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On Closed and Exact Grad grad- and div Div-Complexes,  
Corresponding Compact Embeddings for Tensor Rotations,  
and a Related Decomposition Result for Biharmonic Problems in 3D

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# On Closed and Exact Grad grad- and div Div-Complexes, Corresponding Compact Embeddings for Tensor Rotations, and a Related Decomposition Result for Biharmonic Problems in 3D

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ABSTRACT. It is shown that the first biharmonic boundary value problem on a topologically trivial domain in 3D is equivalent to three (consecutively to solve) second-order problems. This decomposition result is based on a Helmholtz-like decomposition of an involved non-standard Sobolev space of tensor fields and a proper characterization of the operator  $\operatorname{div} \operatorname{Div}$  acting on this space. Similar results for biharmonic problems in 2D and their impact on the construction and analysis of finite element methods have been recently published in [14]. The discussion of the kernel of  $\operatorname{div} \operatorname{Div}$  leads to (de Rham-like) closed and exact Hilbert complexes, the  $\operatorname{div} \operatorname{Div}$ -complex and its adjoint the Grad grad-complex, involving spaces of trace-free and symmetric tensor fields. For these tensor fields we show Helmholtz type decompositions and, most importantly, new compact embedding results. Almost all our results hold and are formulated for general bounded strong Lipschitz domains of arbitrary topology. There is no reasonable doubt that our results extend to strong Lipschitz domains in  $\mathbb{R}^N$ .

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## 1. INTRODUCTION

In [14] it was shown that the fourth-order biharmonic boundary value problem

$$(1.1) \quad \Delta^2 u = f \quad \text{in } \Omega, \quad u = \partial_n u = 0 \quad \text{on } \Gamma,$$

where  $\Omega$  is a bounded and simply connected domain in  $\mathbb{R}^2$  with a (strong) Lipschitz boundary  $\Gamma$ , can be decomposed into three second-order problems. The first problem is a Poisson problem for an auxiliary scalar field  $p$

$$\Delta p = f \quad \text{in } \Omega, \quad p = 0 \quad \text{on } \Gamma,$$

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the second problem is a linear elasticity problem for an auxiliary vector field  $V$

$$-\operatorname{Div} \varepsilon(V) = -\operatorname{Div}(\operatorname{sym Grad} V) = \operatorname{grad} p \quad \text{in } \Omega, \quad (\operatorname{sym Grad} V) n = -p n = 0 \quad \text{on } \Gamma,$$

i.e.,

$$\operatorname{Div}(\operatorname{sym Grad} V + p \mathbf{I}) = 0 \quad \text{in } \Omega, \quad (\operatorname{sym Grad} V + p \mathbf{I}) n = 0 \quad \text{on } \Gamma,$$

and, finally, the third problem is a Poisson problem for the original scalar field  $u$

$$\Delta u = 2p + \operatorname{div} V \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \Gamma.$$

Here  $f$  is a given right-hand side,  $\Delta$ ,  $n$ , and  $\partial_n$  denote the Laplace operator, the outward normal vector to the boundary, and the derivative in this direction, respectively. The differential operators  $\operatorname{grad}$ ,  $\operatorname{div}$ , and (for later use)  $\operatorname{rot}$  denote the gradient of a scalar field and the divergence and rotation of a vector field, the corresponding capitalized differential operators  $\operatorname{Grad}$ ,  $\operatorname{Div}$ , and  $\operatorname{Rot}$  denote the row-wise application of  $\operatorname{grad}$  to a vector field,  $\operatorname{div}$  and  $\operatorname{rot}$  to a tensor field. The prefix  $\operatorname{sym}$  is used for the symmetric part of a matrix, for the skew-symmetric part we use the prefix  $\operatorname{skw}$ . This decomposition is of triangular structure, i.e., the first problem is a well-posed second-order problem in  $p$ , the second problem is a well-posed second-order problem in  $V$  for given  $p$ , and the third problem is a well-posed second-order problem in  $u$  for given  $p$  and  $V$ . This allows to solve them consecutively analytically or numerically by means of techniques for second-order problems.

This is - in the first place - a new analytic result for fourth-order problems. But it also has interesting implications for discretization methods applied to (1.1). It allows to re-interpret known finite element methods as well as to construct new discretization methods for (1.1) by exploiting the decomposable structure of the problem. In particular, it was shown in [14] that the Hellan-Herrmann-Johnson mixed method (see [8, 9, 13]) for (1.1) allows a similar decomposition as the continuous problem, which leads to a new and simpler assembling procedure for the discretization matrix and to more efficient solution techniques for the discretized problem. Moreover, a novel conforming variant of the Hellan-Herrmann-Johnson mixed method was found based on the decomposition.

The aim of this paper is to derive a similar decomposition result for biharmonic problems on bounded and topologically trivial three-dimensional domains  $\Omega$  with a (strong) Lipschitz boundary  $\Gamma$ . For this we proceed as in [14] and reformulate (1.1) using  $\Delta^2 = \operatorname{div} \operatorname{Div} \operatorname{Grad} \operatorname{grad}$  as a mixed problem by introducing the (negative) Hessian of the original scalar field  $u$  as an auxiliary tensor field

$$(1.2) \quad \mathbf{M} = -\operatorname{Grad} \operatorname{grad} u.$$

Then the biharmonic differential equation reads

$$(1.3) \quad -\operatorname{div} \operatorname{Div} \mathbf{M} = f \quad \text{in } \Omega.$$

For an appropriate non-standard Sobolev space for  $\mathbf{M}$  it can be shown that the mixed problem in  $\mathbf{M}$  and  $u$  is well-posed. Then the decomposition of the biharmonic problem follows from a regular decomposition of this non-standard Sobolev space. This part of the analysis carries over completely from the two-dimensional case to the three-dimensional case and is shortly recalled in Section 4. To efficiently utilize this regular decomposition for the decomposition of the biharmonic problem an appropriate characterization of the kernel of the operator  $\operatorname{div} \operatorname{Div}$  is required, which is well understood for the two-dimensional case, see, e.g., [3, 11, 14]. Its extension to the three-dimensional case is the central topic of this paper. We expect - as in the two-dimensional case - similar interesting implications for the study of appropriate discretization methods for four-order problems in the three-dimensional case.

The paper is organized as follows. After some preliminaries in Section 2 and introducing our general functional analytical setting, we will discuss the relevant unbounded linear operators, show closed and exact Hilbert complex properties, and present a suitable representation of the kernel of  $\operatorname{div} \operatorname{Div}$  for the three-dimensional case in Section 3.1 for topologically trivial domains. In Section 3.2 we extend the results to (strong) Lipschitz domains based on two new and crucial compact embeddings. Based on the representation of the kernel of  $\operatorname{div} \operatorname{Div}$  a decomposition of the three-dimensional biharmonic problem into three (consecutively to solve) second-order problems will be derived in Section 4. The proofs of some useful identities are presented in an appendix.

## 2. PRELIMINARIES

We start by recalling some basic concepts and abstract results from functional analysis concerning Helmholtz decompositions, closed ranges, Friedrichs/Poincaré type estimates, and bounded or even compact inverse operators. Since we will need both the Banach space setting for bounded linear operators as well as the Hilbert space setting for (possibly unbounded) closed and densely defined linear operators, we will shortly recall these two variants.

**2.1. Functional Analysis Toolbox.** Let  $X$  and  $Y$  be real Banach spaces. With  $BL(X, Y)$  we introduce the space of bounded linear operators mapping  $X$  to  $Y$ . The dual spaces of  $X$  and  $Y$  are denoted by  $X' := BL(X, \mathbb{R})$  and  $Y' := BL(Y, \mathbb{R})$ . For a given  $A \in BL(X, Y)$  we write  $A' \in BL(Y', X')$  for its Banach space dual or adjoint operator defined by  $A' y'(x) := y'(Ax)$  for all  $y' \in Y'$  and all  $x \in X$ . Norms and duality in  $X$  resp.  $X'$  are denoted by  $|\cdot|_X$ ,  $|\cdot|_{X'}$ , and  $\langle \cdot, \cdot \rangle_{X'}$ .

Suppose  $H_1$  and  $H_2$  are Hilbert spaces. For a (possibly unbounded) densely defined linear operator  $A : D(A) \subset H_1 \rightarrow H_2$  we recall that its Hilbert space dual or adjoint  $A^* : D(A^*) \subset H_2 \rightarrow H_1$  can be defined via its Banach space adjoint  $A'$  and the Riesz isomorphisms of  $H_1$  and  $H_2$  or directly as follows:  $y \in D(A^*)$  if and only if  $y \in H_2$  and

$$\exists f \in H_1 \quad \forall x \in D(A) \quad \langle Ax, y \rangle_{H_2} = \langle x, f \rangle_{H_1}.$$

In this case we define  $A^* y := f$ . We note that  $A^*$  has maximal domain of definition and that  $A^*$  is characterized by

$$\forall x \in D(A) \quad \forall y \in D(A^*) \quad \langle Ax, y \rangle_{H_2} = \langle x, A^* y \rangle_{H_1}.$$

Here  $\langle \cdot, \cdot \rangle_H$  denotes the scalar product in a Hilbert space  $H$  and  $D$  is used for the domain of definition of a linear operator. Additionally, we introduce the notation  $N$  for the kernel or null space and  $R$  for the range of a linear operator.

Let  $A : D(A) \subset H_1 \rightarrow H_2$  be a (possibly unbounded) closed and densely defined linear operator on two Hilbert spaces  $H_1$  and  $H_2$  with adjoint  $A^* : D(A^*) \subset H_2 \rightarrow H_1$ . Note  $(A^*)^* = \overline{A} = A$ , i.e.,  $(A, A^*)$  is a dual pair. By the projection theorem the Helmholtz type decompositions

$$(2.1) \quad H_1 = N(A) \oplus_{H_1} \overline{R(A^*)}, \quad H_2 = N(A^*) \oplus_{H_2} \overline{R(A)}$$

hold and we can define the reduced operators

$$\begin{aligned} \mathcal{A} &:= A|_{\overline{R(A^*)}} : D(\mathcal{A}) \subset \overline{R(A^*)} \rightarrow \overline{R(A)}, & D(\mathcal{A}) &:= D(A) \cap N(A)^{\perp_{H_1}} = D(A) \cap \overline{R(A^*)}, \\ \mathcal{A}^* &:= A^*|_{\overline{R(A)}} : D(\mathcal{A}^*) \subset \overline{R(A)} \rightarrow \overline{R(A^*)}, & D(\mathcal{A}^*) &:= D(A^*) \cap N(A^*)^{\perp_{H_2}} = D(A^*) \cap \overline{R(A)}, \end{aligned}$$

which are also closed and densely defined linear operators. We note that  $\mathcal{A}$  and  $\mathcal{A}^*$  are indeed adjoint to each other, i.e.,  $(\mathcal{A}, \mathcal{A}^*)$  is a dual pair as well. Now the inverse operators

$$\mathcal{A}^{-1} : R(\mathcal{A}) \rightarrow D(\mathcal{A}), \quad (\mathcal{A}^*)^{-1} : R(\mathcal{A}^*) \rightarrow D(\mathcal{A}^*)$$

exist and they are bijective, since  $\mathcal{A}$  and  $\mathcal{A}^*$  are injective by definition. Furthermore, by (2.1) we have the refined Helmholtz type decompositions

$$(2.2) \quad D(A) = N(A) \oplus_{H_1} D(\mathcal{A}), \quad D(A^*) = N(A^*) \oplus_{H_2} D(\mathcal{A}^*)$$

and thus we obtain for the ranges

$$(2.3) \quad R(A) = R(\mathcal{A}), \quad R(A^*) = R(\mathcal{A}^*).$$

By the closed range theorem and the closed graph theorem we get immediately the following.

**Lemma 2.1.** *The following assertions are equivalent:*

- (i)  $\exists c_A \in (0, \infty) \quad \forall x \in D(\mathcal{A}) \quad |x|_{H_1} \leq c_A |Ax|_{H_2}$
- (i\*)  $\exists c_{A^*} \in (0, \infty) \quad \forall y \in D(\mathcal{A}^*) \quad |y|_{H_2} \leq c_{A^*} |A^*y|_{H_1}$
- (ii)  $R(A) = R(\mathcal{A})$  is closed in  $H_2$ .
- (ii\*)  $R(A^*) = R(\mathcal{A}^*)$  is closed in  $H_1$ .
- (iii)  $\mathcal{A}^{-1} : R(\mathcal{A}) \rightarrow D(\mathcal{A})$  is continuous and bijective with norm bounded by  $(1 + c_A^2)^{1/2}$ .

(iii\*)  $(\mathcal{A}^*)^{-1} : R(\mathcal{A}^*) \rightarrow D(\mathcal{A}^*)$  is continuous and bijective with norm bounded by  $(1 + c_{\mathcal{A}^*}^2)^{1/2}$ .

In case that one of the assertions of Lemma 2.1 is true, e.g.,  $R(\mathcal{A})$  is closed, we have

$$\begin{aligned} H_1 &= N(\mathcal{A}) \oplus_{H_1} R(\mathcal{A}^*), & H_2 &= N(\mathcal{A}^*) \oplus_{H_2} R(\mathcal{A}), \\ D(\mathcal{A}) &= N(\mathcal{A}) \oplus_{H_1} D(\mathcal{A}), & D(\mathcal{A}^*) &= N(\mathcal{A}^*) \oplus_{H_2} D(\mathcal{A}^*), \\ D(\mathcal{A}) &= D(\mathcal{A}) \cap R(\mathcal{A}^*), & D(\mathcal{A}^*) &= D(\mathcal{A}^*) \cap R(\mathcal{A}). \end{aligned}$$

For the “best” constants  $c_{\mathcal{A}}$ ,  $c_{\mathcal{A}^*}$  we have the following lemma.

**Lemma 2.2.** *The Rayleigh quotients*

$$\frac{1}{c_{\mathcal{A}}} := \inf_{0 \neq x \in D(\mathcal{A})} \frac{|Ax|_{H_2}}{|x|_{H_1}} = \inf_{0 \neq y \in D(\mathcal{A}^*)} \frac{|A^*y|_{H_1}}{|y|_{H_2}} =: \frac{1}{c_{\mathcal{A}^*}}$$

coincide, i.e.,  $c_{\mathcal{A}} = c_{\mathcal{A}^*}$ , if either  $c_{\mathcal{A}}$  or  $c_{\mathcal{A}^*}$  exists in  $(0, \infty)$ . Otherwise they also coincide, i.e., it holds  $c_{\mathcal{A}} = c_{\mathcal{A}^*} = \infty$ .

From now on and throughout this paper, we always pick the best possible constants in the various Friedrichs/Poincaré type estimates.

A standard indirect argument shows the following.

**Lemma 2.3.** *Let  $D(\mathcal{A}) = D(\mathcal{A}) \cap \overline{R(\mathcal{A}^*)} \hookrightarrow H_1$  be compact. Then the assertions of Lemma 2.1 hold. Moreover, the inverse operators*

$$\mathcal{A}^{-1} : R(\mathcal{A}) \rightarrow R(\mathcal{A}^*), \quad (\mathcal{A}^*)^{-1} : R(\mathcal{A}^*) \rightarrow R(\mathcal{A})$$

are compact with norms  $|\mathcal{A}^{-1}|_{R(\mathcal{A}), R(\mathcal{A}^*)} = |(\mathcal{A}^*)^{-1}|_{R(\mathcal{A}^*), R(\mathcal{A})} = c_{\mathcal{A}}$ .

Moreover, we have

**Lemma 2.4.**  *$D(\mathcal{A}) \hookrightarrow H_1$  is compact, if and only if  $D(\mathcal{A}^*) \hookrightarrow H_2$  is compact.*

Now, let  $A_0 : D(A_0) \subset H_0 \rightarrow H_1$  and  $A_1 : D(A_1) \subset H_1 \rightarrow H_2$  be (possibly unbounded) closed and densely defined linear operators on three Hilbert spaces  $H_0$ ,  $H_1$  and  $H_2$  with adjoints  $A_0^* : D(A_0^*) \subset H_1 \rightarrow H_0$  and  $A_1^* : D(A_1^*) \subset H_2 \rightarrow H_1$  as well as reduced operators  $\mathcal{A}_0$ ,  $\mathcal{A}_0^*$ , and  $\mathcal{A}_1$ ,  $\mathcal{A}_1^*$ . Furthermore, we assume the sequence or complex property of  $A_0$  and  $A_1$ , that is,  $A_1 A_0 = 0$ , i.e.,

$$(2.4) \quad R(A_0) \subset N(A_1).$$

Then also  $A_0^* A_1^* = 0$ , i.e.,  $R(A_1^*) \subset N(A_0^*)$ . The Helmholtz type decompositions of (2.1) for  $\mathcal{A} = A_1$  and  $\mathcal{A} = A_0$  read

$$(2.5) \quad H_1 = N(A_1) \oplus_{H_1} \overline{R(A_1^*)}, \quad H_1 = N(A_0^*) \oplus_{H_1} \overline{R(A_0)}$$

and by (2.4) we see

$$(2.6) \quad N(A_0^*) = N_{0,1} \oplus_{H_1} \overline{R(A_1^*)}, \quad N(A_1) = N_{0,1} \oplus_{H_1} \overline{R(A_0)}, \quad N_{0,1} := N(A_1) \cap N(A_0^*)$$

yielding the refined Helmholtz type decomposition

$$(2.7) \quad H_1 = \overline{R(A_0)} \oplus_{H_1} N_{0,1} \oplus_{H_1} \overline{R(A_1^*)}, \quad R(A_0) = R(\mathcal{A}_0), \quad R(A_1^*) = R(\mathcal{A}_1^*).$$

The previous results of this section imply immediately the following.

**Lemma 2.5.** *Let  $A_0$ ,  $A_1$  be as introduced before with  $A_1 A_0 = 0$ , i.e., (2.4). Moreover, let  $R(A_0)$  and  $R(A_1)$  be closed. Then, the assertions of Lemma 2.1 and Lemma 2.2 hold for  $\mathcal{A}_0$  and  $\mathcal{A}_1$ . Moreover, the refined Helmholtz type decompositions*

$$\begin{aligned} H_1 &= R(A_0) \oplus_{H_1} N_{0,1} \oplus_{H_1} R(A_1^*), & N_{0,1} &= N(A_1) \cap N(A_0^*), \\ N(A_1) &= R(A_0) \oplus_{H_1} N_{0,1}, & N(A_0^*) &= N_{0,1} \oplus_{H_1} R(A_1^*), \\ D(A_1) &= R(A_0) \oplus_{H_1} N_{0,1} \oplus_{H_1} D(\mathcal{A}_1), & D(A_0^*) &= D(\mathcal{A}_0^*) \oplus_{H_1} N_{0,1} \oplus_{H_1} R(A_1^*), \end{aligned}$$

$$D(A_1) \cap D(A_0^*) = D(\mathcal{A}_0^*) \oplus_{\mathbf{H}_1} N_{0,1} \oplus_{\mathbf{H}_1} D(\mathcal{A}_1)$$

hold. Especially,  $R(A_0)$ ,  $R(A_0^*)$ ,  $R(A_1)$ , and  $R(A_1^*)$  are closed, the respective inverse operators, i.e.,

$$\begin{aligned} \mathcal{A}_0^{-1} : R(A_0) &\rightarrow D(\mathcal{A}_0), & \mathcal{A}_1^{-1} : R(A_1) &\rightarrow D(\mathcal{A}_1), \\ (\mathcal{A}_0^*)^{-1} : R(A_0^*) &\rightarrow D(\mathcal{A}_0^*), & (\mathcal{A}_1^*)^{-1} : R(A_1^*) &\rightarrow D(\mathcal{A}_1^*), \end{aligned}$$

are continuous, and there exist positive constants  $c_{A_0}$ ,  $c_{A_1}$ , such that the Friedrichs/Poincaré type estimates

$$\begin{aligned} \forall x \in D(\mathcal{A}_0) & \quad |x|_{\mathbf{H}_0} \leq c_{A_0} |A_0 x|_{\mathbf{H}_1}, & \forall y \in D(\mathcal{A}_1) & \quad |y|_{\mathbf{H}_1} \leq c_{A_1} |A_1 y|_{\mathbf{H}_2}, \\ \forall y \in D(\mathcal{A}_0^*) & \quad |y|_{\mathbf{H}_1} \leq c_{A_0} |A_0^* y|_{\mathbf{H}_0}, & \forall z \in D(\mathcal{A}_1^*) & \quad |z|_{\mathbf{H}_2} \leq c_{A_1} |A_1^* z|_{\mathbf{H}_1} \end{aligned}$$

hold.

**Remark 2.6.** Note that  $R(A_0)$  resp.  $R(A_1)$  is closed, if e.g.  $D(\mathcal{A}_0) \hookrightarrow \mathbf{H}_0$  resp.  $D(\mathcal{A}_1) \hookrightarrow \mathbf{H}_1$  is compact. In this case, the respective inverse operators, i.e.,

$$\begin{aligned} \mathcal{A}_0^{-1} : R(A_0) &\rightarrow R(\mathcal{A}_0^*), & \mathcal{A}_1^{-1} : R(A_1) &\rightarrow R(\mathcal{A}_1^*), \\ (\mathcal{A}_0^*)^{-1} : R(A_0^*) &\rightarrow R(A_0), & (\mathcal{A}_1^*)^{-1} : R(A_1^*) &\rightarrow R(A_1), \end{aligned}$$

are compact.

Observe  $D(A_1) = D(A_1) \cap \overline{R(A_1^*)} \subset D(A_1) \cap N(A_0^*) \subset D(A_1) \cap D(A_0^*)$ . Utilizing the Helmholtz type decompositions of Lemma 2.5 we immediately have:

**Lemma 2.7.** The embeddings  $D(\mathcal{A}_0) \hookrightarrow \mathbf{H}_0$ ,  $D(\mathcal{A}_1) \hookrightarrow \mathbf{H}_1$ , and  $N_{0,1} \hookrightarrow \mathbf{H}_1$  are compact, if and only if the embedding  $D(A_1) \cap D(A_0^*) \hookrightarrow \mathbf{H}_1$  is compact. In this case  $N_{0,1}$  has finite dimension.

**Remark 2.8.** The assumptions in Lemma 2.5 on  $A_0$  and  $A_1$  are equivalent to the assumption that

$$D(A_0) \subset \mathbf{H}_0 \xrightarrow{A_0} D(A_1) \subset \mathbf{H}_1 \xrightarrow{A_1} \mathbf{H}_2$$

is a closed Hilbert complex, meaning that the ranges are closed. As a result of the previous lemmas, the adjoint complex

$$\mathbf{H}_0 \xleftarrow{A_0^*} D(A_0^*) \subset \mathbf{H}_1 \xleftarrow{A_1^*} D(A_1^*) \subset \mathbf{H}_2.$$

is a closed Hilbert complex as well.

We can summarize.

**Theorem 2.9.** Let  $A_0, A_1$  be as introduced before, i.e., having the complex property  $A_1 A_0 = 0$ , i.e.,  $R(A_0) \subset N(A_1)$ . Moreover, let  $D(A_1) \cap D(A_0^*) \hookrightarrow \mathbf{H}_1$  be compact. Then the assertions of Lemma 2.5 hold,  $N_{0,1}$  is finite dimensional and the corresponding inverse operators are continuous resp. compact. Especially, all ranges are closed and the corresponding Friedrichs/Poincaré type estimates hold.

A special situation is the following.

**Lemma 2.10.** Let  $A_0, A_1$  be as introduced before with  $R(A_0) = N(A_1)$  and  $R(A_1)$  closed in  $\mathbf{H}_2$ . Then  $R(A_0^*)$  and  $R(A_1^*)$  are closed as well, and the simplified Helmholtz type decompositions

$$\begin{aligned} \mathbf{H}_1 &= R(A_0) \oplus_{\mathbf{H}_1} R(A_1^*), & N_{0,1} &= \{0\}, \\ N(A_1) &= R(A_0) = R(\mathcal{A}_0), & N(A_0^*) &= R(A_1^*) = R(\mathcal{A}_1^*), \\ D(A_1) &= R(A_0) \oplus_{\mathbf{H}_1} D(\mathcal{A}_1), & D(A_0^*) &= D(\mathcal{A}_0^*) \oplus_{\mathbf{H}_1} R(A_1^*), \\ D(A_1) \cap D(A_0^*) &= D(\mathcal{A}_0^*) \oplus_{\mathbf{H}_1} D(\mathcal{A}_1) \end{aligned}$$

are valid. Moreover, the respective inverse operators are continuous and the corresponding Friedrichs/Poincaré type estimates hold.

**Remark 2.11.** Note that  $R(A_1^*) = N(A_0^*)$  and  $R(A_0^*)$  closed are equivalent assumptions for Lemma 2.10 to hold.

**Lemma 2.12.** *Let  $A_0, A_1$  be as introduced before with the sequence property (2.4), i.e.,  $R(A_0) \subset N(A_1)$ . If the embedding  $D(A_1) \cap D(A_0^*) \hookrightarrow H_1$  is compact and  $N_{0,1} = \{0\}$ , then the assumptions of Lemma 2.10 are satisfied.*

**Remark 2.13.** *The assumptions in Lemma 2.10 on  $A_0$  and  $A_1$  are equivalent to the assumption that*

$$D(A_0) \subset H_0 \xrightarrow{A_0} D(A_1) \subset H_1 \xrightarrow{A_1} H_2$$

*is a closed and exact Hilbert complex. By Lemma 2.10 the adjoint complex*

$$H_0 \xleftarrow{A_0^*} D(A_0^*) \subset H_1 \xleftarrow{A_1^*} D(A_1^*) \subset H_2.$$

*is a closed and exact Hilbert complex as well.*

Parts of Lemma 2.10 hold also in the Banach space setting. As a direct consequence of the closed range theorem and the closed graph theorem the following abstract result holds.

**Lemma 2.14.** *Let  $X_0, X_1, X_2$  be Banach spaces and suppose  $A_0 \in BL(X_0, X_1)$ ,  $A_1 \in BL(X_1, X_2)$  with  $R(A_0) = N(A_1)$  and that  $R(A_1)$  is closed in  $X_2$ . Then  $R(A_0')$  is closed in  $X_0'$  and  $R(A_1') = N(A_0')$ . Moreover,  $(A_1')^{-1} \in BL(R(A_1'), R(A_1)')$ .*

Note that in the latter context we consider the operators

$$A_1 : X_1 \longrightarrow R(A_1), \quad A_1' : R(A_1)' \longrightarrow R(A_1') \quad (A_1')^{-1} : R(A_1') \longrightarrow R(A_1)',$$

with  $N(A_1') = R(A_1)^\circ = \{0\}$ .

**Remark 2.15.** *The conditions on  $A_0$  and  $A_1$  in Lemma 2.14 are identical to the assumption that*

$$X_0 \xrightarrow{A_0} X_1 \xrightarrow{A_1} X_2$$

*is a closed and exact complex of Banach spaces. The consequences of Lemma 2.14 can be rephrased as follows. The adjoint complex of Banach spaces*

$$X_0' \xleftarrow{A_0'} X_1' \xleftarrow{A_1'} X_2'$$

*is closed and exact as well.*

**Lemma 2.16.**  *$(A_1')^{-1} \in BL(R(A_1'), R(A_1)')$  is equivalent to*

$$(2.8) \quad \exists c_{A_1'} > 0 \quad \forall y' \in R(A_1)' \quad |y'|_{R(A_1)'} \leq c_{A_1'} |A_1' y'|_{X_1'}.$$

*For the best constant  $c_{A_1'}$ , (2.8) is equivalent to the general inf-sup-condition*

$$(2.9) \quad 0 < \frac{1}{c_{A_1'}} = \inf_{0 \neq y' \in R(A_1)'} \sup_{0 \neq x \in X_1} \frac{\langle y', A_1 x \rangle_{R(A_1)'}}{|y'|_{R(A_1)'} |x|_{X_1}}.$$

*In the special case that  $X_2 = H_2$  is a Hilbert space the closed subspace  $R(A_1)$  is isometrically isomorphic to  $R(A_1)'$  and we obtain the following form of the inf-sup-condition*

$$(2.10) \quad 0 < \frac{1}{c_{A_1'}} = \inf_{0 \neq y \in R(A_1)} \sup_{0 \neq x \in X_1} \frac{\langle y, A_1 x \rangle_{H_2}}{|y|_{H_2} |x|_{X_1}}.$$

The results collected in this section are well-known in functional analysis. We refer to [1] for a presentation of some results of this section from a numerical analysis perspective.



**2.2. Sobolev Spaces.** Next we introduce our notations for several classes of Sobolev spaces on a bounded domain  $\Omega \subset \mathbb{R}^3$ . Let  $m \in \mathbb{N}_0$ . We denote by  $L^2(\Omega)$  and  $H^m(\Omega)$  the standard Lebesgue and Sobolev spaces and write  $H^0(\Omega) = L^2(\Omega)$ . Our notation of spaces will not indicate whether the elements are scalar functions or vector fields. For the rotation and divergence we define the Sobolev spaces

$$\mathbf{R}(\Omega) := \{V \in L^2(\Omega) : \text{rot } V \in L^2(\Omega)\}, \quad \mathbf{D}(\Omega) := \{V \in L^2(\Omega) : \text{div } V \in L^2(\Omega)\}$$

with the respective graph norms, where rot and div have to be understood in the distributional or weak sense. We introduce spaces with boundary conditions in the weak sense in the natural way by

$$\overset{\circ}{\mathbf{H}}^m(\Omega) := \overline{\overset{\circ}{\mathbf{C}}^\infty(\Omega)}^{\mathbf{H}^m(\Omega)}, \quad \overset{\circ}{\mathbf{R}}(\Omega) := \overline{\overset{\circ}{\mathbf{C}}^\infty(\Omega)}^{\mathbf{R}(\Omega)}, \quad \overset{\circ}{\mathbf{D}}(\Omega) := \overline{\overset{\circ}{\mathbf{C}}^\infty(\Omega)}^{\mathbf{D}(\Omega)},$$

i.e., as closures of test functions or fields under the respective graph norms, which generalizes homogeneous scalar, tangential and normal boundary conditions, respectively. We also introduce the well known dual spaces

$$\mathbf{H}^{-m}(\Omega) := (\overset{\circ}{\mathbf{H}}^m(\Omega))'$$

with the standard dual or operator norm defined by

$$|u|_{\mathbf{H}^{-m}(\Omega)} := \sup_{0 \neq \varphi \in \overset{\circ}{\mathbf{H}}^m(\Omega)} \frac{\langle u, \varphi \rangle_{\mathbf{H}^{-m}(\Omega)}}{|\varphi|_{\overset{\circ}{\mathbf{H}}^m(\Omega)}} \quad \text{for } u \in \mathbf{H}^{-m}(\Omega),$$

where we recall the duality pairing  $\langle \cdot, \cdot \rangle_{\mathbf{H}^{-m}(\Omega)}$  in  $(\mathbf{H}^{-m}(\Omega), \overset{\circ}{\mathbf{H}}^m(\Omega))$ . Moreover, we define with respective graph norms

$$\begin{aligned} \mathbf{R}^{-m}(\Omega) &:= \{V \in \mathbf{H}^{-m}(\Omega) : \text{rot } V \in \mathbf{H}^{-m}(\Omega)\}, \\ \mathbf{D}^{-m}(\Omega) &:= \{V \in \mathbf{H}^{-m}(\Omega) : \text{div } V \in \mathbf{H}^{-m}(\Omega)\}. \end{aligned}$$

A vanishing differential operator will be indicated by a zero at the lower right corner of the spaces, e.g.,

$$\begin{aligned} \mathbf{R}_0(\Omega) &= \{V \in \mathbf{R}(\Omega) : \text{rot } V = 0\}, & \overset{\circ}{\mathbf{D}}_0(\Omega) &= \{V \in \overset{\circ}{\mathbf{D}}(\Omega) : \text{div } V = 0\}, \\ \mathbf{R}_0^{-m}(\Omega) &= \{V \in \mathbf{R}^{-m}(\Omega) : \text{rot } V = 0\}, & \mathbf{D}_0^{-1}(\Omega) &= \{V \in \mathbf{D}^{-1}(\Omega) : \text{div } V = 0\}. \end{aligned}$$

Let us also introduce

$$\mathbf{L}_0^2(\Omega) := \{u \in L^2(\Omega) : u \perp_{L^2(\Omega)} \mathbb{R}\} = \{u \in L^2(\Omega) : \int_{\Omega} u = 0\},$$

where  $\perp_{L^2(\Omega)}$  denotes orthogonality in  $L^2(\Omega)$ .

**Remark 2.17.** *Other widely used notations for the spaces  $\mathbf{R}(\Omega)$ ,  $\overset{\circ}{\mathbf{R}}(\Omega)$ ,  $\mathbf{R}^{-m}(\Omega)$ ,  $\mathbf{R}_0(\Omega)$ ,  $\dots$  are  $\mathbf{H}(\text{rot}, \Omega)$ ,  $\mathbf{H}_0(\text{rot}, \Omega)$ ,  $\mathbf{H}^{-m}(\text{rot}, \Omega)$ ,  $\mathbf{H}(\text{rot } 0, \Omega)$ ,  $\dots$ , respectively. Similarly, alternative notations for  $\mathbf{D}(\Omega)$ ,  $\overset{\circ}{\mathbf{D}}(\Omega)$ ,  $\mathbf{D}^{-m}(\Omega)$ ,  $\mathbf{D}_0(\Omega)$ ,  $\dots$  are  $\mathbf{H}(\text{div}, \Omega)$ ,  $\mathbf{H}_0(\text{div}, \Omega)$ ,  $\mathbf{H}^{-m}(\text{div}, \Omega)$ ,  $\mathbf{H}(\text{div } 0, \Omega)$ ,  $\dots$ , respectively.*

**2.3. General Assumptions.** We will impose the following regularity and topology assumptions on our domain  $\Omega$ .

**Definition 2.18.** *Let  $\Omega$  be an open subset of  $\mathbb{R}^3$  with boundary  $\Gamma := \partial\Omega$ . We will call  $\Omega$*

- (i) *strong Lipschitz, if  $\Gamma$  is locally a graph of a Lipschitz function  $\psi : U \subset \mathbb{R}^2 \rightarrow \mathbb{R}$ ,*
- (ii) *topologically trivial, if  $\Omega$  is simply connected with connected boundary  $\Gamma$ .*

**General Assumption 2.19.** *From now on and throughout this paper it is assumed that  $\Omega \subset \mathbb{R}^3$  is a bounded strong Lipschitz domain.*

If the domain  $\Omega$  has to be topologically trivial, we will always indicate this in the respective result. Note that several results will hold for arbitrary open subsets  $\Omega$  of  $\mathbb{R}^3$ . All results are valid for bounded and topologically trivial strong Lipschitz domains  $\Omega \subset \mathbb{R}^3$ . Nevertheless, most of the results will remain true for bounded strong Lipschitz domains  $\Omega \subset \mathbb{R}^3$ .

**2.4. Vector Analysis.** In this last part of the preliminary section we summarize and prove several results related to scalar and vector potentials of various smoothness, corresponding Friedrichs/Poincaré type estimates, and related Helmholtz decompositions of  $L^2(\Omega)$  and other Hilbert and Sobolev spaces. This is a first application of the functional analysis toolbox Section 2.1 for the operators  $\overset{\circ}{\text{grad}}$ ,  $\overset{\circ}{\text{rot}}$ ,  $\overset{\circ}{\text{div}}$ , and their adjoints  $-\text{div}$ ,  $\text{rot}$ ,  $-\text{grad}$ . Although these are well known facts, we recall and collect them here, as we will use later similar techniques to obtain related results for the more complicated operators  $\overset{\circ}{\text{Gradgrad}}$ ,  $\overset{\circ}{\text{Rot}}_{\mathbb{S}}$ ,  $\overset{\circ}{\text{Div}}_{\mathbb{T}}$ , and their adjoints  $\text{div Div}_{\mathbb{S}}$ ,  $\text{sym Rot}_{\mathbb{T}}$ ,  $-\text{dev Grad}$ . Let

$$\begin{aligned} A_0 &:= \overset{\circ}{\text{grad}} : \overset{\circ}{\mathbf{H}}^1(\Omega) \subset L^2(\Omega) \longrightarrow L^2(\Omega), \\ A_1 &:= \overset{\circ}{\text{rot}} : \overset{\circ}{\mathbf{R}}(\Omega) \subset L^2(\Omega) \longrightarrow L^2(\Omega), \\ A_2 &:= \overset{\circ}{\text{div}} : \overset{\circ}{\mathbf{D}}(\Omega) \subset L^2(\Omega) \longrightarrow L^2(\Omega). \end{aligned}$$

Then  $A_0$ ,  $A_1$ , and  $A_2$  are unbounded, densely defined, and closed linear operators with adjoints

$$\begin{aligned} A_0^* &= \overset{\circ}{\text{grad}}^* = -\text{div} : \mathbf{D}(\Omega) \subset L^2(\Omega) \longrightarrow L^2(\Omega), \\ A_1^* &= \overset{\circ}{\text{rot}}^* = \text{rot} : \mathbf{R}(\Omega) \subset L^2(\Omega) \longrightarrow L^2(\Omega), \\ A_2^* &= \overset{\circ}{\text{div}}^* = -\text{grad} : \mathbf{H}^1(\Omega) \subset L^2(\Omega) \longrightarrow L^2(\Omega) \end{aligned}$$

and the sequence or complex properties

$$\begin{aligned} R(A_0) &= \overset{\circ}{\text{grad}} \overset{\circ}{\mathbf{H}}^1(\Omega) \subset \overset{\circ}{\mathbf{R}}_0(\Omega) = N(A_1), & R(A_1^*) &= \text{rot } \mathbf{R}(\Omega) \subset \mathbf{D}_0(\Omega) = N(A_0^*), \\ R(A_1) &= \overset{\circ}{\text{rot}} \overset{\circ}{\mathbf{R}}(\Omega) \subset \overset{\circ}{\mathbf{D}}_0(\Omega) = N(A_2), & R(A_2^*) &= \text{grad } \mathbf{H}^1(\Omega) \subset \mathbf{R}_0(\Omega) = N(A_1^*) \end{aligned}$$

hold. Note  $N(A_0) = \{0\}$  and  $N(A_2^*) = \mathbb{R}$ . Moreover, the embeddings

$$D(A_1) \cap D(A_0^*) = \overset{\circ}{\mathbf{R}}(\Omega) \cap \mathbf{D}(\Omega) \hookrightarrow L^2(\Omega), \quad D(A_2) \cap D(A_1^*) = \overset{\circ}{\mathbf{D}}(\Omega) \cap \mathbf{R}(\Omega) \hookrightarrow L^2(\Omega)$$

are compact. The latter compact embeddings are called Maxwell compactness properties or Weck's selection theorems. The first proof for strong Lipschitz domains (uniform cone like domains) avoiding smoothness of  $\Gamma$  was given by Weck in [27]. Generally, Weck's selection theorems hold e.g. for weak Lipschitz domains, see [22], or even for more general domains with  $p$ -cusps or antennas, see [28, 23]. See also [26] for a different proof in the case of a strong Lipschitz domain. Weck's selection theorem for mixed boundary conditions has been proved in [12] for strong Lipschitz domains and recently in [2] for weak Lipschitz domains. Similar to Rellich's selection theorem, i.e., the compact embedding of  $\overset{\circ}{\mathbf{H}}^1(\Omega)$  resp.  $\mathbf{H}^1(\Omega)$  into  $L^2(\Omega)$ , it is crucial that the domain  $\Omega$  is bounded. Finally, the kernels

$$N(A_1) \cap N(A_0^*) = \overset{\circ}{\mathbf{R}}_0(\Omega) \cap \mathbf{D}_0(\Omega) =: \mathcal{H}_{\mathbf{D}}(\Omega) \quad \text{resp.} \quad N(A_2) \cap N(A_1^*) = \overset{\circ}{\mathbf{D}}_0(\Omega) \cap \mathbf{R}_0(\Omega) =: \mathcal{H}_{\mathbf{N}}(\Omega),$$

are finite dimensional, as the unit balls are compact, i.e., the spaces of Dirichlet or Neumann fields are finite dimensional. More precisely, the dimension of the Dirichlet resp. Neumann fields depends on the topology or cohomology of  $\Omega$ , i.e., second resp. first Betti number, see e.g. [20, 21]. Especially we have

$$\mathcal{H}_{\mathbf{D}}(\Omega) = \{0\}, \text{ if } \Gamma \text{ is connected,} \quad \mathcal{H}_{\mathbf{N}}(\Omega) = \{0\}, \text{ if } \Omega \text{ is simply connected.}$$

**Remark 2.20.** *Our general assumption on  $\Omega$  to be bounded and strong Lipschitz ensures that Weck's selection theorems (and thus also Rellich's) hold. The additional assumption that  $\Omega$  is also topologically trivial excludes the existence of non-trivial Dirichlet or Neumann fields, as  $\Omega$  is simply connected with a connected boundary  $\Gamma$ .*

By the results of the functional analysis toolbox Section 2.1 we see that all ranges are closed with

$$R(A_0) = R(\mathcal{A}_0), \quad R(A_1) = R(\mathcal{A}_1), \quad R(A_2) = R(\mathcal{A}_2),$$

$$R(\mathcal{A}_0^*) = R(\mathcal{A}_0^*), \quad R(\mathcal{A}_1^*) = R(\mathcal{A}_1^*), \quad R(\mathcal{A}_2^*) = R(\mathcal{A}_2^*),$$

i.e., the ranges

$$(2.11) \quad \begin{aligned} \mathring{\text{grad}} \mathring{H}^1(\Omega), & & \text{grad } H^1(\Omega) &= \text{grad} (H^1(\Omega) \cap L_0^2(\Omega)), \\ \mathring{\text{rot}} \mathring{R}(\Omega) &= \mathring{\text{rot}} (\mathring{R}(\Omega) \cap \text{rot } R(\Omega)), & \text{rot } R(\Omega) &= \text{rot} (R(\Omega) \cap \mathring{\text{rot}} \mathring{R}(\Omega)), \\ \mathring{\text{div}} \mathring{D}(\Omega) &= \mathring{\text{div}} (\mathring{D}(\Omega) \cap \text{grad } H^1(\Omega)), & \text{div } D(\Omega) &= \text{div} (D(\Omega) \cap \mathring{\text{grad}} \mathring{H}^1(\Omega)) \end{aligned}$$

are closed, and the reduced operators are

$$\begin{aligned} \mathcal{A}_0 &= \mathring{\text{grad}} : \mathring{H}^1(\Omega) \subset L^2(\Omega) \longrightarrow \mathring{\text{grad}} \mathring{H}^1(\Omega), \\ \mathcal{A}_1 &= \mathring{\text{rot}} : \mathring{R}(\Omega) \cap \text{rot } R(\Omega) \subset \text{rot } R(\Omega) \longrightarrow \mathring{\text{rot}} \mathring{R}(\Omega), \\ \mathcal{A}_2 &= \mathring{\text{div}} : \mathring{D}(\Omega) \cap \text{grad } H^1(\Omega) \subset \text{grad } H^1(\Omega) \longrightarrow L_0^2(\Omega), \\ \mathcal{A}_0^* &= -\text{div} : D(\Omega) \cap \mathring{\text{grad}} \mathring{H}^1(\Omega) \subset \text{grad } \mathring{H}^1(\Omega) \longrightarrow L^2(\Omega), \\ \mathcal{A}_1^* &= \text{rot} : R(\Omega) \cap \mathring{\text{rot}} \mathring{R}(\Omega) \subset \mathring{\text{rot}} \mathring{R}(\Omega) \longrightarrow \text{rot } R(\Omega), \\ \mathcal{A}_2^* &= -\text{grad} : H^1(\Omega) \cap L_0^2(\Omega) \subset L_0^2(\Omega) \longrightarrow \text{grad } H^1(\Omega). \end{aligned}$$

Moreover, we have the following well known Helmholtz decompositions of  $L^2$ -vector fields into irrotational and solenoidal vector fields, corresponding Friedrichs/Poincaré type estimates and continuous or compact inverse operators.

**Lemma 2.21.** *The Helmholtz decompositions*

$$\begin{aligned} L^2(\Omega) &= \mathring{\text{div}} \mathring{D}(\Omega) \oplus_{L^2(\Omega)} \mathbb{R}, \quad \mathring{\text{div}} \mathring{D}(\Omega) = L_0^2(\Omega), \\ L^2(\Omega) &= \text{div } D(\Omega), \\ L^2(\Omega) &= \mathring{\text{grad}} \mathring{H}^1(\Omega) \oplus_{L^2(\Omega)} D_0(\Omega) \\ &= \mathring{R}_0(\Omega) \oplus_{L^2(\Omega)} \text{rot } R(\Omega) \\ &= \mathring{\text{grad}} \mathring{H}^1(\Omega) \oplus_{L^2(\Omega)} \mathcal{H}_D(\Omega) \oplus_{L^2(\Omega)} \text{rot } R(\Omega), \\ L^2(\Omega) &= \text{grad } H^1(\Omega) \oplus_{L^2(\Omega)} \mathring{D}_0(\Omega) \\ &= R_0(\Omega) \oplus_{L^2(\Omega)} \mathring{\text{rot}} \mathring{R}(\Omega) \\ &= \text{grad } H^1(\Omega) \oplus_{L^2(\Omega)} \mathcal{H}_N(\Omega) \oplus_{L^2(\Omega)} \mathring{\text{rot}} \mathring{R}(\Omega) \end{aligned}$$

*hold. Moreover, (2.11) is true for the respective ranges and the “better” potentials in (2.11) are uniquely determined and depend continuously in the right hand sides. If  $\Gamma$  is connected, it holds  $\mathcal{H}_D(\Omega) = \{0\}$  and, e.g.,*

$$L^2(\Omega) = \mathring{R}_0(\Omega) \oplus D_0(\Omega) \quad \text{and} \quad \mathring{R}_0(\Omega) = \mathring{\text{grad}} \mathring{H}^1(\Omega), \quad D_0(\Omega) = \text{rot } R(\Omega) = \text{rot} (R(\Omega) \cap \mathring{D}_0(\Omega)).$$

*If  $\Omega$  is simply connected, it holds  $\mathcal{H}_N(\Omega) = \{0\}$  and, e.g.,*

$$L^2(\Omega) = R_0(\Omega) \oplus \mathring{D}_0(\Omega) \quad \text{and} \quad R_0(\Omega) = \text{grad } H^1(\Omega), \quad \mathring{D}_0(\Omega) = \mathring{\text{rot}} \mathring{R}(\Omega) = \mathring{\text{rot}} (\mathring{R}(\Omega) \cap D_0(\Omega)).$$

**Lemma 2.22.** *The following Friedrichs/Poincaré type estimates hold. There exist positive constants  $c_g$ ,  $c_r$ ,  $c_d$ , such that*

$$\begin{aligned}
\forall u \in \mathring{H}^1(\Omega) & & |u|_{L^2(\Omega)} & \leq c_g |\operatorname{grad} u|_{L^2(\Omega)}, \\
\forall V \in D(\Omega) \cap \mathring{\operatorname{grad}} \mathring{H}^1(\Omega) & & |V|_{L^2(\Omega)} & \leq c_g |\operatorname{div} V|_{L^2(\Omega)}, \\
\forall V \in \mathring{R}(\Omega) \cap \operatorname{rot} R(\Omega) & & |V|_{L^2(\Omega)} & \leq c_r |\operatorname{rot} V|_{L^2(\Omega)}, \\
\forall V \in R(\Omega) \cap \mathring{\operatorname{rot}} \mathring{R}(\Omega) & & |V|_{L^2(\Omega)} & \leq c_r |\operatorname{rot} V|_{L^2(\Omega)}, \\
\forall V \in \mathring{D}(\Omega) \cap \operatorname{grad} H^1(\Omega) & & |V|_{L^2(\Omega)} & \leq c_d |\operatorname{div} V|_{L^2(\Omega)}, \\
\forall u \in H^1(\Omega) \cap L_0^2(\Omega) & & |u|_{L^2(\Omega)} & \leq c_d |\operatorname{grad} u|_{L^2(\Omega)}.
\end{aligned}$$

Moreover, the reduced versions of the operators

$$\mathring{\operatorname{grad}}, \quad \mathring{\operatorname{rot}}, \quad \mathring{\operatorname{div}}, \quad \operatorname{grad}, \quad \operatorname{rot}, \quad \operatorname{div}$$

have continuous resp. compact inverse operators

$$\begin{aligned}
\mathring{\operatorname{grad}}^{-1} : \mathring{\operatorname{grad}} \mathring{H}^1(\Omega) & \longrightarrow \mathring{H}^1(\Omega), & \mathring{\operatorname{grad}}^{-1} : \mathring{\operatorname{grad}} \mathring{H}^1(\Omega) & \longrightarrow L^2(\Omega), \\
\operatorname{div}^{-1} : L^2(\Omega) & \longrightarrow D(\Omega) \cap \mathring{\operatorname{grad}} \mathring{H}^1(\Omega), & \operatorname{div}^{-1} : L^2(\Omega) & \longrightarrow \mathring{\operatorname{grad}} \mathring{H}^1(\Omega) \subset L^2(\Omega), \\
\mathring{\operatorname{rot}}^{-1} : \mathring{\operatorname{rot}} \mathring{R}(\Omega) & \longrightarrow \mathring{R}(\Omega) \cap \operatorname{rot} R(\Omega), & \mathring{\operatorname{rot}}^{-1} : \mathring{\operatorname{rot}} \mathring{R}(\Omega) & \longrightarrow \operatorname{rot} R(\Omega) \subset L^2(\Omega), \\
\operatorname{rot}^{-1} : \operatorname{rot} R(\Omega) & \longrightarrow R(\Omega) \cap \mathring{\operatorname{rot}} \mathring{R}(\Omega), & \operatorname{rot}^{-1} : \operatorname{rot} R(\Omega) & \longrightarrow \mathring{\operatorname{rot}} \mathring{R}(\Omega) \subset L^2(\Omega), \\
\mathring{\operatorname{div}}^{-1} : L_0^2(\Omega) & \longrightarrow \mathring{D}(\Omega) \cap \operatorname{grad} H^1(\Omega), & \mathring{\operatorname{div}}^{-1} : L_0^2(\Omega) & \longrightarrow \operatorname{grad} H^1(\Omega) \subset L^2(\Omega), \\
\operatorname{grad}^{-1} : \operatorname{grad} H^1(\Omega) & \longrightarrow H^1(\Omega) \cap L_0^2(\Omega), & \operatorname{grad}^{-1} : \operatorname{grad} H^1(\Omega) & \longrightarrow L_0^2(\Omega),
\end{aligned}$$

with norms  $(1 + c_g^2)^{1/2}$ ,  $(1 + c_r^2)^{1/2}$ ,  $(1 + c_d^2)^{1/2}$  resp.  $c_g$ ,  $c_r$ ,  $c_d$ . In other words, the operators

$$\begin{aligned}
\mathring{\operatorname{grad}} : \mathring{H}^1(\Omega) & \longrightarrow \mathring{\operatorname{grad}} \mathring{H}^1(\Omega), & \operatorname{div} : D(\Omega) \cap \mathring{\operatorname{grad}} \mathring{H}^1(\Omega) & \longrightarrow L^2(\Omega), \\
u & \longmapsto \operatorname{grad} u & V & \longmapsto \operatorname{div} V \\
\mathring{\operatorname{rot}} : \mathring{R}(\Omega) \cap \operatorname{rot} R(\Omega) & \longrightarrow \mathring{\operatorname{rot}} \mathring{R}(\Omega), & \operatorname{rot} : R(\Omega) \cap \mathring{\operatorname{rot}} \mathring{R}(\Omega) & \longrightarrow \operatorname{rot} R(\Omega), \\
V & \longmapsto \operatorname{rot} V & V & \longmapsto \operatorname{rot} V \\
\mathring{\operatorname{div}} : \mathring{D}(\Omega) \cap \operatorname{grad} H^1(\Omega) & \longrightarrow L_0^2(\Omega), & \operatorname{grad} : H^1(\Omega) \cap L_0^2(\Omega) & \longrightarrow \operatorname{grad} H^1(\Omega), \\
V & \longmapsto \operatorname{div} V & u & \longmapsto \operatorname{grad} u
\end{aligned}$$

are topological isomorphisms. If  $\Omega$  is topologically trivial, then

$$\begin{aligned}
\mathring{\operatorname{grad}} : \mathring{H}^1(\Omega) & \longrightarrow \mathring{R}_0(\Omega), & \operatorname{div} : D(\Omega) \cap \mathring{R}_0(\Omega) & \longrightarrow L^2(\Omega), \\
u & \longmapsto \operatorname{grad} u & V & \longmapsto \operatorname{div} V \\
(2.12) \quad \mathring{\operatorname{rot}} : \mathring{R}(\Omega) \cap D_0(\Omega) & \longrightarrow \mathring{D}_0(\Omega), & \operatorname{rot} : R(\Omega) \cap \mathring{D}_0(\Omega) & \longrightarrow D_0(\Omega), \\
V & \longmapsto \operatorname{rot} V & V & \longmapsto \operatorname{rot} V \\
\mathring{\operatorname{div}} : \mathring{D}(\Omega) \cap R_0(\Omega) & \longrightarrow L_0^2(\Omega), & \operatorname{grad} : H^1(\Omega) \cap L_0^2(\Omega) & \longrightarrow R_0(\Omega), \\
V & \longmapsto \operatorname{div} V & u & \longmapsto \operatorname{grad} u
\end{aligned}$$

are topological isomorphisms.

**Remark 2.23.** Recently it has been shown in [17, 18, 19], that for bounded and convex  $\Omega \subset \mathbb{R}^3$  it holds

$$c_r \leq c_d \leq \frac{\text{diam } \Omega}{\pi},$$

i.e., the Maxwell constant  $c_r$  can be estimated from above by the Friedrichs/Poincaré constant.

**Remark 2.24.** Some of the previous results can be formulated equivalently in terms of complexes: The sequence

$$\{0\} \xrightarrow{0} \mathring{H}^1(\Omega) \xrightarrow{\mathring{\text{grad}}} \mathring{R}(\Omega) \xrightarrow{\mathring{\text{rot}}} \mathring{D}(\Omega) \xrightarrow{\mathring{\text{div}}} \mathring{L}^2(\Omega) \xrightarrow{\pi_{\mathbb{R}}} \mathbb{R}$$

and thus also its dual or adjoint sequence

$$\{0\} \xleftarrow{0} \mathring{L}^2(\Omega) \xleftarrow{-\mathring{\text{div}}} \mathring{D}(\Omega) \xleftarrow{\mathring{\text{rot}}} \mathring{R}(\Omega) \xleftarrow{-\mathring{\text{grad}}} \mathring{H}^1(\Omega) \xleftarrow{\iota_{\mathbb{R}}} \mathbb{R}$$

are closed Hilbert complexes. Here  $\pi_{\mathbb{R}} : \mathring{L}^2(\Omega) \rightarrow \mathbb{R}$  denotes the orthogonal projector onto  $\mathbb{R}$  with adjoint  $\pi_{\mathbb{R}}^* = \iota_{\mathbb{R}} : \mathbb{R} \rightarrow \mathring{L}^2(\Omega)$ , the canonical embedding. If  $\Omega$  is additionally topologically trivial, then the complexes are also exact. These complexes are widely known as de Rham complexes.

Let  $\Omega$  be additionally topologically trivial. For irrotational vector fields in  $\mathring{H}^m(\Omega)$  resp.  $H^m(\Omega)$  we have smooth potentials, which follows immediately by  $\mathring{R}_0(\Omega) = \mathring{\text{grad}} \mathring{H}^1(\Omega)$  resp.  $R_0(\Omega) = \text{grad } H^1(\Omega)$  from the previous lemma.

**Lemma 2.25.** Let  $\Omega$  be additionally topologically trivial and  $m \in \mathbb{N}_0$ . Then

$$\mathring{H}^m(\Omega) \cap \mathring{R}_0(\Omega) = \mathring{\text{grad}} \mathring{H}^{m+1}(\Omega), \quad H^m(\Omega) \cap R_0(\Omega) = \text{grad } H^{m+1}(\Omega)$$

hold with linear and continuous potential operators  $P_{\mathring{\text{grad}}}$ ,  $P_{\text{grad}}$ .

So, for each  $V \in \mathring{H}^m(\Omega) \cap \mathring{R}_0(\Omega)$ , we have  $V = \mathring{\text{grad}} u$  for the potential  $u = P_{\mathring{\text{grad}}} \circ V \in \mathring{H}^{m+1}(\Omega)$  and, analogously, for each  $V \in H^m(\Omega) \cap R_0(\Omega)$ , it holds  $V = \text{grad } u$  for the potential  $u = P_{\text{grad}} V \in H^{m+1}(\Omega)$ . Note that the potential in  $H^{m+1}(\Omega)$  is uniquely determined only up to a constant.

For solenoidal vector fields in  $\mathring{H}^m(\Omega)$  resp.  $H^m(\Omega)$  we have smooth potentials, too.

**Lemma 2.26.** Let  $\Omega$  be additionally topologically trivial and  $m \in \mathbb{N}_0$ . Then

$$\mathring{H}^m(\Omega) \cap \mathring{D}_0(\Omega) = \mathring{\text{rot}} \mathring{H}^{m+1}(\Omega), \quad H^m(\Omega) \cap D_0(\Omega) = \text{rot } H^{m+1}(\Omega)$$

hold with linear and continuous potential operators  $P_{\mathring{\text{rot}}}$ ,  $P_{\text{rot}}$ .

For a proof see, e.g., [6, Corollary 4.7] or with slight modifications the generalized lifting lemma [10, Corollary 5.4] for the case  $d = 3$ ,  $k = m$ ,  $l = 2$ . Moreover, the potential in  $\mathring{H}^{m+1}(\Omega)$  resp.  $H^{m+1}(\Omega)$  is no longer uniquely determined.

For the divergence operator we have the following result.

**Lemma 2.27.** Let  $m \in \mathbb{N}_0$ . Then

$$\mathring{H}^m(\Omega) \cap \mathring{L}_0^2(\Omega) = \mathring{\text{div}} \mathring{H}^{m+1}(\Omega), \quad H^m(\Omega) = \text{div } H^{m+1}(\Omega)$$

hold with linear and continuous potential operators  $P_{\mathring{\text{div}}}$ ,  $P_{\text{div}}$ .

Again, the potential in  $\mathring{H}^{m+1}(\Omega)$  resp.  $H^{m+1}(\Omega)$  is no longer uniquely determined. Also Lemma 2.25 resp. Lemma 2.27 has been proved in [6, Corollary 4.7(b)] and in [10, Corollary 5.4] for the case  $d = 3$ ,  $k = m$ ,  $l = 1$  resp.  $d = 3$ ,  $k = m$ ,  $l = 3$ .

**Remark 2.28.** Lemma 2.27, which shows a classical result on the solvability and on the properties of the solution operator of the divergence equation, is an important tool in fluid dynamics, i.e., in the theory of Stokes or Navier-Stokes equations. The potential operator is often called Bogovskiĭ operator, see [4, 5] for the original works and also [7, p. 179, Theorem III.3.3], [25, Lemma 2.1.1]. Moreover, there are also versions of Lemma 2.25 and Lemma 2.26, if  $\Omega$  is not topologically trivial, which we will not need in the paper at hand.

**Remark 2.29.** A closer inspection of Lemma 2.25 and Lemma 2.26 and their proofs shows, that these results extend to general topologies as well. More precisely we have:

(i) It holds

$$\begin{aligned}\mathring{H}^m(\Omega) \cap \mathring{\text{grad}} \mathring{H}^1(\Omega) &= \mathring{H}^m(\Omega) \cap \mathring{R}_0(\Omega) \cap \mathcal{H}_D(\Omega)^\perp = \mathring{\text{grad}} \mathring{H}^{m+1}(\Omega), \\ H^m(\Omega) \cap \text{grad} H^1(\Omega) &= H^m(\Omega) \cap R_0(\Omega) \cap \mathcal{H}_N(\Omega)^\perp = \text{grad} H^{m+1}(\Omega)\end{aligned}$$

with linear and continuous potential operators  $P_{\mathring{\text{grad}}}$ ,  $P_{\text{grad}}$ .

(ii) It holds

$$\begin{aligned}\mathring{H}^m(\Omega) \cap \mathring{\text{rot}} \mathring{R}(\Omega) &= \mathring{H}^m(\Omega) \cap \mathring{D}_0(\Omega) \cap \mathcal{H}_N(\Omega)^\perp = \mathring{\text{rot}} \mathring{H}^{m+1}(\Omega), \\ H^m(\Omega) \cap \text{rot} R(\Omega) &= H^m(\Omega) \cap D_0(\Omega) \cap \mathcal{H}_D(\Omega)^\perp = \text{rot} H^{m+1}(\Omega)\end{aligned}$$

with linear and continuous potential operators  $P_{\mathring{\text{rot}}}$ ,  $P_{\text{rot}}$ .

Using the latter three results and Lemma 2.14, irrotational and solenoidal vector fields in  $H^{-m}(\Omega)$  can be characterized.

**Corollary 2.30.** Let  $\Omega$  be additionally topologically trivial and  $m \in \mathbb{N}$ . Then

$$R_0^{-m}(\Omega) = \text{grad} H^{-m+1}(\Omega) = \text{grad} (\mathring{H}^{m-1}(\Omega) \cap L_0^2(\Omega))'$$

is closed in  $H^{-m}(\Omega)$  with continuous inverse, i.e.,  $\text{grad}^{-1} \in BL(R_0^{-m}(\Omega), (\mathring{H}^{m-1}(\Omega) \cap L_0^2(\Omega))')$ . Especially for  $m = 1$ ,

$$R_0^{-1}(\Omega) = \text{grad} L^2(\Omega) = \text{grad} L_0^2(\Omega)$$

is closed in  $H^{-1}(\Omega)$  with continuous inverse  $\text{grad}^{-1} \in BL(R_0^{-1}(\Omega), L_0^2(\Omega))$  and uniquely determined potential in  $L_0^2(\Omega)$ . Moreover,

$$\exists c_{\mathbf{g},-1} > 0 \quad \forall u \in L_0^2(\Omega) \quad |u|_{L^2(\Omega)} \leq c_{\mathbf{g},-1} |\text{grad} u|_{H^{-1}(\Omega)} \leq \sqrt{3} c_{\mathbf{g},-1} |u|_{L^2(\Omega)}$$

and the inf-sup-condition

$$0 < \frac{1}{c_{\mathbf{g},-1}} = \inf_{0 \neq u \in L_0^2(\Omega)} \frac{|\text{grad} u|_{H^{-1}(\Omega)}}{|u|_{L^2(\Omega)}} = \inf_{0 \neq u \in L_0^2(\Omega)} \sup_{0 \neq V \in \mathring{H}^1(\Omega)} \frac{\langle u, \text{div} V \rangle_{L^2(\Omega)}}{|u|_{L^2(\Omega)} |\text{Grad} V|_{L^2(\Omega)}}.$$

holds.

*Proof.* Let  $X_0 := \mathring{H}^{m+1}(\Omega)$ ,  $X_1 := \mathring{H}^m(\Omega)$ ,  $X_2 := \mathring{H}^{m-1}(\Omega)$  and

$$A_0 := \mathring{\text{rot}} : \mathring{H}^{m+1}(\Omega) \rightarrow \mathring{H}^m(\Omega), \quad A_1 := -\mathring{\text{div}} : \mathring{H}^m(\Omega) \rightarrow \mathring{H}^{m-1}(\Omega).$$

These linear operators are bounded,  $R(A_0) = \mathring{\text{rot}} \mathring{H}^{m+1}(\Omega) = \mathring{H}^m(\Omega) \cap \mathring{D}_0(\Omega) = N(A_1)$  by Lemma 2.26, and  $R(A_1) = \mathring{\text{div}} \mathring{H}^m(\Omega) = \mathring{H}^{m-1}(\Omega) \cap L_0^2(\Omega)$  by Lemma 2.27. Therefore,  $R(A_1)$  is closed. For the adjoint operators we get

$$A_0' = \mathring{\text{rot}} = \mathring{\text{rot}}' : H^{-m}(\Omega) \rightarrow H^{-m-1}(\Omega), \quad A_1' = \text{grad} = -\mathring{\text{div}}' : H^{-m+1}(\Omega) \rightarrow H^{-m}(\Omega)$$

and obtain from Lemma 2.14 that

$$R_0^{-m}(\Omega) = N(A_0') = R(A_1') = \text{grad} H^{-m+1}(\Omega)$$

is closed and

$$\text{grad}^{-1} = (A'_1)^{-1} \in BL(R(A'_1), R(A_1)') = BL(\mathring{R}_0^{-m}(\Omega), (\mathring{H}^{m-1}(\Omega) \cap \mathring{L}_0^2(\Omega))'),$$

which completes the proof for general  $m$ . If  $m = 1$ , we get the assertions about the Friedrichs/Poincaré/Nečas inequality and inf-sup-condition by Lemma 2.16, i.e., (2.8) and (2.10).  $\square$

**Corollary 2.31.** *Let  $\Omega$  be additionally topologically trivial and  $m \in \mathbb{N}$ . Then*

$$D_0^{-m}(\Omega) = \text{rot } H^{-m+1}(\Omega) = \text{rot } (\mathring{H}^{m-1}(\Omega) \cap \mathring{D}_0(\Omega))'$$

is closed in  $H^{-m}(\Omega)$  with continuous inverse, i.e.,  $\text{rot}^{-1} \in BL(D_0^{-m}(\Omega), (\mathring{H}^{m-1}(\Omega) \cap \mathring{D}_0(\Omega))')$ . Especially for  $m = 1$ ,

$$D_0^{-1}(\Omega) = \text{rot } L^2(\Omega) = \text{rot } \mathring{D}_0(\Omega)$$

is closed in  $H^{-1}(\Omega)$  with continuous inverse  $\text{rot}^{-1} \in BL(D_0^{-1}(\Omega), \mathring{D}_0(\Omega))$  and uniquely determined potential in  $\mathring{D}_0(\Omega)$ . Moreover,

$$\exists c_{r,-1} > 0 \quad \forall V \in \mathring{D}_0(\Omega) \quad |V|_{L^2(\Omega)} \leq c_{r,-1} |\text{rot } V|_{H^{-1}(\Omega)} \leq \sqrt{2} c_{r,-1} |V|_{L^2(\Omega)}$$

and the inf-sup-condition

$$0 < \frac{1}{c_{r,-1}} = \inf_{0 \neq V \in \mathring{D}_0(\Omega)} \frac{|\text{rot } V|_{H^{-1}(\Omega)}}{|V|_{L^2(\Omega)}} = \inf_{0 \neq V \in \mathring{D}_0(\Omega)} \sup_{0 \neq H \in \mathring{H}^1(\Omega)} \frac{\langle V, \text{rot } H \rangle_{L^2(\Omega)}}{|V|_{L^2(\Omega)} |\text{Grad } H|_{L^2(\Omega)}}.$$

holds.

*Proof.* Let  $X_0 := \mathring{H}^{m+1}(\Omega)$ ,  $X_1 := \mathring{H}^m(\Omega)$ ,  $X_2 := \mathring{H}^{m-1}(\Omega)$  and

$$A_0 := \text{grad} : \mathring{H}^{m+1}(\Omega) \rightarrow \mathring{H}^m(\Omega), \quad A_1 := \text{rot} : \mathring{H}^m(\Omega) \rightarrow \mathring{H}^{m-1}(\Omega).$$

These linear operators are bounded,  $R(A_0) = \text{grad } \mathring{H}^{m+1}(\Omega) = \mathring{H}^m(\Omega) \cap \mathring{R}_0(\Omega) = N(A_1)$  by Lemma 2.25, and  $R(A_1) = \text{rot } \mathring{H}^m(\Omega) = \mathring{H}^{m-1}(\Omega) \cap \mathring{D}_0(\Omega)$  by Lemma 2.26. Therefore,  $R(A_1)$  is closed. For the adjoint operators we get

$$A'_0 = -\text{div} = \mathring{\text{grad}}' : H^{-m}(\Omega) \rightarrow H^{-m-1}(\Omega), \quad A'_1 = \text{rot} = \mathring{\text{rot}}' : H^{-m+1}(\Omega) \rightarrow H^{-m}(\Omega)$$

and obtain from Lemma 2.14 that

$$D_0^{-m}(\Omega) = N(A'_0) = R(A'_1) = \text{rot } H^{-m+1}(\Omega)$$

is closed and

$$\text{rot}^{-1} = (A'_1)^{-1} \in BL(R(A'_1), R(A_1)') = BL(D_0^{-m}(\Omega), (\mathring{H}^{m-1}(\Omega) \cap \mathring{D}_0(\Omega))'),$$

which completes the proof for general  $m$ . If  $m = 1$ , we get the assertions about the Friedrichs/Poincaré/Nečas inequality and inf-sup-condition by Lemma 2.16, i.e., (2.8) and (2.10).  $\square$

Let us present the corresponding result for the divergence as well.

**Corollary 2.32.** *Let  $\Omega$  be additionally topologically trivial and  $m \in \mathbb{N}$ . Then*

$$H^{-m}(\Omega) = \text{div } H^{-m+1}(\Omega) = \text{div } (\mathring{H}^{m-1}(\Omega) \cap \mathring{R}_0(\Omega))'$$

(is closed in  $H^{-m}(\Omega)$ ) with continuous inverse, i.e.,  $\text{div}^{-1} \in BL(H^{-m}(\Omega), (\mathring{H}^{m-1}(\Omega) \cap \mathring{R}_0(\Omega))')$ . Especially for  $m = 1$ ,

$$H^{-1}(\Omega) = \text{div } L^2(\Omega) = \text{div } \mathring{R}_0(\Omega)$$

(is closed in  $\mathbf{H}^{-1}(\Omega)$ ) with continuous inverse  $\operatorname{div}^{-1} \in BL(\mathbf{H}^{-1}(\Omega), \mathring{\mathbf{R}}_0(\Omega))$  and uniquely determined potential in  $\mathring{\mathbf{R}}_0(\Omega)$ . Moreover,

$$\exists c_{d,-1} > 0 \quad \forall V \in \mathring{\mathbf{R}}_0(\Omega) \quad |V|_{\mathbf{L}^2(\Omega)} \leq c_{d,-1} |\operatorname{div} V|_{\mathbf{H}^{-1}(\Omega)} \leq c_{d,-1} |V|_{\mathbf{L}^2(\Omega)}$$

and the inf-sup-condition

$$0 < \frac{1}{c_{d,-1}} = \inf_{0 \neq V \in \mathring{\mathbf{R}}_0(\Omega)} \frac{|\operatorname{div} V|_{\mathbf{H}^{-1}(\Omega)}}{|V|_{\mathbf{L}^2(\Omega)}} = \inf_{0 \neq V \in \mathring{\mathbf{D}}_0(\Omega)} \sup_{0 \neq u \in \mathring{\mathbf{H}}^1(\Omega)} \frac{\langle V, \operatorname{grad} u \rangle_{\mathbf{L}^2(\Omega)}}{|V|_{\mathbf{L}^2(\Omega)} |\operatorname{grad} u|_{\mathbf{L}^2(\Omega)}}.$$

holds.

*Proof.* Let  $\mathbf{X}_1 := \mathring{\mathbf{H}}^m(\Omega)$ ,  $\mathbf{X}_2 := \mathring{\mathbf{H}}^{m-1}(\Omega)$  and  $\mathbf{A}_1 := -\operatorname{grad} : \mathring{\mathbf{H}}^m(\Omega) \rightarrow \mathring{\mathbf{H}}^{m-1}(\Omega)$ .  $\mathbf{A}_1$  is linear and bounded with  $R(\mathbf{A}_1) = \operatorname{grad} \mathring{\mathbf{H}}^m(\Omega) = \mathring{\mathbf{H}}^{m-1}(\Omega) \cap \mathring{\mathbf{R}}_0(\Omega)$  by Lemma 2.25. Therefore,  $R(\mathbf{A}_1)$  is closed. The adjoint is  $\mathbf{A}'_1 = \operatorname{div} = -\operatorname{grad}' : \mathbf{H}^{-m+1}(\Omega) \rightarrow \mathbf{H}^{-m}(\Omega)$  with closed range  $R(\mathbf{A}'_1) = \operatorname{div} \mathbf{H}^{-m+1}(\Omega)$  by the closed range theorem. Moreover,  $N(\mathbf{A}_1) = \{0\}$ . Hence  $\mathbf{A}'_1$  is surjective as  $\mathbf{A}_1$  is injective, i.e.,

$$\mathbf{H}^{-m}(\Omega) = N(\mathbf{A}_1)^\circ = R(\mathbf{A}'_1) = \operatorname{div} \mathbf{H}^{-m+1}(\Omega).$$

As  $\mathbf{A}_1$  is also surjective onto its range,  $\mathbf{A}'_1 = \operatorname{div} : \mathbf{H}^{-m+1}(\Omega) \rightarrow R(\mathbf{A}'_1)$  is bijective. By the bounded inverse theorem we get

$$\operatorname{div}^{-1} = (\mathbf{A}'_1)^{-1} \in BL(R(\mathbf{A}'_1), R(\mathbf{A}_1)') = BL(\mathbf{H}^{-m}(\Omega), (\mathring{\mathbf{H}}^{m-1}(\Omega) \cap \mathring{\mathbf{R}}_0(\Omega))'),$$

which completes the proof for general  $m$ . If  $m = 1$ , we get the assertions about the Friedrichs/Poincaré/Nečas inequality and inf-sup-condition by Lemma 2.16, i.e., (2.8) and (2.10).  $\square$

**Remark 2.33.** *The results of the latter three lemmas and corollaries can be formulated equivalently in terms of complexes: Let  $\Omega$  be additionally topologically trivial. Then the sequence*

$$\mathring{\mathbf{H}}^{m+1}(\Omega) \xrightarrow{\operatorname{grad}} \mathring{\mathbf{H}}^m(\Omega) \xrightarrow{\operatorname{rot}} \mathring{\mathbf{H}}^{m-1}(\Omega) \xrightarrow{\operatorname{div}} \mathring{\mathbf{H}}^{m-2}(\Omega)$$

and thus also its dual or adjoint sequence

$$\mathbf{H}^{-m-1}(\Omega) \xleftarrow{-\operatorname{div}} \mathbf{H}^{-m}(\Omega) \xleftarrow{-\operatorname{rot}} \mathbf{H}^{-m+1}(\Omega) \xleftarrow{-\operatorname{grad}} \mathbf{H}^{-m+2}(\Omega)$$

are closed and exact Banach complexes.

### 3. THE Grad grad- AND div Div-COMPLEXES

We will use the following standard notations from linear algebra. For vectors  $a, b \in \mathbb{R}^3$  and matrices  $A, B \in \mathbb{R}^{3 \times 3}$  the expressions  $a \cdot b$  and  $A : B$  denote the inner product of vectors and the Frobenius inner product of matrices, respectively. For a vector  $a \in \mathbb{R}^3$  with components  $a_i$  for  $i = 1, 2, 3$  the matrix  $\operatorname{spn} a \in \mathbb{R}^{3 \times 3}$  is defined by

$$\operatorname{spn} a = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}.$$

Observe that  $(\operatorname{spn} a) b = a \times b$  for  $a, b \in \mathbb{R}^3$ , where  $a \times b$  denotes the exterior product of vectors. The exterior product  $a \times B$  of a vector  $a \in \mathbb{R}^3$  and a matrix  $B \in \mathbb{R}^{3 \times 3}$  is defined as the matrix which is obtained by applying the exterior product row-wise. Note that  $\operatorname{spn}$  is a bijective mapping from  $\mathbb{R}^3$  to the set of skew-symmetric matrices in  $\mathbb{R}^{3 \times 3}$  with the inverse mapping  $\operatorname{spn}^{-1}$ . In addition to  $\operatorname{sym} A$  and  $\operatorname{skw} A$  for the symmetric part and the skew-symmetric part of a matrix  $A$ , we use  $\operatorname{dev} A$  and  $\operatorname{tr} A$  for denoting the deviatoric part and the trace of a matrix  $A$ . Finally, the set of symmetric matrices in  $\mathbb{R}^{3 \times 3}$  is denoted by  $\mathbb{S}$ , the set of matrices in  $\mathbb{R}^{3 \times 3}$  with vanishing trace is denoted by  $\mathbb{T}$ .

In this section we need several spaces of tensor fields. The spaces

$$\mathring{\mathbf{C}}^\infty(\Omega), \quad \mathbf{L}^2(\Omega), \quad \mathbf{H}^1(\Omega), \quad \mathring{\mathbf{H}}^1(\Omega), \quad \mathbf{D}(\Omega), \quad \mathring{\mathbf{D}}(\Omega), \quad \mathring{\mathbf{R}}_0(\Omega), \quad \dots$$



are introduced as those spaces of tensor fields, whose rows are in the corresponding spaces of vector fields  $\mathring{C}^\infty(\Omega)$ ,  $\mathbf{L}^2(\Omega)$ ,  $\mathbf{H}^1(\Omega)$ ,  $\mathring{\mathbf{H}}^1(\Omega)$ ,  $\mathbf{D}(\Omega)$ ,  $\mathring{\mathbf{D}}(\Omega)$ ,  $\mathring{\mathbf{R}}_0(\Omega)$ ,  $\dots$ , respectively. Additionally, we will need spaces allowing for a deviatoric gradient, a symmetric rotation, and a double divergence, i.e.,

$$\begin{aligned} \mathbf{G}_{\text{dev}}(\Omega) &:= \{V \in \mathbf{L}^2(\Omega) : \text{dev Grad } V \in \mathbf{L}^2(\Omega)\}, & \mathbf{G}_{\text{dev},0}(\Omega) &:= \{V \in \mathbf{L}^2(\Omega) : \text{dev Grad } V = 0\}, \\ \mathbf{R}_{\text{sym}}(\Omega) &:= \{\mathbf{E} \in \mathbf{L}^2(\Omega) : \text{sym Rot } \mathbf{E} \in \mathbf{L}^2(\Omega)\}, & \mathbf{R}_{\text{sym},0}(\Omega) &:= \{\mathbf{E} \in \mathbf{L}^2(\Omega) : \text{sym Rot } \mathbf{E} = 0\}, \\ \mathbf{DD}(\Omega) &:= \{\mathbf{M} \in \mathbf{L}^2(\Omega) : \text{div Div } \mathbf{M} \in \mathbf{L}^2(\Omega)\}, & \mathbf{DD}_0(\Omega) &:= \{\mathbf{M} \in \mathbf{L}^2(\Omega) : \text{div Div } \mathbf{M} = 0\}. \end{aligned}$$

Moreover, we introduce various spaces of symmetric tensor fields without prescribed boundary conditions, i.e.,

$$\mathbf{L}^2(\Omega, \mathbb{S}) := \{\mathbf{M} \in \mathbf{L}^2(\Omega) : \mathbf{M}^\top = \mathbf{M}\}, \quad \mathbf{DD}(\Omega, \mathbb{S}) := \mathbf{DD}(\Omega) \cap \mathbf{L}^2(\Omega, \mathbb{S}), \quad \dots,$$

and with homogeneous boundary conditions as closures of symmetric test tensor fields, i.e.,

$$\mathring{\mathbf{R}}(\Omega, \mathbb{S}) := \overline{\mathring{\mathbf{C}}^\infty(\Omega) \cap \mathbf{L}^2(\Omega, \mathbb{S})}^{\mathbf{R}(\Omega)}, \quad \dots,$$

as well as spaces of tensor fields with vanishing trace and without prescribed boundary conditions, i.e.,

$$\mathbf{L}^2(\Omega, \mathbb{T}) := \{\mathbf{E} \in \mathbf{L}^2(\Omega) : \text{tr } \mathbf{E} = 0\}, \quad \mathbf{H}^1(\Omega, \mathbb{T}) := \mathbf{H}^1(\Omega) \cap \mathbf{L}^2(\Omega, \mathbb{T}), \quad \dots,$$

and with homogeneous boundary conditions as closures of trace-free test tensor fields, i.e.,

$$\mathring{\mathbf{D}}(\Omega, \mathbb{T}) := \overline{\mathring{\mathbf{C}}^\infty(\Omega) \cap \mathbf{L}^2(\Omega, \mathbb{T})}^{\mathbf{D}(\Omega)}, \quad \dots$$

We note

$$\mathring{\mathbf{R}}(\Omega, \mathbb{S}) \subset \mathring{\mathbf{R}}(\Omega) \cap \mathbf{L}^2(\Omega, \mathbb{S}), \quad \mathring{\mathbf{D}}(\Omega, \mathbb{T}) \subset \mathring{\mathbf{D}}(\Omega) \cap \mathbf{L}^2(\Omega, \mathbb{T}), \quad \dots$$

Let us also mention that

$$\text{dev Grad } \mathbf{G}_{\text{dev}}(\Omega) \subset \mathbf{L}^2(\Omega, \mathbb{T}), \quad \text{sym Rot } \mathbf{R}_{\text{sym}}(\Omega) \subset \mathbf{L}^2(\Omega, \mathbb{S})$$

hold. This can be seen as follows. Pick  $\Phi \in \mathbf{G}_{\text{dev}}(\Omega)$  with  $\mathbf{E} := \text{dev Grad } \Phi$  and  $\Phi \in \mathbf{R}_{\text{sym}}(\Omega)$  with  $\mathbf{M} := \text{sym Rot } \Phi$ . Then for all  $\psi \in \mathring{\mathbf{C}}^\infty(\Omega)$  and  $\Psi \in \mathring{\mathbf{C}}^\infty(\Omega)$

$$\begin{aligned} \langle \text{tr } \mathbf{E}, \psi \rangle_{\mathbf{L}^2(\Omega)} &= \langle \mathbf{E}, \psi \mathbf{I} \rangle_{\mathbf{L}^2(\Omega)} = -\langle \Phi, \text{Div dev } \psi \mathbf{I} \rangle_{\mathbf{L}^2(\Omega)} = 0, \\ \langle \text{skw } \mathbf{M}, \Psi \rangle_{\mathbf{L}^2(\Omega)} &= \langle \mathbf{M}, \text{skw } \Psi \rangle_{\mathbf{L}^2(\Omega)} = \langle \Phi, \text{Rot sym skw } \Psi \rangle_{\mathbf{L}^2(\Omega)} = 0. \end{aligned}$$

Before we proceed we need a few technical lemmas.

**Lemma 3.1.** *For any distributional vector field  $V$  it holds for  $i, j, k = 1, \dots, 3$*

$$\partial_k(\text{Grad } V)_{ij} = \begin{cases} \partial_k(\text{dev Grad } V)_{ij} & , \text{ if } i \neq j, \\ \partial_j(\text{dev Grad } V)_{ik} & , \text{ if } i \neq k, \\ \frac{3}{2} \partial_i(\text{dev Grad } V)_{ii} + \frac{1}{2} \sum_{l \neq i} \partial_l(\text{dev Grad } V)_{li} & , \text{ if } i = j = k. \end{cases}$$

*Proof.* Let  $\Phi \in \mathring{\mathbf{C}}^\infty(\mathbb{R}^3)$  be a vector field. We want to express the second derivatives of  $\Phi$  by the derivatives of the deviatoric part of the Jacobian, i.e., of  $\text{dev Grad } \Phi$ . Recall that we have  $\text{dev } \mathbf{E} = \mathbf{E} - \frac{1}{3}(\text{tr } \mathbf{E}) \mathbf{I}$  for a tensor  $\mathbf{E}$ . Hence  $\text{dev Grad } \Phi$  coincides with  $\text{Grad } \Phi$  outside the diagonal entries, i.e., we have  $(\text{Grad } \Phi)_{ij} = (\text{dev Grad } \Phi)_{ij}$  for  $i \neq j$ . Hence, looking at second derivatives, we see immediately

$$\begin{aligned} \partial_k \partial_j \Phi_i &= \partial_k(\text{Grad } \Phi)_{ij} = \partial_k(\text{dev Grad } \Phi)_{ij} & \text{for } i \neq j, \\ \partial_k \partial_j \Phi_i &= \partial_j \partial_k \Phi_i = \partial_j(\text{Grad } \Phi)_{ik} = \partial_j(\text{dev Grad } \Phi)_{ik} & \text{for } i \neq k. \end{aligned}$$

Thus it remains to represent  $\partial_i^2 \Phi_i$  by the derivatives of  $\text{dev Grad } \Phi$ . By

$$\partial_i^2 \Phi_i = \partial_i(\text{Grad } \Phi)_{ii} = \partial_i(\text{dev Grad } \Phi)_{ii} + \frac{1}{3} \partial_i \text{div } \Phi$$

we get

$$\frac{2}{3} \partial_i^2 \Phi_i = \partial_i(\operatorname{dev Grad} \Phi)_{ii} + \frac{1}{3} \sum_{l \neq i} \partial_i \partial_l \Phi_l = \partial_i(\operatorname{dev Grad} \Phi)_{ii} + \frac{1}{3} \sum_{l \neq i} \partial_l(\operatorname{dev Grad} \Phi)_{li},$$

yielding the stated result for test vector fields. Testing extends the formulas to distributions, which finishes the proof.  $\square$

We note that the latter trick is similar to the well known fact that second derivatives of a vector field can always be written as derivatives of the symmetric gradient of the vector field, leading by Nečas estimate to Korn's second and first inequalities. We will now do the same for the operator  $\operatorname{dev Grad}$ .

**Lemma 3.2.** *It holds:*

(i) *There exists  $c > 0$ , such that for all vector fields  $V \in \mathbf{H}^1(\Omega)$*

$$|\operatorname{Grad} V|_{\mathbf{L}^2(\Omega)} \leq c (|V|_{\mathbf{L}^2(\Omega)} + |\operatorname{dev Grad} V|_{\mathbf{L}^2(\Omega)}).$$

(ii)  $\mathbf{G}_{\operatorname{dev}}(\Omega) = \mathbf{H}^1(\Omega)$ .

(iii) *For  $\operatorname{dev Grad} : \mathbf{G}_{\operatorname{dev}}(\Omega) \subset \mathbf{L}^2(\Omega) \longrightarrow \mathbf{L}^2(\Omega, \mathbb{T})$  it holds  $D(\operatorname{dev Grad}) = \mathbf{G}_{\operatorname{dev}}(\Omega) = \mathbf{H}^1(\Omega)$ , and the kernel of  $\operatorname{dev Grad}$  equals the space of (global) shape functions of the lowest order Raviart-Thomas elements, i.e.,*

$$N(\operatorname{dev Grad}) = \mathbf{G}_{\operatorname{dev},0}(\Omega) = \operatorname{RT}_0 := \{P : P(x) = ax + b, a \in \mathbb{R}, b \in \mathbb{R}^3\},$$

*which dimension is  $\dim \operatorname{RT}_0 = 4$ .*

(iv) *There exists  $c > 0$ , such that for all vector fields  $V \in \mathbf{H}^1(\Omega) \cap \operatorname{RT}_0^{\perp \mathbf{L}^2(\Omega)}$*

$$|V|_{\mathbf{H}^1(\Omega)} \leq c |\operatorname{dev Grad} V|_{\mathbf{L}^2(\Omega)}.$$

*Proof.* Let  $V \in \mathbf{H}^1(\Omega)$ . By the latter lemma and Nečas estimate, i.e.,

$$\exists c > 0 \quad \forall u \in \mathbf{L}^2(\Omega) \quad c |u|_{\mathbf{L}^2(\Omega)} \leq |\operatorname{grad} u|_{\mathbf{H}^{-1}(\Omega)} + |u|_{\mathbf{H}^{-1}(\Omega)} \leq (\sqrt{3} + 1) |u|_{\mathbf{L}^2(\Omega)},$$

we get

$$\begin{aligned} |\operatorname{Grad} V|_{\mathbf{L}^2(\Omega)} &\leq c \left( \sum_{k=1}^3 |\partial_k \operatorname{Grad} V|_{\mathbf{H}^{-1}(\Omega)} + |\operatorname{Grad} V|_{\mathbf{H}^{-1}(\Omega)} \right) \\ &\leq c \left( \sum_{k=1}^3 |\partial_k \operatorname{dev Grad} V|_{\mathbf{H}^{-1}(\Omega)} + |\operatorname{Grad} V|_{\mathbf{H}^{-1}(\Omega)} \right) \\ &\leq c (|\operatorname{dev Grad} V|_{\mathbf{L}^2(\Omega)} + |V|_{\mathbf{L}^2(\Omega)}), \end{aligned}$$

which shows (i). As  $\Omega$  has the segment property and by standard mollification we obtain that restrictions of  $\mathring{C}^\infty(\mathbb{R}^3)$ -vector fields are dense in  $\mathbf{G}_{\operatorname{dev}}(\Omega)$ . Especially  $\mathbf{H}^1(\Omega)$  is dense in  $\mathbf{G}_{\operatorname{dev}}(\Omega)$ . Let  $V \in \mathbf{G}_{\operatorname{dev}}(\Omega)$  and  $(V_n) \subset \mathbf{H}^1(\Omega)$  with  $V_n \rightarrow V$  in  $\mathbf{G}_{\operatorname{dev}}(\Omega)$ . By (i)  $(V_n)$  is a Cauchy sequence in  $\mathbf{H}^1(\Omega)$  converging to  $V$  in  $\mathbf{H}^1(\Omega)$ , which proves  $V \in \mathbf{H}^1(\Omega)$  and hence (ii). For  $P \in \operatorname{RT}_0$  it holds  $\operatorname{dev Grad} P = a \operatorname{dev} \mathbf{I} = 0$ . Let  $\operatorname{dev Grad} V = 0$  for some vector field  $V \in \mathbf{G}_{\operatorname{dev}}(\Omega) = \mathbf{H}^1(\Omega)$ . By Lemma 3.1 we get  $\partial_k \operatorname{Grad} V = 0$  for all  $k = 1, \dots, 3$ , and therefore  $V(x) = Ax + b$  for some matrix  $A \in \mathbb{R}^{3 \times 3}$  and vector  $b \in \mathbb{R}^3$ . Then  $0 = \operatorname{dev Grad} V = \operatorname{dev} A$ , if and only if  $A = \frac{1}{3}(\operatorname{tr} A) \mathbf{I}$ , which shows (iii). If (iv) was wrong, there exists a sequence  $(V_n) \subset \mathbf{H}^1(\Omega) \cap \operatorname{RT}_0^{\perp \mathbf{L}^2(\Omega)}$  with  $|V_n|_{\mathbf{H}^1(\Omega)} = 1$  and  $\operatorname{dev Grad} V_n \rightarrow 0$ . As  $(V_n)$  is bounded in  $\mathbf{H}^1(\Omega)$ , by Rellich's selection theorem there exists a subsequence, again denoted by  $(V_n)$ , and some  $V \in \mathbf{L}^2(\Omega)$  with  $V_n \rightarrow V$  in  $\mathbf{L}^2(\Omega)$ . By (i),  $(V_n)$  is a Cauchy sequence in  $\mathbf{H}^1(\Omega)$ . Hence  $V_n \rightarrow V$  in  $\mathbf{H}^1(\Omega)$  and  $V \in \mathbf{H}^1(\Omega) \cap \operatorname{RT}_0^{\perp \mathbf{L}^2(\Omega)}$ . As  $0 \leftarrow \operatorname{dev Grad} V_n \rightarrow \operatorname{dev Grad} V$ , we have by (iii)  $V \in \operatorname{RT}_0 \cap \operatorname{RT}_0^{\perp \mathbf{L}^2(\Omega)} = \{0\}$ , a contradiction to  $1 = |V_n|_{\mathbf{H}^1(\Omega)} \rightarrow 0$ . The proof is complete.  $\square$

We recall the following well-known result.

**Lemma 3.3.** Let  $\mathring{\mathbf{G}}\mathring{\mathbf{G}}(\Omega) := \{u \in \mathbf{L}^2(\Omega) : \text{Grad grad } u \in \mathbf{L}^2(\Omega)\}$  and  $\mathring{\mathbf{G}}\mathring{\mathbf{G}}(\Omega) := \overline{\mathring{\mathbf{C}}^\infty(\Omega)}^{\mathring{\mathbf{G}}\mathring{\mathbf{G}}(\Omega)}$ . Then

$$\mathring{\mathbf{G}}\mathring{\mathbf{G}}(\Omega) = \mathring{\mathbf{H}}^2(\Omega), \quad \mathring{\mathbf{G}}\mathring{\mathbf{G}}_0(\Omega) = \{0\},$$

and there exists  $c > 0$  such that for all  $u \in \mathring{\mathbf{H}}^2(\Omega)$

$$|u|_{\mathring{\mathbf{H}}^2(\Omega)} \leq c |\text{Grad grad } u|_{\mathbf{L}^2(\Omega)} = c |\Delta u|_{\mathbf{L}^2(\Omega)}.$$

It holds  $c \leq \sqrt{1 + c_{\mathring{\mathbf{g}}}^2(1 + c_{\mathring{\mathbf{g}}}^2)} \leq 1 + c_{\mathring{\mathbf{g}}}^2$ .

By straight forward calculations and standard arguments for distributions, see the Appendix, we get the following.

**Lemma 3.4.** It holds:

- (i)  $\text{skw Grad grad } \mathbf{H}^2(\Omega) = 0$ , i.e., Hessians are symmetric.
- (ii)  $\text{tr Rot } \mathbf{R}(\Omega, \mathbb{S}) = 0$ , i.e., rotations of symmetric tensors are trace free.

These formulas extend to distributions as well.

With Lemma 3.3 and Lemma 3.4 let us now consider the linear operators

$$(3.1) \quad \mathbf{A}_0 := \text{Grad grad} : \mathring{\mathbf{G}}\mathring{\mathbf{G}}(\Omega) = \mathring{\mathbf{H}}^2(\Omega) \subset \mathbf{L}^2(\Omega) \longrightarrow \mathbf{L}^2(\Omega, \mathbb{S}), \quad u \mapsto \text{Grad grad } u,$$

$$(3.2) \quad \mathbf{A}_1 := \text{Rot}_{\mathbb{S}} : \mathring{\mathbf{R}}(\Omega, \mathbb{S}) \subset \mathbf{L}^2(\Omega, \mathbb{S}) \longrightarrow \mathbf{L}^2(\Omega, \mathbb{T}), \quad \mathbf{M} \mapsto \text{Rot } \mathbf{M},$$

$$(3.3) \quad \mathbf{A}_2 := \text{Div}_{\mathbb{T}} : \mathring{\mathbf{D}}(\Omega, \mathbb{T}) \subset \mathbf{L}^2(\Omega, \mathbb{T}) \longrightarrow \mathbf{L}^2(\Omega), \quad \mathbf{E} \mapsto \text{Div } \mathbf{E}.$$

These are well and densely defined and closed. Closedness is clear. For densely definedness we look e.g. at  $\mathring{\text{Rot}}_{\mathbb{S}}$ . For  $\mathbf{M} \in \mathbf{L}^2(\Omega, \mathbb{S})$  pick  $(\mathring{\Phi}_n) \subset \mathring{\mathbf{C}}^\infty(\Omega)$  with  $\mathring{\Phi}_n \rightarrow \mathbf{M}$  in  $\mathbf{L}^2(\Omega)$ . Then

$$|\mathbf{M} - \text{sym } \mathring{\Phi}_n|_{\mathbf{L}^2(\Omega)}^2 + |\text{skw } \mathring{\Phi}_n|_{\mathbf{L}^2(\Omega)}^2 = |\mathbf{M} - \mathring{\Phi}_n|_{\mathbf{L}^2(\Omega)}^2 \rightarrow 0,$$

showing  $(\text{sym } \mathring{\Phi}_n) \subset \mathring{\mathbf{C}}^\infty(\Omega) \cap \mathbf{L}^2(\Omega, \mathbb{S}) \subset \mathring{\mathbf{R}}(\Omega, \mathbb{S})$  and  $\text{sym } \mathring{\Phi}_n \rightarrow \mathbf{M}$  in  $\mathbf{L}^2(\Omega, \mathbb{S})$ . By Lemma 3.3 the kernels are

$$N(\text{Grad grad}) = \{0\}, \quad N(\text{Rot}_{\mathbb{S}}) = \mathring{\mathbf{R}}_0(\Omega, \mathbb{S}), \quad N(\text{Div}_{\mathbb{T}}) = \mathring{\mathbf{D}}_0(\Omega, \mathbb{T}).$$

**Lemma 3.5.** The adjoints of (3.1), (3.2), (3.3) are

$$\mathbf{A}_0^* = (\text{Grad grad})^* = \text{div Div}_{\mathbb{S}} : \mathring{\mathbf{D}}\mathring{\mathbf{D}}(\Omega, \mathbb{S}) \subset \mathbf{L}^2(\Omega, \mathbb{S}) \longrightarrow \mathbf{L}^2(\Omega), \quad \mathbf{M} \mapsto \text{div Div } \mathbf{M},$$

$$\mathbf{A}_1^* = (\text{Rot}_{\mathbb{S}})^* = \text{sym Rot}_{\mathbb{T}} : \mathbf{R}_{\text{sym},0}(\Omega, \mathbb{T}) \subset \mathbf{L}^2(\Omega, \mathbb{T}) \longrightarrow \mathbf{L}^2(\Omega, \mathbb{S}), \quad \mathbf{E} \mapsto \text{sym Rot } \mathbf{E},$$

$$\mathbf{A}_2^* = (\text{Div}_{\mathbb{T}})^* = -\text{dev Grad} : \mathbf{G}_{\text{dev}}(\Omega) = \mathbf{H}^1(\Omega) \subset \mathbf{L}^2(\Omega) \longrightarrow \mathbf{L}^2(\Omega, \mathbb{T}), \quad V \mapsto -\text{dev Grad } V.$$

with kernels

$$N(\text{div Div}_{\mathbb{S}}) = \mathring{\mathbf{D}}\mathring{\mathbf{D}}_0(\Omega, \mathbb{S}), \quad N(\text{sym Rot}_{\mathbb{T}}) = \mathbf{R}_{\text{sym},0}(\Omega, \mathbb{T}), \quad N(\text{dev Grad}) = \mathbf{R}\mathbf{T}_0.$$

*Proof.* We have  $\mathbf{M} \in D((\text{Grad grad})^*) \subset \mathbf{L}^2(\Omega, \mathbb{S})$  and  $(\text{Grad grad})^* \mathbf{M} = u \in \mathbf{L}^2(\Omega)$ , if and only if  $\mathbf{M} \in \mathbf{L}^2(\Omega, \mathbb{S})$  and there exists  $u \in \mathbf{L}^2(\Omega)$ , such that

$$\begin{aligned} \forall \varphi \in D(\text{Grad grad}) = \mathring{\mathbf{H}}^2(\Omega) & \quad \langle \text{Grad grad } \varphi, \mathbf{M} \rangle_{\mathbf{L}^2(\Omega, \mathbb{S})} = \langle \varphi, u \rangle_{\mathbf{L}^2(\Omega)} \\ \Leftrightarrow \forall \varphi \in \mathring{\mathbf{C}}^\infty(\Omega) & \quad \langle \text{Grad grad } \varphi, \mathbf{M} \rangle_{\mathbf{L}^2(\Omega)} = \langle \varphi, u \rangle_{\mathbf{L}^2(\Omega)}, \end{aligned}$$

if and only if  $\mathbf{M} \in \mathbf{DD}(\Omega) \cap \mathbf{L}^2(\Omega, \mathbb{S}) = \mathbf{DD}(\Omega, \mathbb{S})$  and  $\operatorname{div} \operatorname{Div} \mathbf{M} = u$ . Moreover, we observe that  $\mathbf{E} \in D((\operatorname{Rot}_{\mathbb{S}})^*) \subset \mathbf{L}^2(\Omega, \mathbb{T})$  and  $(\operatorname{Rot}_{\mathbb{S}})^* \mathbf{E} = \mathbf{M} \in \mathbf{L}^2(\Omega, \mathbb{S})$ , if and only if  $\mathbf{E} \in \mathbf{L}^2(\Omega, \mathbb{T})$  and there exists  $\mathbf{M} \in \mathbf{L}^2(\Omega, \mathbb{S})$ , such that (note  $\operatorname{sym}^2 = \operatorname{sym}$ )

$$\begin{aligned} & \forall \Phi \in D(\operatorname{Rot}_{\mathbb{S}}) = \mathring{\mathbf{R}}(\Omega, \mathbb{S}) & \langle \operatorname{Rot} \Phi, \mathbf{E} \rangle_{\mathbf{L}^2(\Omega, \mathbb{T})} &= \langle \Phi, \mathbf{M} \rangle_{\mathbf{L}^2(\Omega, \mathbb{S})} \\ \Leftrightarrow & \forall \Phi \in \mathring{\mathbf{C}}^\infty(\Omega) \cap \mathbf{L}^2(\Omega, \mathbb{S}) & \langle \operatorname{Rot} \operatorname{sym} \Phi, \mathbf{E} \rangle_{\mathbf{L}^2(\Omega)} &= \langle \operatorname{sym} \Phi, \mathbf{M} \rangle_{\mathbf{L}^2(\Omega)} \\ \Leftrightarrow & \forall \Phi \in \mathring{\mathbf{C}}^\infty(\Omega) & \langle \operatorname{Rot} \operatorname{sym} \Phi, \mathbf{E} \rangle_{\mathbf{L}^2(\Omega)} &= \langle \operatorname{sym} \Phi, \mathbf{M} \rangle_{\mathbf{L}^2(\Omega)} \\ \Leftrightarrow & \forall \Phi \in \mathring{\mathbf{C}}^\infty(\Omega) & \langle \operatorname{Rot} \operatorname{sym} \Phi, \mathbf{E} \rangle_{\mathbf{L}^2(\Omega)} &= \langle \Phi, \mathbf{M} \rangle_{\mathbf{L}^2(\Omega)}, \end{aligned}$$

if and only if  $\mathbf{E} \in \mathbf{R}_{\operatorname{sym}}(\Omega) \cap \mathbf{L}^2(\Omega, \mathbb{T}) = \mathbf{R}_{\operatorname{sym}}(\Omega, \mathbb{T})$  and  $\operatorname{sym} \operatorname{Rot} \mathbf{E} = \mathbf{M}$ . Similarly, we see that  $V \in D((\operatorname{Div}_{\mathbb{T}})^*) \subset \mathbf{L}^2(\Omega)$  and  $(\operatorname{Div}_{\mathbb{T}})^* V = \mathbf{E} \in \mathbf{L}^2(\Omega, \mathbb{T})$ , if and only if  $V \in \mathbf{L}^2(\Omega)$  and there exists  $\mathbf{E} \in \mathbf{L}^2(\Omega, \mathbb{T})$ , such that (note  $\operatorname{dev}^2 = \operatorname{dev}$ )

$$\begin{aligned} & \forall \Phi \in D(\operatorname{Div}_{\mathbb{S}}) = \mathring{\mathbf{D}}(\Omega, \mathbb{T}) & \langle \operatorname{Div} \Phi, V \rangle_{\mathbf{L}^2(\Omega)} &= \langle \Phi, \mathbf{E} \rangle_{\mathbf{L}^2(\Omega, \mathbb{T})} \\ \Leftrightarrow & \forall \Phi \in \mathring{\mathbf{C}}^\infty(\Omega) \cap \mathbf{L}^2(\Omega, \mathbb{T}) & \langle \operatorname{Div} \operatorname{dev} \Phi, V \rangle_{\mathbf{L}^2(\Omega)} &= \langle \operatorname{dev} \Phi, \mathbf{E} \rangle_{\mathbf{L}^2(\Omega)} \\ \Leftrightarrow & \forall \Phi \in \mathring{\mathbf{C}}^\infty(\Omega) & \langle \operatorname{Div} \operatorname{dev} \Phi, V \rangle_{\mathbf{L}^2(\Omega)} &= \langle \operatorname{dev} \Phi, \mathbf{E} \rangle_{\mathbf{L}^2(\Omega)} \\ \Leftrightarrow & \forall \Phi \in \mathring{\mathbf{C}}^\infty(\Omega) & \langle \operatorname{Div} \operatorname{dev} \Phi, V \rangle_{\mathbf{L}^2(\Omega)} &= \langle \Phi, \mathbf{E} \rangle_{\mathbf{L}^2(\Omega)}, \end{aligned}$$

if and only if  $V \in \mathbf{G}_{\operatorname{dev}}(\Omega) = \mathbf{H}^1(\Omega)$  and  $-\operatorname{dev} \operatorname{Grad} V = \mathbf{E}$  using Lemma 3.2. Lemma 3.2 also shows  $N(\operatorname{dev} \operatorname{Grad}) = \mathbf{G}_{\operatorname{dev}, 0}(\Omega) = \operatorname{RT}_0$ , completing the proof.  $\square$

**Remark 3.6.** Note that, e.g., the second order operator  $\operatorname{Grad} \overset{\circ}{\operatorname{grad}}$  is “one” operator and not a composition of the two first order operators  $\operatorname{Grad}$  and  $\overset{\circ}{\operatorname{grad}}$ . Similarly the operator  $\operatorname{div} \operatorname{Div}_{\mathbb{S}}$ ,  $\operatorname{sym} \operatorname{Rot}_{\mathbb{T}}$ , resp.  $\operatorname{dev} \operatorname{Grad}$  has to be understood as “one” operator.

We observe the following complex properties for  $A_0, A_1, A_2$ , and  $A_0^*, A_1^*, A_2^*$ .

**Lemma 3.7.** It holds

$$\operatorname{Rot}_{\mathbb{S}} \overset{\circ}{\operatorname{Grad}} \overset{\circ}{\operatorname{grad}} = 0, \quad \operatorname{Div}_{\mathbb{T}} \overset{\circ}{\operatorname{Rot}}_{\mathbb{S}} = 0, \quad \operatorname{div} \operatorname{Div}_{\mathbb{S}} \operatorname{sym} \operatorname{Rot}_{\mathbb{T}} = 0, \quad \operatorname{sym} \operatorname{Rot}_{\mathbb{T}} \operatorname{dev} \operatorname{Grad} = 0,$$

i.e.,

$$\begin{aligned} R(\operatorname{Grad} \overset{\circ}{\operatorname{grad}}) &\subset N(\operatorname{Rot}_{\mathbb{S}}), & R(\operatorname{sym} \operatorname{Rot}_{\mathbb{T}}) &\subset N(\operatorname{div} \operatorname{Div}_{\mathbb{S}}), \\ R(\operatorname{Rot}_{\mathbb{S}}) &\subset N(\operatorname{Div}_{\mathbb{T}}), & R(\operatorname{dev} \operatorname{Grad}) &\subset N(\operatorname{sym} \operatorname{Rot}_{\mathbb{T}}). \end{aligned}$$

*Proof.* For  $\mathbf{E} = \operatorname{Rot} \mathbf{M} \in R(\operatorname{Rot}_{\mathbb{S}})$  with  $\mathbf{M} \in D(\operatorname{Rot}_{\mathbb{S}})$  there exists a sequence  $(\mathbf{M}_n) \subset \mathring{\mathbf{C}}^\infty(\Omega) \cap \mathbf{L}^2(\Omega, \mathbb{S})$  such that  $\mathbf{M}_n \rightarrow \mathbf{M}$  in the graph norm of  $D(\operatorname{Rot}_{\mathbb{S}})$ . As

$$\operatorname{Rot}(\mathring{\mathbf{C}}^\infty(\Omega) \cap \mathbf{L}^2(\Omega, \mathbb{S})) \subset \mathring{\mathbf{C}}^\infty(\Omega) \cap \mathbf{L}^2(\Omega, \mathbb{T}) \cap \mathbf{D}_0(\Omega) \subset N(\operatorname{Div}_{\mathbb{T}})$$

we have  $\mathbf{E} \in N(\operatorname{Div}_{\mathbb{T}})$  since  $\mathbf{E} \leftarrow \operatorname{Rot} \mathbf{M}_n \in N(\operatorname{Div}_{\mathbb{T}})$ . Hence  $R(\operatorname{Rot}_{\mathbb{S}}) \subset N(\operatorname{Div}_{\mathbb{T}})$ , i.e.,  $\operatorname{Div}_{\mathbb{T}} \operatorname{Rot}_{\mathbb{S}} = 0$  and for the adjoints we have  $\operatorname{sym} \operatorname{Rot}_{\mathbb{T}} \operatorname{dev} \operatorname{Grad} = 0$ . Analogously we see the other two inclusions.  $\square$

**Remark 3.8.** The latter considerations show that the sequence

$$\{0\} \xrightarrow{0} \mathring{\mathbf{H}}^2(\Omega) \xrightarrow{\operatorname{Grad} \overset{\circ}{\operatorname{grad}}} \mathring{\mathbf{R}}(\Omega; \mathbb{S}) \xrightarrow{\operatorname{Rot}_{\mathbb{S}}} \mathring{\mathbf{D}}(\Omega, \mathbb{T}) \xrightarrow{\operatorname{Div}_{\mathbb{T}}} \mathbf{L}^2(\Omega) \xrightarrow{\pi_{\operatorname{RT}_0}} \operatorname{RT}_0$$

and thus also its dual or adjoint sequence

$$\{0\} \xleftarrow{0} \mathbf{L}^2(\Omega) \xleftarrow{\text{div Div}_{\mathbb{S}}} \mathbf{DD}(\Omega, \mathbb{S}) \xleftarrow{\text{sym Rot}_{\mathbb{T}}} \mathbf{R}_{\text{sym}}(\Omega, \mathbb{T}) \xleftarrow{-\text{dev Grad}} \mathbf{H}^1(\Omega) \xleftarrow{\iota_{\text{RT}_0}} \text{RT}_0$$

are Hilbert complexes. Here  $\pi_{\text{RT}_0} : \mathbf{L}^2(\Omega) \rightarrow \text{RT}_0$  denotes the orthogonal projector onto  $\text{RT}_0$  with adjoint  $\pi_{\text{RT}_0}^* = \iota_{\text{RT}_0} : \text{RT}_0 \rightarrow \mathbf{L}^2(\Omega)$ , the canonical embedding. The first complex might be called Grad grad-complex and the second one div Div-complex.

**3.1. Topologically Trivial Domains.** We start with a useful lemma, which will be shown in the Appendix, collecting a few differential identities, which will be utilized in the proof of the subsequent main theorem.

**Lemma 3.9.** *Let  $u$ ,  $V$ , and  $\mathbf{E}$  be distributional scalar, vector, and tensor fields. Then*

- (i)  $2 \text{skw Grad } V = \text{spn rot } V$ ,
- (ii)  $\text{Rot spn } V = (\text{div } V) \mathbf{I} - (\text{Grad } V)^\top$  and, as a consequence,  $\text{tr Rot spn } V = 2 \text{div } V$ ,
- (iii)  $\text{Div}(u \mathbf{I}) = \text{grad } u$  and  $\text{Rot}(u \mathbf{I}) = -\text{spn grad } u$ ,
- (iv)  $2 \text{grad div } V = 3 \text{Div}(\text{dev}(\text{Grad } V)^\top)$ ,
- (v)  $\text{skw Rot } \mathbf{E} = \text{spn } H$  and  $\text{Div}(\text{sym Rot } \mathbf{E}) = \text{rot } H$  with  $2H = \text{Div } \mathbf{E}^\top - \text{grad}(\text{tr } \mathbf{E})$ ,
- (vi)  $\text{Div}(\text{spn } V) = -\text{rot } V$ .

Observe that we already know that  $N(\text{Grad grad}) = \{0\}$  and  $N(\text{dev Grad}) = \text{RT}_0$ . If the topology of the underlying domain is trivial, we will now characterize the remaining kernels and the ranges of the linear operators Grad grad, Rot<sub>ℝ</sub>, Div<sub>ℝ</sub>, and dev Grad, sym Rot<sub>ℝ</sub>, div Div<sub>ℝ</sub>.

**Theorem 3.10.** *Let  $\Omega$  be additionally topologically trivial. Then*

- (i)  $\mathring{\mathbf{R}}_0(\Omega, \mathbb{S}) = N(\mathring{\text{Rot}}_{\mathbb{S}}) = R(\mathring{\text{Grad grad}}) = \text{Grad grad } \mathring{\mathbf{H}}^2(\Omega)$ ,
- (ii)  $\mathring{\mathbf{D}}_0(\Omega, \mathbb{T}) = N(\mathring{\text{Div}}_{\mathbb{T}}) = R(\mathring{\text{Rot}}_{\mathbb{S}}) = \text{Rot } \mathring{\mathbf{H}}^1(\Omega, \mathbb{S})$ ,
- (iii)  $\text{RT}_0^{\perp \mathbf{L}^2(\Omega)} = N(\pi_{\text{RT}_0}) = R(\mathring{\text{Div}}_{\mathbb{T}}) = \text{Div } \mathring{\mathbf{H}}^1(\Omega, \mathbb{T})$ ,
- (iv)  $\mathbf{R}_{\text{sym},0}(\Omega, \mathbb{T}) = N(\text{sym Rot}_{\mathbb{T}}) = R(\text{dev Grad}) = \text{dev Grad } \mathbf{H}^1(\Omega)$ ,
- (v)  $\mathbf{DD}_0(\Omega, \mathbb{S}) = N(\text{div Div}_{\mathbb{S}}) = R(\text{sym Rot}_{\mathbb{T}}) = \text{sym Rot } \mathbf{H}^1(\Omega, \mathbb{T})$ ,
- (vi)  $\mathbf{L}^2(\Omega) = N(0) = R(\text{div Div}_{\mathbb{S}}) = \text{div Div } \mathbf{H}^2(\Omega, \mathbb{S})$ .

The corresponding linear and continuous (regular) potential operators are given by

$$\begin{aligned} P_{\text{Grad grad}} &= P_{\text{grad}} P_{\text{Grad}} : \mathring{\mathbf{R}}_0(\Omega, \mathbb{S}) \longrightarrow \mathring{\mathbf{H}}^2(\Omega), \\ P_{\text{Rot}_{\mathbb{S}}} &= \text{sym} \left( 1 - 2 \text{Grad } P_{\text{rot}} \text{spn}^{-1} \text{skw} \right) P_{\text{Rot}} : \mathring{\mathbf{D}}_0(\Omega, \mathbb{T}) \longrightarrow \mathring{\mathbf{H}}^1(\Omega, \mathbb{S}), \\ P_{\text{Div}_{\mathbb{T}}} &= \text{dev} \left( 1 + \frac{1}{2} \text{Grad}^\top P_{\text{div}} \text{tr} \right) P_{\text{Div}} : \text{RT}_0^{\perp \mathbf{L}^2(\Omega)} \longrightarrow \mathring{\mathbf{H}}^1(\Omega, \mathbb{T}), \\ P_{\text{dev Grad}} &= \text{Grad}^{-1} \left( 1 + \frac{1}{2} (\text{grad}^{-1} \text{Div}(\cdot)^\top) \mathbf{I} \right) : \mathbf{R}_{\text{sym},0}(\Omega, \mathbb{T}) \longrightarrow \mathbf{H}^1(\Omega), \\ P_{\text{sym Rot}_{\mathbb{T}}} &= \text{dev } P_{\text{Rot}} \left( 1 + \text{spn rot}^{-1} \text{Div} \right) : \mathbf{DD}_0(\Omega, \mathbb{S}) \longrightarrow \mathbf{H}^1(\Omega, \mathbb{T}), \\ P_{\text{div Div}_{\mathbb{S}}} &= \text{sym } P_{\text{Div}} P_{\text{div}} : \mathbf{L}^2(\Omega) \longrightarrow \mathbf{H}^2(\Omega, \mathbb{S}). \end{aligned}$$

**Remark 3.11.** *We note that*

$$\mathbf{H}^1(\Omega, \mathbb{S}) = \text{sym } \mathbf{H}^1(\Omega), \quad \mathbf{H}^1(\Omega, \mathbb{T}) = \text{dev } \mathbf{H}^1(\Omega), \quad \mathring{\mathbf{H}}^1(\Omega, \mathbb{S}) = \text{sym } \mathring{\mathbf{H}}^1(\Omega), \quad \mathring{\mathbf{H}}^1(\Omega, \mathbb{T}) = \text{dev } \mathring{\mathbf{H}}^1(\Omega)$$

as, e.g.,  $\text{dev } \mathbf{H}^1(\Omega) \subset \mathbf{H}^1(\Omega, \mathbb{T}) = \text{dev } \mathbf{H}^1(\Omega, \mathbb{T}) \subset \text{dev } \mathbf{H}^1(\Omega)$ . The same holds for the corresponding spaces of skew-symmetric tensor fields as well. Moreover:

- (i) *Theorem 3.10 holds also for the other set of canonical boundary conditions, which follows directly from the proof.*
- (ii) *A closer inspection shows, that for (iii) and (vi), i.e.,  $P_{\text{Div}_T}$  and  $P_{\text{div Div}_S}$ , only the potential operators corresponding to the divergence, i.e.,  $P_{\text{div}}$ ,  $P_{\text{Div}}$ ,  $P_{\text{Div}}$ , are involved. As Lemma 2.27 does not need any topological assumptions, (iii) and (vi), together with the representations of the potential operators, hold for general topologies as well.*

*Proof of Theorem 3.10.* Note that by Lemma 3.2 (iii), Lemma 3.3, and Lemma 3.7 all inclusions of the type  $R(\dots) \subset N(\dots)$  easily follow. Therefore it suffices to show that  $N(\dots)$  is included in the corresponding space appearing at the end of each line in (i) - (vi), which itself is obviously included in  $R(\dots)$ . Throughout the proof we will frequently use the formulas of Lemma 3.9.

ad (i): Let  $\mathbf{M} \in \mathring{\mathbf{R}}_0(\Omega, \mathbb{S}) = N(\text{Rot}_S)$ . Applying Lemma 2.25 for  $m = 0$  row-wise, there is a vector field  $V := P_{\text{Grad}} \circ \mathbf{M} \in \mathring{\mathbf{H}}^1(\Omega)$  with  $\mathbf{M} = \text{Grad } V$ . Since  $\text{skw } \mathbf{M} = 0$  and  $2 \text{skw } \text{Grad } V = \text{spn rot } V$ , it follows that  $\text{rot } V = 0$ . By Lemma 2.25 for  $m = 1$  there is a function  $u := P_{\text{grad}} \circ V \in \mathring{\mathbf{H}}^2(\Omega)$  with  $V = \text{grad } u$ . Hence  $\mathbf{M} = \text{Grad } V = \text{Grad grad } u \in \text{Grad grad } \mathring{\mathbf{H}}^2(\Omega)$ . So  $\mathring{\mathbf{R}}_0(\Omega, \mathbb{S}) \subset \text{Grad grad } \mathring{\mathbf{H}}^2(\Omega)$ , which completes the proof of (i). Note that

$$P_{\text{Grad grad}} \circ \mathbf{M} := u = P_{\text{grad}} \circ P_{\text{Grad}} \circ \mathbf{M} \in \mathring{\mathbf{H}}^2(\Omega),$$

from which it directly follows that  $P_{\text{Grad grad}} \circ$  is linear and bounded.

ad (ii): Let  $\mathbf{E} \in \mathring{\mathbf{D}}_0(\Omega, \mathbb{T}) = N(\text{Div}_T)$ . Then there is a tensor field  $\mathbf{N} := P_{\text{Rot}} \circ \mathbf{E} \in \mathring{\mathbf{H}}^1(\Omega)$  with  $\mathbf{E} = \text{Rot } \mathbf{N}$ , see Lemma 2.26 for  $m = 0$  applied row-wise. Since  $\text{tr } \mathbf{E} = 0$  and  $\text{tr Rot sym } \mathbf{N} = 0$ , it follows that  $\text{tr Rot skw } \mathbf{N} = 0$ . Now let  $V := \text{spn}^{-1} \text{skw } \mathbf{N} \in \mathring{\mathbf{H}}^1(\Omega)$ , i.e.,  $\text{skw } \mathbf{N} = \text{spn } V$ . Since  $\text{tr Rot spn } V = 2 \text{div } V$ , it follows that  $\text{div } V = 0$ . Therefore, there is a vector field  $H := P_{\text{rot}} \circ V \in \mathring{\mathbf{H}}^2(\Omega)$  such that  $V = \text{rot } H$ , see Lemma 2.26 for  $m = 1$ . So we have

$$\text{Rot skw } \mathbf{N} = \text{Rot spn rot } H = 2 \text{Rot skw Grad } H = -2 \text{Rot sym Grad } H.$$

Hence

$$\mathbf{E} = \text{Rot } \mathbf{N} = \text{Rot sym } \mathbf{N} + \text{Rot skw } \mathbf{N} = \text{Rot } \mathbf{M}, \quad \mathbf{M} := \text{sym } \mathbf{N} - 2 \text{sym Grad } H \in \mathring{\mathbf{H}}^1(\Omega, \mathbb{S}),$$

So  $\mathring{\mathbf{D}}_0(\Omega, \mathbb{T}) \subset \text{Rot } \mathring{\mathbf{H}}^1(\Omega, \mathbb{S})$ , which completes the proof of (ii). Note that

$$\begin{aligned} P_{\text{Rot}_S} \circ \mathbf{E} &:= \mathbf{M} = \text{sym } P_{\text{Rot}} \circ \mathbf{E} - 2 \text{sym Grad } (P_{\text{rot}} \circ \text{spn}^{-1} \text{skw } P_{\text{Rot}} \circ \mathbf{E}) \\ &= \text{sym } (1 - 2 \text{Grad } P_{\text{rot}} \circ \text{spn}^{-1} \text{skw}) P_{\text{Rot}} \circ \mathbf{E} \in \mathring{\mathbf{H}}^1(\Omega, \mathbb{S}), \end{aligned}$$

from which it directly follows that  $P_{\text{Rot}_S} \circ$  is linear and bounded.

ad (iii): Let  $V \in \text{RT}_0^{\perp L^2(\Omega)} = N(\pi_{\text{RT}_0})$ . As  $V \in (\mathbb{R}^3)^{\perp L^2(\Omega)}$ , there is a tensor field  $\mathbf{F} = P_{\text{Div}} \circ V \in \mathring{\mathbf{H}}^1(\Omega)$  with  $V = \text{Div } \mathbf{F}$ , see Lemma 2.27 for  $m = 0$  applied row-wise. We have  $\text{Div } \mathbf{F} \in \text{RT}_0^{\perp L^2(\Omega)}$  as well as  $\text{Div dev } \mathbf{F} \in \text{RT}_0^{\perp L^2(\Omega)}$ . Hence  $\text{grad}(\text{tr } \mathbf{F}) = \text{Div}((\text{tr } \mathbf{F}) \mathbf{I}) \in \text{RT}_0^{\perp L^2(\Omega)}$ , which implies  $\text{tr } \mathbf{F} \in \mathring{\mathbf{H}}^1(\Omega) \cap L_0^2(\Omega)$ . Therefore, there is a vector field  $H := P_{\text{div}} \circ \text{tr } \mathbf{F} \in \mathring{\mathbf{H}}^2(\Omega)$  with  $\text{tr } \mathbf{F} = \text{div } H$ , see Lemma 2.27 for  $m = 1$ . Thus

$$\text{Div}((\text{tr } \mathbf{F}) \mathbf{I}) = \text{grad div } H = \frac{3}{2} \text{Div}(\text{dev}(\text{Grad } H)^\top).$$

Hence

$$V = \text{Div } \mathbf{F} = \text{Div dev } \mathbf{F} + \frac{1}{3} \text{Div}((\text{tr } \mathbf{F}) \mathbf{I}) = \text{Div } \mathbf{E}, \quad \mathbf{E} := \text{dev}(\mathbf{F} + \frac{1}{2}(\text{Grad } H)^\top) \in \mathring{\mathbf{H}}^1(\Omega, \mathbb{T}).$$

So  $\text{RT}_0^{\perp L^2(\Omega)} \subset \text{Div } \mathring{\mathbf{H}}^1(\Omega, \mathbb{T})$ , which completes the proof of (iii). Note that

$$\begin{aligned} P_{\text{Div}_{\mathbb{T}}}^{\circ} V &:= \mathbf{E} = \text{dev} \left( P_{\text{Div}}^{\circ} V + \frac{1}{2} (\text{Grad } P_{\text{div}}^{\circ} \text{tr } P_{\text{Div}}^{\circ} V)^{\top} \right) \\ &= \text{dev} \left( 1 + \frac{1}{2} \text{Grad}^{\top} P_{\text{div}}^{\circ} \text{tr} \right) P_{\text{Div}}^{\circ} V \in \mathring{\mathbf{H}}^1(\Omega, \mathbb{T}), \end{aligned}$$

from which it directly follows that  $P_{\text{Div}_{\mathbb{T}}}^{\circ}$  is linear and bounded.

ad (iv): Let  $\mathbf{E} \in \mathbf{R}_{\text{sym},0}(\Omega, \mathbb{T}) = N(\text{sym Rot}_{\mathbb{T}})$ . Then (trivially)  $\text{Div sym Rot } \mathbf{E} = 0$  and it follows

$$\text{rot } H = 0 \quad \text{with} \quad H := \frac{1}{2} (\text{Div } \mathbf{E}^{\top} - \text{grad}(\text{tr } \mathbf{E})) = \frac{1}{2} \text{Div } \mathbf{E}^{\top}$$

and

$$(3.4) \quad \text{skw Rot } \mathbf{E} = \text{spn } H.$$

So  $H \in \mathbf{R}_0^{-1}(\Omega)$ . Therefore, there is a unique scalar field  $u := \text{grad}^{-1} H \in \mathbf{L}_0^2(\Omega)$ , such that

$$H = \text{grad } u,$$

see Corollary 2.30 for  $m = 1$ . As  $\text{Rot}(u \mathbf{I}) = -\text{spn grad } u$  implies  $\text{sym Rot}(u \mathbf{I}) = 0$ , we see

$$\mathbf{F} := \mathbf{E} + u \mathbf{I} \in \mathbf{R}_{\text{sym},0}(\Omega).$$

Moreover, by (3.4)

$$\text{skw Rot } \mathbf{F} = \text{skw Rot } \mathbf{E} + \text{skw Rot}(u \mathbf{I}) = \text{spn } H - \text{spn grad } u = 0.$$

Hence  $\mathbf{F} \in \mathbf{R}_0(\Omega)$ . Therefore, there is a unique vector field  $V := \text{Grad}^{-1} \mathbf{F} \in \mathbf{H}^1(\Omega) \cap \mathbf{L}_0^2(\Omega)$ , such that  $\mathbf{F} = \text{Grad } V$ , see Lemma 2.25 for  $m = 0$ . So we have

$$\mathbf{E} = \text{Grad } V - u \mathbf{I}.$$

From the additional condition  $\text{tr } \mathbf{E} = 0$  it follows that  $3u = \text{tr Grad } V = \text{div } V$  leading to

$$\mathbf{E} = \text{dev Grad } V, \quad V \in \mathbf{H}^1(\Omega).$$

So  $\mathbf{R}_{\text{sym},0}(\Omega, \mathbb{T}) \subset \text{dev Grad } \mathbf{H}^1(\Omega)$ , which completes the proof of (iv). Note that

$$\begin{aligned} P_{\text{dev Grad}} \mathbf{E} &:= V = \text{Grad}^{-1} \left( \mathbf{E} + \frac{1}{2} (\text{grad}^{-1} \text{Div } \mathbf{E}^{\top}) \mathbf{I} \right) \\ &= \text{Grad}^{-1} \left( 1 + \frac{1}{2} (\text{grad}^{-1} \text{Div}(\cdot)^{\top}) \mathbf{I} \right) \mathbf{E} \in \mathbf{H}^1(\Omega), \end{aligned}$$

from which it directly follows that  $P_{\text{dev Grad}}$  is linear and bounded.

ad (v): Let  $\mathbf{M} \in \mathbf{DD}_0(\Omega, \mathbb{S}) = N(\text{div Div}_{\mathbb{S}})$ . So  $\text{Div } \mathbf{M} \in \mathbf{D}_0^{-1}(\Omega)$  and there is a unique vector field  $V := \text{rot}^{-1} \text{Div } \mathbf{M} \in \mathring{\mathbf{D}}_0(\Omega)$ , such that

$$\text{Div } \mathbf{M} = \text{rot } V = -\text{Div}(\text{spn } V),$$

see Corollary 2.31 for  $m = 1$ . Hence  $\text{Div}(\mathbf{M} + \text{spn } V) = 0$ , i.e.,  $\mathbf{M} + \text{spn } V \in \mathbf{D}_0(\Omega)$ , and by Lemma 2.26 there is a tensor field  $\mathbf{F} := \text{P}_{\text{Rot}}(\mathbf{M} + \text{spn } V) \in \mathbf{H}^1(\Omega)$ , such that

$$\mathbf{M} + \text{spn } V = \text{Rot } \mathbf{F}.$$

Observe that  $\mathbf{M}$  is symmetric and  $\text{spn } V$  is skew-symmetric. Thus

$$\mathbf{M} = \text{sym Rot } \mathbf{F} \quad \text{and} \quad \text{spn } V = \text{skw Rot } \mathbf{F}, \quad \mathbf{F} \in \mathbf{H}^1(\Omega),$$

and hence

$$\mathbf{M} = \text{sym Rot } \mathbf{F} = \text{sym Rot } \mathbf{E} \quad \text{with} \quad \mathbf{E} := \text{dev } \mathbf{F} \in \mathbf{H}^1(\Omega, \mathbb{T}),$$

as  $\text{dev } \mathbf{F} = \mathbf{F} - \frac{1}{3}(\text{tr } \mathbf{F}) \mathbf{I}$  and  $\text{sym Rot}((\text{tr } \mathbf{F}) \mathbf{I}) = 0$ . So  $\mathbf{DD}_0(\Omega, \mathbb{S}) \subset \text{sym Rot } \mathbf{H}^1(\Omega, \mathbb{T})$ , which completes the proof of (v). Note that

$$P_{\text{sym Rot}_{\mathbb{T}}} \mathbf{M} := \mathbf{E} = \text{dev } \text{P}_{\text{Rot}}(\mathbf{M} + \text{spn rot}^{-1} \text{Div } \mathbf{M})$$

$$= \operatorname{dev} P_{\operatorname{Rot}} (1 + \operatorname{spn} \operatorname{rot}^{-1} \operatorname{Div}) \mathbf{M} \in \mathbf{H}^1(\Omega, \mathbb{T}),$$

from which it directly follows that  $P_{\operatorname{sym} \operatorname{Rot}_{\mathbb{T}}}$  is linear and bounded.

ad (vi): Let  $u \in \mathbf{L}^2(\Omega) = N(0)$ . Then there is a vector field  $V = P_{\operatorname{div}} u \in \mathbf{H}^1(\Omega)$  with  $u = \operatorname{div} V$ , see Lemma 2.27 for  $m = 0$ , and a tensor field  $\mathbf{N} = P_{\operatorname{Div}} V \in \mathbf{H}^2(\Omega)$  such that  $V = \operatorname{Div} \mathbf{N}$ , see Lemma 2.27 for  $m = 1$  applied row-wise. Since  $\operatorname{div} \operatorname{Div} \operatorname{skw} \mathbf{N} = 0$ , it follows that

$$u = \operatorname{div} \operatorname{Div} \mathbf{N} = \operatorname{div} \operatorname{Div} \mathbf{M} \quad \text{with} \quad \mathbf{M} := \operatorname{sym} \mathbf{N} \in \mathbf{H}^2(\Omega, \mathbb{S}).$$

So  $\mathbf{L}^2(\Omega) \subset \operatorname{div} \operatorname{Div} \mathbf{H}^2(\Omega, \mathbb{S})$ , which completes the proof of (vi). Note that

$$P_{\operatorname{div} \operatorname{Div}_{\mathbb{S}}} u := \mathbf{M} = \operatorname{sym} P_{\operatorname{Div}} P_{\operatorname{div}} u \in \mathbf{H}^2(\Omega, \mathbb{S}),$$

from which it directly follows that  $P_{\operatorname{div} \operatorname{Div}_{\mathbb{S}}}$  is linear and bounded.  $\square$

Provided that the domain  $\Omega$  has trivial topology, Theorem 3.10 implies that the densely defined, closed and unbounded linear operators  $\operatorname{Grad} \overset{\circ}{\operatorname{grad}}$ ,  $\overset{\circ}{\operatorname{Rot}}_{\mathbb{S}}$ ,  $\overset{\circ}{\operatorname{Div}}_{\mathbb{T}}$ , and their adjoints  $\operatorname{div} \operatorname{Div}_{\mathbb{S}}$ ,  $\operatorname{sym} \operatorname{Rot}_{\mathbb{T}}$ ,  $\operatorname{dev} \operatorname{Grad}$  have closed ranges and that all relevant cohomology groups are trivial, as

$$\begin{aligned} N(\operatorname{Grad} \overset{\circ}{\operatorname{grad}}) \cap N(0) &= \{0\} \cap \mathbf{L}^2(\Omega) = \{0\}, \\ N(\overset{\circ}{\operatorname{Rot}}_{\mathbb{S}}) \cap N(\operatorname{div} \operatorname{Div}_{\mathbb{S}}) &= \overset{\circ}{\mathbf{R}}_0(\Omega, \mathbb{S}) \cap \operatorname{DD}_0(\Omega, \mathbb{S}) = \overset{\circ}{\mathbf{R}}_0(\Omega, \mathbb{S}) \cap \operatorname{sym} \operatorname{Rot} \mathbf{H}^1(\Omega, \mathbb{T}) \\ &= N(\overset{\circ}{\operatorname{Rot}}_{\mathbb{S}}) \cap R(\operatorname{sym} \operatorname{Rot}_{\mathbb{T}}) = \{0\}, \\ N(\overset{\circ}{\operatorname{Div}}_{\mathbb{T}}) \cap N(\operatorname{sym} \operatorname{Rot}_{\mathbb{T}}) &= \overset{\circ}{\mathbf{D}}_0(\Omega, \mathbb{T}) \cap \mathbf{R}_{\operatorname{sym},0}(\Omega, \mathbb{T}) = \overset{\circ}{\mathbf{D}}_0(\Omega, \mathbb{T}) \cap \operatorname{dev} \operatorname{Grad} \mathbf{H}^1(\Omega) \\ &= N(\overset{\circ}{\operatorname{Div}}_{\mathbb{T}}) \cap R(\operatorname{dev} \operatorname{Grad}) = \{0\}, \\ N(\pi_{\operatorname{RT}_0}) \cap N(\operatorname{dev} \operatorname{Grad}) &= \operatorname{RT}_0^{\perp \mathbf{L}^2(\Omega)} \cap \operatorname{RT}_0 = \{0\}. \end{aligned}$$

In this case, the reduced operators are

$$\begin{aligned} \mathcal{A}_0 &= \operatorname{Grad} \overset{\circ}{\operatorname{grad}} : \overset{\circ}{\mathbf{H}}^2(\Omega) \subset \mathbf{L}^2(\Omega) \longrightarrow \overset{\circ}{\mathbf{R}}_0(\Omega, \mathbb{S}), \\ \mathcal{A}_1 &= \overset{\circ}{\operatorname{Rot}}_{\mathbb{S}} : \overset{\circ}{\mathbf{R}}(\Omega, \mathbb{S}) \cap \operatorname{DD}_0(\Omega, \mathbb{S}) \subset \operatorname{DD}_0(\Omega, \mathbb{S}) \longrightarrow \overset{\circ}{\mathbf{D}}_0(\Omega, \mathbb{T}), \\ \mathcal{A}_2 &= \overset{\circ}{\operatorname{Div}}_{\mathbb{T}} : \overset{\circ}{\mathbf{D}}(\Omega, \mathbb{T}) \cap \mathbf{R}_{\operatorname{sym},0}(\Omega, \mathbb{T}) \subset \mathbf{R}_{\operatorname{sym},0}(\Omega, \mathbb{T}) \longrightarrow \operatorname{RT}_0^{\perp \mathbf{L}^2(\Omega)}, \\ \mathcal{A}_0^* &= \operatorname{div} \operatorname{Div}_{\mathbb{S}} : \operatorname{DD}(\Omega, \mathbb{S}) \cap \overset{\circ}{\mathbf{R}}_0(\Omega, \mathbb{S}) \subset \overset{\circ}{\mathbf{R}}_0(\Omega, \mathbb{S}) \longrightarrow \mathbf{L}^2(\Omega), \\ \mathcal{A}_1^* &= \operatorname{sym} \operatorname{Rot}_{\mathbb{T}} : \mathbf{R}_{\operatorname{sym}}(\Omega, \mathbb{T}) \cap \overset{\circ}{\mathbf{D}}_0(\Omega, \mathbb{T}) \subset \overset{\circ}{\mathbf{D}}_0(\Omega, \mathbb{T}) \longrightarrow \operatorname{DD}_0(\Omega, \mathbb{S}), \\ \mathcal{A}_2^* &= -\operatorname{dev} \operatorname{Grad} : \mathbf{H}^1(\Omega) \cap \operatorname{RT}_0^{\perp \mathbf{L}^2(\Omega)} \subset \operatorname{RT}_0^{\perp \mathbf{L}^2(\Omega)} \longrightarrow \mathbf{R}_{\operatorname{sym},0}(\Omega, \mathbb{T}) \end{aligned}$$

as

$$R(\operatorname{div} \operatorname{Div}_{\mathbb{S}}) = \mathbf{L}^2(\Omega), \quad R(\overset{\circ}{\operatorname{Div}}_{\mathbb{T}}) = \operatorname{RT}_0^{\perp \mathbf{L}^2(\Omega)}.$$

The functional analysis toolbox Section 2.1, e.g., Lemma 2.10, immediately lead to the following implications about Helmholtz type decompositions, Friedrichs/Poincaré type estimates and continuous inverse operators.

**Theorem 3.12.** *Let  $\Omega$  be additionally topologically trivial. Then all occurring ranges are closed and all related cohomology groups are trivial. Moreover, the Helmholtz type decompositions*

$$\mathbf{L}^2(\Omega, \mathbb{S}) = \overset{\circ}{\mathbf{R}}_0(\Omega, \mathbb{S}) \oplus_{\mathbf{L}^2(\Omega, \mathbb{S})} \operatorname{DD}_0(\Omega, \mathbb{S}), \quad \mathbf{L}^2(\Omega, \mathbb{T}) = \overset{\circ}{\mathbf{D}}_0(\Omega, \mathbb{T}) \oplus_{\mathbf{L}^2(\Omega, \mathbb{T})} \mathbf{R}_{\operatorname{sym},0}(\Omega, \mathbb{T})$$

hold. The kernels can be represented by the following closed ranges

$$\overset{\circ}{\mathbf{R}}_0(\Omega, \mathbb{S}) = \operatorname{Grad} \operatorname{grad} \overset{\circ}{\mathbf{H}}^2(\Omega),$$



$$\begin{aligned}\text{sym Rot } \mathbf{H}^1(\Omega, \mathbb{T}) &= \mathbf{DD}_0(\Omega, \mathbb{S}) = \text{sym Rot } \mathbf{R}_{\text{sym}}(\Omega, \mathbb{T}) = \text{sym Rot } (\mathbf{R}_{\text{sym}}(\Omega, \mathbb{T}) \cap \mathring{\mathbf{D}}_0(\Omega, \mathbb{T})), \\ \text{Rot } \mathring{\mathbf{H}}^1(\Omega, \mathbb{S}) &= \mathring{\mathbf{D}}_0(\Omega, \mathbb{T}) = \text{Rot } \mathring{\mathbf{R}}(\Omega, \mathbb{S}) = \text{Rot } (\mathring{\mathbf{R}}(\Omega, \mathbb{S}) \cap \mathbf{DD}_0(\Omega, \mathbb{S})), \\ \mathbf{R}_{\text{sym},0}(\Omega, \mathbb{T}) &= \text{dev Grad } \mathbf{H}^1(\Omega) = \text{dev Grad } (\mathbf{H}^1(\Omega) \cap \mathbf{RT}_0^{\perp L^2(\Omega)}),\end{aligned}$$

and it holds

$$\begin{aligned}\text{div Div } \mathbf{H}^2(\Omega, \mathbb{S}) &= \mathbf{L}^2(\Omega) = \text{div Div } \mathbf{DD}(\Omega, \mathbb{S}) = \text{div Div } (\mathbf{DD}(\Omega, \mathbb{S}) \cap \mathring{\mathbf{R}}_0(\Omega, \mathbb{S})), \\ \text{Div } \mathring{\mathbf{H}}^1(\Omega, \mathbb{T}) &= \mathbf{RT}_0^{\perp L^2(\Omega)} = N(\pi_{\mathbf{RT}_0}) = \text{Div } \mathring{\mathbf{D}}(\Omega, \mathbb{T}) = \text{Div } (\mathring{\mathbf{D}}(\Omega, \mathbb{T}) \cap \mathbf{R}_{\text{sym},0}(\Omega, \mathbb{T})).\end{aligned}$$

All potentials depend continuously on the data. The potentials on the very right hand sides are uniquely determined. There exist positive constants  $c_{\mathbf{Gg}}$ ,  $c_{\mathbf{D}}$ ,  $c_{\mathbf{R}}$  such that the Friedrichs/Poincaré type estimates

$$\begin{aligned}\forall u \in \mathring{\mathbf{H}}^2(\Omega) & \quad |u|_{\mathbf{L}^2(\Omega)} \leq c_{\mathbf{Gg}} |\text{Grad grad } u|_{\mathbf{L}^2(\Omega)}, \\ \forall \mathbf{M} \in \mathbf{DD}(\Omega, \mathbb{S}) \cap \mathring{\mathbf{R}}_0(\Omega, \mathbb{S}) & \quad |\mathbf{M}|_{\mathbf{L}^2(\Omega)} \leq c_{\mathbf{Gg}} |\text{div Div } \mathbf{M}|_{\mathbf{L}^2(\Omega)}, \\ \forall \mathbf{E} \in \mathring{\mathbf{D}}(\Omega, \mathbb{T}) \cap \mathbf{R}_{\text{sym},0}(\Omega, \mathbb{T}) & \quad |\mathbf{E}|_{\mathbf{L}^2(\Omega)} \leq c_{\mathbf{D}} |\text{Div } \mathbf{E}|_{\mathbf{L}^2(\Omega)}, \\ \forall V \in \mathbf{H}^1(\Omega) \cap \mathbf{RT}_0^{\perp L^2(\Omega)} & \quad |V|_{\mathbf{L}^2(\Omega)} \leq c_{\mathbf{D}} |\text{dev Grad } V|_{\mathbf{L}^2(\Omega)}, \\ \forall \mathbf{M} \in \mathring{\mathbf{R}}(\Omega, \mathbb{S}) \cap \mathbf{DD}_0(\Omega, \mathbb{S}) & \quad |\mathbf{M}|_{\mathbf{L}^2(\Omega)} \leq c_{\mathbf{R}} |\text{Rot } \mathbf{M}|_{\mathbf{L}^2(\Omega)}, \\ \forall \mathbf{E} \in \mathbf{R}_{\text{sym}}(\Omega, \mathbb{T}) \cap \mathring{\mathbf{D}}_0(\Omega, \mathbb{T}) & \quad |\mathbf{E}|_{\mathbf{L}^2(\Omega)} \leq c_{\mathbf{R}} |\text{sym Rot } \mathbf{E}|_{\mathbf{L}^2(\Omega)}\end{aligned}$$

hold. Moreover, the reduced versions of the operators

$$\text{Grad grad}, \quad \text{div Div}_{\mathbb{S}}, \quad \text{Div}_{\mathbb{T}}, \quad \text{dev Grad}, \quad \text{Rot}_{\mathbb{S}}, \quad \text{sym Rot}_{\mathbb{T}}$$

have continuous inverse operators

$$\begin{aligned}(\text{Grad grad})^{-1} : \mathring{\mathbf{R}}_0(\Omega, \mathbb{S}) &\longrightarrow \mathring{\mathbf{H}}^2(\Omega), \\ (\text{div Div}_{\mathbb{S}})^{-1} : \mathbf{L}^2(\Omega) &\longrightarrow \mathbf{DD}(\Omega, \mathbb{S}) \cap \mathring{\mathbf{R}}_0(\Omega, \mathbb{S}), \\ (\text{Div}_{\mathbb{T}})^{-1} : \mathbf{RT}_0^{\perp L^2(\Omega)} &\longrightarrow \mathring{\mathbf{D}}(\Omega, \mathbb{T}) \cap \mathbf{R}_{\text{sym},0}(\Omega, \mathbb{T}), \\ (\text{dev Grad})^{-1} : \mathbf{R}_{\text{sym},0}(\Omega, \mathbb{T}) &\longrightarrow \mathbf{H}^1(\Omega) \cap \mathbf{RT}_0^{\perp L^2(\Omega)}, \\ (\text{Rot}_{\mathbb{S}})^{-1} : \mathring{\mathbf{D}}_0(\Omega, \mathbb{T}) &\longrightarrow \mathring{\mathbf{R}}(\Omega, \mathbb{S}) \cap \mathbf{DD}_0(\Omega, \mathbb{S}), \\ (\text{sym Rot}_{\mathbb{T}})^{-1} : \mathbf{DD}_0(\Omega, \mathbb{S}) &\longrightarrow \mathbf{R}_{\text{sym}}(\Omega, \mathbb{T}) \cap \mathring{\mathbf{D}}_0(\Omega, \mathbb{T})\end{aligned}$$

with norms  $(1 + c_{\mathbf{Gg}}^2)^{1/2}$ ,  $(1 + c_{\mathbf{D}}^2)^{1/2}$ , resp.  $(1 + c_{\mathbf{R}}^2)^{1/2}$ .

**Remark 3.13.** Let  $\Omega$  be additionally topologically trivial. The Friedrichs/Poincaré type estimate for  $\text{Rot } \mathbf{M}$  in the latter theorem can be slightly sharpened. Utilizing Lemma 3.4 we observe  $\text{tr Rot } \mathbf{M} = 0$  and thus  $\text{dev Rot } \mathbf{M} = \text{Rot } \mathbf{M}$  for  $\mathbf{M} \in \mathbf{R}(\Omega, \mathbb{S})$ . Hence

$$\forall \mathbf{M} \in \mathring{\mathbf{R}}(\Omega, \mathbb{S}) \cap \mathbf{DD}_0(\Omega, \mathbb{S}) \quad |\mathbf{M}|_{\mathbf{L}^2(\Omega)} \leq c_{\mathbf{R}} |\text{dev Rot } \mathbf{M}|_{\mathbf{L}^2(\Omega)}.$$

Similarly and trivially we see

$$\forall u \in \mathring{\mathbf{H}}^2(\Omega) \quad |u|_{\mathbf{L}^2(\Omega)} \leq c_{\mathbf{Gg}} |\text{sym Grad grad } u|_{\mathbf{L}^2(\Omega)}.$$

Recalling Remark 3.8 we have the following result.

**Remark 3.14.** Let  $\Omega$  be additionally topologically trivial. Theorem 3.10 and Theorem 3.12 easily lead to the following results in terms of complexes: The sequence

$$\{0\} \xrightarrow{0} \mathring{H}^2(\Omega) \xrightarrow{\text{Grad grad}} \mathring{R}(\Omega; \mathbb{S}) \xrightarrow{\text{Rot}_{\mathbb{S}}} \mathring{D}(\Omega, \mathbb{T}) \xrightarrow{\text{Div}_{\mathbb{T}}} \mathbf{L}^2(\Omega) \xrightarrow{\pi_{\text{RT}_0}} \text{RT}_0$$

and thus also its dual or adjoint sequence

$$\{0\} \xleftarrow{0} \mathbf{L}^2(\Omega) \xleftarrow{\text{div Div}_{\mathbb{S}}} \mathbf{DD}(\Omega, \mathbb{S}) \xleftarrow{\text{sym Rot}_{\mathbb{T}}} \mathbf{R}_{\text{sym}}(\Omega, \mathbb{T}) \xleftarrow{-\text{dev Grad}} \mathbf{H}^1(\Omega) \xleftarrow{\iota_{\text{RT}_0}} \text{RT}_0$$

are closed and exact Hilbert complexes.

**Remark 3.15.** The part

$$\{0\} \xrightarrow{0} \mathring{H}^2(\Omega) \xrightarrow{\text{Grad grad}} \mathring{R}(\Omega; \mathbb{S}) \xrightarrow{\text{Rot}_{\mathbb{S}}} \mathbf{L}^2(\Omega)$$

of the Hilbert complex from above and the related adjoint complex

$$\{0\} \xleftarrow{0} \mathbf{L}^2(\Omega) \xleftarrow{\text{div Div}_{\mathbb{S}}} \mathbf{DD}(\Omega, \mathbb{S}) \xleftarrow{\text{sym Rot}_{\mathbb{T}}} \mathbf{R}_{\text{sym}}(\Omega, \mathbb{T})$$

have been discussed in [24] for problems in general relativity.

**Remark 3.16.** In 2D and under similar assumptions we obtain by completely analogous but much simpler arguments that the Hilbert complexes

$$\begin{aligned} \{0\} &\xrightarrow{0} \mathring{H}^2(\Omega) \xrightarrow{\text{Grad grad}} \mathring{R}(\Omega; \mathbb{S}) \xrightarrow{\text{Rot}_{\mathbb{S}}} \mathbf{L}^2(\Omega) \xrightarrow{\pi_{\text{RT}_0}} \text{RT}_0, \\ \{0\} &\xleftarrow{0} \mathbf{L}^2(\Omega) \xleftarrow{\text{div Div}_{\mathbb{S}}} \mathbf{DD}(\Omega, \mathbb{S}) \xleftarrow{\text{sym Rot}} \mathbf{H}^1(\Omega) \xleftarrow{\iota_{\text{RT}_0}} \text{RT}_0 \end{aligned}$$

are dual to each other, closed and exact. Contrary to the 3D case, the operator  $\text{Rot}_{\mathbb{S}}$  maps a tensor field to a vector field and the operator  $\text{sym Rot} \cong \text{sym Grad}$  is applied row-wise to a vector field and maps this vector field to a tensor field. The associated Helmholtz decomposition is

$$\mathbf{L}^2(\Omega, \mathbb{S}) = \mathring{R}_0(\Omega, \mathbb{S}) \oplus_{\mathbf{L}^2(\Omega, \mathbb{S})} \mathbf{DD}_0(\Omega, \mathbb{S})$$

with

$$\mathring{R}_0(\Omega, \mathbb{S}) = \text{Grad grad } \mathring{H}^2(\Omega), \quad \mathbf{DD}_0(\Omega, \mathbb{S}) = \text{sym Rot } \mathbf{H}^1(\Omega).$$

Theorem 3.10 leads to the following so called regular decompositions.

**Theorem 3.17.** Let  $\Omega$  be additionally topologically trivial. Then

$$\begin{aligned} \mathring{R}(\Omega, \mathbb{S}) &= \mathring{H}^1(\Omega, \mathbb{S}) + \mathring{R}_0(\Omega, \mathbb{S}), & \mathring{R}_0(\Omega, \mathbb{S}) &= \text{Grad grad } \mathring{H}^2(\Omega), \\ \mathring{D}(\Omega, \mathbb{T}) &= \mathring{H}^1(\Omega, \mathbb{T}) + \mathring{D}_0(\Omega, \mathbb{T}), & \mathring{D}_0(\Omega, \mathbb{T}) &= \text{Rot } \mathring{H}^1(\Omega, \mathbb{S}), \\ \mathbf{R}_{\text{sym}}(\Omega, \mathbb{T}) &= \mathbf{H}^1(\Omega, \mathbb{T}) + \mathbf{R}_{\text{sym},0}(\Omega, \mathbb{T}), & \mathbf{R}_{\text{sym},0}(\Omega, \mathbb{T}) &= \text{dev Grad } \mathbf{H}^1(\Omega), \\ \mathbf{DD}(\Omega, \mathbb{S}) &= \mathbf{H}^2(\Omega, \mathbb{S}) + \mathbf{DD}_0(\Omega, \mathbb{S}), & \mathbf{DD}_0(\Omega, \mathbb{S}) &= \text{sym Rot } \mathbf{H}^1(\Omega, \mathbb{T}) \end{aligned}$$

with linear and continuous decomposition resp. potential operators

$$\begin{aligned} P_{\mathring{R}(\Omega, \mathbb{S}), \mathring{H}^1(\Omega, \mathbb{S})} &: \mathring{R}(\Omega, \mathbb{S}) \longrightarrow \mathring{H}^1(\Omega, \mathbb{S}), & P_{\mathring{R}(\Omega, \mathbb{S}), \mathring{H}^2(\Omega)} &: \mathring{R}(\Omega, \mathbb{S}) \longrightarrow \mathring{H}^2(\Omega), \\ P_{\mathring{D}(\Omega, \mathbb{T}), \mathring{H}^1(\Omega, \mathbb{T})} &: \mathring{D}(\Omega, \mathbb{T}) \longrightarrow \mathring{H}^1(\Omega, \mathbb{T}), & P_{\mathring{D}(\Omega, \mathbb{T}), \mathring{H}^1(\Omega, \mathbb{S})} &: \mathring{D}(\Omega, \mathbb{T}) \longrightarrow \mathring{H}^1(\Omega, \mathbb{S}), \\ P_{\mathbf{R}_{\text{sym}}(\Omega, \mathbb{T}), \mathbf{H}^1(\Omega, \mathbb{T})} &: \mathbf{R}_{\text{sym}}(\Omega, \mathbb{T}) \longrightarrow \mathbf{H}^1(\Omega, \mathbb{T}), & P_{\mathbf{R}_{\text{sym}}(\Omega, \mathbb{T}), \mathbf{H}^1(\Omega)} &: \mathbf{R}_{\text{sym}}(\Omega, \mathbb{T}) \longrightarrow \mathbf{H}^1(\Omega), \\ P_{\mathbf{DD}(\Omega, \mathbb{S}), \mathbf{H}^2(\Omega, \mathbb{S})} &: \mathbf{DD}(\Omega, \mathbb{S}) \longrightarrow \mathbf{H}^2(\Omega, \mathbb{S}), & P_{\mathbf{DD}(\Omega, \mathbb{S}), \mathbf{H}^1(\Omega, \mathbb{T})} &: \mathbf{DD}(\Omega, \mathbb{S}) \longrightarrow \mathbf{H}^1(\Omega, \mathbb{T}). \end{aligned}$$

*Proof.* Let, e.g.,  $\mathbf{E} \in \mathbf{R}_{\text{sym}}(\Omega, \mathbb{T})$ . Then

$$\text{sym Rot } \mathbf{E} \in \mathbf{DD}_0(\Omega, \mathbb{S}) = \text{sym Rot } \mathbf{H}^1(\Omega, \mathbb{T})$$

with linear and continuous potential operator  $P_{\text{sym Rot } \mathbb{T}} : \mathbf{DD}_0(\Omega, \mathbb{S}) \longrightarrow \mathbf{H}^1(\Omega, \mathbb{T})$  by Theorem 3.10. Thus, there is  $\tilde{\mathbf{E}} \in \mathbf{H}^1(\Omega, \mathbb{T})$  depending linearly and continuously on  $\mathbf{E}$  with  $\text{sym Rot } \tilde{\mathbf{E}} = \text{sym Rot } \mathbf{E}$ . Hence,

$$\mathbf{E} - \tilde{\mathbf{E}} \in \mathbf{R}_{\text{sym},0}(\Omega, \mathbb{T}) = \text{dev Grad } \mathbf{H}^1(\Omega)$$

with linear and continuous potential operator  $P_{\text{dev Grad}} : \mathbf{R}_{\text{sym},0}(\Omega, \mathbb{T}) \longrightarrow \mathbf{H}^1(\Omega)$  by Theorem 3.10. Hence, there exists  $V \in \mathbf{H}^1(\Omega)$  with  $\mathbf{E} - \tilde{\mathbf{E}} = \text{dev Grad } V$  and  $V$  depends linearly and continuously on  $\mathbf{E}$ . The other assertions are proved analogously.  $\square$

**3.2. General Bounded Strong Lipschitz Domains.** In this section we consider bounded strong Lipschitz domains  $\Omega$  of general topology and will extend results of the previous section as follows. The Grad grad- and the Div div-complexes remain closed and all associated cohomology groups are finite-dimensional. Moreover, the respective inverse operators are continuous resp. compact, and corresponding Friedrichs/Poincaré type estimates hold. We will show this by verifying the compactness properties of Lemma 2.7 for the various linear operators of the complexes. Then Lemma 2.5, Remark 2.6, and Theorem 2.9 immediately lead to the desired results. Using Rellich's selection theorem we have the following compact embeddings

$$D(\text{Grad } \overset{\circ}{\text{grad}}) \cap D(0) = \overset{\circ}{\mathbf{H}}^2(\Omega) \xrightarrow{\text{cpt}} \mathbf{L}^2(\Omega),$$

$$D(\pi_{\mathbb{RT}_0}) \cap D(\text{dev Grad}) = \mathbf{H}^1(\Omega) \xrightarrow{\text{cpt}} \mathbf{L}^2(\Omega).$$

The two missing compactness results that would immediately lead to the desired results are

$$(3.5) \quad D(\text{Rot}_{\mathbb{S}}) \cap D(\text{div Div}_{\mathbb{S}}) = \overset{\circ}{\mathbf{R}}(\Omega, \mathbb{S}) \cap \mathbf{DD}(\Omega, \mathbb{S}) \xrightarrow{\text{cpt}} \mathbf{L}^2(\Omega, \mathbb{S}),$$

$$(3.6) \quad D(\text{Div}_{\mathbb{T}}) \cap D(\text{sym Rot}_{\mathbb{T}}) = \overset{\circ}{\mathbf{D}}(\Omega, \mathbb{T}) \cap \mathbf{R}_{\text{sym}}(\Omega, \mathbb{T}) \xrightarrow{\text{cpt}} \mathbf{L}^2(\Omega, \mathbb{T}).$$

The main aim of this section is to show the compactness of the two crucial embeddings (3.5)-(3.6). As a first step we consider a trivial topology.

**Lemma 3.18.** *Let  $\Omega$  be additionally topologically trivial. Then the embeddings (3.5) and (3.6), i.e.,*

$$\overset{\circ}{\mathbf{R}}(\Omega, \mathbb{S}) \cap \mathbf{DD}(\Omega, \mathbb{S}) \hookrightarrow \mathbf{L}^2(\Omega, \mathbb{S}), \quad \mathbf{R}_{\text{sym}}(\Omega, \mathbb{T}) \cap \overset{\circ}{\mathbf{D}}(\Omega, \mathbb{T}) \hookrightarrow \mathbf{L}^2(\Omega, \mathbb{T}),$$

*are compact.*

*Proof.* Let  $(\mathbf{M}_n)$  be a bounded sequence in  $\overset{\circ}{\mathbf{R}}(\Omega, \mathbb{S}) \cap \mathbf{DD}(\Omega, \mathbb{S})$ . By Theorem 3.12 and Theorem 3.10 we have

$$\overset{\circ}{\mathbf{R}}(\Omega, \mathbb{S}) \cap \mathbf{DD}(\Omega, \mathbb{S}) = (\overset{\circ}{\mathbf{R}}_0(\Omega, \mathbb{S}) \cap \mathbf{DD}(\Omega, \mathbb{S})) \oplus_{\mathbf{L}^2(\Omega, \mathbb{S})} (\overset{\circ}{\mathbf{R}}(\Omega, \mathbb{S}) \cap \mathbf{DD}_0(\Omega, \mathbb{S})),$$

$$\overset{\circ}{\mathbf{R}}_0(\Omega, \mathbb{S}) = \text{Grad grad } \overset{\circ}{\mathbf{H}}^2(\Omega),$$

$$\mathbf{DD}_0(\Omega, \mathbb{S}) = \text{sym Rot } \mathbf{H}^1(\Omega, \mathbb{T})$$

with linear and continuous potential operators. Therefore, we can decompose

$$\overset{\circ}{\mathbf{R}}(\Omega, \mathbb{S}) \cap \mathbf{DD}(\Omega, \mathbb{S}) \ni \mathbf{M}_n = \mathbf{M}_{n,r} + \mathbf{M}_{n,d} \in (\overset{\circ}{\mathbf{R}}_0(\Omega, \mathbb{S}) \cap \mathbf{DD}(\Omega, \mathbb{S})) \oplus_{\mathbf{L}^2(\Omega, \mathbb{S})} (\overset{\circ}{\mathbf{R}}(\Omega, \mathbb{S}) \cap \mathbf{DD}_0(\Omega, \mathbb{S}))$$

with  $\mathbf{M}_{n,r} \in \text{Grad grad } \overset{\circ}{\mathbf{H}}^2(\Omega) \cap \mathbf{DD}(\Omega, \mathbb{S})$ ,  $\text{Rot } \mathbf{M}_{n,d} = \text{Rot } \mathbf{M}_n$ , and  $\mathbf{M}_{n,r} = \text{Grad grad } u_n$ ,  $u_n \in \overset{\circ}{\mathbf{H}}^2(\Omega)$ , as well as  $\mathbf{M}_{n,d} \in \overset{\circ}{\mathbf{R}}(\Omega, \mathbb{S}) \cap \text{sym Rot } \mathbf{H}^1(\Omega, \mathbb{T})$ ,  $\text{div Div } \mathbf{M}_{n,r} = \text{div Div } \mathbf{M}_n$ , and  $\mathbf{M}_{n,d} = \text{sym Rot } \mathbf{E}_n$ ,  $\mathbf{E}_n \in \mathbf{H}^1(\Omega, \mathbb{T})$ , and both  $u_n$  and  $\mathbf{E}_n$  depend continuously on  $\mathbf{M}_n$ , i.e.,

$$|u_n|_{\mathbf{H}^2(\Omega)} \leq c |\mathbf{M}_{n,r}|_{\mathbf{L}^2(\Omega)} \leq c |\mathbf{M}_n|_{\mathbf{L}^2(\Omega)}, \quad |\mathbf{E}_n|_{\mathbf{H}^1(\Omega)} \leq c |\mathbf{M}_{n,d}|_{\mathbf{L}^2(\Omega)} \leq c |\mathbf{M}_n|_{\mathbf{L}^2(\Omega)}.$$

By Rellich's selection theorem, there exist subsequences, again denoted by  $(u_n)$  and  $(\mathbf{E}_n)$ , such that  $(u_n)$  converges in  $H^1(\Omega)$  and  $(\mathbf{E}_n)$  converges in  $\mathbf{L}^2(\Omega)$ . Thus with  $\mathbf{M}_{n,m} := \mathbf{M}_n - \mathbf{M}_m$ , and similarly for  $\mathbf{M}_{n,m,r}$ ,  $\mathbf{M}_{n,m,d}$ ,  $u_{n,m}$ ,  $\mathbf{E}_{n,m}$ , we see

$$\begin{aligned} |\mathbf{M}_{n,m,r}|_{\mathbf{L}^2(\Omega)}^2 &= \langle \mathbf{M}_{n,m,r}, \text{Grad grad } u_{n,m} \rangle_{\mathbf{L}^2(\Omega)} = \langle \text{div Div } \mathbf{M}_{n,m,r}, u_{n,m} \rangle_{\mathbf{L}^2(\Omega)} \\ &= \langle \text{div Div } \mathbf{M}_{n,m}, u_{n,m} \rangle_{\mathbf{L}^2(\Omega)} \leq c |u_{n,m}|_{\mathbf{L}^2(\Omega)}, \\ |\mathbf{M}_{n,m,d}|_{\mathbf{L}^2(\Omega)}^2 &= \langle \mathbf{M}_{n,m,d}, \text{sym Rot } \mathbf{E}_{n,m} \rangle_{\mathbf{L}^2(\Omega)} = \langle \text{Rot } \mathbf{M}_{n,m,d}, \mathbf{E}_{n,m} \rangle_{\mathbf{L}^2(\Omega)} \\ &= \langle \text{Rot } \mathbf{M}_{n,m}, \mathbf{E}_{n,m} \rangle_{\mathbf{L}^2(\Omega)} \leq c |\mathbf{E}_{n,m}|_{\mathbf{L}^2(\Omega)}. \end{aligned}$$

Hence,  $(\mathbf{M}_n)$  is a Cauchy sequence in  $\mathbf{L}^2(\Omega, \mathbb{S})$ . So

$$\mathring{\mathbf{R}}(\Omega, \mathbb{S}) \cap \text{DD}(\Omega, \mathbb{S}) \hookrightarrow \mathbf{L}^2(\Omega, \mathbb{S})$$

is compact. To show the second compact embedding, let  $(\mathbf{E}_n) \subset \mathbf{R}_{\text{sym}}(\Omega, \mathbb{T}) \cap \mathring{\mathbf{D}}(\Omega, \mathbb{T})$  be a bounded sequence. By Theorem 3.12 and Theorem 3.10 we have

$$\begin{aligned} \mathbf{R}_{\text{sym}}(\Omega, \mathbb{T}) \cap \mathring{\mathbf{D}}(\Omega, \mathbb{T}) &= (\mathbf{R}_{\text{sym},0}(\Omega, \mathbb{T}) \cap \mathring{\mathbf{D}}(\Omega, \mathbb{T})) \oplus_{\mathbf{L}^2(\Omega, \mathbb{T})} (\mathbf{R}_{\text{sym}}(\Omega, \mathbb{T}) \cap \mathring{\mathbf{D}}_0(\Omega, \mathbb{T})), \\ \mathbf{R}_{\text{sym},0}(\Omega, \mathbb{T}) &= \text{dev Grad } H^1(\Omega), \\ \mathring{\mathbf{D}}_0(\Omega, \mathbb{T}) &= \text{Rot } \mathring{\mathbf{H}}^1(\Omega, \mathbb{S}) \end{aligned}$$

with linear and continuous potential operators. Therefore, we can decompose

$$\mathbf{R}_{\text{sym}}(\Omega, \mathbb{T}) \cap \mathring{\mathbf{D}}(\Omega, \mathbb{T}) \ni \mathbf{E}_n = \mathbf{E}_{n,r} + \mathbf{E}_{n,d} \in (\mathbf{R}_{\text{sym},0}(\Omega, \mathbb{T}) \cap \mathring{\mathbf{D}}(\Omega, \mathbb{T})) \oplus_{\mathbf{L}^2(\Omega, \mathbb{T})} (\mathbf{R}_{\text{sym}}(\Omega, \mathbb{T}) \cap \mathring{\mathbf{D}}_0(\Omega, \mathbb{T}))$$

with  $\mathbf{E}_{n,r} \in \text{dev Grad } H^1(\Omega) \cap \mathring{\mathbf{D}}(\Omega, \mathbb{T})$ ,  $\text{sym Rot } \mathbf{E}_{n,d} = \text{sym Rot } \mathbf{E}_n$ ,  $\mathbf{E}_{n,r} = \text{dev Grad } V_n$ ,  $V_n \in H^1(\Omega)$ , as well as  $\mathbf{E}_{n,d} \in \mathbf{R}_{\text{sym}}(\Omega, \mathbb{T}) \cap \text{Rot } \mathring{\mathbf{H}}^1(\Omega, \mathbb{S})$ ,  $\text{Div } \mathbf{E}_{n,r} = \text{Div } \mathbf{E}_n$ , and  $\mathbf{E}_{n,d} = \text{Rot } \mathbf{M}_n$ ,  $\mathbf{M}_n \in \mathring{\mathbf{H}}^1(\Omega, \mathbb{S})$ , and both  $V_n$  and  $\mathbf{M}_n$  depend continuously on  $\mathbf{E}_n$ , i.e.,

$$|V_n|_{H^1(\Omega)} \leq c |\mathbf{E}_{n,r}|_{\mathbf{L}^2(\Omega)} \leq c |\mathbf{E}_n|_{\mathbf{L}^2(\Omega)}, \quad |\mathbf{M}_n|_{\mathring{\mathbf{H}}^1(\Omega)} \leq c |\mathbf{E}_{n,d}|_{\mathbf{L}^2(\Omega)} \leq c |\mathbf{E}_n|_{\mathbf{L}^2(\Omega)}.$$

By Rellich's selection theorem, there exist subsequences, again denoted by  $(V_n)$  and  $(\mathbf{M}_n)$ , such that  $(V_n)$  converges in  $L^2(\Omega)$  and  $(\mathbf{M}_n)$  converges in  $\mathbf{L}^2(\Omega)$ . Thus with  $\mathbf{E}_{n,m} := \mathbf{E}_n - \mathbf{E}_m$ , and similarly for  $\mathbf{E}_{n,m,r}$ ,  $\mathbf{E}_{n,m,d}$ ,  $V_{n,m}$ ,  $\mathbf{M}_{n,m}$ , we see

$$\begin{aligned} |\mathbf{E}_{n,m,r}|_{\mathbf{L}^2(\Omega)}^2 &= \langle \mathbf{E}_{n,m,r}, \text{dev Grad } V_{n,m} \rangle_{\mathbf{L}^2(\Omega)} = -\langle \text{Div } \mathbf{E}_{n,m,r}, V_{n,m} \rangle_{\mathbf{L}^2(\Omega)} \\ &= -\langle \text{Div } \mathbf{E}_{n,m}, V_{n,m} \rangle_{\mathbf{L}^2(\Omega)} \leq c |V_{n,m}|_{\mathbf{L}^2(\Omega)}, \\ |\mathbf{E}_{n,m,d}|_{\mathbf{L}^2(\Omega)}^2 &= \langle \mathbf{E}_{n,m,d}, \text{Rot } \mathbf{M}_{n,m} \rangle_{\mathbf{L}^2(\Omega)} = \langle \text{sym Rot } \mathbf{E}_{n,m,d}, \mathbf{M}_{n,m} \rangle_{\mathbf{L}^2(\Omega)} \\ &= \langle \text{sym Rot } \mathbf{E}_{n,m}, \mathbf{M}_{n,m} \rangle_{\mathbf{L}^2(\Omega)} \leq c |\mathbf{M}_{n,m}|_{\mathbf{L}^2(\Omega)}. \end{aligned}$$

Note, that here the symmetry of  $\mathbf{M}_{n,m}$  is crucial. Finally,  $(\mathbf{E}_n)$  is a Cauchy sequence in  $\mathbf{L}^2(\Omega, \mathbb{T})$ . So

$$\mathbf{R}_{\text{sym}}(\Omega, \mathbb{T}) \cap \mathring{\mathbf{D}}(\Omega, \mathbb{T}) \hookrightarrow \mathbf{L}^2(\Omega, \mathbb{T})$$

is compact.  $\square$

For general topologies we will use a partition of unity argument. The next lemma, which we will prove in the Appendix, provides the necessary tools for this.

**Lemma 3.19.** *Let  $\varphi \in \mathring{\mathbf{C}}^\infty(\mathbb{R}^3)$ .*

(i) *If  $\mathbf{M} \in \mathring{\mathbf{R}}(\Omega)$  resp.  $\mathring{\mathbf{R}}(\Omega, \mathbb{S})$  resp.  $\mathring{\mathbf{R}}(\Omega, \mathbb{T})$ , then  $\varphi \mathbf{M} \in \mathring{\mathbf{R}}(\Omega)$  resp.  $\mathring{\mathbf{R}}(\Omega, \mathbb{S})$  resp.  $\mathring{\mathbf{R}}(\Omega, \mathbb{T})$  and*

$$(3.7) \quad \text{Rot}(\varphi \mathbf{M}) = \varphi \text{Rot } \mathbf{M} + \text{grad } \varphi \times \mathbf{M}.$$

(ii) If  $\mathbf{M} \in \mathbf{R}(\Omega)$  resp.  $\mathbf{R}(\Omega, \mathbb{S})$  resp.  $\mathbf{R}(\Omega, \mathbb{T})$ , then  $\varphi \mathbf{M} \in \mathbf{R}(\Omega)$  resp.  $\mathbf{R}(\Omega, \mathbb{S})$  resp.  $\mathbf{R}(\Omega, \mathbb{T})$  and (3.7) holds.

(iii) If  $\mathbf{E} \in \mathring{\mathbf{D}}(\Omega)$  resp.  $\mathring{\mathbf{D}}(\Omega, \mathbb{T})$  resp.  $\mathring{\mathbf{D}}(\Omega, \mathbb{S})$ , then  $\varphi \mathbf{E} \in \mathring{\mathbf{D}}(\Omega)$  resp.  $\mathring{\mathbf{D}}(\Omega, \mathbb{T})$  resp.  $\mathring{\mathbf{D}}(\Omega, \mathbb{S})$  and

$$(3.8) \quad \text{Div}(\varphi \mathbf{E}) = \varphi \text{Div } \mathbf{E} + \text{grad } \varphi \cdot \mathbf{E}.$$

(iv) If  $\mathbf{E} \in \mathbf{D}(\Omega)$  resp.  $\mathbf{D}(\Omega, \mathbb{T})$  resp.  $\mathbf{D}(\Omega, \mathbb{S})$ , then  $\varphi \mathbf{E} \in \mathbf{D}(\Omega)$  resp.  $\mathbf{D}(\Omega, \mathbb{T})$  resp.  $\mathbf{D}(\Omega, \mathbb{S})$  and (3.8) holds.

(v) If  $\mathbf{E} \in \mathbf{R}_{\text{sym}}(\Omega, \mathbb{T})$ , then  $\varphi \mathbf{E} \in \mathbf{R}_{\text{sym}}(\Omega, \mathbb{T})$  and

$$\text{sym Rot}(\varphi \mathbf{E}) = \varphi \text{sym Rot } \mathbf{E} + \text{sym}(\text{grad } \varphi \times \mathbf{E}).$$

(vi) If  $\mathbf{M} \in \mathbf{DD}(\Omega, \mathbb{S})$ , then  $\varphi \mathbf{M} \in \mathbf{DD}^{0,-1}(\Omega, \mathbb{S})$  and

$$\text{div Div}(\varphi \mathbf{M}) = \varphi \text{div Div } \mathbf{M} + 2 \text{grad } \varphi \cdot \text{Div } \mathbf{M} + \text{tr}(\text{Grad grad } \varphi \cdot \mathbf{M}).$$

By mollifying these formulas extend to  $\varphi \in \mathring{\mathbf{C}}^{0,1}(\mathbb{R}^3)$  resp.  $\varphi \in \mathring{\mathbf{C}}^{1,1}(\mathbb{R}^3)$ .

Here  $\text{grad } \varphi \times$  resp.  $\text{grad } \varphi \cdot$  is applied row-wise to a tensor  $\mathbf{M}$  and we see  $\text{grad } \varphi \cdot \mathbf{M} = \mathbf{M} \text{grad } \varphi$ . Moreover, we introduce

$$\mathbf{DD}^{0,-1}(\Omega, \mathbb{S}) = \{\mathbf{M} \in \mathbf{L}^2(\Omega, \mathbb{S}) : \text{div Div } \mathbf{M} \in \mathbf{H}^{-1}(\Omega)\}.$$

Another auxiliary result required for the compactness proof is contained in the next lemma.

**Lemma 3.20.** *The regular (type) decomposition*

$$\mathbf{DD}^{0,-1}(\Omega, \mathbb{S}) = \mathring{\mathbf{H}}^1(\Omega) \cdot \mathbf{I} \dot{+} \mathbf{DD}_0(\Omega, \mathbb{S})$$

holds, where  $\dot{+}$  denotes the direct sum. More precisely, for each  $\mathbf{M} \in \mathbf{DD}^{0,-1}(\Omega, \mathbb{S})$  there are unique  $u \in \mathring{\mathbf{H}}^1(\Omega)$  and  $\mathbf{M}_0 \in \mathbf{DD}_0(\Omega, \mathbb{S})$  such that  $\mathbf{M} = u \mathbf{I} + \mathbf{M}_0$ . The scalar function  $u \in \mathring{\mathbf{H}}^1(\Omega)$  is given as the unique solution of the Dirichlet-Poisson problem

$$\langle \text{grad } u, \text{grad } \varphi \rangle_{\mathbf{L}^2(\Omega)} = -\langle \text{div Div } \mathbf{M}, \varphi \rangle_{\mathbf{H}^{-1}(\Omega)} \quad \text{for all } \varphi \in \mathring{\mathbf{H}}^1(\Omega),$$

and the decomposition is continuous, more precisely there exists  $c > 0$ , such that

$$|u|_{\mathbf{H}^1(\Omega)} \leq c |\text{div Div } \mathbf{M}|_{\mathbf{H}^{-1}(\Omega)}, \quad |\mathbf{M} - u \mathbf{I}|_{\mathbf{L}^2(\Omega)} \leq c |\mathbf{M}|_{\mathbf{DD}^{0,-1}(\Omega, \mathbb{S})}.$$

*Proof.* The unique solution  $u \in \mathring{\mathbf{H}}^1(\Omega)$  satisfies

$$\mathbf{H}^{-1}(\Omega) \ni \text{div Div } u \mathbf{I} = \text{div grad } u = \text{div Div } \mathbf{M},$$

i.e.,  $\mathbf{M}_0 := \mathbf{M} - u \mathbf{I} \in \mathbf{DD}_0(\Omega, \mathbb{S})$ , which shows the decomposition. Moreover,

$$|u|_{\mathbf{H}^1(\Omega)} \leq (1 + c_{\mathbf{g}}^2) |\text{div Div } \mathbf{M}|_{\mathbf{H}^{-1}(\Omega)}$$

shows, that  $u$  depends continuously on  $\mathbf{M}$  and hence also  $\mathbf{M}_0$  since

$$|\mathbf{M}_0|_{\mathbf{L}^2(\Omega)} \leq |\mathbf{M}|_{\mathbf{L}^2(\Omega)} + |u|_{\mathbf{L}^2(\Omega)} \leq \sqrt{2} (1 + c_{\mathbf{g}}^2) |\mathbf{M}|_{\mathbf{DD}^{0,-1}(\Omega, \mathbb{S})}.$$

Let  $u \mathbf{I} \in \mathbf{DD}_0(\Omega, \mathbb{S})$  with  $u \in \mathring{\mathbf{H}}^1(\Omega)$ . Then  $0 = \text{div Div } u \mathbf{I} = \text{div grad } u = \Delta u$ , yielding  $u = 0$ . Hence, the decomposition is direct, completing the proof.  $\square$

**Lemma 3.21.** *The embeddings (3.5)-(3.6), i.e.,*

$$\mathring{\mathbf{R}}(\Omega, \mathbb{S}) \cap \mathbf{DD}(\Omega, \mathbb{S}) \hookrightarrow \mathbf{L}^2(\Omega, \mathbb{S}), \quad \mathbf{R}_{\text{sym}}(\Omega, \mathbb{T}) \cap \mathring{\mathbf{D}}(\Omega, \mathbb{T}) \hookrightarrow \mathbf{L}^2(\Omega, \mathbb{T}),$$

are compact.

*Proof.* Let  $(U_i)$  be an open covering of  $\bar{\Omega}$ , such that  $\Omega_i := \Omega \cap U_i$  is topologically trivial for all  $i$ . As  $\bar{\Omega}$  is compact, there is a finite subcovering denoted by  $(U_i)_{i=1, \dots, I}$  with  $I \in \mathbb{N}$ . Let  $(\varphi_i)$  with  $\varphi_i \in \mathring{C}^\infty(U_i)$  be a partition of unity subordinate to  $(U_i)$ . Suppose  $(\mathbf{E}_n) \subset \mathbf{R}_{\text{sym}}(\Omega, \mathbb{T}) \cap \mathring{\mathbf{D}}(\Omega, \mathbb{T})$  is a bounded sequence. Then  $\mathbf{E}_n = \sum_{i=1}^I \varphi_i \mathbf{E}_n$  and  $(\varphi_i \mathbf{E}_n) \subset \mathbf{R}_{\text{sym}}(\Omega_i, \mathbb{T}) \cap \mathring{\mathbf{D}}(\Omega_i, \mathbb{T})$  is a bounded sequence for all  $i$  by Lemma 3.19. As  $\Omega_i$  is topologically trivial, there exists a subsequence, again denoted by  $(\varphi_i \mathbf{E}_n)$ , which is a Cauchy sequence in  $\mathbf{L}^2(\Omega_i)$  by Lemma 3.18. Picking successively subsequences yields that  $(\varphi_i \mathbf{E}_n)$  is a Cauchy sequence in  $\mathbf{L}^2(\Omega_i)$  for all  $i$ . Hence  $(\mathbf{E}_n)$  is a Cauchy sequence in  $\mathbf{L}^2(\Omega)$ . So the second embedding of the lemma is compact. Let  $(\mathbf{M}_n) \subset \mathring{\mathbf{R}}(\Omega, \mathbb{S}) \cap \mathbf{DD}(\Omega, \mathbb{S})$  be a bounded sequence. Then  $\mathbf{M}_n = \sum_{i=1}^I \varphi_i \mathbf{M}_n$  and  $(\varphi_i \mathbf{M}_n) \subset \mathring{\mathbf{R}}(\Omega_i, \mathbb{S}) \cap \mathbf{DD}^{0,-1}(\Omega_i, \mathbb{S})$  is a bounded sequence for all  $i$  by Lemma 3.19 as  $|\text{Div } \mathbf{M}_n|_{\mathbf{H}^{-1}(\Omega)} \leq |\mathbf{M}_n|_{\mathbf{L}^2(\Omega)}$ . Using Lemma 3.20 we decompose

$$\varphi_i \mathbf{M}_n = u_{i,n} \mathbf{I} + \mathbf{M}_{0,i,n} \in \mathring{\mathbf{H}}^1(\Omega_i) \cdot \mathbf{I} + (\mathring{\mathbf{R}}(\Omega_i, \mathbb{S}) \cap \mathbf{DD}_0(\Omega_i, \mathbb{S})).$$

Moreover,  $(u_{i,n})$  is bounded in  $\mathring{\mathbf{H}}^1(\Omega_i)$  and  $(\mathbf{M}_{0,i,n})$  is bounded in  $(\mathring{\mathbf{R}}(\Omega_i, \mathbb{S}) \cap \mathbf{DD}_0(\Omega_i, \mathbb{S}))$ . By Rellich's selection theorem and Lemma 3.18 as well as picking successively subsequences we get that  $(\varphi_i \mathbf{M}_n)$  is a Cauchy sequence in  $\mathbf{L}^2(\Omega_i)$  for all  $i$ . Hence  $(\mathbf{M}_n)$  is a Cauchy sequence in  $\mathbf{L}^2(\Omega)$ , showing that the first embedding of the lemma is also compact and finishing the proof.  $\square$

Utilizing the crucial compact embeddings of Lemma 3.21, we can apply the functional analysis toolbox Section 2.1 to the (linear, densely defined, and closed 'complex') operators  $\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_0^*, \mathcal{A}_1^*, \mathcal{A}_2^*$ . In this general case the reduced operators are

$$\begin{aligned} \mathcal{A}_0 &= \text{Grad grad} : \mathring{\mathbf{H}}^2(\Omega) \subset \mathbf{L}^2(\Omega) \longrightarrow \overline{\text{Grad grad } \mathring{\mathbf{H}}^2(\Omega)}, \\ \mathcal{A}_1 &= \text{Rot}_{\mathbb{S}} : \mathring{\mathbf{R}}(\Omega, \mathbb{S}) \cap \text{sym Rot } \mathbf{R}_{\text{sym}}(\Omega, \mathbb{T}) \subset \text{sym Rot } \mathbf{R}_{\text{sym}}(\Omega, \mathbb{T}) \longrightarrow \overline{\text{Rot } \mathring{\mathbf{R}}(\Omega, \mathbb{S})}, \\ \mathcal{A}_2 &= \text{Div}_{\mathbb{T}} : \mathring{\mathbf{D}}(\Omega, \mathbb{T}) \cap \text{dev Grad } \mathbf{H}^1(\Omega) \subset \text{dev Grad } \mathbf{H}^1(\Omega) \longrightarrow \overline{\text{RT}_0^{\perp \mathbf{L}^2(\Omega)}}, \\ \mathcal{A}_0^* &= \text{div Div}_{\mathbb{S}} : \mathbf{DD}(\Omega, \mathbb{S}) \cap \text{Grad grad } \mathring{\mathbf{H}}^2(\Omega) \subset \text{Grad grad } \mathring{\mathbf{H}}^2(\Omega) \longrightarrow \mathbf{L}^2(\Omega), \\ \mathcal{A}_1^* &= \text{sym Rot}_{\mathbb{T}} : \mathbf{R}_{\text{sym}}(\Omega, \mathbb{T}) \cap \text{Rot } \mathring{\mathbf{R}}(\Omega, \mathbb{S}) \subset \text{Rot } \mathring{\mathbf{R}}(\Omega, \mathbb{S}) \longrightarrow \overline{\text{sym Rot } \mathbf{R}_{\text{sym}}(\Omega, \mathbb{T})}, \\ \mathcal{A}_2^* &= -\text{dev Grad} : \mathbf{H}^1(\Omega) \cap \text{RT}_0^{\perp \mathbf{L}^2(\Omega)} \subset \text{RT}_0^{\perp \mathbf{L}^2(\Omega)} \longrightarrow \overline{\text{dev Grad } \mathbf{H}^1(\Omega)} \end{aligned}$$

as

$$\begin{aligned} \overline{\text{div Div } \mathbf{DD}(\Omega, \mathbb{S})} &= \overline{R(\text{div Div}_{\mathbb{S}})} = N(\text{Grad grad})^{\perp \mathbf{L}^2(\Omega)} = \{0\}^{\perp \mathbf{L}^2(\Omega)} = \mathbf{L}^2(\Omega), \\ \overline{\text{Div } \mathring{\mathbf{D}}(\Omega, \mathbb{T})} &= \overline{R(\text{Div}_{\mathbb{T}})} = N(\text{dev Grad})^{\perp \mathbf{L}^2(\Omega)} = \text{RT}_0^{\perp \mathbf{L}^2(\Omega)}. \end{aligned}$$

Note that by the compact embeddings of Lemma 3.21 all ranges are actually closed and we can skip the closure bars. We obtain the following theorem.

**Theorem 3.22.** *It holds:*

(i) *The ranges*

$$\begin{aligned} R(\text{Grad grad}) &= \text{Grad grad } \mathring{\mathbf{H}}^2(\Omega), \\ \mathbf{L}^2(\Omega) &= R(\text{div Div}_{\mathbb{S}}) = \text{div Div } \mathbf{DD}(\Omega, \mathbb{S}) = \text{div Div } (\mathbf{DD}(\Omega, \mathbb{S}) \cap \text{Grad grad } \mathring{\mathbf{H}}^2(\Omega)), \\ R(\text{Rot}_{\mathbb{S}}) &= \text{Rot } \mathring{\mathbf{R}}(\Omega, \mathbb{S}) = \text{Rot } (\mathring{\mathbf{R}}(\Omega, \mathbb{S}) \cap \text{sym Rot } \mathbf{R}_{\text{sym}}(\Omega, \mathbb{T})), \\ R(\text{sym Rot}_{\mathbb{T}}) &= \text{sym Rot } \mathbf{R}_{\text{sym}}(\Omega, \mathbb{T}) = \text{sym Rot } (\mathbf{R}_{\text{sym}}(\Omega, \mathbb{T}) \cap \text{Rot } \mathring{\mathbf{R}}(\Omega, \mathbb{S})), \end{aligned}$$

$$\mathbf{RT}_0^{\perp \mathbf{L}^2(\Omega)} = R(\mathring{\mathbf{Div}}_{\mathbb{T}}) = \mathbf{Div} \mathring{\mathbf{D}}(\Omega, \mathbb{T}) = \mathbf{Div} (\mathring{\mathbf{D}}(\Omega, \mathbb{T}) \cap \text{dev Grad } \mathbf{H}^1(\Omega)),$$

$$R(\text{dev Grad}) = \text{dev Grad } \mathbf{H}^1(\Omega) = \text{dev Grad} (\mathbf{H}^1(\Omega) \cap \mathbf{RT}_0^{\perp \mathbf{L}^2(\Omega)})$$

are closed. The more regular potentials on the right hand sides are uniquely determined and depend linearly and continuously on the data, see (v).

(ii) The cohomology groups

$$\mathcal{H}_{\mathbf{D}}(\Omega, \mathbb{S}) := \mathring{\mathbf{R}}_0(\Omega, \mathbb{S}) \cap \mathbf{DD}_0(\Omega, \mathbb{S}), \quad \mathcal{H}_{\mathbf{N}}(\Omega, \mathbb{T}) := \mathring{\mathbf{D}}_0(\Omega, \mathbb{T}) \cap \mathbf{R}_{\text{sym},0}(\Omega, \mathbb{T})$$

are finite dimensional and may be called Dirichlet resp. Neumann tensor fields.

(iii) The Hilbert complexes from Remark 3.8, i.e.,

$$\{0\} \xrightarrow{0} \mathring{\mathbf{H}}^2(\Omega) \xrightarrow{\text{Grad grad}} \mathring{\mathbf{R}}(\Omega; \mathbb{S}) \xrightarrow{\text{Rot}_{\mathbb{S}}} \mathring{\mathbf{D}}(\Omega, \mathbb{T}) \xrightarrow{\mathring{\mathbf{Div}}_{\mathbb{T}}} \mathbf{L}^2(\Omega) \xrightarrow{\pi_{\mathbf{RT}_0}} \mathbf{RT}_0$$

and its adjoint

$$\{0\} \xleftarrow{0} \mathbf{L}^2(\Omega) \xleftarrow{\text{div Div}_{\mathbb{S}}} \mathbf{DD}(\Omega, \mathbb{S}) \xleftarrow{\text{sym Rot}_{\mathbb{T}}} \mathbf{R}_{\text{sym}}(\Omega, \mathbb{T}) \xleftarrow{-\text{dev Grad}} \mathbf{H}^1(\Omega) \xleftarrow{\iota_{\mathbf{RT}_0}} \mathbf{RT}_0,$$

are closed. They are also exact, if and only if  $\mathcal{H}_{\mathbf{D}}(\Omega, \mathbb{S}) = \{0\}$ ,  $\mathcal{H}_{\mathbf{N}}(\Omega, \mathbb{T}) = \{0\}$ . The latter holds, if  $\Omega$  is topologically trivial.

(iv) The Helmholtz type decompositions

$$\begin{aligned} \mathbf{L}^2(\Omega, \mathbb{S}) &= \text{Grad grad } \mathring{\mathbf{H}}^2(\Omega) \oplus_{\mathbf{L}^2(\Omega, \mathbb{S})} \mathbf{DD}_0(\Omega, \mathbb{S}) \\ &= \mathring{\mathbf{R}}_0(\Omega, \mathbb{S}) \oplus_{\mathbf{L}^2(\Omega, \mathbb{S})} \text{sym Rot } \mathbf{R}_{\text{sym}}(\Omega, \mathbb{T}) \\ &= \text{Grad grad } \mathring{\mathbf{H}}^2(\Omega) \oplus_{\mathbf{L}^2(\Omega, \mathbb{S})} \mathcal{H}_{\mathbf{D}}(\Omega, \mathbb{S}) \oplus_{\mathbf{L}^2(\Omega, \mathbb{S})} \text{sym Rot } \mathbf{R}_{\text{sym}}(\Omega, \mathbb{T}), \\ \mathbf{L}^2(\Omega, \mathbb{T}) &= \text{Rot } \mathring{\mathbf{R}}(\Omega, \mathbb{S}) \oplus_{\mathbf{L}^2(\Omega, \mathbb{T})} \mathbf{R}_{\text{sym},0}(\Omega, \mathbb{T}) \\ &= \mathring{\mathbf{D}}_0(\Omega, \mathbb{T}) \oplus_{\mathbf{L}^2(\Omega, \mathbb{T})} \text{dev Grad } \mathbf{H}^1(\Omega) \\ &= \text{Rot } \mathring{\mathbf{R}}(\Omega, \mathbb{S}) \oplus_{\mathbf{L}^2(\Omega, \mathbb{T})} \mathcal{H}_{\mathbf{N}}(\Omega, \mathbb{T}) \oplus_{\mathbf{L}^2(\Omega, \mathbb{T})} \text{dev Grad } \mathbf{H}^1(\Omega) \end{aligned}$$

are valid.

(v) There exist positive constants  $c_{\mathbf{Gg}}$ ,  $c_{\mathbf{D}}$ ,  $c_{\mathbf{R}}$ , such that the Friedrichs/Poincaré type estimates

$$\begin{aligned} \forall u \in \mathring{\mathbf{H}}^2(\Omega) & \quad |u|_{\mathbf{L}^2(\Omega)} \leq c_{\mathbf{Gg}} |\text{Grad grad } u|_{\mathbf{L}^2(\Omega)}, \\ \forall \mathbf{M} \in \mathbf{DD}(\Omega, \mathbb{S}) \cap \text{Grad grad } \mathring{\mathbf{H}}^2(\Omega) & \quad |\mathbf{M}|_{\mathbf{L}^2(\Omega)} \leq c_{\mathbf{Gg}} |\text{div Div } \mathbf{M}|_{\mathbf{L}^2(\Omega)}, \\ \forall \mathbf{E} \in \mathring{\mathbf{D}}(\Omega, \mathbb{T}) \cap \text{dev Grad } \mathbf{H}^1(\Omega) & \quad |\mathbf{E}|_{\mathbf{L}^2(\Omega)} \leq c_{\mathbf{D}} |\text{Div } \mathbf{E}|_{\mathbf{L}^2(\Omega)}, \\ \forall V \in \mathbf{H}^1(\Omega) \cap \mathbf{RT}_0^{\perp \mathbf{L}^2(\Omega)} & \quad |V|_{\mathbf{L}^2(\Omega)} \leq c_{\mathbf{D}} |\text{dev Grad } V|_{\mathbf{L}^2(\Omega)}, \\ \forall \mathbf{M} \in \mathring{\mathbf{R}}(\Omega, \mathbb{S}) \cap \text{sym Rot } \mathbf{R}_{\text{sym}}(\Omega, \mathbb{T}) & \quad |\mathbf{M}|_{\mathbf{L}^2(\Omega)} \leq c_{\mathbf{R}} |\text{Rot } \mathbf{M}|_{\mathbf{L}^2(\Omega)}, \\ \forall \mathbf{E} \in \mathbf{R}_{\text{sym}}(\Omega, \mathbb{T}) \cap \text{Rot } \mathring{\mathbf{R}}(\Omega, \mathbb{S}) & \quad |\mathbf{E}|_{\mathbf{L}^2(\Omega)} \leq c_{\mathbf{R}} |\text{sym Rot } \mathbf{E}|_{\mathbf{L}^2(\Omega)} \end{aligned}$$

hold<sup>†</sup>.

<sup>†</sup>Note  $\text{Rot } \mathbf{M} = \text{dev Rot } \mathbf{M}$  for  $\mathbf{M} \in \mathbf{R}(\Omega, \mathbb{S})$  and thus for all  $\mathbf{M} \in \mathring{\mathbf{R}}(\Omega, \mathbb{S}) \cap \text{sym Rot } \mathbf{R}_{\text{sym}}(\Omega, \mathbb{T})$

$$|\mathbf{M}|_{\mathbf{L}^2(\Omega)} \leq c_{\mathbf{R}} |\text{Rot } \mathbf{M}|_{\mathbf{L}^2(\Omega)} = c_{\mathbf{R}} |\text{dev Rot } \mathbf{M}|_{\mathbf{L}^2(\Omega)}.$$

(vi) *The inverse operators*

$$\begin{aligned}
(\mathring{\text{Grad grad}})^{-1} &: \text{Grad grad } \mathring{\mathbf{H}}^2(\Omega) \longrightarrow \mathring{\mathbf{H}}^2(\Omega), \\
(\text{div Div}_{\mathbb{S}})^{-1} &: \mathbf{L}^2(\Omega) \longrightarrow \mathbf{DD}(\Omega, \mathbb{S}) \cap \text{Grad grad } \mathring{\mathbf{H}}^2(\Omega), \\
(\mathring{\text{Div}}_{\mathbb{T}})^{-1} &: \text{RT}_0^{\perp \mathbf{L}^2(\Omega)} \longrightarrow \mathring{\mathbf{D}}(\Omega, \mathbb{T}) \cap \text{dev Grad } \mathbf{H}^1(\Omega), \\
(\text{dev Grad})^{-1} &: \text{dev Grad } \mathbf{H}^1(\Omega) \longrightarrow \mathbf{H}^1(\Omega) \cap \text{RT}_0^{\perp \mathbf{L}^2(\Omega)}, \\
(\mathring{\text{Rot}}_{\mathbb{S}})^{-1} &: \text{Rot } \mathring{\mathbf{R}}(\Omega, \mathbb{S}) \longrightarrow \mathring{\mathbf{R}}(\Omega, \mathbb{S}) \cap \text{sym Rot } \mathbf{R}_{\text{sym}}(\Omega, \mathbb{T}), \\
(\text{sym Rot}_{\mathbb{T}})^{-1} &: \text{sym Rot } \mathbf{R}_{\text{sym}}(\Omega, \mathbb{T}) \longrightarrow \mathbf{R}_{\text{sym}}(\Omega, \mathbb{T}) \cap \text{Rot } \mathring{\mathbf{R}}(\Omega, \mathbb{S})
\end{aligned}$$

are continuous with norms  $(1 + c_{\mathbb{G}\mathbb{g}}^2)^{1/2}$  resp.  $(1 + c_{\mathbb{D}}^2)^{1/2}$ , resp.  $(1 + c_{\mathbb{R}}^2)^{1/2}$ , and their modifications

$$\begin{aligned}
(\mathring{\text{Grad grad}})^{-1} &: \text{Grad grad } \mathring{\mathbf{H}}^2(\Omega) \longrightarrow \mathring{\mathbf{H}}^1(\Omega) \subset \mathbf{L}^2(\Omega), \\
(\text{div Div}_{\mathbb{S}})^{-1} &: \mathbf{L}^2(\Omega) \longrightarrow \text{Grad grad } \mathring{\mathbf{H}}^2(\Omega) \subset \mathbf{L}^2(\Omega, \mathbb{S}), \\
(\mathring{\text{Div}}_{\mathbb{T}})^{-1} &: \text{RT}_0^{\perp \mathbf{L}^2(\Omega)} \longrightarrow \text{dev Grad } \mathbf{H}^1(\Omega) \subset \mathbf{L}^2(\Omega, \mathbb{T}), \\
(\text{dev Grad})^{-1} &: \text{dev Grad } \mathbf{H}^1(\Omega) \longrightarrow \text{RT}_0^{\perp \mathbf{L}^2(\Omega)} \subset \mathbf{L}^2(\Omega), \\
(\mathring{\text{Rot}}_{\mathbb{S}})^{-1} &: \text{Rot } \mathring{\mathbf{R}}(\Omega, \mathbb{S}) \longrightarrow \text{sym Rot } \mathbf{R}_{\text{sym}}(\Omega, \mathbb{T}) \subset \mathbf{L}^2(\Omega, \mathbb{S}), \\
(\text{sym Rot}_{\mathbb{T}})^{-1} &: \text{sym Rot } \mathbf{R}_{\text{sym}}(\Omega, \mathbb{T}) \longrightarrow \text{Rot } \mathring{\mathbf{R}}(\Omega, \mathbb{S}) \subset \mathbf{L}^2(\Omega, \mathbb{T})
\end{aligned}$$

are compact with norms  $c_{\mathbb{G}\mathbb{g}}$ ,  $c_{\mathbb{D}}$ , resp.  $c_{\mathbb{R}}$ .

We note

$$\begin{aligned}
\mathring{\mathbf{R}}_0(\Omega, \mathbb{S}) &= \text{Grad grad } \mathring{\mathbf{H}}^2(\Omega) \oplus_{\mathbf{L}^2(\Omega, \mathbb{S})} \mathcal{H}_{\mathbb{D}}(\Omega, \mathbb{S}), \\
\mathbf{DD}_0(\Omega, \mathbb{S}) &= \text{sym Rot } \mathbf{R}_{\text{sym}}(\Omega, \mathbb{T}) \oplus_{\mathbf{L}^2(\Omega, \mathbb{S})} \mathcal{H}_{\mathbb{D}}(\Omega, \mathbb{S}), \\
\mathring{\mathbf{D}}_0(\Omega, \mathbb{T}) &= \text{Rot } \mathring{\mathbf{R}}(\Omega, \mathbb{S}) \oplus_{\mathbf{L}^2(\Omega, \mathbb{T})} \mathcal{H}_{\mathbb{N}}(\Omega, \mathbb{T}), \\
\mathbf{R}_{\text{sym},0}(\Omega, \mathbb{T}) &= \text{dev Grad } \mathbf{H}^1(\Omega) \oplus_{\mathbf{L}^2(\Omega, \mathbb{T})} \mathcal{H}_{\mathbb{N}}(\Omega, \mathbb{T}).
\end{aligned}$$

Finally, even parts of Theorem 3.10 and Theorem 3.17 extend to the general case, i.e., we have regular potentials and regular decompositions for bounded strong Lipschitz domains as well.

**Theorem 3.23.** *The regular decompositions*

$$\begin{aligned}
\text{(i)} \quad & \mathring{\mathbf{R}}(\Omega, \mathbb{S}) = \mathring{\mathbf{H}}^1(\Omega, \mathbb{S}) + \text{Grad grad } \mathring{\mathbf{H}}^2(\Omega), \\
\text{(ii)} \quad & \mathring{\mathbf{D}}(\Omega, \mathbb{T}) = \mathring{\mathbf{H}}^1(\Omega, \mathbb{T}) + \text{Rot } \mathring{\mathbf{H}}^1(\Omega, \mathbb{S}), \\
\text{(iii)} \quad & \mathbf{R}_{\text{sym}}(\Omega, \mathbb{T}) = \mathbf{H}^1(\Omega, \mathbb{T}) + \text{dev Grad } \mathbf{H}^1(\Omega), \\
\text{(iv)} \quad & \mathbf{DD}(\Omega, \mathbb{S}) = \mathbf{H}^2(\Omega, \mathbb{S}) + \mathbf{DD}_0(\Omega, \mathbb{S})
\end{aligned}$$

hold with linear and continuous (regular) potential operators.

*Proof.* As in the proof of Lemma 3.21, let  $(U_i)$  be an open covering of  $\overline{\Omega}$ , such that  $\Omega_i := \Omega \cap U_i$  is topologically trivial for all  $i$ . As  $\overline{\Omega}$  is compact, there is a finite subcovering denoted by  $(U_i)_{i=1, \dots, I}$  with  $I \in \mathbb{N}$ . Let  $(\varphi_i)$  with  $\varphi_i \in \mathring{C}^\infty(U_i)$  be a partition of unity subordinate to  $(U_i)$  and let additionally



$\phi_i \in \mathring{C}^\infty(U_i)$  with  $\phi_i|_{\text{supp } \phi_i} = 1$ . To prove (i), suppose  $\mathbf{M} \in \mathring{\mathbf{R}}(\Omega, \mathbb{S})$ . By Lemma 3.19 and Theorem 3.17 we have

$$\varphi_i \mathbf{M} \in \mathring{\mathbf{R}}(\Omega_i, \mathbb{S}) = \mathring{\mathbf{H}}^1(\Omega_i, \mathbb{S}) + \mathring{\mathbf{R}}_0(\Omega_i, \mathbb{S}) = \mathring{\mathbf{H}}^1(\Omega_i, \mathbb{S}) + \text{Grad grad } \mathring{\mathbf{H}}^2(\Omega_i).$$

Hence,  $\varphi_i \mathbf{M} = \mathbf{M}_i + \text{Grad grad } u_i$  with  $\mathbf{M}_i \in \mathring{\mathbf{H}}^1(\Omega_i, \mathbb{S})$  and  $u_i \in \mathring{\mathbf{H}}^2(\Omega_i)$ . Let  $\hat{\mathbf{M}}_i$  and  $\hat{u}_i$  denote the extensions by zero of  $\mathbf{M}_i$  and  $u_i$ . Then  $\hat{\mathbf{M}}_i \in \mathring{\mathbf{H}}^1(\Omega, \mathbb{S})$  and  $\hat{u}_i \in \mathring{\mathbf{H}}^2(\Omega)$ . Thus

$$\mathbf{M} = \sum_i \varphi_i \mathbf{M} = \sum_i \hat{\mathbf{M}}_i + \text{Grad grad } \sum_i \hat{u}_i \in \mathring{\mathbf{H}}^1(\Omega, \mathbb{S}) + \text{Grad grad } \mathring{\mathbf{H}}^2(\Omega),$$

and all applied operations are continuous. Similarly we proof (ii). To show (iii), let  $\mathbf{E} \in \mathbf{R}_{\text{sym}}(\Omega, \mathbb{T})$ . By Lemma 3.19 and Theorem 3.17 we have

$$\varphi_i \mathbf{E} \in \mathbf{R}_{\text{sym}}(\Omega_i, \mathbb{T}) = \mathbf{H}^1(\Omega_i, \mathbb{T}) + \mathbf{R}_{\text{sym},0}(\Omega_i, \mathbb{T}) = \mathbf{H}^1(\Omega_i, \mathbb{T}) + \text{dev Grad } \mathbf{H}^1(\Omega_i).$$

Hence,  $\varphi_i \mathbf{E} = \mathbf{E}_i + \text{dev Grad } V_i$  with  $\mathbf{E}_i \in \mathbf{H}^1(\Omega_i, \mathbb{T})$  and  $V_i \in \mathbf{H}^1(\Omega_i)$ . In  $\Omega_i$  we observe

$$\begin{aligned} \varphi_i \mathbf{E} &= \phi_i \varphi_i \mathbf{E} = \phi_i \mathbf{E}_i + \phi_i \text{dev Grad } V_i \\ &= \phi_i \mathbf{E}_i - \text{dev}(V_i \cdot \text{grad}^\top \phi_i) + \text{dev Grad}(\phi_i V_i) \in \mathbf{H}^1(\Omega_i, \mathbb{T}) + \text{dev Grad } \mathbf{H}^1(\Omega_i). \end{aligned}$$

Let  $\hat{\mathbf{E}}_i$  and  $\hat{V}_i$  denote the extensions by zero of  $\phi_i \mathbf{E}_i - \text{dev}(V_i \cdot \text{grad}^\top \phi_i)$  and  $\phi_i V_i$ . Then  $\hat{\mathbf{E}}_i \in \mathbf{H}^1(\Omega, \mathbb{T})$  and  $\hat{V}_i \in \mathbf{H}^1(\Omega)$ . Thus

$$\mathbf{E} = \sum_i \varphi_i \mathbf{E} = \sum_i \hat{\mathbf{E}}_i + \text{dev Grad } \sum_i \hat{V}_i \in \mathbf{H}^1(\Omega, \mathbb{T}) + \text{dev Grad } \mathbf{H}^1(\Omega),$$

and all applied operations are continuous. To show (iv), let  $\mathbf{M} \in \mathbf{DD}(\Omega, \mathbb{S})$ . Then  $\text{div Div } \mathbf{M} \in \mathbf{L}^2(\Omega)$  and by Theorem 3.10 and Remark 3.11 (ii) there is some  $\tilde{\mathbf{M}} \in \mathbf{H}^2(\Omega, \mathbb{S})$ , together with a linear and continuous potential operator, with  $\text{div Div } \tilde{\mathbf{M}} = \text{div Div } \mathbf{M}$ . Therefore, we have  $\mathbf{M} - \tilde{\mathbf{M}} \in \mathbf{DD}_0(\Omega, \mathbb{S})$ , completing the proof.  $\square$

Applying  $\mathring{\text{Rot}}_{\mathbb{S}}$ ,  $\mathring{\text{Div}}_{\mathbb{T}}$ , and  $\text{sym Rot}_{\mathbb{T}}$ ,  $\text{div Div}_{\mathbb{S}}$  to the latter regular decompositions we get the following regular potentials.

**Theorem 3.24.** *It holds*

- (i)  $R(\mathring{\text{Rot}}_{\mathbb{S}}) = \text{Rot } \mathring{\mathbf{R}}(\Omega, \mathbb{S}) = \text{Rot } \mathring{\mathbf{H}}^1(\Omega, \mathbb{S}),$
- (ii)  $R\mathring{\text{Div}}_0^{\perp \mathbf{L}^2(\Omega)} = R(\mathring{\text{Div}}_{\mathbb{T}}) = \text{Div } \mathring{\mathbf{D}}(\Omega, \mathbb{T}) = \text{Div } \mathring{\mathbf{H}}^1(\Omega, \mathbb{T}),$
- (iii)  $R(\text{sym Rot}_{\mathbb{T}}) = \text{sym Rot } \mathbf{R}_{\text{sym}}(\Omega, \mathbb{T}) = \text{sym Rot } \mathbf{H}^1(\Omega, \mathbb{T}),$
- (iv)  $\mathbf{L}^2(\Omega) = R(\text{div Div}_{\mathbb{S}}) = \text{div Div } \mathbf{DD}(\Omega, \mathbb{S}) = \text{div Div } \mathbf{H}^2(\Omega, \mathbb{S})$

*with corresponding linear and continuous (regular) potential operators (on the right hand sides).*

**Remark 3.25.** *While the results about the regular potentials in Theorem 3.24 hold in full generality for all operators, one may wonder that the regular decompositions from Theorem 3.23 hold in full generality only for (i)-(iii), but not for (iv), i.e., we just have in (iv)*

$$\mathbf{DD}(\Omega, \mathbb{S}) = \mathbf{H}^2(\Omega, \mathbb{S}) + \mathbf{DD}_0(\Omega, \mathbb{S}) \supset \mathbf{H}^2(\Omega, \mathbb{S}) + \text{sym Rot } \mathbf{H}^1(\Omega, \mathbb{T}).$$

*The reason for the failure of the partition of unity argument from the proof of Theorem 3.23 is the following: Let  $\mathbf{M} \in \mathbf{DD}(\Omega, \mathbb{S})$ . By Lemma 3.19 (vi) we just get  $\varphi_i \mathbf{M} \in \mathbf{DD}^{0,-1}(\Omega_i, \mathbb{S})$ , see also the proof of Lemma 3.21. Using Lemma 3.20 and Theorem 3.17 we can decompose*

$$\varphi_i \mathbf{M} = u_i \mathbf{I} + \text{sym Rot } \mathbf{E}_i \in \mathring{\mathbf{H}}^1(\Omega_i) \cdot \mathbf{I} \dot{+} \text{sym Rot } \mathbf{H}^1(\Omega_i, \mathbb{T})$$

*as  $\mathbf{DD}_0(\Omega_i, \mathbb{S}) = \text{sym Rot } \mathbf{H}^1(\Omega_i, \mathbb{T})$ . In  $\Omega_i$  we observe*

$$\varphi_i \mathbf{M} = \phi_i \varphi_i \mathbf{M} = \phi_i u_i \mathbf{I} + \phi_i \text{sym Rot } \mathbf{E}_i$$

$$= \phi_i u_i \mathbf{I} - \text{sym}(\text{grad } \phi_i \times \mathbf{E}_i) + \text{sym Rot}(\phi_i \mathbf{E}_i) \in \mathbf{H}^1(\Omega_i, \mathbb{S}) + \text{sym Rot } \mathbf{H}^1(\Omega_i, \mathbb{T}).$$

Let  $\hat{\mathbf{M}}_i$  and  $\hat{\mathbf{E}}_i$  denote the extensions by zero of  $\phi_i u_i \mathbf{I} - \text{sym}(\text{grad } \phi_i \times \mathbf{E}_i)$  and  $\phi_i \mathbf{E}_i$ . Then  $\hat{\mathbf{M}}_i \in \mathbf{H}^1(\Omega, \mathbb{S})$  and  $\hat{\mathbf{E}}_i \in \mathbf{H}^1(\Omega, \mathbb{T})$  and thus

$$\mathbf{M} = \sum_i \varphi_i \mathbf{M} = \sum_i \hat{\mathbf{M}}_i + \text{sym Rot} \sum_i \hat{\mathbf{E}}_i \in \mathbf{H}^1(\Omega, \mathbb{S}) + \text{sym Rot } \mathbf{H}^1(\Omega, \mathbb{T}),$$

and all applied operations are continuous. Therefore, we obtain

$$\mathbf{H}^2(\Omega, \mathbb{S}) + \text{sym Rot } \mathbf{H}^1(\Omega, \mathbb{T}) \subset \mathbf{H}^2(\Omega, \mathbb{S}) + \mathbf{DD}_0(\Omega, \mathbb{S}) = \mathbf{DD}(\Omega, \mathbb{S}) \subset \mathbf{H}^1(\Omega, \mathbb{S}) + \text{sym Rot } \mathbf{H}^1(\Omega, \mathbb{T}).$$

So we have lost one Sobolev order in the summand  $\mathbf{H}^1(\Omega, \mathbb{S})$ .

#### 4. APPLICATION TO BIHARMONIC PROBLEMS

By  $\Delta^2 = \text{div Div Grad grad}$ , a standard (primal) variational formulation of (1.1) in  $\mathbb{R}^3$  reads as follows: For given  $f \in \mathbf{H}^{-2}(\Omega)$ , find  $u \in \mathring{\mathbf{H}}^2(\Omega)$  such that

$$(4.1) \quad \langle \text{Grad grad } u, \text{Grad grad } \phi \rangle_{\mathbf{L}^2(\Omega)} = \langle f, \phi \rangle_{\mathbf{H}^{-2}(\Omega)} \quad \text{for all } \phi \in \mathring{\mathbf{H}}^2(\Omega).$$

Existence, uniqueness, and continuous dependence on  $f$  of a solution to (4.1) is guaranteed by the theorem of Lax-Milgram, see, e.g., [16, 15] or Lemma 3.3. Note that then

$$\mathbf{M} := \text{Grad grad } u \in \mathring{\mathbf{R}}_0(\Omega, \mathbb{S}) \ominus_{\mathbf{L}^2(\Omega, \mathbb{S})} \mathcal{H}_D(\Omega, \mathbb{S}) \subset \mathbf{L}^2(\Omega, \mathbb{S})$$

with  $\text{div Div } \mathbf{M} = f \in \mathbf{H}^{-2}(\Omega)$ . In other words the operator

$$(4.2) \quad \text{div Div} : \mathbf{L}^2(\Omega, \mathbb{S}) \rightarrow \mathbf{H}^{-2}(\Omega)$$

is surjective and

$$(4.3) \quad \text{div Div} : \mathring{\mathbf{R}}_0(\Omega, \mathbb{S}) \ominus_{\mathbf{L}^2(\Omega, \mathbb{S})} \mathcal{H}_D(\Omega, \mathbb{S}) \rightarrow \mathbf{H}^{-2}(\Omega)$$

is bijective and even a topological isomorphism by the bounded inverse theorem. For our decomposition result we need the following variant of the Hilbert complex from Theorem 3.22.

$$\text{RT}_0 \xrightarrow{\iota_{\text{RT}_0}} \mathbf{H}^1(\Omega) \xrightarrow{-\text{dev Grad}} \mathbf{R}_{\text{sym}}(\Omega, \mathbb{T}) \xrightarrow{\text{sym Rot}_{\mathbb{T}}} \mathbf{DD}^{0,-1}(\Omega, \mathbb{S}) \xrightarrow{\text{div Div}_{\mathbb{S}}} \mathbf{H}^{-1}(\Omega) \xrightarrow{0} \{0\},$$

where we recall  $\mathbf{DD}^{0,-1}(\Omega, \mathbb{S})$  from Lemma 3.20. This is obviously also a closed Hilbert complex as  $\text{div Div} : \mathbf{DD}^{0,-1}(\Omega, \mathbb{S}) \rightarrow \mathbf{H}^{-1}(\Omega)$  is surjective as well by (4.2). Observe that

$$\mathbf{H}^1(\Omega, \mathbb{S}) \subset \mathbf{DD}^{0,-1}(\Omega, \mathbb{S}) \subset \mathbf{L}^2(\Omega, \mathbb{S}).$$

For right-hand sides  $f \in \mathbf{H}^{-1}(\Omega)$  we consider the following mixed variational problem for  $u$  and the Hessian  $\mathbf{M}$  of  $u$ : Find  $\mathbf{M} \in \mathbf{DD}^{0,-1}(\Omega, \mathbb{S})$  and  $u \in \mathring{\mathbf{H}}^1(\Omega)$  such that

$$(4.4) \quad \langle \mathbf{M}, \mathbf{\Psi} \rangle_{\mathbf{L}^2(\Omega)} + \langle u, \text{div Div } \mathbf{\Psi} \rangle_{\mathbf{H}^{-1}(\Omega)} = 0 \quad \text{for all } \mathbf{\Psi} \in \mathbf{DD}^{0,-1}(\Omega, \mathbb{S}),$$

$$(4.5) \quad \langle \text{div Div } \mathbf{M}, \psi \rangle_{\mathbf{H}^{-1}(\Omega)} = -\langle f, \psi \rangle_{\mathbf{H}^{-1}(\Omega)} \quad \text{for all } \psi \in \mathring{\mathbf{H}}^1(\Omega).$$

The first row and the second row of this mixed problem are variational formulations of (1.2) and (1.3), respectively. We recall the following two results related to these mixed problems from [14].

**Theorem 4.1.** *Let  $f \in \mathbf{H}^{-1}(\Omega)$ . Then:*

(i) *Problem (4.4)-(4.5) is a well-posed saddle point problem.*

(ii) *The variational problems (4.1) and (4.4)-(4.5) are equivalent, i.e., if  $u \in \mathring{\mathbf{H}}^2(\Omega)$  solves (4.1), then  $\mathbf{M} = -\text{Grad grad } u$  lies in  $\mathbf{DD}^{0,-1}(\Omega, \mathbb{S})$  and  $(\mathbf{M}, u)$  solves (4.4)-(4.5). And, vice versa, if  $(\mathbf{M}, u) \in \mathbf{DD}^{0,-1}(\Omega, \mathbb{S}) \times \mathring{\mathbf{H}}^1(\Omega)$  solves (4.4)-(4.5), then  $u \in \mathring{\mathbf{H}}^2(\Omega)$  with  $\text{Grad grad } u = -\mathbf{M}$  and  $u$  solves (4.1).*

Although only two-dimensional biharmonic problems were considered in [14], the proof of the latter theorem is completely identical for the three-dimensional case. The same holds for Lemma 3.20.

*Proof.* To show (i), we first note that  $(\Phi, \Psi) \mapsto \langle \Phi, \Psi \rangle_{\mathbf{L}^2(\Omega)}$  is coercive over the kernel of (4.5), i.e., for  $\Phi \in \mathbf{DD}_0(\Omega, \mathbb{S})$  we have  $\langle \Phi, \Phi \rangle_{\mathbf{L}^2(\Omega)} = |\Phi|_{\mathbf{L}^2(\Omega)}^2 = |\Phi|_{\mathbf{DD}^{0,-1}(\Omega, \mathbb{S})}^2$ . Moreover, the inf-sup-condition holds, as

$$\begin{aligned} & \inf_{0 \neq \varphi \in \mathring{\mathbf{H}}^1(\Omega)} \sup_{0 \neq \Phi \in \mathbf{DD}^{0,-1}(\Omega, \mathbb{S})} \frac{\langle \varphi, \operatorname{div} \operatorname{Div} \Phi \rangle_{\mathbf{H}^{-1}(\Omega)}}{|\operatorname{grad} \varphi|_{\mathbf{L}^2(\Omega)} |\Phi|_{\mathbf{DD}^{0,-1}(\Omega, \mathbb{S})}} \\ & \geq \inf_{0 \neq \varphi \in \mathring{\mathbf{H}}^1(\Omega)} \frac{-\langle \varphi, \operatorname{div} \operatorname{Div} \varphi \mathbf{I} \rangle_{\mathbf{H}^{-1}(\Omega)}}{|\operatorname{grad} \varphi|_{\mathbf{L}^2(\Omega)} |\varphi \mathbf{I}|_{\mathbf{DD}^{0,-1}(\Omega, \mathbb{S})}} = \inf_{0 \neq \varphi \in \mathring{\mathbf{H}}^1(\Omega)} \frac{|\operatorname{grad} \varphi|_{\mathbf{L}^2(\Omega)}}{\left( |\varphi \mathbf{I}|_{\mathbf{L}^2(\Omega)}^2 + |\operatorname{div} \operatorname{Div}(\varphi \mathbf{I})|_{\mathbf{H}^{-1}(\Omega)}^2 \right)^{1/2}} \\ & = \inf_{0 \neq \varphi \in \mathring{\mathbf{H}}^1(\Omega)} \frac{|\operatorname{grad} \varphi|_{\mathbf{L}^2(\Omega)}}{\left( 3|\varphi|_{\mathbf{L}^2(\Omega)}^2 + |\operatorname{grad} \varphi|_{\mathbf{L}^2(\Omega)}^2 \right)^{1/2}} \geq (3c_{\mathbb{g}}^2 + 1)^{-1/2} \end{aligned}$$

by choosing  $\Phi := -\varphi \mathbf{I} \in \mathring{\mathbf{H}}^1(\Omega) \cdot \mathbf{I} \subset \mathbf{DD}^{0,-1}(\Omega, \mathbb{S})$  and observing

$$\begin{aligned} -\langle \varphi, \operatorname{div} \operatorname{Div}(\varphi \mathbf{I}) \rangle_{\mathbf{H}^{-1}(\Omega)} &= -\langle \varphi, \operatorname{div} \operatorname{grad} \varphi \rangle_{\mathbf{H}^{-1}(\Omega)} = |\operatorname{grad} \varphi|_{\mathbf{L}^2(\Omega)}^2, \\ |\operatorname{div} \operatorname{Div}(\varphi \mathbf{I})|_{\mathbf{H}^{-1}(\Omega)} &= \sup_{0 \neq \phi \in \mathring{\mathbf{H}}^1(\Omega)} \frac{\langle \phi, \operatorname{div} \operatorname{grad} \varphi \rangle_{\mathbf{H}^{-1}(\Omega)}}{|\operatorname{grad} \phi|_{\mathbf{L}^2(\Omega)}} \\ &= \sup_{0 \neq \phi \in \mathring{\mathbf{H}}^1(\Omega)} \frac{\langle \operatorname{grad} \phi, \operatorname{grad} \varphi \rangle_{\mathbf{L}^2(\Omega)}}{|\operatorname{grad} \phi|_{\mathbf{L}^2(\Omega)}} = |\operatorname{grad} \varphi|_{\mathbf{L}^2(\Omega)}. \end{aligned}$$

Note that both the primal problem (4.1) and the mixed problem (4.4)-(4.5) are well-posed. So, it suffices to show the first part of (ii) only. The reverse direction follows then automatically. Let  $u \in \mathring{\mathbf{H}}^2(\Omega)$  solve (4.1). Then  $\mathbf{M} := -\operatorname{Grad} \operatorname{grad} u \in \mathbf{DD}^{0,-1}(\Omega, \mathbb{S})$  with  $\operatorname{div} \operatorname{Div} \mathbf{M} = -f$  in  $\mathbf{H}^{-2}(\Omega)$  and hence in  $\mathbf{H}^{-1}(\Omega)$ . Thus (4.5) holds. Moreover, for  $\Psi \in \mathbf{DD}^{0,-1}(\Omega, \mathbb{S})$  we see

$$\langle \mathbf{M}, \Psi \rangle_{\mathbf{L}^2(\Omega)} = -\langle \operatorname{Grad} \operatorname{grad} u, \Psi \rangle_{\mathbf{L}^2(\Omega)} = -\langle u, \operatorname{div} \operatorname{Div} \Psi \rangle_{\mathbf{H}^{-2}(\Omega)} = -\langle u, \operatorname{div} \operatorname{Div} \Psi \rangle_{\mathbf{H}^{-1}(\Omega)}$$

and hence (4.4) is true. Therefore,  $(\mathbf{M}, u)$  solves (4.4)-(4.5).  $\square$

**Remark 4.2.** For convenience of the reader, we give additionally a proof of the other direction as well: If  $(\mathbf{M}, u)$  in  $\mathbf{DD}^{0,-1}(\Omega, \mathbb{S}) \times \mathring{\mathbf{H}}^1(\Omega)$  solves (4.4)-(4.5), then  $\operatorname{div} \operatorname{Div} \mathbf{M} = -f$  in  $\mathbf{H}^{-1}(\Omega)$  and (4.4) holds. Especially, (4.4) holds for  $\Psi \in \mathbf{H}^2(\Omega, \mathbb{S}) \subset \mathbf{H}^1(\Omega, \mathbb{S}) \subset \mathbf{DD}^{0,-1}(\Omega, \mathbb{S})$ , i.e.,

$$(4.6) \quad -\langle \mathbf{M}, \Psi \rangle_{\mathbf{L}^2(\Omega)} = \langle u, \operatorname{div} \operatorname{Div} \Psi \rangle_{\mathbf{H}^{-1}(\Omega)} = \langle u, \operatorname{div} \operatorname{Div} \Psi \rangle_{\mathbf{L}^2(\Omega)}.$$

But then (4.6) holds for all  $\Psi \in \mathbf{H}^2(\Omega)$  as  $\operatorname{sym} \Psi \in \mathbf{H}^2(\Omega, \mathbb{S})$  and

$$(4.7) \quad -\langle \mathbf{M}, \Psi \rangle_{\mathbf{L}^2(\Omega)} = -\langle \mathbf{M}, \operatorname{sym} \Psi \rangle_{\mathbf{L}^2(\Omega)} = \langle u, \operatorname{div} \operatorname{Div} \operatorname{sym} \Psi \rangle_{\mathbf{L}^2(\Omega)} = \langle u, \operatorname{div} \operatorname{Div} \Psi \rangle_{\mathbf{L}^2(\Omega)},$$

since  $\operatorname{div} \operatorname{Div} \operatorname{skw} \Psi = 0$  by

$$\langle \operatorname{div} \operatorname{Div} \operatorname{skw} \Psi, \phi \rangle_{\mathbf{L}^2(\Omega)} = \langle \operatorname{skw} \Psi, \operatorname{Grad} \operatorname{grad} \phi \rangle_{\mathbf{L}^2(\Omega)} = 0$$

for all  $\phi \in \mathring{\mathbf{C}}^\infty(\Omega)$ . (4.7) yields that  $u \in \mathring{\mathbf{H}}^2(\Omega)$  with  $\operatorname{Grad} \operatorname{grad} u = -\mathbf{M}$ . Finally, for all  $\phi \in \mathring{\mathbf{H}}^2(\Omega)$

$$\langle \operatorname{Grad} \operatorname{grad} u, \operatorname{Grad} \operatorname{grad} \phi \rangle_{\mathbf{L}^2(\Omega)} = -\langle \mathbf{M}, \operatorname{Grad} \operatorname{grad} \phi \rangle_{\mathbf{L}^2(\Omega)} = -\langle \operatorname{div} \operatorname{Div} \mathbf{M}, \phi \rangle_{\mathbf{H}^{-2}(\Omega)} = \langle f, \phi \rangle_{\mathbf{H}^{-2}(\Omega)},$$

showing that  $u \in \mathring{\mathbf{H}}^2(\Omega)$  solves (4.1).

We note that the decomposition of  $\mathbf{DD}^{0,-1}(\Omega, \mathbb{S})$  in Lemma 3.20 is different to the Helmholtz type decomposition of the larger space  $\mathbf{L}^2(\Omega, \mathbb{S})$  in Theorem 3.12 and Theorem 3.22 and does not involve the Hessian of scalar functions in  $\mathring{\mathbf{H}}^2(\Omega)$ . Using the decomposition of  $\mathbf{DD}^{0,-1}(\Omega, \mathbb{S})$  in Lemma 3.20, we have the following decomposition result for the biharmonic problem. Let  $(\mathbf{M}, u) \in \mathbf{DD}^{0,-1}(\Omega, \mathbb{S}) \times \mathring{\mathbf{H}}^1(\Omega)$  be the unique solution of (4.4)-(4.5). Using Lemma 3.20 we have the following direct decompositions for  $\mathbf{M}, \mathbf{\Psi} \in \mathbf{DD}^{0,-1}(\Omega, \mathbb{S})$

$$\mathbf{M} = p\mathbf{I} + \mathbf{M}_0, \quad \mathbf{\Psi} = \varphi\mathbf{I} + \mathbf{\Psi}_0, \quad p, \varphi \in \mathring{\mathbf{H}}^1(\Omega), \quad \mathbf{M}_0, \mathbf{\Psi}_0 \in \mathbf{DD}_0(\Omega, \mathbb{S}).$$

This allows to rewrite (4.4)-(4.5) equivalently in terms of  $(p, \mathbf{M}_0, u)$  and for all  $(\varphi, \mathbf{\Psi}_0, \psi)$ , i.e.,

$$\begin{aligned} \langle p\mathbf{I}, \varphi\mathbf{I} \rangle_{\mathbf{L}^2(\Omega)} + \langle \mathbf{M}_0, \mathbf{\Psi}_0 \rangle_{\mathbf{L}^2(\Omega)} + \langle p\mathbf{I}, \mathbf{\Psi}_0 \rangle_{\mathbf{L}^2(\Omega)} + \langle \mathbf{M}_0, \varphi\mathbf{I} \rangle_{\mathbf{L}^2(\Omega)} + \langle u, \operatorname{div} \operatorname{Div}(\varphi\mathbf{I}) \rangle_{\mathbf{H}^{-1}(\Omega)} &= 0, \\ \langle \operatorname{div} \operatorname{Div}(p\mathbf{I}), \psi \rangle_{\mathbf{H}^{-1}(\Omega)} &= -\langle f, \psi \rangle_{\mathbf{H}^{-1}(\Omega)} \end{aligned}$$

or equivalently

$$\begin{aligned} \langle \operatorname{grad} u, \operatorname{grad} \varphi \rangle_{\mathbf{L}^2(\Omega)} + 3\langle p, \varphi \rangle_{\mathbf{L}^2(\Omega)} + \langle \mathbf{M}_0, \mathbf{\Psi}_0 \rangle_{\mathbf{L}^2(\Omega)} + \langle p, \operatorname{tr} \mathbf{\Psi}_0 \rangle_{\mathbf{L}^2(\Omega)} + \langle \operatorname{tr} \mathbf{M}_0, \varphi \rangle_{\mathbf{L}^2(\Omega)} &= 0, \\ \langle \operatorname{grad} p, \operatorname{grad} \psi \rangle_{\mathbf{L}^2(\Omega)} &= -\langle f, \psi \rangle_{\mathbf{H}^{-1}(\Omega)}, \end{aligned}$$

which leads to the equivalent system

$$\begin{aligned} \langle \operatorname{grad} u, \operatorname{grad} \varphi \rangle_{\mathbf{L}^2(\Omega)} + 3\langle p, \varphi \rangle_{\mathbf{L}^2(\Omega)} + \langle \operatorname{tr} \mathbf{M}_0, \varphi \rangle_{\mathbf{L}^2(\Omega)} &= 0, \\ + \langle \mathbf{M}_0, \mathbf{\Psi}_0 \rangle_{\mathbf{L}^2(\Omega)} + \langle p, \operatorname{tr} \mathbf{\Psi}_0 \rangle_{\mathbf{L}^2(\Omega)} &= 0, \\ \langle \operatorname{grad} p, \operatorname{grad} \psi \rangle_{\mathbf{L}^2(\Omega)} &= -\langle f, \psi \rangle_{\mathbf{H}^{-1}(\Omega)}. \end{aligned}$$

**Theorem 4.3.** *The variational problem (4.4)-(4.5) is equivalent to the following well-posed and uniquely solvable variational problem. For  $f \in \mathbf{H}^{-1}(\Omega)$  find  $p \in \mathring{\mathbf{H}}^1(\Omega)$ ,  $\mathbf{M}_0 \in \mathbf{DD}_0(\Omega, \mathbb{S})$ , and  $u \in \mathring{\mathbf{H}}^1(\Omega)$  such that*

$$(4.8) \quad \langle \operatorname{grad} u, \operatorname{grad} \varphi \rangle_{\mathbf{L}^2(\Omega)} + \langle \operatorname{tr} \mathbf{M}_0, \varphi \rangle_{\mathbf{L}^2(\Omega)} + 3\langle p, \varphi \rangle_{\mathbf{L}^2(\Omega)} = 0,$$

$$(4.9) \quad \langle \mathbf{M}_0, \mathbf{\Psi}_0 \rangle_{\mathbf{L}^2(\Omega)} + \langle p, \operatorname{tr} \mathbf{\Psi}_0 \rangle_{\mathbf{L}^2(\Omega)} = 0,$$

$$(4.10) \quad \langle \operatorname{grad} p, \operatorname{grad} \psi \rangle_{\mathbf{L}^2(\Omega)} = -\langle f, \psi \rangle_{\mathbf{H}^{-1}(\Omega)}$$

for all  $\psi \in \mathring{\mathbf{H}}^1(\Omega)$ ,  $\mathbf{\Psi}_0 \in \mathbf{DD}_0(\Omega, \mathbb{S})$ , and  $\varphi \in \mathring{\mathbf{H}}^1(\Omega)$ . Moreover, the unique solution  $(\mathbf{M}, u)$  of (4.4)-(4.5) is given by  $\mathbf{M} := p\mathbf{I} + \mathbf{M}_0$  and  $u$  for the unique solution  $(p, \mathbf{M}_0, u)$  of (4.11)-(4.13).

If  $\Omega$  is additionally topologically trivial, then by Theorem 3.12 or Theorem 3.22

$$\mathbf{DD}_0(\Omega, \mathbb{S}) = \operatorname{sym} \operatorname{Rot} \mathbf{R}_{\operatorname{sym}}(\Omega, \mathbb{T}) = \operatorname{sym} \operatorname{Rot} (\mathbf{R}_{\operatorname{sym}}(\Omega, \mathbb{T}) \cap \mathring{\mathbf{D}}_0(\Omega, \mathbb{T}))$$

and we obtain the following result.

**Theorem 4.4.** *Let  $\Omega$  be additionally topologically trivial. The variational problem (4.4)-(4.5) is equivalent to the following well-posed and uniquely solvable variational problem. For  $f \in \mathbf{H}^{-1}(\Omega)$  find  $p \in \mathring{\mathbf{H}}^1(\Omega)$ ,  $\mathbf{E} \in \mathbf{R}_{\operatorname{sym}}(\Omega, \mathbb{T}) \cap \mathring{\mathbf{D}}_0(\Omega, \mathbb{T})$ , and  $u \in \mathring{\mathbf{H}}^1(\Omega)$  such that*

$$(4.11) \quad \langle \operatorname{grad} u, \operatorname{grad} \varphi \rangle_{\mathbf{L}^2(\Omega)} + \langle \operatorname{tr} \operatorname{sym} \operatorname{Rot} \mathbf{E}, \varphi \rangle_{\mathbf{L}^2(\Omega)} + 3\langle p, \varphi \rangle_{\mathbf{L}^2(\Omega)} = 0,$$

$$(4.12) \quad \langle \operatorname{sym} \operatorname{Rot} \mathbf{E}, \operatorname{sym} \operatorname{Rot} \mathbf{\Phi} \rangle_{\mathbf{L}^2(\Omega)} + \langle p, \operatorname{tr} \operatorname{sym} \operatorname{Rot} \mathbf{\Phi} \rangle_{\mathbf{L}^2(\Omega)} = 0,$$

$$(4.13) \quad \langle \operatorname{grad} p, \operatorname{grad} \psi \rangle_{\mathbf{L}^2(\Omega)} = -\langle f, \psi \rangle_{\mathbf{H}^{-1}(\Omega)}$$

for all  $\psi \in \mathring{\mathbf{H}}^1(\Omega)$ ,  $\mathbf{\Phi} \in \mathbf{R}_{\operatorname{sym}}(\Omega, \mathbb{T}) \cap \mathring{\mathbf{D}}_0(\Omega, \mathbb{T})$ , and  $\varphi \in \mathring{\mathbf{H}}^1(\Omega)$ . Moreover, the unique solution  $(\mathbf{M}, u)$  of (4.4)-(4.5) is given by  $\mathbf{M} := p\mathbf{I} + \operatorname{sym} \operatorname{Rot} \mathbf{E}$  and  $u$  for the unique solution  $(p, \mathbf{E}, u)$  of (4.11)-(4.13).

Note that, e.g.,  $\langle \operatorname{tr} \operatorname{sym} \operatorname{Rot} \mathbf{E}, \varphi \rangle_{\mathbf{L}^2(\Omega)} = \langle \operatorname{sym} \operatorname{Rot} \mathbf{E}, \varphi\mathbf{I} \rangle_{\mathbf{L}^2(\Omega)}$  and  $3\langle p, \varphi \rangle_{\mathbf{L}^2(\Omega)} = \langle p\mathbf{I}, \varphi\mathbf{I} \rangle_{\mathbf{L}^2(\Omega)}$ .

*Proof.* (4.4)-(4.5) is equivalent to (4.8)-(4.10) and hence also to (4.11)-(4.13), if the latter system is well-posed. By Theorem 3.12 or Theorem 3.22 the bilinear form  $\langle \text{sym Rot } \cdot, \text{sym Rot } \cdot \rangle_{\mathbf{L}^2(\Omega)}$  is coercive over  $\mathbf{R}_{\text{sym}}(\Omega, \mathbb{T}) \cap \mathring{\mathbf{D}}_0(\Omega, \mathbb{T})$ , which shows the consecutive unique solvability of (4.11)-(4.13).  $\square$

The three problems in the previous theorem are weak formulations of the following three second-order problems in strong form. A homogeneous Dirichlet Poisson problem for the auxiliary scalar function  $p$

$$\Delta p = f \quad \text{in } \Omega, \quad p = 0 \quad \text{on } \Gamma,$$

a second-order inhomogeneous Neumann type Rot sym Rot-Div-system for the auxiliary tensor field  $\mathbf{E}$

$$\begin{aligned} \text{tr } \mathbf{E} = 0, \quad \text{Rot sym Rot } \mathbf{E} = -\text{Rot}(p \mathbf{I}) = \text{spn grad } p, \quad \text{Div } \mathbf{E} = 0 & \quad \text{in } \Omega, \\ n \times \text{sym Rot } \mathbf{E} = -n \times p \mathbf{I} = p \text{spn } n = \mathbf{0}, \quad \mathbf{E} n = 0 & \quad \text{on } \Gamma, \end{aligned}$$

and, finally, a homogeneous Dirichlet Poisson problem for the original scalar function  $u$

$$\Delta u = 3p + \text{tr sym Rot } \mathbf{E} = \text{tr}(p \mathbf{I} + \text{sym Rot } \mathbf{E}) \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \Gamma.$$

In other words, the system (4.11)-(4.13) has triangular structure

$$\begin{bmatrix} 3 & \text{tr sym Rot } \mathbb{T} & -\mathring{\Delta} \\ \mathring{\text{Rot}}_{\mathbb{S}}(\cdot \mathbf{I}) & \mathring{\text{Rot}}_{\mathbb{S}} \text{sym Rot } \mathbb{T} & 0 \\ -\mathring{\Delta} & 0 & 0 \end{bmatrix} \begin{bmatrix} p \\ \mathbf{E} \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ \mathbf{0} \\ -f \end{bmatrix}$$

with  $(\text{tr sym Rot } \mathbb{T})^* = \mathring{\text{Rot}}_{\mathbb{S}}(\cdot \mathbf{I})$ . Indeed we see that  $\mathbf{E} \in \mathbf{R}_{\text{sym}}(\Omega, \mathbb{T}) \cap \mathring{\mathbf{D}}_0(\Omega, \mathbb{T})$  with

$$\langle \text{sym Rot } \mathbf{E}, \text{sym Rot } \Phi \rangle_{\mathbf{L}^2(\Omega)} + \langle p, \text{tr sym Rot } \Phi \rangle_{\mathbf{L}^2(\Omega)} = 0$$

for all  $\Phi \in \mathbf{R}_{\text{sym}}(\Omega, \mathbb{T}) \cap \mathring{\mathbf{D}}_0(\Omega, \mathbb{T})$  is equivalent to  $\mathbf{E} \in \mathbf{R}_{\text{sym}}(\Omega, \mathbb{T}) \cap \mathring{\mathbf{D}}_0(\Omega, \mathbb{T})$  and

$$(4.14) \quad \langle \text{sym Rot } \mathbf{E} + p \mathbf{I}, \text{sym Rot } \Phi \rangle_{\mathbf{L}^2(\Omega)} = 0$$

for all  $\Phi \in \mathbf{R}_{\text{sym}}(\Omega, \mathbb{T})$  as by Theorem 3.12

$$(4.15) \quad \text{sym Rot}(\mathbf{R}_{\text{sym}}(\Omega, \mathbb{T}) \cap \mathring{\mathbf{D}}_0(\Omega, \mathbb{T})) = \text{sym Rot } \mathbf{R}_{\text{sym}}(\Omega, \mathbb{T}).$$

Now (4.14) shows that

$$\text{sym Rot } \mathbf{E} + p \mathbf{I} \in D(\text{sym Rot } \mathbb{T}^*) = D(\mathring{\text{Rot}}_{\mathbb{S}}) = \mathring{\mathbf{R}}(\Omega, \mathbb{S})$$

with  $\text{Rot}(\text{sym Rot } \mathbf{E} + p \mathbf{I}) = 0$ .

Finally, we want to get rid of the complicated space  $\mathbf{R}_{\text{sym}}(\Omega, \mathbb{T}) \cap \mathring{\mathbf{D}}_0(\Omega, \mathbb{T})$  in the variational formulation in Theorem 4.4, which might be very complicated to implement in forthcoming numerical applications using finite elements due to the solenoidal and homogeneous normal boundary conditions. For given  $p \in \mathring{\mathbf{H}}^1(\Omega)$  the part (4.12), i.e., find  $\mathbf{E} \in \mathbf{R}_{\text{sym}}(\Omega, \mathbb{T}) \cap \mathring{\mathbf{D}}_0(\Omega, \mathbb{T})$  such that

$$(4.16) \quad \langle \text{sym Rot } \mathbf{E}, \text{sym Rot } \Phi \rangle_{\mathbf{L}^2(\Omega)} + \langle p, \text{tr sym Rot } \Phi \rangle_{\mathbf{L}^2(\Omega)} = 0$$

for all  $\Phi \in \mathbf{R}_{\text{sym}}(\Omega, \mathbb{T}) \cap \mathring{\mathbf{D}}_0(\Omega, \mathbb{T})$ , of the variational system (4.11)-(4.13), has also a saddle point structure. By Theorem 3.12 we have (4.15) as well as

$$\mathring{\mathbf{D}}_0(\Omega, \mathbb{T}) = \mathbf{R}_{\text{sym},0}(\Omega, \mathbb{T})^{\perp \mathbf{L}^2(\Omega, \mathbb{T})} = \text{dev Grad}(\mathbf{H}^1(\Omega) \cap \mathbf{RT}_0^{\perp \mathbf{L}^2(\Omega)})^{\perp \mathbf{L}^2(\Omega)}.$$

Hence (4.16) is equivalent to find  $\mathbf{E} \in \mathbf{R}_{\text{sym}}(\Omega, \mathbb{T})$  such that

$$(4.17) \quad \langle \text{sym Rot } \mathbf{E}, \text{sym Rot } \Phi \rangle_{\mathbf{L}^2(\Omega)} + \langle p, \text{tr sym Rot } \Phi \rangle_{\mathbf{L}^2(\Omega)} = 0,$$

$$(4.18) \quad \langle \mathbf{E}, \text{dev Grad } \Phi \rangle_{\mathbf{L}^2(\Omega)} = 0$$

for all  $\Phi \in \mathbf{R}_{\text{sym}}(\Omega, \mathbb{T})$  and  $\Phi \in \mathbf{H}^1(\Omega) \cap \text{RT}_0^{\perp \text{L}^2(\Omega)}$ . Observe that

$$(\mathbf{E}, V) := (\mathbf{E}, 0) \in \mathbf{R}_{\text{sym}}(\Omega, \mathbb{T}) \times (\mathbf{H}^1(\Omega) \cap \text{RT}_0^{\perp \text{L}^2(\Omega)})$$

solves the modified variational system

$$(4.19) \quad \langle \text{sym Rot } \mathbf{E}, \text{sym Rot } \Phi \rangle_{\text{L}^2(\Omega)} + \langle \Phi, \text{dev Grad } V \rangle_{\text{L}^2(\Omega)} = -\langle p, \text{tr sym Rot } \Phi \rangle_{\text{L}^2(\Omega)},$$

$$(4.20) \quad \langle \mathbf{E}, \text{dev Grad } \Phi \rangle_{\text{L}^2(\Omega)} = 0$$

for all  $\Phi \in \mathbf{R}_{\text{sym}}(\Omega, \mathbb{T})$  and  $\Phi \in \mathbf{H}^1(\Omega) \cap \text{RT}_0^{\perp \text{L}^2(\Omega)}$ . On the other hand, any solution

$$(\mathbf{E}, V) \in \mathbf{R}_{\text{sym}}(\Omega, \mathbb{T}) \times (\mathbf{H}^1(\Omega) \cap \text{RT}_0^{\perp \text{L}^2(\Omega)})$$

of (4.19)-(4.20) satisfies  $V = 0$ , as (4.19) tested with

$$\Phi := \text{dev Grad } V \in \text{dev Grad } \mathbf{H}^1(\Omega) = \mathbf{R}_{\text{sym},0}(\Omega, \mathbb{T})$$

shows  $\text{dev Grad } V = 0$  and thus  $V \in \text{RT}_0$  by Lemma 3.2 yielding  $V = 0$ . Note that (4.19)-(4.20) has the saddle point structure

$$\begin{bmatrix} \overset{\circ}{\text{Rot}}_{\mathbb{S}} \text{sym Rot}_{\mathbb{T}} & \text{dev Grad} \\ -\overset{\circ}{\text{Div}}_{\mathbb{T}} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{E} \\ V \end{bmatrix} = \begin{bmatrix} -\overset{\circ}{\text{Rot}}_{\mathbb{S}}(v \cdot \mathbf{I}) \\ 0 \end{bmatrix}, \quad (\text{dev Grad})^* = -\overset{\circ}{\text{Div}}_{\mathbb{T}}.$$

We obtain the following theorem.

**Theorem 4.5.** *Let  $\Omega$  be additionally topologically trivial. The variational problem (4.11)-(4.13) is equivalent to the following well-posed and uniquely solvable variational system. For  $f \in \mathbf{H}^{-1}(\Omega)$  find  $p \in \overset{\circ}{\mathbf{H}}^1(\Omega)$ ,  $\mathbf{E} \in \mathbf{R}_{\text{sym}}(\Omega, \mathbb{T})$ ,  $V \in \mathbf{H}^1(\Omega) \cap \text{RT}_0^{\perp \text{L}^2(\Omega)}$ , and  $u \in \overset{\circ}{\mathbf{H}}^1(\Omega)$  such that*

$$(4.21) \quad \langle \text{grad } u, \text{grad } \varphi \rangle_{\text{L}^2(\Omega)} + \langle \text{tr sym Rot } \mathbf{E}, \varphi \rangle_{\text{L}^2(\Omega)} + 3\langle p, \varphi \rangle_{\text{L}^2(\Omega)} = 0,$$

$$(4.22) \quad \langle \text{sym Rot } \mathbf{E}, \text{sym Rot } \Phi \rangle_{\text{L}^2(\Omega)} + \langle \Phi, \text{dev Grad } V \rangle_{\text{L}^2(\Omega)} + \langle p, \text{tr sym Rot } \Phi \rangle_{\text{L}^2(\Omega)} = 0,$$

$$(4.23) \quad \langle \mathbf{E}, \text{dev Grad } \Phi \rangle_{\text{L}^2(\Omega)} = 0,$$

$$(4.24) \quad \langle \text{grad } p, \text{grad } \psi \rangle_{\text{L}^2(\Omega)} = -\langle f, \psi \rangle_{\mathbf{H}^{-1}(\Omega)}$$

for all  $\psi \in \overset{\circ}{\mathbf{H}}^1(\Omega)$ ,  $\Phi \in \mathbf{R}_{\text{sym}}(\Omega, \mathbb{T})$ ,  $\Phi \in \mathbf{H}^1(\Omega) \cap \text{RT}_0^{\perp \text{L}^2(\Omega)}$ , and  $\varphi \in \overset{\circ}{\mathbf{H}}^1(\Omega)$ . Moreover, the unique solution  $(p, \mathbf{E}, V, u)$  of (4.21)-(4.24) satisfies  $V = 0$  and  $(p, \mathbf{E}, u)$  is the unique solution of (4.11)-(4.13).

Note that the system (4.21)-(4.24) has the block triangular saddle point structure

$$\begin{bmatrix} 3 & \text{tr sym Rot}_{\mathbb{T}} & 0 & -\overset{\circ}{\Delta} \\ \overset{\circ}{\text{Rot}}_{\mathbb{S}}(\cdot \mathbf{I}) & \overset{\circ}{\text{Rot}}_{\mathbb{S}} \text{sym Rot}_{\mathbb{T}} & \text{dev Grad} & 0 \\ 0 & -\overset{\circ}{\text{Div}}_{\mathbb{T}} & 0 & 0 \\ -\overset{\circ}{\Delta} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} p \\ \mathbf{E} \\ V \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ \mathbf{0} \\ 0 \\ -f \end{bmatrix}.$$

with  $(\text{tr sym Rot}_{\mathbb{T}})^* = \overset{\circ}{\text{Rot}}_{\mathbb{S}}(\cdot) \mathbf{I}$  and  $(\text{dev Grad})^* = -\overset{\circ}{\text{Div}}_{\mathbb{T}}$ .

*Proof.* We only have to show well-posedness of the partial system (4.22)-(4.23). First note that by Theorem 3.12 the bilinear form  $\langle \text{sym Rot } \cdot, \text{sym Rot } \cdot \rangle_{\text{L}^2(\Omega)}$  is coercive over  $\mathbf{R}_{\text{sym}}(\Omega, \mathbb{T}) \cap \overset{\circ}{\mathbf{D}}_0(\Omega, \mathbb{T})$ , which equals the kernel of (4.23). Indeed it follows from (4.23) that

$$\mathbf{E} \in \text{dev Grad } (\mathbf{H}^1(\Omega) \cap \text{RT}_0^{\perp \text{L}^2(\Omega)})^{\perp \text{L}^2(\Omega)} = \overset{\circ}{\mathbf{D}}_0(\Omega, \mathbb{T}).$$

Moreover, the inf-sup-condition is satisfied as by picking for fixed  $0 \neq \Phi \in \mathbf{H}^1(\Omega) \cap \mathbf{RT}_0^{\perp \mathbf{L}^2(\Omega)}$  the tensor  $\Phi := \text{dev Grad } \Phi \in \text{dev Grad } \mathbf{H}^1(\Omega) = \mathbf{R}_{\text{sym},0}(\Omega, \mathbb{T})$  we have

$$\inf_{\substack{0 \neq \Phi \in \mathbf{H}^1(\Omega), \\ \Phi \perp_{\mathbf{L}^2(\Omega)} \mathbf{RT}_0}} \sup_{\Phi \in \mathbf{R}_{\text{sym}}(\Omega, \mathbb{T})} \frac{\langle \Phi, \text{dev Grad } \Phi \rangle_{\mathbf{L}^2(\Omega)}}{|\Phi|_{\mathbf{R}_{\text{sym}}(\Omega, \mathbb{T})} |\Phi|_{\mathbf{H}^1(\Omega)}} \geq \inf_{\substack{0 \neq \Phi \in \mathbf{H}^1(\Omega), \\ \Phi \perp_{\mathbf{L}^2(\Omega)} \mathbf{RT}_0}} \frac{|\text{dev Grad } \Phi|_{\mathbf{L}^2(\Omega)}}{|\Phi|_{\mathbf{H}^1(\Omega)}} \geq \frac{1}{c}$$

by Lemma 3.2 (iv).  $\square$

**Remark 4.6.** *The corresponding result for the two-dimensional case is completely analogous with the exception that the tensor potential  $\mathbf{E} \in \mathbf{R}_{\text{sym}}(\Omega, \mathbb{T}) \cap \mathring{\mathbf{D}}_0(\Omega, \mathbb{T})$  is to be replaced by a much simpler vector potential  $N \in \mathbf{H}^1(\Omega)$ . Furthermore, observe that*

$$\langle \text{sym Rot } N, \text{sym Rot } \Phi \rangle_{\mathbf{L}^2(\Omega)} = \langle \text{sym Grad}^{\perp} N, \text{sym Grad}^{\perp} \Phi \rangle_{\mathbf{L}^2(\Omega)}$$

holds for vector fields  $N, \Phi \in \mathbf{H}^1(\Omega)$ . Here the superscript  $\perp$  denotes the rotation of a vector field by  $90^\circ$ . Note that the complicated second-order inhomogeneous Neumann type Rot sym Rot-Div-system for the auxiliary tensor field  $\mathbf{E}$  is replaced in 2D by a much simpler inhomogeneous Neumann linear elasticity problem, where the standard Sobolev space  $\mathbf{H}^1(\Omega)$  resp.  $\mathbf{H}^1(\Omega) \cap \mathbf{RM}^{\perp \mathbf{L}^2(\Omega)}$  can be used. Here RM denotes the space of rigid motions. This yields the decomposition result in [14] for the two-dimensional case, which was shortly mentioned in the introduction.

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### APPENDIX A. PROOFS OF SOME USEFUL IDENTITIES

Note that for  $a, b \in \mathbb{R}^3$  and  $A \in \mathbb{R}^{3 \times 3}$

$$(A.1) \quad \text{spn } a : \text{spn } b = 2 a \cdot b, \quad \text{skw } A = \frac{1}{2} \text{spn} \begin{bmatrix} A_{32} - A_{23} \\ A_{13} - A_{31} \\ A_{21} - A_{12} \end{bmatrix}$$

hold and hence for skew-symmetric  $A$

$$(A.2) \quad \text{spn } a : A = \text{spn } a : \text{spn } \text{spn}^{-1} A = 2 a \cdot \text{spn}^{-1} A,$$

i.e.,  $\text{spn}^* = 2 \text{spn}^{-1}$ . Moreover, we have for two matrices  $A, B$

$$A^\top : B = \text{tr}(AB) = \text{tr}(BA) = B^\top : A = A : B^\top.$$

The assertions of Lemma 3.4 and Lemma 3.9 are contained in the assertions of the following lemma.

**Lemma A.1.** *For smooth functions, vector fields and tensor fields we have*

- (i)  $\text{skw Grad grad } u = 0$ ,
- (ii)  $\text{div Div } \mathbf{M} = 0$ , if  $\mathbf{M}$  is skew-symmetric,
- (iii)  $\text{Rot}(u \mathbf{I}) = -\text{spn grad } u$ ,
- (iv)  $\text{tr Rot } \mathbf{M} = 2 \text{div}(\text{spn}^{-1} \text{skw } \mathbf{M})$ ,  
especially  $\text{tr Rot } \mathbf{M} = 0$ , if  $\mathbf{M}$  is symmetric,
- (v)  $\text{Div}(u \mathbf{I}) = \text{grad } u$ ,
- (vi)  $\text{tr Grad } V = \text{div } V$ ,
- (vii)  $\text{Div}(\text{spn } V) = -\text{rot } V$ ,  
especially  $\text{Div}(\text{skw } \mathbf{M}) = -\text{rot } V$  for  $V = \text{spn}^{-1} \text{skw } \mathbf{M}$ ,
- (viii)  $\text{Rot}(\text{spn } V) = (\text{div } V) \mathbf{I} - (\text{Grad } V)^\top$ ,  
especially  $\text{Rot skw } \mathbf{M} = (\text{div } V) \mathbf{I} - (\text{Grad } V)^\top$  for  $V = \text{spn}^{-1} \text{skw } \mathbf{M}$ ,
- (ix)  $\text{skw Grad } V = \frac{1}{2} \text{spn rot } V$  and  $\text{Rot}(\text{sym Grad } V) = -\text{Rot}(\text{skw Grad } V) = -\frac{1}{2} \text{Rot}(\text{spn rot } V)$ ,
- (x)  $\text{skw Rot } \mathbf{M} = \text{spn } V$  and  $\text{Div}(\text{sym Rot } \mathbf{M}) = -\text{Div}(\text{skw Rot } \mathbf{M}) = \text{rot } V$   
with  $V = \frac{1}{2}(\text{Div } \mathbf{M}^\top - \text{grad}(\text{tr } \mathbf{M}))$ ,  
especially  $\text{Div}(\text{sym Rot } \mathbf{M}) = -\text{Div}(\text{skw Rot } \mathbf{M}) = \frac{1}{2} \text{rot Div } \mathbf{M}^\top$ , if  $\text{tr } \mathbf{M} = 0$ ,
- (xi)  $\text{grad div } V = \frac{3}{2} \text{Div dev}(\text{Grad } V)^\top$ .

*These formulas hold for distributions as well.*

*Proof.* (i)-(ix) and the first identity in (x) follow by elementary calculations. For the second identity in (x) observe that  $0 = \text{Div Rot } \mathbf{M} = \text{Div}(\text{sym Rot } \mathbf{M}) + \text{Div}(\text{skw Rot } \mathbf{M})$  for  $\mathbf{M} \in \mathring{\mathbf{C}}^\infty(\mathbb{R}^3)$  and hence, using the first identity in (x) and (vii), we obtain

$$\text{Div}(\text{sym Rot } \mathbf{M}) = -\text{Div}(\text{skw Rot } \mathbf{M}) = -\text{Div}(\text{spn } V) = \text{rot } V.$$



To see (xi) we compute

$$\begin{aligned} 0 &= \text{Div Rot spn } V = \text{Div}((\text{div } V) \mathbf{I}) - \text{Div}(\text{Grad } V)^\top \\ &= \text{Div}((\text{div } V) \mathbf{I}) - \text{Div dev}(\text{Grad } V)^\top - \frac{1}{3} \text{Div}((\text{tr}(\text{Grad } V)^\top) \mathbf{I}) \\ &= \frac{2}{3} \text{Div}((\text{div } V) \mathbf{I}) - \text{Div dev}(\text{Grad } V)^\top = \frac{2}{3} \text{grad div } V - \text{Div dev}(\text{Grad } V)^\top. \end{aligned}$$

Therefore, the stated formulas hold in the smooth case. By density these formulas extend to  $u$ ,  $V$ , and  $\mathbf{M}$  in respective Sobolev spaces. Let us give proofs for distributions as well. For this, let  $m \in \mathbb{N}_0$  and  $u \in \mathbf{H}^{-m}(\Omega)$ ,  $V \in \mathbf{H}^{-m}(\Omega)$ ,  $\mathbf{M} \in \mathbf{H}^{-m}(\Omega)$  and  $\varphi \in \mathring{C}^\infty(\Omega)$ ,  $\Phi \in \mathring{C}^\infty(\Omega)$ , and  $\Phi \in \mathring{C}^\infty(\Omega)$ . By

$$\langle u, \partial_i \partial_j \varphi \rangle_{\mathbf{H}^{-m}(\Omega)} = \langle u, \partial_j \partial_i \varphi \rangle_{\mathbf{H}^{-m}(\Omega)}, \quad \text{or (with (ii))} \quad \langle u, \text{div Div skw } \Phi \rangle_{\mathbf{H}^{-m}(\Omega)} = 0$$

we see that  $\text{Grad grad } u \in \mathbf{H}^{-m-2}(\Omega)$  is symmetric and hence (i). Note that the formal adjoint is  $(\text{skw Grad grad})^* = \text{div Div skw}$ . If  $\mathbf{M}$  is skew-symmetric we have  $\langle \mathbf{M}, \text{Grad grad } \varphi \rangle_{\mathbf{H}^{-m}(\Omega)} = 0$ , i.e., (ii).

We compute with (iv)

$$\begin{aligned} \langle u \mathbf{I}, \text{Rot } \Phi \rangle_{\mathbf{H}^{-m}(\Omega)} &= \langle u, \text{tr}(\text{Rot } \Phi) \rangle_{\mathbf{H}^{-m}(\Omega)} = 2 \langle u, \text{div}(\text{spn}^{-1} \text{skw } \Phi) \rangle_{\mathbf{H}^{-m}(\Omega)} \\ &= - \langle \text{spn grad } u, \text{skw } \Phi \rangle_{\mathbf{H}^{-m-1}(\Omega)} = - \langle \text{spn grad } u, \Phi \rangle_{\mathbf{H}^{-m-1}(\Omega)}, \end{aligned}$$

showing (iii). Formally,  $(\text{tr Rot})^* = \text{Rot}(\cdot \mathbf{I})$ . Hence by (iii)

$$\begin{aligned} \langle \mathbf{M}, \text{Rot}(\varphi \mathbf{I}) \rangle_{\mathbf{H}^{-m}(\Omega)} &= - \langle \mathbf{M}, \text{spn grad } \varphi \rangle_{\mathbf{H}^{-m}(\Omega)} = - \langle \text{skw } \mathbf{M}, \text{spn grad } \varphi \rangle_{\mathbf{H}^{-m}(\Omega)} \\ &= -2 \langle \text{spn}^{-1} \text{skw } \mathbf{M}, \text{grad } \varphi \rangle_{\mathbf{H}^{-m}(\Omega)} = 2 \langle \text{div spn}^{-1} \text{skw } \mathbf{M}, \varphi \rangle_{\mathbf{H}^{-m-1}(\Omega)}, \end{aligned}$$

yielding (iv). (v) follows by

$$- \langle u \mathbf{I}, \text{Grad } \Phi \rangle_{\mathbf{H}^{-m}(\Omega)} = - \langle u, \text{tr}(\text{Grad } \Phi) \rangle_{\mathbf{H}^{-m}(\Omega)} = - \langle u, \text{div } \Phi \rangle_{\mathbf{H}^{-m}(\Omega)}.$$

Formally,  $(\text{tr Grad})^* = - \text{Div}(\cdot \mathbf{I})$ . Thus by (v)

$$- \langle V, \text{Div}(\varphi \mathbf{I}) \rangle_{\mathbf{H}^{-m}(\Omega)} = - \langle V, \text{grad } \varphi \rangle_{\mathbf{H}^{-m}(\Omega)} = \langle \text{div } V, \varphi \rangle_{\mathbf{H}^{-m-1}(\Omega)},$$

yielding (vi). We have the formal adjoint  $(\text{Div spn})^* = (\text{Div skw spn})^* = -2 \text{spn}^{-1} \text{skw Grad}$ , and by the formula  $2 \text{skw Grad } \Phi = \text{spn rot } \Phi$  from (ix), we obtain (vii), i.e.,

$$-2 \langle V, \text{spn}^{-1} \text{skw Grad } \Phi \rangle_{\mathbf{H}^{-m}(\Omega)} = - \langle V, \text{rot } \Phi \rangle_{\mathbf{H}^{-m}(\Omega)}.$$

Using the formal adjoint  $(\text{Rot spn})^* = 2 \text{spn}^{-1} \text{skw Rot}$  we calculate with (x)

$$\begin{aligned} 2 \langle V, \text{spn}^{-1} \text{skw Rot } \Phi \rangle_{\mathbf{H}^{-m}(\Omega)} &= \langle V, \text{Div } \Phi^\top - \text{grad}(\text{tr } \Phi) \rangle_{\mathbf{H}^{-m}(\Omega)} \\ &= - \langle \text{Grad } V, \Phi^\top \rangle_{\mathbf{H}^{-m-1}(\Omega)} + \langle \text{div } V, \text{tr } \Phi \rangle_{\mathbf{H}^{-m-1}(\Omega)}, \end{aligned}$$

i.e., (viii) holds. Formally  $(\text{skw Grad})^* = - \text{Div skw}$ . Using (vii) we see

$$- \langle V, \text{Div skw } \Phi \rangle_{\mathbf{H}^{-m}(\Omega)} = \langle V, \text{rot spn}^{-1} \text{skw } \Phi \rangle_{\mathbf{H}^{-m}(\Omega)} = \frac{1}{2} \langle \text{spn rot } V, \text{skw } \Phi \rangle_{\mathbf{H}^{-m-1}(\Omega)},$$

which proves (ix). We compute by (viii)

$$\begin{aligned} \langle \mathbf{M}, \text{Rot skw } \Phi \rangle_{\mathbf{H}^{-m}(\Omega)} &= \langle \text{tr } \mathbf{M}, \text{div}(\text{spn}^{-1} \text{skw } \Phi) \rangle_{\mathbf{H}^{-m}(\Omega)} - \langle \mathbf{M}^\top, \text{Grad}(\text{spn}^{-1} \text{skw } \Phi) \rangle_{\mathbf{H}^{-m}(\Omega)} \\ &= - \langle \text{grad}(\text{tr } \mathbf{M}), \text{spn}^{-1} \text{skw } \Phi \rangle_{\mathbf{H}^{-m-1}(\Omega)} + \langle \text{Div } \mathbf{M}^\top, \text{spn}^{-1} \text{skw } \Phi \rangle_{\mathbf{H}^{-m-1}(\Omega)} \\ &= - \frac{1}{2} \langle \text{spn}(\text{grad tr } \mathbf{M}), \text{skw } \Phi \rangle_{\mathbf{H}^{-m-1}(\Omega)} + \frac{1}{2} \langle \text{spn Div } \mathbf{M}^\top, \text{skw } \Phi \rangle_{\mathbf{H}^{-m-1}(\Omega)}, \end{aligned}$$

showing the first formula in (x) and the second one follows by  $\text{Div Rot} = 0$  and (vii). To prove (xi) we observe

$$\langle V, \text{Div}(\text{dev Grad } \Phi)^\top \rangle_{\mathbf{H}^{-m}(\Omega)} = \langle V, \text{Div dev}(\text{Grad } \Phi)^\top \rangle_{\mathbf{H}^{-m}(\Omega)} = \frac{2}{3} \langle V, \text{grad div } \Phi \rangle_{\mathbf{H}^{-m}(\Omega)},$$

completing the proof.  $\square$

*Proof of Lemma 3.19.* For  $\mathbf{M} \in \mathring{\mathbf{R}}(\Omega, \mathbb{S})$  there exists a sequence  $(\Phi_n) \subset \mathring{\mathbf{C}}^\infty(\Omega) \cap \mathbf{L}^2(\Omega, \mathbb{S})$  with  $\Phi_n \rightarrow \mathbf{M}$  in  $\mathbf{R}(\Omega)$ . But then  $(\varphi\Phi_n) \subset \mathring{\mathbf{C}}^\infty(\Omega) \cap \mathbf{L}^2(\Omega, \mathbb{S})$  with  $\varphi\Phi_n \rightarrow \varphi\mathbf{M}$  in  $\mathbf{R}(\Omega)$ , proving  $\varphi\mathbf{M} \in \mathring{\mathbf{R}}(\Omega, \mathbb{S})$ , as we have  $\text{Rot}(\varphi\Phi_n) = \varphi \text{Rot} \Phi_n + \text{grad} \varphi \times \Phi_n$ . This formula also shows for  $\Psi \in \mathring{\mathbf{C}}^\infty(\Omega)$  (note  $\varphi\Psi \in \mathring{\mathbf{C}}^\infty(\Omega)$ )

$$(A.3) \quad \begin{aligned} \langle \varphi\mathbf{M}, \text{Rot} \Psi \rangle_{\mathbf{L}^2(\Omega)} &= \langle \mathbf{M}, \varphi \text{Rot} \Psi \rangle_{\mathbf{L}^2(\Omega)} = \langle \mathbf{M}, \text{Rot}(\varphi\Psi) \rangle_{\mathbf{L}^2(\Omega)} - \langle \mathbf{M}, \text{grad} \varphi \times \Psi \rangle_{\mathbf{L}^2(\Omega)} \\ &= \langle \text{Rot} \mathbf{M}, \varphi\Psi \rangle_{\mathbf{L}^2(\Omega)} + \langle \text{grad} \varphi \times \mathbf{M}, \Psi \rangle_{\mathbf{L}^2(\Omega)}, \end{aligned}$$

and thus  $\text{Rot}(\varphi\mathbf{M}) = \varphi \text{Rot} \mathbf{M} + \text{grad} \varphi \times \mathbf{M}$ . Analogously we prove the other cases of (i). Similarly we show (iii) using the formula  $\text{Div}(\varphi\Phi_n) = \varphi \text{Div} \Phi_n + \text{grad} \varphi \cdot \Phi_n$ . To show (ii), let  $\mathbf{M} \in \mathbf{R}(\Omega, \mathbb{S})$ . Then  $\varphi\mathbf{M} \in \mathbf{L}^2(\Omega, \mathbb{S})$  and (A.3) shows  $\varphi\mathbf{M} \in \mathbf{R}(\Omega, \mathbb{S})$  with the desired formula. Analogously the other cases of (ii) follow. Similarly we prove (iv). Let  $\mathbf{E} \in \mathbf{R}_{\text{sym}}(\Omega, \mathbb{T})$  and  $\Phi \in \mathring{\mathbf{C}}^\infty(\Omega)$ . Then  $\varphi\mathbf{E} \in \mathbf{L}^2(\Omega, \mathbb{T})$  and with  $\varphi\Phi \in \mathring{\mathbf{C}}^\infty(\Omega)$  we get

$$\begin{aligned} \langle \varphi\mathbf{E}, \text{Rot sym} \Phi \rangle_{\mathbf{L}^2(\Omega)} &= \langle \mathbf{E}, \varphi \text{Rot sym} \Phi \rangle_{\mathbf{L}^2(\Omega)} = \langle \mathbf{E}, \text{Rot sym}(\varphi\Phi) \rangle_{\mathbf{L}^2(\Omega)} - \langle \mathbf{E}, \text{grad} \varphi \times \text{sym} \Phi \rangle_{\mathbf{L}^2(\Omega)} \\ &= \langle \text{sym Rot} \mathbf{E}, \varphi\Phi \rangle_{\mathbf{L}^2(\Omega)} + \langle \text{grad} \varphi \times \mathbf{E}, \text{sym} \Phi \rangle_{\mathbf{L}^2(\Omega)}, \end{aligned}$$

which shows  $\varphi\mathbf{E} \in \mathbf{R}_{\text{sym}}(\Omega, \mathbb{T})$  and  $\text{sym Rot}(\varphi\mathbf{E}) = \varphi \text{sym Rot} \mathbf{E} + \text{sym}(\text{grad} \varphi \times \mathbf{E})$  and hence (v). To prove (vi), let  $\mathbf{M} \in \mathbf{DD}(\Omega, \mathbb{S})$  and  $\phi \in \mathring{\mathbf{C}}^\infty(\Omega)$ . Then  $\varphi\mathbf{M} \in \mathbf{L}^2(\Omega, \mathbb{S})$  and we compute by

$$\begin{aligned} \text{Grad grad}(\varphi\phi) &= \varphi \text{Grad grad} \phi + \phi \text{Grad grad} \varphi + 2 \text{sym}((\text{grad} \varphi)(\text{grad} \phi)^\top), \\ (\text{grad} \varphi)(\text{grad} \phi)^\top &= \text{Grad}(\phi \text{grad} \varphi) - \phi \text{Grad grad} \varphi \end{aligned}$$

the identity

$$\text{Grad grad}(\varphi\phi) = \varphi \text{Grad grad} \phi - \phi \text{Grad grad} \varphi + 2 \text{sym}(\text{Grad}(\phi \text{grad} \varphi)).$$

Finally with  $\varphi\phi \in \mathring{\mathbf{C}}^\infty(\Omega)$  we get

$$\begin{aligned} \langle \varphi\mathbf{M}, \text{Grad grad} \phi \rangle_{\mathbf{L}^2(\Omega)} &= \langle \mathbf{M}, \varphi \text{Grad grad} \phi \rangle_{\mathbf{L}^2(\Omega)} \\ &= \langle \mathbf{M}, \text{Grad grad}(\varphi\phi) \rangle_{\mathbf{L}^2(\Omega)} + \langle \mathbf{M}, \phi \text{Grad grad} \varphi \rangle_{\mathbf{L}^2(\Omega)} - 2 \langle \mathbf{M}, \text{sym}(\text{Grad}(\phi \text{grad} \varphi)) \rangle_{\mathbf{L}^2(\Omega)} \\ &= \langle \text{div Div} \mathbf{M}, \varphi\phi \rangle_{\mathbf{L}^2(\Omega)} + \langle \mathbf{M} : \text{Grad grad} \varphi, \phi \rangle_{\mathbf{L}^2(\Omega)} - 2 \langle \mathbf{M}, \text{Grad}(\phi \text{grad} \varphi) \rangle_{\mathbf{L}^2(\Omega)} \\ &= \langle \varphi \text{div Div} \mathbf{M}, \phi \rangle_{\mathbf{L}^2(\Omega)} + \langle \text{tr}(\mathbf{M} \cdot \text{Grad grad} \varphi), \phi \rangle_{\mathbf{L}^2(\Omega)} + 2 \underbrace{\langle \text{Div} \mathbf{M}, \phi \text{grad} \varphi \rangle_{\mathbf{H}^{-1}(\Omega)}}_{= \langle \text{Div} \mathbf{M} \cdot \text{grad} \varphi, \phi \rangle_{\mathbf{H}^{-1}(\Omega)}}, \end{aligned}$$

which shows (vi), i.e.,  $\varphi\mathbf{M} \in \mathbf{DD}^{0,-1}(\Omega, \mathbb{S})$  and

$$\text{div Div}(\varphi\mathbf{M}) = \varphi \text{div Div} \mathbf{M} + 2 \text{grad} \varphi \cdot \text{Div} \mathbf{M} + \text{tr}(\text{Grad grad} \varphi \cdot \mathbf{M}).$$

The proof is finished.  $\square$

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