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On Closed and Exact Grad grad- and div Div-Complexes, Corresponding Compact Embeddings for Symmetric Rotations, and a Related Decomposition Result for Biharmonic Problems in 3D

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On Closed and Exact Grad grad- and div Div-Complexes, Corresponding Compact Embeddings for Symmetric Rotations, and a Related Decomposition Result for Biharmonic Problems in 3D

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Abstract. It is shown that the first biharmonic boundary value problem on a topologically simple domain in 3D is equivalent to three (consecutively to solve) second-order problems. This decomposition result is based on a Helmholtz-like decomposition of an involved non-standard Sobolev space of tensor fields and a proper characterization of the operator div Div acting on this space. Similar results for biharmonic problems in 2D and their impact on the construction and analysis of finite element methods have been recently published in [14]. The discussion of the kernel of div Div leads to (de Rham-like) closed and exact Hilbert complexes, the div Div-complex and its adjoint the Grad grad-complex, involving spaces of trace-free and symmetric tensor fields. For these tensor fields we show Helmholtz type decompositions and, most importantly, new compact embedding results. Almost all our results hold and are formulated for general bounded strong Lipschitz domains of arbitrary topology. There is no reasonable doubt that our results extend to strong Lipschitz domains in $\mathbb{R}^N$.

Contents

1. Introduction

In [14] it was shown that the fourth-order biharmonic boundary value problem

$$\Delta^2 u = f \quad \text{in } \Omega, \qquad u = \partial_n u = 0 \quad \text{on } \Gamma, \tag{1.1}$$

where Ω is a bounded and simply connected domain in $\mathbb{R}^2$ with a (strong) Lipschitz boundary[i] Γ, can be decomposed into three second-order problems. The first problem is a Poisson problem for an auxiliary

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[i]Locally Γ is a graph of a Lipschitz function.

scalar field p

$$\Delta p = f \quad \text{in } \Omega, \qquad p = 0 \quad \text{on } \Gamma,$$

the second problem is a linear elasticity problem for an auxiliary vector field E

$$-\operatorname{Div} \varepsilon(E) = -\operatorname{Div}(\operatorname{sym}\operatorname{Grad} E) = \operatorname{grad} p \quad \text{in } \Omega, \qquad (\operatorname{sym}\operatorname{Grad} E)\, n = -p\, n = 0 \quad \text{on } \Gamma,$$

i.e.,

$$\operatorname{Div}(\operatorname{sym}\operatorname{Grad} E + p\,\mathbf{I}) = 0 \quad \text{in } \Omega, \qquad (\operatorname{sym}\operatorname{Grad} E + p\,\mathbf{I})\, n = 0 \quad \text{on } \Gamma,$$

and, finally, the third problem is a Poisson problem for the original scalar field u

$$\Delta u = 2\,p + \operatorname{div} E \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \Gamma.$$

Here f is a given right-hand side, Δ, n, and ∂_n denote the Laplace operator, the outward normal vector to the boundary, and the derivative in this direction, respectively. The differential operators grad, div, and (for later use) rot denote the gradient of a scalar field and the divergence and rotation of a vector field, the corresponding capitalized differential operators Grad, Div, and Rot denote the row-wise application of grad to a vector field, div and rot to a tensor field. The prefix sym is used for the symmetric part of a matrix, for the skew-symmetric part we use the prefix skw. This decomposition is of triangular structure, i.e., the first problem is a well-posed second-order problem in p, the second problem is a well-posed second-order problem in E for given p, and the third problem is a well-posed second-order problem in u for given p and E. This allows to solve them consecutely analytically or numerically by means of techniques for second-order problems.

This is - in the first place - a new analytic result for fourth-order problems. But it also has interesting implications for discretization methods applied to (1.1). It allows to re-interpret known finite element methods as well as to construct new discretization methods for (1.1) by exploiting the decomposable structure of the problem. In particular, it was shown in [14] that the Hellan-Herrmann-Johnson mixed method (see [8, 9, 13]) for (1.1) allows a similar decomposition as the continuous problem, which leads to a new and simpler assembling procedure for the discretization matrix and to more efficient solution techniques for the discretized problem. Moreover, a novel conforming variant of the Hellan-Herrmann-Johnson mixed method was found based on the decomposition.

The aim of this paper is to derive a similar decomposition result for biharmonic problems on bounded and topologically simple three-dimensional domains Ω with a (strong) Lipschitz boundary Γ. For this we proceed as in [14] and reformulate (1.1) using $\Delta^2 = \operatorname{div}\operatorname{Div}\operatorname{Grad}\operatorname{grad}$ as a mixed problem by introducing the (negative) Hessian of the original scalar field u as an auxiliary tensor field

$$\mathbf{M} = -\operatorname{Grad}\operatorname{grad} u. \tag{1.2}$$

Then the biharmonic diffential equation reads

$$-\operatorname{div}\operatorname{Div}\mathbf{M} = f \quad \text{in } \Omega. \tag{1.3}$$

For an appropriate non-standard Sobolev space for $\mathbf{M}$ it can be shown that the mixed problem in $\mathbf{M}$ and u is well-posed. Then the decomposition of the biharmonic problem follows from a Helmholtz-like decomposition of this non-standard Sobolev space. This part of the analysis carries over completely from the two-dimensional case to the three-dimensional case and is shortly recalled in Secion 6. To efficiently utilize this Helmholtz-like decomposition for the decomposition of the biharmonic problem an appropriate characterization of the kernel of the operator div Div is required, which is well understood for the two-dimensional case, see, e.g., [3, 11, 14]. Its extension to the three-dimensional case is the central topic of this paper. We expect - as in the two-dimensional case - similar interesting implications for the study of appropriate discretization methods for four-order problems in the three-dimensional case.

The paper is organized as follows. After some preliminaries in Section 2 and introducing our general functional analytical setting, we will discuss the relevant unbounded linear operators, show closed and exact Hilbert complex properties, and present a suitable representation of the kernel of div Div for the three-dimensional case in Section 3. We also prove Helmholtz type decompositions and two new and crucial compact embeddings. Based on the representation of the kernel of div Div a decomposition of the three-dimensional biharmonic problem into three (consecutely to solve) second-order problems will be derived in Section 6. The proofs of some useful identities are presented in an appendix.

2. Preliminaries

We start by recalling some basic concepts and abstract results from functional analysis concerning Helmholtz decompositions, closed ranges, Friedrichs/Poincaré type estimates, and bounded or even compact inverse operators. Since we will need both the Banach space setting for bounded linear operators as well as the Hilbert space setting for (possibly unbounded) closed and densely defined linear operators, we will shortly recall these two variants.

2.1. **Functional Analysis Toolbox.** Let X and Y be real Banach spaces. With $BL(\mathsf{X},\mathsf{Y})$ we introduce the space of bounded linear operators mapping X to Y. The dual spaces of X and Y are denoted by $\mathsf{X}' := BL(\mathsf{X},\mathbb{R})$ and $\mathsf{Y}' := BL(\mathsf{Y},\mathbb{R})$. For a given $\mathrm{A} \in BL(\mathsf{X},\mathsf{Y})$ we write $\mathrm{A}' \in BL(\mathsf{Y}',\mathsf{X}')$ for its Banach space dual or adjoint operator defined by $\mathrm{A}'\,y'(x) := y'(\mathrm{A}\,x)$ for all $y' \in \mathsf{Y}'$ and all $x \in \mathsf{X}$. Norms and duality in X resp. X' are denoted by $|\cdot|_{\mathsf{X}}$, $|\cdot|_{\mathsf{X}'}$, and $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathsf{X}'}$.

Suppose H_1 and H_2 are Hilbert spaces. For a (possibly unbounded) densely defined linear operator $\mathrm{A} : D(\mathrm{A}) \subset \mathsf{H}_1 \to \mathsf{H}_2$ we recall that its Hilbert space dual or adjoint $\mathrm{A}^* : D(\mathrm{A}^*) \subset \mathsf{H}_2 \to \mathsf{H}_1$ can be defined via its Banach space adjoint A' and the Riesz isomorphisms of H_1 and H_2 or directly as follows: $y \in D(\mathrm{A}^*)$ if and only if $y \in \mathsf{H}_2$ and

$$\exists\, f \in \mathsf{H}_1 \quad \forall\, x \in D(\mathrm{A}) \quad \langle \mathrm{A}\,x, y\rangle_{\mathsf{H}_2} = \langle x, f\rangle_{\mathsf{H}_1}.$$

In this case we define $\mathrm{A}^*\,y := f$. We note that A^* has maximal domain of definition and that A^* is characterized by

$$\forall\, x \in D(\mathrm{A}) \quad \forall\, y \in D(\mathrm{A}^*) \qquad \langle \mathrm{A}\,x, y\rangle_{\mathsf{H}_2} = \langle x, \mathrm{A}^*\,y\rangle_{\mathsf{H}_1}.$$

Here $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathsf{H}}$ denotes the scalar product in a Hilbert space H and D is used for the domain of definition of a linear operator. Additionally, we introduce the notation N for the kernel or null space and R for the range of a linear operator.

Let $\mathrm{A} : D(\mathrm{A}) \subset \mathsf{H}_1 \to \mathsf{H}_2$ be a (possibly unbounded) closed and densely defined linear operator on two Hilbert spaces H_1 and H_2 with adjoint $\mathrm{A}^* : D(\mathrm{A}^*) \subset \mathsf{H}_2 \to \mathsf{H}_1$. Note $(\mathrm{A}^*)^* = \overline{\mathrm{A}} = \mathrm{A}$, i.e., $(\mathrm{A}, \mathrm{A}^*)$ is a dual pair. By the projection theorem the Helmholtz type decompositions

$$\mathsf{H}_1 = N(\mathrm{A}) \oplus_{\mathsf{H}_1} \overline{R(\mathrm{A}^*)}, \quad \mathsf{H}_2 = N(\mathrm{A}^*) \oplus_{\mathsf{H}_2} \overline{R(\mathrm{A})} \tag{2.1}$$

hold and we can define the reduced operators

$$\mathcal{A} := \mathrm{A}\,|_{\overline{R(\mathrm{A}^*)}} : D(\mathcal{A}) \subset \overline{R(\mathrm{A}^*)} \to \overline{R(\mathrm{A})}, \qquad D(\mathcal{A}) := D(\mathrm{A}) \cap \overline{R(\mathrm{A}^*)} = D(\mathrm{A}) \cap N(\mathrm{A})^{\perp_{\mathsf{H}_1}},$$

$$\mathcal{A}^* := \mathrm{A}^*\,|_{\overline{R(\mathrm{A})}} : D(\mathcal{A}^*) \subset \overline{R(\mathrm{A})} \to \overline{R(\mathrm{A}^*)}, \qquad D(\mathcal{A}^*) := D(\mathrm{A}^*) \cap \overline{R(\mathrm{A})} = D(\mathrm{A}^*) \cap N(\mathrm{A}^*)^{\perp_{\mathsf{H}_2}},$$

which are also closed and densely defined linear operators. We note that $\mathcal{A}$ and $\mathcal{A}^*$ are indeed adjoint to each other, i.e., $(\mathcal{A}, \mathcal{A}^*)$ is a dual pair as well. Now the inverse operators

$$\mathcal{A}^{-1} : R(\mathrm{A}) \to D(\mathcal{A}), \qquad (\mathcal{A}^*)^{-1} : R(\mathrm{A}^*) \to D(\mathcal{A}^*)$$

exist and they are bijective, since $\mathcal{A}$ and $\mathcal{A}^*$ are injective by definition. Furthermore, by (2.1) we have the refined Helmholtz type decompositions

$$D(\mathrm{A}) = N(\mathrm{A}) \oplus_{\mathsf{H}_1} D(\mathcal{A}), \qquad D(\mathrm{A}^*) = N(\mathrm{A}^*) \oplus_{\mathsf{H}_2} D(\mathcal{A}^*) \tag{2.2}$$

and thus we obtain for the ranges

$$R(\mathrm{A}) = R(\mathcal{A}), \qquad R(\mathrm{A}^*) = R(\mathcal{A}^*). \tag{2.3}$$

By the closed range theorem or the bounded inverse theorem we get immediately the following.

Lemma 2.1. *The following assertions are equivalent:*

- **(i)** $\exists\, c_{\mathrm{A}} \in (0,\infty) \quad \forall\, x \in D(\mathcal{A}) \qquad |x|_{\mathsf{H}_1} \leq c_{\mathrm{A}} |\,\mathrm{A}\,x|_{\mathsf{H}_2}$
- **(i*)** $\exists\, c_{\mathrm{A}^*} \in (0,\infty) \quad \forall\, y \in D(\mathcal{A}^*) \qquad |y|_{\mathsf{H}_2} \leq c_{\mathrm{A}^*} |\,\mathrm{A}^*\,y|_{\mathsf{H}_1}$
- **(ii)** $R(\mathrm{A}) = R(\mathcal{A})$ *is closed in* H_2.
- **(ii*)** $R(\mathrm{A}^*) = R(\mathcal{A}^*)$ *is closed in* H_1.
- **(iii)** $\mathcal{A}^{-1} : R(\mathrm{A}) \to D(\mathcal{A})$ *is continuous and bijective with norm bounded by* $(1 + c_{\mathrm{A}}^2)^{1/2}$.
- **(iii*)** $(\mathcal{A}^*)^{-1} : R(\mathrm{A}^*) \to D(\mathcal{A}^*)$ *is continuous and bijective with norm bounded by* $(1 + c_{\mathrm{A}^*}^2)^{1/2}$.

In case that one of the assertions of Lemma 2.1 is true, e.g., $R(\mathrm{A})$ is closed, we have

$$\begin{aligned} \mathsf{H}_1 &= N(\mathrm{A}) \oplus_{\mathsf{H}_1} R(\mathrm{A}^*), & \mathsf{H}_2 &= N(\mathrm{A}^*) \oplus_{\mathsf{H}_2} R(\mathrm{A}), \\ D(\mathrm{A}) &= N(\mathrm{A}) \oplus_{\mathsf{H}_1} D(\mathcal{A}), & D(\mathrm{A}^*) &= N(\mathrm{A}^*) \oplus_{\mathsf{H}_2} D(\mathcal{A}^*), \\ D(\mathcal{A}) &= D(\mathrm{A}) \cap R(\mathrm{A}^*), & D(\mathcal{A}^*) &= D(\mathrm{A}^*) \cap R(\mathrm{A}). \end{aligned}$$

For the "best" constants c_{A}, c_{A^*} we have the following lemma.

Lemma 2.2. *The Rayleigh quotients*

$$\frac{1}{c_{\mathrm{A}}} := \inf_{0 \neq x \in D(\mathcal{A})} \frac{|\,\mathrm{A}\,x|_{\mathsf{H}_2}}{|x|_{\mathsf{H}_1}} = \inf_{0 \neq y \in D(\mathcal{A}^*)} \frac{|\,\mathrm{A}^*\,y|_{\mathsf{H}_1}}{|y|_{\mathsf{H}_2}} =: \frac{1}{c_{\mathrm{A}^*}}$$

coincide, i.e., $c_{\mathrm{A}} = c_{\mathrm{A}^*}$, if either c_{A} or c_{A^*} exists in $(0, \infty)$. Otherwise they also coincide, i.e., it holds $c_{\mathrm{A}} = c_{\mathrm{A}^*} = \infty$.

From now on and throughout this paper, we always pick the best possible constants in the various Friedrichs/Poincaré type estimates.

Proof. Let, e.g., c_{A^*} exist in $(0, \infty)$. By Lemma 2.1 also c_{A} exists in $(0, \infty)$, $R(\mathrm{A}^*)$ is closed in H_1, and for any $0 \neq x \in D(\mathcal{A}) = D(\mathrm{A}) \cap R(\mathrm{A}^*)$ there exists $y \in D(\mathcal{A}^*)$ with $x = \mathrm{A}^*\,y$. More precisely, $y := (\mathcal{A}^*)^{-1}x \in D(\mathcal{A}^*)$ is uniquely determined and we have $|y|_{\mathsf{H}_2} \leq c_{\mathrm{A}^*}|\,\mathrm{A}^*\,y|_{\mathsf{H}_1}$. But then

$$|x|_{\mathsf{H}_1}^2 = \langle x, \mathrm{A}^*\,y\rangle_{\mathsf{H}_1} = \langle \mathrm{A}\,x, y\rangle_{\mathsf{H}_2} \leq |\,\mathrm{A}\,x|_{\mathsf{H}_2}|y|_{\mathsf{H}_2} \leq c_{\mathrm{A}^*}|\,\mathrm{A}\,x|_{\mathsf{H}_2}|\,\mathrm{A}^*\,y|_{\mathsf{H}_1},$$

yielding $|x|_{\mathsf{H}_1} \leq c_{\mathrm{A}^*}|\,\mathrm{A}\,x|_{\mathsf{H}_2}$. Therefore, $0 < c_{\mathrm{A}} \leq c_{\mathrm{A}^*}$ and by symmetry we obtain $c_{\mathrm{A}} = c_{\mathrm{A}^*}$. □

A standard indirect argument shows the following.

Lemma 2.3. *Let $D(\mathcal{A}) = D(\mathrm{A}) \cap \overline{R(\mathrm{A}^*)} \hookrightarrow \mathsf{H}_1$ be compact. Then the assertions of Lemma 2.1 hold. Moreover, the inverse operators*

$$\mathcal{A}^{-1} : R(\mathrm{A}) \to R(\mathrm{A}^*), \quad (\mathcal{A}^*)^{-1} : R(\mathrm{A}^*) \to R(\mathrm{A})$$

are compact with norms $\left|\,\mathcal{A}^{-1}\,\right|_{R(\mathrm{A}),R(\mathrm{A}^*)} = \left|(\mathcal{A}^*)^{-1}\right|_{R(\mathrm{A}^*),R(\mathrm{A})} = c_{\mathrm{A}}$.

Proof. If, e.g., Lemma 2.1 (i) was wrong, there exists a sequence $(x_n) \subset D(\mathcal{A})$ with $|x_n|_{\mathsf{H}_1} = 1$ and $\mathrm{A}\,x_n \to 0$. As (x_n) is bounded in $D(\mathcal{A})$ we can extract a subsequence, again denoted by (x_n), with $x_n \to x \in \mathsf{H}_1$ in H_1. Since A is closed, we have $x \in D(\mathrm{A})$ and $\mathrm{A}\,x = 0$. Hence $x \in N(\mathrm{A})$. On the other hand, $(x_n) \subset D(\mathcal{A}) \subset \overline{R(\mathrm{A}^*)} = N(\mathrm{A})^{\perp}$ implies $x \in N(\mathrm{A})^{\perp}$. Thus $x = 0$, in contradiction to $1 = |x_n|_{\mathsf{H}_1} \to |x|_{\mathsf{H}_1} = 0$. □

Lemma 2.4. *$D(\mathcal{A}) \hookrightarrow \mathsf{H}_1$ is compact, if and only if $D(\mathcal{A}^*) \hookrightarrow \mathsf{H}_2$ is compact.*

Proof. By symmetry it is enough to show one direction. Let $D(\mathcal{A}) \hookrightarrow \mathsf{H}_1$ be compact and let $(y_n) \subset D(\mathcal{A}^*)$ be a bounded sequence. By Lemma 2.1 and Lemma 2.3 we get $y_n = \mathrm{A}\,x_n$ with $(x_n) \subset D(\mathcal{A})$, which is bounded in $D(\mathcal{A})$ by Lemma 2.1 (i). Hence we may extract a subsequence, again denoted by (x_n) converging in H_1. Thus with $x_{n,m} := x_n - x_m$ and $y_{n,m} := y_n - y_m$ we see

$$|y_{n,m}|_{\mathsf{H}_2}^2 = \big\langle y_{n,m}, \mathrm{A}(x_{n,m})\big\rangle_{\mathsf{H}_2} = \big\langle\, \mathrm{A}^*(y_{n,m}), x_{n,m}\big\rangle_{\mathsf{H}_1} \leq c\,|x_{n,m}|_{\mathsf{H}_1},$$

and hence (y_n) is a Cauchy sequence in H_2. □

Now, let $\mathrm{A}_0 : D(\mathrm{A}_0) \subset \mathsf{H}_0 \to \mathsf{H}_1$ and $\mathrm{A}_1 : D(\mathrm{A}_1) \subset \mathsf{H}_1 \to \mathsf{H}_2$ be (possibly unbounded) closed and densely defined linear operators on three Hilbert spaces H_0, H_1 and H_2 with adjoints $\mathrm{A}_0^* : D(\mathrm{A}_0^*) \subset \mathsf{H}_1 \to \mathsf{H}_0$ and $\mathrm{A}_1^* : D(\mathrm{A}_1^*) \subset \mathsf{H}_2 \to \mathsf{H}_1$ as well as reduced operators $\mathcal{A}_0$, $\mathcal{A}_0^*$, and $\mathcal{A}_1$, $\mathcal{A}_1^*$. Furthermore, we assume the sequence property of A_0 and A_1, that is, $\mathrm{A}_1\,\mathrm{A}_0 = 0$, i.e.,

$$R(\mathrm{A}_0) \subset N(\mathrm{A}_1). \tag{2.4}$$

Then also $\mathrm{A}_0^*\,\mathrm{A}_1^* = 0$, i.e., $R(\mathrm{A}_1^*) \subset N(\mathrm{A}_0^*)$. The Helmholtz type decompositions of (2.1) for $\mathrm{A} = \mathrm{A}_1$ and $\mathrm{A} = \mathrm{A}_0$ read

$$\mathsf{H}_1 = N(\mathrm{A}_1) \oplus_{\mathsf{H}_1} \overline{R(\mathrm{A}_1^*)}, \qquad \mathsf{H}_1 = N(\mathrm{A}_0^*) \oplus_{\mathsf{H}_1} \overline{R(\mathrm{A}_0)} \tag{2.5}$$

and by (2.4) we see

(2.6) $$N(\mathrm{A}_0^*) = N_{0,1} \oplus_{\mathsf{H}_1} \overline{R(\mathrm{A}_1^*)}, \qquad N(\mathrm{A}_1) = N_{0,1} \oplus_{\mathsf{H}_1} \overline{R(\mathrm{A}_0)}, \qquad N_{0,1} := N(\mathrm{A}_1) \cap N(\mathrm{A}_0^*)$$

yielding the refined Helmholtz type decomposition

(2.7) $$\mathsf{H}_1 = \overline{R(\mathrm{A}_0)} \oplus_{\mathsf{H}_1} N_{0,1} \oplus_{\mathsf{H}_1} \overline{R(\mathrm{A}_1^*)}.$$

The previous results of this section imply immediately the following.

Lemma 2.5. *Let A_0, A_1 be as introduced before with $\mathrm{A}_1 \mathrm{A}_0 = 0$, i.e., (2.4). Moreover, let $R(\mathrm{A}_0)$ and $R(\mathrm{A}_1)$ be closed. Then, the assertions of Lemma 2.1 and Lemma 2.2 hold for A_0 and A_1. Moreover, the refined Helmholtz type decompositions*

$$\begin{aligned} \mathsf{H}_1 &= R(\mathrm{A}_0) \oplus_{\mathsf{H}_1} N_{0,1} \oplus_{\mathsf{H}_1} R(\mathrm{A}_1^*), & N_{0,1} &= N(\mathrm{A}_1) \cap N(\mathrm{A}_0^*), \\ N(\mathrm{A}_1) &= R(\mathrm{A}_0) \oplus_{\mathsf{H}_1} N_{0,1}, & N(\mathrm{A}_0^*) &= N_{0,1} \oplus_{\mathsf{H}_1} R(\mathrm{A}_1^*), \\ R(\mathcal{A}_0) = R(\mathrm{A}_0) &= N(\mathrm{A}_1) \ominus_{\mathsf{H}_1} N_{0,1}, & R(\mathcal{A}_1^*) = R(\mathrm{A}_1^*) &= N(\mathrm{A}_0^*) \ominus_{\mathsf{H}_1} N_{0,1}, \\ D(\mathrm{A}_1) &= R(\mathrm{A}_0) \oplus_{\mathsf{H}_1} N_{0,1} \oplus_{\mathsf{H}_1} D(\mathcal{A}_1), & D(\mathrm{A}_0^*) &= D(\mathcal{A}_0^*) \oplus_{\mathsf{H}_1} N_{0,1} \oplus_{\mathsf{H}_1} R(\mathrm{A}_1^*), \\ D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) &= D(\mathcal{A}_0^*) \oplus_{\mathsf{H}_1} N_{0,1} \oplus_{\mathsf{H}_1} D(\mathcal{A}_1) \end{aligned}$$

hold. Especially, $R(\mathrm{A}_0)$, $R(\mathrm{A}_0^*)$, $R(\mathrm{A}_1)$, and $R(\mathrm{A}_1^*)$ are closed, the respective inverse operators are continuous and there exist positive constants c_{A_0}, c_{A_1}, such that the Friedrichs/Poincaré type estimates

$$\begin{aligned} &\forall\, x \in D(\mathcal{A}_0) && |x|_{\mathsf{H}_0} \leq c_{\mathrm{A}_0} |\,\mathrm{A}_0\, x|_{\mathsf{H}_1}, && \forall\, y \in D(\mathcal{A}_1) && |y|_{\mathsf{H}_1} \leq c_{\mathrm{A}_1} |\,\mathrm{A}_1\, y|_{\mathsf{H}_2}, \\ &\forall\, y \in D(\mathcal{A}_0^*) && |y|_{\mathsf{H}_1} \leq c_{\mathrm{A}_0} |\,\mathrm{A}_0^*\, y|_{\mathsf{H}_0}, && \forall\, z \in D(\mathcal{A}_1^*) && |z|_{\mathsf{H}_2} \leq c_{\mathrm{A}_1} |\,\mathrm{A}_1^*\, z|_{\mathsf{H}_1} \end{aligned}$$

hold.

Remark 2.6. *Note that $R(\mathrm{A}_0)$ resp. $R(\mathrm{A}_1)$ is closed, if e.g. $D(\mathcal{A}_0) \hookrightarrow \mathsf{H}_0$ resp. $D(\mathcal{A}_1) \hookrightarrow \mathsf{H}_1$ is compact.*

Remark 2.7. *Observe $D(\mathcal{A}_1) = D(\mathrm{A}_1) \cap \overline{R(\mathrm{A}_1^*)} \subset D(\mathrm{A}_1) \cap N(\mathrm{A}_0^*) \subset D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*)$. Utilizing the Helmholtz type decompositions of Lemma 2.5 we can immediately show: The embeddings $D(\mathcal{A}_0) \hookrightarrow \mathsf{H}_0$, $D(\mathcal{A}_1) \hookrightarrow \mathsf{H}_1$, and $N_{0,1} \hookrightarrow \mathsf{H}_1$ are compact, if and only if the embedding $D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) \hookrightarrow \mathsf{H}_1$ is compact. In this case $N_{0,1}$ has finite dimension.*

Remark 2.8. *The assumptions in Lemma 2.5 on A_0 and A_1 are equivalent to the assumption that*

$$D(\mathrm{A}_0) \subset \mathsf{H}_0 \xrightarrow{\ \mathrm{A}_0\ } D(\mathrm{A}_1) \subset \mathsf{H}_1 \xrightarrow{\ \mathrm{A}_1\ } \mathsf{H}_2$$

is a closed Hilbert complex, meaning that the ranges are closed. As a result of the previous lemmas, the adjoint complex

$$\mathsf{H}_0 \xleftarrow{\ \mathrm{A}_0^*\ } D(\mathrm{A}_0^*) \subset \mathsf{H}_1 \xleftarrow{\ \mathrm{A}_1^*\ } D(\mathrm{A}_1^*) \subset \mathsf{H}_2.$$

is a closed Hilbert complex as well.

A special situation is the following.

Lemma 2.9. *Let A_0, A_1 be as introduced before with $R(\mathrm{A}_0) = N(\mathrm{A}_1)$ and $R(\mathrm{A}_1)$ closed in H_2. Then $R(\mathrm{A}_0^*)$ is closed in H_0 and the simplified Helmholtz type decompositions*

$$\begin{aligned} \mathsf{H}_1 &= R(\mathrm{A}_0) \oplus_{\mathsf{H}_1} R(\mathrm{A}_1^*), & N_{0,1} &= \{0\}, \\ N(\mathrm{A}_1) &= R(\mathrm{A}_0) = R(\mathcal{A}_0), & N(\mathrm{A}_0^*) &= R(\mathrm{A}_1^*) = R(\mathcal{A}_1^*), \\ D(\mathrm{A}_1) &= R(\mathrm{A}_0) \oplus_{\mathsf{H}_1} D(\mathcal{A}_1), & D(\mathrm{A}_0^*) &= D(\mathcal{A}_0^*) \oplus_{\mathsf{H}_1} R(\mathrm{A}_1^*), \\ D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) &= D(\mathcal{A}_0^*) \oplus_{\mathsf{H}_1} D(\mathcal{A}_1) \end{aligned}$$

are valid. Moreover, the respective inverse operators are continuous and the corresponding Friedrichs/Poincaré type estimates hold.

Remark 2.10. *Note that* $R(\mathrm{A}_1^*) = N(\mathrm{A}_0^*)$ *and* $R(\mathrm{A}_0^*)$ *closed are equivalent assumptions for Lemma 2.9 to hold.*

Lemma 2.11. *Let* A_0, A_1 *be as introduced before with the sequence property* (2.4), *i.e.,* $R(\mathrm{A}_0) \subset N(\mathrm{A}_1)$. *If the embedding* $D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) \hookrightarrow \mathsf{H}_1$ *is compact and* $N_{0,1} = \{0\}$, *then the assumptions of Lemma 2.9 are satisfied.*

Remark 2.12. *The assumptions in Lemma 2.9 on* A_0 *and* A_1 *are equivalent to the assumption that*

$$D(\mathrm{A}_0) \subset \mathsf{H}_0 \xrightarrow{\mathrm{A}_0} D(\mathrm{A}_1) \subset \mathsf{H}_1 \xrightarrow{\mathrm{A}_1} \mathsf{H}_2$$

is a closed and exact Hilbert complex. By Lemma 2.9 the adjoint complex

$$\mathsf{H}_0 \xleftarrow{\mathrm{A}_0^*} D(\mathrm{A}_0^*) \subset \mathsf{H}_1 \xleftarrow{\mathrm{A}_1^*} D(\mathrm{A}_1^*) \subset \mathsf{H}_2.$$

is a closed and exact Hilbert complex as well.

Parts of Lemma 2.9 hold also in the Banach space setting. As a direct consequence of the closed range theorem and the bounded inverse theorem the following abstract result holds.

Lemma 2.13. *Let* X_0, X_1, X_2 *be Banach spaces and suppose* $\mathrm{A}_0 \in BL(\mathsf{X}_0, \mathsf{X}_1)$, $\mathrm{A}_1 \in BL(\mathsf{X}_1, \mathsf{X}_2)$ *with* $R(\mathrm{A}_0) = N(\mathrm{A}_1)$ *and that* $R(\mathrm{A}_1)$ *is closed in* X_2. *Then* $R(\mathrm{A}_0')$ *is closed in* X_0' *and* $R(\mathrm{A}_1') = N(\mathrm{A}_0')$. *Moreover,* $(\mathrm{A}_1')^{-1} \in BL\big(R(\mathrm{A}_1'), R(\mathrm{A}_1)'\big)$.

Note that in the latter context we consider the operators

$$\mathrm{A}_1 : \mathsf{X}_1 \longrightarrow R(\mathrm{A}_1), \qquad \mathrm{A}_1' : R(\mathrm{A}_1)' \longrightarrow R(\mathrm{A}_1') \qquad (\mathrm{A}_1')^{-1} : R(\mathrm{A}_1') \longrightarrow R(\mathrm{A}_1)',$$

with $N(\mathrm{A}_1') = R(\mathrm{A}_1)^\circ = \{0\}$.

Remark 2.14. *The conditions on* A_0 *and* A_1 *in Lemma 2.13 are identical to the assumption that*

$$\mathsf{X}_0 \xrightarrow{\mathrm{A}_0} \mathsf{X}_1 \xrightarrow{\mathrm{A}_1} \mathsf{X}_2$$

is a closed and exact complex of Banach spaces. The consequences of Lemma 2.13 can be rephrased as follows. The adjoint complex of Banach spaces

$$\mathsf{X}_0' \xleftarrow{\mathrm{A}_0'} \mathsf{X}_1' \xleftarrow{\mathrm{A}_1'} \mathsf{X}_2'$$

is closed and exact as well.

We refer to [1] for a presentation of some results of this section from a numerical analysis perspective.

Remark 2.15. $(\mathrm{A}_1')^{-1} \in BL\big(R(\mathrm{A}_1'), R(\mathrm{A}_1)'\big)$ *is equivalent to*

$$\exists\, c_{\mathrm{A}_1'} > 0 \quad \forall\, x \in R(\mathrm{A}_1') \qquad |(\mathrm{A}_1')^{-1} x|_{R(\mathrm{A}_1)'} \leq c_{\mathrm{A}_1'} |x|_{\mathsf{X}_1'}. \tag{2.8}$$

We have $\mathrm{A}_1'(\mathrm{A}_1')^{-1} = \mathrm{id}$ *on* $R(\mathrm{A}_1')$. *Moreover,* $(\mathrm{A}_1')^{-1}\,\mathrm{A}_1' = \mathrm{id}$ *holds on* $R(\mathrm{A}_1)'$ *since for* $y \in R(\mathrm{A}_1)'$ *and* $x := \mathrm{A}_1'\, y$ *we get* $y - (\mathrm{A}_1')^{-1} x \in N(\mathrm{A}_1') = \{0\}$ *and hence* $y = (\mathrm{A}_1')^{-1} x$ *as* A_1' *is injective. Therefore* (2.8) *is equivalent to*

$$\exists\, c_{\mathrm{A}_1'} > 0 \quad \forall\, y \in R(\mathrm{A}_1)' \qquad |y|_{R(\mathrm{A}_1)'} \leq c_{\mathrm{A}_1'} |\,\mathrm{A}_1'\, y|_{\mathsf{X}_1'} \tag{2.9}$$

with the same (best) constant $c_{\mathrm{A}_1'}$. *Moreover, for* $y \in \mathsf{X}_2'$

$$|\,\mathrm{A}_1'\, y|_{\mathsf{X}_1'} = \sup_{0 \neq x \in \mathsf{X}_1} \frac{\langle \mathrm{A}_1'\, y, x\rangle_{\mathsf{X}_1'}}{|x|_{\mathsf{X}_1}} = \sup_{0 \neq x \in \mathsf{X}_1} \frac{\langle y, \mathrm{A}_1\, x\rangle_{\mathsf{X}_2'}}{|x|_{\mathsf{X}_1}},$$

and hence for $y \in R(\mathrm{A}_1)'$

$$|\,\mathrm{A}_1'\, y|_{\mathsf{X}_1'} = \sup_{0 \neq x \in \mathsf{X}_1} \frac{\langle y, \mathrm{A}_1\, x\rangle_{R(\mathrm{A}_1)'}}{|x|_{\mathsf{X}_1}}.$$

Choosing the best constant $c_{\mathrm{A}_1'}$, (2.9) *is equivalent to the general inf-sup-condition*

$$0 < \frac{1}{c_{\mathrm{A}_1'}} = \inf_{0 \neq y \in R(\mathrm{A}_1)'} \frac{|\,\mathrm{A}_1'\, y|_{\mathsf{X}_1'}}{|y|_{R(\mathrm{A}_1)'}} = \inf_{0 \neq y \in R(\mathrm{A}_1)'} \sup_{0 \neq x \in \mathsf{X}_1} \frac{\langle y, \mathrm{A}_1\, x\rangle_{R(\mathrm{A}_1)'}}{|y|_{R(\mathrm{A}_1)'} |x|_{\mathsf{X}_1}}. \tag{2.10}$$

In the special case that $\mathsf{X}_2 = \mathsf{H}_2$ *is a Hilbert space and that* H_2' *is identified with* H_2 *and hence* $R(\mathrm{A}_1)'$ *with* $R(\mathrm{A}_1)$ *as a closed subspace of* H_2*, we obtain the following inf-sup-condition*

$$0 < \frac{1}{c_{\mathrm{A}_1'}} = \inf_{0\neq y\in R(\mathrm{A}_1)} \frac{|\,\mathrm{A}_1'\,y|_{\mathsf{X}_1'}}{|y|_{\mathsf{H}_2}} = \inf_{0\neq y\in R(\mathrm{A}_1)} \sup_{0\neq x\in\mathsf{X}_1} \frac{\langle y, \mathrm{A}_1\,x\rangle_{\mathsf{H}_2}}{|y|_{\mathsf{H}_2}|x|_{\mathsf{X}_1}}. \tag{2.11}$$

2.2. **Sobolev Spaces.** Next we introduce our notations for several classes of Sobolev spaces on a bounded domain $\Omega \subset \mathbb{R}^3$. Let $m \in \mathbb{N}_0$. We denote by $\mathsf{L}^2(\Omega)$ and $\mathsf{H}^m(\Omega)$ the standard Lebesgue and Sobolev spaces and write $\mathsf{H}^0(\Omega) = \mathsf{L}^2(\Omega)$. Our notation of spaces will not indicate whether the elements are scalar functions or vector fields. For the rotation and divergence we define the Sobolev spaces

$$\mathsf{R}(\Omega) := \big\{E \in \mathsf{L}^2(\Omega) \,:\, \operatorname{rot} E \in \mathsf{L}^2(\Omega)\big\}, \quad \mathsf{D}(\Omega) := \big\{E \in \mathsf{L}^2(\Omega) \,:\, \operatorname{div} E \in \mathsf{L}^2(\Omega)\big\}$$

with the respective graph norms, where rot and div have to be understood in the distributional or weak sense. We introduce spaces with boundary conditions in the weak sense in the natural way by

$$\mathring{\mathsf{H}}^m(\Omega) := \overline{\mathring{\mathsf{C}}^\infty(\Omega)}^{\mathsf{H}^m(\Omega)}, \quad \mathring{\mathsf{R}}(\Omega) := \overline{\mathring{\mathsf{C}}^\infty(\Omega)}^{\mathsf{R}(\Omega)}, \quad \mathring{\mathsf{D}}(\Omega) := \overline{\mathring{\mathsf{C}}^\infty(\Omega)}^{\mathsf{D}(\Omega)},$$

i.e., as closures of test functions or fields under the respective graph norms, which generalizes homogeneous scalar, tangential and normal boundary conditions, respectively. We also introduce the well known dual spaces

$$\mathsf{H}^{-m}(\Omega) := \big(\mathring{\mathsf{H}}^m(\Omega)\big)'$$

with the standard dual or operator norm defined by

$$|u|_{\mathsf{H}^{-m}(\Omega)} := \sup_{0\neq\varphi\in\mathring{\mathsf{H}}^m(\Omega)} \frac{\langle u, \varphi\rangle_{\mathsf{H}^{-m}(\Omega)}}{|\varphi|_{\mathring{\mathsf{H}}^m(\Omega)}} \qquad \text{for } u \in \mathsf{H}^{-m}(\Omega),$$

where we recall the duality pairing $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathsf{H}^{-m}(\Omega)}$ in $\big(\mathsf{H}^{-m}(\Omega), \mathring{\mathsf{H}}^m(\Omega)\big)$. Moreover, we define with respective graph norms

$$\mathsf{R}^{-m}(\Omega) := \big\{E \in \mathsf{H}^{-m}(\Omega) \,:\, \operatorname{rot} E \in \mathsf{H}^{-m}(\Omega)\big\},$$
$$\mathsf{D}^{-m}(\Omega) := \big\{E \in \mathsf{H}^{-m}(\Omega) \,:\, \operatorname{div} E \in \mathsf{H}^{-m}(\Omega)\big\}.$$

A vanishing differential operator will be indicated by a zero at the lower right corner of the spaces, e.g.,

$$\mathsf{R}_0(\Omega) = \big\{E \in \mathsf{R}(\Omega) \,:\, \operatorname{rot} E = 0\big\}, \qquad \mathring{\mathsf{D}}_0(\Omega) = \big\{E \in \mathring{\mathsf{D}}(\Omega) \,:\, \operatorname{div} E = 0\big\},$$
$$\mathsf{R}_0^{-m}(\Omega) = \big\{E \in \mathsf{R}^{-m}(\Omega) \,:\, \operatorname{rot} E = 0\big\}, \qquad \mathsf{D}_0^{-1}(\Omega) = \big\{E \in \mathsf{D}^{-1}(\Omega) \,:\, \operatorname{div} E = 0\big\}.$$

Let us also introduce

$$\mathsf{L}_0^2(\Omega) := \big\{u \in \mathsf{L}^2(\Omega) \,:\, u \perp_{\mathsf{L}^2(\Omega)} \mathbb{R}\big\} = \big\{u \in \mathsf{L}^2(\Omega) \,:\, \int_\Omega u = 0\big\},$$

where $\perp_{\mathsf{L}^2(\Omega)}$ denotes orthogonality in $\mathsf{L}^2(\Omega)$.

2.3. **General Assumptions.** We will impose the following regularity and topology assumptions on our domain Ω.

Definition 2.16. *Let* Ω *be an open subset of* $\mathbb{R}^3$ *with boundary* $\Gamma := \partial\,\Omega$*. We will call* Ω

- **(i)** *strong Lipschitz, if* Γ *is locally a graph of a Lipschitz function* $\psi : U \subset \mathbb{R}^2 \to \mathbb{R}$*,*
- **(ii)** *topologically trivial, if* Ω *is simply connected with connected boundary* Γ*.*

General Assumption 2.17. *From now on and throughout this paper it is assumed that* $\Omega \subset \mathbb{R}^3$ *is a bounded strong Lipschitz domain.*

If the domain Ω has to be topologically trivial, we will always indicate this in the respective result. Note that several results will hold for arbitrary open subsets Ω of $\mathbb{R}^3$. All results are valid for bounded and topologically trivial strong Lipschitz domains $\Omega \subset \mathbb{R}^3$. Nevertheless, most of the results will remain true for bounded strong Lipschitz domains $\Omega \subset \mathbb{R}^3$.

2.4. **Vector Analysis.** In this last part of the preliminary section we summarize and prove several results related to scalar and vector potentials of various smoothness, corresponding Friedrichs/Poincaré type estimates, and related Helmholtz decompositions of $\mathsf{L}^2(\Omega)$ and other Hilbert and Sobolev spaces. This is a first application of the functional analysis toolbox Section 2.1 for the operators $\mathring{\operatorname{grad}}$, $\mathring{\operatorname{rot}}$, $\mathring{\operatorname{div}}$, and their adjoints $-\operatorname{div}$, rot, $-\operatorname{grad}$. Although these are well known facts, we recall and collect them here, as we will use later similar techniques to obtain related results for the more complicated operators $\mathring{\operatorname{Grad}}\operatorname{grad}$, $\mathring{\operatorname{Rot}}_{\mathbb{S}}$, $\mathring{\operatorname{Div}}_{\mathbb{T}}$, and their adjoints $\operatorname{div}\operatorname{Div}_{\mathbb{S}}$, $\operatorname{sym}\operatorname{Rot}_{\mathbb{T}}$, $-\operatorname{dev}\operatorname{Grad}$. Let

$$\mathrm{A}_0 := \mathring{\operatorname{grad}} : \mathring{\mathsf{H}}^1(\Omega) \subset \mathsf{L}^2(\Omega) \longrightarrow \mathsf{L}^2(\Omega),$$
$$\mathrm{A}_1 := \mathring{\operatorname{rot}} : \mathring{\mathsf{R}}(\Omega) \subset \mathsf{L}^2(\Omega) \longrightarrow \mathsf{L}^2(\Omega),$$
$$\mathrm{A}_2 := \mathring{\operatorname{div}} : \mathring{\mathsf{D}}(\Omega) \subset \mathsf{L}^2(\Omega) \longrightarrow \mathsf{L}^2(\Omega).$$

Then A_0, A_1, and A_2 are unbounded, densely defined, and closed linear operators with adjoints

$$\mathrm{A}_0^* = \mathring{\operatorname{grad}}^* = -\operatorname{div} : \mathsf{D}(\Omega) \subset \mathsf{L}^2(\Omega) \longrightarrow \mathsf{L}^2(\Omega),$$
$$\mathrm{A}_1^* := \mathring{\operatorname{rot}}^* = \operatorname{rot} : \mathsf{R}(\Omega) \subset \mathsf{L}^2(\Omega) \longrightarrow \mathsf{L}^2(\Omega),$$
$$\mathrm{A}_2^* := \mathring{\operatorname{div}}^* = -\operatorname{grad} : \mathsf{H}^1(\Omega) \subset \mathsf{L}^2(\Omega) \longrightarrow \mathsf{L}^2(\Omega)$$

and the sequence properties

$$R(\mathrm{A}_0) = \mathring{\operatorname{grad}}\,\mathring{\mathsf{H}}^1(\Omega) \subset \mathring{\mathsf{R}}_0(\Omega) = N(\mathrm{A}_1), \qquad R(\mathrm{A}_1^*) = \operatorname{rot}\mathsf{R}(\Omega) \subset \mathsf{D}_0(\Omega) = N(\mathrm{A}_0^*),$$
$$R(\mathrm{A}_1) = \mathring{\operatorname{rot}}\,\mathring{\mathsf{R}}(\Omega) \subset \mathring{\mathsf{D}}_0(\Omega) = N(\mathrm{A}_2), \qquad R(\mathrm{A}_2^*) = \operatorname{grad}\mathsf{H}^1(\Omega) \subset \mathsf{R}_0(\Omega) = N(\mathrm{A}_1^*)$$

hold. Note $N(\mathrm{A}_0) = \{0\}$ and $N(\mathrm{A}_2^*) = \mathbb{R}$. Moreover, the embeddings

$$D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) = \mathring{\mathsf{R}}(\Omega) \cap \mathsf{D}(\Omega) \hookrightarrow \mathsf{L}^2(\Omega), \qquad D(\mathrm{A}_2) \cap D(\mathrm{A}_1^*) = \mathring{\mathsf{D}}(\Omega) \cap \mathsf{R}(\Omega) \hookrightarrow \mathsf{L}^2(\Omega)$$

are compact. The latter compact embeddings are called Maxwell compactness properties or Weck's selection theorems. The first proof for strong Lipschitz domains (uniform cone like domains) avoiding smoothness of Γ was given by Weck in [27]. Generally, Weck's selection theorems hold e.g. for weak Lipschitz domains, see [22], or even for more general domains with p-cusps or antennas, see [28, 23]. See also [26] for a different proof in the case of a strong Lipschitz domain. Weck's selection theorem for mixed boundary conditions has been proved in [12] for strong Lipschitz domains and recently in [2] for weak Lipschitz domains. Similar to Rellich's selection theorem, i.e., the compact embedding of $\mathring{\mathsf{H}}^1(\Omega)$ resp. $\mathsf{H}^1(\Omega)$ into $\mathsf{L}^2(\Omega)$, it is crucial that the domain Ω is bounded. Finally, the kernels

$$N(\mathrm{A}_1) \cap N(\mathrm{A}_0^*) = \mathring{\mathsf{R}}_0(\Omega) \cap \mathsf{D}_0(\Omega) =: \mathcal{H}_{\mathsf{D}}(\Omega) \quad \text{resp.} \quad N(\mathrm{A}_2) \cap N(\mathrm{A}_1^*) = \mathring{\mathsf{D}}_0(\Omega) \cap \mathsf{R}_0(\Omega) =: \mathcal{H}_{\mathsf{N}}(\Omega),$$

are finite dimensional, as the unit balls are compact, i.e., the spaces of Dirichlet or Neumann fields are finite dimensional. More precisely, the dimension of the Dirichlet resp. Neumann fields depends on the topology or cohomology of Ω, i.e., second resp. first Betti number, see e.g. [20, 21]. Especially we have

$$\mathcal{H}_{\mathsf{D}}(\Omega) = \{0\}, \text{ if } \Gamma \text{ is connected}, \qquad \mathcal{H}_{\mathsf{N}}(\Omega) = \{0\}, \text{ if } \Omega \text{ is simply connected}.$$

Remark 2.18. *Our general assumption on Ω to be bounded and strong Lipschitz ensures that Weck's selection theorems (and thus also Rellich's) hold. The additional assumption that Ω is also topologically trivial excludes the existence of non-trivial Dirichlet or Neumann fields, as Ω is simply connected with a connected boundary Γ.*

By the results of the functional analysis toolbox Section 2.1 we see that all ranges are closed with

$$R(\mathrm{A}_0) = R(\mathcal{A}_0), \qquad R(\mathrm{A}_1) = R(\mathcal{A}_1), \qquad R(\mathrm{A}_2) = R(\mathcal{A}_2),$$
$$R(\mathrm{A}_0^*) = R(\mathcal{A}_0^*), \qquad R(\mathrm{A}_1^*) = R(\mathcal{A}_1^*), \qquad R(\mathrm{A}_2^*) = R(\mathcal{A}_2^*),$$

i.e., the ranges

$$
\begin{aligned}
&\mathring{\operatorname{grad}}\,\mathring{\mathsf{H}}^1(\Omega), && \operatorname{grad}\mathsf{H}^1(\Omega)=\operatorname{grad}\big(\mathsf{H}^1(\Omega)\cap\mathsf{L}^2_0(\Omega)\big),\\
&\mathring{\operatorname{rot}}\,\mathring{\mathsf{R}}(\Omega)=\mathring{\operatorname{rot}}\big(\mathring{\mathsf{R}}(\Omega)\cap\operatorname{rot}\mathsf{R}(\Omega)\big), && \operatorname{rot}\mathsf{R}(\Omega)=\operatorname{rot}\big(\mathsf{R}(\Omega)\cap\mathring{\operatorname{rot}}\,\mathring{\mathsf{R}}(\Omega)\big),\\
&\mathring{\operatorname{div}}\,\mathring{\mathsf{D}}(\Omega)=\mathring{\operatorname{div}}\big(\mathring{\mathsf{D}}(\Omega)\cap\operatorname{grad}\mathsf{H}^1(\Omega)\big), && \operatorname{div}\mathsf{D}(\Omega)=\operatorname{div}\big(\mathsf{D}(\Omega)\cap\mathring{\operatorname{grad}}\,\mathring{\mathsf{H}}^1(\Omega)\big)
\end{aligned}
\tag{2.12}
$$

are closed, and the reduced operators are

$$
\begin{aligned}
\mathcal{A}_0&:=\mathring{\operatorname{grad}}:\mathring{\mathsf{H}}^1(\Omega)\subset\mathsf{L}^2(\Omega)\longrightarrow\mathring{\operatorname{grad}}\,\mathring{\mathsf{H}}^1(\Omega),\\
\mathcal{A}_1&:=\mathring{\operatorname{rot}}:\mathring{\mathsf{R}}(\Omega)\cap\operatorname{rot}\mathsf{R}(\Omega)\subset\operatorname{rot}\mathsf{R}(\Omega)\longrightarrow\operatorname{rot}\mathring{\mathsf{R}}(\Omega),\\
\mathcal{A}_2&:=\mathring{\operatorname{div}}:\mathring{\mathsf{D}}(\Omega)\cap\operatorname{grad}\mathsf{H}^1(\Omega)\subset\operatorname{grad}\mathsf{H}^1(\Omega)\longrightarrow\mathsf{L}^2_0(\Omega),\\
\mathcal{A}_0^*&:=-\operatorname{div}:\mathsf{D}(\Omega)\cap\operatorname{grad}\mathring{\mathsf{H}}^1(\Omega)\subset\operatorname{grad}\mathring{\mathsf{H}}^1(\Omega)\longrightarrow\mathsf{L}^2(\Omega),\\
\mathcal{A}_1^*&:=\operatorname{rot}:\mathsf{R}(\Omega)\cap\mathring{\operatorname{rot}}\,\mathring{\mathsf{R}}(\Omega)\subset\mathring{\operatorname{rot}}\,\mathring{\mathsf{R}}(\Omega)\longrightarrow\operatorname{rot}\mathsf{R}(\Omega),\\
\mathcal{A}_2^*&:=-\operatorname{grad}:\mathsf{H}^1(\Omega)\cap\mathsf{L}^2_0(\Omega)\subset\mathsf{L}^2_0(\Omega)\longrightarrow\operatorname{grad}\mathsf{H}^1(\Omega).
\end{aligned}
$$

Moreover, we have the following well known Helmholtz decompositions of L^2-vector fields into irrotational and solenoidal vector fields, corresponding Friedrichs/Poincaré type estimates and continuous or compact inverse operators.

Lemma 2.19. *The Helmholtz decompositions*

$$
\begin{aligned}
\mathsf{L}^2(\Omega)&=\mathring{\operatorname{div}}\,\mathring{\mathsf{D}}(\Omega)\oplus_{\mathsf{L}^2(\Omega)}\mathbb{R},\quad \mathring{\operatorname{div}}\,\mathring{\mathsf{D}}(\Omega)=\mathsf{L}^2_0(\Omega),\\
\mathsf{L}^2(\Omega)&=\operatorname{div}\mathsf{D}(\Omega),\\
\mathsf{L}^2(\Omega)&=\mathring{\operatorname{grad}}\,\mathring{\mathsf{H}}^1(\Omega)\oplus_{\mathsf{L}^2(\Omega)}\mathsf{D}_0(\Omega)\\
&=\mathring{\mathsf{R}}_0(\Omega)\oplus_{\mathsf{L}^2(\Omega)}\operatorname{rot}\mathsf{R}(\Omega)\\
&=\operatorname{grad}\mathring{\mathsf{H}}^1(\Omega)\oplus_{\mathsf{L}^2(\Omega)}\mathcal{H}_{\mathsf{D}}(\Omega)\oplus_{\mathsf{L}^2(\Omega)}\operatorname{rot}\mathsf{R}(\Omega),\\
\mathsf{L}^2(\Omega)&=\operatorname{grad}\mathsf{H}^1(\Omega)\oplus_{\mathsf{L}^2(\Omega)}\mathring{\mathsf{D}}_0(\Omega)\\
&=\mathsf{R}_0(\Omega)\oplus_{\mathsf{L}^2(\Omega)}\mathring{\operatorname{rot}}\,\mathring{\mathsf{R}}(\Omega)\\
&=\operatorname{grad}\mathsf{H}^1(\Omega)\oplus_{\mathsf{L}^2(\Omega)}\mathcal{H}_{\mathsf{N}}(\Omega)\oplus_{\mathsf{L}^2(\Omega)}\mathring{\operatorname{rot}}\,\mathring{\mathsf{R}}(\Omega)
\end{aligned}
$$

holds with

$$
\begin{aligned}
&\mathring{\operatorname{grad}}\,\mathring{\mathsf{H}}^1(\Omega)=\mathring{\mathsf{R}}_0(\Omega)\ominus_{\mathsf{L}^2(\Omega)}\mathcal{H}_{\mathsf{D}}(\Omega), && \operatorname{grad}\mathsf{H}^1(\Omega)=\mathsf{R}_0(\Omega)\ominus_{\mathsf{L}^2(\Omega)}\mathcal{H}_{\mathsf{N}}(\Omega)\\
&\mathring{\operatorname{rot}}\,\mathring{\mathsf{R}}(\Omega)=\mathring{\mathsf{D}}_0(\Omega)\ominus_{\mathsf{L}^2(\Omega)}\mathcal{H}_{\mathsf{N}}(\Omega), && \operatorname{rot}\mathsf{R}(\Omega)=\mathsf{D}_0(\Omega)\ominus_{\mathsf{L}^2(\Omega)}\mathcal{H}_{\mathsf{D}}(\Omega).
\end{aligned}
$$

Moreover, (2.12) *is true for the respective ranges and the "better" potentials in* (2.12) *are uniquely determined and depend continuously in the right hand sides. If Γ is connected, it holds $\mathcal{H}_{\mathsf{D}}(\Omega)=\{0\}$ and, e.g.,*

$$
\mathsf{L}^2(\Omega)=\mathring{\mathsf{R}}_0(\Omega)\oplus\mathsf{D}_0(\Omega)\quad\textit{and}\quad\mathring{\mathsf{R}}_0(\Omega)=\mathring{\operatorname{grad}}\,\mathring{\mathsf{H}}^1(\Omega),\quad\mathsf{D}_0(\Omega)=\operatorname{rot}\mathsf{R}(\Omega)=\operatorname{rot}\big(\mathsf{R}(\Omega)\cap\mathring{\mathsf{D}}_0(\Omega)\big).
$$

If Ω is simply connected, it holds $\mathcal{H}_{\mathsf{N}}(\Omega)=\{0\}$ and, e.g.,

$$
\mathsf{L}^2(\Omega)=\mathsf{R}_0(\Omega)\oplus\mathring{\mathsf{D}}_0(\Omega)\quad\textit{and}\quad\mathsf{R}_0(\Omega)=\operatorname{grad}\mathsf{H}^1(\Omega),\quad\mathring{\mathsf{D}}_0(\Omega)=\mathring{\operatorname{rot}}\,\mathring{\mathsf{R}}(\Omega)=\mathring{\operatorname{rot}}\big(\mathring{\mathsf{R}}(\Omega)\cap\mathsf{D}_0(\Omega)\big).
$$

Lemma 2.20. *The following Friedrichs/Poincaré type estimates hold. There exist positive constants c_{g}, c_{r}, c_{d}, such that*

$$\begin{aligned}
&\forall\, u \in \mathring{\mathsf{H}}^1(\Omega) && |u|_{\mathsf{L}^2(\Omega)} \leq c_{\mathsf{g}}\, |\operatorname{grad} u|_{\mathsf{L}^2(\Omega)},\\
&\forall\, E \in \mathsf{D}(\Omega) \cap \mathring{\operatorname{grad}}\, \mathring{\mathsf{H}}^1(\Omega) && |E|_{\mathsf{L}^2(\Omega)} \leq c_{\mathsf{g}}\, |\operatorname{div} E|_{\mathsf{L}^2(\Omega)},\\
&\forall\, E \in \mathring{\mathsf{R}}(\Omega) \cap \operatorname{rot} \mathsf{R}(\Omega) && |E|_{\mathsf{L}^2(\Omega)} \leq c_{\mathsf{r}}\, |\operatorname{rot} E|_{\mathsf{L}^2(\Omega)},\\
&\forall\, E \in \mathsf{R}(\Omega) \cap \mathring{\operatorname{rot}}\, \mathring{\mathsf{R}}(\Omega) && |E|_{\mathsf{L}^2(\Omega)} \leq c_{\mathsf{r}}\, |\operatorname{rot} E|_{\mathsf{L}^2(\Omega)},\\
&\forall\, E \in \mathring{\mathsf{D}}(\Omega) \cap \operatorname{grad} \mathsf{H}^1(\Omega) && |E|_{\mathsf{L}^2(\Omega)} \leq c_{\mathsf{d}}\, |\operatorname{div} E|_{\mathsf{L}^2(\Omega)},\\
&\forall\, u \in \mathsf{H}^1(\Omega) \cap \mathsf{L}^2_0(\Omega) && |u|_{\mathsf{L}^2(\Omega)} \leq c_{\mathsf{d}}\, |\operatorname{grad} u|_{\mathsf{L}^2(\Omega)}.
\end{aligned}$$

Moreover, the reduced versions of the operators

$$\mathring{\operatorname{grad}}, \quad \mathring{\operatorname{rot}}, \quad \mathring{\operatorname{div}}, \quad \operatorname{grad}, \quad \operatorname{rot}, \quad \operatorname{div}$$

have continuous resp. compact inverse operators

$$\begin{aligned}
\mathring{\operatorname{grad}}^{-1} &: \mathring{\operatorname{grad}}\, \mathring{\mathsf{H}}^1(\Omega) \longrightarrow \mathring{\mathsf{H}}^1(\Omega), & \mathring{\operatorname{grad}}^{-1} &: \mathring{\operatorname{grad}}\, \mathring{\mathsf{H}}^1(\Omega) \longrightarrow \mathsf{L}^2(\Omega),\\
\operatorname{div}^{-1} &: \mathsf{L}^2(\Omega) \longrightarrow \mathsf{D}(\Omega) \cap \mathring{\operatorname{grad}}\, \mathring{\mathsf{H}}^1(\Omega), & \operatorname{div}^{-1} &: \mathsf{L}^2(\Omega) \longrightarrow \mathring{\operatorname{grad}}\, \mathring{\mathsf{H}}^1(\Omega) \subset \mathsf{L}^2(\Omega),\\
\mathring{\operatorname{rot}}^{-1} &: \mathring{\operatorname{rot}}\, \mathring{\mathsf{R}}(\Omega) \longrightarrow \mathring{\mathsf{R}}(\Omega) \cap \operatorname{rot} \mathsf{R}(\Omega), & \mathring{\operatorname{rot}}^{-1} &: \mathring{\operatorname{rot}}\, \mathring{\mathsf{R}}(\Omega) \longrightarrow \operatorname{rot} \mathsf{R}(\Omega) \subset \mathsf{L}^2(\Omega),\\
\operatorname{rot}^{-1} &: \operatorname{rot} \mathsf{R}(\Omega) \longrightarrow \mathsf{R}(\Omega) \cap \mathring{\operatorname{rot}}\, \mathring{\mathsf{R}}(\Omega), & \operatorname{rot}^{-1} &: \operatorname{rot} \mathsf{R}(\Omega) \longrightarrow \mathring{\operatorname{rot}}\, \mathring{\mathsf{R}}(\Omega) \subset \mathsf{L}^2(\Omega),\\
\mathring{\operatorname{div}}^{-1} &: \mathsf{L}^2_0(\Omega) \longrightarrow \mathring{\mathsf{D}}(\Omega) \cap \operatorname{grad} \mathsf{H}^1(\Omega), & \mathring{\operatorname{div}}^{-1} &: \mathsf{L}^2_0(\Omega) \longrightarrow \operatorname{grad} \mathsf{H}^1(\Omega) \subset \mathsf{L}^2(\Omega),\\
\operatorname{grad}^{-1} &: \operatorname{grad} \mathsf{H}^1(\Omega) \longrightarrow \mathsf{H}^1(\Omega) \cap \mathsf{L}^2_0(\Omega), & \operatorname{grad}^{-1} &: \operatorname{grad} \mathsf{H}^1(\Omega) \longrightarrow \mathsf{L}^2_0(\Omega),
\end{aligned}$$

with norms $(1+c_{\mathsf{g}}^2)^{1/2}$, $(1+c_{\mathsf{r}}^2)^{1/2}$, $(1+c_{\mathsf{d}}^2)^{1/2}$ resp. c_{g}, c_{r}, c_{d}. In other words, the operators

$$\begin{aligned}
\mathring{\operatorname{grad}} : \mathring{\mathsf{H}}^1(\Omega) &\longrightarrow \mathring{\operatorname{grad}}\, \mathring{\mathsf{H}}^1(\Omega), & \operatorname{div} : \mathsf{D}(\Omega) \cap \mathring{\operatorname{grad}}\, \mathring{\mathsf{H}}^1(\Omega) &\longrightarrow \mathsf{L}^2(\Omega),\\
u &\longmapsto \operatorname{grad} u & E &\longmapsto \operatorname{div} E\\
\mathring{\operatorname{rot}} : \mathring{\mathsf{R}}(\Omega) \cap \operatorname{rot} \mathsf{R}(\Omega) &\longrightarrow \mathring{\operatorname{rot}}\, \mathring{\mathsf{R}}(\Omega), & \operatorname{rot} : \mathsf{R}(\Omega) \cap \mathring{\operatorname{rot}}\, \mathring{\mathsf{R}}(\Omega) &\longrightarrow \operatorname{rot} \mathsf{R}(\Omega),\\
E &\longmapsto \operatorname{rot} E & E &\longmapsto \operatorname{rot} E\\
\mathring{\operatorname{div}} : \mathring{\mathsf{D}}(\Omega) \cap \operatorname{grad} \mathsf{H}^1(\Omega) &\longrightarrow \mathsf{L}^2_0(\Omega), & \operatorname{grad} : \mathsf{H}^1(\Omega) \cap \mathsf{L}^2_0(\Omega) &\longrightarrow \operatorname{grad} \mathsf{H}^1(\Omega),\\
E &\longmapsto \operatorname{div} E & u &\longmapsto \operatorname{grad} u
\end{aligned}$$

are topological isomorphisms. If Ω is topologically trivial, then

$$\begin{aligned}
\mathring{\operatorname{grad}} : \mathring{\mathsf{H}}^1(\Omega) &\longrightarrow \mathring{\mathsf{R}}_0(\Omega), & \operatorname{div} : \mathsf{D}(\Omega) \cap \mathring{\mathsf{R}}_0(\Omega) &\longrightarrow \mathsf{L}^2(\Omega),\\
u &\longmapsto \operatorname{grad} u & E &\longmapsto \operatorname{div} E\\
\mathring{\operatorname{rot}} : \mathring{\mathsf{R}}(\Omega) \cap \mathsf{D}_0(\Omega) &\longrightarrow \mathring{\mathsf{D}}_0(\Omega), & \operatorname{rot} : \mathsf{R}(\Omega) \cap \mathring{\mathsf{D}}_0(\Omega) &\longrightarrow \mathsf{D}_0(\Omega),\\
E &\longmapsto \operatorname{rot} E & E &\longmapsto \operatorname{rot} E\\
\mathring{\operatorname{div}} : \mathring{\mathsf{D}}(\Omega) \cap \mathsf{R}_0(\Omega) &\longrightarrow \mathsf{L}^2_0(\Omega), & \operatorname{grad} : \mathsf{H}^1(\Omega) \cap \mathsf{L}^2_0(\Omega) &\longrightarrow \mathsf{R}_0(\Omega),\\
E &\longmapsto \operatorname{div} E & u &\longmapsto \operatorname{grad} u
\end{aligned} \tag{2.13}$$

are topological isomorphisms.

Remark 2.21. *Recently it has been shown in* [17, 18, 19], *that for bounded and convex* $\Omega \subset \mathbb{R}^3$ *it holds*

$$c_{\mathsf{r}} \le c_{\mathsf{d}} \le \frac{\operatorname{diam}\Omega}{\pi},$$

i.e., the Maxwell constant c_{r} *can be estimates from above by the Poincaré constant.*

Remark 2.22. *Some of the previous results can be formulated equivalently in terms of complexes: The sequence*

$$\{0\} \xrightarrow{\;0\;} \mathring{\mathsf{H}}^1(\Omega) \xrightarrow{\;\mathring{\operatorname{grad}}\;} \mathring{\mathsf{R}}(\Omega) \xrightarrow{\;\mathring{\operatorname{rot}}\;} \mathring{\mathsf{D}}(\Omega) \xrightarrow{\;\mathring{\operatorname{div}}\;} \mathsf{L}^2(\Omega) \xrightarrow{\;\pi_{\mathbb{R}}\;} \mathbb{R}$$

and thus also its dual or adjoint sequence

$$\{0\} \xleftarrow{\;0\;} \mathsf{L}^2(\Omega) \xleftarrow{\;-\operatorname{div}\;} \mathsf{D}(\Omega) \xleftarrow{\;\operatorname{rot}\;} \mathsf{R}(\Omega) \xleftarrow{\;-\operatorname{grad}\;} \mathsf{H}^1(\Omega) \xleftarrow{\;\iota_{\mathbb{R}}\;} \mathbb{R}$$

are closed Hilbert complexes. Here $\pi_{\mathbb{R}} : \mathsf{L}^2(\Omega) \to \mathbb{R}$ *denotes the orthogonal projector onto* $\mathbb{R}$ *with adjoint* $\pi_{\mathbb{R}}^* = \iota_{\mathbb{R}} : \mathbb{R} \to \mathsf{L}^2(\Omega)$, *the canonical embedding. If* Ω *is additionally topologically trivial, then the complexes are also exact.*

Let Ω be additionally topologically trivial. For irrotational vector fields in $\mathring{\mathsf{H}}^m(\Omega)$ resp. $\mathsf{H}^m(\Omega)$ we have smooth potentials, which follows immediately by $\mathring{\mathsf{R}}_0(\Omega) = \mathring{\operatorname{grad}}\,\mathring{\mathsf{H}}^1(\Omega)$ resp. $\mathsf{R}_0(\Omega) = \operatorname{grad}\mathsf{H}^1(\Omega)$ from the previous lemma.

Lemma 2.23. *Let* Ω *be additionally topologically trivial and* $m \in \mathbb{N}_0$. *Then*

$$\mathring{\mathsf{H}}^m(\Omega) \cap \mathring{\mathsf{R}}_0(\Omega) = \mathring{\operatorname{grad}}\,\mathring{\mathsf{H}}^{m+1}(\Omega), \qquad \mathsf{H}^m(\Omega) \cap \mathsf{R}_0(\Omega) = \operatorname{grad}\mathsf{H}^{m+1}(\Omega)$$

hold with linear and continuous potential operators $\mathrm{P}_{\mathring{\operatorname{grad}}}$, $\mathrm{P}_{\operatorname{grad}}$.

Note that the potential in $\mathsf{H}^{m+1}(\Omega)$ is uniquely determined only up to a constant. For solenoidal vector fields in $\mathring{\mathsf{H}}^m(\Omega)$ resp. $\mathsf{H}^m(\Omega)$ we have smooth potentials, too.

Lemma 2.24. *Let* Ω *be additionally topologically trivial and* $m \in \mathbb{N}_0$. *Then*

$$\mathring{\mathsf{H}}^m(\Omega) \cap \mathring{\mathsf{D}}_0(\Omega) = \mathring{\operatorname{rot}}\,\mathring{\mathsf{H}}^{m+1}(\Omega), \qquad \mathsf{H}^m(\Omega) \cap \mathsf{D}_0(\Omega) = \operatorname{rot}\mathsf{H}^{m+1}(\Omega)$$

hold with linear and continuous potential operators $\mathrm{P}_{\mathring{\operatorname{rot}}}$, $\mathrm{P}_{\operatorname{rot}}$.

For a proof see e.g. [6, Corollary 4.7] or with slight modifications the generalized lifting lemma [10, Corollary 5.4] for the case $d = 3$, $k = m$, $l = 2$. Moreover, the potential in $\mathring{\mathsf{H}}^{m+1}(\Omega)$ resp. $\mathsf{H}^{m+1}(\Omega)$ is no longer uniquely determined. For the divergence operator we have the following result.

Lemma 2.25. *Let* $m \in \mathbb{N}_0$. *Then*

$$\mathring{\mathsf{H}}^m(\Omega) \cap \mathsf{L}^2_0(\Omega) = \mathring{\operatorname{div}}\,\mathring{\mathsf{H}}^{m+1}(\Omega), \qquad \mathsf{H}^m(\Omega) = \operatorname{div}\mathsf{H}^{m+1}(\Omega)$$

hold with linear and continuous potential operators $\mathrm{P}_{\mathring{\operatorname{div}}}$, $\mathrm{P}_{\operatorname{div}}$*.*.

Again, the potential in $\mathring{\mathsf{H}}^{m+1}(\Omega)$ resp. $\mathsf{H}^{m+1}(\Omega)$ is no longer uniquely determined. Also Lemma 2.23 resp. Lemma 2.25 has been proved in [6, Corollary 4.7(b)] and in [10, Corollary 5.4] for the case $d = 3$, $k = m$, $l = 1$ resp. $d = 3$, $k = m$, $l = 3$.

Remark 2.26. *Lemma 2.25, which shows a classical result on the solvability and on the properties of the solution operator of the divergence equation, is an important tool in fluid dynamics, i.e., in the theory of Stokes or Navier-Stokes equations. The potential operator is often called Bogovskiĭ operator, see* [4, 5] *for the original works and also* [7, p. 179, Theorem III.3.3], [25, Lemma 2.1.1]. *Moreover, there are also versions of Lemma 2.23 and Lemma 2.24, if* Ω *is not topologically trivial, which we will not need in the paper at hand.*

Using the latter three results and Lemma 2.13, irrotational and solenoidal vector fields in $\mathsf{H}^{-m}(\Omega)$ can be characterized.

Corollary 2.27. *Let Ω be additionally topologically trivial and $m \in \mathbb{N}$. Then*

$$\mathsf{R}_0^{-m}(\Omega) = \operatorname{grad} \mathsf{H}^{-m+1}(\Omega) = \operatorname{grad} \big(\mathring{\mathsf{H}}^{m-1}(\Omega) \cap \mathsf{L}_0^2(\Omega)\big)'$$

is closed in $\mathsf{H}^{-m}(\Omega)$ with continuous inverse, i.e., $\operatorname{grad}^{-1} \in BL\big(\mathsf{R}_0^{-m}(\Omega), (\mathring{\mathsf{H}}^{m-1}(\Omega) \cap \mathsf{L}_0^2(\Omega))'\big)$. *Especially for $m = 1$,*

$$\mathsf{R}_0^{-1}(\Omega) = \operatorname{grad} \mathsf{L}^2(\Omega) = \operatorname{grad} \mathsf{L}_0^2(\Omega)$$

is closed in $\mathsf{H}^{-1}(\Omega)$ with continuous inverse $\operatorname{grad}^{-1} \in BL\big(\mathsf{R}_0^{-1}(\Omega), \mathsf{L}_0^2(\Omega)\big)$ *and uniquely determined potential in $\mathsf{L}_0^2(\Omega)$. Moreover,*

$$\exists\, c_{\mathsf{g},-1} > 0 \quad \forall\, u \in \mathsf{L}_0^2(\Omega) \qquad |u|_{\mathsf{L}^2(\Omega)} \leq c_{\mathsf{g},-1} |\operatorname{grad} u|_{\mathsf{H}^{-1}(\Omega)} \leq \sqrt{3}\, c_{\mathsf{g},-1} |u|_{\mathsf{L}^2(\Omega)}$$

and the inf-sup-condition

$$0 < \frac{1}{c_{\mathsf{g},-1}} = \inf_{0 \neq u \in \mathsf{L}_0^2(\Omega)} \frac{|\operatorname{grad} u|_{\mathsf{H}^{-1}(\Omega)}}{|u|_{\mathsf{L}^2(\Omega)}} = \inf_{0 \neq u \in \mathsf{L}_0^2(\Omega)} \sup_{0 \neq E \in \mathring{\mathsf{H}}^1(\Omega)} \frac{\langle u, \operatorname{div} E\rangle_{\mathsf{L}^2(\Omega)}}{|u|_{\mathsf{L}^2(\Omega)} |\operatorname{Grad} E|_{\mathsf{L}^2(\Omega)}}.$$

holds.

Proof. Let $\mathsf{X}_0 := \mathring{\mathsf{H}}^{m+1}(\Omega)$, $\mathsf{X}_1 := \mathring{\mathsf{H}}^{m}(\Omega)$, $\mathsf{X}_2 := \mathring{\mathsf{H}}^{m-1}(\Omega)$ and

$$\mathrm{A}_0 := \mathring{\operatorname{rot}} : \mathring{\mathsf{H}}^{m+1}(\Omega) \to \mathring{\mathsf{H}}^{m}(\Omega), \quad \mathrm{A}_1 := -\mathring{\operatorname{div}} : \mathring{\mathsf{H}}^{m}(\Omega) \to \mathring{\mathsf{H}}^{m-1}(\Omega).$$

These linear operators are bounded, $R(\mathrm{A}_0) = \mathring{\operatorname{rot}}\, \mathring{\mathsf{H}}^{m+1}(\Omega) = \mathring{\mathsf{H}}^{m}(\Omega) \cap \mathring{\mathsf{D}}_0(\Omega) = N(\mathrm{A}_1)$ by Lemma 2.24, and $R(\mathrm{A}_1) = \mathring{\operatorname{div}}\, \mathring{\mathsf{H}}^{m}(\Omega) = \mathring{\mathsf{H}}^{m-1}(\Omega) \cap \mathsf{L}_0^2(\Omega)$ by Lemma 2.25. Therefore, $R(\mathrm{A}_1)$ is closed. For the adjoint operators we get

$$\mathrm{A}_0' = \operatorname{rot} = \mathring{\operatorname{rot}}\,' : \mathsf{H}^{-m}(\Omega) \to \mathsf{H}^{-m-1}(\Omega), \quad \mathrm{A}_1' = \operatorname{grad} = -\mathring{\operatorname{div}}\,' : \mathsf{H}^{-m+1}(\Omega) \to \mathsf{H}^{-m}(\Omega)$$

and obtain from Lemma 2.13 that

$$\mathsf{R}_0^{-m}(\Omega) = N(\mathrm{A}_0') = R(\mathrm{A}_1') = \operatorname{grad} \mathsf{H}^{-m+1}(\Omega)$$

is closed and

$$\operatorname{grad}^{-1} = (\mathrm{A}_1')^{-1} \in BL\big(R(\mathrm{A}_1'), R(\mathrm{A}_1)'\big) = BL\big(\mathsf{R}_0^{-m}(\Omega), (\mathring{\mathsf{H}}^{m-1}(\Omega) \cap \mathsf{L}_0^2(\Omega))'\big).$$

which completes the proof for general m. If $m = 1$, we identify $(\mathring{\mathsf{H}}^{0}(\Omega) \cap \mathsf{L}_0^2(\Omega))' = \mathsf{L}_0^2(\Omega)' = \mathsf{L}_0^2(\Omega)$ and get the assertions about the Friedrichs/Poincaré/Nečas inequality and inf-sup-condition by Remark 2.15, i.e., (2.9) and (2.11). □

Corollary 2.28. *Let Ω be additionally topologically trivial and $m \in \mathbb{N}$. Then*

$$\mathsf{D}_0^{-m}(\Omega) = \operatorname{rot} \mathsf{H}^{-m+1}(\Omega) = \operatorname{rot} \big(\mathring{\mathsf{H}}^{m-1}(\Omega) \cap \mathring{\mathsf{D}}_0(\Omega)\big)'$$

is closed in $\mathsf{H}^{-m}(\Omega)$ with continuous inverse, i.e., $\operatorname{rot}^{-1} \in BL\big(\mathsf{D}_0^{-m}(\Omega), (\mathring{\mathsf{H}}^{m-1}(\Omega) \cap \mathring{\mathsf{D}}_0(\Omega))'\big)$. *Especially for $m = 1$,*

$$\mathsf{D}_0^{-1}(\Omega) = \operatorname{rot} \mathsf{L}^2(\Omega) = \operatorname{rot} \mathring{\mathsf{D}}_0(\Omega)$$

is closed in $\mathsf{H}^{-1}(\Omega)$ with continuous inverse $\operatorname{rot}^{-1} \in BL\big(\mathsf{D}_0^{-1}(\Omega), \mathring{\mathsf{D}}_0(\Omega)\big)$ *and uniquely determined potential in $\mathring{\mathsf{D}}_0(\Omega)$. Moreover,*

$$\exists\, c_{\mathsf{r},-1} > 0 \quad \forall\, E \in \mathring{\mathsf{D}}_0(\Omega) \qquad |E|_{\mathsf{L}^2(\Omega)} \leq c_{\mathsf{r},-1} |\operatorname{rot} E|_{\mathsf{H}^{-1}(\Omega)} \leq \sqrt{2}\, c_{\mathsf{r},-1} |E|_{\mathsf{L}^2(\Omega)}$$

and the inf-sup-condition

$$0 < \frac{1}{c_{\mathsf{r},-1}} = \inf_{0\neq E\in\mathring{\mathsf{D}}_0(\Omega)} \frac{|\operatorname{rot} E|_{\mathsf{H}^{-1}(\Omega)}}{|E|_{\mathsf{L}^2(\Omega)}} = \inf_{0\neq E\in\mathring{\mathsf{D}}_0(\Omega)} \sup_{0\neq E\in\mathring{\mathsf{H}}^1(\Omega)} \frac{\langle E, \operatorname{rot} H\rangle_{\mathsf{L}^2(\Omega)}}{|E|_{\mathsf{L}^2(\Omega)}|\operatorname{Grad} H|_{\mathsf{L}^2(\Omega)}}.$$

holds.

Proof. Let $\mathsf{X}_0 := \mathring{\mathsf{H}}^{m+1}(\Omega)$, $\mathsf{X}_1 := \mathring{\mathsf{H}}^{m}(\Omega)$, $\mathsf{X}_2 := \mathring{\mathsf{H}}^{m-1}(\Omega)$ and

$$\mathrm{A}_0 := \mathring{\operatorname{grad}} : \mathring{\mathsf{H}}^{m+1}(\Omega) \to \mathring{\mathsf{H}}^{m}(\Omega), \quad \mathrm{A}_1 := \mathring{\operatorname{rot}} : \mathring{\mathsf{H}}^{m}(\Omega) \to \mathring{\mathsf{H}}^{m-1}(\Omega).$$

These linear operators are bounded, $R(\mathrm{A}_0) = \mathring{\operatorname{grad}}\,\mathring{\mathsf{H}}^{m+1}(\Omega) = \mathring{\mathsf{H}}^{m}(\Omega) \cap \mathring{\mathsf{R}}_0(\Omega) = N(\mathrm{A}_1)$ by Lemma 2.23, and $R(\mathrm{A}_1) = \operatorname{rot}\mathring{\mathsf{H}}^{m}(\Omega) = \mathring{\mathsf{H}}^{m-1}(\Omega) \cap \mathring{\mathsf{D}}_0(\Omega)$ by Lemma 2.24. Therefore, $R(\mathrm{A}_1)$ is closed. For the adjoint operators we get

$$\mathrm{A}_0' = -\operatorname{div} = \mathring{\operatorname{grad}}\,' : \mathsf{H}^{-m}(\Omega) \to \mathsf{H}^{-m-1}(\Omega), \quad \mathrm{A}_1' = \operatorname{rot} = \mathring{\operatorname{rot}}\,' : \mathsf{H}^{-m+1}(\Omega) \to \mathsf{H}^{-m}(\Omega)$$

and obtain from Lemma 2.13 that

$$\mathsf{D}_0^{-m}(\Omega) = N(\mathrm{A}_0') = R(\mathrm{A}_1') = \operatorname{rot}\mathsf{H}^{-m+1}(\Omega)$$

is closed and

$$\operatorname{rot}^{-1} = (\mathrm{A}_1')^{-1} \in BL\big(R(\mathrm{A}_1'), R(\mathrm{A}_1)'\big) = BL\big(\mathsf{D}_0^{-m}(\Omega), (\mathring{\mathsf{H}}^{m-1}(\Omega) \cap \mathring{\mathsf{D}}_0(\Omega))'\big),$$

which completes the proof for general m. If $m = 1$, we identify $(\mathring{\mathsf{H}}^{0}(\Omega) \cap \mathring{\mathsf{D}}_0(\Omega))' = \mathring{\mathsf{D}}_0(\Omega)' = \mathring{\mathsf{D}}_0(\Omega)$ and get the assertions about the Friedrichs/Poincaré/Nečas inequality and inf-sup-condition by Remark 2.15, i.e., (2.9) and (2.11). □

For completeness let us present the corresponding result for the divergence as well.

Corollary 2.29. *Let Ω be additionally topologically trivial and $m \in \mathbb{N}$. Then*

$$\mathsf{H}^{-m}(\Omega) = \operatorname{div}\mathsf{H}^{-m+1}(\Omega) = \operatorname{div}\big(\mathring{\mathsf{H}}^{m-1}(\Omega) \cap \mathring{\mathsf{R}}_0(\Omega)\big)'$$

(is closed in $\mathsf{H}^{-m}(\Omega)$) with continuous inverse, i.e., $\operatorname{div}^{-1} \in BL\big(\mathsf{H}^{-m}(\Omega), (\mathring{\mathsf{H}}^{m-1}(\Omega) \cap \mathring{\mathsf{R}}_0(\Omega))'\big)$. Especially for $m = 1$,

$$\mathsf{H}^{-1}(\Omega) = \operatorname{div}\mathsf{L}^2(\Omega) = \operatorname{div}\mathring{\mathsf{R}}_0(\Omega)$$

(is closed in $\mathsf{H}^{-1}(\Omega)$) with continuous inverse $\operatorname{div}^{-1} \in BL\big(\mathsf{H}^{-1}(\Omega), \mathring{\mathsf{R}}_0(\Omega)\big)$ and uniquely determined potential in $\mathring{\mathsf{R}}_0(\Omega)$. Moreover,

$$\exists\, c_{\mathsf{d},-1} > 0 \quad \forall\, E \in \mathring{\mathsf{R}}_0(\Omega) \qquad |E|_{\mathsf{L}^2(\Omega)} \leq c_{\mathsf{d},-1}|\operatorname{div} E|_{\mathsf{H}^{-1}(\Omega)} \leq c_{\mathsf{d},-1}\,|E|_{\mathsf{L}^2(\Omega)}$$

and the inf-sup-condition

$$0 < \frac{1}{c_{\mathsf{d},-1}} = \inf_{0\neq E\in\mathring{\mathsf{R}}_0(\Omega)} \frac{|\operatorname{div} E|_{\mathsf{H}^{-1}(\Omega)}}{|E|_{\mathsf{L}^2(\Omega)}} = \inf_{0\neq E\in\mathring{\mathsf{D}}_0(\Omega)} \sup_{0\neq u\in\mathring{\mathsf{H}}^1(\Omega)} \frac{\langle E, \operatorname{grad} u\rangle_{\mathsf{L}^2(\Omega)}}{|E|_{\mathsf{L}^2(\Omega)}|\operatorname{grad} u|_{\mathsf{L}^2(\Omega)}}.$$

holds.

Proof. Let $\mathsf{X}_1 := \mathring{\mathsf{H}}^{m}(\Omega)$, $\mathsf{X}_2 := \mathring{\mathsf{H}}^{m-1}(\Omega)$ and $\mathrm{A}_1 := -\mathring{\operatorname{grad}} : \mathring{\mathsf{H}}^{m}(\Omega) \to \mathring{\mathsf{H}}^{m-1}(\Omega)$. A_1 is linear and bounded with $R(\mathrm{A}_1) = \operatorname{grad}\mathring{\mathsf{H}}^{m}(\Omega) = \mathring{\mathsf{H}}^{m-1}(\Omega) \cap \mathring{\mathsf{R}}_0(\Omega)$ by Lemma 2.23. Therefore, $R(\mathrm{A}_1)$ is closed. The adjoint is $\mathrm{A}_1' = \operatorname{div} = -\mathring{\operatorname{grad}}\,' : \mathsf{H}^{-m+1}(\Omega) \to \mathsf{H}^{-m}(\Omega)$ with closed range $R(\mathrm{A}_1') = \operatorname{div}\mathsf{H}^{-m+1}(\Omega)$ by the closed range theorem. Moreover, $N(\mathrm{A}_1) = \{0\}$. Hence A_1' is surjective as A_1 is injective, i.e.,

$$\mathsf{H}^{-m}(\Omega) = N(\mathrm{A}_1)^\circ = R(\mathrm{A}_1') = \operatorname{div}\mathsf{H}^{-m+1}(\Omega).$$

As A_1 is also surjective onto its range, $\mathrm{A}_1' = \operatorname{div} : \mathsf{H}^{-m+1}(\Omega) \to R(\mathrm{A}_1')$ is bijective. By the bounded inverse theorem we get

$$\operatorname{div}^{-1} = (\mathrm{A}_1')^{-1} \in BL\big(R(\mathrm{A}_1'), R(\mathrm{A}_1)'\big) = BL\big(\mathsf{H}^{-m}(\Omega), (\mathring{\mathsf{H}}^{m-1}(\Omega) \cap \mathring{\mathsf{R}}_0(\Omega))'\big),$$

which completes the proof[ii] for general m. If $m = 1$, we identify $(\mathring{\mathsf{H}}^0(\Omega) \cap \mathring{\mathsf{R}}_0(\Omega))' = \mathring{\mathsf{R}}_0(\Omega)' = \mathring{\mathsf{R}}_0(\Omega)$ and get the assertions about the Friedrichs/Poincaré/Nečas inequality and inf-sup-condition by Remark 2.15, i.e., (2.9) and (2.11). □

Remark 2.30. *The results of the latter three lemmas and corollaries can be formulated equivalently in terms of complexes: Let Ω be additionally topologically trivial and $m \in \mathbb{N}$. Then the sequence*

$$\{0\} \xrightarrow{\;0\;} \mathring{\mathsf{H}}^{m+1}(\Omega) \xrightarrow{\;\mathring{\operatorname{grad}}\;} \mathring{\mathsf{H}}^{m}(\Omega) \xrightarrow{\;\mathring{\operatorname{rot}}\;} \mathring{\mathsf{H}}^{m-1}(\Omega) \xrightarrow{\;\mathring{\operatorname{div}}\;} \mathring{\mathsf{H}}^{m-2}(\Omega) \xrightarrow{\;\pi_{\mathbb{R}}\;} \mathbb{R}$$

and thus also its dual or adjoint sequence

$$\{0\} \xleftarrow{\;0\;} \mathsf{H}^{-m-1}(\Omega) \xleftarrow{\;-\operatorname{div}\;} \mathsf{H}^{-m}(\Omega) \xleftarrow{\;\operatorname{rot}\;} \mathsf{H}^{-m+1}(\Omega) \xleftarrow{\;-\operatorname{grad}\;} \mathsf{H}^{-m+2}(\Omega) \xleftarrow{\;\iota_{\mathbb{R}}\;} \mathbb{R}$$

are closed and exact Banach complexes.

3. The Grad grad- and div Div-Complexes

We will use the following standard notations from linear algebra. For vectors $a, b \in \mathbb{R}^3$ and matrices $A, B \in \mathbb{R}^{3\times3}$ the expressions $a \cdot b$ and $A : B$ denote the inner product of vectors and the Frobenius inner product of matrices, respectively. For a vector $a \in \mathbb{R}^3$ with components a_i for $i = 1, 2, 3$ the matrix $\operatorname{spn} a \in \mathbb{R}^{3\times3}$ is defined by

$$\operatorname{spn} a = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}.$$

Observe that $(\operatorname{spn} a)\, b = a \times b$ for $a, b \in \mathbb{R}^3$, where $a \times b$ denotes the exterior product of vectors. The exterior product $a \times B$ of a vector $a \in \mathbb{R}^3$ and a matrix $B \in \mathbb{R}^{3\times3}$ is defined as the matrix which is obtained by applying the exterior product row-wise. In addition to $\operatorname{sym} A$ and $\operatorname{skw} A$ for the symmetric part and the skew-symmetric part of a matrix A, we use $\operatorname{dev} A$ and $\operatorname{tr} A$ for denoting the deviatoric part and the trace of a matrix A. Finally, the set of symmetric matrices in $\mathbb{R}^{3\times3}$ is denoted by $\mathbb{S}$, the set of matrices in $\mathbb{R}^{3\times3}$ with vanishing trace is denoted by $\mathbb{T}$.

In this section we need several spaces of tensor fields. The spaces

$$\mathring{\mathbf{C}}^\infty(\Omega), \quad \mathbf{L}^2(\Omega), \quad \mathbf{H}^1(\Omega), \quad \mathring{\mathbf{H}}^1(\Omega), \quad \mathbf{D}(\Omega), \quad \mathring{\mathbf{D}}(\Omega), \quad \mathring{\mathbf{R}}_0(\Omega), \quad \dots$$

are introduced as those spaces of tensor fields, whose rows are in the corresponding spaces of vector fields $\mathring{\mathsf{C}}^\infty(\Omega)$, $\mathsf{L}^2(\Omega)$, $\mathsf{H}^1(\Omega)$, $\mathring{\mathsf{H}}^1(\Omega)$, $\mathsf{D}(\Omega)$, $\mathring{\mathsf{D}}(\Omega)$, $\mathring{\mathsf{R}}_0(\Omega)$, …, respectively. Additionally, we will need spaces allowing for a deviatoric gradient, a symmetric rotation, and a double divergence, i.e.,

$$\mathsf{G}_{\operatorname{dev}}(\Omega) := \big\{E \in \mathsf{L}^2(\Omega) \,:\, \operatorname{dev} \operatorname{Grad} E \in \mathbf{L}^2(\Omega)\big\}, \qquad \mathsf{G}_{\operatorname{dev},0}(\Omega) := \big\{E \in \mathsf{L}^2(\Omega) \,:\, \operatorname{dev} \operatorname{Grad} E = 0\big\},$$

[ii]An alternative proof using Lemma 2.13: Let $\mathsf{X}_0 := \mathring{\mathsf{H}}^{m+1}(\Omega)$, $\mathsf{X}_1 := \mathring{\mathsf{H}}^{m}(\Omega)$, $\mathsf{X}_2 := \mathring{\mathsf{H}}^{m-1}(\Omega)$ and

$$\mathrm{A}_0 := 0 : \mathring{\mathsf{H}}^{m+1}(\Omega) \to \mathring{\mathsf{H}}^{m}(\Omega), \quad \mathrm{A}_1 := -\mathring{\operatorname{grad}} : \mathring{\mathsf{H}}^{m}(\Omega) \to \mathring{\mathsf{H}}^{m-1}(\Omega).$$

These linear operators are bounded, $R(\mathrm{A}_0) = \{0\} = N(\mathrm{A}_1)$, and $R(\mathrm{A}_1) = \operatorname{grad} \mathring{\mathsf{H}}^m(\Omega) = \mathring{\mathsf{H}}^{m-1}(\Omega) \cap \mathring{\mathsf{R}}_0(\Omega)$ by Lemma 2.23. Therefore, $R(\mathrm{A}_1)$ is closed. For the adjoint operators we get

$$\mathrm{A}_0' = 0 = 0' : \mathsf{H}^{-m}(\Omega) \to \mathsf{H}^{-m-1}(\Omega), \quad \mathrm{A}_1' = \operatorname{div} = -\mathring{\operatorname{grad}}\,' : \mathsf{H}^{-m+1}(\Omega) \to \mathsf{H}^{-m}(\Omega)$$

and obtain from Lemma 2.13 that

$$\mathsf{H}^{-m}(\Omega) = N(\mathrm{A}_0') = R(\mathrm{A}_1') = \operatorname{div} \mathsf{H}^{-m+1}(\Omega)$$

is closed and

$$\operatorname{div}^{-1} = (\mathrm{A}_1')^{-1} \in BL\big(R(\mathrm{A}_1'), R(\mathrm{A}_1)'\big) = BL\big(\mathsf{H}^{-m}(\Omega), (\mathring{\mathsf{H}}^{m-1}(\Omega) \cap \mathring{\mathsf{R}}_0(\Omega))'\big).$$

$$\mathsf{R}_{\mathrm{sym}}(\Omega) := \big\{\mathbf{N} \in \mathsf{L}^2(\Omega) \,:\, \operatorname{sym}\operatorname{Rot}\mathbf{N} \in \mathsf{L}^2(\Omega)\big\}, \qquad \mathsf{R}_{\mathrm{sym},0}(\Omega) := \big\{\mathbf{N} \in \mathsf{L}^2(\Omega) \,:\, \operatorname{sym}\operatorname{Rot}\mathbf{N} = 0\big\},$$
$$\mathsf{D}\mathbf{D}(\Omega) := \big\{\mathbf{M} \in \mathsf{L}^2(\Omega) \,:\, \operatorname{div}\operatorname{Div}\mathbf{M} \in \mathsf{L}^2(\Omega)\big\}, \qquad \mathsf{D}\mathbf{D}_0(\Omega) := \big\{\mathbf{M} \in \mathsf{L}^2(\Omega) \,:\, \operatorname{div}\operatorname{Div}\mathbf{M} = 0\big\}.$$

Moreover, we introduce various spaces of symmetric[iii] tensor fields without prescribed boundary conditions, i.e.,

$$\mathsf{L}^2(\Omega,\mathbb{S}) := \{\mathbf{M} \in \mathsf{L}^2(\Omega) \,:\, \mathbf{M}^\top = \mathbf{M}\}, \quad \mathsf{D}\mathbf{D}(\Omega,\mathbb{S}) := \mathsf{D}\mathbf{D}(\Omega) \cap \mathsf{L}^2(\Omega,\mathbb{S}), \quad \dots,$$

and with homogeneous boundary conditions as closures of symmetric test tensor fields, i.e.,

$$\mathring{\mathsf{R}}(\Omega,\mathbb{S}) := \overline{\mathring{\mathsf{C}}^\infty(\Omega) \cap \mathsf{L}^2(\Omega,\mathbb{S})}^{\mathsf{R}(\Omega)}, \quad \dots,$$

as well as spaces of tensor fields with vanishing trace[iv] and without prescribed boundary conditions, i.e.,

$$\mathsf{L}^2(\Omega,\mathbb{T}) := \{\mathbf{N} \in \mathsf{L}^2(\Omega) \,:\, \operatorname{tr}\mathbf{N} = 0\}, \quad \mathsf{H}^1(\Omega,\mathbb{T}) := \mathsf{H}^1(\Omega) \cap \mathsf{L}^2(\Omega,\mathbb{T}), \quad \dots,$$

and with homogeneous boundary conditions as closures of trace-free test tensor fields, i.e.,

$$\mathring{\mathsf{D}}(\Omega,\mathbb{T}) := \overline{\mathring{\mathsf{C}}^\infty(\Omega) \cap \mathsf{L}^2(\Omega,\mathbb{T})}^{\mathsf{D}(\Omega)}, \quad \dots.$$

We note

$$\mathring{\mathsf{R}}(\Omega,\mathbb{S}) \subset \mathring{\mathsf{R}}(\Omega) \cap \mathsf{L}^2(\Omega,\mathbb{S}), \quad \mathring{\mathsf{D}}(\Omega,\mathbb{T}) \subset \mathring{\mathsf{D}}(\Omega) \cap \mathsf{L}^2(\Omega,\mathbb{T}), \quad \dots.$$

Let us also mention that

$$\operatorname{dev}\operatorname{Grad}\mathsf{G}_{\mathrm{dev}}(\Omega) \subset \mathsf{L}^2(\Omega,\mathbb{T}), \quad \operatorname{sym}\operatorname{Rot}\mathsf{R}_{\mathrm{sym}}(\Omega) \subset \mathsf{L}^2(\Omega,\mathbb{S})$$

hold. This can be seen as follows. Pick $\Phi \in \mathsf{G}_{\mathrm{dev}}(\Omega)$ with $\mathbf{N} := \operatorname{dev}\operatorname{Grad}\Phi$ and $\mathbf{\Phi} \in \mathsf{R}_{\mathrm{sym}}(\Omega)$ with $\mathbf{M} := \operatorname{sym}\operatorname{Rot}\mathbf{\Phi}$. Then for all $\psi \in \mathring{\mathsf{C}}^\infty(\Omega)$ and $\mathbf{\Psi} \in \mathring{\mathsf{C}}^\infty(\Omega)$

$$\langle \operatorname{tr}\mathbf{N}, \psi\rangle_{\mathsf{L}^2(\Omega)} = \langle \mathbf{N}, \psi\,\mathbf{I}\rangle_{\mathsf{L}^2(\Omega)} = -\langle \Phi, \operatorname{Div}\operatorname{dev}\psi\,\mathbf{I}\rangle_{\mathsf{L}^2(\Omega)} = 0,$$
$$\langle \operatorname{skw}\mathbf{M}, \mathbf{\Psi}\rangle_{\mathsf{L}^2(\Omega)} = \langle \mathbf{M}, \operatorname{skw}\mathbf{\Psi}\rangle_{\mathsf{L}^2(\Omega)} = \langle \mathbf{\Phi}, \operatorname{Rot}\operatorname{sym}\operatorname{skw}\mathbf{\Psi}\rangle_{\mathsf{L}^2(\Omega)} = 0.$$

Before we proceed we need a few technical lemmas.

Lemma 3.1. *For any distributional vector field E it holds for $i, j, k = 1, \dots, 3$*

$$\partial_k(\operatorname{Grad}E)_{ij} = \begin{cases} \partial_k(\operatorname{dev}\operatorname{Grad}E)_{ij} & , \text{ if } i \neq j, \\ \partial_j(\operatorname{dev}\operatorname{Grad}E)_{ik} & , \text{ if } i \neq k, \\ \dfrac{3}{2}\,\partial_i(\operatorname{dev}\operatorname{Grad}E)_{ii} + \dfrac{1}{2}\displaystyle\sum_{l\neq i}\partial_l(\operatorname{dev}\operatorname{Grad}E)_{li} & , \text{ if } i = j = k. \end{cases}$$

Proof. Let $\Phi \in \mathring{\mathsf{C}}^\infty(\mathbb{R}^3)$ be a vector field. We want to express the second derivatives of Φ by the derivatives of the deviatoric part of the Jacobian, i.e., of $\operatorname{dev}\operatorname{Grad}\Phi$. Recall that we have $\operatorname{dev}\mathbf{M} = \mathbf{M} - \frac{1}{3}(\operatorname{tr}\mathbf{M})\,\mathbf{I}$ for a tensor $\mathbf{M}$. Hence $\operatorname{dev}\operatorname{Grad}\Phi$ coincides with $\operatorname{Grad}\Phi$ outside the diagonal entries, i.e., we have $(\operatorname{Grad}\Phi)_{ij} = (\operatorname{dev}\operatorname{Grad}\Phi)_{ij}$ for $i \neq j$. Hence, looking at second derivatives, we see immediately

$$\partial_k\,\partial_j\,\Phi_i = \partial_k(\operatorname{Grad}\Phi)_{ij} = \partial_k(\operatorname{dev}\operatorname{Grad}\Phi)_{ij} \qquad \text{for } i \neq j,$$
$$\partial_k\,\partial_j\,\Phi_i = \partial_j\,\partial_k\,\Phi_i = \partial_j(\operatorname{Grad}\Phi)_{ik} = \partial_j(\operatorname{dev}\operatorname{Grad}\Phi)_{ik} \qquad \text{for } i \neq k.$$

Thus it remains to represent $\partial_i^2\,\Phi_i$ by the derivatives of $\operatorname{dev}\operatorname{Grad}\Phi$. By

$$\partial_i^2\,\Phi_i = \partial_i(\operatorname{Grad}\Phi)_{ii} = \partial_i(\operatorname{dev}\operatorname{Grad}\Phi)_{ii} + \frac{1}{3}\,\partial_i\operatorname{div}\Phi$$

[iii]By $\operatorname{sym}\mathsf{L}^2(\Omega) \subset \mathsf{L}^2(\Omega,\mathbb{S}) = \operatorname{sym}\mathsf{L}^2(\Omega,\mathbb{S}) \subset \operatorname{sym}\mathsf{L}^2(\Omega)$ we see $\mathsf{L}^2(\Omega,\mathbb{S}) = \operatorname{sym}\mathsf{L}^2(\Omega)$.

[iv]By $\operatorname{dev}\mathsf{L}^2(\Omega) \subset \mathsf{L}^2(\Omega,\mathbb{T}) = \operatorname{dev}\mathsf{L}^2(\Omega,\mathbb{T}) \subset \operatorname{dev}\mathsf{L}^2(\Omega)$ we see $\mathsf{L}^2(\Omega,\mathbb{T}) = \operatorname{dev}\mathsf{L}^2(\Omega)$.

we get

$$\frac{2}{3}\,\partial_i^2\,\Phi_i = \partial_i(\operatorname{dev}\operatorname{Grad}\Phi)_{ii} + \frac{1}{3}\sum_{l\neq i}\partial_i\,\partial_l\,\Phi_l = \partial_i(\operatorname{dev}\operatorname{Grad}\Phi)_{ii} + \frac{1}{3}\sum_{l\neq i}\partial_l(\operatorname{dev}\operatorname{Grad}\Phi)_{li},$$

yielding the stated result for test vector fields. Testing extends the formulas to distributions, which finishes the proof. □

We note that the latter trick is similar to the well known fact that second derivatives of a vector field can always be written as derivatives of the symmetric gradient of the vector field, leading by Nečas estimate to Korn's second and first inequalities. We will now do the same for the operator dev Grad.

Lemma 3.2. *It holds:*

(i) *There exists $c > 0$, such that for all vector fields $E \in \mathsf{H}^1(\Omega)$*

$$|\operatorname{Grad} E|_{\mathsf{L}^2(\Omega)} \leq c\,\big(|E|_{\mathsf{L}^2(\Omega)} + |\operatorname{dev}\operatorname{Grad} E|_{\mathsf{L}^2(\Omega)}\big).$$

(ii) $\mathsf{G}_{\operatorname{dev}}(\Omega) = \mathsf{H}^1(\Omega)$.

(iii) *For* $\operatorname{dev}\operatorname{Grad} : \mathsf{G}_{\operatorname{dev}}(\Omega) \subset \mathsf{L}^2(\Omega) \longrightarrow \mathbf{L}^2(\Omega,\mathbb{T})$ *it holds* $D(\operatorname{dev}\operatorname{Grad}) = \mathsf{G}_{\operatorname{dev}}(\Omega) = \mathsf{H}^1(\Omega)$, *and the kernel of* dev Grad *equals the space of (global) shape functions of the lowest order Raviart-Thomas elements, i.e.,*

$$N(\operatorname{dev}\operatorname{Grad}) = \mathsf{G}_{\operatorname{dev},0}(\Omega) = \mathsf{RT}_0 := \{P \,:\, P(x) = a\,x + b,\ a \in \mathbb{R},\ b \in \mathbb{R}^3\},$$

which dimension is $\dim \mathsf{RT}_0 = 4$.

(iv) *There exists $c > 0$, such that for all vector fields $E \in \mathsf{H}^1(\Omega) \cap \mathsf{RT}_0^{\perp_{\mathsf{L}^2(\Omega)}}$*

$$|E|_{\mathsf{H}^1(\Omega)} \leq c\,|\operatorname{dev}\operatorname{Grad} E|_{\mathsf{L}^2(\Omega)}.$$

Proof. Let $E \in \mathsf{H}^1(\Omega)$. By the latter lemma and Nečas estimate, i.e.,

$$\exists\, c > 0 \quad \forall\, u \in \mathsf{L}^2(\Omega) \qquad c\,|u|_{\mathsf{L}^2(\Omega)} \leq |\operatorname{grad} u|_{\mathsf{H}^{-1}(\Omega)} + |u|_{\mathsf{H}^{-1}(\Omega)} \leq (\sqrt{3}+1)|u|_{\mathsf{L}^2(\Omega)},$$

we get

$$\begin{aligned}
|\operatorname{Grad} E|_{\mathbf{L}^2(\Omega)} &\leq c\,\Big(\sum_{k=1}^{3}|\,\partial_k \operatorname{Grad} E|_{\mathbf{H}^{-1}(\Omega)} + |\operatorname{Grad} E|_{\mathbf{H}^{-1}(\Omega)}\Big)\\
&\leq c\,\Big(\sum_{k=1}^{3}|\,\partial_k \operatorname{dev}\operatorname{Grad} E|_{\mathbf{H}^{-1}(\Omega)} + |\operatorname{Grad} E|_{\mathbf{H}^{-1}(\Omega)}\Big)\\
&\leq c\,\big(|\operatorname{dev}\operatorname{Grad} E|_{\mathbf{L}^2(\Omega)} + |E|_{\mathbf{L}^2(\Omega)}\big),
\end{aligned}$$

which shows (i). As Ω has the segment property and by standard mollification we obtain that restrictions of $\mathring{\mathsf{C}}^\infty(\mathbb{R}^3)$-vector fields are dense in $\mathsf{G}_{\operatorname{dev}}(\Omega)$. Especially $\mathsf{H}^1(\Omega)$ is dense in $\mathsf{G}_{\operatorname{dev}}(\Omega)$. Let $E \in \mathsf{G}_{\operatorname{dev}}(\Omega)$ and $(E_n) \subset \mathsf{H}^1(\Omega)$ with $E_n \to E$ in $\mathsf{G}_{\operatorname{dev}}(\Omega)$. By (i) (E_n) is a Cauchy sequence in $\mathsf{H}^1(\Omega)$ converging to E in $\mathsf{H}^1(\Omega)$, which proves $E \in \mathsf{H}^1(\Omega)$ and hence (ii). For $P \in \mathsf{RT}_0$ it holds $\operatorname{dev}\operatorname{Grad} P = a \operatorname{dev}\mathbf{I} = 0$. Let $\operatorname{dev}\operatorname{Grad} E = 0$ for some vector field $E \in \mathsf{G}_{\operatorname{dev}}(\Omega) = \mathsf{H}^1(\Omega)$. By Lemma 3.1 we get $\partial_k \operatorname{Grad} E = 0$ for all $k = 1,\ldots,3$, and therefore $E(x) = A\,x + b$ for some matrix $A \in \mathbb{R}^{3\times 3}$ and vector $b \in \mathbb{R}^3$. Then $0 = \operatorname{dev}\operatorname{Grad} E = \operatorname{dev} A$, if and only if $A = \frac{1}{3}(\operatorname{tr} A)\,\mathbf{I}$, which shows (iii). If (iv) was wrong, there exists a sequence $(E_n) \subset \mathsf{H}^1(\Omega) \cap \mathsf{RT}_0^{\perp_{\mathsf{L}^2(\Omega)}}$ with $|E_n|_{\mathsf{H}^1(\Omega)} = 1$ and $\operatorname{dev}\operatorname{Grad} E_n \to 0$. As (E_n) is bounded in $\mathsf{H}^1(\Omega)$, by Rellich's selection theorem there exists a subsequence, again denoted by (E_n), and some $E \in \mathsf{L}^2(\Omega)$ with $E_n \to E$ in $\mathsf{L}^2(\Omega)$. By (i), (E_n) is a Cauchy sequence in $\mathsf{H}^1(\Omega)$. Hence $E_n \to E$ in $\mathsf{H}^1(\Omega)$ and $E \in \mathsf{H}^1(\Omega) \cap \mathsf{RT}_0^{\perp_{\mathsf{L}^2(\Omega)}}$. As $0 \leftarrow \operatorname{dev}\operatorname{Grad} E_n \to \operatorname{dev}\operatorname{Grad} E$, we have by (iii) $E \in \mathsf{RT}_0 \cap \mathsf{RT}_0^{\perp_{\mathsf{L}^2(\Omega)}} = \{0\}$, a contradiction to $1 = |E_n|_{\mathsf{H}^1(\Omega)} \to 0$. The proof is complete. □

Lemma 3.3. *Let* $\mathsf{GG}(\Omega) := \{u \in \mathsf{L}^2(\Omega) : \operatorname{Grad}\operatorname{grad} u \in \mathbf{L}^2(\Omega)\}$ *and* $\mathring{\mathsf{GG}}(\Omega) := \overline{\mathring{\mathsf{C}}^\infty(\Omega)}^{\mathsf{GG}(\Omega)}$. *Then*

$$\mathring{\mathsf{GG}}(\Omega) = \mathring{\mathsf{H}}^2(\Omega)$$

and there exists $c > 0$ *such that for all* $u \in \mathring{\mathsf{H}}^2(\Omega)$

$$|u|_{\mathsf{H}^2(\Omega)} \leq c\,|\Delta u|_{\mathsf{L}^2(\Omega)}.$$

Proof. Let $\varphi \in \mathring{\mathsf{C}}^\infty(\Omega)$. Then by the Friedrichs/Poincaré inequality, i.e., $|\varphi|_{\mathsf{L}^2(\Omega)} \leq c_{\mathsf{g}}\,|\operatorname{grad}\varphi|_{\mathsf{L}^2(\Omega)}$, and

$$|\operatorname{grad}\varphi|^2_{\mathsf{L}^2(\Omega)} = -\langle \varphi, \Delta\varphi\rangle_{\mathsf{L}^2(\Omega)} \leq c_{\mathsf{g}}\,|\operatorname{grad}\varphi|_{\mathsf{L}^2(\Omega)}|\Delta\varphi|_{\mathsf{L}^2(\Omega)},$$

we obtain $|\operatorname{grad}\varphi|_{\mathsf{L}^2(\Omega)} \leq c_{\mathsf{g}}\,|\Delta\varphi|_{\mathsf{L}^2(\Omega)}$. Moreover, by

$$\sum_{i,j=1}^{3} |\partial_i\,\partial_j\,\varphi|^2_{\mathsf{L}^2(\Omega)} = \sum_{i,j=1}^{3} \langle \partial_i\,\partial_i\,\varphi, \partial_i\,\partial_i\,\varphi\rangle_{\mathsf{L}^2(\Omega)} = |\Delta\varphi|^2_{\mathsf{L}^2(\Omega)}$$

we see

$$|\varphi|^2_{\mathsf{H}^2(\Omega)} \leq \big(1 + c_{\mathsf{g}}^2(1 + c_{\mathsf{g}}^2)\big)|\Delta\varphi|^2_{\mathsf{L}^2(\Omega)}. \tag{3.1}$$

Let $u \in \mathring{\mathsf{GG}}(\Omega)$ and $(u_n) \subset \mathring{\mathsf{C}}^\infty(\Omega)$ with $u_n \to u$ in $\mathsf{GG}(\Omega)$. By (3.1), (u_n) is a Cauchy sequence in $\mathsf{H}^2(\Omega)$ converging to u in $\mathsf{H}^2(\Omega)$, which proves $u \in \mathring{\mathsf{H}}^2(\Omega)$. By continuity, (3.1) holds for $u \in \mathring{\mathsf{H}}^2(\Omega)$, which finishes the proof. □

By straight forward calculations and standard arguments for distributions, see the Appendix, we get the following.

Lemma 3.4. *It holds:*

- **(i)** $\operatorname{skw}\operatorname{Grad}\operatorname{grad}\mathsf{H}^2(\Omega) = 0$, *i.e., Hessians are symmetric.*
- **(ii)** $\operatorname{tr}\operatorname{Rot}\mathbf{R}(\Omega, \mathbb{S}) = 0$, *i.e., rotations of symmetric tensors are trace free.*

These formulas extend to distributions as well.

With Lemma 3.3 and Lemma 3.4 let us now consider the linear operators

$$\mathring{\operatorname{Grad}}\operatorname{grad} : \mathring{\mathsf{GG}}(\Omega) = \mathring{\mathsf{H}}^2(\Omega) \subset \mathsf{L}^2(\Omega) \longrightarrow \mathbf{L}^2(\Omega, \mathbb{S}), \qquad u \mapsto \operatorname{Grad}\operatorname{grad} u, \tag{3.2}$$

$$\mathring{\operatorname{Rot}}_{\mathbb{S}} : \mathring{\mathbf{R}}(\Omega, \mathbb{S}) \subset \mathbf{L}^2(\Omega, \mathbb{S}) \longrightarrow \mathbf{L}^2(\Omega, \mathbb{T}), \qquad \mathbf{M} \mapsto \operatorname{Rot}\mathbf{M}, \tag{3.3}$$

$$\mathring{\operatorname{Div}}_{\mathbb{T}} : \mathring{\mathbf{D}}(\Omega, \mathbb{T}) \subset \mathbf{L}^2(\Omega, \mathbb{T}) \longrightarrow \mathsf{L}^2(\Omega), \qquad \mathbf{N} \mapsto \operatorname{Div}\mathbf{N}. \tag{3.4}$$

These are well and densely defined and closed. Closedness is clear. For densely definedness we look e.g. at $\mathring{\operatorname{Rot}}_{\mathbb{S}}$. For $\mathbf{M} \in \mathbf{L}^2(\Omega, \mathbb{S})$ pick $(\mathbf{\Phi}_n) \subset \mathring{\mathbf{C}}^\infty(\Omega)$ with $\mathbf{\Phi}_n \to \mathbf{M}$ in $\mathbf{L}^2(\Omega)$. Then

$$|\mathbf{M} - \operatorname{sym}\mathbf{\Phi}_n|^2_{\mathbf{L}^2(\Omega)} + |\operatorname{skw}\mathbf{\Phi}_n|^2_{\mathbf{L}^2(\Omega)} = |\mathbf{M} - \mathbf{\Phi}_n|^2_{\mathbf{L}^2(\Omega)} \to 0,$$

showing $(\operatorname{sym}\mathbf{\Phi}_n) \subset \mathring{\mathbf{C}}^\infty(\Omega) \cap \mathbf{L}^2(\Omega, \mathbb{S}) \subset \mathring{\mathbf{R}}(\Omega, \mathbb{S})$ and $\operatorname{sym}\mathbf{\Phi}_n \to \mathbf{M}$ in $\mathbf{L}^2(\Omega, \mathbb{S})$. By Lemma 3.3 the kernels are

$$N(\mathring{\operatorname{Grad}}\operatorname{grad}) = \{0\}, \qquad N(\mathring{\operatorname{Rot}}_{\mathbb{S}}) = \mathring{\mathbf{R}}_0(\Omega, \mathbb{S}), \qquad N(\mathring{\operatorname{Div}}_{\mathbb{T}}) = \mathring{\mathbf{D}}_0(\Omega, \mathbb{T}).$$

Lemma 3.5. *The adjoints of* (3.2), (3.3), (3.4) *are*

$$(\mathring{\operatorname{Grad}}\operatorname{grad})^* = \operatorname{div}\operatorname{Div}_{\mathbb{S}} : \mathsf{D}\mathbf{D}(\Omega, \mathbb{S}) \subset \mathbf{L}^2(\Omega, \mathbb{S}) \longrightarrow \mathsf{L}^2(\Omega), \qquad \mathbf{M} \mapsto \operatorname{div}\operatorname{Div}\mathbf{M},$$

$$(\mathring{\operatorname{Rot}}_{\mathbb{S}})^* = \operatorname{sym}\operatorname{Rot}_{\mathbb{T}} : \mathbf{R}_{\mathrm{sym}}(\Omega, \mathbb{T}) \subset \mathbf{L}^2(\Omega, \mathbb{T}) \longrightarrow \mathbf{L}^2(\Omega, \mathbb{S}), \qquad \mathbf{N} \mapsto \operatorname{sym}\operatorname{Rot}\mathbf{N},$$

$$(\mathring{\operatorname{Div}}_{\mathbb{T}})^* = -\operatorname{dev}\operatorname{Grad} : \mathsf{G}_{\mathrm{dev}}(\Omega) = \mathsf{H}^1(\Omega) \subset \mathsf{L}^2(\Omega) \longrightarrow \mathbf{L}^2(\Omega, \mathbb{T}), \qquad E \mapsto -\operatorname{dev}\operatorname{Grad} E.$$

with kernels

$$N(\operatorname{div}\operatorname{Div}_{\mathbb{S}})=\mathsf{D}\mathbf{D}_0(\Omega,\mathbb{S}),\qquad N(\operatorname{sym}\operatorname{Rot}_{\mathbb{T}})=\mathbf{R}_{\mathrm{sym},0}(\Omega,\mathbb{T}),\qquad N(\operatorname{dev}\operatorname{Grad})=\mathsf{RT}_0.$$

Proof. We have $\mathbf{M}\in D\big((\operatorname{Grad}\mathring{}\operatorname{grad})^*\big)\subset\mathbf{L}^2(\Omega,\mathbb{S})$ and $(\operatorname{Grad}\mathring{}\operatorname{grad})^*\mathbf{M}=u\in\mathsf{L}^2(\Omega)$, if and only if $\mathbf{M}\in\mathbf{L}^2(\Omega,\mathbb{S})$ and there exists $u\in\mathsf{L}^2(\Omega)$, such that

$$\begin{aligned}
&\forall\,\varphi\in D(\operatorname{Grad}\mathring{}\operatorname{grad})=\mathring{\mathsf{H}}^2(\Omega) && \langle\operatorname{Grad}\operatorname{grad}\varphi,\mathbf{M}\rangle_{\mathbf{L}^2(\Omega,\mathbb{S})}=\langle\varphi,u\rangle_{\mathsf{L}^2(\Omega)}\\
\Leftrightarrow\quad&\forall\,\varphi\in\mathring{\mathsf{C}}^\infty(\Omega) && \langle\operatorname{Grad}\operatorname{grad}\varphi,\mathbf{M}\rangle_{\mathbf{L}^2(\Omega)}=\langle\varphi,u\rangle_{\mathsf{L}^2(\Omega)},
\end{aligned}$$

if and only if $\mathbf{M}\in\mathsf{D}\mathbf{D}(\Omega)\cap\mathbf{L}^2(\Omega,\mathbb{S})=\mathsf{D}\mathbf{D}(\Omega,\mathbb{S})$ and $\operatorname{div}\operatorname{Div}\mathbf{M}=u$. Moreover, we observe that $\mathbf{N}\in D\big((\mathring{\operatorname{Rot}}_{\mathbb{S}})^*\big)\subset\mathbf{L}^2(\Omega,\mathbb{T})$ and $(\mathring{\operatorname{Rot}}_{\mathbb{S}})^*\mathbf{N}=\mathbf{M}\in\mathbf{L}^2(\Omega,\mathbb{S})$, if and only if $\mathbf{N}\in\mathbf{L}^2(\Omega,\mathbb{T})$ and there exists $\mathbf{M}\in\mathbf{L}^2(\Omega,\mathbb{S})$, such that (note $\operatorname{sym}^2=\operatorname{sym}$)

$$\begin{aligned}
&\forall\,\mathbf{\Phi}\in D(\mathring{\operatorname{Rot}}_{\mathbb{S}})=\mathring{\mathbf{R}}(\Omega,\mathbb{S}) && \langle\operatorname{Rot}\mathbf{\Phi},\mathbf{N}\rangle_{\mathbf{L}^2(\Omega,\mathbb{T})}=\langle\mathbf{\Phi},\mathbf{M}\rangle_{\mathbf{L}^2(\Omega,\mathbb{S})}\\
\Leftrightarrow\quad&\forall\,\mathbf{\Phi}\in\mathring{\mathbf{C}}^\infty(\Omega)\cap\mathbf{L}^2(\Omega,\mathbb{S}) && \langle\operatorname{Rot}\operatorname{sym}\mathbf{\Phi},\mathbf{N}\rangle_{\mathbf{L}^2(\Omega)}=\langle\operatorname{sym}\mathbf{\Phi},\mathbf{M}\rangle_{\mathbf{L}^2(\Omega)}\\
\Leftrightarrow\quad&\forall\,\mathbf{\Phi}\in\mathring{\mathbf{C}}^\infty(\Omega) && \langle\operatorname{Rot}\operatorname{sym}\mathbf{\Phi},\mathbf{N}\rangle_{\mathbf{L}^2(\Omega)}=\langle\operatorname{sym}\mathbf{\Phi},\mathbf{M}\rangle_{\mathbf{L}^2(\Omega)}\\
\Leftrightarrow\quad&\forall\,\mathbf{\Phi}\in\mathring{\mathbf{C}}^\infty(\Omega) && \langle\operatorname{Rot}\operatorname{sym}\mathbf{\Phi},\mathbf{N}\rangle_{\mathbf{L}^2(\Omega)}=\langle\mathbf{\Phi},\mathbf{M}\rangle_{\mathbf{L}^2(\Omega)},
\end{aligned}$$

if and only if $\mathbf{N}\in\mathbf{R}_{\mathrm{sym}}(\Omega)\cap\mathbf{L}^2(\Omega,\mathbb{T})=\mathbf{R}_{\mathrm{sym}}(\Omega,\mathbb{T})$ and $\operatorname{sym}\operatorname{Rot}\mathbf{N}=\mathbf{M}$. Similarly, we see that $E\in D\big((\mathring{\operatorname{Div}}_{\mathbb{T}})^*\big)\subset\mathsf{L}^2(\Omega)$ and $(\mathring{\operatorname{Div}}_{\mathbb{T}})^*E=\mathbf{N}\in\mathbf{L}^2(\Omega,\mathbb{T})$, if and only if $E\in\mathsf{L}^2(\Omega)$ and there exists $\mathbf{N}\in\mathbf{L}^2(\Omega,\mathbb{T})$, such that (note $\operatorname{dev}^2=\operatorname{dev}$)

$$\begin{aligned}
&\forall\,\mathbf{\Phi}\in D(\mathring{\operatorname{Div}}_{\mathbb{S}})=\mathring{\mathbf{D}}(\Omega,\mathbb{T}) && \langle\operatorname{Div}\mathbf{\Phi},E\rangle_{\mathsf{L}^2(\Omega)}=\langle\mathbf{\Phi},\mathbf{N}\rangle_{\mathbf{L}^2(\Omega,\mathbb{T})}\\
\Leftrightarrow\quad&\forall\,\mathbf{\Phi}\in\mathring{\mathbf{C}}^\infty(\Omega)\cap\mathbf{L}^2(\Omega,\mathbb{T}) && \langle\operatorname{Div}\operatorname{dev}\mathbf{\Phi},E\rangle_{\mathsf{L}^2(\Omega)}=\langle\operatorname{dev}\mathbf{\Phi},\mathbf{N}\rangle_{\mathbf{L}^2(\Omega)}\\
\Leftrightarrow\quad&\forall\,\mathbf{\Phi}\in\mathring{\mathbf{C}}^\infty(\Omega) && \langle\operatorname{Div}\operatorname{dev}\mathbf{\Phi},E\rangle_{\mathsf{L}^2(\Omega)}=\langle\operatorname{dev}\mathbf{\Phi},\mathbf{N}\rangle_{\mathbf{L}^2(\Omega)}\\
\Leftrightarrow\quad&\forall\,\mathbf{\Phi}\in\mathring{\mathbf{C}}^\infty(\Omega) && \langle\operatorname{Div}\operatorname{dev}\mathbf{\Phi},E\rangle_{\mathsf{L}^2(\Omega)}=\langle\mathbf{\Phi},\mathbf{N}\rangle_{\mathbf{L}^2(\Omega)},
\end{aligned}$$

if and only if $E\in\mathsf{G}_{\mathrm{dev}}(\Omega)=\mathsf{H}^1(\Omega)$ and $-\operatorname{dev}\operatorname{Grad}E=\mathbf{N}$ using Lemma 3.2. Lemma 3.2 also shows $N(\operatorname{dev}\operatorname{Grad})=\mathsf{G}_{\mathrm{dev},0}(\Omega)=\mathsf{RT}_0$, completing the proof. □

Remark 3.6. *Note that, e.g., the second order operator* $\operatorname{Grad}\mathring{}\operatorname{grad}$ *is "one" operator and not a composition of the two first order operators* $\mathring{\operatorname{Grad}}$ *and* $\mathring{\operatorname{grad}}$*. Similarly the operator* $\operatorname{div}\operatorname{Div}_{\mathbb{S}}$*,* $\operatorname{sym}\operatorname{Rot}_{\mathbb{T}}$*, resp.* $\operatorname{dev}\operatorname{Grad}$ *has to be understood as "one" operator.*

Lemma 3.7. *The ranges*

$$R(\operatorname{Grad}\mathring{}\operatorname{grad})=\operatorname{Grad}\operatorname{grad}\mathring{\mathsf{H}}^2(\Omega),\qquad R(\operatorname{div}\operatorname{Div}_{\mathbb{S}})=\operatorname{div}\operatorname{Div}\mathsf{D}\mathbf{D}(\Omega,\mathbb{S}),$$
$$R(\operatorname{dev}\operatorname{Grad})=\operatorname{dev}\operatorname{Grad}\mathsf{H}^1(\Omega)=\operatorname{dev}\operatorname{Grad}\big(\mathsf{H}^1(\Omega)\cap\mathsf{RT}_0^{\perp_{\mathsf{L}^2(\Omega)}}\big),\qquad R(\mathring{\operatorname{Div}}_{\mathbb{T}})=\operatorname{Div}\mathring{\mathbf{D}}(\Omega,\mathbb{T})$$

are closed in $\mathbf{L}^2(\Omega)$ *resp.* $\mathsf{L}^2(\Omega)$.

Proof. For $\mathbf{M}\in\overline{\operatorname{Grad}\operatorname{grad}\mathring{\mathsf{H}}^2(\Omega)}^{\mathbf{L}^2(\Omega)}$ there is a sequence $(u_n)\subset\mathring{\mathsf{H}}^2(\Omega)$ with $\operatorname{Grad}\operatorname{grad}u_n\to\mathbf{M}$. By Lemma 3.3, (u_n) is a Cauchy sequence in $\mathsf{H}^2(\Omega)$, converging to some $u\in\mathring{\mathsf{H}}^2(\Omega)$ in $\mathsf{H}^2(\Omega)$. Therefore, $\mathbf{M}\leftarrow\operatorname{Grad}\operatorname{grad}u_n\to\operatorname{Grad}\operatorname{grad}u\in\operatorname{Grad}\operatorname{grad}\mathring{\mathsf{H}}^2(\Omega)$. Similarly for $\mathbf{N}\in\overline{\operatorname{dev}\operatorname{Grad}\mathsf{H}^1(\Omega)}^{\mathbf{L}^2(\Omega)}$ there is a sequence $(E_n) \subset \mathsf{H}^1(\Omega)$ with $\operatorname{dev}\operatorname{Grad} E_n \to \mathbf{N}$. Let $\pi_{\mathsf{RT}_0} : \mathsf{L}^2(\Omega) \to \mathsf{RT}_0$ be the orthogonal projector onto RT_0 subject to the orthogonal decomposition

$$\mathsf{L}^2(\Omega) = \mathsf{RT}_0 \oplus_{\mathsf{L}^2(\Omega)} \mathsf{RT}_0^{\perp_{\mathsf{L}^2(\Omega)}}.$$

Then $H_n := (1 - \pi_{\mathsf{RT}_0})E_n \in \mathsf{H}^1(\Omega) \cap \mathsf{RT}_0^{\perp_{\mathsf{L}^2(\Omega)}}$ with $\operatorname{dev}\operatorname{Grad} H_n = \operatorname{dev}\operatorname{Grad} E_n \to \mathbf{N}$. By Lemma 3.2 (iv), (H_n) is a Cauchy sequence in $\mathsf{H}^1(\Omega)$, converging to some $H \in \mathsf{H}^1(\Omega) \cap \mathsf{RT}_0^{\perp_{\mathsf{L}^2(\Omega)}}$ in $\mathsf{H}^1(\Omega)$. Therefore, $\mathbf{N} \leftarrow \operatorname{dev}\operatorname{Grad} H_n \to \operatorname{dev}\operatorname{Grad} H \in \operatorname{dev}\operatorname{Grad}\big(\mathsf{H}^1(\Omega) \cap \mathsf{RT}_0^{\perp_{\mathsf{L}^2(\Omega)}}\big)$. As $R(\mathring{\operatorname{Grad}}\operatorname{grad})$ and $R(\operatorname{dev}\operatorname{Grad})$ are closed, so are the ranges of the adjoints by the closed range theorem, finishing the proof. □

Using the results of the functional analysis toolbox Section 2.1, e.g., (2.1), (2.5), (2.6), (2.7), for the densely defined, closed and unbounded linear operators $\mathring{\operatorname{Grad}}\operatorname{grad}$, $\mathring{\operatorname{Rot}}_{\mathbb{S}}$, $\mathring{\operatorname{Div}}_{\mathbb{T}}$, and their adjoints $\operatorname{div}\operatorname{Div}_{\mathbb{S}}$, $\operatorname{sym}\operatorname{Rot}_{\mathbb{T}}$, $\operatorname{dev}\operatorname{Grad}$ as well as the corresponding sequence properties, i.e.,

$$\mathring{\operatorname{Rot}}_{\mathbb{S}}\mathring{\operatorname{Grad}}\operatorname{grad} = 0, \qquad \mathring{\operatorname{Div}}_{\mathbb{T}}\mathring{\operatorname{Rot}}_{\mathbb{S}} = 0$$

and hence also for the adjoints

$$\operatorname{div}\operatorname{Div}_{\mathbb{S}}\operatorname{sym}\operatorname{Rot}_{\mathbb{T}} = 0, \qquad \operatorname{sym}\operatorname{Rot}_{\mathbb{T}}\operatorname{dev}\operatorname{Grad} = 0,$$

we have the following results for the ranges and corresponding Helmholtz type decompositions.

Lemma 3.8. *For the ranges it holds*

$$\begin{aligned}
\mathsf{D}\mathbf{D}_0(\Omega,\mathbb{S})^{\perp_{\mathsf{L}^2(\Omega)}} &= N(\operatorname{div}\operatorname{Div}_{\mathbb{S}})^{\perp_{\mathsf{L}^2(\Omega)}} = R(\mathring{\operatorname{Grad}}\operatorname{grad}) \subset N(\mathring{\operatorname{Rot}}_{\mathbb{S}}) = \mathring{\mathbf{R}}_0(\Omega,\mathbb{S}),\\
\mathsf{L}^2(\Omega) = \{0\}^{\perp_{\mathsf{L}^2(\Omega)}} &= N(\mathring{\operatorname{Grad}}\operatorname{grad})^{\perp_{\mathsf{L}^2(\Omega)}} = R(\operatorname{div}\operatorname{Div}_{\mathbb{S}}),\\
\mathbf{R}_{\mathrm{sym},0}(\Omega,\mathbb{T})^{\perp_{\mathsf{L}^2(\Omega)}} &= N(\operatorname{sym}\operatorname{Rot}_{\mathbb{T}})^{\perp_{\mathsf{L}^2(\Omega)}} = \overline{R(\mathring{\operatorname{Rot}}_{\mathbb{S}})}^{\mathsf{L}^2(\Omega)} \subset N(\mathring{\operatorname{Div}}_{\mathbb{T}}) = \mathring{\mathbf{D}}_0(\Omega,\mathbb{T}),\\
\mathring{\mathbf{R}}_0(\Omega,\mathbb{S})^{\perp_{\mathsf{L}^2(\Omega)}} &= N(\mathring{\operatorname{Rot}}_{\mathbb{S}})^{\perp_{\mathsf{L}^2(\Omega)}} = \overline{R(\operatorname{sym}\operatorname{Rot}_{\mathbb{T}})}^{\mathsf{L}^2(\Omega)} \subset N(\operatorname{div}\operatorname{Div}_{\mathbb{S}}) = \mathsf{D}\mathbf{D}_0(\Omega,\mathbb{S}),\\
\mathsf{RT}_0^{\perp_{\mathsf{L}^2(\Omega)}} &= N(\operatorname{dev}\operatorname{Grad})^{\perp_{\mathsf{L}^2(\Omega)}} = R(\mathring{\operatorname{Div}}_{\mathbb{T}}),\\
\mathring{\mathbf{D}}_0(\Omega,\mathbb{T})^{\perp_{\mathsf{L}^2(\Omega)}} &= N(\mathring{\operatorname{Div}}_{\mathbb{T}})^{\perp_{\mathsf{L}^2(\Omega)}} = R(\operatorname{dev}\operatorname{Grad}) \subset N(\operatorname{sym}\operatorname{Rot}_{\mathbb{T}}) = \mathbf{R}_{\mathrm{sym},0}(\Omega,\mathbb{T}),
\end{aligned}$$

and we have

$$\begin{aligned}
\mathsf{L}^2(\Omega) = R(\operatorname{div}\operatorname{Div}_{\mathbb{S}}) = \operatorname{div}\operatorname{Div}\mathsf{D}\mathbf{D}(\Omega,\mathbb{S}) &= \operatorname{div}\operatorname{Div}\big(\mathsf{D}\mathbf{D}(\Omega,\mathbb{S}) \cap R(\mathring{\operatorname{Grad}}\operatorname{grad})\big)\\
&= \operatorname{div}\operatorname{Div}\big(\mathsf{D}\mathbf{D}(\Omega,\mathbb{S}) \cap \operatorname{Grad}\operatorname{grad}\mathring{\mathsf{H}}^2(\Omega)\big),\\
R(\mathring{\operatorname{Rot}}_{\mathbb{S}}) = \operatorname{Rot}\mathring{\mathbf{R}}(\Omega,\mathbb{S}) &= \operatorname{Rot}\big(\mathring{\mathbf{R}}(\Omega,\mathbb{S}) \cap \overline{R(\operatorname{sym}\operatorname{Rot}_{\mathbb{T}})}^{\mathsf{L}^2(\Omega)}\big)\\
&= \operatorname{Rot}\big(\mathring{\mathbf{R}}(\Omega,\mathbb{S}) \cap \overline{\operatorname{sym}\operatorname{Rot}\mathbf{R}_{\mathrm{sym}}(\Omega,\mathbb{T})}^{\mathsf{L}^2(\Omega)}\big),\\
R(\operatorname{sym}\operatorname{Rot}_{\mathbb{T}}) = \operatorname{sym}\operatorname{Rot}\mathbf{R}_{\mathrm{sym}}(\Omega,\mathbb{T}) &= \operatorname{sym}\operatorname{Rot}\big(\mathbf{R}_{\mathrm{sym}}(\Omega,\mathbb{T}) \cap \overline{R(\mathring{\operatorname{Rot}}_{\mathbb{S}})}^{\mathsf{L}^2(\Omega)}\big)\\
&= \operatorname{sym}\operatorname{Rot}\big(\mathbf{R}_{\mathrm{sym}}(\Omega,\mathbb{T}) \cap \overline{\operatorname{Rot}\mathring{\mathbf{R}}(\Omega,\mathbb{S})}^{\mathsf{L}^2(\Omega)}\big),\\
\mathsf{RT}_0^{\perp_{\mathsf{L}^2(\Omega)}} = R(\mathring{\operatorname{Div}}_{\mathbb{T}}) = \operatorname{Div}\mathring{\mathbf{D}}(\Omega,\mathbb{T}) &= \operatorname{Div}\big(\mathring{\mathbf{D}}(\Omega,\mathbb{T}) \cap R(\operatorname{dev}\operatorname{Grad})\big)\\
&= \operatorname{Div}\big(\mathring{\mathbf{D}}(\Omega,\mathbb{T}) \cap \operatorname{dev}\operatorname{Grad}\mathsf{H}^1(\Omega)\big),\\
R(\operatorname{dev}\operatorname{Grad}) = \operatorname{dev}\operatorname{Grad}\mathsf{H}^1(\Omega) &= \operatorname{dev}\operatorname{Grad}\big(\mathsf{H}^1(\Omega) \cap R(\mathring{\operatorname{Div}}_{\mathbb{T}})\big)
\end{aligned}$$

$$= \operatorname{dev}\operatorname{Grad}\big(\mathsf{H}^1(\Omega)\cap\operatorname{Div}\mathring{\mathbf{D}}(\Omega,\mathbb{T})\big)$$
$$= \operatorname{dev}\operatorname{Grad}\big(\mathsf{H}^1(\Omega)\cap\mathsf{RT}_0^{\perp_{\mathsf{L}^2(\Omega)}}\big).$$

These more regular potentials on the right hand sides are uniquely determined.

Remark 3.9. *Lemma 3.8 can be formulated equivalently in terms of complexes: The sequence*

$$\{0\}\xrightarrow{\;0\;}\mathring{\mathsf{H}}^2(\Omega)\xrightarrow{\operatorname{Grad}\mathring{\operatorname{grad}}}\mathring{\mathbf{R}}(\Omega;\mathbb{S})\xrightarrow{\mathring{\operatorname{Rot}}_{\mathbb{S}}}\mathring{\mathbf{D}}(\Omega,\mathbb{T})\xrightarrow{\mathring{\operatorname{Div}}_{\mathbb{T}}}\mathsf{L}^2(\Omega)\xrightarrow{\pi_{\mathsf{RT}_0}}\mathsf{RT}_0$$

and thus also its dual or adjoint sequence

$$\{0\}\xleftarrow{\;0\;}\mathsf{L}^2(\Omega)\xleftarrow{\operatorname{div}\operatorname{Div}_{\mathbb{S}}}\mathsf{D}\mathbf{D}(\Omega,\mathbb{S})\xleftarrow{\operatorname{sym}\operatorname{Rot}_{\mathbb{T}}}\mathbf{R}_{\operatorname{sym}}(\Omega,\mathbb{T})\xleftarrow{-\operatorname{dev}\operatorname{Grad}}\mathsf{H}^1(\Omega)\xleftarrow{\iota_{\mathsf{RT}_0}}\mathsf{RT}_0$$

are Hilbert complexes. As before, $\pi_{\mathsf{RT}_0}:\mathsf{L}^2(\Omega)\to\mathsf{RT}_0$ *denotes the orthogonal projector onto* RT_0 *with adjoint* $\pi_{\mathsf{RT}_0}^*=\iota_{\mathsf{RT}_0}:\mathsf{RT}_0\to\mathsf{L}^2(\Omega)$*, the canonical embedding. Except of the central operators*

$$\mathring{\operatorname{Rot}}_{\mathbb{S}}\qquad\text{and}\qquad(\mathring{\operatorname{Rot}}_{\mathbb{S}})^*=\operatorname{sym}\operatorname{Rot}_{\mathbb{T}}$$

the sequences is already closed.

Lemma 3.10. *The following Helmholtz type decompositions hold.*

$$\begin{aligned}
\mathsf{L}^2(\Omega)&=\operatorname{div}\operatorname{Div}\mathsf{D}\mathbf{D}(\Omega,\mathbb{S}),\\
\mathsf{L}^2(\Omega)&=\operatorname{Div}\mathring{\mathbf{D}}(\Omega,\mathbb{T})\oplus_{\mathsf{L}^2(\Omega)}\mathsf{RT}_0,\qquad \operatorname{Div}\mathring{\mathbf{D}}(\Omega,\mathbb{T})=\mathsf{RT}_0^{\perp_{\mathsf{L}^2(\Omega)}}=N(\pi_{\mathsf{RT}_0}),\\
\mathbf{L}^2(\Omega,\mathbb{S})&=\operatorname{Grad}\operatorname{grad}\mathring{\mathsf{H}}^2(\Omega)\oplus_{\mathbf{L}^2(\Omega,\mathbb{S})}\mathsf{D}\mathbf{D}_0(\Omega,\mathbb{S})\\
&=\mathring{\mathbf{R}}_0(\Omega,\mathbb{S})\oplus_{\mathbf{L}^2(\Omega,\mathbb{S})}\overline{\operatorname{sym}\operatorname{Rot}\mathbf{R}_{\operatorname{sym}}(\Omega,\mathbb{T})}^{\mathbf{L}^2(\Omega)}\\
&=\operatorname{Grad}\operatorname{grad}\mathring{\mathsf{H}}^2(\Omega)\oplus_{\mathbf{L}^2(\Omega,\mathbb{S})}\big(\mathring{\mathbf{R}}_0(\Omega,\mathbb{S})\cap\mathsf{D}\mathbf{D}_0(\Omega,\mathbb{S})\big)\oplus_{\mathbf{L}^2(\Omega,\mathbb{S})}\overline{\operatorname{sym}\operatorname{Rot}\mathbf{R}_{\operatorname{sym}}(\Omega,\mathbb{T})}^{\mathbf{L}^2(\Omega)},\\
\mathbf{L}^2(\Omega,\mathbb{T})&=\overline{\operatorname{Rot}\mathring{\mathbf{R}}(\Omega,\mathbb{S})}^{\mathbf{L}^2(\Omega)}\oplus_{\mathbf{L}^2(\Omega,\mathbb{T})}\mathbf{R}_{\operatorname{sym},0}(\Omega,\mathbb{T})\\
&=\mathring{\mathbf{D}}_0(\Omega,\mathbb{T})\oplus_{\mathbf{L}^2(\Omega,\mathbb{T})}\operatorname{dev}\operatorname{Grad}\mathsf{H}^1(\Omega)\\
&=\overline{\operatorname{Rot}\mathring{\mathbf{R}}(\Omega,\mathbb{S})}^{\mathbf{L}^2(\Omega)}\oplus_{\mathbf{L}^2(\Omega,\mathbb{T})}\big(\mathring{\mathbf{D}}_0(\Omega,\mathbb{T})\cap\mathbf{R}_{\operatorname{sym},0}(\Omega,\mathbb{T})\big)\oplus_{\mathbf{L}^2(\Omega,\mathbb{T})}\operatorname{dev}\operatorname{Grad}\mathsf{H}^1(\Omega).
\end{aligned}$$

For the operators with already closed ranges we can apply Lemma 2.1 and Lemma 2.2 of the functional analysis toolbox Section 2.1.

Lemma 3.11. *The following Friedrichs/Poincaré type estimates hold: There exist positive constants* c_{Gg}*,* c_{D}*, such that*

$$\begin{aligned}
&\forall\,u\in\mathring{\mathsf{H}}^2(\Omega) && |u|_{\mathsf{L}^2(\Omega)}\le c_{\mathsf{Gg}}\,|\operatorname{Grad}\operatorname{grad}u|_{\mathbf{L}^2(\Omega)},\\
&\forall\,\mathbf{M}\in\mathsf{D}\mathbf{D}(\Omega,\mathbb{S})\cap\operatorname{Grad}\operatorname{grad}\mathring{\mathsf{H}}^2(\Omega) && |\mathbf{M}|_{\mathbf{L}^2(\Omega)}\le c_{\mathsf{Gg}}\,|\operatorname{div}\operatorname{Div}\mathbf{M}|_{\mathsf{L}^2(\Omega)},\\
&\forall\,E\in\mathsf{H}^1(\Omega)\cap\mathsf{RT}_0^{\perp_{\mathsf{L}^2(\Omega)}} && |E|_{\mathsf{L}^2(\Omega)}\le c_{\mathsf{D}}\,|\operatorname{dev}\operatorname{Grad}E|_{\mathbf{L}^2(\Omega)},\\
&\forall\,\mathbf{N}\in\mathring{\mathbf{D}}(\Omega,\mathbb{T})\cap\operatorname{dev}\operatorname{Grad}\mathsf{H}^1(\Omega) && |\mathbf{N}|_{\mathbf{L}^2(\Omega)}\le c_{\mathsf{D}}\,|\operatorname{Div}\mathbf{N}|_{\mathsf{L}^2(\Omega)}.
\end{aligned}$$

Moreover, the reduced versions of the operators $\operatorname{Grad}\mathring{\operatorname{grad}}$*,* $\operatorname{div}\operatorname{Div}_{\mathbb{S}}$*,* $\operatorname{dev}\operatorname{Grad}$*,* $\mathring{\operatorname{Div}}_{\mathbb{T}}$ *have continuous inverse operators*

$$(\operatorname{Grad}\mathring{\operatorname{grad}})^{-1}:\operatorname{Grad}\operatorname{grad}\mathring{\mathsf{H}}^2(\Omega)\longrightarrow\mathring{\mathsf{H}}^2(\Omega),$$

$$(\operatorname{div}\operatorname{Div}_{\mathbb{S}})^{-1} : \mathsf{L}^2(\Omega) \longrightarrow \mathsf{D}\mathbf{D}(\Omega,\mathbb{S}) \cap \operatorname{Grad}\operatorname{grad}\mathring{\mathsf{H}}^2(\Omega),$$
$$(\operatorname{dev}\operatorname{Grad})^{-1} : \operatorname{dev}\operatorname{Grad}\mathsf{H}^1(\Omega) \longrightarrow \mathsf{H}^1(\Omega) \cap \mathsf{RT}_0^{\perp_{\mathsf{L}^2(\Omega)}},$$
$$(\mathring{\operatorname{Div}}_{\mathbb{T}})^{-1} : \mathsf{RT}_0^{\perp_{\mathsf{L}^2(\Omega)}} \longrightarrow \mathring{\mathbf{D}}(\Omega,\mathbb{T}) \cap \operatorname{dev}\operatorname{Grad}\mathsf{H}^1(\Omega)$$

with norms $(1+c_{\mathsf{Gg}}^2)^{1/2}$ *resp.* $(1+c_{\mathsf{D}}^2)^{1/2}$.

We note that stronger versions of the Friedrichs/Poincaré estimates for $\mathring{\operatorname{Grad}}\operatorname{grad}$ and $\operatorname{dev}\operatorname{Grad}$ have already been proved in Lemma 3.2 (iv) and Lemma 3.3. It remains to show that the ranges

$$R(\mathring{\operatorname{Rot}}_{\mathbb{S}}) = \operatorname{Rot}\mathring{\mathbf{R}}(\Omega,\mathbb{S}), \qquad R(\operatorname{sym}\operatorname{Rot}_{\mathbb{T}}) = \operatorname{sym}\operatorname{Rot}\mathbf{R}_{\operatorname{sym}}(\Omega,\mathbb{T}) \tag{3.5}$$

are closed as well, where, again by the closed range theorem, it is again sufficient to show that one of them is closed as $(\mathrm{A},\mathrm{A}^*) = (\mathring{\operatorname{Rot}}_{\mathbb{S}}, \operatorname{sym}\operatorname{Rot}_{\mathbb{T}})$ is a dual pair. By the considerations in the functional analysis toolbox Section 2.1, one possibility of proving this is to show the compactness of one of the embeddings (then the other one follows automatically, see the Lemma 2.4)

$$\begin{aligned} \mathring{\mathbf{R}}(\Omega,\mathbb{S}) \cap \overline{\operatorname{sym}\operatorname{Rot}\mathbf{R}_{\operatorname{sym}}(\Omega,\mathbb{T})}^{\mathbf{L}^2(\Omega)} &\hookrightarrow \mathbf{L}^2(\Omega,\mathbb{S}),\\ \mathbf{R}_{\operatorname{sym}}(\Omega,\mathbb{T}) \cap \overline{\operatorname{Rot}\mathring{\mathbf{R}}(\Omega,\mathbb{S})}^{\mathbf{L}^2(\Omega)} &\hookrightarrow \mathbf{L}^2(\Omega,\mathbb{T}) \end{aligned} \tag{3.6}$$

or, even better, of one of the embeddings

$$\begin{aligned} \mathring{\mathbf{R}}(\Omega,\mathbb{S}) \cap \mathsf{D}\mathbf{D}_0(\Omega,\mathbb{S}) \subset \mathring{\mathbf{R}}(\Omega,\mathbb{S}) \cap \mathsf{D}\mathbf{D}(\Omega,\mathbb{S}) &\hookrightarrow \mathbf{L}^2(\Omega,\mathbb{S}),\\ \mathbf{R}_{\operatorname{sym}}(\Omega,\mathbb{T}) \cap \mathring{\mathbf{D}}_0(\Omega,\mathbb{T}) \subset \mathbf{R}_{\operatorname{sym}}(\Omega,\mathbb{T}) \cap \mathring{\mathbf{D}}(\Omega,\mathbb{T}) &\hookrightarrow \mathbf{L}^2(\Omega,\mathbb{T}). \end{aligned} \tag{3.7}$$

Indeed we will show in the next section that (3.7) are compact. As a first step, we will prove directly the closedness of the ranges (3.5) by showing that they are kernels, if the topology of the underlying domain is trivial. In the next lemma, which will be shown in the Appendix, a few identities are collected, which will be used in the proof of the subsequent theorem.

Lemma 3.12. *Let u, E, and $\mathbf{M}$ be distributional scalar, vector, and tensor fields. Then*

- **(i)** $\operatorname{Div}(u\,\mathbf{I}) = \operatorname{grad} u$ *and* $\operatorname{Rot}(u\,\mathbf{I}) = -\operatorname{spn}\operatorname{grad} u$,
- **(ii)** $\operatorname{Div}(\operatorname{spn} E) = -\operatorname{rot} E$,
- **(iii)** $\operatorname{skw}\operatorname{Rot}\mathbf{N} = \operatorname{spn} H$ *and* $\operatorname{Div}(\operatorname{sym}\operatorname{Rot}\mathbf{N}) = \operatorname{rot} H$ *with* $H = \frac{1}{2}\big(\operatorname{Div}\mathbf{N}^\top - \operatorname{grad}(\operatorname{tr}\mathbf{N})\big)$.

We can characterize the kernels of $\operatorname{div}\operatorname{Div}_{\mathbb{S}}$ and $\operatorname{sym}\operatorname{Rot}_{\mathbb{T}}$.

Lemma 3.13. *Let Ω be additionally topologically trivial. Then*

- **(i)** $N(\operatorname{div}\operatorname{Div}_{\mathbb{S}}) = \mathsf{D}\mathbf{D}_0(\Omega,\mathbb{S}) = \operatorname{sym}\operatorname{Rot}\mathbf{R}_{\operatorname{sym}}(\Omega,\mathbb{T}) = \operatorname{sym}\operatorname{Rot}\mathbf{H}^1(\Omega,\mathbb{T}) = \operatorname{sym}\operatorname{Rot}\big(\mathbf{R}(\Omega)\cap\mathring{\mathbf{D}}_0(\Omega)\big)$, *especially the range* $R(\operatorname{sym}\operatorname{Rot}_{\mathbb{T}}) = \operatorname{sym}\operatorname{Rot}\mathbf{R}_{\operatorname{sym}}(\Omega,\mathbb{T})$ *is closed.*
- **(ii)** $N(\operatorname{sym}\operatorname{Rot}_{\mathbb{T}}) = \mathbf{R}_{\operatorname{sym},0}(\Omega,\mathbb{T}) = \operatorname{dev}\operatorname{Grad}\mathsf{H}^1(\Omega) = \operatorname{dev}\operatorname{Grad}\big(\mathsf{H}^1(\Omega)\cap\mathsf{RT}_0^{\perp_{\mathsf{L}^2(\Omega)}}\big)$ *and the potential in* $\mathsf{H}^1(\Omega)\cap\mathsf{RT}_0^{\perp_{\mathsf{L}^2(\Omega)}}$ *is uniquely determined.*

The corresponding potential operators, e.g.,

$$\mathrm{P}_{\operatorname{sym}\operatorname{Rot}} : \mathsf{D}\mathbf{D}_0(\Omega,\mathbb{S}) \longrightarrow \mathbf{H}^1(\Omega,\mathbb{T}), \qquad \mathrm{P}_{\operatorname{dev}\operatorname{Grad}} : \mathbf{R}_{\operatorname{sym},0}(\Omega,\mathbb{T}) \longrightarrow \mathsf{H}^1(\Omega),$$

are linear and continuous.

Note that[v]

$$\mathsf{D}\mathbf{D}_0(\Omega,\mathbb{S}) = \operatorname{sym}\operatorname{Rot}\operatorname{dev}\mathbf{H}^1(\Omega) \quad \text{as} \quad \mathbf{H}^1(\Omega,\mathbb{T}) = \operatorname{dev}\mathbf{H}^1(\Omega).$$

[v]By $\operatorname{dev}\mathbf{H}^1(\Omega) \subset \mathbf{H}^1(\Omega,\mathbb{T}) = \operatorname{dev}\mathbf{H}^1(\Omega,\mathbb{T}) \subset \operatorname{dev}\mathbf{H}^1(\Omega)$ we see $\mathbf{H}^1(\Omega,\mathbb{T}) = \operatorname{dev}\mathbf{H}^1(\Omega)$.

Proof. By Lemma 3.8 we already know

$$\operatorname{sym}\operatorname{Rot}\mathsf{H}^1(\Omega,\mathbb{T})\subset\operatorname{sym}\operatorname{Rot}\mathsf{R}_{\mathrm{sym}}(\Omega,\mathbb{T})=R(\operatorname{sym}\operatorname{Rot}_{\mathbb{T}})\subset N(\operatorname{div}\operatorname{Div}_{\mathbb{S}})=\mathsf{D}\mathsf{D}_0(\Omega,\mathbb{S}).$$

Moreover, for $\mathbf{\Phi}\in\mathsf{R}_{\mathrm{sym}}(\Omega)$ we have

$$\langle\operatorname{div}\operatorname{Div}\operatorname{sym}\operatorname{Rot}\mathbf{\Phi},\varphi\rangle_{\mathsf{H}^{-2}(\Omega)}=\langle\operatorname{sym}\operatorname{Rot}\mathbf{\Phi},\operatorname{Grad}\operatorname{grad}\varphi\rangle_{\mathsf{L}^2(\Omega)}=\langle\mathbf{\Phi},\operatorname{Rot}\operatorname{sym}\operatorname{Grad}\operatorname{grad}\varphi\rangle_0=0$$

for all $\varphi\in\mathring{\mathsf{C}}^\infty(\Omega)$ and hence by continuity for all $\varphi\in\mathring{\mathsf{H}}^2(\Omega)$, which shows $\operatorname{sym}\operatorname{Rot}\mathsf{R}_{\mathrm{sym}}(\Omega)\subset\mathsf{D}\mathsf{D}_0(\Omega,\mathbb{S})$. Hence

$$\operatorname{sym}\operatorname{Rot}\big(\mathsf{R}(\Omega)\cap\mathring{\mathsf{D}}_0(\Omega)\big)\subset\operatorname{sym}\operatorname{Rot}\mathsf{R}(\Omega)\subset\operatorname{sym}\operatorname{Rot}\mathsf{R}_{\mathrm{sym}}(\Omega)\subset\mathsf{D}\mathsf{D}_0(\Omega,\mathbb{S}).$$

Let $\mathbf{M}\in\mathsf{D}\mathsf{D}_0(\Omega,\mathbb{S})$. So $\operatorname{Div}\mathbf{M}\in\mathsf{D}_0^{-1}(\Omega)$ and there is a unique vector field $E:=\operatorname{rot}^{-1}\operatorname{Div}\mathbf{M}\in\mathring{\mathsf{D}}_0(\Omega)$, depending linearly and continuously on $\mathbf{M}$, i.e.,

$$|E|_{\mathsf{L}^2(\Omega)}\le c\,|\operatorname{Div}\mathbf{M}|_{\mathsf{H}^{-1}(\Omega)}\le c\,|\mathbf{M}|_{\mathsf{L}^2(\Omega)},$$

such that

$$\operatorname{Div}\mathbf{M}=\operatorname{rot}E=-\operatorname{Div}(\operatorname{spn}E),$$

see Corollary 2.28 for $m=1$, and Lemma 3.12 (ii). Hence $\operatorname{Div}(\mathbf{M}+\operatorname{spn}E)=0$, i.e., $\mathbf{M}+\operatorname{spn}E\in\mathsf{D}_0(\Omega)$, and by Lemma 2.24 there is a tensor field $\mathbf{\Phi}:=\mathrm{P}_{\mathrm{Rot}}(\mathbf{M}+\operatorname{spn}E)\in\mathsf{H}^1(\Omega)$, depending linearly and continuously on $\mathbf{M}$, i.e.,

$$|\mathbf{\Phi}|_{\mathsf{H}^1(\Omega)}\le c\,\big(|\mathbf{M}|_{\mathsf{L}^2(\Omega)}+|E|_{\mathsf{L}^2(\Omega)}\big)\le c\,|\mathbf{M}|_{\mathsf{L}^2(\Omega)},$$

such that

$$\mathbf{M}+\operatorname{spn}E=\operatorname{Rot}\mathbf{\Phi}.$$

Note that we can also choose $\mathbf{\Phi}:=\operatorname{Rot}^{-1}(\mathbf{M}+\operatorname{spn}E)\in\mathsf{R}(\Omega)\cap\mathring{\mathsf{D}}_0(\Omega)$ by (2.13) and that in this case $\mathbf{\Phi}$ is uniquely determined and depends also linearly and continuously on $\mathbf{M}$, i.e.,

$$|\mathbf{\Phi}|_{\mathsf{R}(\Omega)}\le c\,\big(|\mathbf{M}|_{\mathsf{L}^2(\Omega)}+|E|_{\mathsf{L}^2(\Omega)}\big)\le c\,|\mathbf{M}|_{\mathsf{L}^2(\Omega)}.$$

Observe that $\mathbf{M}$ is symmetric and $\operatorname{spn}E$ is skew-symmetric. Thus

$$\mathbf{M}=\operatorname{sym}\operatorname{Rot}\mathbf{\Phi}\quad\text{and}\quad\operatorname{spn}E=\operatorname{skw}\operatorname{Rot}\mathbf{\Phi},$$

which completes the proof of (i) for the potential $\mathbf{\Phi}\in\mathsf{R}(\Omega)\cap\mathring{\mathsf{D}}_0(\Omega)$ as we have shown

$$\mathsf{D}\mathsf{D}_0(\Omega,\mathbb{S})\subset\operatorname{sym}\operatorname{Rot}\big(\mathsf{R}(\Omega)\cap\mathring{\mathsf{D}}_0(\Omega)\big)$$

with a linear and continuous potential operator. If we choose the potential $\mathbf{\Phi}\in\mathsf{H}^1(\Omega)$, we see

$$\mathbf{M}=\operatorname{sym}\operatorname{Rot}\mathbf{\Phi}=\operatorname{sym}\operatorname{Rot}\tilde{\mathbf{\Phi}}\quad\text{with}\quad\tilde{\mathbf{\Phi}}:=\operatorname{dev}\mathbf{\Phi}\in\mathsf{H}^1(\Omega,\mathbb{T}),$$

as $\operatorname{dev}\mathbf{\Phi}=\mathbf{\Phi}-\frac{1}{3}(\operatorname{tr}\mathbf{\Phi})\,\mathbf{I}$ and $\operatorname{Rot}((\operatorname{tr}\mathbf{\Phi})\,\mathbf{I})=-\operatorname{spn}\operatorname{grad}(\operatorname{tr}\mathbf{\Phi})$ is skew-symmetric by the second identity in Lemma 3.12 (i), which completes the proof of part (i) since we have proved

$$\mathsf{D}\mathsf{D}_0(\Omega,\mathbb{S})\subset\operatorname{sym}\operatorname{Rot}\mathsf{H}^1(\Omega,\mathbb{T})=\operatorname{sym}\operatorname{Rot}\operatorname{dev}\mathsf{H}^1(\Omega)$$

with a linear and continuous potential operator. Let us note

$$\mathbf{\Phi}=\operatorname{Rot}^{-1}(\mathbf{M}+\operatorname{spn}\operatorname{rot}^{-1}\operatorname{Div}\mathbf{M})\in\mathsf{R}(\Omega)\cap\mathring{\mathsf{D}}_0(\Omega),$$
$$\tilde{\mathbf{\Phi}}=\operatorname{dev}\mathrm{P}_{\mathrm{Rot}}(\mathbf{M}+\operatorname{spn}\operatorname{rot}^{-1}\operatorname{Div}\mathbf{M})\in\mathsf{H}^1(\Omega,\mathbb{T}).$$

Moreover, the Helmholtz projection $\tilde{\tilde{\mathbf{\Phi}}}\in\mathsf{R}(\Omega)\cap\mathring{\mathsf{D}}_0(\Omega)$ of $\tilde{\mathbf{\Phi}}$ onto $\mathring{\mathsf{D}}_0(\Omega)$, see Lemma 2.19, gives another potential $\tilde{\tilde{\mathbf{\Phi}}}$ with the same properties as $\mathbf{\Phi}\in\mathsf{R}(\Omega)\cap\mathring{\mathsf{D}}_0(\Omega)$ itself.

Let us prove (ii). We know already by Lemma 3.7 and Lemma 3.8 that

$$\operatorname{dev}\operatorname{Grad}\big(\mathsf{H}^1(\Omega)\cap\mathsf{RT}_0^{\perp_{\mathsf{L}^2(\Omega)}}\big)=\operatorname{dev}\operatorname{Grad}\mathsf{H}^1(\Omega)=R(\operatorname{dev}\operatorname{Grad})\subset N(\operatorname{sym}\operatorname{Rot}_{\mathbb{T}})=\mathsf{R}_{\mathrm{sym},0}(\Omega,\mathbb{T}),$$

with unique potentials in $\mathsf{H}^1(\Omega) \cap \mathsf{RT}_0^{\perp_{\mathsf{L}^2(\Omega)}}$, and that the ranges are closed. Let $\mathbf{N} \in \mathbf{R}_{\mathrm{sym},0}(\Omega, \mathbb{T})$. Then (trivially) $\operatorname{Div} \operatorname{sym} \operatorname{Rot} \mathbf{N} = 0$ and it follows from Lemma 3.12 (iii) that

$$\operatorname{rot} H = 0 \quad \text{with} \quad H := \frac{1}{2}\big(\operatorname{Div} \mathbf{N}^\top - \operatorname{grad}(\operatorname{tr} \mathbf{N})\big) = \frac{1}{2} \operatorname{Div} \mathbf{N}^\top$$

and

$$\operatorname{skw} \operatorname{Rot} \mathbf{N} = \operatorname{spn} H. \tag{3.8}$$

So $H \in \mathsf{R}_0^{-1}(\Omega)$. Therefore, there is a unique scalar field $u := \operatorname{grad}^{-1} H \in \mathsf{L}_0^2(\Omega)$, depending linearly and continuously on H resp. $\mathbf{N}$, i.e.,

$$|u|_{\mathsf{L}^2(\Omega)} \leq c\,|H|_{\mathsf{H}^{-1}(\Omega)} \leq c\,|\mathbf{N}|_{\mathbf{L}^2(\Omega)},$$

such that

$$H = \operatorname{grad} u,$$

see Corollary 2.27 for $m = 1$. By the second identity in Lemma 3.12 (i) we see $\operatorname{Rot}(u\,\mathbf{I}) = -\operatorname{spn} \operatorname{grad} u$ and thus $\operatorname{sym} \operatorname{Rot}(u\,\mathbf{I}) = 0$, implying

$$\hat{\mathbf{N}} := \mathbf{N} + u\,\mathbf{I} \in \mathbf{R}_{\mathrm{sym},0}(\Omega).$$

Moreover, by (3.8)

$$\operatorname{skw} \operatorname{Rot} \hat{\mathbf{N}} = \operatorname{skw} \operatorname{Rot} \mathbf{N} + \operatorname{skw} \operatorname{Rot}(u\,\mathbf{I}) = \operatorname{spn} H - \operatorname{spn} \operatorname{grad} u = 0.$$

Hence $\hat{\mathbf{N}} \in \mathbf{R}_0(\Omega)$. Therefore, there is a unique vector field $\Phi := \operatorname{Grad}^{-1} \hat{\mathbf{N}} \in \mathsf{H}^1(\Omega)$, depending linearly and continuously on $\mathbf{N}$, i.e.,

$$|\Phi|_{\mathsf{H}^1(\Omega)} \leq c\,|\hat{\mathbf{N}}|_{\mathbf{L}^2(\Omega)} \leq c\,|\mathbf{N}|_{\mathbf{L}^2(\Omega)},$$

such that $\hat{\mathbf{N}} = \operatorname{Grad} \Phi$, see (2.13) or Lemma 2.23. So we have

$$\mathbf{N} = \operatorname{Grad} \Phi - u\,\mathbf{I}.$$

From the additional condition $\operatorname{tr} \mathbf{N} = 0$ it follows that $3\,u = \operatorname{tr} \operatorname{Grad} \Phi = \operatorname{div} \Phi$ leading to

$$\mathbf{N} = \operatorname{dev} \operatorname{Grad} \Phi, \quad \Phi \in \mathsf{H}^1(\Omega),$$

which completes the proof as we have shown

$$\mathbf{R}_{\mathrm{sym},0}(\Omega, \mathbb{T}) \subset \operatorname{dev} \operatorname{Grad} \mathsf{H}^1(\Omega) = \operatorname{dev} \operatorname{Grad} \big(\mathsf{H}^1(\Omega) \cap \mathsf{RT}_0^{\perp_{\mathsf{L}^2(\Omega)}}\big)$$

and by Lemma 3.2 (iv) the potential in $\mathsf{H}^1(\Omega) \cap \mathsf{RT}_0^{\perp_{\mathsf{L}^2(\Omega)}}$ depends linearly and continuously on the right hand side. Note that $\Phi = \operatorname{Grad}^{-1} \big(\mathbf{N} + \frac{1}{2}(\operatorname{grad}^{-1} \operatorname{Div} \mathbf{N}^\top)\,\mathbf{I}\big) \in \mathsf{H}^1(\Omega)$. □

Now, if Ω has trivial topology, we can improve Lemma 3.13 and the potentials, the Helmholtz decompositions, as well as the Friedrichs/Poincaré estimates of Lemma 3.8, Lemma 3.10, and Lemma 3.11. Let Ω be additionally topologically trivial. By Lemma 3.13 and the closed range theorem, see (3.5), the ranges

$$R(\operatorname{sym} \operatorname{Rot}_{\mathbb{T}}) = \operatorname{sym} \operatorname{Rot} \mathbf{R}_{\mathrm{sym}}(\Omega, \mathbb{T}), \quad R(\mathring{\operatorname{Rot}}_{\mathbb{S}}) = \operatorname{Rot} \mathring{\mathbf{R}}(\Omega, \mathbb{S})$$

are closed in $\mathbf{L}^2(\Omega)$. Lemma 3.8 shows

$$R(\operatorname{sym} \operatorname{Rot}_{\mathbb{T}}) = \operatorname{sym} \operatorname{Rot} \mathbf{R}_{\mathrm{sym}}(\Omega, \mathbb{T}) = \operatorname{sym} \operatorname{Rot} \big(\mathbf{R}_{\mathrm{sym}}(\Omega, \mathbb{T}) \cap \operatorname{Rot} \mathring{\mathbf{R}}(\Omega, \mathbb{S})\big),$$
$$R(\mathring{\operatorname{Rot}}_{\mathbb{S}}) = \operatorname{Rot} \mathring{\mathbf{R}}(\Omega, \mathbb{S}) = \operatorname{Rot} \big(\mathring{\mathbf{R}}(\Omega, \mathbb{S}) \cap \operatorname{sym} \operatorname{Rot} \mathbf{R}_{\mathrm{sym}}(\Omega, \mathbb{T})\big),$$

where the potentials on the right hand sides are uniquely determined. As in Lemma 3.11 two important Friedrichs/Poincaré type estimates follow, i.e., There exists $c_{\mathsf{R}} > 0$, such that

$$\forall\,\mathbf{N} \in \mathbf{R}_{\mathrm{sym}}(\Omega, \mathbb{T}) \cap \operatorname{Rot} \mathring{\mathbf{R}}(\Omega, \mathbb{S}) \qquad |\mathbf{N}|_{\mathbf{L}^2(\Omega)} \leq c_{\mathsf{R}}\,|\operatorname{sym} \operatorname{Rot} \mathbf{N}|_{\mathbf{L}^2(\Omega)},$$
$$\forall\,\mathbf{M} \in \mathring{\mathbf{R}}(\Omega, \mathbb{S}) \cap \operatorname{sym} \operatorname{Rot} \mathbf{R}_{\mathrm{sym}}(\Omega, \mathbb{T}) \qquad |\mathbf{M}|_{\mathbf{L}^2(\Omega)} \leq c_{\mathsf{R}}\,|\operatorname{Rot} \mathbf{M}|_{\mathbf{L}^2(\Omega)}.$$

Moreover, the reduced versions of $\operatorname{sym}\operatorname{Rot}_{\mathbb{T}}$ and $\mathring{\operatorname{Rot}}_{\mathbb{S}}$ have continuous inverse operators

$$(\operatorname{sym}\operatorname{Rot}_{\mathbb{T}})^{-1} : \operatorname{sym}\operatorname{Rot}\mathbf{R}_{\mathrm{sym}}(\Omega,\mathbb{T}) \longrightarrow \mathbf{R}_{\mathrm{sym}}(\Omega,\mathbb{T}) \cap \operatorname{Rot}\mathring{\mathbf{R}}(\Omega,\mathbb{S}),$$
$$(\mathring{\operatorname{Rot}}_{\mathbb{S}})^{-1} : \operatorname{Rot}\mathring{\mathbf{R}}(\Omega,\mathbb{S}) \longrightarrow \mathring{\mathbf{R}}(\Omega,\mathbb{S}) \cap \operatorname{sym}\operatorname{Rot}\mathbf{R}_{\mathrm{sym}}(\Omega,\mathbb{T})$$

with norms $(1+c_{\mathsf{R}}^2)^{1/2}$. By Lemma 3.10 and Lemma 3.13 we obtain

$$\mathring{\mathbf{R}}_0(\Omega,\mathbb{S}) \cap \mathsf{D}\mathbf{D}_0(\Omega,\mathbb{S}) = \mathring{\mathbf{R}}_0(\Omega,\mathbb{S}) \cap \operatorname{sym}\operatorname{Rot}\mathbf{R}_{\mathrm{sym}}(\Omega,\mathbb{T}) = N(\mathring{\operatorname{Rot}}_{\mathbb{S}}) \cap R(\operatorname{sym}\operatorname{Rot}_{\mathbb{T}}) = \{0\},$$
$$\mathring{\mathbf{D}}_0(\Omega,\mathbb{T}) \cap \mathbf{R}_{\mathrm{sym},0}(\Omega,\mathbb{T}) = \mathring{\mathbf{D}}_0(\Omega,\mathbb{T}) \cap \operatorname{dev}\operatorname{Grad}\mathsf{H}^1(\Omega) = N(\mathring{\operatorname{Div}}_{\mathbb{T}}) \cap R(\operatorname{dev}\operatorname{Grad}) = \{0\},$$

i.e., the cohomology groups are trivial. By Lemma 3.10 we get improved Helmholtz decompositions

$$\begin{aligned}
\mathbf{L}^2(\Omega,\mathbb{S}) &= \mathring{\mathbf{R}}_0(\Omega,\mathbb{S}) \oplus_{\mathbf{L}^2(\Omega,\mathbb{S})} \mathsf{D}\mathbf{D}_0(\Omega,\mathbb{S}), & \mathbf{L}^2(\Omega,\mathbb{T}) &= \mathring{\mathbf{D}}_0(\Omega,\mathbb{T}) \oplus_{\mathbf{L}^2(\Omega,\mathbb{T})} \mathbf{R}_{\mathrm{sym},0}(\Omega,\mathbb{T}),\\
\mathring{\mathbf{R}}_0(\Omega,\mathbb{S}) &= \operatorname{Grad}\operatorname{grad}\mathring{\mathsf{H}}^2(\Omega), & \mathring{\mathbf{D}}_0(\Omega,\mathbb{T}) &= \operatorname{Rot}\mathring{\mathbf{R}}(\Omega,\mathbb{S}),\\
\mathsf{D}\mathbf{D}_0(\Omega,\mathbb{S}) &= \operatorname{sym}\operatorname{Rot}\mathbf{R}_{\mathrm{sym}}(\Omega,\mathbb{T}), & \mathbf{R}_{\mathrm{sym},0}(\Omega,\mathbb{T}) &= \operatorname{dev}\operatorname{Grad}\mathsf{H}^1(\Omega).
\end{aligned} \tag{3.9}$$

Let us summarize our main results of this section.

Theorem 3.14. *Let Ω be additionally topologically trivial. Then the cohomology groups are trivial, i.e.,*

$$\mathring{\mathbf{R}}_0(\Omega,\mathbb{S}) \cap \mathsf{D}\mathbf{D}_0(\Omega,\mathbb{S}) = \{0\}, \qquad \mathring{\mathbf{D}}_0(\Omega,\mathbb{T}) \cap \mathbf{R}_{\mathrm{sym},0}(\Omega,\mathbb{T}) = \{0\},$$

and the Helmholtz type decompositions (3.9)*, i.e.,*

$$\mathbf{L}^2(\Omega,\mathbb{S}) = \mathring{\mathbf{R}}_0(\Omega,\mathbb{S}) \oplus_{\mathbf{L}^2(\Omega,\mathbb{S})} \mathsf{D}\mathbf{D}_0(\Omega,\mathbb{S}), \qquad \mathbf{L}^2(\Omega,\mathbb{T}) = \mathring{\mathbf{D}}_0(\Omega,\mathbb{T}) \oplus_{\mathbf{L}^2(\Omega,\mathbb{T})} \mathbf{R}_{\mathrm{sym},0}(\Omega,\mathbb{T}),$$

hold. The ranges

$$\begin{aligned}
\mathring{\mathbf{R}}_0(\Omega,\mathbb{S}) &= \operatorname{Grad}\operatorname{grad}\mathring{\mathsf{H}}^2(\Omega),\\
\mathsf{L}^2(\Omega) &= \operatorname{div}\operatorname{Div}\mathsf{D}\mathbf{D}(\Omega,\mathbb{S}) = \operatorname{div}\operatorname{Div}\big(\mathsf{D}\mathbf{D}(\Omega,\mathbb{S}) \cap \mathring{\mathbf{R}}_0(\Omega,\mathbb{S})\big),\\
\mathring{\mathbf{D}}_0(\Omega,\mathbb{T}) &= \operatorname{Rot}\mathring{\mathbf{R}}(\Omega,\mathbb{S}) = \operatorname{Rot}\big(\mathring{\mathbf{R}}(\Omega,\mathbb{S}) \cap \mathsf{D}\mathbf{D}_0(\Omega,\mathbb{S})\big),\\
\mathsf{D}\mathbf{D}_0(\Omega,\mathbb{S}) &= \operatorname{sym}\operatorname{Rot}\mathbf{R}_{\mathrm{sym}}(\Omega,\mathbb{T}) = \operatorname{sym}\operatorname{Rot}\big(\mathbf{R}_{\mathrm{sym}}(\Omega,\mathbb{T}) \cap \mathring{\mathbf{D}}_0(\Omega,\mathbb{T})\big),\\
\mathsf{RT}_0^{\perp_{\mathsf{L}^2(\Omega)}} &= \operatorname{Div}\mathring{\mathbf{D}}(\Omega,\mathbb{T}) = \operatorname{Div}\big(\mathring{\mathbf{D}}(\Omega,\mathbb{T}) \cap \mathbf{R}_{\mathrm{sym},0}(\Omega,\mathbb{T})\big),\\
\mathbf{R}_{\mathrm{sym},0}(\Omega,\mathbb{T}) &= \operatorname{dev}\operatorname{Grad}\mathsf{H}^1(\Omega) = \operatorname{dev}\operatorname{Grad}\big(\mathsf{H}^1(\Omega) \cap \mathsf{RT}_0^{\perp_{\mathsf{L}^2(\Omega)}}\big).
\end{aligned}$$

are closed. The more regular potentials on the right hand sides are uniquely determined and depend linearly and continuously on the data. Especially, there exist positive constants c_{Gg}, c_{D}, c_{R} such that the Friedrichs/Poincaré type estimates

$$\begin{aligned}
&\forall\, u \in \mathring{\mathsf{H}}^2(\Omega) & |u|_{\mathsf{L}^2(\Omega)} &\le c_{\mathsf{Gg}}\,|\operatorname{Grad}\operatorname{grad} u|_{\mathbf{L}^2(\Omega)},\\
&\forall\, \mathbf{M} \in \mathsf{D}\mathbf{D}(\Omega,\mathbb{S}) \cap \mathring{\mathbf{R}}_0(\Omega,\mathbb{S}) & |\mathbf{M}|_{\mathbf{L}^2(\Omega)} &\le c_{\mathsf{Gg}}\,|\operatorname{div}\operatorname{Div}\mathbf{M}|_{\mathsf{L}^2(\Omega)},\\
&\forall\, \mathbf{N} \in \mathring{\mathbf{D}}(\Omega,\mathbb{T}) \cap \mathbf{R}_{\mathrm{sym},0}(\Omega,\mathbb{T}) & |\mathbf{N}|_{\mathbf{L}^2(\Omega)} &\le c_{\mathsf{D}}\,|\operatorname{Div}\mathbf{N}|_{\mathsf{L}^2(\Omega)},\\
&\forall\, E \in \mathsf{H}^1(\Omega) \cap \mathsf{RT}_0^{\perp_{\mathsf{L}^2(\Omega)}} & |E|_{\mathsf{L}^2(\Omega)} &\le c_{\mathsf{D}}\,|\operatorname{dev}\operatorname{Grad} E|_{\mathbf{L}^2(\Omega)},\\
&\forall\, \mathbf{M} \in \mathring{\mathbf{R}}(\Omega,\mathbb{S}) \cap \mathsf{D}\mathbf{D}_0(\Omega,\mathbb{S}) & |\mathbf{M}|_{\mathbf{L}^2(\Omega)} &\le c_{\mathsf{R}}\,|\operatorname{Rot}\mathbf{M}|_{\mathbf{L}^2(\Omega)},\\
&\forall\, \mathbf{N} \in \mathbf{R}_{\mathrm{sym}}(\Omega,\mathbb{T}) \cap \mathring{\mathbf{D}}_0(\Omega,\mathbb{T}) & |\mathbf{N}|_{\mathbf{L}^2(\Omega)} &\le c_{\mathsf{R}}\,|\operatorname{sym}\operatorname{Rot}\mathbf{N}|_{\mathbf{L}^2(\Omega)}
\end{aligned}$$

hold. Moreover, the reduced versions of the operators

$$\mathring{\operatorname{Grad}}\operatorname{grad},\quad \operatorname{div}\operatorname{Div}_{\mathbb{S}},\quad \mathring{\operatorname{Div}}_{\mathbb{T}},\quad \operatorname{dev}\operatorname{Grad},\quad \mathring{\operatorname{Rot}}_{\mathbb{S}},\quad \operatorname{sym}\operatorname{Rot}_{\mathbb{T}}$$

have continuous inverse operators

$$\begin{aligned}
(\mathring{\operatorname{Grad}}\operatorname{grad})^{-1} : \mathring{\mathbf{R}}_0(\Omega,\mathbb{S}) &\longrightarrow \mathring{\mathsf{H}}^2(\Omega),\\
(\operatorname{div}\operatorname{Div}_{\mathbb{S}})^{-1} : \mathsf{L}^2(\Omega) &\longrightarrow \mathsf{D}\mathbf{D}(\Omega,\mathbb{S}) \cap \mathring{\mathbf{R}}_0(\Omega,\mathbb{S}),\\
(\mathring{\operatorname{Div}}_{\mathbb{T}})^{-1} : \mathsf{RT}_0^{\perp_{\mathsf{L}^2(\Omega)}} &\longrightarrow \mathring{\mathbf{D}}(\Omega,\mathbb{T}) \cap \mathbf{R}_{\operatorname{sym},0}(\Omega,\mathbb{T}),\\
(\operatorname{dev}\operatorname{Grad})^{-1} : \mathbf{R}_{\operatorname{sym},0}(\Omega,\mathbb{T}) &\longrightarrow \mathsf{H}^1(\Omega) \cap \mathsf{RT}_0^{\perp_{\mathsf{L}^2(\Omega)}},\\
(\mathring{\operatorname{Rot}}_{\mathbb{S}})^{-1} : \mathring{\mathbf{D}}_0(\Omega,\mathbb{T}) &\longrightarrow \mathring{\mathbf{R}}(\Omega,\mathbb{S}) \cap \mathsf{D}\mathbf{D}_0(\Omega,\mathbb{S}),\\
(\operatorname{sym}\operatorname{Rot}_{\mathbb{T}})^{-1} : \mathsf{D}\mathbf{D}_0(\Omega,\mathbb{S}) &\longrightarrow \mathbf{R}_{\operatorname{sym}}(\Omega,\mathbb{T}) \cap \mathring{\mathbf{D}}_0(\Omega,\mathbb{T})
\end{aligned}$$

with norms $(1+c_{\mathsf{Gg}}^2)^{1/2}$ *resp.* $(1+c_{\mathsf{D}}^2)^{1/2}$, *resp.* $(1+c_{\mathsf{R}}^2)^{1/2}$.

Recalling Remark 3.9 we have the following result.

Remark 3.15. *Let* Ω *be additionally topologically trivial. Theorem 3.14 easily leads to the following equivalent results in terms of complexes: The sequence*

$$\{0\} \xrightarrow{\;0\;} \mathring{\mathsf{H}}^2(\Omega) \xrightarrow{\mathring{\operatorname{Grad}}\operatorname{grad}} \mathring{\mathbf{R}}(\Omega;\mathbb{S}) \xrightarrow{\mathring{\operatorname{Rot}}_{\mathbb{S}}} \mathring{\mathbf{D}}(\Omega,\mathbb{T}) \xrightarrow{\mathring{\operatorname{Div}}_{\mathbb{T}}} \mathsf{L}^2(\Omega) \xrightarrow{\pi_{\mathsf{RT}_0}} \mathsf{RT}_0$$

and thus also its dual or adjoint sequence

$$\{0\} \xleftarrow{\;0\;} \mathsf{L}^2(\Omega) \xleftarrow{\operatorname{div}\operatorname{Div}_{\mathbb{S}}} \mathsf{D}\mathbf{D}(\Omega,\mathbb{S}) \xleftarrow{\operatorname{sym}\operatorname{Rot}_{\mathbb{T}}} \mathbf{R}_{\operatorname{sym}}(\Omega,\mathbb{T}) \xleftarrow{-\operatorname{dev}\operatorname{Grad}} \mathsf{H}^1(\Omega) \xleftarrow{\iota_{\mathsf{RT}_0}} \mathsf{RT}_0$$

are closed and exact Hilbert complexes. The first complex might by called $\mathring{\operatorname{Grad}}\operatorname{grad}$*-complex and the second one* $\operatorname{div}\operatorname{Div}$*-complex.*

Remark 3.16. *The part*

$$\{0\} \xrightarrow{\;0\;} \mathring{\mathsf{H}}^2(\Omega) \xrightarrow{\mathring{\operatorname{Grad}}\operatorname{grad}} \mathring{\mathbf{R}}(\Omega;\mathbb{S}) \xrightarrow{\mathring{\operatorname{Rot}}_{\mathbb{S}}} \mathbf{L}^2(\Omega)$$

of the Hilbert complex from above and the related adjoint complex

$$\{0\} \xleftarrow{\;0\;} \mathsf{L}^2(\Omega) \xleftarrow{\operatorname{div}\operatorname{Div}_{\mathbb{S}}} \mathsf{D}\mathbf{D}(\Omega,\mathbb{S}) \xleftarrow{\operatorname{sym}\operatorname{Rot}_{\mathbb{T}}} \mathbf{R}_{\operatorname{sym}}(\Omega,\mathbb{T})$$

have been discussed in [24] *for problems in general relativity.*

Remark 3.17. *In 2D and under similar assumptions we obtain by completely analogous but much simpler arguments that the Hilbert complexes*

$$\begin{aligned}
&\{0\} \xrightarrow{\;0\;} \mathring{\mathsf{H}}^2(\Omega) \xrightarrow{\mathring{\operatorname{Grad}}\operatorname{grad}} \mathring{\mathbf{R}}(\Omega;\mathbb{S}) \xrightarrow{\mathring{\operatorname{Rot}}_{\mathbb{S}}} \mathsf{L}^2(\Omega) \xrightarrow{\pi_{\mathsf{RT}_0}} \mathsf{RT}_0,\\
&\{0\} \xleftarrow{\;0\;} \mathsf{L}^2(\Omega) \xleftarrow{\operatorname{div}\operatorname{Div}_{\mathbb{S}}} \mathsf{D}\mathbf{D}(\Omega,\mathbb{S}) \xleftarrow{\operatorname{sym}\operatorname{Rot}} \mathsf{H}^1(\Omega) \xleftarrow{\iota_{\mathsf{RT}_0}} \mathsf{RT}_0
\end{aligned}$$

are dual to each other, closed and exact. Contrary to the 3D case, the operator $\mathring{\operatorname{Rot}}_{\mathbb{S}}$ *maps a tensor field to a vector field and the operator* $\operatorname{sym}\operatorname{Rot}$ *is applied row-wise to a vector field and maps this vector field to a tensor field. The associated Helmholtz decomposition is*

$$\mathbf{L}^2(\Omega,\mathbb{S}) = \mathring{\mathbf{R}}_0(\Omega,\mathbb{S}) \oplus_{\mathbf{L}^2(\Omega,\mathbb{S})} \mathsf{D}\mathbf{D}_0(\Omega,\mathbb{S})$$

with

$$\mathring{\mathbf{R}}_0(\Omega,\mathbb{S}) = \operatorname{Grad}\operatorname{grad}\mathring{\mathsf{H}}^2(\Omega),\quad \mathsf{D}\mathbf{D}_0(\Omega,\mathbb{S}) = \operatorname{sym}\operatorname{Rot}\mathsf{H}^1(\Omega).$$

4. Compact Embeddings for Symmetric Rotations

We will show that indeed the embeddings (3.6) and (3.7) are compact.

Lemma 4.1. *Let Ω be additionally topologically trivial. Then the embeddings* (3.7)*, i.e.,*

$$\mathring{\mathsf{R}}(\Omega,\mathbb{S})\cap\mathsf{D}\mathbf{D}(\Omega,\mathbb{S})\hookrightarrow\mathsf{L}^2(\Omega,\mathbb{S}),\qquad \mathsf{R}_{\mathrm{sym}}(\Omega,\mathbb{T})\cap\mathring{\mathsf{D}}(\Omega,\mathbb{T})\hookrightarrow\mathsf{L}^2(\Omega,\mathbb{T}),$$

and (3.6) *are compact.*

Proof. Let $(\mathbf{M}_n)$ be a bounded sequence in $\mathring{\mathsf{R}}(\Omega,\mathbb{S})\cap\mathsf{D}\mathbf{D}(\Omega,\mathbb{S})$. By Theorem 3.14 and Lemma 3.13 we have

$$\begin{aligned}\mathring{\mathsf{R}}(\Omega,\mathbb{S})\cap\mathsf{D}\mathbf{D}(\Omega,\mathbb{S})&=\big(\mathring{\mathsf{R}}_0(\Omega,\mathbb{S})\cap\mathsf{D}\mathbf{D}(\Omega,\mathbb{S})\big)\oplus_{\mathsf{L}^2(\Omega,\mathbb{S})}\big(\mathring{\mathsf{R}}(\Omega,\mathbb{S})\cap\mathsf{D}\mathbf{D}_0(\Omega,\mathbb{S})\big),\\ \mathring{\mathsf{R}}_0(\Omega,\mathbb{S})&=\operatorname{Grad}\operatorname{grad}\mathring{\mathsf{H}}^2(\Omega),\\ \mathsf{D}\mathbf{D}_0(\Omega,\mathbb{S})&=\operatorname{sym}\operatorname{Rot}\mathsf{H}^1(\Omega)\end{aligned}$$

with linear and continuous potential operators. Therefore, we can decompose

$$\mathring{\mathsf{R}}(\Omega,\mathbb{S})\cap\mathsf{D}\mathbf{D}(\Omega,\mathbb{S})\ni\mathbf{M}_n=\mathbf{M}_{n,\mathsf{r}}+\mathbf{M}_{n,\mathsf{d}}\in\big(\mathring{\mathsf{R}}_0(\Omega,\mathbb{S})\cap\mathsf{D}\mathbf{D}(\Omega,\mathbb{S})\big)\oplus_{\mathsf{L}^2(\Omega,\mathbb{S})}\big(\mathring{\mathsf{R}}(\Omega,\mathbb{S})\cap\mathsf{D}\mathbf{D}_0(\Omega,\mathbb{S})\big)$$

with $\mathbf{M}_{n,\mathsf{r}}\in\operatorname{Grad}\operatorname{grad}\mathring{\mathsf{H}}^2(\Omega)\cap\mathsf{D}\mathbf{D}(\Omega,\mathbb{S})$, $\operatorname{Rot}\mathbf{M}_{n,\mathsf{d}}=\operatorname{Rot}\mathbf{M}_n$, and $\mathbf{M}_{n,\mathsf{r}}=\operatorname{Grad}\operatorname{grad}u_n$, $u_n\in\mathring{\mathsf{H}}^2(\Omega)$, as well as $\mathbf{M}_{n,\mathsf{d}}\in\mathring{\mathsf{R}}(\Omega,\mathbb{S})\cap\operatorname{sym}\operatorname{Rot}\mathsf{H}^1(\Omega)$, $\operatorname{div}\operatorname{Div}\mathbf{M}_{n,\mathsf{r}}=\operatorname{div}\operatorname{Div}\mathbf{M}_n$, and $\mathbf{M}_{n,\mathsf{d}}=\operatorname{sym}\operatorname{Rot}\mathbf{N}_n$, $\mathbf{N}_n\in\mathsf{H}^1(\Omega)$, and both u_n and $\mathbf{N}_n$ depend continuously on $\mathbf{M}_n$, i.e.,

$$|u_n|_{\mathsf{H}^2(\Omega)}\leq c\,|\mathbf{M}_{n,\mathsf{r}}|_{\mathsf{L}^2(\Omega)}\leq c\,|\mathbf{M}_n|_{\mathsf{L}^2(\Omega)},\qquad|\mathbf{N}_n|_{\mathsf{H}^1(\Omega)}\leq c\,|\mathbf{M}_{n,\mathsf{d}}|_{\mathsf{L}^2(\Omega)}\leq c\,|\mathbf{M}_n|_{\mathsf{L}^2(\Omega)}.$$

By Rellich's selection theorem, there exist subsequences, again denoted by (u_n) and $(\mathbf{N}_n)$, such that (u_n) converges in $\mathsf{H}^1(\Omega)$ and $(\mathbf{N}_n)$ converges in $\mathsf{L}^2(\Omega)$. Thus with $\mathbf{M}_{n,m}:=\mathbf{M}_n-\mathbf{M}_m$, and similarly for $\mathbf{M}_{n,m,\mathsf{r}}$, $\mathbf{M}_{n,m,\mathsf{d}}$, $u_{n,m}$, $\mathbf{N}_{n,m}$, we see

$$\begin{aligned}|\mathbf{M}_{n,m,\mathsf{r}}|^2_{\mathsf{L}^2(\Omega)}&=\langle\mathbf{M}_{n,m,\mathsf{r}},\operatorname{Grad}\operatorname{grad}u_{n,m}\rangle_{\mathsf{L}^2(\Omega)}=\langle\operatorname{div}\operatorname{Div}\mathbf{M}_{n,m,\mathsf{r}},u_{n,m}\rangle_{\mathsf{L}^2(\Omega)}\\&=\langle\operatorname{div}\operatorname{Div}\mathbf{M}_{n,m},u_{n,m}\rangle_{\mathsf{L}^2(\Omega)}\leq c\,|u_{n,m}|_{\mathsf{L}^2(\Omega)},\\|\mathbf{M}_{n,m,\mathsf{d}}|^2_{\mathsf{L}^2(\Omega)}&=\langle\mathbf{M}_{n,m,\mathsf{d}},\operatorname{sym}\operatorname{Rot}\mathbf{N}_{n,m}\rangle_{\mathsf{L}^2(\Omega)}=\langle\operatorname{Rot}\mathbf{M}_{n,m,\mathsf{d}},\mathbf{N}_{n,m}\rangle_{\mathsf{L}^2(\Omega)}\\&=\langle\operatorname{Rot}\mathbf{M}_{n,m},\mathbf{N}_{n,m}\rangle_{\mathsf{L}^2(\Omega)}\leq c\,|\mathbf{N}_{n,m}|_{\mathsf{L}^2(\Omega)}.\end{aligned}$$

Hence, $(\mathbf{M}_n)$ is a Cauchy sequence in $\mathsf{L}^2(\Omega,\mathbb{S})$. So

$$\mathring{\mathsf{R}}(\Omega,\mathbb{S})\cap\mathsf{D}\mathbf{D}(\Omega,\mathbb{S})\hookrightarrow\mathsf{L}^2(\Omega,\mathbb{S})\tag{4.1}$$

is compact. To show the second compact embedding, let $(\mathbf{N}_n)\subset\mathsf{R}_{\mathrm{sym}}(\Omega,\mathbb{T})\cap\mathring{\mathsf{D}}(\Omega,\mathbb{T})$ be a bounded sequence. By Theorem 3.14 and Lemma 3.13 we have

$$\begin{aligned}\mathsf{R}_{\mathrm{sym}}(\Omega,\mathbb{T})\cap\mathring{\mathsf{D}}(\Omega,\mathbb{T})&=\big(\mathsf{R}_{\mathrm{sym},0}(\Omega,\mathbb{T})\cap\mathring{\mathsf{D}}(\Omega,\mathbb{T})\big)\oplus_{\mathsf{L}^2(\Omega,\mathbb{T})}\big(\mathsf{R}_{\mathrm{sym}}(\Omega,\mathbb{T})\cap\mathring{\mathsf{D}}_0(\Omega,\mathbb{T})\big),\\\mathsf{R}_{\mathrm{sym},0}(\Omega,\mathbb{T})&=\operatorname{dev}\operatorname{Grad}\mathsf{H}^1(\Omega),\\\mathring{\mathsf{D}}_0(\Omega,\mathbb{T})&=\operatorname{Rot}\big(\mathring{\mathsf{R}}(\Omega,\mathbb{S})\cap\mathsf{D}\mathbf{D}_0(\Omega,\mathbb{S})\big)\end{aligned}$$

with linear and continuous potential operators. Therefore, we can decompose

$$\mathsf{R}_{\mathrm{sym}}(\Omega,\mathbb{T})\cap\mathring{\mathsf{D}}(\Omega,\mathbb{T})\ni\mathbf{N}_n=\mathbf{N}_{n,\mathsf{r}}+\mathbf{N}_{n,\mathsf{d}}\in\big(\mathsf{R}_{\mathrm{sym},0}(\Omega,\mathbb{T})\cap\mathring{\mathsf{D}}(\Omega,\mathbb{T})\big)\oplus_{\mathsf{L}^2(\Omega,\mathbb{T})}\big(\mathsf{R}_{\mathrm{sym}}(\Omega,\mathbb{T})\cap\mathring{\mathsf{D}}_0(\Omega,\mathbb{T})\big)$$

with $\mathbf{N}_{n,\mathsf{r}} \in \operatorname{dev}\operatorname{Grad}\mathsf{H}^1(\Omega) \cap \mathring{\mathbf{D}}(\Omega,\mathbb{T})$, $\operatorname{sym}\operatorname{Rot}\mathbf{N}_{n,\mathsf{d}} = \operatorname{sym}\operatorname{Rot}\mathbf{N}_n$, $\mathbf{N}_{n,r} = \operatorname{dev}\operatorname{Grad}E_n$, $E_n \in \mathsf{H}^1(\Omega)$, as well as $\mathbf{N}_{n,\mathsf{d}} \in \mathbf{R}_{\mathrm{sym}}(\Omega,\mathbb{T}) \cap \operatorname{Rot}\big(\mathring{\mathbf{R}}(\Omega,\mathbb{S}) \cap \mathsf{D}\mathbf{D}_0(\Omega,\mathbb{S})\big)$, $\operatorname{Div}\mathbf{N}_{n,\mathsf{r}} = \operatorname{Div}\mathbf{N}_n$, and $\mathbf{N}_{n,\mathsf{d}} = \operatorname{Rot}\mathbf{M}_n$, $\mathbf{M}_n \in \mathring{\mathbf{R}}(\Omega,\mathbb{S}) \cap \mathsf{D}\mathbf{D}_0(\Omega,\mathbb{S})$, and both E_n and $\mathbf{M}_n$ depend continuously on $\mathbf{N}_n$, i.e.,

$$|E_n|_{\mathsf{H}^1(\Omega)} \leq c\,|\mathbf{N}_{n,\mathsf{r}}|_{\mathbf{L}^2(\Omega)} \leq c\,|\mathbf{N}_n|_{\mathbf{L}^2(\Omega)}, \qquad |\mathbf{M}_n|_{\mathbf{R}(\Omega)} \leq c\,|\mathbf{N}_{n,\mathsf{d}}|_{\mathbf{L}^2(\Omega)} \leq c\,|\mathbf{N}_n|_{\mathbf{L}^2(\Omega)}.$$

By Rellich's selection theorem and the previously proved compact embedding (4.1), there exist subsequences, again denoted by (E_n) and $(\mathbf{M}_n)$, such that (E_n) converges in $\mathsf{L}^2(\Omega)$ and $(\mathbf{M}_n)$ converges in $\mathbf{L}^2(\Omega)$. Thus with $\mathbf{N}_{n,m} := \mathbf{N}_n - \mathbf{N}_m$, and similarly for $\mathbf{N}_{n,m,\mathsf{r}}$, $\mathbf{N}_{n,m,\mathsf{d}}$, $E_{n,m}$, $\mathbf{M}_{n,m}$, we see

$$\begin{aligned}
|\mathbf{N}_{n,m,\mathsf{r}}|^2_{\mathbf{L}^2(\Omega)} &= \langle \mathbf{N}_{n,m,\mathsf{r}}, \operatorname{dev}\operatorname{Grad}E_{n,m}\rangle_{\mathbf{L}^2(\Omega)} = -\langle \operatorname{Div}\mathbf{N}_{n,m,\mathsf{r}}, E_{n,m}\rangle_{\mathsf{L}^2(\Omega)}\\
&= -\langle \operatorname{Div}\mathbf{N}_{n,m}, E_{n,m}\rangle_{\mathsf{L}^2(\Omega)} \leq c\,|E_{n,m}|_{\mathsf{L}^2(\Omega)},\\
|\mathbf{N}_{n,m,\mathsf{d}}|^2_{\mathbf{L}^2(\Omega)} &= \langle \mathbf{N}_{n,m,\mathsf{d}}, \operatorname{Rot}\mathbf{M}_{n,m}\rangle_{\mathbf{L}^2(\Omega)} = \langle \operatorname{sym}\operatorname{Rot}\mathbf{N}_{n,m,\mathsf{d}}, \mathbf{M}_{n,m}\rangle_{\mathbf{L}^2(\Omega)}\\
&= \langle \operatorname{sym}\operatorname{Rot}\mathbf{N}_{n,m}, \mathbf{M}_{n,m}\rangle_{\mathbf{L}^2(\Omega)} \leq c\,|\mathbf{M}_{n,m}|_{\mathbf{L}^2(\Omega)}.
\end{aligned}$$

Note, that here the symmetry of $\mathbf{M}_{n,m}$ is crucial. Finally, $(\mathbf{N}_n)$ is a Cauchy sequence in $\mathbf{L}^2(\Omega,\mathbb{T})$. So

$$\mathbf{R}_{\mathrm{sym}}(\Omega,\mathbb{T}) \cap \mathring{\mathbf{D}}(\Omega,\mathbb{T}) \hookrightarrow \mathbf{L}^2(\Omega,\mathbb{T}) \tag{4.2}$$

is compact. □

In the Appendix we will prove the following lemma.

Lemma 4.2. *Let* $\varphi \in \mathring{\mathsf{C}}^\infty(\mathbb{R}^3)$.

- **(i)** *If* $\mathbf{M} \in \mathring{\mathbf{R}}(\Omega)$ *resp.* $\mathring{\mathbf{R}}(\Omega,\mathbb{S})$ *resp.* $\mathring{\mathbf{R}}(\Omega,\mathbb{T})$*, then* $\varphi\mathbf{M} \in \mathring{\mathbf{R}}(\Omega)$ *resp.* $\mathring{\mathbf{R}}(\Omega,\mathbb{S})$ *resp.* $\mathring{\mathbf{R}}(\Omega,\mathbb{T})$ *and*
$$\operatorname{Rot}(\varphi\mathbf{M}) = \varphi\operatorname{Rot}\mathbf{M} + \operatorname{grad}\varphi \times \mathbf{M}. \tag{4.3}$$
- **(ii)** *If* $\mathbf{M} \in \mathbf{R}(\Omega)$ *resp.* $\mathbf{R}(\Omega,\mathbb{S})$ *resp.* $\mathbf{R}(\Omega,\mathbb{T})$*, then* $\varphi\mathbf{M} \in \mathbf{R}(\Omega)$ *resp.* $\mathbf{R}(\Omega,\mathbb{S})$ *resp.* $\mathbf{R}(\Omega,\mathbb{T})$ *and* (4.3) *holds.*
- **(iii)** *If* $\mathbf{N} \in \mathring{\mathbf{D}}(\Omega)$ *resp.* $\mathring{\mathbf{D}}(\Omega,\mathbb{T})$ *resp.* $\mathring{\mathbf{D}}(\Omega,\mathbb{S})$*, then* $\varphi\mathbf{N} \in \mathring{\mathbf{D}}(\Omega)$ *resp.* $\mathring{\mathbf{D}}(\Omega,\mathbb{T})$ *resp.* $\mathring{\mathbf{D}}(\Omega,\mathbb{S})$ *and*
$$\operatorname{Div}(\varphi\mathbf{N}) = \varphi\operatorname{Div}\mathbf{N} + \operatorname{grad}\varphi \cdot \mathbf{N}. \tag{4.4}$$
- **(iv)** *If* $\mathbf{N} \in \mathbf{D}(\Omega)$ *resp.* $\mathbf{D}(\Omega,\mathbb{T})$ *resp.* $\mathbf{D}(\Omega,\mathbb{S})$*, then* $\varphi\mathbf{N} \in \mathbf{D}(\Omega)$ *resp.* $\mathbf{D}(\Omega,\mathbb{T})$ *resp.* $\mathbf{D}(\Omega,\mathbb{S})$ *and* (4.4) *holds.*
- **(v)** *If* $\mathbf{N} \in \mathbf{R}_{\mathrm{sym}}(\Omega,\mathbb{T})$*, then* $\varphi\mathbf{N} \in \mathbf{R}_{\mathrm{sym}}(\Omega,\mathbb{T})$ *and*
$$\operatorname{sym}\operatorname{Rot}(\varphi\mathbf{N}) = \varphi\operatorname{sym}\operatorname{Rot}\mathbf{N} + \operatorname{sym}(\operatorname{grad}\varphi \times \mathbf{N}).$$
- **(vi)** *If* $\mathbf{M} \in \mathsf{D}\mathbf{D}(\Omega,\mathbb{S})$*, then* $\varphi\mathbf{M} \in \mathsf{D}\mathbf{D}^{0,-1}(\Omega,\mathbb{S})$ *and*
$$\operatorname{div}\operatorname{Div}(\varphi\mathbf{M}) = \varphi\operatorname{div}\operatorname{Div}\mathbf{M} + 2\operatorname{grad}\varphi\cdot\operatorname{Div}\mathbf{M} + \operatorname{tr}(\operatorname{Grad}\operatorname{grad}\varphi\cdot\mathbf{M}).$$

By mollifying these formulas extend to $\varphi \in \mathring{\mathsf{C}}^{0,1}(\mathbb{R}^3)$.

Here $\operatorname{grad}\varphi\times$ resp. $\operatorname{grad}\varphi\cdot$ is applied rows-wise to a tensor $\mathbf{M}$ and we see $\operatorname{grad}\varphi\cdot\mathbf{M} = \mathbf{M}\operatorname{grad}\varphi$. Moreover, we introduce

$$\mathsf{D}\mathbf{D}^{0,-1}(\Omega,\mathbb{S}) = \{\mathbf{M} \in \mathbf{L}^2(\Omega,\mathbb{S}) \,:\, \operatorname{div}\operatorname{Div}\mathbf{M} \in \mathsf{H}^{-1}(\Omega)\}.$$

Lemma 4.3. *The Helmholtz type decomposition*

$$\mathsf{D}\mathbf{D}^{0,-1}(\Omega,\mathbb{S}) = \mathring{\mathsf{H}}^1(\Omega)\cdot\mathbf{I} \dotplus \mathsf{D}\mathbf{D}_0(\Omega,\mathbb{S})$$

holds, where $\dotplus$ *denotes the direct sum. More precisely, for each* $\mathbf{M} \in \mathsf{D}\mathbf{D}^{0,-1}(\Omega,\mathbb{S})$ *there are unique* $u \in \overset{\circ}{\mathsf{H}}{}^1(\Omega)$ *and* $\mathbf{M}_0 \in \mathsf{D}\mathbf{D}_0(\Omega,\mathbb{S})$ *such that* $\mathbf{M} = u\,\mathbf{I} + \mathbf{M}_0$*. The scalar function* $u \in \overset{\circ}{\mathsf{H}}{}^1(\Omega)$ *is given as the unique solution of the Dirichlet Poisson problem*

$$\langle \operatorname{grad} u, \operatorname{grad} \varphi\rangle_{\mathsf{L}^2(\Omega)} = -\langle \operatorname{div} \operatorname{Div} \mathbf{M}, \varphi\rangle_{\mathsf{H}^{-1}(\Omega)} \quad \text{for all} \quad \varphi \in \overset{\circ}{\mathsf{H}}{}^1(\Omega),$$

and the decomposition is continuous, more precisely there exists $c > 0$*, such that*

$$|u|_{\mathsf{H}^1(\Omega)} \leq c\,|\operatorname{div} \operatorname{Div} \mathbf{M}|_{\mathsf{H}^{-1}(\Omega)}, \qquad |\mathbf{M} - u\,\mathbf{I}|_{\mathsf{L}^2(\Omega)} \leq c\,|\mathbf{M}|_{\mathsf{D}\mathbf{D}^{0,-1}(\Omega,\mathbb{S})}.$$

Proof. The unique solution $u \in \overset{\circ}{\mathsf{H}}{}^1(\Omega)$ satisfies

$$\mathsf{H}^{-1}(\Omega) \ni \operatorname{div} \operatorname{Div} u\,\mathbf{I} = \operatorname{div} \operatorname{grad} u = \operatorname{div} \operatorname{Div} \mathbf{M},$$

i.e., $\mathbf{M}_0 := \mathbf{M} - u\,\mathbf{I} \in \mathsf{D}\mathbf{D}_0(\Omega,\mathbb{S})$, which shows the decomposition. Moreover,

$$|u|_{\mathsf{H}^1(\Omega)} \leq (1 + c_{\mathsf{g}}^2)\,|\operatorname{div} \operatorname{Div} \mathbf{M}|_{\mathsf{H}^{-1}(\Omega)}$$

shows, that u depends continuously on $\mathbf{M}$ and hence also $\mathbf{M}_0$ since

$$|\mathbf{M}_0|_{\mathsf{L}^2(\Omega)} \leq |\mathbf{M}|_{\mathsf{L}^2(\Omega)} + |u|_{\mathsf{L}^2(\Omega)} \leq \sqrt{2}\,(1 + c_{\mathsf{g}}^2)\,|\mathbf{M}|_{\mathsf{D}\mathbf{D}^{0,-1}(\Omega,\mathbb{S})}.$$

Let $u\,\mathbf{I} \in \mathsf{D}\mathbf{D}_0(\Omega,\mathbb{S})$ with $u \in \overset{\circ}{\mathsf{H}}{}^1(\Omega)$. Then $0 = \operatorname{div} \operatorname{Div} u\,\mathbf{I} = \operatorname{div} \operatorname{grad} u = \Delta u$, yielding $u = 0$. Hence, the decomposition is direct, completing the proof. □

Lemma 4.4. *The embeddings* (3.7)*, i.e.,*

$$\overset{\circ}{\mathbf{R}}(\Omega,\mathbb{S}) \cap \mathsf{D}\mathbf{D}(\Omega,\mathbb{S}) \hookrightarrow \mathbf{L}^2(\Omega,\mathbb{S}), \qquad \mathbf{R}_{\mathrm{sym}}(\Omega,\mathbb{T}) \cap \overset{\circ}{\mathbf{D}}(\Omega,\mathbb{T}) \hookrightarrow \mathbf{L}^2(\Omega,\mathbb{T}),$$

and (3.6) *are compact.*

Proof. Let (U_i) be an open covering of $\overline{\Omega}$, such that $\Omega_i := \Omega \cap U_i$ is topologically trivial for all i. As $\overline{\Omega}$ is compact, there is a finite subcovering denoted by $(U_i)_{i=1,\dots,I}$ with $I \in \mathbb{N}$. Let (φ_i) with $\varphi_i \in \overset{\circ}{\mathsf{C}}{}^\infty(U_i)$ be a partition of unity subordinate to (U_i). Suppose $(\mathbf{N}_n) \subset \mathbf{R}_{\mathrm{sym}}(\Omega,\mathbb{T}) \cap \overset{\circ}{\mathbf{D}}(\Omega,\mathbb{T})$ is a bounded sequence. Then $\mathbf{N}_n = \sum_{i=1}^I \varphi_i \mathbf{N}_n$ and $(\varphi_i \mathbf{N}_n) \subset \mathbf{R}_{\mathrm{sym}}(\Omega_i,\mathbb{T}) \cap \overset{\circ}{\mathbf{D}}(\Omega_i,\mathbb{T})$ is a bounded sequence for all i by Lemma 4.2. As Ω_i is topologically trivial, there exists a subsequence again denoted by $(\varphi_i \mathbf{N}_n)$, which is a Cauchy sequence in $\mathbf{L}^2(\Omega_i)$ by Lemma 4.1. Picking successively subsequences yields that $(\varphi_i \mathbf{N}_n)$ is a Cauchy sequence in $\mathbf{L}^2(\Omega_i)$ for all i. Hence $(\mathbf{N}_n)$ is a Cauchy sequence in $\mathbf{L}^2(\Omega)$. So the second embedding of the lemma is compact. Let $(\mathbf{M}_n) \subset \overset{\circ}{\mathbf{R}}(\Omega,\mathbb{S}) \cap \mathsf{D}\mathbf{D}(\Omega,\mathbb{S})$ be a bounded sequence. Then $\mathbf{M}_n = \sum_{i=1}^I \varphi_i \mathbf{M}_n$ and $(\varphi_i \mathbf{M}_n) \subset \overset{\circ}{\mathbf{R}}(\Omega_i,\mathbb{S}) \cap \mathsf{D}\mathbf{D}^{0,-1}(\Omega_i,\mathbb{S})$ is a bounded sequence for all i by Lemma 4.2 as $|\operatorname{Div} \mathbf{M}_n|_{\mathsf{H}^{-1}(\Omega)} \leq |\mathbf{M}_n|_{\mathsf{L}^2(\Omega)}$. Using Lemma 4.3 we decompose

$$\varphi_i \mathbf{M}_n = u_{i,n}\,\mathbf{I} + \mathbf{M}_{0,i,n} \in \overset{\circ}{\mathsf{H}}{}^1(\Omega_i) \cdot \mathbf{I} \dotplus \big(\overset{\circ}{\mathbf{R}}(\Omega_i,\mathbb{S}) \cap \mathsf{D}\mathbf{D}_0(\Omega_i,\mathbb{S})\big).$$

Moreover, $(u_{i,n})$ is bounded in $\overset{\circ}{\mathsf{H}}{}^1(\Omega_i)$ and $(\mathbf{M}_{0,i,n})$ is bounded in $\big(\overset{\circ}{\mathbf{R}}(\Omega_i,\mathbb{S}) \cap \mathsf{D}\mathbf{D}_0(\Omega_i,\mathbb{S})\big)$. By Rellich's selection theorem and Lemma 4.1 as well as picking successively subsequences we get that $(\varphi_i \mathbf{M}_n)$ is a Cauchy sequence in $\mathbf{L}^2(\Omega_i)$ for all i. Hence $(\mathbf{M}_n)$ is a Cauchy sequence in $\mathbf{L}^2(\Omega)$, showing that the first embedding of the lemma is also compact, finishing the proof. □

5. The Grad grad- and div Div-Complexes Revisited

Utilizing the crucial compact embeddings of Lemma 4.4, we can apply the functional analysis toolbox Section 2.1 to the operators $\overset{\circ}{\operatorname{Rot}}_{\mathbb{S}}$ and $(\overset{\circ}{\operatorname{Rot}}_{\mathbb{S}})^* = \operatorname{sym} \operatorname{Rot}_{\mathbb{T}}$ as well, and obtain the following theorem.

Theorem 5.1. *The ranges*

$$R(\mathring{\operatorname{Rot}}_{\mathbb{S}})=\operatorname{Rot}\mathring{\mathsf{R}}(\Omega,\mathbb{S}),\qquad R(\operatorname{sym}\operatorname{Rot}_{\mathbb{T}})=\operatorname{sym}\operatorname{Rot}\mathsf{R}_{\mathrm{sym}}(\Omega,\mathbb{T})$$

are closed and the cohomology groups

$$\mathcal{H}_{\mathsf{DD}}(\Omega,\mathbb{S}):=\mathring{\mathsf{R}}_0(\Omega,\mathbb{S})\cap\mathsf{D}\mathbf{D}_0(\Omega,\mathbb{S}),\qquad \mathcal{H}_{\mathrm{sym}}(\Omega,\mathbb{T}):=\mathring{\mathsf{D}}_0(\Omega,\mathbb{T})\cap\mathsf{R}_{\mathrm{sym},0}(\Omega,\mathbb{T})$$

are finite dimensional. Moreover, Lemma 3.8 and Lemma 3.10 hold and all occurring ranges are closed. The sequences in Remark 3.9, i.e.,

$$\{0\}\xrightarrow{\;0\;}\mathring{\mathsf{H}}^2(\Omega)\xrightarrow{\operatorname{Grad}\mathring{\operatorname{grad}}}\mathring{\mathsf{R}}(\Omega;\mathbb{S})\xrightarrow{\mathring{\operatorname{Rot}}_{\mathbb{S}}}\mathring{\mathsf{D}}(\Omega,\mathbb{T})\xrightarrow{\mathring{\operatorname{Div}}_{\mathbb{T}}}\mathsf{L}^2(\Omega)\xrightarrow{\pi_{\mathsf{RT}_0}}\mathsf{RT}_0$$

and its adjoint

$$\{0\}\xleftarrow{\;0\;}\mathsf{L}^2(\Omega)\xleftarrow{\operatorname{div}\operatorname{Div}_{\mathbb{S}}}\mathsf{D}\mathbf{D}(\Omega,\mathbb{S})\xleftarrow{\operatorname{sym}\operatorname{Rot}_{\mathbb{T}}}\mathsf{R}_{\mathrm{sym}}(\Omega,\mathbb{T})\xleftarrow{-\operatorname{dev}\operatorname{Grad}}\mathsf{H}^1(\Omega)\xleftarrow{\iota_{\mathsf{RT}_0}}\mathsf{RT}_0,$$

are closed. They are also exact and $\mathcal{H}_{\mathsf{DD}}(\Omega,\mathbb{S})=\{0\}$, $\mathcal{H}_{\mathrm{sym}}(\Omega,\mathbb{T})=\{0\}$ hold, if Ω is topologically trivial. Especially, the Helmholtz type decompositions

$$\begin{aligned}
\mathsf{L}^2(\Omega,\mathbb{S})&=\operatorname{Grad}\operatorname{grad}\mathring{\mathsf{H}}^2(\Omega)\oplus_{\mathsf{L}^2(\Omega,\mathbb{S})}\mathsf{D}\mathbf{D}_0(\Omega,\mathbb{S})\\
&=\mathring{\mathsf{R}}_0(\Omega,\mathbb{S})\oplus_{\mathsf{L}^2(\Omega,\mathbb{S})}\operatorname{sym}\operatorname{Rot}\mathsf{R}_{\mathrm{sym}}(\Omega,\mathbb{T})\\
&=\operatorname{Grad}\operatorname{grad}\mathring{\mathsf{H}}^2(\Omega)\oplus_{\mathsf{L}^2(\Omega,\mathbb{S})}\mathcal{H}_{\mathsf{DD}}(\Omega,\mathbb{S})\oplus_{\mathsf{L}^2(\Omega,\mathbb{S})}\operatorname{sym}\operatorname{Rot}\mathsf{R}_{\mathrm{sym}}(\Omega,\mathbb{T}),\\
\mathsf{L}^2(\Omega,\mathbb{T})&=\operatorname{Rot}\mathring{\mathsf{R}}(\Omega,\mathbb{S})\oplus_{\mathsf{L}^2(\Omega,\mathbb{T})}\mathsf{R}_{\mathrm{sym},0}(\Omega,\mathbb{T})\\
&=\mathring{\mathsf{D}}_0(\Omega,\mathbb{T})\oplus_{\mathsf{L}^2(\Omega,\mathbb{T})}\operatorname{dev}\operatorname{Grad}\mathsf{H}^1(\Omega)\\
&=\operatorname{Rot}\mathring{\mathsf{R}}(\Omega,\mathbb{S})\oplus_{\mathsf{L}^2(\Omega,\mathbb{T})}\mathcal{H}_{\mathrm{sym}}(\Omega,\mathbb{T})\oplus_{\mathsf{L}^2(\Omega,\mathbb{T})}\operatorname{dev}\operatorname{Grad}\mathsf{H}^1(\Omega)
\end{aligned}$$

are valid. The ranges

$$\begin{aligned}
\mathring{\mathsf{R}}_0(\Omega,\mathbb{S})\ominus_{\mathsf{L}^2(\Omega,\mathbb{S})}\mathcal{H}_{\mathsf{DD}}(\Omega,\mathbb{S})&=\operatorname{Grad}\operatorname{grad}\mathring{\mathsf{H}}^2(\Omega),\\
\mathsf{L}^2(\Omega)&=\operatorname{div}\operatorname{Div}\mathsf{D}\mathbf{D}(\Omega,\mathbb{S})=\operatorname{div}\operatorname{Div}\big(\mathsf{D}\mathbf{D}(\Omega,\mathbb{S})\cap\operatorname{Grad}\operatorname{grad}\mathring{\mathsf{H}}^2(\Omega)\big),\\
\mathring{\mathsf{D}}_0(\Omega,\mathbb{T})\ominus_{\mathsf{L}^2(\Omega,\mathbb{T})}\mathcal{H}_{\mathrm{sym}}(\Omega,\mathbb{T})&=\operatorname{Rot}\mathring{\mathsf{R}}(\Omega,\mathbb{S})=\operatorname{Rot}\big(\mathring{\mathsf{R}}(\Omega,\mathbb{S})\cap\operatorname{sym}\operatorname{Rot}\mathsf{R}_{\mathrm{sym}}(\Omega,\mathbb{T})\big),\\
\mathsf{D}\mathbf{D}_0(\Omega,\mathbb{S})\ominus_{\mathsf{L}^2(\Omega,\mathbb{S})}\mathcal{H}_{\mathsf{DD}}(\Omega,\mathbb{S})&=\operatorname{sym}\operatorname{Rot}\mathsf{R}_{\mathrm{sym}}(\Omega,\mathbb{T})=\operatorname{sym}\operatorname{Rot}\big(\mathsf{R}_{\mathrm{sym}}(\Omega,\mathbb{T})\cap\operatorname{Rot}\mathring{\mathsf{R}}(\Omega,\mathbb{S})\big),\\
\mathsf{RT}_0^{\perp_{\mathsf{L}^2(\Omega)}}&=\operatorname{Div}\mathring{\mathsf{D}}(\Omega,\mathbb{T})=\operatorname{Div}\big(\mathring{\mathsf{D}}(\Omega,\mathbb{T})\cap\operatorname{dev}\operatorname{Grad}\mathsf{H}^1(\Omega)\big),\\
\mathsf{R}_{\mathrm{sym},0}(\Omega,\mathbb{T})\ominus_{\mathsf{L}^2(\Omega,\mathbb{T})}\mathcal{H}_{\mathrm{sym}}(\Omega,\mathbb{T})&=\operatorname{dev}\operatorname{Grad}\mathsf{H}^1(\Omega)=\operatorname{dev}\operatorname{Grad}\big(\mathsf{H}^1(\Omega)\cap\mathsf{RT}_0^{\perp_{\mathsf{L}^2(\Omega)}}\big)
\end{aligned}$$

are closed. The more regular potentials on the right hand sides are uniquely determined and depend linearly and continuously on the data. Especially, there exist positive constants c_{Gg}, c_{D}, c_{R} such that the Friedrichs/Poincaré type estimates

$$\begin{aligned}
&\forall\,u\in\mathring{\mathsf{H}}^2(\Omega) && |u|_{\mathsf{L}^2(\Omega)}\leq c_{\mathsf{Gg}}\,|\operatorname{Grad}\operatorname{grad}u|_{\mathsf{L}^2(\Omega)},\\
&\forall\,\mathbf{M}\in\mathsf{D}\mathbf{D}(\Omega,\mathbb{S})\cap\operatorname{Grad}\operatorname{grad}\mathring{\mathsf{H}}^2(\Omega) && |\mathbf{M}|_{\mathsf{L}^2(\Omega)}\leq c_{\mathsf{Gg}}\,|\operatorname{div}\operatorname{Div}\mathbf{M}|_{\mathsf{L}^2(\Omega)},\\
&\forall\,\mathbf{N}\in\mathring{\mathsf{D}}(\Omega,\mathbb{T})\cap\operatorname{dev}\operatorname{Grad}\mathsf{H}^1(\Omega) && |\mathbf{N}|_{\mathsf{L}^2(\Omega)}\leq c_{\mathsf{D}}\,|\operatorname{Div}\mathbf{N}|_{\mathsf{L}^2(\Omega)},\\
&\forall\,E\in\mathsf{H}^1(\Omega)\cap\mathsf{RT}_0^{\perp_{\mathsf{L}^2(\Omega)}} && |E|_{\mathsf{L}^2(\Omega)}\leq c_{\mathsf{D}}\,|\operatorname{dev}\operatorname{Grad}E|_{\mathsf{L}^2(\Omega)},
\end{aligned}$$

$$\forall\,\mathbf{M}\in\mathring{\mathsf{R}}(\Omega,\mathbb{S})\cap\operatorname{sym}\operatorname{Rot}\mathsf{R}_{\mathrm{sym}}(\Omega,\mathbb{T})\qquad |\mathbf{M}|_{\mathsf{L}^2(\Omega)}\le c_{\mathsf{R}}\,|\operatorname{Rot}\mathbf{M}|_{\mathsf{L}^2(\Omega)},$$

$$\forall\,\mathbf{N}\in\mathsf{R}_{\mathrm{sym}}(\Omega,\mathbb{T})\cap\operatorname{Rot}\mathring{\mathsf{R}}(\Omega,\mathbb{S})\qquad |\mathbf{N}|_{\mathsf{L}^2(\Omega)}\le c_{\mathsf{R}}\,|\operatorname{sym}\operatorname{Rot}\mathbf{N}|_{\mathsf{L}^2(\Omega)}$$

hold. Moreover, the inverse operators

$$\begin{aligned}
(\mathring{\operatorname{Grad}}\operatorname{grad})^{-1}&:\operatorname{Grad}\operatorname{grad}\mathring{\mathsf{H}}^2(\Omega)\longrightarrow\mathring{\mathsf{H}}^2(\Omega),\\
(\operatorname{div}\operatorname{Div}_{\mathbb{S}})^{-1}&:\mathsf{L}^2(\Omega)\longrightarrow\mathsf{D}\mathbf{D}(\Omega,\mathbb{S})\cap\operatorname{Grad}\operatorname{grad}\mathring{\mathsf{H}}^2(\Omega),\\
(\mathring{\operatorname{Div}}_{\mathbb{T}})^{-1}&:\mathsf{RT}_0^{\perp_{\mathsf{L}^2(\Omega)}}\longrightarrow\mathring{\mathbf{D}}(\Omega,\mathbb{T})\cap\operatorname{dev}\operatorname{Grad}\mathsf{H}^1(\Omega),\\
(\operatorname{dev}\operatorname{Grad})^{-1}&:\operatorname{dev}\operatorname{Grad}\mathsf{H}^1(\Omega)\longrightarrow\mathsf{H}^1(\Omega)\cap\mathsf{RT}_0^{\perp_{\mathsf{L}^2(\Omega)}},\\
(\mathring{\operatorname{Rot}}_{\mathbb{S}})^{-1}&:\operatorname{Rot}\mathring{\mathsf{R}}(\Omega,\mathbb{S})\longrightarrow\mathring{\mathsf{R}}(\Omega,\mathbb{S})\cap\operatorname{sym}\operatorname{Rot}\mathsf{R}_{\mathrm{sym}}(\Omega,\mathbb{T}),\\
(\operatorname{sym}\operatorname{Rot}_{\mathbb{T}})^{-1}&:\operatorname{sym}\operatorname{Rot}\mathsf{R}_{\mathrm{sym}}(\Omega,\mathbb{T})\longrightarrow\mathsf{R}_{\mathrm{sym}}(\Omega,\mathbb{T})\cap\operatorname{Rot}\mathring{\mathsf{R}}(\Omega,\mathbb{S})
\end{aligned}$$

are continuous with norms $(1+c_{\mathsf{Gg}}^2)^{1/2}$ *resp.* $(1+c_{\mathsf{D}}^2)^{1/2}$*, resp.* $(1+c_{\mathsf{R}}^2)^{1/2}$ *and their modifications*

$$\begin{aligned}
(\mathring{\operatorname{Grad}}\operatorname{grad})^{-1}&:\operatorname{Grad}\operatorname{grad}\mathring{\mathsf{H}}^2(\Omega)\longrightarrow\mathring{\mathsf{H}}^1(\Omega)\subset\mathsf{L}^2(\Omega),\\
(\operatorname{div}\operatorname{Div}_{\mathbb{S}})^{-1}&:\mathsf{L}^2(\Omega)\longrightarrow\operatorname{Grad}\operatorname{grad}\mathring{\mathsf{H}}^2(\Omega)\subset\mathsf{L}^2(\Omega,\mathbb{S}),\\
(\mathring{\operatorname{Div}}_{\mathbb{T}})^{-1}&:\mathsf{RT}_0^{\perp_{\mathsf{L}^2(\Omega)}}\longrightarrow\operatorname{dev}\operatorname{Grad}\mathsf{H}^1(\Omega)\subset\mathsf{L}^2(\Omega,\mathbb{T}),\\
(\operatorname{dev}\operatorname{Grad})^{-1}&:\operatorname{dev}\operatorname{Grad}\mathsf{H}^1(\Omega)\longrightarrow\mathsf{RT}_0^{\perp_{\mathsf{L}^2(\Omega)}}\subset\mathsf{L}^2(\Omega),\\
(\mathring{\operatorname{Rot}}_{\mathbb{S}})^{-1}&:\operatorname{Rot}\mathring{\mathsf{R}}(\Omega,\mathbb{S})\longrightarrow\operatorname{sym}\operatorname{Rot}\mathsf{R}_{\mathrm{sym}}(\Omega,\mathbb{T})\subset\mathsf{L}^2(\Omega,\mathbb{S}),\\
(\operatorname{sym}\operatorname{Rot}_{\mathbb{T}})^{-1}&:\operatorname{sym}\operatorname{Rot}\mathsf{R}_{\mathrm{sym}}(\Omega,\mathbb{T})\longrightarrow\operatorname{Rot}\mathring{\mathsf{R}}(\Omega,\mathbb{S})\subset\mathsf{L}^2(\Omega,\mathbb{T})
\end{aligned}$$

are compact.

6. Application to Biharmonic Problems

By $\Delta^2=\operatorname{div}\operatorname{Div}\operatorname{Grad}\operatorname{grad}$, a standard (primal) variational formulation of (1.1) in $\mathbb{R}^3$ reads as follows: For given $f\in\mathsf{H}^{-2}(\Omega)$, find $u\in\mathring{\mathsf{H}}^2(\Omega)$ such that

$$\langle\operatorname{Grad}\operatorname{grad}u,\operatorname{Grad}\operatorname{grad}\phi\rangle_{\mathsf{L}^2(\Omega)}=\langle f,\phi\rangle_{\mathsf{H}^{-2}(\Omega)}\quad\text{for all}\quad\phi\in\mathring{\mathsf{H}}^2(\Omega).\tag{6.1}$$

Existence, uniqueness, and continuous dependence on f of a solution to (6.1) is guaranteed by the theorem of Lax-Milgram, see, e.g., [16, 15] or Lemma 3.3. Note that then

$$\mathbf{M}:=\operatorname{Grad}\operatorname{grad}u\in\mathring{\mathsf{R}}_0(\Omega,\mathbb{S})\ominus_{\mathsf{L}^2(\Omega,\mathbb{S})}\mathcal{H}_{\mathsf{D}\mathbf{D}}(\Omega,\mathbb{S})\subset\mathsf{L}^2(\Omega,\mathbb{S})$$

with $\operatorname{div}\operatorname{Div}\mathbf{M}=f\in\mathsf{H}^{-2}(\Omega)$. In other words the operator

$$\operatorname{div}\operatorname{Div}:\mathsf{L}^2(\Omega,\mathbb{S})\to\mathsf{H}^{-2}(\Omega)\tag{6.2}$$

is surjective and

$$\operatorname{div}\operatorname{Div}:\mathring{\mathsf{R}}_0(\Omega,\mathbb{S})\ominus_{\mathsf{L}^2(\Omega,\mathbb{S})}\mathcal{H}_{\mathsf{D}\mathbf{D}}(\Omega,\mathbb{S})\to\mathsf{H}^{-2}(\Omega)\tag{6.3}$$

is bijective and even a topological isomorphism by the bounded inverse theorem. For our decomposition result we need the following variant of the Hilbert complex from Theorem 5.1.

$$\mathsf{RT}_0\xrightarrow{\iota_{\mathsf{RT}_0}}\mathsf{H}^1(\Omega)\xrightarrow{-\operatorname{dev}\operatorname{Grad}}\mathsf{R}_{\mathrm{sym}}(\Omega,\mathbb{T})\xrightarrow{\operatorname{sym}\operatorname{Rot}_{\mathbb{T}}}\mathsf{D}\mathbf{D}^{0,-1}(\Omega,\mathbb{S})\xrightarrow{\operatorname{div}\operatorname{Div}_{\mathbb{S}}}\mathsf{H}^{-1}(\Omega)\xrightarrow{0}\{0\},$$

where we recall $\mathsf{D}\mathbf{D}^{0,-1}(\Omega,\mathbb{S})$ from Lemma 4.3. This is obviously also a closed Hilbert complex as $\operatorname{div}\operatorname{Div}:\mathsf{D}\mathbf{D}^{0,-1}(\Omega,\mathbb{S})\to\mathsf{H}^{-1}(\Omega)$ is surjective as well by (6.2). Observe that

$$\mathbf{H}^1(\Omega,\mathbb{S})\subset\mathsf{D}\mathbf{D}^{0,-1}(\Omega,\mathbb{S})\subset\mathbf{L}^2(\Omega,\mathbb{S}).$$

For right-hand sides $f\in\mathsf{H}^{-1}(\Omega)$ we consider the following mixed variational problem for u and the Hessian $\mathbf{M}$ of u: Find $\mathbf{M}\in\mathsf{D}\mathbf{D}^{0,-1}(\Omega,\mathbb{S})$ and $u\in\mathring{\mathsf{H}}^1(\Omega)$ such that

$$\langle\mathbf{M},\mathbf{\Psi}\rangle_{\mathsf{L}^2(\Omega)}+\langle u,\operatorname{div}\operatorname{Div}\mathbf{\Psi}\rangle_{\mathsf{H}^{-1}(\Omega)}=0\qquad\text{for all}\qquad\mathbf{\Psi}\in\mathsf{D}\mathbf{D}^{0,-1}(\Omega,\mathbb{S}),\tag{6.4}$$

$$\langle\operatorname{div}\operatorname{Div}\mathbf{M},\psi\rangle_{\mathsf{H}^{-1}(\Omega)}=-\langle f,\psi\rangle_{\mathsf{H}^{-1}(\Omega)}\qquad\text{for all}\qquad\psi\in\mathring{\mathsf{H}}^1(\Omega).\tag{6.5}$$

The first row and the second row of this mixed problem are variational formulations of (1.2) and (1.3), respectively. We recall the following two results related to these mixed problems from [14].

Theorem 6.1. *Let $f\in\mathsf{H}^{-1}(\Omega)$. Then:*

- **(i)** *Problem (6.4)-(6.5) is a well-posed saddle point problem.*
- **(ii)** *The variational problems (6.1) and (6.4)-(6.5) are equivalent, i.e., if $u\in\mathring{\mathsf{H}}^2(\Omega)$ solves (6.1), then $\mathbf{M}=-\operatorname{Grad}\operatorname{grad}u$ lies in $\mathsf{D}\mathbf{D}^{0,-1}(\Omega,\mathbb{S})$ with $\operatorname{div}\operatorname{Div}\mathbf{M}=-f$ and $(\mathbf{M},u)$ solves (6.4)-(6.5). And, vice versa, if $(\mathbf{M},u)\in\mathsf{D}\mathbf{D}^{0,-1}(\Omega,\mathbb{S})\times\mathring{\mathsf{H}}^1(\Omega)$ solves (6.4)-(6.5), then $u\in\mathring{\mathsf{H}}^2(\Omega)$ with $\operatorname{Grad}\operatorname{grad}u=-\mathbf{M}$ and u solves (6.1).*

Although only two-dimensional biharmonic problems were considered in [14], the proof of the latter theorem is completely identical for the three-dimensional case. The same holds for Lemma 4.3.

Proof. To show (i), we first note that $(\mathbf{\Phi},\mathbf{\Psi})\mapsto\langle\mathbf{\Phi},\mathbf{\Psi}\rangle_{\mathsf{L}^2(\Omega)}$ is coercive over the kernel of (6.5), i.e., for $\mathbf{\Phi}\in\mathsf{D}\mathbf{D}_0(\Omega,\mathbb{S})$ we have $\langle\mathbf{\Phi},\mathbf{\Phi}\rangle_{\mathsf{L}^2(\Omega)}=|\mathbf{\Phi}|^2_{\mathsf{L}^2(\Omega)}=|\mathbf{\Phi}|^2_{\mathsf{D}\mathbf{D}^{0,-1}(\Omega,\mathbb{S})}$. Moreover, the inf-sup-condition holds, as

$$\begin{aligned}
&\inf_{0\neq\varphi\in\mathring{\mathsf{H}}^1(\Omega)}\ \sup_{0\neq\mathbf{\Phi}\in\mathsf{D}\mathbf{D}^{0,-1}(\Omega,\mathbb{S})}\frac{\langle\varphi,\operatorname{div}\operatorname{Div}\mathbf{\Phi}\rangle_{\mathsf{H}^{-1}(\Omega)}}{|\operatorname{grad}\varphi|_{\mathsf{L}^2(\Omega)}|\mathbf{\Phi}|_{\mathsf{D}\mathbf{D}^{0,-1}(\Omega,\mathbb{S})}}\\
&\geq\inf_{0\neq\varphi\in\mathring{\mathsf{H}}^1(\Omega)}\frac{-\langle\varphi,\operatorname{div}\operatorname{Div}\varphi\,\mathbf{I}\rangle_{\mathsf{H}^{-1}(\Omega)}}{|\operatorname{grad}\varphi|_{\mathsf{L}^2(\Omega)}|\varphi\,\mathbf{I}|_{\mathsf{D}\mathbf{D}^{0,-1}(\Omega,\mathbb{S})}}=\inf_{0\neq\varphi\in\mathring{\mathsf{H}}^1(\Omega)}\frac{|\operatorname{grad}\varphi|_{\mathsf{L}^2(\Omega)}}{\big(|\varphi\,\mathbf{I}|^2_{\mathsf{L}^2(\Omega)}+|\operatorname{div}\operatorname{Div}(\varphi\,\mathbf{I})|^2_{\mathsf{H}^{-1}(\Omega)}\big)^{1/2}}\\
&=\inf_{0\neq\varphi\in\mathring{\mathsf{H}}^1(\Omega)}\frac{|\operatorname{grad}\varphi|_{\mathsf{L}^2(\Omega)}}{\big(3|\varphi|^2_{\mathsf{L}^2(\Omega)}+|\operatorname{grad}\varphi|^2_{\mathsf{L}^2(\Omega)}\big)^{1/2}}\geq(3\,c_{\mathsf{g}}^2+1)^{-1/2}
\end{aligned}$$

by choosing $\mathbf{\Phi}:=-\varphi\,\mathbf{I}\in\mathring{\mathsf{H}}^1(\Omega)\cdot\mathbf{I}\subset\mathsf{D}\mathbf{D}^{0,-1}(\Omega,\mathbb{S})$ and observing

$$\begin{aligned}
-\langle\varphi,\operatorname{div}\operatorname{Div}(\varphi\,\mathbf{I})\rangle_{\mathsf{H}^{-1}(\Omega)}&=-\langle\varphi,\operatorname{div}\operatorname{grad}\varphi\rangle_{\mathsf{H}^{-1}(\Omega)}=|\operatorname{grad}\varphi|^2_{\mathsf{L}^2(\Omega)},\\
|\operatorname{div}\operatorname{Div}(\varphi\,\mathbf{I})|_{\mathsf{H}^{-1}(\Omega)}&=\sup_{0\neq\phi\in\mathring{\mathsf{H}}^1(\Omega)}\frac{\langle\phi,\operatorname{div}\operatorname{grad}\varphi\rangle_{\mathsf{H}^{-1}(\Omega)}}{|\operatorname{grad}\phi|_{\mathsf{L}^2(\Omega)}}\\
&=\sup_{0\neq\phi\in\mathring{\mathsf{H}}^1(\Omega)}\frac{\langle\operatorname{grad}\phi,\operatorname{grad}\varphi\rangle_{\mathsf{L}^2(\Omega)}}{|\operatorname{grad}\phi|_{\mathsf{L}^2(\Omega)}}=|\operatorname{grad}\varphi|_{\mathsf{L}^2(\Omega)}.
\end{aligned}$$

To show (ii), let $u\in\mathring{\mathsf{H}}^2(\Omega)$ solve (6.1). Then $\mathbf{M}:=-\operatorname{Grad}\operatorname{grad}u\in\mathsf{D}\mathbf{D}^{0,-1}(\Omega,\mathbb{S})$ with $\operatorname{div}\operatorname{Div}\mathbf{M}=-f$ in $\mathsf{H}^{-2}(\Omega)$ and hence in $\mathsf{H}^{-1}(\Omega)$. Thus (6.5) holds. Moreover, for $\mathbf{\Psi}\in\mathsf{D}\mathbf{D}^{0,-1}(\Omega,\mathbb{S})$ we see

$$\langle\mathbf{M},\mathbf{\Psi}\rangle_{\mathsf{L}^2(\Omega)}=-\langle\operatorname{Grad}\operatorname{grad}u,\mathbf{\Psi}\rangle_{\mathsf{L}^2(\Omega)}=-\langle u,\operatorname{div}\operatorname{Div}\mathbf{\Psi}\rangle_{\mathsf{H}^{-2}(\Omega)}=-\langle u,\operatorname{div}\operatorname{Div}\mathbf{\Psi}\rangle_{\mathsf{H}^{-1}(\Omega)}$$

and hence (6.4) is true. Therefore, $(\mathbf{M},u)$ solves (6.4)-(6.5). On the other hand, if the solution $(\mathbf{M},u)$ in $\mathsf{D}\mathbf{D}^{0,-1}(\Omega,\mathbb{S})\times\mathring{\mathsf{H}}^1(\Omega)$ solves (6.4)-(6.5), then $\operatorname{div}\operatorname{Div}\mathbf{M}=-f$ in $\mathsf{H}^{-1}(\Omega)$ and (6.4) holds. Especially, (6.4) holds for $\mathbf{\Psi}\in\mathbf{H}^2(\Omega,\mathbb{S})\subset\mathbf{H}^1(\Omega,\mathbb{S})\subset\mathsf{D}\mathbf{D}^{0,-1}(\Omega,\mathbb{S})$, i.e.,

$$-\langle\mathbf{M},\mathbf{\Psi}\rangle_{\mathsf{L}^2(\Omega)}=\langle u,\operatorname{div}\operatorname{Div}\mathbf{\Psi}\rangle_{\mathsf{H}^{-1}(\Omega)}=\langle u,\operatorname{div}\operatorname{Div}\mathbf{\Psi}\rangle_{\mathsf{L}^2(\Omega)}. \tag{6.6}$$

But then (6.6) holds for all $\mathbf{\Psi}\in\mathbf{H}^2(\Omega)$ as $\operatorname{sym}\mathbf{\Psi}\in\mathbf{H}^2(\Omega,\mathbb{S})$ and

$$-\langle\mathbf{M},\mathbf{\Psi}\rangle_{\mathsf{L}^2(\Omega)}=-\langle\mathbf{M},\operatorname{sym}\mathbf{\Psi}\rangle_{\mathsf{L}^2(\Omega)}=\langle u,\operatorname{div}\operatorname{Div}\operatorname{sym}\mathbf{\Psi}\rangle_{\mathsf{L}^2(\Omega)}=\langle u,\operatorname{div}\operatorname{Div}\mathbf{\Psi}\rangle_{\mathsf{L}^2(\Omega)}, \tag{6.7}$$

since $\operatorname{div}\operatorname{Div}\operatorname{skw}\mathbf{\Psi}=0$ by

$$\langle\operatorname{div}\operatorname{Div}\operatorname{skw}\mathbf{\Psi},\phi\rangle_{\mathsf{L}^2(\Omega)}=\langle\operatorname{skw}\mathbf{\Psi},\operatorname{Grad}\operatorname{grad}\phi\rangle_{\mathsf{L}^2(\Omega)}=0$$

for all $\phi\in\mathring{\mathsf{C}}^\infty(\Omega)$. (6.7) yields that $u\in\mathring{\mathsf{H}}^2(\Omega)$ with $\operatorname{Grad}\operatorname{grad}u=-\mathbf{M}$. Finally, for all $\phi\in\mathring{\mathsf{H}}^2(\Omega)$

$$\langle\operatorname{Grad}\operatorname{grad}u,\operatorname{Grad}\operatorname{grad}\phi\rangle_{\mathsf{L}^2(\Omega)}=-\langle\mathbf{M},\operatorname{Grad}\operatorname{grad}\phi\rangle_{\mathsf{L}^2(\Omega)}=-\langle\operatorname{div}\operatorname{Div}\mathbf{M},\phi\rangle_{\mathsf{H}^{-2}(\Omega)}=\langle f,\phi\rangle_{\mathsf{H}^{-2}(\Omega)},$$

showing that $u\in\mathring{\mathsf{H}}^2(\Omega)$ solves (6.1). □

We note that the decomposition of $\mathsf{D}\mathbf{D}^{0,-1}(\Omega,\mathbb{S})$ in Lemma 4.3 is different to the Helmholtz type decomposition of the larger space $\mathbf{L}^2(\Omega,\mathbb{S})$ in Theorem 3.14 and Theorem 5.1 and does not involve the Hessian of scalar functions in $\mathring{\mathsf{H}}^2(\Omega)$. Using the decomposition of $\mathsf{D}\mathbf{D}^{0,-1}(\Omega,\mathbb{S})$ in Lemma 4.3, we have the following decomposition result for the biharmonic problem. Let $(\mathbf{M},u)\in\mathsf{D}\mathbf{D}^{0,-1}(\Omega,\mathbb{S})\times\mathring{\mathsf{H}}^1(\Omega)$ be the unique solution[vi] of (6.4)-(6.5). Using Lemma 4.3 we have the following direct decompositions for $\mathbf{M},\mathbf{\Psi}\in\mathsf{D}\mathbf{D}^{0,-1}(\Omega,\mathbb{S})$

$$\mathbf{M}=p\,\mathbf{I}+\mathbf{M}_0,\quad\mathbf{\Psi}=\varphi\,\mathbf{I}+\mathbf{\Psi}_0,\qquad p,\varphi\in\mathring{\mathsf{H}}^1(\Omega),\quad\mathbf{M}_0,\mathbf{\Psi}_0\in\mathsf{D}\mathbf{D}_0(\Omega,\mathbb{S}).$$

This allows to rewrite (6.4)-(6.5) equivalently in terms of $(p,\mathbf{M}_0,u)$ and for all $(\varphi,\mathbf{\Psi}_0,\psi)$, i.e.,

$$\begin{aligned}\langle p\,\mathbf{I},\varphi\,\mathbf{I}\rangle_{\mathsf{L}^2(\Omega)}+\langle\mathbf{M}_0,\mathbf{\Psi}_0\rangle_{\mathsf{L}^2(\Omega)}+\langle p\,\mathbf{I},\mathbf{\Psi}_0\rangle_{\mathsf{L}^2(\Omega)}+\langle\mathbf{M}_0,\varphi\,\mathbf{I}\rangle_{\mathsf{L}^2(\Omega)}+\langle u,\operatorname{div}\operatorname{Div}(\varphi\,\mathbf{I})\rangle_{\mathsf{H}^{-1}(\Omega)}&=0,\\ \langle\operatorname{div}\operatorname{Div}(p\,\mathbf{I}),\psi\rangle_{\mathsf{H}^{-1}(\Omega)}&=-\langle f,\psi\rangle_{\mathsf{H}^{-1}(\Omega)}\end{aligned}$$

or equivalently

$$\begin{aligned}\langle\operatorname{grad}u,\operatorname{grad}\varphi\rangle_{\mathsf{L}^2(\Omega)}+3\langle p,\varphi\rangle_{\mathsf{L}^2(\Omega)}+\langle\mathbf{M}_0,\mathbf{\Psi}_0\rangle_{\mathsf{L}^2(\Omega)}+\langle p,\operatorname{tr}\mathbf{\Psi}_0\rangle_{\mathsf{L}^2(\Omega)}+\langle\operatorname{tr}\mathbf{M}_0,\varphi\rangle_{\mathsf{L}^2(\Omega)}&=0,\\ \langle\operatorname{grad}p,\operatorname{grad}\psi\rangle_{\mathsf{L}^2(\Omega)}&=-\langle f,\psi\rangle_{\mathsf{H}^{-1}(\Omega)},\end{aligned}$$

which leads to the equivalent system

$$\begin{aligned}\langle\operatorname{grad}u,\operatorname{grad}\varphi\rangle_{\mathsf{L}^2(\Omega)}+3\langle p,\varphi\rangle_{\mathsf{L}^2(\Omega)}+\langle\operatorname{tr}\mathbf{M}_0,\varphi\rangle_{\mathsf{L}^2(\Omega)}&=0,\\ +\langle\mathbf{M}_0,\mathbf{\Psi}_0\rangle_{\mathsf{L}^2(\Omega)}+\langle p,\operatorname{tr}\mathbf{\Psi}_0\rangle_{\mathsf{L}^2(\Omega)}&=0,\\ \langle\operatorname{grad}p,\operatorname{grad}\psi\rangle_{\mathsf{L}^2(\Omega)}&=-\langle f,\psi\rangle_{\mathsf{H}^{-1}(\Omega)}.\end{aligned}$$

Theorem 6.2. *The variational problem* (6.4)-(6.5) *is equivalent to the following well-posed and uniquely solvable variational problem. For* $f\in\mathsf{H}^{-1}(\Omega)$ *find* $p\in\mathring{\mathsf{H}}^1(\Omega)$, $\mathbf{M}_0\in\mathsf{D}\mathbf{D}_0(\Omega,\mathbb{S})$, *and* $u\in\mathring{\mathsf{H}}^1(\Omega)$ *such that*

$$\langle\operatorname{grad}u,\operatorname{grad}\varphi\rangle_{\mathsf{L}^2(\Omega)}+\langle\operatorname{tr}\mathbf{M}_0,\varphi\rangle_{\mathsf{L}^2(\Omega)}+3\langle p,\varphi\rangle_{\mathsf{L}^2(\Omega)}=0, \tag{6.8}$$

$$\langle\mathbf{M}_0,\mathbf{\Psi}_0\rangle_{\mathsf{L}^2(\Omega)}+\langle p,\operatorname{tr}\mathbf{\Psi}_0\rangle_{\mathsf{L}^2(\Omega)}=0, \tag{6.9}$$

[vi]By Theorem 6.1 we even have $u\in\mathring{\mathsf{H}}^2(\Omega)$ and

$$\begin{aligned}\mathbf{M}=-\operatorname{Grad}\operatorname{grad}u&\in\mathsf{D}\mathbf{D}^{0,-1}(\Omega,\mathbb{S})\cap\operatorname{Grad}\operatorname{grad}\mathring{\mathsf{H}}^2(\Omega)\\ &=\big(\mathsf{D}\mathbf{D}^{0,-1}(\Omega,\mathbb{S})\cap\mathring{\mathbf{R}}_0(\Omega,\mathbb{S})\big)\ominus_{\mathbf{L}^2(\Omega,\mathbb{S})}\mathcal{H}_{\mathsf{D}\mathbf{D}}(\Omega,\mathbb{S})\subset\mathbf{L}^2(\Omega,\mathbb{S})\end{aligned}$$

as $\operatorname{div}\operatorname{Div}\mathbf{M}=-f$.

$$\langle \operatorname{grad} p, \operatorname{grad} \psi \rangle_{\mathsf{L}^2(\Omega)} = -\langle f, \psi \rangle_{\mathsf{H}^{-1}(\Omega)} \tag{6.10}$$

for all $\psi \in \overset{\circ}{\mathsf{H}}{}^1(\Omega)$, $\mathbf{\Psi}_0 \in \mathsf{D}\mathbf{D}_0(\Omega, \mathbb{S})$, *and* $\varphi \in \overset{\circ}{\mathsf{H}}{}^1(\Omega)$. *Moreover, the unique solution* $(\mathbf{M}, u)$ *of* (6.4)-(6.5) *is given by* $\mathbf{M} := p\,\mathbf{I} + \mathbf{M}_0$ *and* u *for the unique solution* $(p, \mathbf{M}_0, u)$ *of* (6.11)-(6.13).

If Ω is additionally topologically trivial, then by Theorem 3.14 or Theorem 5.1

$$\mathsf{D}\mathbf{D}_0(\Omega, \mathbb{S}) = \operatorname{sym}\operatorname{Rot}\mathbf{R}_{\mathrm{sym}}(\Omega, \mathbb{T}) = \operatorname{sym}\operatorname{Rot}\big(\mathbf{R}_{\mathrm{sym}}(\Omega, \mathbb{T}) \cap \overset{\circ}{\mathbf{D}}_0(\Omega, \mathbb{T})\big)$$

and we obtain the following result.

Theorem 6.3. *Let* Ω *be additionally topologically trivial. The variational problem* (6.4)-(6.5) *is equivalent to the following well-posed and uniquely solvable variational problem. For* $f \in \mathsf{H}^{-1}(\Omega)$ *find* $p \in \overset{\circ}{\mathsf{H}}{}^1(\Omega)$, $\mathbf{N} \in \mathbf{R}_{\mathrm{sym}}(\Omega, \mathbb{T}) \cap \overset{\circ}{\mathbf{D}}_0(\Omega, \mathbb{T})$, *and* $u \in \overset{\circ}{\mathsf{H}}{}^1(\Omega)$ *such that*

$$\langle \operatorname{grad} u, \operatorname{grad} \varphi \rangle_{\mathsf{L}^2(\Omega)} + \langle \operatorname{tr}\operatorname{sym}\operatorname{Rot}\mathbf{N}, \varphi \rangle_{\mathsf{L}^2(\Omega)} + 3\langle p, \varphi \rangle_{\mathsf{L}^2(\Omega)} = 0, \tag{6.11}$$

$$\langle \operatorname{sym}\operatorname{Rot}\mathbf{N}, \operatorname{sym}\operatorname{Rot}\mathbf{\Phi} \rangle_{\mathsf{L}^2(\Omega)} + \langle p, \operatorname{tr}\operatorname{sym}\operatorname{Rot}\mathbf{\Phi} \rangle_{\mathsf{L}^2(\Omega)} = 0, \tag{6.12}$$

$$\langle \operatorname{grad} p, \operatorname{grad} \psi \rangle_{\mathsf{L}^2(\Omega)} = -\langle f, \psi \rangle_{\mathsf{H}^{-1}(\Omega)} \tag{6.13}$$

for all $\psi \in \overset{\circ}{\mathsf{H}}{}^1(\Omega)$, $\mathbf{\Phi} \in \mathbf{R}_{\mathrm{sym}}(\Omega, \mathbb{T}) \cap \overset{\circ}{\mathbf{D}}_0(\Omega, \mathbb{T})$, *and* $\varphi \in \overset{\circ}{\mathsf{H}}{}^1(\Omega)$. *Moreover, the unique solution* $(\mathbf{M}, u)$ *of* (6.4)-(6.5) *is given by* $\mathbf{M} := p\,\mathbf{I} + \operatorname{sym}\operatorname{Rot}\mathbf{N}$ *and* u *for the unique solution* $(p, \mathbf{N}, u)$ *of* (6.11)-(6.13).

Note that, e.g., $\langle \operatorname{tr}\operatorname{sym}\operatorname{Rot}\mathbf{N}, \varphi \rangle_{\mathsf{L}^2(\Omega)} = \langle \operatorname{sym}\operatorname{Rot}\mathbf{N}, \varphi\,\mathbf{I} \rangle_{\mathsf{L}^2(\Omega)}$ and $3\langle p, \varphi \rangle_{\mathsf{L}^2(\Omega)} = \langle p\,\mathbf{I}, \varphi\,\mathbf{I} \rangle_{\mathsf{L}^2(\Omega)}$.

Proof. (6.4)-(6.5) is equivalent to (6.8)-(6.10) and hence also to (6.11)-(6.13), if the latter system is well-posed. By Theorem 3.14 or Theorem 5.1 the bilinear form $\langle \operatorname{sym}\operatorname{Rot}\,\cdot\,, \operatorname{sym}\operatorname{Rot}\,\cdot\,\rangle_{\mathsf{L}^2(\Omega)}$ is coercive over $\mathbf{R}_{\mathrm{sym}}(\Omega, \mathbb{T}) \cap \overset{\circ}{\mathbf{D}}_0(\Omega, \mathbb{T})$, which shows the consecutive unique solvability of (6.11)-(6.13). □

The three problems in the previous theorem are weak formulations of the following three second-order problems in strong form. A homogeneous Dirichlet Poisson problem for the auxiliary scalar function p

$$\Delta p = f \quad \text{in } \Omega, \qquad p = 0 \quad \text{on } \Gamma,$$

a second-order inhomogeneous Neumann type Rot sym Rot-Div-system for the auxiliary tensor field $\mathbf{N}$

$$\begin{aligned} \operatorname{tr}\mathbf{N} = 0, \qquad &\operatorname{Rot}\operatorname{sym}\operatorname{Rot}\mathbf{N} = -\operatorname{Rot}(p\,\mathbf{I}) = \operatorname{spn}\operatorname{grad} p, &\quad \operatorname{Div}\mathbf{N} = 0 &\quad \text{in } \Omega,\\ &n \times \operatorname{sym}\operatorname{Rot}\mathbf{N} = -n \times p\,\mathbf{I} = p\operatorname{spn} n = \mathbf{0}, &\quad \mathbf{N}\,n = 0 &\quad \text{on } \Gamma, \end{aligned}$$

and, finally, a homogeneous Dirichlet Poisson problem for the original scalar function u

$$\Delta u = 3p + \operatorname{tr}\operatorname{sym}\operatorname{Rot}\mathbf{N} = \operatorname{tr}(p\,\mathbf{I} + \operatorname{sym}\operatorname{Rot}\mathbf{N}) \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \Gamma.$$

In other words, the system (6.11)-(6.13) has triangular structure

$$\begin{bmatrix} \overset{\circ}{\Delta} & -\operatorname{tr}\operatorname{sym}\operatorname{Rot}_{\mathbb{T}} & -3 \\ 0 & \overset{(\circ)}{\operatorname{Rot}}_{\mathbb{S}}\operatorname{sym}\operatorname{Rot}_{\mathbb{T}} & \overset{(\circ)}{\operatorname{Rot}}_{\mathbb{S}}(\,\cdot\,)\mathbf{I} \\ 0 & 0 & \overset{\circ}{\Delta} \end{bmatrix} \begin{bmatrix} u \\ \mathbf{N} \\ p \end{bmatrix} = \begin{bmatrix} 0 \\ \mathbf{0} \\ f \end{bmatrix}, \qquad (\operatorname{tr}\operatorname{sym}\operatorname{Rot}_{\mathbb{T}})^* = \overset{\circ}{\operatorname{Rot}}_{\mathbb{S}}(\,\cdot\,)\mathbf{I}.$$

Indeed we see that $\mathbf{N} \in \mathbf{R}_{\mathrm{sym}}(\Omega, \mathbb{T}) \cap \overset{\circ}{\mathbf{D}}_0(\Omega, \mathbb{T})$ with

$$\langle \operatorname{sym}\operatorname{Rot}\mathbf{N}, \operatorname{sym}\operatorname{Rot}\mathbf{\Phi} \rangle_{\mathsf{L}^2(\Omega)} + \langle p, \operatorname{tr}\operatorname{sym}\operatorname{Rot}\mathbf{\Phi} \rangle_{\mathsf{L}^2(\Omega)} = 0$$

for all $\mathbf{\Phi} \in \mathbf{R}_{\mathrm{sym}}(\Omega, \mathbb{T}) \cap \overset{\circ}{\mathbf{D}}_0(\Omega, \mathbb{T})$ is equivalent to $\mathbf{N} \in \mathbf{R}_{\mathrm{sym}}(\Omega, \mathbb{T}) \cap \overset{\circ}{\mathbf{D}}_0(\Omega, \mathbb{T})$ and

$$\langle \operatorname{sym}\operatorname{Rot}\mathbf{N} + p\,\mathbf{I}, \operatorname{sym}\operatorname{Rot}\mathbf{\Phi} \rangle_{\mathsf{L}^2(\Omega)} = 0 \tag{6.14}$$

for all $\mathbf{\Phi} \in \mathsf{R}_{\text{sym}}(\Omega, \mathbb{T})$ as by Theorem 3.14

$$\operatorname{sym}\operatorname{Rot}\big(\mathsf{R}_{\text{sym}}(\Omega,\mathbb{T}) \cap \mathring{\mathsf{D}}_0(\Omega,\mathbb{T})\big) = \operatorname{sym}\operatorname{Rot}\mathsf{R}_{\text{sym}}(\Omega,\mathbb{T}). \tag{6.15}$$

Now (6.14) shows that

$$\operatorname{sym}\operatorname{Rot}\mathbf{N} + p\,\mathbf{I} \in D(\operatorname{sym}\operatorname{Rot}_{\mathbb{T}}^*) = D(\mathring{\operatorname{Rot}}_{\mathbb{S}}) = \mathring{\mathsf{R}}(\Omega,\mathbb{S})$$

with $\operatorname{Rot}(\operatorname{sym}\operatorname{Rot}\mathbf{N} + p\,\mathbf{I}) = 0$.

Finally, we want to get rid of the complicated space $\mathsf{R}_{\text{sym}}(\Omega,\mathbb{T}) \cap \mathring{\mathsf{D}}_0(\Omega,\mathbb{T})$ in the variational formulation in Theorem 6.3, which might be very complicated to implement in forthcoming numerical applications using finite elements due to the solenoidal and homogeneous normal boundary conditions. For given $p \in \mathring{\mathsf{H}}^1(\Omega)$ the part (6.12), i.e., find $\mathbf{N} \in \mathsf{R}_{\text{sym}}(\Omega,\mathbb{T}) \cap \mathring{\mathsf{D}}_0(\Omega,\mathbb{T})$ such that

$$\langle \operatorname{sym}\operatorname{Rot}\mathbf{N}, \operatorname{sym}\operatorname{Rot}\mathbf{\Phi}\rangle_{\mathsf{L}^2(\Omega)} + \langle p, \operatorname{tr}\operatorname{sym}\operatorname{Rot}\mathbf{\Phi}\rangle_{\mathsf{L}^2(\Omega)} = 0 \tag{6.16}$$

for all $\mathbf{\Phi} \in \mathsf{R}_{\text{sym}}(\Omega,\mathbb{T}) \cap \mathring{\mathsf{D}}_0(\Omega,\mathbb{T})$, of the variational system (6.11)-(6.13), has also a saddle point structure. By Theorem 3.14 we have (6.15) as well as

$$\mathring{\mathsf{D}}_0(\Omega,\mathbb{T}) = \mathsf{R}_{\text{sym},0}(\Omega,\mathbb{T})^{\perp_{\mathsf{L}^2(\Omega,\mathbb{T})}} = \operatorname{dev}\operatorname{Grad}\big(\mathsf{H}^1(\Omega) \cap \mathsf{RT}_0^{\perp_{\mathsf{L}^2(\Omega)}}\big)^{\perp_{\mathsf{L}^2(\Omega)}}.$$

Hence (6.16) is equivalent to find $\mathbf{N} \in \mathsf{R}_{\text{sym}}(\Omega,\mathbb{T})$ such that

$$\langle \operatorname{sym}\operatorname{Rot}\mathbf{N}, \operatorname{sym}\operatorname{Rot}\mathbf{\Phi}\rangle_{\mathsf{L}^2(\Omega)} + \langle p, \operatorname{tr}\operatorname{sym}\operatorname{Rot}\mathbf{\Phi}\rangle_{\mathsf{L}^2(\Omega)} = 0, \tag{6.17}$$

$$\langle \mathbf{N}, \operatorname{dev}\operatorname{Grad}\Phi\rangle_{\mathsf{L}^2(\Omega)} = 0 \tag{6.18}$$

for all $\mathbf{\Phi} \in \mathsf{R}_{\text{sym}}(\Omega,\mathbb{T})$ and $\Phi \in \mathsf{H}^1(\Omega) \cap \mathsf{RT}_0^{\perp_{\mathsf{L}^2(\Omega)}}$. Observe that

$$(\mathbf{N}, E) := (\mathbf{N}, 0) \in \mathsf{R}_{\text{sym}}(\Omega,\mathbb{T}) \times \big(\mathsf{H}^1(\Omega) \cap \mathsf{RT}_0^{\perp_{\mathsf{L}^2(\Omega)}}\big)$$

solves the modified variational system

$$\langle \operatorname{sym}\operatorname{Rot}\mathbf{N}, \operatorname{sym}\operatorname{Rot}\mathbf{\Phi}\rangle_{\mathsf{L}^2(\Omega)} + \langle \mathbf{\Phi}, \operatorname{dev}\operatorname{Grad}E\rangle_{\mathsf{L}^2(\Omega)} = -\langle p, \operatorname{tr}\operatorname{sym}\operatorname{Rot}\mathbf{\Phi}\rangle_{\mathsf{L}^2(\Omega)}, \tag{6.19}$$

$$\langle \mathbf{N}, \operatorname{dev}\operatorname{Grad}\Phi\rangle_{\mathsf{L}^2(\Omega)} = 0 \tag{6.20}$$

for all $\mathbf{\Phi} \in \mathsf{R}_{\text{sym}}(\Omega,\mathbb{T})$ and $\Phi \in \mathsf{H}^1(\Omega) \cap \mathsf{RT}_0^{\perp_{\mathsf{L}^2(\Omega)}}$. On the other hand, any solution

$$(\mathbf{N}, E) \in \mathsf{R}_{\text{sym}}(\Omega,\mathbb{T}) \times \big(\mathsf{H}^1(\Omega) \cap \mathsf{RT}_0^{\perp_{\mathsf{L}^2(\Omega)}}\big)$$

of (6.19)-(6.20) satisfies $E = 0$, as (6.19) tested with

$$\mathbf{\Phi} := \operatorname{dev}\operatorname{Grad}E \in \operatorname{dev}\operatorname{Grad}\mathsf{H}^1(\Omega) = \mathsf{R}_{\text{sym},0}(\Omega,\mathbb{T})$$

shows $\operatorname{dev}\operatorname{Grad}E = 0$ and thus $E \in \mathsf{RT}_0$ by Lemma 3.2 yielding $E = 0$. Note that (6.19)-(6.20) has the saddle point structure

$$\begin{bmatrix} \overset{(\circ)}{\operatorname{Rot}}_{\mathbb{S}} \operatorname{sym}\operatorname{Rot}_{\mathbb{T}} & \operatorname{dev}\operatorname{Grad} \\ -\mathring{\operatorname{Div}}_{\mathbb{T}} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{N} \\ E \end{bmatrix} = \begin{bmatrix} -\overset{(\circ)}{\operatorname{Rot}}_{\mathbb{S}}(v \cdot \mathbf{I}) \\ 0 \end{bmatrix}, \qquad (\operatorname{dev}\operatorname{Grad})^* = -\mathring{\operatorname{Div}}_{\mathbb{T}}.$$

We obtain the following theorem.

Theorem 6.4. *Let Ω be additionally topologically trivial. The variational problem* (6.11)-(6.13) *is equivalent to the following well-posed and uniquely solvable variational system. For $f \in \mathsf{H}^{-1}(\Omega)$ find $p \in \mathring{\mathsf{H}}^1(\Omega)$, $\mathbf{N} \in \mathsf{R}_{\text{sym}}(\Omega,\mathbb{T})$, $E \in \mathsf{H}^1(\Omega) \cap \mathsf{RT}_0^{\perp_{\mathsf{L}^2(\Omega)}}$, and $u \in \mathring{\mathsf{H}}^1(\Omega)$ such that*

$$\langle \operatorname{grad}u, \operatorname{grad}\varphi\rangle_{\mathsf{L}^2(\Omega)} + \langle \operatorname{tr}\operatorname{sym}\operatorname{Rot}\mathbf{N}, \varphi\rangle_{\mathsf{L}^2(\Omega)} + 3\langle p, \varphi\rangle_{\mathsf{L}^2(\Omega)} = 0, \tag{6.21}$$

$$\langle \operatorname{sym}\operatorname{Rot}\mathbf{N}, \operatorname{sym}\operatorname{Rot}\mathbf{\Phi}\rangle_{\mathsf{L}^2(\Omega)} + \langle \mathbf{\Phi}, \operatorname{dev}\operatorname{Grad}E\rangle_{\mathsf{L}^2(\Omega)} + \langle p, \operatorname{tr}\operatorname{sym}\operatorname{Rot}\mathbf{\Phi}\rangle_{\mathsf{L}^2(\Omega)} = 0, \tag{6.22}$$

$$\langle \mathbf{N}, \operatorname{dev}\operatorname{Grad}\Phi\rangle_{\mathsf{L}^2(\Omega)} = 0, \tag{6.23}$$

$$\langle \operatorname{grad} p, \operatorname{grad}\psi\rangle_{\mathsf{L}^2(\Omega)} = -\langle f, \psi\rangle_{\mathsf{H}^{-1}(\Omega)} \tag{6.24}$$

for all $\psi \in \mathring{\mathsf{H}}^1(\Omega)$, $\boldsymbol{\Phi} \in \mathbf{R}_{\mathrm{sym}}(\Omega,\mathbb{T})$, $\Phi \in \mathsf{H}^1(\Omega) \cap \mathsf{RT}_0^{\perp_{\mathsf{L}^2(\Omega)}}$, *and* $\varphi \in \mathring{\mathsf{H}}^1(\Omega)$. *Moreover, the unique solution* $(p, \mathbf{N}, E, u)$ *of* (6.21)-(6.24) *satisfies* $E = 0$ *and* $(p, \mathbf{N}, u)$ *is the unique solution of* (6.11)-(6.13).

Note that the system (6.21)-(6.24) has the block triangular saddle point structure

$$\begin{bmatrix} \mathring{\Delta} & -\operatorname{tr}\operatorname{sym}\operatorname{Rot}_{\mathbb{T}} & 0 & -3 \\ 0 & \overset{(\circ)}{\operatorname{Rot}}_{\mathbb{S}}\operatorname{sym}\operatorname{Rot}_{\mathbb{T}} & \operatorname{dev}\operatorname{Grad} & \overset{(\circ)}{\operatorname{Rot}}_{\mathbb{S}}(\,\cdot\,)\mathbf{I} \\ 0 & -\mathring{\operatorname{Div}}_{\mathbb{T}} & 0 & 0 \\ 0 & 0 & 0 & \mathring{\Delta} \end{bmatrix} \begin{bmatrix} u \\ \mathbf{N} \\ E \\ p \end{bmatrix} = \begin{bmatrix} 0 \\ \mathbf{0} \\ 0 \\ f \end{bmatrix}.$$

with $(\operatorname{tr}\operatorname{sym}\operatorname{Rot}_{\mathbb{T}})^* = \mathring{\operatorname{Rot}}_{\mathbb{S}}(\,\cdot\,)\mathbf{I}$ and $(\operatorname{dev}\operatorname{Grad})^* = -\mathring{\operatorname{Div}}_{\mathbb{T}}$.

Proof. We only have to show well-posedness of the partial system (6.22)-(6.23). First note that by Theorem 3.14 the bilinear form $\langle \operatorname{sym}\operatorname{Rot}\,\cdot\,, \operatorname{sym}\operatorname{Rot}\,\cdot\,\rangle_{\mathsf{L}^2(\Omega)}$ is coercive over $\mathbf{R}_{\mathrm{sym}}(\Omega,\mathbb{T}) \cap \mathring{\mathbf{D}}_0(\Omega,\mathbb{T})$, which equals the kernel of (6.23). Indeed it follows from (6.23) that

$$\mathbf{N} \in \operatorname{dev}\operatorname{Grad}\big(\mathsf{H}^1(\Omega) \cap \mathsf{RT}_0^{\perp_{\mathsf{L}^2(\Omega)}}\big)^{\perp_{\mathsf{L}^2(\Omega)}} = \mathring{\mathbf{D}}_0(\Omega,\mathbb{T}).$$

Moreover, the inf-sup-condition is satisfied as by picking for fixed $0 \neq \Phi \in \mathsf{H}^1(\Omega) \cap \mathsf{RT}_0^{\perp_{\mathsf{L}^2(\Omega)}}$ the tensor $\boldsymbol{\Phi} := \operatorname{dev}\operatorname{Grad}\Phi \in \operatorname{dev}\operatorname{Grad}\mathsf{H}^1(\Omega) = \mathbf{R}_{\mathrm{sym},0}(\Omega,\mathbb{T})$ we have

$$\inf_{\substack{0\neq\Phi\in\mathsf{H}^1(\Omega),\\ \Phi\perp_{\mathsf{L}^2(\Omega)}\mathsf{RT}_0}} \sup_{\boldsymbol{\Phi}\in\mathbf{R}_{\mathrm{sym}}(\Omega,\mathbb{T})} \frac{\langle \boldsymbol{\Phi}, \operatorname{dev}\operatorname{Grad}\Phi\rangle_{\mathsf{L}^2(\Omega)}}{|\boldsymbol{\Phi}|_{\mathbf{R}_{\mathrm{sym}}(\Omega,\mathbb{T})}|\Phi|_{\mathsf{H}^1(\Omega)}} \geq \inf_{\substack{0\neq\Phi\in\mathsf{H}^1(\Omega),\\ \Phi\perp_{\mathsf{L}^2(\Omega)}\mathsf{RT}_0}} \frac{|\operatorname{dev}\operatorname{Grad}\Phi|_{\mathsf{L}^2(\Omega)}}{|\Phi|_{\mathsf{H}^1(\Omega)}} \geq \frac{1}{c}$$

by Lemma 3.2 (iv). □

Remark 6.5. *The corresponding result for the two-dimensional case is completely analogous with the exception that the tensor potential* $\mathbf{N} \in \mathbf{R}_{\mathrm{sym}}(\Omega,\mathbb{T}) \cap \mathring{\mathbf{D}}_0(\Omega,\mathbb{T})$ *is to be replaced by a much simpler vector potential* $N \in \mathsf{H}^1(\Omega)$. *Furthermore, observe that*

$$\langle \operatorname{sym}\operatorname{Rot} N, \operatorname{sym}\operatorname{Rot}\Phi\rangle_{\mathsf{L}^2(\Omega)} = \langle \operatorname{sym}\operatorname{Grad}^{\perp} N, \operatorname{sym}\operatorname{Grad}^{\perp}\Phi\rangle_{\mathsf{L}^2(\Omega)}$$

holds for vector fields $N, \Phi \in \mathsf{H}^1(\Omega)$. *Here the superscript* $\perp$ *denotes the rotation of a vector field by* $90°$. *Note that the complicated second-order inhomogeneous Neumann type* Rot sym Rot-Div-*system for the auxiliary tensor field* $\mathbf{N}$ *is replaced in 2D by a much simpler inhomogeneous Neumann linear elasticity problem, where the standard Sobolev space* $\mathsf{H}^1(\Omega)$ *resp.* $\mathsf{H}^1(\Omega) \cap \mathsf{RM}^{\perp_{\mathsf{L}^2(\Omega)}}$ *can be used, where* RM *denotes the space of rigid motions. This yields the decomposition result in* [14] *for the two-dimensional case, which was shortly mentioned in the introduction.*

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Appendix A. Proofs of Some Useful Identities

Proof of Lemma 3.4 and Lemma 3.12. Note that for $a, b \in \mathbb{R}^3$ and $A \in \mathbb{R}^{3\times 3}$

$$\operatorname{spn} a : \operatorname{spn} b = 2\, a \cdot b, \quad \operatorname{skw} A = \frac{1}{2} \operatorname{spn} \begin{bmatrix} A_{32} - A_{23} \\ A_{13} - A_{31} \\ A_{21} - A_{12} \end{bmatrix} \tag{A.1}$$

hold and hence for skew-symmetric A

$$\operatorname{spn} a : A = \operatorname{spn} a : \operatorname{spn} \operatorname{spn}^{-1} A = 2\, a \cdot \operatorname{spn}^{-1} A, \tag{A.2}$$

i.e., $\operatorname{spn}^* = 2 \operatorname{spn}^{-1}$. Moreover, we have for two matrices A, B

$$A^\top : B = \operatorname{tr}(AB) = \operatorname{tr}(BA) = B^\top : A = A : B^\top.$$

Lemma A.1. *For smooth functions, vector fields and tensor fields we have*

(i) $\operatorname{skw} \operatorname{Grad} \operatorname{grad} u = 0$,
(ii) $\operatorname{div} \operatorname{Div} \mathbf{M} = 0$, *if* $\mathbf{M}$ *is skew-symmetric,*

(iii) $\operatorname{Rot}(u\,\mathbf{I}) = -\operatorname{spn}\operatorname{grad} u$,
(iv) $\operatorname{tr}\operatorname{Rot}\mathbf{M} = 2\operatorname{div}(\operatorname{spn}^{-1}\operatorname{skw}\mathbf{M})$,
especially $\operatorname{tr}\operatorname{Rot}\mathbf{M} = 0$, *if* $\mathbf{M}$ *is symmetric,*
(v) $\operatorname{Div}(u\,\mathbf{I}) = \operatorname{grad} u$,
(vi) $\operatorname{tr}\operatorname{Grad} E = \operatorname{div} E$,
(vii) $\operatorname{Div}(\operatorname{spn} E) = -\operatorname{rot} E$,
especially $\operatorname{Div}(\operatorname{skw}\mathbf{M}) = -\operatorname{rot} E$ *for* $E = \operatorname{spn}^{-1}\operatorname{skw}\mathbf{M}$,
(viii) $\operatorname{Rot}(\operatorname{spn} E) = (\operatorname{div} E)\,\mathbf{I} - (\operatorname{Grad} E)^\top$,
especially $\operatorname{Rot}\operatorname{skw}\mathbf{M} = (\operatorname{div} E)\,\mathbf{I} - (\operatorname{Grad} E)^\top$ *for* $E = \operatorname{spn}^{-1}\operatorname{skw}\mathbf{M}$,
(ix) $\operatorname{skw}\operatorname{Grad} E = \frac{1}{2}\operatorname{spn}\operatorname{rot} E$ *and* $\operatorname{Rot}(\operatorname{sym}\operatorname{Grad} E) = -\operatorname{Rot}(\operatorname{skw}\operatorname{Grad} E) = -\frac{1}{2}\operatorname{Rot}(\operatorname{spn}\operatorname{rot} E)$,
(x) $\operatorname{skw}\operatorname{Rot}\mathbf{M} = \operatorname{spn} E$ *and* $\operatorname{Div}(\operatorname{sym}\operatorname{Rot}\mathbf{M}) = -\operatorname{Div}(\operatorname{skw}\operatorname{Rot}\mathbf{M}) = \operatorname{rot} E$
with $E = \frac{1}{2}\big(\operatorname{Div}\mathbf{M}^\top - \operatorname{grad}(\operatorname{tr}\mathbf{M})\big)$,
especially $\operatorname{Div}(\operatorname{sym}\operatorname{Rot}\mathbf{M}) = -\operatorname{Div}(\operatorname{skw}\operatorname{Rot}\mathbf{M}) = \frac{1}{2}\operatorname{rot}\operatorname{Div}\mathbf{M}^\top$, *if* $\operatorname{tr}\mathbf{M} = 0$.

These formulas hold for distributions as well.

(i)-(ix) and the first identity in (x) follow by elementary calculations. For the second identity in (x) observe that $0 = \operatorname{Div}\operatorname{Rot}\mathbf{M} = \operatorname{Div}(\operatorname{sym}\operatorname{Rot}\mathbf{M}) + \operatorname{Div}(\operatorname{skw}\operatorname{Rot}\mathbf{M})$ for $\mathbf{M} \in \mathring{\mathsf{C}}^\infty(\mathbb{R}^3)$ and hence, using the first identity in (x) and (vii), we obtain

$$\operatorname{Div}(\operatorname{sym}\operatorname{Rot}\mathbf{M}) = -\operatorname{Div}(\operatorname{skw}\operatorname{Rot}\mathbf{M}) = -\operatorname{Div}(\operatorname{spn} E) = \operatorname{rot} E.$$

Therefore, the stated formulas hold in the smooth case. By density these formulas extend to u, E, and $\mathbf{M}$ in respective Sobolev spaces. Let us give proofs for distributions as well. For this, let $m \in \mathbb{N}_0$ and $u \in \mathsf{H}^{-m}(\Omega)$, $E \in \mathsf{H}^{-m}(\Omega)$, $\mathbf{M} \in \mathbf{H}^{-m}(\Omega)$ and $\varphi \in \mathring{\mathsf{C}}^\infty(\Omega)$, $\Phi \in \mathring{\mathsf{C}}^\infty(\Omega)$, and $\mathbf{\Phi} \in \mathring{\mathbf{C}}^\infty(\Omega)$. By

$$\langle u, \partial_i\,\partial_j\,\varphi\rangle_{\mathsf{H}^{-m}(\Omega)} = \langle u, \partial_j\,\partial_i\,\varphi\rangle_{\mathsf{H}^{-m}(\Omega)}, \quad \text{or (with (ii))} \quad \langle u, \operatorname{div}\operatorname{Div}\operatorname{skw}\mathbf{\Phi}\rangle_{\mathsf{H}^{-m}(\Omega)} = 0$$

we see that $\operatorname{Grad}\operatorname{grad} u \in \mathsf{H}^{-m-2}(\Omega)$ is symmetric and hence (i). Note that the formal adjoint is $(\operatorname{skw}\operatorname{Grad}\operatorname{grad})^* = \operatorname{div}\operatorname{Div}\operatorname{skw}$. If $\mathbf{M}$ is skew-symmetric we have $\langle \mathbf{M}, \operatorname{Grad}\operatorname{grad}\varphi\rangle_{\mathbf{H}^{-m}(\Omega)} = 0$, i.e., (ii). We compute with (iv)

$$\begin{aligned}\langle u\,\mathbf{I}, \operatorname{Rot}\mathbf{\Phi}\rangle_{\mathbf{H}^{-m}(\Omega)} &= \langle u, \operatorname{tr}(\operatorname{Rot}\mathbf{\Phi})\rangle_{\mathsf{H}^{-m}(\Omega)} = 2\langle u, \operatorname{div}(\operatorname{spn}^{-1}\operatorname{skw}\mathbf{\Phi})\rangle_{\mathsf{H}^{-m}(\Omega)}\\ &= -\langle \operatorname{spn}\operatorname{grad} u, \operatorname{skw}\mathbf{\Phi}\rangle_{\mathbf{H}^{-m-1}(\Omega)} = -\langle \operatorname{spn}\operatorname{grad} u, \mathbf{\Phi}\rangle_{\mathbf{H}^{-m-1}(\Omega)},\end{aligned}$$

showing (iii). Formally, $(\operatorname{tr}\operatorname{Rot})^* = \operatorname{Rot}(\,\cdot\,\mathbf{I})$. Hence by (iii)

$$\begin{aligned}\langle \mathbf{M}, \operatorname{Rot}(\varphi\,\mathbf{I})\rangle_{\mathbf{H}^{-m}(\Omega)} &= -\langle \mathbf{M}, \operatorname{spn}\operatorname{grad}\varphi\rangle_{\mathbf{H}^{-m}(\Omega)} = -\langle \operatorname{skw}\mathbf{M}, \operatorname{spn}\operatorname{grad}\varphi\rangle_{\mathbf{H}^{-m}(\Omega)}\\ &= -2\langle \operatorname{spn}^{-1}\operatorname{skw}\mathbf{M}, \operatorname{grad}\varphi\rangle_{\mathsf{H}^{-m}(\Omega)} = 2\langle \operatorname{div}\operatorname{spn}^{-1}\operatorname{skw}\mathbf{M}, \varphi\rangle_{\mathsf{H}^{-m-1}(\Omega)},\end{aligned}$$

yielding (iv). (v) follows by

$$-\langle u\,\mathbf{I}, \operatorname{Grad}\Phi\rangle_{\mathbf{H}^{-m}(\Omega)} = -\langle u, \operatorname{tr}(\operatorname{Grad}\Phi)\rangle_{\mathsf{H}^{-m}(\Omega)} = -\langle u, \operatorname{div}\Phi\rangle_{\mathsf{H}^{-m}(\Omega)}.$$

Formally, $(\operatorname{tr}\operatorname{Grad})^* = -\operatorname{Div}(\,\cdot\,\mathbf{I})$. Thus by (v)

$$-\langle E, \operatorname{Div}(\varphi\,\mathbf{I})\rangle_{\mathsf{H}^{-m}(\Omega)} = -\langle E, \operatorname{grad}\varphi\rangle_{\mathsf{H}^{-m}(\Omega)} = \langle \operatorname{div} E, \varphi\rangle_{\mathsf{H}^{-m-1}(\Omega)},$$

yielding (vi). We have the formal adjoint $(\operatorname{Div}\operatorname{spn})^* = (\operatorname{Div}\operatorname{skw}\operatorname{spn})^* = -2\operatorname{spn}^{-1}\operatorname{skw}\operatorname{Grad}$, and by the formula $2\operatorname{skw}\operatorname{Grad}\Phi = \operatorname{spn}\operatorname{rot}\Phi$ from (ix), we obtain (vii), i.e.,

$$-2\langle E, \operatorname{spn}^{-1}\operatorname{skw}\operatorname{Grad}\Phi\rangle_{\mathsf{H}^{-m}(\Omega)} = -\langle E, \operatorname{rot}\Phi\rangle_{\mathsf{H}^{-m}(\Omega)}.$$

Using the formal adjoint $(\operatorname{Rot}\operatorname{spn})^* = 2\operatorname{spn}^{-1}\operatorname{skw}\operatorname{Rot}$ we calculate with (x)

$$\begin{aligned}2\langle E, \operatorname{spn}^{-1}\operatorname{skw}\operatorname{Rot}\mathbf{\Phi}\rangle_{\mathsf{H}^{-m}(\Omega)} &= \langle E, \operatorname{Div}\mathbf{\Phi}^\top - \operatorname{grad}(\operatorname{tr}\mathbf{\Phi})\rangle_{\mathsf{H}^{-m}(\Omega)}\\ &= -\langle \operatorname{Grad} E, \mathbf{\Phi}^\top\rangle_{\mathbf{H}^{-m-1}(\Omega)} + \langle \operatorname{div} E, \operatorname{tr}\mathbf{\Phi}\rangle_{\mathsf{H}^{-m-1}(\Omega)},\end{aligned}$$

i.e., (viii) holds. Formally $(\operatorname{skw}\operatorname{Grad})^* = -\operatorname{Div}\operatorname{skw}$. Using (vii) we see

$$-\langle E, \operatorname{Div}\operatorname{skw}\mathbf{\Phi}\rangle_{\mathsf{H}^{-m}(\Omega)} = \langle E, \operatorname{rot}\operatorname{spn}^{-1}\operatorname{skw}\mathbf{\Phi}\rangle_{\mathsf{H}^{-m}(\Omega)} = \frac{1}{2}\langle \operatorname{spn}\operatorname{rot} E, \operatorname{skw}\mathbf{\Phi}\rangle_{\mathsf{H}^{-m-1}(\Omega)},$$

which proves (ix). We compute by (viii)

$$\begin{aligned}\langle \mathbf{M}, \operatorname{Rot}\operatorname{skw}\mathbf{\Phi}\rangle_{\mathsf{H}^{-m}(\Omega)} &= \langle \operatorname{tr}\mathbf{M}, \operatorname{div}(\operatorname{spn}^{-1}\operatorname{skw}\mathbf{\Phi})\rangle_{\mathsf{H}^{-m}(\Omega)} - \langle \mathbf{M}^\top, \operatorname{Grad}(\operatorname{spn}^{-1}\operatorname{skw}\mathbf{\Phi})\rangle_{\mathsf{H}^{-m}(\Omega)}\\ &= -\langle \operatorname{grad}(\operatorname{tr}\mathbf{M}), \operatorname{spn}^{-1}\operatorname{skw}\mathbf{\Phi}\rangle_{\mathsf{H}^{-m-1}(\Omega)} + \langle \operatorname{Div}\mathbf{M}^\top, \operatorname{spn}^{-1}\operatorname{skw}\mathbf{\Phi}\rangle_{\mathsf{H}^{-m-1}(\Omega)}\\ &= -\frac{1}{2}\langle \operatorname{spn}(\operatorname{grad}\operatorname{tr}\mathbf{M}), \operatorname{skw}\mathbf{\Phi}\rangle_{\mathsf{H}^{-m-1}(\Omega)} + \frac{1}{2}\langle \operatorname{spn}\operatorname{Div}\mathbf{M}^\top, \operatorname{skw}\mathbf{\Phi}\rangle_{\mathsf{H}^{-m-1}(\Omega)},\end{aligned}$$

showing the first formula in (x) and the second one follows by $\operatorname{Div}\operatorname{Rot} = 0$ and (vii). □

Proof of Lemma 4.2. For $\mathbf{M} \in \mathring{\mathsf{R}}(\Omega, \mathbb{S})$ there exists a sequence $(\mathbf{\Phi}_n) \subset \mathring{\mathsf{C}}^\infty(\Omega) \cap \mathsf{L}^2(\Omega, \mathbb{S})$ with $\mathbf{\Phi}_n \to \mathbf{M}$ in $\mathsf{R}(\Omega)$. But then $(\varphi\mathbf{\Phi}_n) \subset \mathring{\mathsf{C}}^\infty(\Omega) \cap \mathsf{L}^2(\Omega, \mathbb{S})$ with $\varphi\mathbf{\Phi}_n \to \varphi\mathbf{M}$ in $\mathsf{R}(\Omega)$, proving $\varphi\mathbf{M} \in \mathring{\mathsf{R}}(\Omega, \mathbb{S})$, as we have $\operatorname{Rot}(\varphi\mathbf{\Phi}_n) = \varphi\operatorname{Rot}\mathbf{\Phi}_n + \operatorname{grad}\varphi \times \mathbf{\Phi}_n$. This formula also shows for $\mathbf{\Psi} \in \mathring{\mathsf{C}}^\infty(\Omega)$ $\big($note $\varphi\mathbf{\Psi} \in \mathring{\mathsf{C}}^\infty(\Omega)\big)$

$$\begin{aligned}\langle \varphi\mathbf{M}, \operatorname{Rot}\mathbf{\Psi}\rangle_{\mathsf{L}^2(\Omega)} &= \langle \mathbf{M}, \varphi\operatorname{Rot}\mathbf{\Psi}\rangle_{\mathsf{L}^2(\Omega)} = \langle \mathbf{M}, \operatorname{Rot}(\varphi\,\mathbf{\Psi})\rangle_{\mathsf{L}^2(\Omega)} - \langle \mathbf{M}, \operatorname{grad}\varphi \times \mathbf{\Psi}\rangle_{\mathsf{L}^2(\Omega)}\\ &= \langle \operatorname{Rot}\mathbf{M}, \varphi\mathbf{\Psi}\rangle_{\mathsf{L}^2(\Omega)} + \langle \operatorname{grad}\varphi \times \mathbf{M}, \mathbf{\Psi}\rangle_{\mathsf{L}^2(\Omega)},\end{aligned} \tag{A.3}$$

and thus $\operatorname{Rot}(\varphi\mathbf{M}) = \varphi\operatorname{Rot}\mathbf{M} + \operatorname{grad}\varphi \times \mathbf{M}$. Analogously we prove the other cases of (i). Similarly we show (iii) using the formula $\operatorname{Div}(\varphi\mathbf{\Phi}_n) = \varphi\operatorname{Div}\mathbf{\Phi}_n + \operatorname{grad}\varphi \cdot \mathbf{\Phi}_n$. To show (ii), let $\mathbf{M} \in \mathsf{R}(\Omega, \mathbb{S})$. Then $\varphi\mathbf{M} \in \mathsf{L}^2(\Omega, \mathbb{S})$ and (A.3) shows $\varphi\mathbf{M} \in \mathsf{R}(\Omega, \mathbb{S})$ with the desired formula. Analogously the other cases of (ii) follow. Similarly we prove (iv). Let $\mathbf{N} \in \mathsf{R}_{\operatorname{sym}}(\Omega, \mathbb{T})$ and $\mathbf{\Phi} \in \mathring{\mathsf{C}}^\infty(\Omega)$. Then $\varphi\mathbf{N} \in \mathsf{L}^2(\Omega, \mathbb{T})$ and with $\varphi\,\mathbf{\Phi} \in \mathring{\mathsf{C}}^\infty(\Omega)$ we get

$$\begin{aligned}\langle \varphi\mathbf{N}, \operatorname{Rot}\operatorname{sym}\mathbf{\Phi}\rangle_{\mathsf{L}^2(\Omega)} &= \langle \mathbf{N}, \varphi\operatorname{Rot}\operatorname{sym}\mathbf{\Phi}\rangle_{\mathsf{L}^2(\Omega)} = \langle \mathbf{N}, \operatorname{Rot}\operatorname{sym}(\varphi\,\mathbf{\Phi})\rangle_{\mathsf{L}^2(\Omega)} - \langle \mathbf{N}, \operatorname{grad}\varphi \times \operatorname{sym}\mathbf{\Phi}\rangle_{\mathsf{L}^2(\Omega)}\\ &= \langle \operatorname{sym}\operatorname{Rot}\mathbf{N}, \varphi\mathbf{\Phi}\rangle_{\mathsf{L}^2(\Omega)} + \langle \operatorname{grad}\varphi \times \mathbf{N}, \operatorname{sym}\mathbf{\Phi}\rangle_{\mathsf{L}^2(\Omega)},\end{aligned}$$

which shows $\varphi\mathbf{N} \in \mathsf{R}_{\operatorname{sym}}(\Omega, \mathbb{T})$ and $\operatorname{sym}\operatorname{Rot}(\varphi\mathbf{N}) = \varphi\operatorname{sym}\operatorname{Rot}\mathbf{N} + \operatorname{sym}(\operatorname{grad}\varphi \times \mathbf{N})$ and hence (v). To prove (vi), let $\mathbf{M} \in \mathsf{D}\mathbf{D}(\Omega, \mathbb{S})$ and $\phi \in \mathring{\mathsf{C}}^\infty(\Omega)$. Then $\varphi\mathbf{M} \in \mathsf{L}^2(\Omega, \mathbb{S})$ and we compute by

$$\begin{aligned}\operatorname{Grad}\operatorname{grad}(\varphi\,\phi) &= \varphi\operatorname{Grad}\operatorname{grad}\phi + \phi\operatorname{Grad}\operatorname{grad}\varphi + 2\operatorname{sym}\big((\operatorname{grad}\varphi)(\operatorname{grad}\phi)^\top\big),\\ (\operatorname{grad}\varphi)(\operatorname{grad}\phi)^\top &= \operatorname{Grad}(\phi\operatorname{grad}\varphi) - \phi\operatorname{Grad}\operatorname{grad}\varphi\end{aligned}$$

the identity

$$\operatorname{Grad}\operatorname{grad}(\varphi\,\phi) = \varphi\operatorname{Grad}\operatorname{grad}\phi - \phi\operatorname{Grad}\operatorname{grad}\varphi + 2\operatorname{sym}\big(\operatorname{Grad}(\phi\operatorname{grad}\varphi)\big).$$

Finally with $\varphi\phi \in \mathring{\mathsf{C}}^\infty(\Omega)$ we get

$$\begin{aligned}&\langle \varphi\mathbf{M}, \operatorname{Grad}\operatorname{grad}\phi\rangle_{\mathsf{L}^2(\Omega)} = \langle \mathbf{M}, \varphi\operatorname{Grad}\operatorname{grad}\phi\rangle_{\mathsf{L}^2(\Omega)}\\ &= \big\langle \mathbf{M}, \operatorname{Grad}\operatorname{grad}(\varphi\,\phi)\big\rangle_{\mathsf{L}^2(\Omega)} + \langle \mathbf{M}, \phi\operatorname{Grad}\operatorname{grad}\varphi\rangle_{\mathsf{L}^2(\Omega)} - 2\big\langle \mathbf{M}, \operatorname{sym}\big(\operatorname{Grad}(\phi\operatorname{grad}\varphi)\big)\big\rangle_{\mathsf{L}^2(\Omega)}\\ &= \langle \operatorname{div}\operatorname{Div}\mathbf{M}, \varphi\,\phi\rangle_{\mathsf{L}^2(\Omega)} + \langle \mathbf{M} : \operatorname{Grad}\operatorname{grad}\varphi, \phi\rangle_{\mathsf{L}^2(\Omega)} - 2\big\langle \mathbf{M}, \operatorname{Grad}(\phi\operatorname{grad}\varphi)\big\rangle_{\mathsf{L}^2(\Omega)}\\ &= \langle \varphi\,\operatorname{div}\operatorname{Div}\mathbf{M}, \phi\rangle_{\mathsf{L}^2(\Omega)} + \big\langle \operatorname{tr}(\mathbf{M} \cdot \operatorname{Grad}\operatorname{grad}\varphi), \phi\big\rangle_{\mathsf{L}^2(\Omega)} + 2\underbrace{\langle \operatorname{Div}\mathbf{M}, \phi\operatorname{grad}\varphi\rangle_{\mathsf{H}^{-1}(\Omega)}}_{=\langle \operatorname{Div}\mathbf{M} \cdot \operatorname{grad}\varphi, \phi\rangle_{\mathsf{H}^{-1}(\Omega)}},\end{aligned}$$

which shows (vi), i.e., $\varphi\mathbf{M} \in \mathsf{D}\mathbf{D}^{0,-1}(\Omega, \mathbb{S})$ and

$$\operatorname{div}\operatorname{Div}(\varphi\mathbf{M}) = \varphi\operatorname{div}\operatorname{Div}\mathbf{M} + 2\operatorname{grad}\varphi \cdot \operatorname{Div}\mathbf{M} + \operatorname{tr}(\operatorname{Grad}\operatorname{grad}\varphi \cdot \mathbf{M}).$$

The proof is finished. □

Appendix B. A Regular Decomposition for Tensor Fields with Symmetric Rotation

Lemma B.1. *Let Ω be additionally topologically trivial. Then*

$$\mathsf{R}_{\mathrm{sym}}(\Omega,\mathbb{T}) = \mathsf{H}^1(\Omega,\mathbb{T}) + \operatorname{dev}\operatorname{Grad}\mathsf{H}^1(\Omega)$$

with linear and continuous potential operators

$$\mathrm{P}_{\mathsf{H}^1} : \mathsf{R}_{\mathrm{sym}}(\Omega,\mathbb{T}) \longrightarrow \mathsf{H}^1(\Omega,\mathbb{T}), \qquad \mathrm{P}_{\mathsf{H}^1} : \mathsf{R}_{\mathrm{sym}}(\Omega,\mathbb{T}) \longrightarrow \mathsf{H}^1(\Omega).$$

Proof. Let $\mathbf{N} \in \mathsf{R}_{\mathrm{sym}}(\Omega,\mathbb{T})$. Then

$$\operatorname{sym}\operatorname{Rot}\mathbf{N} \in \mathsf{D}\mathsf{D}_0(\Omega,\mathbb{S}) = \operatorname{sym}\operatorname{Rot}\mathsf{H}^1(\Omega,\mathbb{T})$$

with linear and continuous potential operator $\mathrm{P}_{\operatorname{sym}\operatorname{Rot}} : \mathsf{D}\mathsf{D}_0(\Omega,\mathbb{S}) \longrightarrow \mathsf{H}^1(\Omega,\mathbb{T})$ by Lemma 3.13 (i). Thus, there is $\tilde{\mathbf{N}} \in \mathsf{H}^1(\Omega,\mathbb{T})$ depending linearly and continuously on $\mathbf{N}$ with $\operatorname{sym}\operatorname{Rot}\tilde{\mathbf{N}} = \operatorname{sym}\operatorname{Rot}\mathbf{N}$. Hence,

$$\mathbf{N} - \tilde{\mathbf{N}} \in \mathsf{R}_{\mathrm{sym},0}(\Omega,\mathbb{T}) = \operatorname{dev}\operatorname{Grad}\mathsf{H}^1(\Omega)$$

with linear and continuous potential operator $\mathrm{P}_{\operatorname{dev}\operatorname{Grad}} : \mathsf{R}_{\mathrm{sym},0}(\Omega,\mathbb{T}) \longrightarrow \mathsf{H}^1(\Omega)$ by Lemma 3.13 (ii). Finally, there exists $E \in \mathsf{H}^1(\Omega)$ with $\mathbf{N} - \tilde{\mathbf{N}} = \operatorname{dev}\operatorname{Grad} E$ and E depends linearly and continuously on $\mathbf{N}$. □

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