], where $\pi : \mathsf{L}^2(\Omega) \to \mathcal{H}(\Omega)$ denotes the $\mathsf{L}^2(\Omega)$-orthonormal projector onto the Dirichlet-Neumann fields. Recent estimates for the Maxwell constant c_{m} can be found in [14, 15, 16]. Analogously, Rellich's selection theorem (2.3) shows the Friedrichs/Poincaré estimate

$$\exists\, c_{\mathsf{f,p}} > 0 \quad \forall\, u \in \mathring{\mathsf{H}}^1_{\Gamma_t}(\Omega) \qquad |u|_{\mathsf{L}^2(\Omega)} \leq c_{\mathsf{f,p}} |\nabla u|_{\mathsf{L}^2(\Omega)}, \tag{2.9}$$

see [3, Theorem 4.8]. By the projection theorem, applied to the densely defined and closed (unbounded) linear operator

$$\nabla : \mathring{\mathsf{H}}^1_{\Gamma_t}(\Omega) \subset \mathsf{L}^2(\Omega) \longrightarrow \mathsf{L}^2(\Omega)$$

with (Hilbert space) adjoint

$$\nabla^* = -\operatorname{div} : \mathring{\mathsf{D}}_{\Gamma_{\mathsf{n}}}(\Omega) \subset \mathsf{L}^2(\Omega) \longrightarrow \mathsf{L}^2(\Omega),$$

here we need (2.6), we get the simple Helmholtz decomposition

$$\mathsf{L}^2(\Omega) = \nabla \mathring{\mathsf{H}}^1_{\Gamma_{\mathsf{t}}}(\Omega) \oplus_{\mathsf{L}^2(\Omega)} \mathring{\mathsf{D}}_{\Gamma_{\mathsf{n}},0}(\Omega), \tag{2.10}$$

see [3, Theorem 5.3 or (13)], which immediately implies

$$\mathring{\mathsf{R}}_{\Gamma_{\mathsf{t}}}(\Omega) = \nabla \mathring{\mathsf{H}}^1_{\Gamma_{\mathsf{t}}}(\Omega) \oplus_{\mathsf{L}^2(\Omega)} \big(\mathring{\mathsf{R}}_{\Gamma_{\mathsf{t}}}(\Omega) \cap \mathring{\mathsf{D}}_{\Gamma_{\mathsf{n}},0}(\Omega)\big) \tag{2.11}$$

as $\nabla \mathring{\mathsf{H}}^1_{\Gamma_{\mathsf{t}}}(\Omega) \subset \mathring{\mathsf{R}}_{\Gamma_{\mathsf{t}},0}(\Omega)$. Note that the decompositions (2.10) and (2.11) are orthogonal which is denoted by $\oplus_{\mathsf{L}^2(\Omega)}$. By (2.9), the range $\nabla \mathring{\mathsf{H}}^1_{\Gamma_{\mathsf{t}}}(\Omega)$ is closed in $\mathsf{L}^2(\Omega)$, see also [3, Lemma 5.2]. Note that we call (2.10) a simple Helmholtz decomposition, since the refined Helmholtz decomposition

$$\mathsf{L}^2(\Omega) = \nabla \mathring{\mathsf{H}}^1_{\Gamma_{\mathsf{t}}}(\Omega) \oplus_{\mathsf{L}^2(\Omega)} \mathcal{H}(\Omega) \oplus_{\mathsf{L}^2(\Omega)} \operatorname{rot} \mathring{\mathsf{R}}_{\Gamma_{\mathsf{n}}}(\Omega)$$

holds as well, see [3, Theorem 5.3], where also $\operatorname{rot} \mathring{\mathsf{R}}_{\Gamma_{\mathsf{n}}}(\Omega)$ is closed in $\mathsf{L}^2(\Omega)$ as a consequence of (2.7), see [3, Lemma 5.2].

3. The div-rot-Lemma

Theorem 3.1 (global div-rot-lemma). *Let $(E_n) \subset \mathring{\mathsf{R}}_{\Gamma_{\mathsf{t}}}(\Omega)$ and $(H_n) \subset \mathring{\mathsf{D}}_{\Gamma_{\mathsf{n}}}(\Omega)$ be two sequences bounded in $\mathsf{R}(\Omega)$ resp. $\mathsf{D}(\Omega)$. Then there exist $E \in \mathring{\mathsf{R}}_{\Gamma_{\mathsf{t}}}(\Omega)$ and $H \in \mathring{\mathsf{D}}_{\Gamma_{\mathsf{n}}}(\Omega)$ as well as subsequences, again denoted by (E_n) and (H_n), such that (E_n) and (H_n) converge weakly in $\mathsf{R}(\Omega)$ resp. $\mathsf{D}(\Omega)$ to E resp. H together with the convergence of the inner products*

$$\langle E_n, H_n\rangle_{\mathsf{L}^2(\Omega)} \to \langle E, H\rangle_{\mathsf{L}^2(\Omega)}.$$

Proof. We pick subsequences, again denoted by (E_n) and (H_n), such that (E_n) and (H_n) converge weakly in $\mathsf{R}(\Omega)$ resp. $\mathsf{D}(\Omega)$ to E resp. H for some $E \in \mathring{\mathsf{R}}_{\Gamma_{\mathsf{t}}}(\Omega)$ and $H \in \mathring{\mathsf{D}}_{\Gamma_{\mathsf{n}}}(\Omega)$. By the simple Helmholtz decomposition (2.11), we have the orthogonal decomposition $\mathring{\mathsf{R}}_{\Gamma_{\mathsf{t}}}(\Omega) \ni E_n = \nabla u_n + \tilde{E}_n$ with some $u_n \in \mathring{\mathsf{H}}^1_{\Gamma_{\mathsf{t}}}(\Omega)$ and $\tilde{E}_n \in \mathring{\mathsf{R}}_{\Gamma_{\mathsf{t}}}(\Omega) \cap \mathring{\mathsf{D}}_{\Gamma_{\mathsf{n}},0}(\Omega)$. Then (u_n) is bounded in $\mathsf{H}^1(\Omega)$ by orthogonality and the Friedrichs/Poincaré estimate (2.9). $(\tilde{E}_n)$ is bounded in $\mathsf{R}(\Omega) \cap \mathsf{D}(\Omega)$ by orthogonality and $\operatorname{rot} \tilde{E}_n = \operatorname{rot} E_n$, $\operatorname{div} \tilde{E}_n = 0$. Hence, using Rellich's and Weck's selection theorems there exist $u \in \mathring{\mathsf{H}}^1_{\Gamma_{\mathsf{t}}}(\Omega)$ and $\tilde{E} \in \mathring{\mathsf{R}}_{\Gamma_{\mathsf{t}}}(\Omega) \cap \mathring{\mathsf{D}}_{\Gamma_{\mathsf{n}},0}(\Omega)$ and we can extract two subsequences, again denoted by (u_n) and $\tilde{E}_n$, such that $u_n \rightharpoonup u$ in $\mathsf{H}^1(\Omega)$ and $u_n \to u$ in $\mathsf{L}^2(\Omega)$ as well as $\tilde{E}_n \rightharpoonup \tilde{E}$ in $\mathsf{R}(\Omega) \cap \mathsf{D}(\Omega)$ and $\tilde{E}_n \to \tilde{E}$ in $\mathsf{L}^2(\Omega)$. We have $E = \nabla u + \tilde{E}$, giving the simple Helmholtz decomposition for E, as, e.g., for all $\varphi \in \mathring{\mathsf{C}}^\infty(\Omega)$

$$\langle E, \varphi\rangle_{\mathsf{L}^2(\Omega)} \leftarrow \langle E_n, \varphi\rangle_{\mathsf{L}^2(\Omega)} = \langle \nabla u_n, \varphi\rangle_{\mathsf{L}^2(\Omega)} + \langle \tilde{E}_n, \varphi\rangle_{\mathsf{L}^2(\Omega)} \to \langle \nabla u, \varphi\rangle_{\mathsf{L}^2(\Omega)} + \langle \tilde{E}, \varphi\rangle_{\mathsf{L}^2(\Omega)}.$$

Then by (2.4)

$$\begin{aligned}\langle E_n, H_n\rangle_{\mathsf{L}^2(\Omega)} &= \langle \nabla u_n, H_n\rangle_{\mathsf{L}^2(\Omega)} + \langle \tilde{E}_n, H_n\rangle_{\mathsf{L}^2(\Omega)} = -\langle u_n, \operatorname{div} H_n\rangle_{\mathsf{L}^2(\Omega)} + \langle \tilde{E}_n, H_n\rangle_{\mathsf{L}^2(\Omega)}\\ &\to -\langle u, \operatorname{div} H\rangle_{\mathsf{L}^2(\Omega)} + \langle \tilde{E}, H\rangle_{\mathsf{L}^2(\Omega)} = \langle \nabla u, H\rangle_{\mathsf{L}^2(\Omega)} + \langle \tilde{E}, H\rangle_{\mathsf{L}^2(\Omega)} = \langle E, H\rangle_{\mathsf{L}^2(\Omega)},\end{aligned}$$

completing the proof. □

Corollary 3.2 (local div-rot-lemma). *Let $(E_n) \subset \mathsf{R}(\Omega)$ and $(H_n) \subset \mathsf{D}(\Omega)$ be two sequences bounded in $\mathsf{R}(\Omega)$ resp. $\mathsf{D}(\Omega)$. Then there exist $E \in \mathsf{R}(\Omega)$ and $H \in \mathsf{D}(\Omega)$ as well as subsequences, again denoted by (E_n) and (H_n), such that $E_n \rightharpoonup E$ in $\mathsf{R}(\Omega)$ and $H_n \rightharpoonup H$ in $\mathsf{D}(\Omega)$ together with the distributional convergence*

$$\forall\, \varphi \in \mathring{\mathsf{C}}^\infty(\Omega) \qquad \langle \varphi\, E_n, H_n\rangle_{\mathsf{L}^2(\Omega)} \to \langle \varphi\, E, H\rangle_{\mathsf{L}^2(\Omega)}.$$

Proof. Let $\Gamma_{\mathsf{t}} := \Gamma$ and hence $\Gamma_{\mathsf{n}} = \emptyset$. $(\varphi\, E_n)$ is bounded in $\mathring{\mathsf{R}}_\Gamma(\Omega)$ and (H_n) is bounded in $\mathsf{D}(\Omega)$. Theorem 3.1 shows the assertion. □

Remark 3.3. *We note that the boundedness of (E_n) and (H_n) in local spaces is sufficient for Corollary 3.2 to hold. Hence, no regularity or boundedness assumptions on Ω are needed, i.e., Ω can be an arbitrary domain in $\mathbb{R}^3$.*

4. Generalizations

The idea of the proof of Theorem 3.1 can be generalized.

4.1. **Functional Analysis Toolbox.** Let $\mathrm{A} : D(\mathrm{A}) \subset \mathsf{H}_1 \to \mathsf{H}_2$ be a (possibly unbounded) closed and densely defined linear operator on two Hilbert spaces H_1 and H_2 with adjoint $\mathrm{A}^* : D(\mathrm{A}^*) \subset \mathsf{H}_2 \to \mathsf{H}_1$. Note $(\mathrm{A}^*)^* = \overline{\mathrm{A}} = \mathrm{A}$, i.e., $(\mathrm{A}, \mathrm{A}^*)$ is a dual pair. By the projection theorem the Helmholtz type decompositions

$$\mathsf{H}_1 = N(\mathrm{A}) \oplus_{\mathsf{H}_1} \overline{R(\mathrm{A}^*)}, \quad \mathsf{H}_2 = N(\mathrm{A}^*) \oplus_{\mathsf{H}_2} \overline{R(\mathrm{A})} \tag{4.1}$$

hold, where we introduce the notation N for the kernel (or null space) and R for the range of a linear operator. We can define the reduced operators

$$\mathcal{A} := \mathrm{A}\,|_{\overline{R(\mathrm{A}^*)}} : D(\mathcal{A}) \subset \overline{R(\mathrm{A}^*)} \to \overline{R(\mathrm{A})}, \qquad D(\mathcal{A}) := D(\mathrm{A}) \cap N(\mathrm{A})^{\perp_{\mathsf{H}_1}} = D(\mathrm{A}) \cap \overline{R(\mathrm{A}^*)},$$

$$\mathcal{A}^* := \mathrm{A}^*\,|_{\overline{R(\mathrm{A})}} : D(\mathcal{A}^*) \subset \overline{R(\mathrm{A})} \to \overline{R(\mathrm{A}^*)}, \qquad D(\mathcal{A}^*) := D(\mathrm{A}^*) \cap N(\mathrm{A}^*)^{\perp_{\mathsf{H}_2}} = D(\mathrm{A}^*) \cap \overline{R(\mathrm{A})},$$

which are also closed and densely defined linear operators. We note that $\mathcal{A}$ and $\mathcal{A}^*$ are indeed adjoint to each other, i.e., $(\mathcal{A}, \mathcal{A}^*)$ is a dual pair as well. Now the inverse operators

$$\mathcal{A}^{-1} : R(\mathrm{A}) \to D(\mathcal{A}), \qquad (\mathcal{A}^*)^{-1} : R(\mathrm{A}^*) \to D(\mathcal{A}^*)$$

exist and they are bijective, since $\mathcal{A}$ and $\mathcal{A}^*$ are injective by definition. Furthermore, by (4.1) we have the refined Helmholtz type decompositions

$$D(\mathrm{A}) = N(\mathrm{A}) \oplus_{\mathsf{H}_1} D(\mathcal{A}), \qquad D(\mathrm{A}^*) = N(\mathrm{A}^*) \oplus_{\mathsf{H}_2} D(\mathcal{A}^*) \tag{4.2}$$

and thus we obtain for the ranges

$$R(\mathrm{A}) = R(\mathcal{A}), \qquad R(\mathrm{A}^*) = R(\mathcal{A}^*). \tag{4.3}$$

By the closed range theorem and the closed graph theorem we get immediately the following.

Lemma 4.1. *The following assertions are equivalent:*

- **(i)** $\exists\, c_\mathrm{A} \in (0,\infty) \quad \forall\, x \in D(\mathcal{A}) \quad |x|_{\mathsf{H}_1} \leq c_\mathrm{A} |\,\mathrm{A}\,x|_{\mathsf{H}_2}$
- **(i*)** $\exists\, c_{\mathrm{A}^*} \in (0,\infty) \quad \forall\, y \in D(\mathcal{A}^*) \quad |y|_{\mathsf{H}_2} \leq c_{\mathrm{A}^*} |\,\mathrm{A}^*\,y|_{\mathsf{H}_1}$
- **(ii)** $R(\mathrm{A}) = R(\mathcal{A})$ *is closed in* H_2.
- **(ii*)** $R(\mathrm{A}^*) = R(\mathcal{A}^*)$ *is closed in* H_1.
- **(iii)** $\mathcal{A}^{-1} : R(\mathrm{A}) \to D(\mathcal{A})$ *is continuous and bijective with norm bounded by* $(1 + c_\mathrm{A}^2)^{1/2}$.
- **(iii*)** $(\mathcal{A}^*)^{-1} : R(\mathrm{A}^*) \to D(\mathcal{A}^*)$ *is continuous and bijective with norm bounded by* $(1 + c_{\mathrm{A}^*}^2)^{1/2}$.

In case that one of the latter assertions is true, e.g., (ii), $R(\mathrm{A})$ is closed, we have

$$\mathsf{H}_1 = N(\mathrm{A}) \oplus_{\mathsf{H}_1} R(\mathrm{A}^*), \qquad \mathsf{H}_2 = N(\mathrm{A}^*) \oplus_{\mathsf{H}_2} R(\mathrm{A}),$$

$$D(\mathrm{A}) = N(\mathrm{A}) \oplus_{\mathsf{H}_1} D(\mathcal{A}), \qquad D(\mathrm{A}^*) = N(\mathrm{A}^*) \oplus_{\mathsf{H}_2} D(\mathcal{A}^*),$$

$$D(\mathcal{A}) = D(\mathrm{A}) \cap R(\mathrm{A}^*), \qquad D(\mathcal{A}^*) = D(\mathrm{A}^*) \cap R(\mathrm{A}),$$

and

$$\mathcal{A} : D(\mathcal{A}) \subset R(\mathrm{A}^*) \to R(\mathrm{A}), \quad \mathcal{A}^* : D(\mathcal{A}^*) \subset R(\mathrm{A}) \to R(\mathrm{A}^*).$$

Remark 4.2. *For the "best" constants c_A, c_{A^*} the following holds: The Rayleigh quotients*

$$\frac{1}{c_\mathrm{A}} := \inf_{0 \neq x \in D(\mathcal{A})} \frac{|\,\mathrm{A}\,x|_{\mathsf{H}_2}}{|x|_{\mathsf{H}_1}}, \quad \frac{1}{c_{\mathrm{A}^*}} := \inf_{0 \neq y \in D(\mathcal{A}^*)} \frac{|\,\mathrm{A}^*\,y|_{\mathsf{H}_1}}{|y|_{\mathsf{H}_2}}$$

coincide, i.e., $c_\mathrm{A} = c_{\mathrm{A}^*} \in (0,\infty]$.

Lemma 4.3. *The following assertions are equivalent:*

- **(i)** $D(\mathcal{A}) \hookrightarrow\!\!\!\!\rightarrow \mathsf{H}_1$ *is compact.*
- **(i*)** $D(\mathcal{A}^*) \hookrightarrow\!\!\!\!\rightarrow \mathsf{H}_2$ *is compact.*
- **(ii)** $\mathcal{A}^{-1} : R(\mathrm{A}) \to R(\mathrm{A}^*)$ *is compact.*
- **(ii*)** $(\mathcal{A}^*)^{-1} : R(\mathrm{A}^*) \to R(\mathrm{A})$ *is compact.*

If one of these assertions holds true, e.g., (i), $D(\mathcal{A}) \hookrightarrow\!\!\!\!\rightarrow \mathsf{H}_1$ *is compact, then the assertions of Lemma 4.1 and Remark 4.2 hold with* $c_{\mathrm{A}} = c_{\mathrm{A}^*} \in (0, \infty)$*. Especially, the Friedrichs/Poincaré type estimates hold, all ranges are closed and the inverse operators*

$$\mathcal{A}^{-1} : R(\mathrm{A}) \to R(\mathrm{A}^*), \quad (\mathcal{A}^*)^{-1} : R(\mathrm{A}^*) \to R(\mathrm{A})$$

are compact with norms $\left|\mathcal{A}^{-1}\right|_{R(\mathrm{A}),R(\mathrm{A}^*)} = \left|(\mathcal{A}^*)^{-1}\right|_{R(\mathrm{A}^*),R(\mathrm{A})} = c_{\mathrm{A}}$.

Proof. As the other assertions are easily proved or immediately clear by symmtery, we just show that (i), i.e., the compactness of

$$D(\mathcal{A}) = D(\mathrm{A}) \cap \overline{R(\mathrm{A}^*)} \hookrightarrow\!\!\!\!\rightarrow \mathsf{H}_1,$$

implies (i*) as well as Lemma 4.1 (i).

(i)⇒Lemma 4.1 (i): For this we use a standard indirect argument. If Lemma 4.1 (i) was wrong, there exists a sequence $(x_n) \subset D(\mathcal{A})$ with $|x_n|_{\mathsf{H}_1} = 1$ and $\mathrm{A}\, x_n \to 0$. As (x_n) is bounded in $D(\mathcal{A})$ we can extract a subsequence, again denoted by (x_n), with $x_n \to x \in \mathsf{H}_1$ in H_1. Since $\mathcal{A}$ is closed, we have $x \in D(\mathcal{A})$ and $\mathrm{A}\, x = 0$, hence $x \in N(\mathcal{A}) = \{0\}$, in contradiction to $1 = |x_n|_{\mathsf{H}_1} \to |x|_{\mathsf{H}_1} = 0$.

(i)⇒(i*): Let $(y_n) \subset D(\mathcal{A}^*)$ be a bounded sequence. Utilizing Lemma 4.1 (i) and (ii) we obtain $D(\mathcal{A}^*) = D(\mathrm{A}^*) \cap R(\mathcal{A})$ and thus $y_n = \mathrm{A}\, x_n$ with $(x_n) \subset D(\mathcal{A})$, which is bounded in $D(\mathcal{A})$ by Lemma 4.1 (i). Hence we may extract a subsequence, again denoted by (x_n), converging in H_1. Therefore with $x_{n,m} := x_n - x_m$ and $y_{n,m} := y_n - y_m$ we see

$$|y_{n,m}|^2_{\mathsf{H}_2} = \big\langle y_{n,m}, \mathrm{A}(x_{n,m})\big\rangle_{\mathsf{H}_2} = \big\langle \mathrm{A}^*(y_{n,m}), x_{n,m}\big\rangle_{\mathsf{H}_1} \leq c\, |x_{n,m}|_{\mathsf{H}_1},$$

and hence (y_n) is a Cauchy sequence in H_2. □

Now, let $\mathrm{A}_0 : D(\mathrm{A}_0) \subset \mathsf{H}_0 \to \mathsf{H}_1$ and $\mathrm{A}_1 : D(\mathrm{A}_1) \subset \mathsf{H}_1 \to \mathsf{H}_2$ be (possibly unbounded) closed and densely defined linear operators on three Hilbert spaces H_0, H_1, and H_2 with adjoints $\mathrm{A}_0^* : D(\mathrm{A}_0^*) \subset \mathsf{H}_1 \to \mathsf{H}_0$ and $\mathrm{A}_1^* : D(\mathrm{A}_1^*) \subset \mathsf{H}_2 \to \mathsf{H}_1$ as well as reduced operators $\mathcal{A}_0$, $\mathcal{A}_0^*$, and $\mathcal{A}_1$, $\mathcal{A}_1^*$. Furthermore, we assume the sequence or complex property of A_0 and A_1, that is, $\mathrm{A}_1\,\mathrm{A}_0 = 0$, i.e.,

$$R(\mathrm{A}_0) \subset N(\mathrm{A}_1). \tag{4.4}$$

Then also $\mathrm{A}_0^*\,\mathrm{A}_1^* = 0$, i.e., $R(\mathrm{A}_1^*) \subset N(\mathrm{A}_0^*)$. From the Helmholtz type decompositions (4.1) for $\mathrm{A} = \mathrm{A}_0$ and $\mathrm{A} = \mathrm{A}_1$ we get in particular

$$\mathsf{H}_1 = \overline{R(\mathrm{A}_0)} \oplus_{\mathsf{H}_1} N(\mathrm{A}_0^*), \qquad \mathsf{H}_1 = \overline{R(\mathrm{A}_1^*)} \oplus_{\mathsf{H}_1} N(\mathrm{A}_1), \tag{4.5}$$

and the following result for Helmholtz type decompositions:

Lemma 4.4. *Let* $N_{0,1} := N(\mathrm{A}_1) \cap N(\mathrm{A}_0^*)$*. The refined Helmholtz type decompositions*

$$N(\mathrm{A}_1) = \overline{R(\mathrm{A}_0)} \oplus_{\mathsf{H}_1} N_{0,1}, \qquad D(\mathrm{A}_1) = \overline{R(\mathrm{A}_0)} \oplus_{\mathsf{H}_1} \big(D(\mathrm{A}_1) \cap N(\mathrm{A}_0^*)\big), \qquad R(\mathrm{A}_0) = R(\mathcal{A}_0), \tag{4.6}$$

$$N(\mathrm{A}_0^*) = \overline{R(\mathrm{A}_1^*)} \oplus_{\mathsf{H}_1} N_{0,1}, \qquad D(\mathrm{A}_0^*) = \overline{R(\mathrm{A}_1^*)} \oplus_{\mathsf{H}_1} \big(D(\mathrm{A}_0^*) \cap N(\mathrm{A}_1)\big), \qquad R(\mathrm{A}_1^*) = R(\mathcal{A}_1^*), \tag{4.7}$$

and

$$\mathsf{H}_1 = \overline{R(\mathrm{A}_0)} \oplus_{\mathsf{H}_1} N_{0,1} \oplus_{\mathsf{H}_1} \overline{R(\mathrm{A}_1^*)} \tag{4.8}$$

hold, which can be further refined and specialized, e.g., to

$$D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) = D(\mathcal{A}_0^*) \oplus_{\mathsf{H}_1} N_{0,1} \oplus_{\mathsf{H}_1} D(\mathcal{A}_1). \tag{4.9}$$

Proof. By (4.5) and the complex properties we see (4.6) and (4.7), yielding directly (4.8) and (4.9). □

We observe

$$D(\mathcal{A}_1) = D(\mathrm{A}_1) \cap \overline{R(\mathrm{A}_1^*)} \subset D(\mathrm{A}_1) \cap N(\mathrm{A}_0^*) \subset D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*),$$
$$D(\mathcal{A}_0^*) = D(\mathrm{A}_0^*) \cap \overline{R(\mathrm{A}_0)} \subset D(\mathrm{A}_0^*) \cap N(\mathrm{A}_1) \subset D(\mathrm{A}_0^*) \cap D(\mathrm{A}_1),$$

and using the refined Helmholtz type decompositions of Lemma 4.4 as well as the results of Lemma 4.1, Lemma 4.3, and Lemma 4.5, we immediately see:

Lemma 4.5. *The following assertions are equivalent:*

- **(i)** $D(\mathcal{A}_0) \hookrightarrow\!\!\!\!\rightarrow \mathsf{H}_0$*,* $D(\mathcal{A}_1) \hookrightarrow\!\!\!\!\rightarrow \mathsf{H}_1$*, and* $N_{0,1} \hookrightarrow\!\!\!\!\rightarrow \mathsf{H}_1$ *are compact.*
- **(ii)** $D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) \hookrightarrow\!\!\!\!\rightarrow \mathsf{H}_1$ *is compact.*

In this case, the cohomology group $N_{0,1}$ *has finite dimension.*

We summarize:

Theorem 4.6. *Let* $D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) \hookrightarrow\!\!\!\!\rightarrow \mathsf{H}_1$ *be compact. Then* $D(\mathcal{A}_0) \hookrightarrow\!\!\!\!\rightarrow \mathsf{H}_0$*,* $D(\mathcal{A}_1) \hookrightarrow\!\!\!\!\rightarrow \mathsf{H}_1$*, and* $D(\mathcal{A}_0^*) \hookrightarrow\!\!\!\!\rightarrow \mathsf{H}_1$*,* $D(\mathcal{A}_1^*) \hookrightarrow\!\!\!\!\rightarrow \mathsf{H}_2$ *are compact, all ranges* $R(\mathrm{A}_0)$*,* $R(\mathrm{A}_0^*)$*, and* $R(\mathrm{A}_1)$*,* $R(\mathrm{A}_1^*)$ *are closed, and the corresponding Friedrichs/Poincaré type estimates hold, i.e. there exists* $c_{\mathrm{A}_0}, c_{\mathrm{A}_1} \in (0,\infty)$ *such that*

$$\begin{aligned} &\forall\, z \in D(\mathcal{A}_0) && |z|_{\mathsf{H}_0} \le c_{\mathrm{A}_0} |\,\mathrm{A}_0\, z|_{\mathsf{H}_1}, \\ &\forall\, x \in D(\mathcal{A}_0^*) && |x|_{\mathsf{H}_1} \le c_{\mathrm{A}_0} |\,\mathrm{A}_0^*\, x|_{\mathsf{H}_0}, \\ &\forall\, x \in D(\mathcal{A}_1) && |x|_{\mathsf{H}_1} \le c_{\mathrm{A}_1} |\,\mathrm{A}_1\, x|_{\mathsf{H}_2}, \\ &\forall\, y \in D(\mathcal{A}_1^*) && |y|_{\mathsf{H}_2} \le c_{\mathrm{A}_1} |\,\mathrm{A}_1^*\, y|_{\mathsf{H}_1}. \end{aligned} \tag{4.10}$$

Moreover, all refined Helmholtz type decompositions of Lemma 4.4 hold with closed ranges, especially

$$D(\mathrm{A}_1) = R(\mathcal{A}_0) \oplus_{\mathsf{H}_1} \big(D(\mathrm{A}_1) \cap N(\mathrm{A}_0^*)\big). \tag{4.11}$$

Proof. Apply the latter lemmas and remarks to $\mathrm{A} = \mathrm{A}_0$ and $\mathrm{A} = \mathrm{A}_1$. □

4.2. **The A_0^*-A_1-Lemma.** Let A_0 and A_1 be as introduced before satisfying the complex property (4.4), i.e., $\mathrm{A}_1\,\mathrm{A}_0 = 0$ or $R(\mathrm{A}_0) \subset N(\mathrm{A}_1)$. In other words, the primal and dual sequences

$$\begin{aligned} D(\mathrm{A}_0) \subset \mathsf{H}_0 \xrightarrow{\ \mathrm{A}_0\ } D(\mathrm{A}_1) \subset \mathsf{H}_1 \xrightarrow{\ \mathrm{A}_1\ } \mathsf{H}_2, \\ \mathsf{H}_0 \xleftarrow{\ \mathrm{A}_0^*\ } D(\mathrm{A}_0^*) \subset \mathsf{H}_1 \xleftarrow{\ \mathrm{A}_1^*\ } D(\mathrm{A}_1^*) \subset \mathsf{H}_2 \end{aligned} \tag{4.12}$$

are Hilbert complexes of closed and densely defined linear operators. The additional assumption of closed ranges $R(\mathrm{A}_0)$ and $R(\mathrm{A}_1)$ $\big($and hence also closed ranges $R(\mathrm{A}_0^*)$ and $R(\mathrm{A}_1^*)\big)$ is equivalent to calling the Hilbert complexes closed. The complexes are exact, if and only if $N_{0,1} = \{0\}$.

As our main result, the following generalized global div-curl-lemma holds.

Theorem 4.7 (A_0^*-A_1-lemma)**.** *Let* $D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) \hookrightarrow\!\!\!\!\rightarrow \mathsf{H}_1$ *be compact. Moreover, let* $(x_n) \subset D(\mathrm{A}_1)$ *and* $(y_n) \subset D(\mathrm{A}_0^*)$ *be two sequences bounded in* $D(\mathrm{A}_1)$ *resp.* $D(\mathrm{A}_0^*)$*. Then there exist* $x \in D(\mathrm{A}_1)$ *and* $y \in D(\mathrm{A}_0^*)$ *as well as subsequences, again denoted by* (x_n) *and* (y_n)*, such that* (x_n) *and* (y_n) *converge weakly in* $D(\mathrm{A}_1)$ *resp.* $D(\mathrm{A}_0^*)$ *to* x *resp.* y *together with the convergence of the inner products*

$$\langle x_n, y_n\rangle_{\mathsf{H}_1} \to \langle x, y\rangle_{\mathsf{H}_1}.$$

Proof. Note that Theorem 4.6 can be applied. We pick subsequences, again denoted by (x_n) and (y_n), such that (x_n) and (y_n) converge weakly in $D(\mathrm{A}_1)$ resp. $D(\mathrm{A}_0^*)$ to $x \in D(\mathrm{A}_1)$ resp. $y \in D(\mathrm{A}_0^*)$. By (4.11) we get the orthogonal decomposition

$$D(\mathrm{A}_1) \ni x_n = \mathrm{A}_0\, z_n + \tilde{x}_n, \quad z_n \in D(\mathcal{A}_0), \quad \tilde{x}_n \in D(\mathrm{A}_1) \cap N(\mathrm{A}_0^*).$$

(z_n) is bounded in $D(\mathcal{A}_0)$ by orthogonality and the Friedrichs/Poincaré tye estimate (4.10). $(\tilde{x}_n)$ is bounded in $D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*)$ by orthogonality and $\mathrm{A}_1\,\tilde{x}_n = \mathrm{A}_1\, x_n$, $\mathrm{A}_0^*\,\tilde{x}_n = 0$. Using the compact embeddings $D(\mathcal{A}_0) \hookrightarrow\!\!\!\!\rightarrow \mathsf{H}_0$ and $D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) \hookrightarrow\!\!\!\!\rightarrow \mathsf{H}_1$, there exist $z \in D(\mathcal{A}_0)$ and $\tilde{x} \in D(\mathrm{A}_1) \cap N(\mathrm{A}_0^*)$ and we can extract two subsequences, again denoted by (z_n) and $(\tilde{x}_n)$, such that $z_n \rightharpoonup z$ in $D(\mathrm{A}_0)$ and $z_n \to z$ in

H_0 as well as $\tilde{x}_n \rightharpoonup \tilde{x}$ in $D(\mathrm{A}_1)\cap D(\mathrm{A}_0^*)$ and $\tilde{x}_n \to \tilde{x}$ in H_1. We have $x = \mathrm{A}_0\, z + \tilde{x}$, giving the Helmholtz type decomposition for x, as, e.g., for all $\varphi \in \mathsf{H}_1$

$$\langle x, \varphi\rangle_{\mathsf{H}_1} \leftarrow \langle x_n, \varphi\rangle_{\mathsf{H}_1} = \langle \mathrm{A}_0\, z_n, \varphi\rangle_{\mathsf{H}_1} + \langle \tilde{x}_n, \varphi\rangle_{\mathsf{H}_1} \to \langle \mathrm{A}_0\, z, \varphi\rangle_{\mathsf{H}_1} + \langle \tilde{x}, \varphi\rangle_{\mathsf{H}_1}.$$

Finally, we see

$$\begin{aligned}\langle x_n, y_n\rangle_{\mathsf{H}_1} &= \langle \mathrm{A}_0\, z_n, y_n\rangle_{\mathsf{H}_1} + \langle \tilde{x}_n, y_n\rangle_{\mathsf{H}_1} = \langle z_n, \mathrm{A}_0^*\, y_n\rangle_{\mathsf{H}_0} + \langle \tilde{x}_n, y_n\rangle_{\mathsf{H}_1}\\ &\to \langle z, \mathrm{A}_0^*\, y\rangle_{\mathsf{H}_0} + \langle \tilde{x}, y\rangle_{\mathsf{H}_1} = \langle \mathrm{A}_0\, z, y\rangle_{\mathsf{H}_1} + \langle \tilde{x}, y\rangle_{\mathsf{H}_1} = \langle x, y\rangle_{\mathsf{H}_1},\end{aligned}$$

completing the proof. □

Remark 4.8. *By Lemma 4.5 the crucial assumption, i.e., $D(\mathrm{A}_1)\cap D(\mathrm{A}_0^*) \hookrightarrow\!\!\!\!\rightarrow \mathsf{H}_1$ is compact, holds, if and only if $D(\mathcal{A}_0) \hookrightarrow\!\!\!\!\rightarrow \mathsf{H}_0$, $D(\mathcal{A}_1) \hookrightarrow\!\!\!\!\rightarrow \mathsf{H}_1$ are compact and $N_{0,1}$ is finite dimensional. Moreover, as Banach space adjoints we have*

$$\mathsf{H}_0 \cong \mathsf{H}_0' \hookrightarrow\!\!\!\!\rightarrow D(\mathcal{A}_0)' \quad\Leftrightarrow\quad D(\mathcal{A}_0) \hookrightarrow\!\!\!\!\rightarrow \mathsf{H}_0 \quad\Leftrightarrow\quad D(\mathcal{A}_0^*) \hookrightarrow\!\!\!\!\rightarrow \mathsf{H}_1 \quad\Leftrightarrow\quad \mathsf{H}_1 \cong \mathsf{H}_1' \hookrightarrow\!\!\!\!\rightarrow D(\mathcal{A}_0^*)',$$

and

$$\mathsf{H}_1 \cong \mathsf{H}_1' \hookrightarrow\!\!\!\!\rightarrow D(\mathcal{A}_1)' \quad\Leftrightarrow\quad D(\mathcal{A}_1) \hookrightarrow\!\!\!\!\rightarrow \mathsf{H}_1 \quad\Leftrightarrow\quad D(\mathcal{A}_1^*) \hookrightarrow\!\!\!\!\rightarrow \mathsf{H}_2 \quad\Leftrightarrow\quad \mathsf{H}_2 \cong \mathsf{H}_2' \hookrightarrow\!\!\!\!\rightarrow D(\mathcal{A}_1^*)'.$$

Especially, the assumption that $D(\mathrm{A}_1)\cap D(\mathrm{A}_0^*) \hookrightarrow\!\!\!\!\rightarrow \mathsf{H}_1$ is compact, is equivalent to $\dim N_{0,1} < \infty$ *and*

$$\mathsf{H}_0 \hookrightarrow\!\!\!\!\rightarrow D(\mathcal{A}_0)', \qquad \mathsf{H}_2 \hookrightarrow\!\!\!\!\rightarrow D(\mathcal{A}_1^*)'$$

are compact. Thus, we observe that Theorem 4.7 is equivalent to [26, Theorem 2.5] *and that the assumptions of Theorem 4.7 are practically equivalent to those of* [26, Theorem 2.4].

5. Applications

Whenever closed Hilbert complexes like (4.12) together with the corresponding compact embedding $D(\mathrm{A}_1)\cap D(\mathrm{A}_0^*) \hookrightarrow\!\!\!\!\rightarrow \mathsf{H}_1$ occur, we can apply the general A_0^*-A_1-lemma, i.e., Theorem 4.7. In three dimensions we typically have three closed and densely defined linear operators A_0, A_1, and A_2, satisfying the complex properties $R(\mathrm{A}_0) \subset N(\mathrm{A}_1)$ and $R(\mathrm{A}_1) \subset N(\mathrm{A}_2)$, i.e.,

$$\begin{gathered} D(\mathrm{A}_0) \subset \mathsf{H}_0 \xrightarrow{\ \mathrm{A}_0\ } D(\mathrm{A}_1) \subset \mathsf{H}_1 \xrightarrow{\ \mathrm{A}_1\ } D(\mathrm{A}_2) \subset \mathsf{H}_2 \xrightarrow{\ \mathrm{A}_2\ } \mathsf{H}_3,\\ \mathsf{H}_0 \xleftarrow{\ \mathrm{A}_0^*\ } D(\mathrm{A}_0^*) \subset \mathsf{H}_1 \xleftarrow{\ \mathrm{A}_1^*\ } D(\mathrm{A}_1^*) \subset \mathsf{H}_2 \xleftarrow{\ \mathrm{A}_2^*\ } D(\mathrm{A}_2^*) \subset \mathsf{H}_3, \end{gathered} \tag{5.1}$$

together with the crucial compact embeddings

$$D(\mathrm{A}_1)\cap D(\mathrm{A}_0^*) \hookrightarrow\!\!\!\!\rightarrow \mathsf{H}_1, \qquad D(\mathrm{A}_2)\cap D(\mathrm{A}_1^*) \hookrightarrow\!\!\!\!\rightarrow \mathsf{H}_2. \tag{5.2}$$

Let us recall our general assumptions on the underlying domain from Section 2: In the following examples we suppose that $\Omega \subset \mathbb{R}^3$ is a bounded weak Lipschitz domain, see [3, Definition 2.3], with boundary Γ, which is divided into two relatively open weak Lipschitz subsets Γ_{t} and $\Gamma_{\mathsf{n}} := \Gamma \setminus \overline{\Gamma_{\mathsf{t}}}$, see [3, Definition 2.5].

5.1. **The div-rot-Lemma Revisited.** The first and most canonical example is given by the classical operators from vector analysis

$$\begin{aligned} \mathrm{A}_0 &:= \mathring{\nabla}_{\Gamma_{\mathsf{t}}} : \mathring{\mathsf{H}}^1_{\Gamma_{\mathsf{t}}}(\Omega) \subset \mathsf{L}^2(\Omega) \longrightarrow \mathsf{L}^2_\epsilon(\Omega); && u \mapsto \nabla u,\\ \mathrm{A}_1 &:= \mu^{-1}\mathring{\mathrm{rot}}_{\Gamma_{\mathsf{t}}} : \mathring{\mathsf{R}}_{\Gamma_{\mathsf{t}}}(\Omega) \subset \mathsf{L}^2_\epsilon(\Omega) \longrightarrow \mathsf{L}^2_\mu(\Omega); && E \mapsto \mu^{-1}\operatorname{rot} E,\\ \mathrm{A}_2 &:= \mathring{\mathrm{div}}_{\Gamma_{\mathsf{t}}}\mu : \mu^{-1}\mathring{\mathsf{D}}_{\Gamma_{\mathsf{t}}}(\Omega) \subset \mathsf{L}^2_\mu(\Omega) \longrightarrow \mathsf{L}^2(\Omega); && H \mapsto \operatorname{div} \mu H. \end{aligned}$$

A_0, A_1, and A_2 are unbounded, densely defined, and closed linear operators with adjoints

$$\begin{aligned} \mathrm{A}_0^* &= \mathring{\nabla}_{\Gamma_{\mathsf{t}}}^* = -\mathring{\mathrm{div}}_{\Gamma_{\mathsf{n}}}\epsilon : \epsilon^{-1}\mathring{\mathsf{D}}_{\Gamma_{\mathsf{n}}}(\Omega) \subset \mathsf{L}^2_\epsilon(\Omega) \longrightarrow \mathsf{L}^2(\Omega); && H \mapsto -\operatorname{div} \epsilon H,\\ \mathrm{A}_1^* &= (\mu^{-1}\mathring{\mathrm{rot}}_{\Gamma_{\mathsf{t}}})^* = \epsilon^{-1}\mathring{\mathrm{rot}}_{\Gamma_{\mathsf{n}}} : \mathring{\mathsf{R}}_{\Gamma_{\mathsf{n}}}(\Omega) \subset \mathsf{L}^2_\mu(\Omega) \longrightarrow \mathsf{L}^2_\epsilon(\Omega); && E \mapsto \epsilon^{-1}\operatorname{rot} E,\\ \mathrm{A}_2^* &= (\mathring{\mathrm{div}}_{\Gamma_{\mathsf{t}}}\mu)^* = -\mathring{\nabla}_{\Gamma_{\mathsf{n}}} : \mathring{\mathsf{H}}^1_{\Gamma_{\mathsf{n}}}(\Omega) \subset \mathsf{L}^2(\Omega) \longrightarrow \mathsf{L}^2_\mu(\Omega); && u \mapsto -\nabla u. \end{aligned}$$

Here, $\epsilon, \mu : \Omega \to \mathbb{R}^{3\times 3}$ are symmetric and uniformly positive definite $\mathsf{L}^\infty(\Omega)$-tensor fields. The complex properties hold as

$$R(\mathrm{A}_0) = \mathring{\nabla}_{\Gamma_t}\mathring{\mathsf{H}}^1_{\Gamma_t}(\Omega) \subset \mathring{\mathsf{R}}_{\Gamma_t,0}(\Omega) = N(\mathrm{A}_1), \qquad R(\mathrm{A}_1^*) = \epsilon^{-1}\mathring{\mathrm{rot}}_{\Gamma_n}\mathring{\mathsf{R}}_{\Gamma_n}(\Omega) \subset \epsilon^{-1}\mathring{\mathsf{D}}_{\Gamma_n,0}(\Omega) = N(\mathrm{A}_0^*),$$
$$R(\mathrm{A}_1) = \mu^{-1}\mathring{\mathrm{rot}}_{\Gamma_t}\mathring{\mathsf{R}}_{\Gamma_t}(\Omega) \subset \mu^{-1}\mathring{\mathsf{D}}_{\Gamma_t,0}(\Omega) = N(\mathrm{A}_2), \qquad R(\mathrm{A}_2^*) = \mathring{\nabla}_{\Gamma_n}\mathring{\mathsf{H}}^1_{\Gamma_n}(\Omega) \subset \mathring{\mathsf{R}}_{\Gamma_n,0}(\Omega) = N(\mathrm{A}_1^*).$$

Hence, the sequences (5.1) read

$$\mathring{\mathsf{H}}^1_{\Gamma_t}(\Omega) \subset \mathsf{L}^2(\Omega) \xrightarrow{\mathrm{A}_0=\mathring{\nabla}_{\Gamma_t}} \mathring{\mathsf{R}}_{\Gamma_t}(\Omega) \subset \mathsf{L}^2_\epsilon(\Omega) \xrightarrow{\mathrm{A}_1=\mu^{-1}\mathring{\mathrm{rot}}_{\Gamma_t}} \mu^{-1}\mathring{\mathsf{D}}_{\Gamma_t}(\Omega) \subset \mathsf{L}^2_\mu(\Omega) \xrightarrow{\mathrm{A}_2=\mathring{\mathrm{div}}_{\Gamma_t}\mu} \mathsf{L}^2(\Omega),$$
$$\mathsf{L}^2(\Omega) \xleftarrow{\mathrm{A}_0^*=-\mathring{\mathrm{div}}_{\Gamma_n}\epsilon} \epsilon^{-1}\mathring{\mathsf{D}}_{\Gamma_n}(\Omega) \subset \mathsf{L}^2_\epsilon(\Omega) \xleftarrow{\mathrm{A}_1^*=\epsilon^{-1}\mathring{\mathrm{rot}}_{\Gamma_n}} \mathring{\mathsf{R}}_{\Gamma_n}(\Omega) \subset \mathsf{L}^2_\mu(\Omega) \xleftarrow{\mathrm{A}_2^*=-\mathring{\nabla}_{\Gamma_n}} \mathring{\mathsf{H}}^1_{\Gamma_n}(\Omega) \subset \mathsf{L}^2(\Omega).$$

These are the well known Hilbert complexes for electro-magnetics, which are also known as de Rham complexes. Moreover, the crucial embeddings (5.2), i.e.,

$$D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) = \mathring{\mathsf{R}}_{\Gamma_t}(\Omega) \cap \epsilon^{-1}\mathring{\mathsf{D}}_{\Gamma_n}(\Omega) \hookrightarrow\!\!\!\!\rightarrow \mathsf{L}^2(\Omega), \quad D(\mathrm{A}_2) \cap D(\mathrm{A}_1^*) = \mu^{-1}\mathring{\mathsf{D}}_{\Gamma_t}(\Omega) \cap \mathring{\mathsf{R}}_{\Gamma_n}(\Omega) \hookrightarrow\!\!\!\!\rightarrow \mathsf{L}^2(\Omega),$$

are compact by Weck's selection theorem (2.1), see [3, Theorem 4.7]. Indeed, Weck's selection theorems are independent of the material law tensors ϵ or μ. Choosing the pair $(\mathrm{A}_0, \mathrm{A}_1)$ we get by Theorem 4.7 the following:

Theorem 5.1 (global div ϵ-μ^{-1} rot-lemma). *Let $(E_n) \subset \mathring{\mathsf{R}}_{\Gamma_t}(\Omega)$ and $(H_n) \subset \epsilon^{-1}\mathring{\mathsf{D}}_{\Gamma_n}(\Omega)$ be two sequences bounded in $\mathsf{R}(\Omega)$ resp. $\epsilon^{-1}\mathsf{D}(\Omega)$. Then there exist $E \in \mathring{\mathsf{R}}_{\Gamma_t}(\Omega)$ and $H \in \epsilon^{-1}\mathring{\mathsf{D}}_{\Gamma_n}(\Omega)$ as well as subsequences, again denoted by (E_n) and (H_n), such that (E_n) and (H_n) converge weakly in $\mathsf{R}(\Omega)$ resp. $\epsilon^{-1}\mathsf{D}(\Omega)$ to E resp. H together with the convergence of the inner products*

$$\langle E_n, H_n\rangle_{\mathsf{L}^2_\epsilon(\Omega)} \to \langle E, H\rangle_{\mathsf{L}^2_\epsilon(\Omega)}.$$

Remark 5.2. *We note:*

- **(i)** *Considering (E_n) and (ϵH_n) shows that Theorem 5.1 is equivalent to the global* div-curl-*lemma Theorem 3.1.*
- **(ii)** *Theorem 5.1 has a corresponding local version similar to the local* div-curl-*lemma Corollary 3.2 and Remark 3.3, which holds with no regularity or boundedness assumptions on Ω.*
- **(iii)** *Choosing the pair $(\mathrm{A}_1, \mathrm{A}_2)$ we get by Theorem 4.7 a permutation of Theorem 5.1, shortly stating, that for bounded sequences $(E_n) \subset \mu^{-1}\mathring{\mathsf{D}}_{\Gamma_t}(\Omega)$ and $(H_n) \subset \mathring{\mathsf{R}}_{\Gamma_n}(\Omega)$ it holds (after picking subsequences) $\langle E_n, H_n\rangle_{\mathsf{L}^2_\mu(\Omega)} \to \langle E, H\rangle_{\mathsf{L}^2_\mu(\Omega)}$.*

Other examples are the following:

5.2. **Generalized Electro-Magnetics.** Let us allow $\Omega \subset \mathbb{R}^N$ or even to be a smooth Riemannian manifold with Lipschitz boundary and interface submanifolds Γ, and Γ_t, Γ_n. Using the calculus of alternating differential q-forms, we define the exterior derivative d and co-derivative $\delta = \pm * \mathrm{d} *$ as Sobolev derivatives in the distributional weak sense by

$$\mathsf{D}^q(\Omega) := \big\{E \in \mathsf{L}^{2,q}(\Omega) \,:\, \mathrm{d}\,E \in \mathsf{L}^{2,q+1}(\Omega)\big\}, \quad \Delta^{q+1}(\Omega) := \big\{H \in \mathsf{L}^{2,q+1}(\Omega) \,:\, \delta\,H \in \mathsf{L}^{2,q}(\Omega)\big\}.$$

To introduce boundary conditions we define

$$\mathring{\mathrm{d}}^q_{\Gamma_t} : \mathring{\mathsf{D}}^q_{\Gamma_t}(\Omega) := \overline{\mathring{\mathsf{C}}^{\infty,q}_{\Gamma_t}(\Omega)}^{\mathsf{D}^q(\Omega)} \subset \mathsf{L}^{2,q}(\Omega) \longrightarrow \mathsf{L}^{2,q+1}(\Omega); \qquad E \mapsto \mathrm{d}\,E$$

as closure of the classical exterior derivative d acting on test q-forms. $\mathring{\mathrm{d}}^q_{\Gamma_t}$ is an unbounded, densely defined, and closed linear operator with adjoint

$$(\mathring{\mathrm{d}}^q_{\Gamma_t})^* = -\mathring{\delta}^{q+1}_{\Gamma_n} : \mathring{\Delta}^{q+1}_{\Gamma_n}(\Omega) := \overline{\mathring{\mathsf{C}}^{\infty,q+1}_{\Gamma_n}(\Omega)}^{\Delta^{q+1}(\Omega)} \subset \mathsf{L}^{2,q+1}(\Omega) \longrightarrow \mathsf{L}^{2,q}(\Omega); \qquad H \mapsto -\,\delta\,H.$$

Let us introduce

$$\mathrm{A}_0 := \mathring{\mathrm{d}}^{q-1}_{\Gamma_t}, \quad \mathrm{A}_1 := \mathring{\mathrm{d}}^q_{\Gamma_t}, \qquad \mathrm{A}_0^* = \mathring{\delta}^q_{\Gamma_n}, \quad \mathrm{A}_1^* = \mathring{\delta}^{q+1}_{\Gamma_n}.$$

The complex properties hold as, e.g.,

$$R(\mathrm{A}_0)=\mathring{\mathrm{d}}^{q-1}_{\Gamma_t}\mathring{\mathsf{D}}^{q-1}_{\Gamma_t}(\Omega)\subset\mathring{\mathsf{D}}^{q}_{\Gamma_t,0}(\Omega)=N(\mathrm{A}_1),\qquad R(\mathrm{A}_1^*)=\mathring{\delta}^{q+1}_{\Gamma_n}\mathring{\Delta}^{q+1}_{\Gamma_n}(\Omega)\subset\mathring{\Delta}^{q}_{\Gamma_n,0}(\Omega)=N(\mathrm{A}_0^*)$$

by the classical properties $\delta\,\delta=\pm*\mathrm{d}\,\mathrm{d}*=0$. Hence, the sequences (5.1) read

$$\mathring{\mathsf{D}}^{q-1}_{\Gamma_t}(\Omega)\subset\mathsf{L}^{2,q-1}(\Omega)\xrightarrow{\mathrm{A}_0=\mathring{\mathrm{d}}^{q-1}_{\Gamma_t}}\mathring{\mathsf{D}}^{q}_{\Gamma_t}(\Omega)\subset\mathsf{L}^{2,q}(\Omega)\xrightarrow{\mathrm{A}_1=\mathring{\mathrm{d}}^{q}_{\Gamma_t}}\mathsf{L}^{2,q+1}(\Omega),$$

$$\mathsf{L}^{2,q-1}(\Omega)\xleftarrow{\mathrm{A}_0^*=\mathring{\delta}^{q}_{\Gamma_n}}\mathring{\Delta}^{q}_{\Gamma_n}(\Omega)\subset\mathsf{L}^{2,q}(\Omega)\xleftarrow{\mathrm{A}_1^*=\mathring{\delta}^{q+1}_{\Gamma_n}}\mathring{\Delta}^{q+1}_{\Gamma_n}(\Omega)\subset\mathsf{L}^{2,q+1}(\Omega),$$

which are the well known Hilbert complexes for generalized electro-magnetics, i.e., the de Rham complexes. Moreover, the crucial embedding (5.2), i.e.,

$$D(\mathrm{A}_1)\cap D(\mathrm{A}_0^*)=\mathring{\mathsf{D}}^{q}_{\Gamma_t}(\Omega)\cap\mathring{\Delta}^{q}_{\Gamma_n}(\Omega)\hookrightarrow\!\!\!\rightarrow\mathsf{L}^2(\Omega),$$

is compact by a generalization of Weck's selection theorem (2.1), see [4] or the fundamental papers of Weck [28] and Picard [18] for full boundary conditions. Indeed, Weck's selection theorems are independent of the material law tensors ϵ or μ. Theorem 4.7 shows the following result:

Theorem 5.3 (global δ-d-lemma). *Let $(E_n)\subset\mathring{\mathsf{D}}^{q}_{\Gamma_t}(\Omega)$ and $(H_n)\subset\mathring{\Delta}^{q}_{\Gamma_n}(\Omega)$ be two sequences bounded in $\mathsf{D}^q(\Omega)$ resp. $\Delta^q(\Omega)$. Then there exist $E\in\mathring{\mathsf{D}}^{q}_{\Gamma_t}(\Omega)$ and $H\in\mathring{\Delta}^{q}_{\Gamma_n}(\Omega)$ as well as subsequences, again denoted by (E_n) and (H_n), such that (E_n) and (H_n) converge weakly in $\mathsf{D}^q(\Omega)$ resp. $\Delta^q(\Omega)$ to E resp. H together with the convergence of the inner products*

$$\langle E_n,H_n\rangle_{\mathsf{L}^{2,q}(\Omega)}\to\langle E,H\rangle_{\mathsf{L}^{2,q}(\Omega)}.$$

Remark 5.4. *We note:*

- **(i)** *For $N=3$ and $q=1$ (resp. $q=2$) we obtain by Theorem 5.3 again the global* div-curl-*lemma Theorem 3.1.*
- **(ii)** *For $q=0$ (resp. $q=N$) as well as identifying $\mathring{\mathrm{d}}^0_{\Gamma_t}=\mathring{\nabla}_{\Gamma_t}$ and $\mathring{\Delta}^0_{\Gamma_n}(\Omega)=0$ (resp. $\mathring{\mathrm{d}}^N_{\Gamma_t}=0$ and $\mathring{\Delta}^N_{\Gamma_n}(\Omega)=\mathring{\nabla}_{\Gamma_n}$) we get by Theorem 5.3 the following trivial (by Rellich's selection theorem) result: For all bounded sequences $(u_n)\subset\mathring{\mathsf{H}}^1_{\Gamma_t}(\Omega)$ and $(v_n)\subset\mathsf{L}^2(\Omega)$ there exist $u\in\mathring{\mathsf{H}}^1_{\Gamma_t}(\Omega)$ and $v\in\mathsf{L}^2(\Omega)$ as well as subsequences, again denoted by (u_n) and (v_n), such that (u_n) and (v_n) converge weakly in $\mathring{\mathsf{H}}^1_{\Gamma_t}(\Omega)$ resp. $\mathsf{L}^2(\Omega)$ to u resp. v together with the convergence of the inner products $\langle u_n,v_n\rangle_{\mathsf{L}^2(\Omega)}\to\langle u,v\rangle_{\mathsf{L}^2(\Omega)}$.*
- **(iii)** *Theorem 5.3 has a corresponding local version similar to the local* div-curl-*lemma Corollary 3.2 and Remark 3.3, which holds with no regularity or boundedness assumptions on Ω.*

5.3. **Biharmonic Equation and General Relativity, Gravitational Waves.** Let $\Omega\subset\mathbb{R}^3$ enjoy our general assumptions. We introduce symmetric and deviatoric (trace-free) square integrable tensor fields in $\mathsf{L}^2(\Omega;\mathbb{S})$ resp. $\mathsf{L}^2(\Omega;\mathbb{T})$ and as closures of the Hessian $\nabla\nabla$, and Rot, Div (row-wise rot, div), applied to test functions resp. test tensor fields, the linear operators

$$\mathrm{A}_0:=\nabla\mathring{\nabla}_{\Gamma_t}:\mathring{\mathsf{H}}^2_{\Gamma_t}(\Omega):=\overline{\mathring{\mathsf{C}}^\infty_{\Gamma_t}(\Omega)}^{\mathsf{H}^2(\Omega)}\subset\mathsf{L}^2(\Omega)\longrightarrow\mathsf{L}^2(\Omega;\mathbb{S});\qquad u\mapsto\nabla\nabla u,$$

$$\mathrm{A}_1:=\mathring{\mathrm{Rot}}_{\mathbb{S},\Gamma_t}:\mathring{\mathsf{R}}_{\Gamma_t}(\Omega;\mathbb{S}):=\overline{\mathring{\mathsf{C}}^\infty_{\Gamma_t}(\Omega;\mathbb{S})}^{\mathsf{R}(\Omega)}\subset\mathsf{L}^2(\Omega;\mathbb{S})\longrightarrow\mathsf{L}^2(\Omega;\mathbb{T});\qquad S\mapsto\mathrm{Rot}\,S,$$

$$\mathrm{A}_2:=\mathring{\mathrm{Div}}_{\mathbb{T},\Gamma_t}:\mathring{\mathsf{D}}_{\Gamma_t}(\Omega;\mathbb{T}):=\overline{\mathring{\mathsf{C}}^\infty_{\Gamma_t}(\Omega;\mathbb{T})}^{\mathsf{D}(\Omega)}\subset\mathsf{L}^2(\Omega;\mathbb{T})\longrightarrow\mathsf{L}^2(\Omega);\qquad T\mapsto\mathrm{Div}\,T.$$

A_0, A_1, and A_2 are unbounded, densely defined, and closed linear operators with adjoints

$$\mathrm{A}_0^*=(\nabla\mathring{\nabla}_{\Gamma_t})^*=\mathrm{div}\,\mathring{\mathrm{Div}}_{\mathbb{S},\Gamma_n}:\mathring{\mathsf{DD}}_{\Gamma_n}(\Omega;\mathbb{S}):=\overline{\mathring{\mathsf{C}}^\infty_{\Gamma_n}(\Omega;\mathbb{S})}^{\mathsf{DD}(\Omega)}\subset\mathsf{L}^2(\Omega;\mathbb{S})\longrightarrow\mathsf{L}^2(\Omega);\qquad S\mapsto\mathrm{div}\,\mathrm{Div}\,S,$$

$$\mathrm{A}_1^*=\mathring{\mathrm{Rot}}^*_{\mathbb{S},\Gamma_t}=\mathrm{sym}\,\mathring{\mathrm{Rot}}_{\mathbb{T},\Gamma_n}:\mathring{\mathsf{R}}_{\mathrm{sym},\Gamma_n}(\Omega;\mathbb{T}):=\overline{\mathring{\mathsf{C}}^\infty_{\Gamma_n}(\Omega;\mathbb{T})}^{\mathsf{R}_{\mathrm{sym}}(\Omega)}\subset\mathsf{L}^2(\Omega;\mathbb{T})\longrightarrow\mathsf{L}^2(\Omega;\mathbb{S});\qquad T\mapsto\mathrm{sym}\,\mathrm{Rot}\,T,$$

$$\mathrm{A}_2^*=\mathring{\mathrm{Div}}^*_{\mathbb{T},\Gamma_t}=-\,\mathrm{dev}\,\mathring{\nabla}_{\Gamma_n}:\mathring{\mathsf{H}}^1_{\Gamma_n}(\Omega)\subset\mathsf{L}^2(\Omega)\longrightarrow\mathsf{L}^2(\Omega;\mathbb{T});\qquad v\mapsto-\,\mathrm{dev}\,\nabla v,$$

see [17] for details. Note that u, v, and S, T are scalar, vector, and tensor (matrix) fields, respectively. The complex properties hold as

$$R(\mathrm{A}_0) = \nabla\mathring{\nabla}_{\Gamma_t}\mathring{\mathsf{H}}^2_{\Gamma_t}(\Omega) \subset \mathring{\mathsf{R}}_{\Gamma_t,0}(\Omega;\mathbb{S}) = N(\mathrm{A}_1),$$
$$R(\mathrm{A}_1^*) = \operatorname{sym}\mathring{\operatorname{Rot}}_{\mathbb{T},\Gamma_n}\mathring{\mathsf{R}}_{\mathrm{sym},\Gamma_n}(\Omega;\mathbb{T}) \subset \mathring{\mathsf{DD}}_{\Gamma_n,0}(\Omega;\mathbb{S}) = N(\mathrm{A}_0^*),$$
$$R(\mathrm{A}_1) = \mathring{\operatorname{Rot}}_{\mathbb{S},\Gamma_t}\mathring{\mathsf{R}}_{\Gamma_t}(\Omega;\mathbb{S}) \subset \mathring{\mathsf{D}}_{\Gamma_t,0}(\Omega;\mathbb{T}) = N(\mathrm{A}_2),$$
$$R(\mathrm{A}_2^*) = \operatorname{dev}\mathring{\nabla}_{\Gamma_n}\mathring{\mathsf{H}}^1_{\Gamma_n}(\Omega) \subset \mathring{\mathsf{R}}_{\mathrm{sym},\Gamma_n,0}(\Omega;\mathbb{T}) = N(\mathrm{A}_1^*).$$

The sequences (5.1) read

$$\mathring{\mathsf{H}}^2_{\Gamma_t}(\Omega) \subset \mathsf{L}^2(\Omega) \xrightarrow{\mathrm{A}_0=\nabla\mathring{\nabla}_{\Gamma_t}} \mathring{\mathsf{R}}_{\Gamma_t}(\Omega;\mathbb{S}) \subset \mathsf{L}^2(\Omega;\mathbb{S}) \xrightarrow{\mathrm{A}_1=\mathring{\operatorname{Rot}}_{\mathbb{S},\Gamma_t}} \mathring{\mathsf{D}}_{\Gamma_t}(\Omega;\mathbb{T}) \subset \mathsf{L}^2(\Omega;\mathbb{T}) \xrightarrow{\mathrm{A}_2=\mathring{\operatorname{Div}}_{\mathbb{T},\Gamma_t}} \mathsf{L}^2(\Omega),$$

$$\mathsf{L}^2(\Omega) \xleftarrow{\mathrm{A}_0^*=\operatorname{div}\mathring{\operatorname{Div}}_{\mathbb{S},\Gamma_n}} \mathring{\mathsf{DD}}_{\Gamma_n}(\Omega;\mathbb{S}) \subset \mathsf{L}^2(\Omega;\mathbb{S}) \xleftarrow{\mathrm{A}_1^*=\operatorname{sym}\mathring{\operatorname{Rot}}_{\mathbb{T},\Gamma_n}} \mathring{\mathsf{R}}_{\mathrm{sym},\Gamma_n}(\Omega;\mathbb{T}) \subset \mathsf{L}^2(\Omega;\mathbb{T}) \xleftarrow{\mathrm{A}_2^*=-\operatorname{dev}\mathring{\nabla}_{\Gamma_n}} \mathring{\mathsf{H}}^1_{\Gamma_n}(\Omega) \subset \mathsf{L}^2(\Omega).$$

These are the so called Grad grad and div Div complexes, appearing, e.g., in the biharmonic problem or general relativity, see [17] for details. The crucial embeddings (5.2), i.e.,

$$D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) = \mathring{\mathsf{R}}_{\Gamma_t}(\Omega;\mathbb{S}) \cap \mathring{\mathsf{DD}}_{\Gamma_n}(\Omega;\mathbb{S}) \hookrightarrow\!\!\!\!\to \mathsf{L}^2(\Omega;\mathbb{S}),$$
$$D(\mathrm{A}_2) \cap D(\mathrm{A}_1^*) = \mathring{\mathsf{D}}_{\Gamma_t}(\Omega;\mathbb{T}) \cap \mathring{\mathsf{R}}_{\mathrm{sym},\Gamma_n}(\Omega;\mathbb{T}) \hookrightarrow\!\!\!\!\to \mathsf{L}^2(\Omega;\mathbb{T}),$$

are compact by [17, Lemma 3.22]. Choosing the pair $(\mathrm{A}_0, \mathrm{A}_1)$ we get by Theorem 4.7 the following:

Theorem 5.5 (global div Div-Rot-$\mathbb{S}$-lemma)**.** *Let $(S_n) \subset \mathring{\mathsf{R}}_{\Gamma_t}(\Omega;\mathbb{S})$ and $(T_n) \subset \mathring{\mathsf{DD}}_{\Gamma_n}(\Omega;\mathbb{S})$ be two sequences bounded in $\mathsf{R}(\Omega)$ resp. $\mathsf{DD}(\Omega)$. Then there exist $S \in \mathring{\mathsf{R}}_{\Gamma_t}(\Omega;\mathbb{S})$ and $T \in \mathring{\mathsf{DD}}_{\Gamma_n}(\Omega;\mathbb{S})$ as well as subsequences, again denoted by (S_n) and (T_n), such that (S_n) and (T_n) converge weakly in $\mathsf{R}(\Omega)$ resp. $\mathsf{DD}(\Omega)$ to S resp. T together with the convergence of the inner products*

$$\langle S_n, T_n\rangle_{\mathsf{L}^2(\Omega,\mathbb{S})} \to \langle S, T\rangle_{\mathsf{L}^2(\Omega,\mathbb{S})}.$$

For the pair $(\mathrm{A}_1, \mathrm{A}_2)$ Theorem 4.7 implies:

Theorem 5.6 (global sym Rot-Div-$\mathbb{T}$-lemma)**.** *Let $(S_n) \subset \mathring{\mathsf{D}}_{\Gamma_t}(\Omega;\mathbb{T})$ and $(T_n) \subset \mathring{\mathsf{R}}_{\mathrm{sym},\Gamma_n}(\Omega;\mathbb{T})$ be two sequences bounded in $\mathsf{D}(\Omega)$ resp. $\mathsf{R}_{\mathrm{sym}}(\Omega)$. Then there exist $S \in \mathring{\mathsf{D}}_{\Gamma_t}(\Omega;\mathbb{T})$ and $T \in \mathring{\mathsf{R}}_{\mathrm{sym},\Gamma_n}(\Omega;\mathbb{T})$ as well as subsequences, again denoted by (S_n) and (T_n), such that (S_n) and (T_n) converge weakly in $\mathsf{D}(\Omega)$ resp. $\mathsf{R}_{\mathrm{sym}}(\Omega)$ to S resp. T together with the convergence of the inner products*

$$\langle S_n, T_n\rangle_{\mathsf{L}^2(\Omega,\mathbb{T})} \to \langle S, T\rangle_{\mathsf{L}^2(\Omega,\mathbb{T})}.$$

Remark 5.7. *Theorem 5.1 has a corresponding local version similar to the local* div-curl-*lemma Corollary 3.2 and Remark 3.3, which holds with no regularity or boundedness assumptions on Ω.*

5.4. **Linear Elasticity.** Let

$$\mathrm{A}_0 := \operatorname{sym}\mathring{\nabla}_{\Gamma_t} : \mathring{\mathsf{H}}^1_{\Gamma_t}(\Omega) \subset \mathsf{L}^2(\Omega) \longrightarrow \mathsf{L}^2(\Omega;\mathbb{S}); \qquad v \mapsto \operatorname{sym}\nabla v,$$
$$\mathrm{A}_1 := \operatorname{Rot}\mathring{\operatorname{Rot}}^\top_{\mathbb{S},\Gamma_t} : \mathring{\mathsf{RR}}^\top_{\Gamma_t}(\Omega;\mathbb{S}) := \overline{\mathring{\mathsf{C}}^\infty_{\Gamma_t}(\Omega;\mathbb{S})}^{\mathsf{RR}^\top(\Omega)} \subset \mathsf{L}^2(\Omega;\mathbb{S}) \longrightarrow \mathsf{L}^2(\Omega;\mathbb{S}); \qquad S \mapsto \operatorname{Rot}\operatorname{Rot}^\top S,$$
$$\mathrm{A}_2 := \mathring{\operatorname{Div}}_{\mathbb{S},\Gamma_t} : \mathring{\mathsf{D}}_{\Gamma_t}(\Omega;\mathbb{S}) := \overline{\mathring{\mathsf{C}}^\infty_{\Gamma_t}(\Omega;\mathbb{S})}^{\mathsf{D}(\Omega)} \subset \mathsf{L}^2(\Omega;\mathbb{S}) \longrightarrow \mathsf{L}^2(\Omega); \qquad T \mapsto \operatorname{Div} T.$$

A_0, A_1, and A_2 are unbounded, densely defined, and closed linear operators with adjoints

$$\mathrm{A}_0^* = (\operatorname{sym}\mathring{\nabla}_{\Gamma_t})^* = -\mathring{\operatorname{Div}}_{\mathbb{S},\Gamma_n} : \mathring{\mathsf{D}}_{\Gamma_n}(\Omega;\mathbb{S}) \subset \mathsf{L}^2(\Omega;\mathbb{S}) \longrightarrow \mathsf{L}^2(\Omega); \qquad S \mapsto -\operatorname{Div} S,$$
$$\mathrm{A}_1^* = (\operatorname{Rot}\mathring{\operatorname{Rot}}^\top_{\mathbb{S},\Gamma_t})^* = \operatorname{Rot}\mathring{\operatorname{Rot}}^\top_{\mathbb{S},\Gamma_n} : \mathring{\mathsf{RR}}^\top_{\Gamma_n}(\Omega;\mathbb{S}) \subset \mathsf{L}^2(\Omega;\mathbb{S}) \longrightarrow \mathsf{L}^2(\Omega;\mathbb{S}); \qquad T \mapsto \operatorname{Rot}\operatorname{Rot}^\top T,$$
$$\mathrm{A}_2^* = \mathring{\operatorname{Div}}^*_{\mathbb{S},\Gamma_t} = -\operatorname{sym}\mathring{\nabla}_{\Gamma_n} : \mathring{\mathsf{H}}^1_{\Gamma_n}(\Omega) \subset \mathsf{L}^2(\Omega) \longrightarrow \mathsf{L}^2(\Omega;\mathbb{S}); \qquad v \mapsto -\operatorname{sym}\nabla v.$$

Note that v resp. S, T are vector resp. tensor (matrix) fields. The complex properties hold as

$$R(\mathrm{A}_0) = \operatorname{sym} \mathring{\nabla}_{\Gamma_t} \mathring{\mathsf{H}}^1_{\Gamma_t}(\Omega) \subset \mathring{\mathsf{RR}}^\top_{\Gamma_t,0}(\Omega;\mathbb{S}) = N(\mathrm{A}_1),$$

$$R(\mathrm{A}_1^*) = \operatorname{Rot} \mathring{\operatorname{Rot}}^\top_{\mathbb{S},\Gamma_n} \mathring{\mathsf{RR}}^\top_{\Gamma_n}(\Omega;\mathbb{S}) \subset \mathring{\mathsf{D}}_{\Gamma_n,0}(\Omega;\mathbb{S}) = N(\mathrm{A}_0^*),$$

$$R(\mathrm{A}_1) = \operatorname{Rot} \mathring{\operatorname{Rot}}^\top_{\mathbb{S},\Gamma_t} \mathring{\mathsf{RR}}^\top_{\Gamma_t}(\Omega;\mathbb{S}) \subset \mathring{\mathsf{D}}_{\Gamma_t,0}(\Omega;\mathbb{S}) = N(\mathrm{A}_2),$$

$$R(\mathrm{A}_2^*) = \operatorname{sym} \mathring{\nabla}_{\Gamma_n} \mathring{\mathsf{H}}^1_{\Gamma_n}(\Omega) \subset \mathring{\mathsf{RR}}^\top_{\Gamma_n,0}(\Omega;\mathbb{S}) = N(\mathrm{A}_1^*).$$

The sequences (5.1) read

$$\mathring{\mathsf{H}}^1_{\Gamma_t}(\Omega) \subset \mathsf{L}^2(\Omega) \xrightarrow{\mathrm{A}_0 = \operatorname{sym} \mathring{\nabla}_{\Gamma_t}} \mathring{\mathsf{RR}}^\top_{\Gamma_t}(\Omega;\mathbb{S}) \subset \mathsf{L}^2(\Omega;\mathbb{S}) \xrightarrow{\mathrm{A}_1 = \operatorname{Rot} \mathring{\operatorname{Rot}}^\top_{\mathbb{S},\Gamma_t}} \mathring{\mathsf{D}}_{\Gamma_t}(\Omega;\mathbb{S}) \subset \mathsf{L}^2(\Omega;\mathbb{S}) \xrightarrow{\mathrm{A}_2 = \mathring{\operatorname{Div}}_{\mathbb{S},\Gamma_t}} \mathsf{L}^2(\Omega),$$

$$\mathsf{L}^2(\Omega) \xleftarrow{\mathrm{A}_0^* = -\mathring{\operatorname{Div}}_{\mathbb{S},\Gamma_n}} \mathring{\mathsf{D}}_{\Gamma_n}(\Omega;\mathbb{S}) \subset \mathsf{L}^2(\Omega;\mathbb{S}) \xleftarrow{\mathrm{A}_1^* = \operatorname{Rot} \mathring{\operatorname{Rot}}^\top_{\mathbb{S},\Gamma_n}} \mathring{\mathsf{RR}}^\top_{\Gamma_n}(\Omega;\mathbb{S}) \subset \mathsf{L}^2(\Omega;\mathbb{S}) \xleftarrow{\mathrm{A}_2^* = -\operatorname{sym} \mathring{\nabla}_{\Gamma_n}} \mathring{\mathsf{H}}^1_{\Gamma_n}(\Omega) \subset \mathsf{L}^2(\Omega).$$

These are the so called Rot Rot complexes, appearing, e.g., in linear elasticity, see [17]. The crucial embeddings (5.2), i.e.,

$$D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) = \mathring{\mathsf{RR}}^\top_{\Gamma_t}(\Omega;\mathbb{S}) \cap \mathring{\mathsf{D}}_{\Gamma_n}(\Omega;\mathbb{S}) \hookrightarrow\!\!\!\!\rightarrow \mathsf{L}^2(\Omega;\mathbb{S}),$$

$$D(\mathrm{A}_2) \cap D(\mathrm{A}_1^*) = \mathring{\mathsf{D}}_{\Gamma_t}(\Omega;\mathbb{S}) \cap \mathring{\mathsf{RR}}^\top_{\Gamma_n}(\Omega;\mathbb{S}) \hookrightarrow\!\!\!\!\rightarrow \mathsf{L}^2(\Omega;\mathbb{S}),$$

are compact, which can be proved by the same techniques showing [17, Lemma 3.22]. Choosing the pair $(\mathrm{A}_0, \mathrm{A}_1)$ we get by Theorem 4.7 the following:

Theorem 5.8 (global $\operatorname{Rot}\operatorname{Rot}^\top$-Div-$\mathbb{S}$-lemma). *Let $(S_n) \subset \mathring{\mathsf{RR}}^\top_{\Gamma_t}(\Omega;\mathbb{S})$ and $(T_n) \subset \mathring{\mathsf{D}}_{\Gamma_n}(\Omega;\mathbb{S})$ be two sequences bounded in $\mathsf{RR}^\top(\Omega)$ resp. $\mathsf{D}(\Omega)$. Then there exist $S \in \mathring{\mathsf{RR}}^\top_{\Gamma_t}(\Omega;\mathbb{S})$ and $T \in \mathring{\mathsf{D}}_{\Gamma_n}(\Omega;\mathbb{S})$ as well as subsequences, again denoted by (S_n) and (T_n), such that (S_n) and (T_n) converge weakly in $\mathsf{RR}^\top(\Omega)$ resp. $\mathsf{D}(\Omega)$ to S resp. T together with the convergence of the inner products*

$$\langle S_n, T_n \rangle_{\mathsf{L}^2(\Omega,\mathbb{S})} \to \langle S, T \rangle_{\mathsf{L}^2(\Omega,\mathbb{S})}.$$

Remark 5.9. *Let us note:*

- **(i)** *Theorem 4.7 for the pair $(\mathrm{A}_1, \mathrm{A}_2)$ implies the same result just interchanging S_n, T_n and the boundary conditions.*
- **(ii)** *Theorem 5.8 has a corresponding local version similar to the local* div-curl-*lemma Corollary 3.2 and Remark 3.3, which holds with no regularity or boundedness assumptions on Ω.*
- **(iii)** *We emphasize the strong symmetry in the* Rot Rot *complexes of linear elasticity.*

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