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Solution Theory and Functional A Posteriori Error Estimates for General First Order Systems with Applications to Electro-Magneto-Statics

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ABSTRACT. We prove a solution theory and functional a posteriori error estimates for general linear first order systems of type

$$\mathrm{A}_2\, x = f, \qquad \mathrm{A}_1^*\, x = g$$

for two densely defined and closed (possibly unbounded) linear operators A_1 and A_2. As a prototypical application we will discuss the system of electro-magneto statics with mixed tangential and normal boundary conditions

$$\operatorname{rot} E = F, \qquad -\operatorname{div} \varepsilon E = g.$$

Second order systems of type

$$\mathrm{A}_2^*\, \mathrm{A}_2\, x = f, \qquad \mathrm{A}_1^*\, x = g$$

will be considered as well.

CONTENTS

1. INTRODUCTION

For $\ell = 0, \dots, 4$ let H_ℓ be Hilbert spaces and for $\ell = 0, \dots, 3$ let

$$\mathrm{A}_\ell : D(\mathrm{A}_\ell) \subset \mathsf{H}_\ell \to \mathsf{H}_{\ell+1}$$

be densely defined and closed (possibly unbounded) linear operators. Here, $D(\mathrm{A})$ denotes the domain of definition of a linear operator A and we introduce by $N(\mathrm{A})$ and $R(\mathrm{A})$ its kernel and range, respectively. Inner product, norm, orthogonality, orthogonal sum and difference of (or in) an Hilbert space H will be denoted by $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathsf{H}}$, $|\,\cdot\,|_{\mathsf{H}}$, $\perp_{\mathsf{H}}$, and $\oplus_{\mathsf{H}}$, $\ominus_{\mathsf{H}}$, respectively. We note that $D(\mathrm{A})$, equipped with the graph

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inner product, is a Hilbert space itself. Moreover, we assume that the operators A_ℓ satisfy the sequence or complex property, this is for $\ell = 0, \dots, 2$

$$R(\mathrm{A}_\ell) \subset N(\mathrm{A}_{\ell+1}) \tag{1.1}$$

or equivalently $\mathrm{A}_{\ell+1}\,\mathrm{A}_\ell \subset 0$. For $\ell = 0, \dots, 3$ the (Hilbert space) adjoint operators

$$\mathrm{A}_\ell^* : D(\mathrm{A}_\ell^*) \subset \mathsf{H}_{\ell+1} \to \mathsf{H}_\ell$$

defined by the relation

$$\forall\, x \in D(\mathrm{A}_\ell) \quad \forall\, y \in D(\mathrm{A}_\ell^*) \qquad \langle \mathrm{A}_\ell\, x, y\rangle_{\mathsf{H}_{\ell+1}} = \langle x, \mathrm{A}_\ell^*\, y\rangle_{\mathsf{H}_\ell}$$

satisfy the sequence or complex property

$$R(\mathrm{A}_{\ell+1}^*) \subset N(\mathrm{A}_\ell^*), \quad \ell = 0, \dots, 2, \tag{1.2}$$

or equivalently $\mathrm{A}_\ell^*\,\mathrm{A}_{\ell+1}^* \subset 0$. We note $(\mathrm{A}_\ell^*)^* = \overline{\mathrm{A}_\ell} = \mathrm{A}_\ell$, i.e., $(\mathrm{A}_\ell, \mathrm{A}_\ell^*)$ is a dual pair. For $\ell = 1, \dots, 3$ the complex

$$D(\mathrm{A}_{\ell\text{-}1}) \xrightarrow{\mathrm{A}_{\ell\text{-}1}} D(\mathrm{A}_\ell) \xrightarrow{\mathrm{A}_\ell} \mathsf{H}_{\ell+1} \tag{1.3}$$

is called closed, if the ranges $R(\mathrm{A}_{\ell\text{-}1})$ and $R(\mathrm{A}_\ell)$ are closed, and called exact, if $R(\mathrm{A}_{\ell\text{-}1}) = N(\mathrm{A}_\ell)$. By the closed range theorem, (1.3) is closed resp. exact, if and only if the adjoint complex

$$\mathsf{H}_{\ell\text{-}1} \xleftarrow{\mathrm{A}_{\ell\text{-}1}^*} D(\mathrm{A}_{\ell\text{-}1}^*) \xleftarrow{\mathrm{A}_\ell^*} D(\mathrm{A}_\ell^*) \tag{1.4}$$

is closed resp. exact.

The aim of this paper is to prove functional a posteriori error estimates in the spirit of Sergey Repin, see, e.g., [3, 2, 8], for the linear system

$$\begin{aligned} \mathrm{A}_2\, x &= f, \\ \mathrm{A}_1^*\, x &= g, \\ \pi_2\, x &= k \end{aligned} \tag{1.5}$$

with $x \in D_2$, where we define for $\ell = 1, \dots, 3$

$$D_\ell := D(\mathrm{A}_\ell) \cap D(\mathrm{A}_{\ell\text{-}1}^*), \qquad K_\ell := N(\mathrm{A}_\ell) \cap N(\mathrm{A}_{\ell\text{-}1}^*)$$

and $\pi_\ell : \mathsf{H}_\ell \to K_\ell$ denotes the orthonormal projector onto the cohomology group, i.e., the kernel K_ℓ. Obviously, $f \in R(\mathrm{A}_2)$, $g \in R(\mathrm{A}_1^*)$, and $k \in K_2$ are necessary for solvability of (1.5) and there exists at most one solution to (1.5). A proper solution theory for (1.5), i.e., existence of a solution of (1.5) depending continuously on the data, will be given in the next section.

Let $\tilde{x} \in \mathsf{H}_2$ be a possibly non-conforming[i] "approximation" for the exact solution

$$x \in D_2 = D(\mathrm{A}_2) \cap D(\mathrm{A}_1^*)$$

of (1.5). Proving functional a posteriori error estimates for the linear problem (1.5) means, that we will present two-sided estimates for the error

$$e := x - \tilde{x} \in \mathsf{H}_2$$

with the following properties:

① There exist two functionals $\mathcal{M}_\mp$, a lower and an upper bound, such that

$$\forall\, z_i, y_j \qquad \mathcal{M}_-(z_1, \dots, z_I; \tilde{x}, f, g, k) \le |e|_{\mathsf{H}_2} \le \mathcal{M}_+(y_1, \dots, y_J; \tilde{x}, f, g, k), \tag{1.6}$$

were the z_i and the y_j belong to some suitable Hilbert spaces. The functionals $\mathcal{M}_\mp$ are guaranteed lower and upper bounds for the norm of the error $|e|_{\mathsf{H}_2}$ and explicitly computable as long as at least upper bounds for the natural Friedrichs/Poincaré type constants for the operators A_1 and A_2 are known[ii]. The bounds $\mathcal{M}_\mp$ do not depend on the possibly unknown exact solution x, but only on the data, the approximation $\tilde{x}$, and the "free" vectors z_i, y_j.

[i] A conforming "approximation" $\tilde{x}$ belongs to D_2.

[ii] Just needed for the upper bound $\mathcal{M}_+$.

② The lower and upper bound $\mathcal{M}_\mp$ are sharp, i.e.,

$$\max_{z_1,\dots,z_I} \mathcal{M}_-(z_1,\dots,z_I;\tilde{x},f,g,k) = |e|_{\mathsf{H}_2} = \min_{y_1,\dots,y_J} \mathcal{M}_+(y_1,\dots,y_J;\tilde{x},f,g,k). \tag{1.7}$$

③ The minimization over z_i and y_j is "simple", typically a minimization of quadratic functionals.

We will also present functional a posteriori error estimates for linear second order systems such as

$$\begin{aligned} \mathrm{A}_2^* \mathrm{A}_2 x &= f, \\ \mathrm{A}_1^* x &= g, \\ \pi_2 x &= k \end{aligned} \tag{1.8}$$

with $x \in D_2$ such that $\mathrm{A}_2 x \in D(\mathrm{A}_2^*)$, i.e., $x \in D(\mathrm{A}_1^*) \cap D(\mathrm{A}_2^* \mathrm{A}_2)$. These will follow immediately by the theory developed for the first order system (1.5), since the pair $(x,y) \in \big(D(\mathrm{A}_2) \cap D(\mathrm{A}_1^*)\big) \times \big(D(\mathrm{A}_3) \cap D(\mathrm{A}_2^*)\big)$ defined by $y := \mathrm{A}_2 x \in D(\mathrm{A}_2^*) \cap R(\mathrm{A}_2)$ solves the system

$$\begin{aligned} \mathrm{A}_2 x &= y, & \mathrm{A}_3 y &= 0, \\ \mathrm{A}_1^* x &= g, & \mathrm{A}_2^* y &= f, \\ \pi_2 x &= k, & \pi_3 y &= 0. \end{aligned}$$

Analogously, we can treat problems such as

$$\begin{aligned} \mathrm{A}_2^* \mathrm{A}_2 x &= f, \\ \mathrm{A}_1 \mathrm{A}_1^* x &= g, \\ \pi_2 x &= k \end{aligned} \tag{1.9}$$

as well, related to the generalized Hodge-Helmholtz decomposition of $f+g+k \in \mathsf{H}_2$.

Our main applications will be the linear systems of electro-magneto statics as well as related rot rot systems and, as a simple example, the Laplacian.

2. Functional Analysis Tool Box

Let $\ell \in \{0,\dots,3\}$ resp. $\ell \in \{1,\dots,4\}$. By the projection theorem, the Helmholtz type decompositions

$$\mathsf{H}_\ell = N(\mathrm{A}_\ell) \oplus_{\mathsf{H}_\ell} \overline{R(\mathrm{A}_\ell^*)} = \overline{R(\mathrm{A}_{\ell\text{-}1})} \oplus_{\mathsf{H}_\ell} N(\mathrm{A}_{\ell\text{-}1}^*) \tag{2.1}$$

hold. The complex properties (1.1)-(1.2) yield

$$N(\mathrm{A}_\ell) = \overline{R(\mathrm{A}_{\ell\text{-}1})} \oplus_{\mathsf{H}_\ell} K_\ell, \quad N(\mathrm{A}_{\ell\text{-}1}^*) = K_\ell \oplus_{\mathsf{H}_\ell} \overline{R(\mathrm{A}_\ell^*)}, \qquad K_\ell = N(\mathrm{A}_\ell) \cap N(\mathrm{A}_{\ell\text{-}1}^*).$$

Therefore, we get the refined Helmholtz type decomposition

$$\mathsf{H}_\ell = \overline{R(\mathrm{A}_{\ell\text{-}1})} \oplus_{\mathsf{H}_\ell} K_\ell \oplus_{\mathsf{H}_\ell} \overline{R(\mathrm{A}_\ell^*)}. \tag{2.2}$$

Using the Helmholtz type decompositions (2.1) we define the reduced operators

$$\begin{aligned} \mathcal{A}_\ell &:= \mathrm{A}_\ell |_{\overline{R(\mathrm{A}_\ell^*)}} : D(\mathcal{A}_\ell) \subset \overline{R(\mathrm{A}_\ell^*)} \to \overline{R(\mathrm{A}_\ell)}, & D(\mathcal{A}_\ell) &:= D(\mathrm{A}_\ell) \cap \overline{R(\mathrm{A}_\ell^*)} = D(\mathrm{A}_\ell) \cap N(\mathrm{A}_\ell)^{\perp_{\mathsf{H}_\ell}}, \\ \mathcal{A}_\ell^* &:= \mathrm{A}_\ell^* |_{\overline{R(\mathrm{A}_\ell)}} : D(\mathcal{A}_\ell^*) \subset \overline{R(\mathrm{A}_\ell)} \to \overline{R(\mathrm{A}_\ell^*)}, & D(\mathcal{A}_\ell^*) &:= D(\mathrm{A}_\ell^*) \cap \overline{R(\mathrm{A}_\ell)} = D(\mathrm{A}_\ell^*) \cap N(\mathrm{A}_\ell^*)^{\perp_{\mathsf{H}_{\ell+1}}}, \end{aligned}$$

which are also densely defined and closed linear operators. We note that $\mathcal{A}_\ell$ and $\mathcal{A}_\ell^*$ are indeed adjoint to each other, i.e., $(\mathcal{A}_\ell, \mathcal{A}_\ell^*)$ is a dual pair as well. Now the inverse operators

$$(\mathcal{A}_\ell)^{-1} : R(\mathrm{A}_\ell) \to D(\mathcal{A}_\ell), \qquad (\mathcal{A}_\ell^*)^{-1} : R(\mathrm{A}_\ell^*) \to D(\mathcal{A}_\ell^*)$$

exist, since $\mathcal{A}_\ell$ and $\mathcal{A}_\ell^*$ are injective by definition, and they are bijective, as, e.g., for $x \in D(\mathcal{A}_\ell)$ and $y := \mathrm{A}_\ell x \in R(\mathrm{A}_\ell)$ we get $(\mathcal{A}_\ell)^{-1} y = x$ by the injectivity of $\mathcal{A}_\ell$. Furthermore, by the Helmholtz type decompositions (2.1) we have

$$D(\mathrm{A}_\ell) = N(\mathrm{A}_\ell) \oplus_{\mathsf{H}_\ell} D(\mathcal{A}_\ell), \qquad D(\mathrm{A}_\ell^*) = N(\mathrm{A}_\ell^*) \oplus_{\mathsf{H}_\ell} D(\mathcal{A}_\ell^*) \tag{2.3}$$

and thus we obtain for the ranges

$$R(\mathrm{A}_\ell) = R(\mathcal{A}_\ell), \qquad R(\mathrm{A}_\ell^*) = R(\mathcal{A}_\ell^*). \tag{2.4}$$

By the closed range and closed graph theorems we get immediately the following lemma.

Lemma 2.1. *Let $\ell \in \{0, \dots, 3\}$. The following assertions are equivalent:*

- **(i)** $\exists\, c_\ell \in (0, \infty) \quad \forall\, x \in D(\mathcal{A}_\ell) \qquad |x|_{\mathsf{H}_\ell} \leq c_\ell |\,\mathrm{A}_\ell\, x|_{\mathsf{H}_{\ell+1}}$
- **(i*)** $\exists\, c_\ell^* \in (0, \infty) \quad \forall\, y \in D(\mathcal{A}_\ell^*) \qquad |y|_{\mathsf{H}_{\ell+1}} \leq c_\ell^* |\,\mathrm{A}_\ell^*\, y|_{\mathsf{H}_\ell}$
- **(ii)** $R(\mathrm{A}_\ell) = R(\mathcal{A}_\ell)$ *is closed in* $\mathsf{H}_{\ell+1}$.
- **(ii*)** $R(\mathrm{A}_\ell^*) = R(\mathcal{A}_\ell^*)$ *is closed in* H_ℓ.
- **(iii)** $(\mathcal{A}_\ell)^{-1} : R(\mathrm{A}_\ell) \to D(\mathcal{A}_\ell)$ *is continuous and bijective with norm bounded by* $(1 + c_\ell^2)^{1/2}$.
- **(iii*)** $(\mathcal{A}_\ell^*)^{-1} : R(\mathrm{A}_\ell^*) \to D(\mathcal{A}_\ell^*)$ *is continuous and bijective with norm bounded by* $(1 + c_\ell^{*2})^{1/2}$.

Proof. Note that by the closed range theorem (ii) $\Leftrightarrow$ (ii*) holds. Hence, by symmetry it is sufficient to show (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii).

(i)$\Rightarrow$(ii) Pick a sequence $(y_n) \subset R(\mathrm{A}_\ell)$ converging to $y \in \mathsf{H}_{\ell+1}$ in $\mathsf{H}_{\ell+1}$. By (2.4) there exists a sequence $(x_n) \subset D(\mathcal{A}_\ell)$ with $y_n = \mathrm{A}_\ell\, x_n$. (i) implies that (x_n) is a Cauchy sequence in H_ℓ and hence there exists some $x \in \mathsf{H}_\ell$ with $x_n \to x$ in H_ℓ. As A_ℓ is closed, we get $x \in D(\mathrm{A}_\ell)$ and $\mathrm{A}_\ell\, x = y \in R(\mathrm{A}_\ell)$.

(ii)$\Rightarrow$(iii) Note that $(\mathcal{A}_\ell)^{-1} : R(\mathrm{A}_\ell) \to D(\mathcal{A}_\ell)$ is a densely defined and closed linear operator. By (ii), $R(\mathrm{A}_\ell)$ is closed and hence itself a Hilbert space. By the closed graph theorem $(\mathcal{A}_\ell)^{-1}$ is continuous.

(iii)$\Rightarrow$(i) For $x \in D(\mathcal{A}_\ell)$ let $y := \mathrm{A}_\ell\, x \in R(\mathrm{A}_\ell)$. Then $x = (\mathcal{A}_\ell)^{-1} y$ as $\mathcal{A}_\ell$ is injective[iii]. Therefore,

$$|x|_{\mathsf{H}_\ell} = \big|(\mathcal{A}_\ell)^{-1} y\big|_{\mathsf{H}_\ell} \leq \big|(\mathcal{A}_\ell)^{-1}\big|_{R(\mathrm{A}_\ell), R(\mathrm{A}_\ell^*)} |y|_{\mathsf{H}_{\ell+1}} = c_\ell |\,\mathrm{A}_\ell\, x|_{\mathsf{H}_{\ell+1}}$$

with $c_\ell := \big|(\mathcal{A}_\ell)^{-1}\big|_{R(\mathrm{A}_\ell), R(\mathrm{A}_\ell^*)}$.

If (i) holds we have for $y \in R(\mathrm{A}_\ell)$ and $x := (\mathcal{A}_\ell)^{-1} y \in D(\mathcal{A}_\ell)$

$$\big|(\mathcal{A}_\ell)^{-1} y\big|_{\mathsf{H}_\ell} \leq c_\ell |\,\mathrm{A}_\ell\, x|_{\mathsf{H}_{\ell+1}} = c_\ell |y|_{\mathsf{H}_{\ell+1}}$$

and hence

$$\big|(\mathcal{A}_\ell)^{-1}\big|_{R(\mathrm{A}_\ell), R(\mathrm{A}_\ell^*)} = \sup_{0 \neq y \in R(\mathrm{A}_\ell)} \frac{\big|(\mathcal{A}_\ell)^{-1} y\big|_{\mathsf{H}_\ell}}{|y|_{\mathsf{H}_{\ell+1}}} \leq c_\ell,$$

$$\big|(\mathcal{A}_\ell)^{-1}\big|^2_{R(\mathrm{A}_\ell), D(\mathcal{A}_\ell)} = \sup_{0 \neq y \in R(\mathrm{A}_\ell)} \frac{\big|(\mathcal{A}_\ell)^{-1} y\big|^2_{D(\mathrm{A}_\ell)}}{|y|^2_{\mathsf{H}_{\ell+1}}} = \sup_{0 \neq y \in R(\mathrm{A}_\ell)} \frac{\big|(\mathcal{A}_\ell)^{-1} y\big|^2_{\mathsf{H}_\ell} + |y|^2_{\mathsf{H}_{\ell+1}}}{|y|^2_{\mathsf{H}_{\ell+1}}} \leq c_\ell^2 + 1,$$

finishing the proof. □

From now on we assume that we always choose the best Friedrichs/Poincaré type constants c_ℓ, c_ℓ^*, if they exist in $(0, \infty)$, i.e., c_ℓ and c_ℓ^* are given by the Rayleigh quotients

$$\frac{1}{c_\ell} := \inf_{0 \neq x \in D(\mathcal{A}_\ell)} \frac{|\,\mathrm{A}_\ell\, x|_{\mathsf{H}_{\ell+1}}}{|x|_{\mathsf{H}_\ell}}, \qquad \frac{1}{c_\ell^*} := \inf_{0 \neq y \in D(\mathcal{A}_\ell^*)} \frac{|\,\mathrm{A}_\ell^*\, y|_{\mathsf{H}_\ell}}{|y|_{\mathsf{H}_{\ell+1}}}.$$

Moreover, we see

$$c_\ell = \sup_{0 \neq x \in D(\mathcal{A}_\ell)} \frac{|x|_{\mathsf{H}_\ell}}{|\,\mathrm{A}_\ell\, x|_{\mathsf{H}_{\ell+1}}} = \sup_{0 \neq y \in R(\mathrm{A}_\ell)} \frac{\big|(\mathcal{A}_\ell)^{-1} y\big|_{\mathsf{H}_\ell}}{|y|_{\mathsf{H}_{\ell+1}}} = \big|(\mathcal{A}_\ell)^{-1}\big|_{R(\mathrm{A}_\ell), R(\mathrm{A}_\ell^*)},$$

as $0 \neq x \in D(\mathcal{A}_\ell)$ implies $0 \neq \mathrm{A}_\ell\, x$ and for $y := \mathrm{A}_\ell\, x$ with $x \in D(\mathcal{A}_\ell)$ we have $(\mathcal{A}_\ell)^{-1} y = x$, both by the injectivity of $\mathcal{A}_\ell$. Analogously, we get

$$c_\ell^* = \sup_{0 \neq y \in D(\mathcal{A}_\ell^*)} \frac{|y|_{\mathsf{H}_{\ell+1}}}{|\,\mathrm{A}_\ell^*\, y|_{\mathsf{H}_\ell}} = \sup_{0 \neq x \in R(\mathrm{A}_\ell^*)} \frac{\big|(\mathcal{A}_\ell^*)^{-1} x\big|_{\mathsf{H}_{\ell+1}}}{|x|_{\mathsf{H}_\ell}} = \big|(\mathcal{A}_\ell^*)^{-1}\big|_{R(\mathrm{A}_\ell^*), R(\mathrm{A}_\ell)}.$$

[iii] It holds $\mathrm{A}_\ell \big(x - (\mathcal{A}_\ell)^{-1} y\big) = 0$ and thus $x = (\mathcal{A}_\ell)^{-1} y$.

Lemma 2.2. *Let $\ell \in \{0, \ldots, 3\}$. Assume that $c_\ell \in (0, \infty)$ or $c_\ell^* \in (0, \infty)$ exists. Then $c_\ell = c_\ell^*$.*

We note that also in the case $c_\ell = \infty$ or $c_\ell^* = \infty$ we have $c_\ell = c_\ell^* = \infty$.

Proof. Let, e.g., c_ℓ^* exist in $(0, \infty)$. By Lemma 2.1 also c_ℓ exists in $(0, \infty)$ and the ranges $R(\mathrm{A}_\ell) = R(\mathcal{A}_\ell)$ and $R(\mathrm{A}_\ell^*) = R(\mathcal{A}_\ell^*)$ are closed. Then for $x \in D(\mathcal{A}_\ell) = D(\mathrm{A}_\ell) \cap R(\mathrm{A}_\ell^*)$ there is $y \in D(\mathcal{A}_\ell^*)$ with $x = \mathrm{A}_\ell^* y$. More precisely, $y := (\mathcal{A}_\ell^*)^{-1} x \in D(\mathcal{A}_\ell^*)$ is uniquely determined and we have $|y|_{\mathsf{H}_{\ell+1}} \leq c_\ell^* |\mathrm{A}_\ell^* y|_{\mathsf{H}_\ell}$. But then

$$|x|_{\mathsf{H}_\ell}^2 = \langle x, \mathrm{A}_\ell^* y \rangle_{\mathsf{H}_\ell} = \langle \mathrm{A}_\ell x, y \rangle_{\mathsf{H}_{\ell+1}} \leq |\mathrm{A}_\ell x|_{\mathsf{H}_{\ell+1}} |y|_{\mathsf{H}_{\ell+1}} \leq c_\ell^* |\mathrm{A}_\ell x|_{\mathsf{H}_{\ell+1}} |\mathrm{A}_\ell^* y|_{\mathsf{H}_\ell},$$

yielding $|x|_{\mathsf{H}_\ell} \leq c_\ell^* |\mathrm{A}_\ell x|_{\mathsf{H}_{\ell+1}}$. Therefore, $c_\ell \leq c_\ell^*$ and by symmetry we obtain $c_\ell = c_\ell^*$. □

A standard indirect argument shows the following lemma.

Lemma 2.3. *Let $\ell \in \{0, \ldots, 3\}$ and let $D(\mathcal{A}_\ell) = D(\mathrm{A}_\ell) \cap \overline{R(\mathrm{A}_\ell^*)} \hookrightarrow \mathsf{H}_\ell$ be compact. Then the assertions of Lemma 2.1 and Lemma 2.2 hold. Moreover, the inverse operators*

$$\mathcal{A}_\ell^{-1} : R(\mathrm{A}_\ell) \to R(\mathrm{A}_\ell^*), \quad (\mathcal{A}_\ell^*)^{-1} : R(\mathrm{A}_\ell^*) \to R(\mathrm{A}_\ell)$$

are compact with norms $\big|\mathcal{A}_\ell^{-1}\big|_{R(\mathrm{A}_\ell), R(\mathrm{A}_\ell^*)} = \big|(\mathcal{A}_\ell^*)^{-1}\big|_{R(\mathrm{A}_\ell^*), R(\mathrm{A}_\ell)} = c_\ell$.

Proof. If, e.g., Lemma 2.1 (i) was wrong, there exists a sequence $(x_n) \subset D(\mathcal{A}_\ell)$ with $|x_n|_{\mathsf{H}_\ell} = 1$ and $\mathrm{A}_\ell x_n \to 0$. As (x_n) is bounded in $D(\mathcal{A}_\ell)$ we can extract a subsequence, again denoted by (x_n), with $x_n \to x \in \mathsf{H}_\ell$ in H_ℓ. Since A_ℓ is closed, we have $x \in D(\mathrm{A}_\ell)$ and $\mathrm{A}_\ell x = 0$. Hence $x \in N(\mathrm{A}_\ell)$. On the other hand, $(x_n) \subset D(\mathcal{A}_\ell) \subset \overline{R(\mathrm{A}_\ell^*)} = N(\mathrm{A}_\ell)^\perp$ implies $x \in N(\mathrm{A}_\ell)^\perp$. Thus $x = 0$, in contradiction to $1 = |x_n|_{\mathsf{H}_\ell} \to |x|_{\mathsf{H}_\ell} = 0$. □

Lemma 2.4. *Let $\ell \in \{0, \ldots, 3\}$. The embedding $D(\mathcal{A}_\ell) \hookrightarrow \mathsf{H}_\ell$ is compact, if and only if the embedding $D(\mathcal{A}_\ell^*) \hookrightarrow \mathsf{H}_{\ell+1}$ is compact. In this case all assertions of Lemma 2.1 and Lemma 2.2 are valid.*

Proof. By symmetry it is enough to show one direction. Let, e.g., the embedding $D(\mathcal{A}_\ell) \hookrightarrow \mathsf{H}_\ell$ be compact. By Lemma 2.1 and Lemma 2.3, especially $R(\mathrm{A}_\ell) = R(\mathcal{A}_\ell)$ and $R(\mathrm{A}_\ell^*) = R(\mathcal{A}_\ell^*)$ are closed. Let $(y_n) \subset D(\mathcal{A}_\ell^*) = D(\mathrm{A}_\ell^*) \cap R(\mathrm{A}_\ell)$ be a $D(\mathrm{A}_\ell^*)$-bounded sequence. We pick a sequence $(x_n) \subset D(\mathcal{A}_\ell)$ with $y_n = \mathrm{A}_\ell x_n$, i.e., $x_n = (\mathcal{A}_\ell)^{-1} y_n$. As $(\mathcal{A}_\ell)^{-1} : R(\mathrm{A}_\ell) \to D(\mathcal{A}_\ell)$ is continuous, (x_n) is bounded in $D(\mathcal{A}_\ell)$ and thus contains a subsequence, again denoted by (x_n), converging in H_ℓ to some $x \in \mathsf{H}_\ell$. Now

$$|y_n - y_m|_{\mathsf{H}_{\ell+1}}^2 = \big\langle y_n - y_m, \mathrm{A}_\ell (x_n - x_m) \big\rangle_{\mathsf{H}_{\ell+1}} = \big\langle \mathrm{A}_\ell^* (y_n - y_m), x_n - x_m \big\rangle_{\mathsf{H}_\ell} \leq c\, |x_n - x_m|_{\mathsf{H}_\ell}$$

as (y_n) is $D(\mathrm{A}_\ell^*)$-bounded. Finally, we see that (y_n) is a Cauchy sequence in $\mathsf{H}_{\ell+1}$. □

Let us summarize:

Corollary 2.5. *Let $\ell \in \{0, \ldots, 3\}$ and, e.g., let $R(\mathrm{A}_\ell)$ be closed. Then*

$$\frac{1}{c_\ell} = \inf_{0 \neq x \in D(\mathcal{A}_\ell)} \frac{|\mathrm{A}_\ell x|_{\mathsf{H}_{\ell+1}}}{|x|_{\mathsf{H}_\ell}} = \inf_{y \in D(\mathcal{A}_\ell^*)} \frac{|\mathrm{A}_\ell^* y|_{\mathsf{H}_\ell}}{|y|_{\mathsf{H}_{\ell+1}}}$$

exists in $(0, \infty)$. Furthermore:

- **(i)** *The Poincaré type estimates*

$$\forall\, x \in D(\mathcal{A}_\ell) \qquad |x|_{\mathsf{H}_\ell} \leq c_\ell |\mathrm{A}_\ell x|_{\mathsf{H}_{\ell+1}},$$
$$\forall\, y \in D(\mathcal{A}_\ell^*) \qquad |y|_{\mathsf{H}_{\ell+1}} \leq c_\ell |\mathrm{A}_\ell^* y|_{\mathsf{H}_\ell}$$

  hold.
- **(ii)** *The ranges $R(\mathrm{A}_\ell) = R(\mathcal{A}_\ell)$ and $R(\mathrm{A}_\ell^*) = R(\mathcal{A}_\ell^*)$ are closed. Moreover, $D(\mathcal{A}_\ell) = D(\mathrm{A}_\ell) \cap R(\mathrm{A}_\ell^*)$ and $D(\mathcal{A}_\ell^*) = D(\mathrm{A}_\ell^*) \cap R(\mathrm{A}_\ell)$ with*

$$\mathcal{A}_\ell : D(\mathcal{A}_\ell) \subset R(\mathrm{A}_\ell^*) \to R(\mathrm{A}_\ell), \quad \mathcal{A}_\ell^* : D(\mathcal{A}_\ell^*) \subset R(\mathrm{A}_\ell) \to R(\mathrm{A}_\ell^*).$$

(iii) *The Helmholtz type decompositions*

$$\mathsf{H}_\ell = N(\mathrm{A}_\ell) \oplus_{\mathsf{H}_\ell} R(\mathrm{A}_\ell^*), \qquad \mathsf{H}_{\ell+1} = N(\mathrm{A}_\ell^*) \oplus_{\mathsf{H}_{\ell+1}} R(\mathrm{A}_\ell),$$
$$D(\mathrm{A}_\ell) = N(\mathrm{A}_\ell) \oplus_{\mathsf{H}_\ell} D(\mathcal{A}_\ell), \qquad D(\mathrm{A}_\ell^*) = N(\mathrm{A}_\ell^*) \oplus_{\mathsf{H}_{\ell+1}} D(\mathcal{A}_\ell^*)$$

hold.

(iv) *The inverse operators*

$$\mathcal{A}_\ell^{-1} : R(\mathrm{A}_\ell) \to D(\mathcal{A}_\ell), \quad (\mathcal{A}_\ell^*)^{-1} : R(\mathrm{A}_\ell^*) \to D(\mathcal{A}_\ell^*)$$

are continuous and bijective with norms $\left|(\mathcal{A}_\ell)^{-1}\right|_{R(\mathrm{A}_\ell),D(\mathcal{A}_\ell)} = \left|(\mathcal{A}_\ell^*)^{-1}\right|_{R(\mathrm{A}_\ell^*),D(\mathcal{A}_\ell^*)} = (1+c_\ell^2)^{1/2}$ *and* $\left|(\mathcal{A}_\ell)^{-1}\right|_{R(\mathrm{A}_\ell),R(\mathrm{A}_\ell^*)} = \left|(\mathcal{A}_\ell^*)^{-1}\right|_{R(\mathrm{A}_\ell^*),R(\mathrm{A}_\ell)} = c_\ell$.

Corollary 2.6. *Let $\ell \in \{0,\dots,3\}$ and, e.g., let $D(\mathcal{A}_\ell) \hookrightarrow \mathsf{H}_\ell$ be compact. Then $R(\mathrm{A}_\ell)$ is closed and the assertions of Corollary 2.5 hold. Moreover, the inverse operators*

$$\mathcal{A}_\ell^{-1} : R(\mathrm{A}_\ell) \to R(\mathrm{A}_\ell^*), \quad (\mathcal{A}_\ell^*)^{-1} : R(\mathrm{A}_\ell^*) \to R(\mathrm{A}_\ell)$$

are compact.

So far, we did not use the complex property (1.1) except of proving the refined Helmholtz type decomposition (2.2), which we did not need until now. Hence Lemma 2.1, Lemma 2.2, Lemma 2.3, Lemma 2.4, and Corollary 2.5, Corollary 2.6 hold without the complex property (1.1). On the other hand, using (1.1) we obtain the following result:

Lemma 2.7. *Let $\ell \in \{1,\dots,3\}$. Then the refined Helmholtz type decompositions*

$$\mathsf{H}_\ell = \overline{R(\mathrm{A}_{\ell\text{-}1})} \oplus_{\mathsf{H}_\ell} K_\ell \oplus_{\mathsf{H}_\ell} \overline{R(\mathrm{A}_\ell^*)}, \qquad K_\ell = N(\mathrm{A}_\ell) \cap N(\mathrm{A}_{\ell\text{-}1}^*),$$
$$N(\mathrm{A}_\ell) = \overline{R(\mathrm{A}_{\ell\text{-}1})} \oplus_{\mathsf{H}_\ell} K_\ell, \qquad N(\mathrm{A}_{\ell\text{-}1}^*) = K_\ell \oplus_{\mathsf{H}_\ell} \overline{R(\mathrm{A}_\ell^*)},$$
$$\overline{R(\mathcal{A}_{\ell\text{-}1})} = \overline{R(\mathrm{A}_{\ell\text{-}1})} = N(\mathrm{A}_\ell) \ominus_{\mathsf{H}_\ell} K_\ell, \qquad \overline{R(\mathcal{A}_\ell^*)} = \overline{R(\mathrm{A}_\ell^*)} = N(\mathrm{A}_{\ell\text{-}1}^*) \ominus_{\mathsf{H}_\ell} K_\ell,$$
$$D(\mathrm{A}_\ell) = \overline{R(\mathrm{A}_{\ell\text{-}1})} \oplus_{\mathsf{H}_\ell} K_\ell \oplus_{\mathsf{H}_\ell} D(\mathcal{A}_\ell), \qquad D(\mathrm{A}_{\ell\text{-}1}^*) = D(\mathcal{A}_{\ell\text{-}1}^*) \oplus_{\mathsf{H}_\ell} K_\ell \oplus_{\mathsf{H}_\ell} \overline{R(\mathrm{A}_\ell^*)},$$
$$D_\ell = D(\mathcal{A}_{\ell\text{-}1}^*) \oplus_{\mathsf{H}_\ell} K_\ell \oplus_{\mathsf{H}_\ell} D(\mathcal{A}_\ell), \qquad D_\ell = D(\mathrm{A}_\ell) \cap D(\mathrm{A}_{\ell\text{-}1}^*)$$

hold. If the range $R(\mathrm{A}_{\ell\text{-}1})$ or $R(\mathrm{A}_\ell)$ is closed, the respective closure bars can be dropped and the assertions of Corollary 2.5 are valid. Especially, if $R(\mathrm{A}_{\ell\text{-}1})$ and $R(\mathrm{A}_\ell)$ are closed, the assertions of Corollary 2.5 and the refined Helmholtz type decompositions

$$\mathsf{H}_\ell = R(\mathrm{A}_{\ell\text{-}1}) \oplus_{\mathsf{H}_\ell} K_\ell \oplus_{\mathsf{H}_\ell} R(\mathrm{A}_\ell^*), \qquad K_\ell = N(\mathrm{A}_\ell) \cap N(\mathrm{A}_{\ell\text{-}1}^*),$$
$$N(\mathrm{A}_\ell) = R(\mathrm{A}_{\ell\text{-}1}) \oplus_{\mathsf{H}_\ell} K_\ell, \qquad N(\mathrm{A}_{\ell\text{-}1}^*) = K_\ell \oplus_{\mathsf{H}_\ell} R(\mathrm{A}_\ell^*),$$
$$R(\mathcal{A}_{\ell\text{-}1}) = R(\mathrm{A}_{\ell\text{-}1}) = N(\mathrm{A}_\ell) \ominus_{\mathsf{H}_\ell} K_\ell, \qquad R(\mathcal{A}_\ell^*) = R(\mathrm{A}_\ell^*) = N(\mathrm{A}_{\ell\text{-}1}^*) \ominus_{\mathsf{H}_\ell} K_\ell,$$
$$D(\mathrm{A}_\ell) = R(\mathrm{A}_{\ell\text{-}1}) \oplus_{\mathsf{H}_\ell} K_\ell \oplus_{\mathsf{H}_\ell} D(\mathcal{A}_\ell), \qquad D(\mathrm{A}_{\ell\text{-}1}^*) = D(\mathcal{A}_{\ell\text{-}1}^*) \oplus_{\mathsf{H}_\ell} K_\ell \oplus_{\mathsf{H}_\ell} R(\mathrm{A}_\ell^*),$$
$$D_\ell = D(\mathcal{A}_{\ell\text{-}1}^*) \oplus_{\mathsf{H}_\ell} K_\ell \oplus_{\mathsf{H}_\ell} D(\mathcal{A}_\ell), \qquad D_\ell = D(\mathrm{A}_\ell) \cap D(\mathrm{A}_{\ell\text{-}1}^*)$$

hold.

Observe that

$$D(\mathcal{A}_\ell) = D(\mathrm{A}_\ell) \cap \overline{R(\mathrm{A}_\ell^*)} \subset D(\mathrm{A}_\ell) \cap N(\mathrm{A}_{\ell\text{-}1}^*) \subset D(\mathrm{A}_\ell) \cap D(\mathrm{A}_{\ell\text{-}1}^*) = D_\ell,$$
$$D(\mathcal{A}_{\ell\text{-}1}^*) = D(\mathrm{A}_{\ell\text{-}1}^*) \cap \overline{R(\mathrm{A}_{\ell\text{-}1})} \subset D(\mathrm{A}_{\ell\text{-}1}^*) \cap N(\mathrm{A}_\ell) \subset D(\mathrm{A}_{\ell\text{-}1}^*) \cap D(\mathrm{A}_\ell) = D_\ell. \tag{2.5}$$

Lemma 2.8. *Let $\ell \in \{1,\dots,3\}$. The embeddings $D(\mathcal{A}_\ell) \hookrightarrow \mathsf{H}_\ell$, $D(\mathcal{A}_{\ell\text{-}1}) \hookrightarrow \mathsf{H}_{\ell\text{-}1}$, and $K_\ell \hookrightarrow \mathsf{H}_\ell$ are compact, if and only if the embedding $D_\ell \hookrightarrow \mathsf{H}_\ell$ is compact. In this case, K_ℓ has finite dimension.*

Proof. Note that, by Lemma 2.4, $D(\mathcal{A}_{\ell\text{-}1}) \hookrightarrow \mathsf{H}_{\ell\text{-}1}$ is compact, if and only if $D(\mathcal{A}_{\ell\text{-}1}^*) \hookrightarrow \mathsf{H}_\ell$ is compact.

⇒: Let $(x_n) \subset D_\ell$ be a D_ℓ-bounded sequence. By the refined Helmholtz type decomposition of Lemma 2.7 we decompose

$$x_n = a_n^* + k_n + a_n \in D(\mathcal{A}_{\ell\text{-}1}^*) \oplus_{\mathsf{H}_\ell} K_\ell \oplus_{\mathsf{H}_\ell} D(\mathcal{A}_\ell).$$

with $\mathrm{A}_\ell\, x_n = \mathrm{A}_\ell\, a_n$ and $\mathrm{A}_{\ell\text{-}1}^*\, x_n = \mathrm{A}_{\ell\text{-}1}^*\, a_n^*$. Hence (a_n) is bounded in $D(\mathcal{A}_\ell)$ and (a_n^*) is bounded in $D(\mathcal{A}_{\ell\text{-}1}^*)$ and we can extract H_ℓ-converging subsequences of (a_n), (a_n^*), and (k_n).

⇐: If $D_\ell \hookrightarrow \mathsf{H}_\ell$ is compact, so is $K_\ell \hookrightarrow \mathsf{H}_\ell$. Moreover, by (2.5)

$$D(\mathcal{A}_\ell) \subset D_\ell \hookrightarrow \mathsf{H}_\ell, \qquad D(\mathcal{A}_{\ell\text{-}1}^*) \subset D_\ell \hookrightarrow \mathsf{H}_\ell\,.$$

Finally, if $K_\ell \hookrightarrow \mathsf{H}_\ell$ is compact, the unit ball in K_ℓ is compact, showing that K_ℓ has finite dimension. □

Lemma 2.8 implies immediately the following result.

Corollary 2.9. *Let $\ell \in \{1,\dots,3\}$ and let $D_\ell \hookrightarrow \mathsf{H}_\ell$ be compact. Then $R(\mathrm{A}_{\ell\text{-}1})$ and $R(\mathrm{A}_\ell)$ are closed, and, besides the assertions of Corollary 2.6, the refined Helmholtz type decompositions of Lemma 2.7 hold and the cohomology group K_ℓ is finite dimensional.*

Remark 2.10. *Let $\ell \in \{1,\dots,3\}$. Under the assumption that the embedding $D_\ell \hookrightarrow \mathsf{H}_\ell$ is compact, all the assertions of this section hold. Especially, the complex*

$$D(\mathrm{A}_{\ell\text{-}1}) \xrightarrow{\mathrm{A}_{\ell\text{-}1}} D(\mathrm{A}_\ell) \xrightarrow{\mathrm{A}_\ell} \mathsf{H}_{\ell+1}$$

together with its adjoint complex

$$\mathsf{H}_{\ell\text{-}1} \xleftarrow{\mathrm{A}_{\ell\text{-}1}^*} D(\mathrm{A}_{\ell\text{-}1}^*) \xleftarrow{\mathrm{A}_\ell^*} D(\mathrm{A}_\ell^*)$$

is closed. These complexes are even exact, if additionally $K_\ell = \{0\}$.

Defining and recalling the orthonormal projectors

$$\pi_{\mathrm{A}_{\ell\text{-}1}} := \pi_{\overline{R(\mathrm{A}_{\ell\text{-}1})}} : \mathsf{H}_\ell \to \overline{R(\mathrm{A}_{\ell\text{-}1})}, \qquad \pi_{\mathrm{A}_\ell^*} := \pi_{\overline{R(\mathrm{A}_\ell^*)}} : \mathsf{H}_\ell \to \overline{R(\mathrm{A}_\ell^*)}, \qquad \pi_\ell : \mathsf{H}_\ell \to K_\ell, \tag{2.6}$$

we have $\pi_\ell = 1 - \pi_{\mathrm{A}_{\ell\text{-}1}} - \pi_{\mathrm{A}_\ell^*}$ as well as

$$\pi_{\mathrm{A}_{\ell\text{-}1}} \mathsf{H}_\ell = \pi_{\mathrm{A}_{\ell\text{-}1}} D(\mathrm{A}_\ell) = \pi_{\mathrm{A}_{\ell\text{-}1}} N(\mathrm{A}_\ell) = \overline{R(\mathrm{A}_{\ell\text{-}1})} = \overline{R(\mathcal{A}_{\ell\text{-}1})},$$
$$\pi_{\mathrm{A}_\ell^*} \mathsf{H}_\ell = \pi_{\mathrm{A}_\ell^*} D(\mathrm{A}_{\ell\text{-}1}^*) = \pi_{\mathrm{A}_\ell^*} N(\mathrm{A}_{\ell\text{-}1}^*) = \overline{R(\mathrm{A}_\ell^*)} = \overline{R(\mathcal{A}_\ell^*)}$$

and

$$\pi_{\mathrm{A}_{\ell\text{-}1}} D(\mathrm{A}_{\ell\text{-}1}^*) = \pi_{\mathrm{A}_{\ell\text{-}1}} D_\ell = D(\mathcal{A}_{\ell\text{-}1}^*), \qquad \pi_{\mathrm{A}_\ell^*} D(\mathrm{A}_\ell) = \pi_{\mathrm{A}_\ell^*} D_\ell = D(\mathcal{A}_\ell).$$

Moreover

$$\forall\, \xi \in D(\mathrm{A}_{\ell\text{-}1}^*) \qquad \pi_{\mathrm{A}_{\ell\text{-}1}} \xi \in D(\mathcal{A}_{\ell\text{-}1}^*) \qquad \wedge \qquad \mathrm{A}_{\ell\text{-}1}^*\, \pi_{\mathrm{A}_{\ell\text{-}1}} \xi = \mathrm{A}_{\ell\text{-}1}^*\, \xi,$$
$$\forall\, \zeta \in D(\mathrm{A}_\ell) \qquad \pi_{\mathrm{A}_\ell^*} \zeta \in D(\mathcal{A}_\ell) \qquad \wedge \qquad \mathrm{A}_\ell\, \pi_{\mathrm{A}_\ell^*} \zeta = \mathrm{A}_\ell\, \zeta.$$

We also introduce the orthogonal projectors onto the kernels

$$\pi_{N(\mathrm{A}_{\ell\text{-}1}^*)} := 1 - \pi_{\mathrm{A}_{\ell\text{-}1}} : \mathsf{H}_\ell \to N(\mathrm{A}_{\ell\text{-}1}^*), \qquad \pi_{N(\mathrm{A}_\ell)} := 1 - \pi_{\mathrm{A}_\ell^*} : \mathsf{H}_\ell \to N(\mathrm{A}_\ell).$$

3. Solution Theory

From now on and throughout this paper we suppose the following.

General Assumption 3.1. *$R(\mathrm{A}_1)$ and $R(\mathrm{A}_2)$ are closed and K_2 is finite dimensional.*

Remark 3.2. *The General Assumption 3.1 is satisfied, if, e.g., $D_2 \hookrightarrow \mathsf{H}_2$ is compact. The finite dimension of the cohomology group K_2 may be dropped.*

3.1. **First Order Systems.** We recall the linear first order system (1.5) from the introduction: Find $x \in D_2 = D(\mathrm{A}_2) \cap D(\mathrm{A}_1^*)$ such that

$$\begin{aligned} \mathrm{A}_2\, x &= f, \\ \mathrm{A}_1^*\, x &= g, \\ \pi_2\, x &= k. \end{aligned} \tag{3.1}$$

Theorem 3.3. (3.1) *is uniquely solvable in D_2, if and only if $f \in R(\mathrm{A}_2)$, $g \in R(\mathrm{A}_1^*)$, and $k \in K_2$. The unique solution $x \in D_2$ is given by*

$$\begin{aligned} x &:= x_f + x_g + k \in D(\mathcal{A}_2) \oplus_{\mathsf{H}_2} D(\mathcal{A}_1^*) \oplus_{\mathsf{H}_2} K_2 = D_2, \\ x_f &:= (\mathcal{A}_2)^{-1} f \in D(\mathcal{A}_2), \\ x_g &:= (\mathcal{A}_1^*)^{-1} g \in D(\mathcal{A}_1^*) \end{aligned}$$

and depends continuously on the data, i.e., $|x|_{\mathsf{H}_2} \leq c_2 |f|_{\mathsf{H}_3} + c_1 |g|_{\mathsf{H}_1} + |k|_{\mathsf{H}_2}$, as

$$|x_f|_{\mathsf{H}_2} \leq c_2 |f|_{\mathsf{H}_3}, \qquad |x_g|_{\mathsf{H}_2} \leq c_1 |g|_{\mathsf{H}_1}.$$

It holds

$$\pi_{\mathrm{A}_2^*} x = x_f, \qquad \pi_{\mathrm{A}_1} x = x_g, \qquad \pi_2\, x = k, \qquad |x|_{\mathsf{H}_2}^2 = |x_f|_{\mathsf{H}_2}^2 + |x_g|_{\mathsf{H}_2}^2 + |k|_{\mathsf{H}_2}^2.$$

The partial solutions x_f and x_g can be found by the following two variational formulations: There exist unique potentials $y_f \in D(\mathcal{A}_2^*)$ and $z_g \in D(\mathcal{A}_1)$, such that

$$\forall\, \phi \in D(\mathcal{A}_2^*) \qquad \langle \mathrm{A}_2^*\, y_f, \mathrm{A}_2^*\, \phi \rangle_{\mathsf{H}_2} = \langle f, \phi \rangle_{\mathsf{H}_3}, \tag{3.2}$$

$$\forall\, \varphi \in D(\mathcal{A}_1) \qquad \langle \mathrm{A}_1\, z_g, \mathrm{A}_1\, \varphi \rangle_{\mathsf{H}_2} = \langle g, \varphi \rangle_{\mathsf{H}_1}. \tag{3.3}$$

Moreover, (3.2) *and* (3.3) *even hold for all $\phi \in D(\mathrm{A}_2^*)$ and for all $\varphi \in D(\mathrm{A}_1)$, respectively, as $f \in R(\mathrm{A}_2)$ and $g \in R(\mathrm{A}_1^*)$. Hence we have $\mathrm{A}_2^*\, y_f \in D(\mathrm{A}_2) \cap R(\mathrm{A}_2^*) = D(\mathcal{A}_2)$ with $\mathrm{A}_2\, \mathrm{A}_2^*\, y_f = f$ as well as $\mathrm{A}_1\, z_g \in D(\mathrm{A}_1^*) \cap R(\mathrm{A}_1) = D(\mathcal{A}_1^*)$ with $\mathrm{A}_1^*\, \mathrm{A}_1\, z_g = g$, yielding*

$$\mathrm{A}_2^*\, y_f = x_f, \qquad \mathrm{A}_1\, z_g = x_g.$$

Proof. As pointed out in the introduction, we just need to show existence. We use the results of Section 2. Let $f \in R(\mathrm{A}_2)$, $g \in R(\mathrm{A}_1^*)$, $k \in K_2$ and define x, x_f, and x_g according to the theorem. For the orthogonality we refer to Lemma 2.7. Moreover, x_f, x_g, and k solve the linear systems

$$\begin{aligned} \mathrm{A}_2\, x_f &= f, & \mathrm{A}_2\, x_g &= 0, & \mathrm{A}_2\, k &= 0, \\ \mathrm{A}_1^*\, x_f &= 0, & \mathrm{A}_1^*\, x_g &= g, & \mathrm{A}_1^*\, k &= 0, \\ \pi_2\, x_f &= 0, & \pi_2\, x_g &= 0, & \pi_2\, k &= k. \end{aligned}$$

Thus x solves (3.1) and we have by Corollary 2.5 $|x_f|_{\mathsf{H}_2} \leq c_2 |f|_{\mathsf{H}_3}$ and $|x_g|_{\mathsf{H}_2} \leq c_1 |g|_{\mathsf{H}_1}$, which completes the solution theory. To find the variational formulation for $x_f \in D(\mathcal{A}_2) = D(\mathrm{A}_2) \cap R(\mathcal{A}_2^*)$, we observe $x_f = \mathrm{A}_2^*\, y_f$ with $y_f := (\mathcal{A}_2^*)^{-1} x_f \in D(\mathcal{A}_2^*)$ and

$$\forall\, \phi \in D(\mathcal{A}_2^*) \qquad \langle \mathrm{A}_2^*\, y_f, \mathrm{A}_2^*\, \phi \rangle_{\mathsf{H}_2} = \langle x_f, \mathrm{A}_2^*\, \phi \rangle_{\mathsf{H}_2} = \langle \mathrm{A}_2\, x_f, \phi \rangle_{\mathsf{H}_3} = \langle f, \phi \rangle_{\mathsf{H}_3}. \tag{3.4}$$

Using Corollary 2.5 (iii) or Lemma 2.7 we can split any $\phi \in D(\mathrm{A}_2^*) = N(\mathrm{A}_2^*) \oplus_{\mathsf{H}_3} D(\mathcal{A}_2^*)$ into $\phi = \phi_N + \phi_R$ (null space and range) with $\phi_N \in N(\mathrm{A}_2^*)$, $\phi_R \in D(\mathcal{A}_2^*)$, and $\mathrm{A}_2^*\, \phi = \mathrm{A}_2^*\, \phi_R$. Utilizing (3.4) for ϕ_R and orthogonality, i.e., $f \in R(\mathrm{A}_2) = N(\mathrm{A}_2^*)^{\perp_{\mathsf{H}_3}}$, we get

$$\langle \mathrm{A}_2^*\, y_f, \mathrm{A}_2^*\, \phi \rangle_{\mathsf{H}_2} = \langle \mathrm{A}_2^*\, y_f, \mathrm{A}_2^*\, \phi_R \rangle_{\mathsf{H}_2} = \langle f, \phi_R \rangle_{\mathsf{H}_3} = \langle f, \phi \rangle_{\mathsf{H}_3}.$$

Therefore, (3.4) holds for all $\phi \in D(\mathrm{A}_2^*)$. On the other hand, (3.4) is coercive over $D(\mathcal{A}_2^*)$ by the Friedrichs/Poincaré type estimate of Corollary 2.5 (i) and hence a unique $y_f \in D(\mathcal{A}_2^*)$ exists by Riesz' representation theorem (or Lax-Milgram's lemma) solving (3.4). But then (3.4) holds for all $\phi \in D(\mathrm{A}_2^*)$ as well, yielding $\tilde{x}_f := \mathrm{A}_2^*\, y_f \in D(\mathrm{A}_2)$ with $\mathrm{A}_2\, \tilde{x}_f = f$. Since $\tilde{x}_f \in D(\mathrm{A}_2) \cap R(\mathrm{A}_2^*) = D(\mathcal{A}_2) \subset R(\mathrm{A}_2^*)$ we have $\tilde{x}_f = (\mathcal{A}_2)^{-1} f = x_f$ and especially $\mathrm{A}_1^* \tilde{x}_f = 0$ and $\pi_2 \tilde{x}_f = 0$. Analogously, we obtain a variational formulation for x_g as well. □

Remark 3.4. *By orthogonality and with* $\mathrm{A}_2 x = \mathrm{A}_2 x_f = f$ *and* $\mathrm{A}_1^* x = \mathrm{A}_1^* x_g = g$ *we even have*

$$|x|_{\mathsf{H}_2}^2 = |x_f|_{\mathsf{H}_2}^2 + |x_g|_{\mathsf{H}_2}^2 + |k|_{\mathsf{H}_2}^2 \leq c_2^2 |f|_{\mathsf{H}_3}^2 + c_1^2 |g|_{\mathsf{H}_1}^2 + |k|_{\mathsf{H}_2}^2,$$
$$|x|_{D_2}^2 = |x_f|_{\mathsf{H}_2}^2 + |f|_{\mathsf{H}_3}^2 + |x_g|_{\mathsf{H}_2}^2 + |g|_{\mathsf{H}_1}^2 + |k|_{\mathsf{H}_2}^2 \leq (1 + c_2^2)|f|_{\mathsf{H}_3}^2 + (1 + c_1^2)|g|_{\mathsf{H}_1}^2 + |k|_{\mathsf{H}_2}^2.$$

Note that

$$y_f = (\mathcal{A}_2^*)^{-1} x_f = (\mathcal{A}_2^*)^{-1} (\mathcal{A}_2)^{-1} f \in D(\mathcal{A}_2^*), \qquad z_g = (\mathcal{A}_1)^{-1} x_g = (\mathcal{A}_1)^{-1} (\mathcal{A}_1^*)^{-1} g \in D(\mathcal{A}_1)$$

holds with $\mathrm{A}_2 \mathrm{A}_2^* y_f = f$ and $\mathrm{A}_1^* \mathrm{A}_1 z_g = g$. Hence x_f, x_g, k, and y_f, z_g solve the first resp. second order systems

$$\begin{aligned} &\mathrm{A}_2 x_f = f, && \mathrm{A}_2 x_g = 0, && \mathrm{A}_2 k = 0, && \mathrm{A}_2 \mathrm{A}_2^* y_f = f, && \mathrm{A}_1^* \mathrm{A}_1 z_g = g, \\ &\mathrm{A}_1^* x_f = 0, && \mathrm{A}_1^* x_g = g, && \mathrm{A}_1^* k = 0, && \mathrm{A}_3 y_f = 0, && \mathrm{A}_0^* z_g = 0, \\ &\pi_2 x_f = 0, && \pi_2 x_g = 0, && \pi_2 k = k, && \pi_3 y_f = 0, && \pi_1 z_g = 0. \end{aligned}$$

We also emphasize that the variational formulations (3.2)-(3.3) have a saddle point structure. We have already seen that, provided $f \in R(\mathrm{A}_2)$ and $g \in R(\mathrm{A}_1^*)$, the formulations (3.2)-(3.3) are equivalent to the following two problems: Find $y_f \in D(\mathcal{A}_2^*)$ and $z_g \in D(\mathcal{A}_1)$, such that

$$\forall\, \phi \in D(\mathrm{A}_2^*) \qquad \langle \mathrm{A}_2^* y_f, \mathrm{A}_2^* \phi \rangle_{\mathsf{H}_2} = \langle f, \phi \rangle_{\mathsf{H}_3}, \tag{3.5}$$
$$\forall\, \varphi \in D(\mathrm{A}_1) \qquad \langle \mathrm{A}_1 z_g, \mathrm{A}_1 \varphi \rangle_{\mathsf{H}_2} = \langle g, \varphi \rangle_{\mathsf{H}_1}. \tag{3.6}$$

Moreover, $y_f \in D(\mathcal{A}_2^*) = D(\mathrm{A}_2^*) \cap R(\mathrm{A}_2)$ if and only if $y_f \in D(\mathrm{A}_2^*)$ and $y_f \perp_{\mathsf{H}_3} N(\mathrm{A}_2^*)$ as well as $z_g \in D(\mathcal{A}_1) = D(\mathrm{A}_1) \cap R(\mathrm{A}_1^*)$ if and only if $z_g \in D(\mathrm{A}_1)$ and $z_g \perp_{\mathsf{H}_1} N(\mathrm{A}_1)$. Therefore, the variational formulations (3.5)-(3.6) are equivalent to the following two saddle point problems: Find $y_f \in D(\mathrm{A}_2^*)$ and $z_g \in D(\mathrm{A}_1)$, such that

$$\forall\, \phi \in D(\mathrm{A}_2^*) \quad \langle \mathrm{A}_2^* y_f, \mathrm{A}_2^* \phi \rangle_{\mathsf{H}_2} = \langle f, \phi \rangle_{\mathsf{H}_3} \quad \wedge \quad \forall\, \theta \in N(\mathrm{A}_2^*) \quad \langle y_f, \theta \rangle_{\mathsf{H}_3} = 0, \tag{3.7}$$
$$\forall\, \varphi \in D(\mathrm{A}_1) \quad \langle \mathrm{A}_1 z_g, \mathrm{A}_1 \varphi \rangle_{\mathsf{H}_2} = \langle g, \varphi \rangle_{\mathsf{H}_1} \quad \wedge \quad \forall\, \psi \in N(\mathrm{A}_1) \quad \langle z_g, \psi \rangle_{\mathsf{H}_1} = 0. \tag{3.8}$$

Remark 3.5. *The finite dimensionality of K_2 may be dropped. Then all other assertions of Theorem 3.3 and all variational and saddle point formulations remain valid. Note that $R(\mathrm{A}_1)$ and $R(\mathrm{A}_2)$ are closed, if $D(\mathcal{A}_1) \hookrightarrow \mathsf{H}_1$ and $D(\mathcal{A}_2) \hookrightarrow \mathsf{H}_2$ are compact.*

3.1.1. *Trivial Cohomology Groups.* By Lemma 2.7 it holds

$$N(\mathrm{A}_1) = \overline{R(\mathrm{A}_0)} \oplus_{\mathsf{H}_1} K_1, \qquad N(\mathrm{A}_2^*) = \overline{R(\mathrm{A}_3^*)} \oplus_{\mathsf{H}_3} K_3.$$

In the special case, that $R(\mathrm{A}_0)$ and $R(\mathrm{A}_3^*)$ are closed and additionally

$$K_1 = \{0\}, \qquad K_3 = \{0\},$$

we see that the two saddle point problems (3.7)-(3.8) are equivalent to: Find $y_f \in D(\mathrm{A}_2^*)$ and $z_g \in D(\mathrm{A}_1)$, such that

$$\forall\, \phi \in D(\mathrm{A}_2^*) \quad \langle \mathrm{A}_2^* y_f, \mathrm{A}_2^* \phi \rangle_{\mathsf{H}_2} = \langle f, \phi \rangle_{\mathsf{H}_3} \quad \wedge \quad \forall\, \vartheta \in D(\mathrm{A}_3^*) \quad \langle y_f, \mathrm{A}_3^* \vartheta \rangle_{\mathsf{H}_3} = 0, \tag{3.9}$$
$$\forall\, \varphi \in D(\mathrm{A}_1) \quad \langle \mathrm{A}_1 z_g, \mathrm{A}_1 \varphi \rangle_{\mathsf{H}_2} = \langle g, \varphi \rangle_{\mathsf{H}_1} \quad \wedge \quad \forall\, \tau \in D(\mathrm{A}_0) \quad \langle z_g, \mathrm{A}_0 \tau \rangle_{\mathsf{H}_1} = 0. \tag{3.10}$$

Let us consider the following modified system: Find

$$(y_f, v_f) \in D(\mathrm{A}_2^*) \times D(\mathcal{A}_3^*), \qquad (z_g, w_g) \in D(\mathrm{A}_1) \times D(\mathcal{A}_0),$$

such that

$$\forall\, (\phi, \vartheta) \in D(\mathrm{A}_2^*) \times D(\mathcal{A}_3^*) \quad \langle \mathrm{A}_2^* y_f, \mathrm{A}_2^* \phi \rangle_{\mathsf{H}_2} + \langle \phi, \mathrm{A}_3^* v_f \rangle_{\mathsf{H}_3} = \langle f, \phi \rangle_{\mathsf{H}_3} \quad \wedge \quad \langle y_f, \mathrm{A}_3^* \vartheta \rangle_{\mathsf{H}_3} = 0, \tag{3.11}$$
$$\forall\, (\varphi, \tau) \in D(\mathrm{A}_1) \times D(\mathcal{A}_0) \quad \langle \mathrm{A}_1 z_g, \mathrm{A}_1 \varphi \rangle_{\mathsf{H}_2} + \langle \varphi, \mathrm{A}_0 w_g \rangle_{\mathsf{H}_1} = \langle g, \varphi \rangle_{\mathsf{H}_1} \quad \wedge \quad \langle z_g, \mathrm{A}_0 \tau \rangle_{\mathsf{H}_1} = 0. \tag{3.12}$$

The unique solutions y_f, z_g of (3.9)-(3.10) yield solutions $(y_f,0)$, $(z_g,0)$ of (3.11)-(3.12). On the other hand, for any solutions (y_f,v_f), (z_g,w_g) of (3.11)-(3.12) we get $\mathrm{A}_3^* v_f=0$ and $\mathrm{A}_0 w_g=0$ by testing with $\phi:=\mathrm{A}_3^* v_f\in R(\mathrm{A}_3^*)=N(\mathrm{A}_2^*)\subset D(\mathrm{A}_2^*)$ and $\varphi:=\mathrm{A}_0 w_g\in R(\mathrm{A}_0)=N(\mathrm{A}_1)\subset D(\mathrm{A}_1)$ since $f\in R(\mathrm{A}_2)\perp_{\mathsf{H}_3}N(\mathrm{A}_2^*)$ and $g\in R(\mathrm{A}_1^*)\perp_{\mathsf{H}_1}N(\mathrm{A}_1)$, respectively. Hence, as $v_f\in D(\mathcal{A}_3^*)$ and $w_g\in D(\mathcal{A}_0)$ we see $v_f=0$ and $w_g=0$. Thus, y_f, z_g are the unique solutions of (3.9)-(3.10). The latter arguments show that (3.9)-(3.10) and (3.11)-(3.12) are equivalent and both are uniquely solvable. Furthermore, the saddle point formulations (3.11)-(3.12) are accessible by the standard inf-sup-theory: The bilinear forms $\langle\mathrm{A}_2^*\,\cdot,\mathrm{A}_2^*\,\cdot\rangle_{\mathsf{H}_2}$ and $\langle\mathrm{A}_1\,\cdot,\mathrm{A}_1\,\cdot\rangle_{\mathsf{H}_2}$ are coercive over the respective kernels, which are $N(\mathrm{A}_3)=R(\mathrm{A}_2)$ and $N(\mathrm{A}_0^*)=R(\mathrm{A}_1^*)$, i.e., over $D(\mathcal{A}_2^*)$ and $D(\mathcal{A}_1)$, and satisfy the inf-sup-conditions[iv]

$$\inf_{0\neq\vartheta\in D(\mathcal{A}_3^*)}\sup_{0\neq\phi\in D(\mathrm{A}_2^*)}\frac{\langle\phi,\mathrm{A}_3^*\vartheta\rangle_{\mathsf{H}_3}}{|\phi|_{D(\mathrm{A}_2^*)}|\vartheta|_{D(\mathrm{A}_3^*)}}\geq\inf_{0\neq\vartheta\in D(\mathcal{A}_3^*)}\frac{|\mathrm{A}_3^*\vartheta|_{\mathsf{H}_3}}{|\vartheta|_{D(\mathrm{A}_3^*)}}=(c_3^2+1)^{-1/2},$$
$$\inf_{0\neq\tau\in D(\mathcal{A}_0)}\sup_{0\neq\varphi\in D(\mathrm{A}_1)}\frac{\langle\varphi,\mathrm{A}_0\tau\rangle_{\mathsf{H}_1}}{|\varphi|_{D(\mathrm{A}_1)}|\tau|_{D(\mathrm{A}_0)}}\geq\inf_{0\neq\tau\in D(\mathcal{A}_0)}\frac{|\mathrm{A}_0\tau|_{\mathsf{H}_1}}{|\tau|_{D(\mathrm{A}_0)}}=(c_0^2+1)^{-1/2},$$

which follows immediately by choosing $\phi:=\mathrm{A}_3^*\vartheta\in R(\mathrm{A}_3^*)=N(\mathrm{A}_2^*)$ and $\varphi:=\mathrm{A}_0\tau\in R(\mathrm{A}_0)=N(\mathrm{A}_1)$. Now, if $D(\mathcal{A}_3^*)$ and $D(\mathcal{A}_0)$ are still not suitable and provided that the respective cohomology groups are trivial, we can repeat the procedure to obtain additional saddle point formulations for v_f and w_g. Note that (3.11)-(3.12) is equivalent to find $(y_f,v_f,z_g,w_g)\in D(\mathrm{A}_2^*)\times D(\mathcal{A}_3^*)\times D(\mathrm{A}_1)\times D(\mathcal{A}_0)$, such that for all $(\phi,\vartheta,\varphi,\tau)\in D(\mathrm{A}_2^*)\times D(\mathcal{A}_3^*)\times D(\mathrm{A}_1)\times D(\mathcal{A}_0)$

$$\begin{aligned}\langle\mathrm{A}_2^* y_f,\mathrm{A}_2^*\phi\rangle_{\mathsf{H}_2}+\langle\phi,\mathrm{A}_3^* v_f\rangle_{\mathsf{H}_3}+\langle y_f,\mathrm{A}_3^*\vartheta\rangle_{\mathsf{H}_3}+\langle\mathrm{A}_1 z_g,\mathrm{A}_1\varphi\rangle_{\mathsf{H}_2}+\langle\varphi,\mathrm{A}_0 w_g\rangle_{\mathsf{H}_1}+\langle z_g,\mathrm{A}_0\tau\rangle_{\mathsf{H}_1}&\\=\langle f,\phi\rangle_{\mathsf{H}_3}+\langle g,\varphi\rangle_{\mathsf{H}_1}.&\end{aligned}\tag{3.13}$$

3.2. **Second Order Systems.** We recall the linear second order system (1.8), i.e., find[v]

$$x\in\tilde{D}_2:=\big\{\xi\in D_2\,:\,\mathrm{A}_2\xi\in D(\mathrm{A}_2^*)\big\}=\big\{\xi\in D(\mathrm{A}_2)\cap D(\mathrm{A}_1^*)\,:\,\mathrm{A}_2\xi\in D(\mathrm{A}_2^*)\big\}=D(\mathrm{A}_2)\cap D(\mathrm{A}_2^*\mathrm{A}_2)$$

such that

$$\begin{aligned}\mathrm{A}_2^*\mathrm{A}_2 x&=f,\\ \mathrm{A}_1^* x&=g,\\ \pi_2 x&=k.\end{aligned}\tag{3.14}$$

Theorem 3.6. (3.14) *is uniquely solvable in* $\tilde{D}_2$*, if and only if* $f\in R(\mathrm{A}_2^*)$*,* $g\in R(\mathrm{A}_1^*)$*, and* $k\in K_2$*. The unique solution* $x\in\tilde{D}_2$ *is given by*

$$\begin{aligned}x&:=x_f+x_g+k\in\big(D(\mathcal{A}_2)\oplus_{\mathsf{H}_2}D(\mathcal{A}_1^*)\oplus_{\mathsf{H}_2}K_2\big)\cap\tilde{D}_2=\tilde{D}_2,\\ x_f&:=(\mathcal{A}_2)^{-1}(\mathcal{A}_2^*)^{-1}f\in D(\mathcal{A}_2)\cap\tilde{D}_2,\\ x_g&:=(\mathcal{A}_1^*)^{-1}g\in D(\mathcal{A}_1^*)\cap\tilde{D}_2\end{aligned}$$

and depends continuously on the data, i.e., $|x|_{\mathsf{H}_2}\leq c_2^2|f|_{\mathsf{H}_2}+c_1|g|_{\mathsf{H}_1}+|k|_{\mathsf{H}_2}$*, as*

$$|x_f|_{\mathsf{H}_2}\leq c_2^2|f|_{\mathsf{H}_2},\qquad|x_g|_{\mathsf{H}_2}\leq c_1|g|_{\mathsf{H}_1}.$$

It holds

$$\pi_{\mathrm{A}_2^*}x=x_f,\qquad\pi_{\mathrm{A}_1}x=x_g,\qquad\pi_2 x=k,\qquad|x|_{\mathsf{H}_2}^2=|x_f|_{\mathsf{H}_2}^2+|x_g|_{\mathsf{H}_2}^2+|k|_{\mathsf{H}_2}^2.$$

[iv]Note that

$$\inf_{0\neq\vartheta\in D(\mathcal{A}_3^*)}\frac{|\mathrm{A}_3^*\vartheta|_{\mathsf{H}_3}^2}{|\vartheta|_{D(\mathrm{A}_3^*)}^2}=\inf_{0\neq\vartheta\in D(\mathcal{A}_3^*)}\frac{|\mathrm{A}_3^*\vartheta|_{\mathsf{H}_3}^2}{|\vartheta|_{\mathsf{H}_4}^2+|\mathrm{A}_3^*\vartheta|_{\mathsf{H}_3}^2}=\Big(\sup_{0\neq\vartheta\in D(\mathcal{A}_3^*)}\frac{|\vartheta|_{\mathsf{H}_4}^2+|\mathrm{A}_3^*\vartheta|_{\mathsf{H}_3}^2}{|\mathrm{A}_3^*\vartheta|_{\mathsf{H}_3}^2}\Big)^{-1}=\frac{1}{c_3^2+1},$$
$$\inf_{0\neq\tau\in D(\mathcal{A}_0)}\frac{|\mathrm{A}_0\tau|_{\mathsf{H}_1}^2}{|\tau|_{D(\mathrm{A}_0)}^2}=\inf_{0\neq\tau\in D(\mathcal{A}_0)}\frac{|\mathrm{A}_0\tau|_{\mathsf{H}_1}^2}{|\tau|_{\mathsf{H}_0}^2+|\mathrm{A}_0\tau|_{\mathsf{H}_1}^2}=\Big(\sup_{0\neq\tau\in D(\mathcal{A}_0)}\frac{|\tau|_{\mathsf{H}_0}^2+|\mathrm{A}_0\tau|_{\mathsf{H}_1}^2}{|\mathrm{A}_0\tau|_{\mathsf{H}_1}^2}\Big)^{-1}=\frac{1}{c_0^2+1}$$

hold.

[v]We generally define $\tilde{D}_\ell:=\big\{\xi\in D_\ell\,:\,\mathrm{A}_\ell\xi\in D(\mathrm{A}_\ell^*)\big\}=D(\mathrm{A}_\ell)\cap D(\mathrm{A}_\ell^*\mathrm{A}_\ell)$ for $\ell=1,\dots,3$.

The partial solutions x_f and x_g can be found by the following two variational formulations: There exist unique potentials $\tilde{x}_f \in D(\mathcal{A}_2)$ and $z_g \in D(\mathcal{A}_1)$, such that

$$\forall\,\xi \in D(\mathcal{A}_2) \qquad \langle \mathrm{A}_2\,\tilde{x}_f, \mathrm{A}_2\,\xi\rangle_{\mathsf{H}_3} = \langle f, \xi\rangle_{\mathsf{H}_2}, \tag{3.15}$$

$$\forall\,\varphi \in D(\mathcal{A}_1) \qquad \langle \mathrm{A}_1\,z_g, \mathrm{A}_1\,\varphi\rangle_{\mathsf{H}_2} = \langle g, \varphi\rangle_{\mathsf{H}_1}. \tag{3.16}$$

Moreover, (3.15) *and* (3.16) *even hold for all $\xi \in D(\mathrm{A}_2)$ and for all $\varphi \in D(\mathrm{A}_1)$, respectively. Hence $\mathrm{A}_2\,\tilde{x}_f \in D(\mathrm{A}_2^*) \cap R(\mathrm{A}_2) = D(\mathcal{A}_2^*)$ with $\mathrm{A}_2^*\,\mathrm{A}_2\,\tilde{x}_f = f$ and $\mathrm{A}_1\,z_g \in D(\mathrm{A}_1^*) \cap R(\mathrm{A}_1) = D(\mathcal{A}_1^*)$ with $\mathrm{A}_1^*\,\mathrm{A}_1\,z_g = g$, yielding*

$$\tilde{x}_f = x_f, \qquad \mathrm{A}_1\,z_g = x_g.$$

Proof. The necessary conditions are clear. To show uniqueness, let $x \in \tilde{D}_2$ solve

$$\mathrm{A}_2^*\,\mathrm{A}_2\,x = 0, \qquad \mathrm{A}_1^*\,x = 0, \qquad \pi_2\,x = 0.$$

Hence $x \in N(\mathrm{A}_1^*) \cap K_2^{\perp_{\mathsf{H}_2}}$ and also $x \in N(\mathrm{A}_2)$ as $\mathrm{A}_2\,x \in D(\mathrm{A}_2^*)$ and

$$|\,\mathrm{A}_2\,x|_{\mathsf{H}_3}^2 = \langle x, \mathrm{A}_2^*\,\mathrm{A}_2\,x\rangle_{\mathsf{H}_2} = 0,$$

yielding $x \in K_2 \cap K_2^{\perp_{\mathsf{H}_2}} = \{0\}$. To prove existence, let $f \in R(\mathrm{A}_2^*)$, $g \in R(\mathrm{A}_1^*)$, $k \in K_2$ and define x, x_f, and x_g according to the theorem. Again the orthogonality follows directly by Lemma 2.7. Moreover, x_f, x_g, and k solve the linear systems

$$\begin{aligned} \mathrm{A}_2^*\,\mathrm{A}_2\,x_f &= f, & \mathrm{A}_2\,x_g &= 0, & \mathrm{A}_2\,k &= 0,\\ \mathrm{A}_1^*\,x_f &= 0, & \mathrm{A}_1^*\,x_g &= g, & \mathrm{A}_1^*\,k &= 0,\\ \pi_2\,x_f &= 0, & \pi_2\,x_g &= 0, & \pi_2\,k &= k. \end{aligned}$$

Thus x solves (3.14) and we have by Corollary 2.5 $|x_f|_{\mathsf{H}_2} \le c_2|\,\mathrm{A}_2\,x_f|_{\mathsf{H}_3} \le c_2^2|f|_{\mathsf{H}_2}$ and $|x_g|_{\mathsf{H}_2} \le c_1|g|_{\mathsf{H}_1}$, completing the solution theory. That the partial solutions can be obtained by the described variational formulations is clear resp. follows analogously to the proof of Theorem 3.3. □

Remark 3.7. *By orthogonality and with $\mathrm{A}_2\,x = (\mathcal{A}_2^*)^{-1}f$, $\mathrm{A}_2^*\,\mathrm{A}_2\,x = f$, and $\mathrm{A}_1^*\,x = g$ we even have*

$$\begin{aligned} |x|_{\mathsf{H}_2}^2 &= \big|x_f\big|_{\mathsf{H}_2}^2 + \big|x_g\big|_{\mathsf{H}_2}^2 + |k|_{\mathsf{H}_2}^2 \le c_2^4|f|_{\mathsf{H}_2}^2 + c_1^2|g|_{\mathsf{H}_1}^2 + |k|_{\mathsf{H}_2}^2,\\ |x|_{\tilde{D}_2}^2 &= \big|x_f\big|_{\mathsf{H}_2}^2 + \big|\,\mathrm{A}_2\,x\big|_{\mathsf{H}_3}^2 + |f|_{\mathsf{H}_2}^2 + \big|x_g\big|_{\mathsf{H}_2}^2 + |g|_{\mathsf{H}_1}^2 + |k|_{\mathsf{H}_2}^2\\ &\le (1 + c_2^2 + c_2^4)|f|_{\mathsf{H}_3}^2 + (1 + c_1^2)|g|_{\mathsf{H}_1}^2 + |k|_{\mathsf{H}_2}^2. \end{aligned}$$

Remark 3.8. *Since the second order system* (3.14) *decomposes into the two first order systems of shape* (1.5) *resp.* (3.1)*, i.e.,*

$$\begin{aligned} \mathrm{A}_2\,x &= y, & \mathrm{A}_3\,y &= 0,\\ \mathrm{A}_1^*\,x &= g, & \mathrm{A}_2^*\,y &= f,\\ \pi_2\,x &= k, & \pi_3\,y &= 0 \end{aligned}$$

for the pair $(x, y) \in D_2 \times D_3$ with $y := \mathrm{A}_2\,x \in D(\mathrm{A}_2^*) \cap R(\mathrm{A}_2) = D(\mathcal{A}_2^*)$, the solution theory follows directly by Theorem 3.3 as well. One just has to solve and set

$$\begin{aligned} y &:= (\mathcal{A}_2^*)^{-1}f \in D(\mathcal{A}_2^*) \subset R(\mathrm{A}_2),\\ x &:= (\mathcal{A}_2)^{-1}y + (\mathcal{A}_1^*)^{-1}g + k \in \big(D(\mathcal{A}_2) \oplus_{\mathsf{H}_2} D(\mathcal{A}_1^*) \oplus_{\mathsf{H}_2} K_2\big) \cap \tilde{D}_2 = \tilde{D}_2. \end{aligned}$$

Note that

$$\tilde{x}_f = x_f = (\mathcal{A}_2)^{-1}(\mathcal{A}_2^*)^{-1}f \in D(\mathcal{A}_2), \qquad z_g = (\mathcal{A}_1)^{-1}x_g = (\mathcal{A}_1)^{-1}(\mathcal{A}_1^*)^{-1}g \in D(\mathcal{A}_1)$$

holds with $\mathrm{A}_2^*\,\mathrm{A}_2\,x_f = f$ and $\mathrm{A}_1^*\,\mathrm{A}_1\,z_g = g$. Hence x_f, x_g, k, and z_g solve the first resp. second order systems

$$\mathrm{A}_2\,x_f = (\mathcal{A}_2^*)^{-1}f, \qquad \mathrm{A}_2\,x_g = 0, \qquad \mathrm{A}_2\,k = 0, \qquad \mathrm{A}_2^*\,\mathrm{A}_2\,x_f = f, \qquad \mathrm{A}_1^*\,\mathrm{A}_1\,z_g = g,$$

$$\begin{aligned} &\mathrm{A}_1^* x_f = 0, && \mathrm{A}_1^* x_g = g, && \mathrm{A}_1^* k = 0, && \mathrm{A}_1^* x_f = 0, && \mathrm{A}_0^* z_g = 0,\\ &\pi_2 x_f = 0, && \pi_2 x_g = 0, && \pi_2 k = k, && \pi_2 x_f = 0, && \pi_1 z_g = 0. \end{aligned}$$

As before we emphasize that the variational formulations (3.15)-(3.16) have again saddle point structure. Provided $f \in R(\mathrm{A}_2^*)$ and $g \in R(\mathrm{A}_1^*)$ the formulations (3.15)-(3.16) are equivalent to the following two problems: Find $x_f \in D(\mathcal{A}_2)$ and $z_g \in D(\mathcal{A}_1)$, such that

$$\forall\, \xi \in D(\mathrm{A}_2) \qquad \langle \mathrm{A}_2\, x_f, \mathrm{A}_2\, \xi\rangle_{\mathsf{H}_3} = \langle f, \xi\rangle_{\mathsf{H}_2}, \tag{3.17}$$

$$\forall\, \varphi \in D(\mathrm{A}_1) \qquad \langle \mathrm{A}_1\, z_g, \mathrm{A}_1\, \varphi\rangle_{\mathsf{H}_2} = \langle g, \varphi\rangle_{\mathsf{H}_1}. \tag{3.18}$$

Moreover, similar to the first order case, $x_f \in D(\mathcal{A}_2) = D(\mathrm{A}_2) \cap R(\mathrm{A}_2^*)$ if and only if $x_f \in D(\mathrm{A}_2)$ and $x_f \perp_{\mathsf{H}_2} N(\mathrm{A}_2)$ as well as $z_g \in D(\mathcal{A}_1) = D(\mathrm{A}_1) \cap R(\mathrm{A}_1^*)$ if and only if $z_g \in D(\mathrm{A}_1)$ and $z_g \perp_{\mathsf{H}_1} N(\mathrm{A}_1)$. Therefore, the variational formulations (3.17)-(3.18) are equivalent to the following two saddle point problems: Find $x_f \in D(\mathrm{A}_2)$ and $z_g \in D(\mathrm{A}_1)$, such that

$$\forall\, \xi \in D(\mathrm{A}_2) \quad \langle \mathrm{A}_2\, x_f, \mathrm{A}_2\, \xi\rangle_{\mathsf{H}_3} = \langle f, \xi\rangle_{\mathsf{H}_2} \quad \wedge \quad \forall\, \zeta \in N(\mathrm{A}_2) \quad \langle x_f, \zeta\rangle_{\mathsf{H}_2} = 0, \tag{3.19}$$

$$\forall\, \varphi \in D(\mathrm{A}_1) \quad \langle \mathrm{A}_1\, z_g, \mathrm{A}_1\, \varphi\rangle_{\mathsf{H}_2} = \langle g, \varphi\rangle_{\mathsf{H}_1} \quad \wedge \quad \forall\, \psi \in N(\mathrm{A}_1) \quad \langle z_g, \psi\rangle_{\mathsf{H}_1} = 0. \tag{3.20}$$

We emphasize that the considerations leading to (3.9)-(3.10) and (3.11)-(3.12) can be repeated here, giving similar saddle point formulations as well.

Remark 3.9. *Remark 3.5 holds word by word also for Theorem 3.6.*

4. Functional A Posteriori Error Estimates

Having establishes a solution theory including suitable variational formulations, we now turn to the so-called functional a posteriori error estimates. Note that General Assumption 3.1 is supposed to hold.

4.1. **First Order Systems.** Let $x \in D_2$ be the exact solution of (3.1) and $\tilde{x} \in \mathsf{H}_2$, which may be considered as a non-conforming approximation of x. Utilizing the notations from Theorem 3.3 we define and decompose the error

$$\begin{aligned} \mathsf{H}_2 \ni e := x - \tilde{x} = e_{\mathrm{A}_1} + e_{K_2} + e_{\mathrm{A}_2^*} &\in R(\mathrm{A}_1) \oplus_{\mathsf{H}_2} K_2 \oplus_{\mathsf{H}_2} R(\mathrm{A}_2^*),\\ e_{\mathrm{A}_1} := \pi_{\mathrm{A}_1} e = x_g - \pi_{\mathrm{A}_1} \tilde{x} &\in R(\mathrm{A}_1),\\ e_{\mathrm{A}_2^*} := \pi_{\mathrm{A}_2^*} e = x_f - \pi_{\mathrm{A}_2^*} \tilde{x} &\in R(\mathrm{A}_2^*),\\ e_{K_2} := \pi_2\, e = k - \pi_2\, \tilde{x} &\in K_2 \end{aligned} \tag{4.1}$$

using the Helmholtz type decompositions of Lemma 2.7. By orthogonality it holds

$$|e|_{\mathsf{H}_2}^2 = |e_{\mathrm{A}_1}|_{\mathsf{H}_2}^2 + |e_{K_2}|_{\mathsf{H}_2}^2 + |e_{\mathrm{A}_2^*}|_{\mathsf{H}_2}^2. \tag{4.2}$$

4.1.1. *Upper Bounds.* Testing (4.1) with $\mathrm{A}_1\, \varphi$ for $\varphi \in D(\mathcal{A}_1)$ we get for all $\zeta \in D(\mathrm{A}_1^*)$ by orthogonality and Corollary 2.5 (i)

$$\begin{aligned} \langle e_{\mathrm{A}_1}, \mathrm{A}_1\, \varphi\rangle_{\mathsf{H}_2} = \langle e, \mathrm{A}_1\, \varphi\rangle_{\mathsf{H}_2} &= \langle \mathrm{A}_1^*\, x, \varphi\rangle_{\mathsf{H}_1} - \langle \tilde{x} - \zeta + \zeta, \mathrm{A}_1\, \varphi\rangle_{\mathsf{H}_2}\\ &= \langle g - \mathrm{A}_1^*\, \zeta, \varphi\rangle_{\mathsf{H}_1} - \big\langle \pi_{\mathrm{A}_1}(\tilde{x} - \zeta), \mathrm{A}_1\, \varphi\big\rangle_{\mathsf{H}_2}\\ &\le |g - \mathrm{A}_1^*\, \zeta|_{\mathsf{H}_1} |\varphi|_{\mathsf{H}_1} + \big|\pi_{\mathrm{A}_1}(\tilde{x} - \zeta)\big|_{\mathsf{H}_2} |\mathrm{A}_1\, \varphi|_{\mathsf{H}_2}\\ &\le \Big(c_1 |g - \mathrm{A}_1^*\, \zeta|_{\mathsf{H}_1} + \big|\pi_{\mathrm{A}_1}(\tilde{x} - \zeta)\big|_{\mathsf{H}_2}\Big) |\mathrm{A}_1\, \varphi|_{\mathsf{H}_2}. \end{aligned} \tag{4.3}$$

As $e_{\mathrm{A}_1} \in R(\mathrm{A}_1) = R(\mathcal{A}_1)$, we have $e_{\mathrm{A}_1} = \mathrm{A}_1\, \varphi_e$ with $\varphi_e := (\mathcal{A}_1)^{-1} e_{\mathrm{A}_1} \in D(\mathcal{A}_1)$. Choosing $\varphi := \varphi_e$ in (4.3) we obtain

$$\forall\, \zeta \in D(\mathrm{A}_1^*) \qquad |e_{\mathrm{A}_1}|_{\mathsf{H}_2} \le c_1 |g - \mathrm{A}_1^*\, \zeta|_{\mathsf{H}_1} + \big|\pi_{\mathrm{A}_1}(\tilde{x} - \zeta)\big|_{\mathsf{H}_2} \le c_1 |g - \mathrm{A}_1^*\, \zeta|_{\mathsf{H}_1} + |\tilde{x} - \zeta|_{\mathsf{H}_2}. \tag{4.4}$$

Analogously, testing with $\mathrm{A}_2^*\phi$ for $\phi\in D(\mathcal{A}_2^*)$ we get for all $\xi\in D(\mathrm{A}_2)$ by orthogonality and Corollary 2.5 (i)

$$\begin{aligned}\langle e_{\mathrm{A}_2^*},\mathrm{A}_2^*\phi\rangle_{\mathsf{H}_2}&=\langle e,\mathrm{A}_2^*\phi\rangle_{\mathsf{H}_2}=\langle \mathrm{A}_2\,x,\phi\rangle_{\mathsf{H}_3}-\langle\tilde{x}-\xi+\xi,\mathrm{A}_2^*\phi\rangle_{\mathsf{H}_2}\\&=\langle f-\mathrm{A}_2\,\xi,\phi\rangle_{\mathsf{H}_3}-\big\langle\pi_{\mathrm{A}_2^*}(\tilde{x}-\xi),\mathrm{A}_2^*\phi\big\rangle_{\mathsf{H}_2}\\&\leq|f-\mathrm{A}_2\,\xi|_{\mathsf{H}_3}|\phi|_{\mathsf{H}_3}+\big|\pi_{\mathrm{A}_2^*}(\tilde{x}-\xi)\big|_{\mathsf{H}_2}|\,\mathrm{A}_2^*\phi|_{\mathsf{H}_2}\\&\leq\Big(c_2|f-\mathrm{A}_2\,\xi|_{\mathsf{H}_3}+\big|\pi_{\mathrm{A}_2^*}(\tilde{x}-\xi)\big|_{\mathsf{H}_2}\Big)|\,\mathrm{A}_2^*\phi|_{\mathsf{H}_2}.\end{aligned}\tag{4.5}$$

As $e_{\mathrm{A}_2^*}\in R(\mathrm{A}_2^*)=R(\mathcal{A}_2^*)$, we have $e_{\mathrm{A}_2^*}=\mathrm{A}_2^*\phi_e$ with $\phi_e:=(\mathcal{A}_2^*)^{-1}e_{\mathrm{A}_2^*}\in D(\mathcal{A}_2^*)$. Choosing $\phi:=\phi_e$ in (4.5) we obtain

$$\forall\,\xi\in D(\mathrm{A}_2)\qquad |e_{\mathrm{A}_2^*}|_{\mathsf{H}_2}\leq c_2|f-\mathrm{A}_2\,\xi|_{\mathsf{H}_3}+\big|\pi_{\mathrm{A}_2^*}(\tilde{x}-\xi)\big|_{\mathsf{H}_2}\leq c_2|f-\mathrm{A}_2\,\xi|_{\mathsf{H}_3}+|\tilde{x}-\xi|_{\mathsf{H}_2}.\tag{4.6}$$

Finally, for all $\varphi\in D(\mathrm{A}_1)$ and all $\phi\in D(\mathrm{A}_2^*)$ we get by orthogonality

$$|e_{K_2}|_{\mathsf{H}_2}^2=\langle e_{K_2},k-\pi_2\,\tilde{x}+\mathrm{A}_1\,\varphi+\mathrm{A}_2^*\phi\rangle_{\mathsf{H}_2}=\langle e_{K_2},k-\tilde{x}+\mathrm{A}_1\,\varphi+\mathrm{A}_2^*\phi\rangle_{\mathsf{H}_2}\tag{4.7}$$

and thus

$$\forall\,\varphi\in D(\mathrm{A}_1)\quad\forall\,\phi\in D(\mathrm{A}_2^*)\qquad |e_{K_2}|_{\mathsf{H}_2}\leq|k-\tilde{x}+\mathrm{A}_1\,\varphi+\mathrm{A}_2^*\phi|_{\mathsf{H}_2}.\tag{4.8}$$

Let us summarize:

Theorem 4.1. *Let $x\in D_2$ be the exact solution of* (3.1) *and $\tilde{x}\in\mathsf{H}_2$. Then the following estimates hold for the error $e=x-\tilde{x}$ defined in* (4.1)*:*

- **(i)** *The error decomposes according to* (4.1)-(4.2)*, i.e.,*
$$e=e_{\mathrm{A}_1}+e_{K_2}+e_{\mathrm{A}_2^*}\in R(\mathrm{A}_1)\oplus_{\mathsf{H}_2}K_2\oplus_{\mathsf{H}_2}R(\mathrm{A}_2^*),\qquad |e|_{\mathsf{H}_2}^2=|e_{\mathrm{A}_1}|_{\mathsf{H}_2}^2+|e_{K_2}|_{\mathsf{H}_2}^2+|e_{\mathrm{A}_2^*}|_{\mathsf{H}_2}^2.$$
- **(ii)** *The projection $e_{\mathrm{A}_1}=\pi_{\mathrm{A}_1}e=x_g-\pi_{\mathrm{A}_1}\tilde{x}\in R(\mathrm{A}_1)$ satisfies*
$$|e_{\mathrm{A}_1}|_{\mathsf{H}_2}=\min_{\zeta\in D(\mathrm{A}_1^*)}\big(c_1|\,\mathrm{A}_1^*\zeta-g|_{\mathsf{H}_1}+|\zeta-\tilde{x}|_{\mathsf{H}_2}\big)$$
and the minimum is attained at
$$\hat{\zeta}:=e_{\mathrm{A}_1}+\tilde{x}=\pi_{\mathrm{A}_1}e+\tilde{x}=-(1-\pi_{\mathrm{A}_1})e+x=-\pi_{N(\mathrm{A}_1^*)}e+x\in D(\mathrm{A}_1^*)$$
since $\mathrm{A}_1^*\hat{\zeta}=\mathrm{A}_1^*x=g$.
- **(iii)** *The projection $e_{\mathrm{A}_2^*}=\pi_{\mathrm{A}_2^*}e=x_f-\pi_{\mathrm{A}_2^*}\tilde{x}\in R(\mathrm{A}_2^*)$ satisfies*
$$|e_{\mathrm{A}_2^*}|_{\mathsf{H}_2}=\min_{\xi\in D(\mathrm{A}_2)}\big(c_2|\,\mathrm{A}_2\,\xi-f|_{\mathsf{H}_3}+|\xi-\tilde{x}|_{\mathsf{H}_2}\big)$$
and the minimum is attained at
$$\hat{\xi}:=e_{\mathrm{A}_2^*}+\tilde{x}=\pi_{\mathrm{A}_2^*}e+\tilde{x}=-(1-\pi_{\mathrm{A}_2^*})e+x=-\pi_{N(\mathrm{A}_2)}e+x\in D(\mathrm{A}_2)$$
since $\mathrm{A}_2\,\hat{\xi}=\mathrm{A}_2\,x=f$.
- **(iv)** *The projection $e_{K_2}=\pi_2\,e=k-\pi_2\,\tilde{x}\in K_2$ satisfies*
$$|e_{K_2}|_{\mathsf{H}_2}=\min_{\varphi\in D(\mathrm{A}_1)}\min_{\phi\in D(\mathrm{A}_2^*)}|k-\tilde{x}+\mathrm{A}_1\,\varphi+\mathrm{A}_2^*\phi|_{\mathsf{H}_2}$$
and the minimum is attained at
$$\hat{\varphi}:=(\mathcal{A}_1)^{-1}\pi_{\mathrm{A}_1}\tilde{x}\in D(\mathcal{A}_1),\qquad\hat{\phi}:=(\mathcal{A}_2^*)^{-1}\pi_{\mathrm{A}_2^*}\tilde{x}\in D(\mathcal{A}_2^*)$$
since $\mathrm{A}_1\,\hat{\varphi}+\mathrm{A}_2^*\hat{\phi}=(\pi_{\mathrm{A}_1}+\pi_{\mathrm{A}_2^*})\tilde{x}=(1-\pi_2)\tilde{x}$.

For conforming approximations we get:

Corollary 4.2. *Let the assumptions of Theorem 4.1 be satisfied.*

(i) *If* $\tilde{x} \in D(\mathrm{A}_1^*)$, *then* $e \in D(\mathrm{A}_1^*)$ *and hence* $e_{\mathrm{A}_1} = \pi_{\mathrm{A}_1} e \in D(\mathcal{A}_1^*)$ *with* $\mathrm{A}_1^* e_{\mathrm{A}_1} = \mathrm{A}_1^* e$ *and*

$$|e_{\mathrm{A}_1}|_{\mathsf{H}_2} \leq c_1 |\,\mathrm{A}_1^* \tilde{x} - g|_{\mathsf{H}_1} = c_1 |\,\mathrm{A}_1^* e|_{\mathsf{H}_1}$$

by setting $\zeta := \tilde{x}$, *which also follows directly by the Friedrichs/Poincaré type estimate.*

(ii) *If* $\tilde{x} \in D(\mathrm{A}_2)$, *then* $e \in D(\mathrm{A}_2)$ *and hence* $e_{\mathrm{A}_2^*} = \pi_{\mathrm{A}_2^*} e \in D(\mathcal{A}_2)$ *with* $\mathrm{A}_2 e_{\mathrm{A}_2^*} = \mathrm{A}_2 e$ *and*

$$|e_{\mathrm{A}_2^*}|_{\mathsf{H}_2} \leq c_2 |\,\mathrm{A}_2 \tilde{x} - f|_{\mathsf{H}_3} = c_2 |\,\mathrm{A}_2 e|_{\mathsf{H}_3}$$

by setting $\xi := \tilde{x}$, *which also follows directly by the Friedrichs/Poincaré type estimate.*

(iii) *If* $\tilde{x} \in D_2$, *then* $e \in D_2$ *and*

$$\begin{aligned} |e|_{D_2}^2 &= |e_{\mathrm{A}_1}|_{\mathsf{H}_2}^2 + |e_{K_2}|_{\mathsf{H}_2}^2 + |e_{\mathrm{A}_2^*}|_{\mathsf{H}_2}^2 + |\,\mathrm{A}_2 e|_{\mathsf{H}_3}^2 + |\,\mathrm{A}_1^* e|_{\mathsf{H}_1}^2 \\ &\leq |e_{K_2}|_{\mathsf{H}_2}^2 + (1 + c_2^2)|\,\mathrm{A}_2 e|_{\mathsf{H}_3}^2 + (1 + c_1^2)|\,\mathrm{A}_1^* e|_{\mathsf{H}_1}^2 \end{aligned}$$

with

$$e_{K_2} = k - \pi_2 \tilde{x}, \qquad \mathrm{A}_2 e = f - \mathrm{A}_2 \tilde{x}, \qquad \mathrm{A}_1^* e = g - \mathrm{A}_1^* \tilde{x},$$

which again also follows immediately by the Friedrichs/Poincaré type estimates.

Remark 4.3. *Corollary 4.2 (iii) shows, that for very conforming* $\tilde{x} \in D_2$ *the weighted least squares functional*

$$\mathcal{F}(\tilde{x}) := |k - \pi_2 \tilde{x}|_{\mathsf{H}_2}^2 + (1 + c_2^2)|\,\mathrm{A}_2 \tilde{x} - f|_{\mathsf{H}_3}^2 + (1 + c_1^2)|\,\mathrm{A}_1^* \tilde{x} - g|_{\mathsf{H}_1}^2$$

is equivalent to the conforming error, i.e.,

$$|e|_{D_2}^2 \leq \mathcal{F}(\tilde{x}) \leq (1 + \max\{c_1, c_2\}^2)|e|_{D_2}^2.$$

Recalling the variational resp. saddle point formulations (3.5)-(3.6) resp. (3.7)-(3.8) and that the partial solutions are given by

$$x_f = \mathrm{A}_2^* y_f \in D(\mathcal{A}_2), \qquad x_g = \mathrm{A}_1 z_g \in D(\mathcal{A}_1^*),$$

a possible numerical method, using these variational formulations in some finite dimensional subspaces to find $\tilde{y}_f \in D(\mathrm{A}_2^*)$ and $\tilde{z}_g \in D(\mathrm{A}_1)$, such as the finite element method, will always ensure

$$\tilde{x}_f := \mathrm{A}_2^* \tilde{y}_f \in R(\mathrm{A}_2^*) = N(\mathrm{A}_2)^{\perp_{\mathsf{H}_2}} \subset N(\mathrm{A}_1^*), \quad \tilde{x}_g := \mathrm{A}_1 \tilde{z}_g \in R(\mathrm{A}_1) = N(\mathrm{A}_1^*)^{\perp_{\mathsf{H}_2}} \subset N(\mathrm{A}_2)$$

and thus

$$\tilde{x}_\perp := \tilde{x}_f + \tilde{x}_g \in R(\mathrm{A}_2^*) \oplus_{\mathsf{H}_2} R(\mathrm{A}_1) = K_2^{\perp_{\mathsf{H}_2}},$$

but maybe not $\tilde{x}_f \in D(\mathrm{A}_2)$ or $\tilde{x}_g \in D(\mathrm{A}_1^*)$. Therefore, a reasonable assumption for our non-conforming approximations is

$$\tilde{x} = \tilde{x}_\perp + k, \qquad \tilde{x}_\perp \in K_2^{\perp_{\mathsf{H}_2}},$$

with $e_{K_2} = \pi_2 e = \pi_2(x - \tilde{x}) = -\pi_2 \tilde{x}_\perp = 0$.

Corollary 4.4. *Let* $x \in D_2$ *be the exact solution of* (3.1) *and* $\tilde{x} := k + \tilde{x}_\perp$ *with some* $\tilde{x}_\perp \in K_2^{\perp_{\mathsf{H}_2}}$. *Then for the error* e *defined in* (4.1) *it holds:*

(i) *According to* (4.1)-(4.2) *the error decomposes, i.e.,*

$$e = x - \tilde{x} = x_f + x_g - \tilde{x}_\perp = e_{\mathrm{A}_1} + e_{\mathrm{A}_2^*} \in R(\mathrm{A}_1) \oplus_{\mathsf{H}_2} R(\mathrm{A}_2^*) = K_2^{\perp_{\mathsf{H}_2}}, \qquad e_{K_2} = 0,$$

and $|e|_{\mathsf{H}_2}^2 = |e_{\mathrm{A}_1}|_{\mathsf{H}_2}^2 + |e_{\mathrm{A}_2^*}|_{\mathsf{H}_2}^2$. *Hence there is no error in the "kernel" part.*

(ii) *The projection* $e_{\mathrm{A}_1} = \pi_{\mathrm{A}_1} e = x_g - \pi_{\mathrm{A}_1} \tilde{x} = x_g - \pi_{\mathrm{A}_1} \tilde{x}_\perp \in R(\mathrm{A}_1)$ *satisfies*

$$\begin{aligned} |e_{\mathrm{A}_1}|_{\mathsf{H}_2} &= \min_{\zeta \in D(\mathrm{A}_1^*)} \big(c_1 |\,\mathrm{A}_1^* \zeta - g|_{\mathsf{H}_1} + |\zeta - \tilde{x}|_{\mathsf{H}_2}\big) \\ &= \min_{\zeta \in D(\mathrm{A}_1^*)} \big(c_1 |\,\mathrm{A}_1^* \zeta - g|_{\mathsf{H}_1} + |\zeta - \tilde{x}_\perp|_{\mathsf{H}_2}\big) \end{aligned}$$

(exchanging ζ *by* $\zeta + k$*) and the minima are attained at*

$$\begin{aligned} \hat{\zeta} &:= e_{\mathrm{A}_1} + \tilde{x} = \pi_{\mathrm{A}_1} e + \tilde{x} = -(1 - \pi_{\mathrm{A}_1}) e + x = -\pi_{N(\mathrm{A}_1^*)} e + x \in D(\mathrm{A}_1^*), \\ \hat{\zeta}_\perp &:= e_{\mathrm{A}_1} + \tilde{x}_\perp = \pi_{\mathrm{A}_1} e + \tilde{x}_\perp = -(1 - \pi_{\mathrm{A}_1}) e + x - k = -\pi_{N(\mathrm{A}_1^*)} e + x - k \in D(\mathrm{A}_1^*) \end{aligned}$$

since $\mathrm{A}_1^*\,\hat{\zeta}_\perp = \mathrm{A}_1^*\,\hat{\zeta} = \mathrm{A}_1^*\,x = g$.

(iii) *The projection* $e_{\mathrm{A}_2^*} = \pi_{\mathrm{A}_2^*}e = x_f - \pi_{\mathrm{A}_2^*}\tilde{x} = x_f - \pi_{\mathrm{A}_2^*}\tilde{x}_\perp \in R(\mathrm{A}_2^*)$ *satisfies*

$$|e_{\mathrm{A}_2^*}|_{\mathsf{H}_2} = \min_{\xi \in D(\mathrm{A}_2)} \big(c_2|\,\mathrm{A}_2\,\xi - f|_{\mathsf{H}_3} + |\xi - \tilde{x}|_{\mathsf{H}_2}\big)$$
$$= \min_{\xi \in D(\mathrm{A}_2)} \big(c_2|\,\mathrm{A}_2\,\xi - f|_{\mathsf{H}_3} + |\xi - \tilde{x}_\perp|_{\mathsf{H}_2}\big)$$

(exchanging ξ *by* $\xi + k$*) and the minima are attained at*

$$\hat{\xi} := e_{\mathrm{A}_2^*} + \tilde{x} = \pi_{\mathrm{A}_2^*}e + \tilde{x} = -(1 - \pi_{\mathrm{A}_2^*})e + x = -\pi_{N(\mathrm{A}_2)}e + x \in D(\mathrm{A}_2),$$
$$\hat{\xi}_\perp := e_{\mathrm{A}_2^*} + \tilde{x}_\perp = \pi_{\mathrm{A}_2^*}e + \tilde{x}_\perp = -(1 - \pi_{\mathrm{A}_2^*})e + x - k = -\pi_{N(\mathrm{A}_2)}e + x - k \in D(\mathrm{A}_2)$$

since $\mathrm{A}_2\,\hat{\xi}_\perp = \mathrm{A}_2\,\hat{\xi} = \mathrm{A}_2\,x = f$.

4.1.2. *Lower Bounds.* In any Hilbert space H we have

$$\forall\,\hat{h} \in \mathsf{H} \qquad |\hat{h}|_{\mathsf{H}}^2 = \max_{h \in \mathsf{H}} \big(2\langle \hat{h}, h\rangle_{\mathsf{H}} - |h|_{\mathsf{H}}^2\big) \tag{4.9}$$

and the maximum is attained at $\hat{h}$. We recall (4.1) and (4.2), especially

$$|e|_{\mathsf{H}_2}^2 = |e_{\mathrm{A}_1}|_{\mathsf{H}_2}^2 + |e_{K_2}|_{\mathsf{H}_2}^2 + |e_{\mathrm{A}_2^*}|_{\mathsf{H}_2}^2.$$

Using (4.9) for $\mathsf{H} = R(\mathrm{A}_1)$ and orthogonality we get

$$|e_{\mathrm{A}_1}|_{\mathsf{H}_2}^2 = \max_{\varphi \in D(\mathrm{A}_1)} \big(2\langle e_{\mathrm{A}_1}, \mathrm{A}_1\,\varphi\rangle_{\mathsf{H}_2} - |\,\mathrm{A}_1\,\varphi|_{\mathsf{H}_2}^2\big)$$
$$= \max_{\varphi \in D(\mathrm{A}_1)} \big(2\langle e, \mathrm{A}_1\,\varphi\rangle_{\mathsf{H}_2} - |\,\mathrm{A}_1\,\varphi|_{\mathsf{H}_2}^2\big)$$
$$= \max_{\varphi \in D(\mathrm{A}_1)} \big(2\langle g, \varphi\rangle_{\mathsf{H}_1} - \langle 2\tilde{x} + \mathrm{A}_1\,\varphi, \mathrm{A}_1\,\varphi\rangle_{\mathsf{H}_2}\big)$$

and the maximum is attained at $\hat{\varphi} \in D(\mathrm{A}_1)$ with $\mathrm{A}_1\,\hat{\varphi} = e_{\mathrm{A}_1}$. Analogously for $\mathsf{H} = R(\mathrm{A}_2^*)$

$$|e_{\mathrm{A}_2^*}|_{\mathsf{H}_2}^2 = \max_{\phi \in D(\mathrm{A}_2^*)} \big(2\langle f, \phi\rangle_{\mathsf{H}_3} - \langle 2\tilde{x} + \mathrm{A}_2^*\,\phi, \mathrm{A}_2^*\,\phi\rangle_{\mathsf{H}_2}\big)$$

and the maximum is attained at $\hat{\phi} \in D(\mathrm{A}_2^*)$ with $\mathrm{A}_2^*\,\hat{\phi} = e_{\mathrm{A}_2^*}$. Finally for $\mathsf{H} = K_2$ and by orthogonality

$$|e_{K_2}|_{\mathsf{H}_2}^2 = \max_{\theta \in K_2} \big(2\langle e_{K_2}, \theta\rangle_{\mathsf{H}_2} - |\theta|_{\mathsf{H}_2}^2\big) = \max_{\theta \in K_2} \big\langle 2(k - \tilde{x}) - \theta, \theta\big\rangle_{\mathsf{H}_2}$$

and the maximum is attained at $\hat{\theta} = e_{K_2}$.

Theorem 4.5. *Let* $x \in D_2$ *be the exact solution of* (3.1) *and* $\tilde{x} \in \mathsf{H}_2$. *Then the following estimates hold for the error* $e = x - \tilde{x}$ *defined in* (4.1)*:*

(i) *The error decomposes according to* (4.1)-(4.2)*, i.e.,*

$$e = e_{\mathrm{A}_1} + e_{K_2} + e_{\mathrm{A}_2^*} \in R(\mathrm{A}_1) \oplus_{\mathsf{H}_2} K_2 \oplus_{\mathsf{H}_2} R(\mathrm{A}_2^*), \qquad |e|_{\mathsf{H}_2}^2 = |e_{\mathrm{A}_1}|_{\mathsf{H}_2}^2 + |e_{K_2}|_{\mathsf{H}_2}^2 + |e_{\mathrm{A}_2^*}|_{\mathsf{H}_2}^2.$$

(ii) *The projection* $e_{\mathrm{A}_1} = \pi_{\mathrm{A}_1}e = x_g - \pi_{\mathrm{A}_1}\tilde{x} \in R(\mathrm{A}_1)$ *satisfies*

$$|e_{\mathrm{A}_1}|_{\mathsf{H}_2}^2 = \max_{\varphi \in D(\mathrm{A}_1)} \big(2\langle g, \varphi\rangle_{\mathsf{H}_1} - \langle 2\tilde{x} + \mathrm{A}_1\,\varphi, \mathrm{A}_1\,\varphi\rangle_{\mathsf{H}_2}\big)$$

and the maximum is attained at, e.g., $\hat{\varphi} := (\mathcal{A}_1)^{-1}e_{\mathrm{A}_1} \in D(\mathcal{A}_1)$.

(iii) *The projection* $e_{\mathrm{A}_2^*} = \pi_{\mathrm{A}_2^*}e = x_f - \pi_{\mathrm{A}_2^*}\tilde{x} \in R(\mathrm{A}_2^*)$ *satisfies*

$$|e_{\mathrm{A}_2^*}|_{\mathsf{H}_2}^2 = \max_{\phi \in D(\mathrm{A}_2^*)} \big(2\langle f, \phi\rangle_{\mathsf{H}_3} - \langle 2\tilde{x} + \mathrm{A}_2^*\,\phi, \mathrm{A}_2^*\,\phi\rangle_{\mathsf{H}_2}\big)$$

and the maximum is attained at, e.g., $\hat{\phi} := (\mathcal{A}_2^*)^{-1}e_{\mathrm{A}_2^*} \in D(\mathcal{A}_2^*)$.

(iv) *The projection* $e_{K_2} = \pi_2 e = k - \pi_2 \tilde{x} \in K_2$ *satisfies*

$$|e_{K_2}|^2_{\mathsf{H}_2} = \max_{\theta \in K_2} \big\langle 2(k - \tilde{x}) - \theta, \theta \big\rangle_{\mathsf{H}_2}$$

and the maximum is attained at $\hat{\theta} := e_{K_2} \in K_2$.

If $\tilde{x} := k + \tilde{x}_\perp$ *with some* $\tilde{x}_\perp \in K_2^{\perp_{\mathsf{H}_2}}$, *see Corollary 4.4, then* $e_{K_2} = 0$, *and in (ii) and (iii)* $\tilde{x}$ *can be replaced by* $\tilde{x}_\perp$ *as* $k \perp_{\mathsf{H}_2} R(\mathrm{A}_1) \oplus_{\mathsf{H}_2} R(\mathrm{A}_2^*)$.

4.1.3. *Two-Sided Bounds.* We summarize our results from the latter sections.

Corollary 4.6. *Let* $x \in D_2$ *be the exact solution of* (3.1) *and* $\tilde{x} \in \mathsf{H}_2$. *Then the following estimates hold for the error* $e = x - \tilde{x}$ *defined in* (4.1)*:*

(i) *The error decomposes according to* (4.1)-(4.2)*, i.e.,*

$$e = e_{\mathrm{A}_1} + e_{K_2} + e_{\mathrm{A}_2^*} \in R(\mathrm{A}_1) \oplus_{\mathsf{H}_2} K_2 \oplus_{\mathsf{H}_2} R(\mathrm{A}_2^*), \qquad |e|^2_{\mathsf{H}_2} = |e_{\mathrm{A}_1}|^2_{\mathsf{H}_2} + |e_{K_2}|^2_{\mathsf{H}_2} + |e_{\mathrm{A}_2^*}|^2_{\mathsf{H}_2}.$$

(ii) *The projection* $e_{\mathrm{A}_1} = \pi_{\mathrm{A}_1} e = x_g - \pi_{\mathrm{A}_1} \tilde{x} \in R(\mathrm{A}_1)$ *satisfies*

$$\begin{aligned}|e_{\mathrm{A}_1}|^2_{\mathsf{H}_2} &= \min_{\zeta \in D(\mathrm{A}_1^*)} \big(c_1 |\,\mathrm{A}_1^* \zeta - g|_{\mathsf{H}_1} + |\zeta - \tilde{x}|_{\mathsf{H}_2}\big)^2 \\ &= \max_{\varphi \in D(\mathrm{A}_1)} \big(2\langle g, \varphi\rangle_{\mathsf{H}_1} - \langle 2\tilde{x} + \mathrm{A}_1 \varphi, \mathrm{A}_1 \varphi\rangle_{\mathsf{H}_2}\big)\end{aligned}$$

and the minimum resp. maximum is attained at

$$\hat{\zeta} := e_{\mathrm{A}_1} + \tilde{x} \in D(\mathrm{A}_1^*), \qquad \hat{\varphi} := (\mathcal{A}_1)^{-1} e_{\mathrm{A}_1} \in D(\mathcal{A}_1)$$

with $\mathrm{A}_1^* \hat{\zeta} = \mathrm{A}_1^* x = g$.

(iii) *The projection* $e_{\mathrm{A}_2^*} = \pi_{\mathrm{A}_2^*} e = x_f - \pi_{\mathrm{A}_2^*} \tilde{x} \in R(\mathrm{A}_2^*)$ *satisfies*

$$\begin{aligned}|e_{\mathrm{A}_2^*}|^2_{\mathsf{H}_2} &= \min_{\xi \in D(\mathrm{A}_2)} \big(c_2 |\,\mathrm{A}_2 \xi - f|_{\mathsf{H}_3} + |\xi - \tilde{x}|_{\mathsf{H}_2}\big)^2 \\ &= \max_{\phi \in D(\mathrm{A}_2^*)} \big(2\langle f, \phi\rangle_{\mathsf{H}_3} - \langle 2\tilde{x} + \mathrm{A}_2^* \phi, \mathrm{A}_2^* \phi\rangle_{\mathsf{H}_2}\big)\end{aligned}$$

and the minimum resp. maximum is attained at

$$\hat{\xi} := e_{\mathrm{A}_2^*} + \tilde{x} \in D(\mathrm{A}_2), \qquad \hat{\phi} := (\mathcal{A}_2^*)^{-1} e_{\mathrm{A}_2^*} \in D(\mathcal{A}_2^*)$$

with $\mathrm{A}_2 \hat{\xi} = \mathrm{A}_2 x = f$.

(iv) *The projection* $e_{K_2} = \pi_2 e = k - \pi_2 \tilde{x} \in K_2$ *satisfies*

$$\begin{aligned}|e_{K_2}|^2_{\mathsf{H}_2} &= \min_{\varphi \in D(\mathrm{A}_1)} \min_{\phi \in D(\mathrm{A}_2^*)} |k - \tilde{x} + \mathrm{A}_1 \varphi + \mathrm{A}_2^* \phi|^2_{\mathsf{H}_2} \\ &= \max_{\theta \in K_2} \big\langle 2(k - \tilde{x}) - \theta, \theta \big\rangle_{\mathsf{H}_2}\end{aligned}$$

and the minimum resp. maximum is attained at

$$\hat{\varphi} := (\mathcal{A}_1)^{-1} \pi_{\mathrm{A}_1} \tilde{x} \in D(\mathcal{A}_1), \qquad \hat{\phi} := (\mathcal{A}_2^*)^{-1} \pi_{\mathrm{A}_2^*} \tilde{x} \in D(\mathcal{A}_2^*), \qquad \hat{\theta} := e_{K_2} \in K_2$$

with $\mathrm{A}_1 \hat{\varphi} + \mathrm{A}_2^* \hat{\phi} = (\pi_{\mathrm{A}_1} + \pi_{\mathrm{A}_2^*}) \tilde{x} = (1 - \pi_2) \tilde{x}$.

If $\tilde{x} := k + \tilde{x}_\perp$ *with some* $\tilde{x}_\perp \in K_2^{\perp_{\mathsf{H}_2}}$, *see Corollary 4.4, then* $e_{K_2} = 0$, *and in (ii) and (iii)* $\tilde{x}$ *can be replaced by* $\tilde{x}_\perp$. *In this case, for the attaining minima it holds*

$$\hat{\zeta}_\perp := e_{\mathrm{A}_1} + \tilde{x}_\perp \in D(\mathrm{A}_1^*), \qquad \hat{\xi}_\perp := e_{\mathrm{A}_2^*} + \tilde{x}_\perp \in D(\mathrm{A}_2).$$

4.2. **Second Order Systems.** Let $x \in \tilde{D}_2$ be the exact solution of (3.14). Recalling Remark 3.8 we introduce the additional quantity $y := \mathrm{A}_2\, x \in D(\mathcal{A}_2^*)$. Then (3.14) decomposes into two first order systems of shape (1.5) resp. (3.1), i.e.,

$$\begin{aligned} \mathrm{A}_2\, x &= y, & \mathrm{A}_3\, y &= 0,\\ \mathrm{A}_1^*\, x &= g, & \mathrm{A}_2^*\, y &= f,\\ \pi_2\, x &= k, & \pi_3\, y &= 0 \end{aligned}$$

for the pair $(x, y) \in D_2 \times D_3$. Hence, we can immediately apply our results for the first order systems. Let $\tilde{x} \in \mathsf{H}_2$ and $\tilde{y} \in \mathsf{H}_3$, which may be considered as non-conforming approximations of x and y, respectively. Utilizing the notations from Theorem 3.6 we define and decompose the errors

$$\begin{aligned} \mathsf{H}_2 \ni e &:= x - \tilde{x} = e_{\mathrm{A}_1} + e_{K_2} + e_{\mathrm{A}_2^*} \in R(\mathrm{A}_1) \oplus_{\mathsf{H}_2} K_2 \oplus_{\mathsf{H}_2} R(\mathrm{A}_2^*),\\ \mathsf{H}_3 \ni h &:= y - \tilde{y} = h_{\mathrm{A}_2} + h_{K_3} + h_{\mathrm{A}_3^*} \in R(\mathrm{A}_2) \oplus_{\mathsf{H}_3} K_3 \oplus_{\mathsf{H}_3} R(\mathrm{A}_3^*), \end{aligned} \tag{4.10}$$

$$\begin{aligned} e_{\mathrm{A}_1} &:= \pi_{\mathrm{A}_1} e = x_g - \pi_{\mathrm{A}_1}\tilde{x} \in R(\mathrm{A}_1), & h_{\mathrm{A}_2} &:= \pi_{\mathrm{A}_2} h = y - \pi_{\mathrm{A}_2}\tilde{y} \in R(\mathrm{A}_2),\\ e_{\mathrm{A}_2^*} &:= \pi_{\mathrm{A}_2^*} e = x_y - \pi_{\mathrm{A}_2^*}\tilde{x} \in R(\mathrm{A}_2^*), & h_{\mathrm{A}_3^*} &:= \pi_{\mathrm{A}_3^*} h = -\pi_{\mathrm{A}_3^*}\tilde{y} \in R(\mathrm{A}_3^*),\\ e_{K_2} &:= \pi_2\, e = k - \pi_2\, \tilde{x} \in K_2, & h_{K_3} &:= \pi_3\, e = -\,\pi_3\, \tilde{y} \in K_3 \end{aligned}$$

using the Helmholtz type decompositions of Lemma 2.7 and noting $\pi_{\mathrm{A}_2} y = y$ as $y \in R(\mathrm{A}_2)$. By orthogonality it holds

$$|e|_{\mathsf{H}_2}^2 = |e_{\mathrm{A}_1}|_{\mathsf{H}_2}^2 + |e_{K_2}|_{\mathsf{H}_2}^2 + |e_{\mathrm{A}_2^*}|_{\mathsf{H}_2}^2, \qquad |h|_{\mathsf{H}_3}^2 = |h_{\mathrm{A}_2}|_{\mathsf{H}_3}^2 + |h_{K_3}|_{\mathsf{H}_3}^2 + |h_{\mathrm{A}_3^*}|_{\mathsf{H}_3}^2. \tag{4.11}$$

Therefore, the results of the latter section can be applied to e_{A_1}, e_{K_2}, $e_{\mathrm{A}_2^*}$, h_{A_2}, h_{K_3}, $h_{\mathrm{A}_3^*}$. Especially, by Corollary 4.6 we obtain

$$|e_{\mathrm{A}_1}|_{\mathsf{H}_2}^2 = \min_{\zeta \in D(\mathrm{A}_1^*)} \big(c_1 |\,\mathrm{A}_1^*\, \zeta - g|_{\mathsf{H}_1} + |\zeta - \tilde{x}|_{\mathsf{H}_2}\big)^2 = \max_{\varphi \in D(\mathrm{A}_1)} \big(2\langle g, \varphi\rangle_{\mathsf{H}_1} - \langle 2\tilde{x} + \mathrm{A}_1\, \varphi, \mathrm{A}_1\, \varphi\rangle_{\mathsf{H}_2}\big) \tag{4.12}$$

and the minimum resp. maximum is attained at $\hat{\zeta} = e_{\mathrm{A}_1} + \tilde{x} \in D(\mathrm{A}_1^*)$ and $\hat{\varphi} = (\mathcal{A}_1)^{-1} e_{\mathrm{A}_1} \in D(\mathcal{A}_1)$ with $\mathrm{A}_1^*\, \hat{\zeta} = \mathrm{A}_1^*\, x = g$,

$$|e_{\mathrm{A}_2^*}|_{\mathsf{H}_2}^2 = \min_{\xi \in D(\mathrm{A}_2)} \big(c_2 |\,\mathrm{A}_2\, \xi - y|_{\mathsf{H}_3} + |\xi - \tilde{x}|_{\mathsf{H}_2}\big)^2 = \max_{\phi \in D(\mathrm{A}_2^*)} \big(2\langle y, \phi\rangle_{\mathsf{H}_3} - \langle 2\tilde{x} + \mathrm{A}_2^*\, \phi, \mathrm{A}_2^*\, \phi\rangle_{\mathsf{H}_2}\big) \tag{4.13}$$

and the minimum resp. maximum is attained at $\hat{\xi} = e_{\mathrm{A}_2^*} + \tilde{x} \in D(\mathrm{A}_2)$ and $\hat{\phi} = (\mathcal{A}_2^*)^{-1} e_{\mathrm{A}_2^*} \in D(\mathcal{A}_2^*)$ with $\mathrm{A}_2\, \hat{\xi} = \mathrm{A}_2\, x = y$,

$$|e_{K_2}|_{\mathsf{H}_2}^2 = \min_{\varphi \in D(\mathrm{A}_1)} \min_{\phi \in D(\mathrm{A}_2^*)} |k - \tilde{x} + \mathrm{A}_1\, \varphi + \mathrm{A}_2^*\, \phi|_{\mathsf{H}_2}^2 = \max_{\theta \in K_2} \big\langle 2(k - \tilde{x}) - \theta, \theta\big\rangle_{\mathsf{H}_2} \tag{4.14}$$

and the minimum resp. maximum is attained at $\hat{\varphi} = (\mathcal{A}_1)^{-1}\pi_{\mathrm{A}_1}\tilde{x} \in D(\mathcal{A}_1)$, $\hat{\phi} = (\mathcal{A}_2^*)^{-1}\pi_{\mathrm{A}_2^*}\tilde{x} \in D(\mathcal{A}_2^*)$, and $\hat{\theta} = e_{K_2} \in K_2$ with $\mathrm{A}_1\, \hat{\varphi} + \mathrm{A}_2^*\, \hat{\phi} = (\pi_{\mathrm{A}_1} + \pi_{\mathrm{A}_2^*})\tilde{x} = (1 - \pi_2)\tilde{x}$. If $\tilde{x} = k + \tilde{x}_\perp$ with some $\tilde{x}_\perp \in K_2^{\perp_{\mathsf{H}_2}}$, then $e_{K_2} = 0$, and $\tilde{x}$ can be replaced by $\tilde{x}_\perp$. If the General Assumption 3.1 holds also for A_3, i.e., $R(\mathrm{A}_3)$ is closed and (not neccessarily) K_3 is finite dimensional, we get the corresponding results for h_{A_2}, h_{K_3}, $h_{\mathrm{A}_3^*}$ as well. Replacing A_1 by A_2 and A_2 by A_3, Corollary 4.6 yields

$$|h_{\mathrm{A}_2}|_{\mathsf{H}_3}^2 = \min_{\zeta \in D(\mathrm{A}_2^*)} \big(c_2 |\,\mathrm{A}_2^*\, \zeta - f|_{\mathsf{H}_2} + |\zeta - \tilde{y}|_{\mathsf{H}_3}\big)^2 = \max_{\varphi \in D(\mathrm{A}_2)} \big(2\langle f, \varphi\rangle_{\mathsf{H}_2} - \langle 2\tilde{y} + \mathrm{A}_2\, \varphi, \mathrm{A}_2\, \varphi\rangle_{\mathsf{H}_3}\big) \tag{4.15}$$

and the minimum resp. maximum is attained at $\hat{\zeta} = h_{\mathrm{A}_2} + \tilde{y} \in D(\mathrm{A}_2^*)$ and $\hat{\varphi} = (\mathcal{A}_2)^{-1} h_{\mathrm{A}_2} \in D(\mathcal{A}_2)$ with $\mathrm{A}_2^*\, \hat{\zeta} = \mathrm{A}_2^*\, y = f$,

$$|h_{\mathrm{A}_3^*}|_{\mathsf{H}_3}^2 = \min_{\xi \in D(\mathrm{A}_3)} \big(c_3 |\,\mathrm{A}_3\, \xi|_{\mathsf{H}_4} + |\xi - \tilde{y}|_{\mathsf{H}_3}\big)^2 = \max_{\phi \in D(\mathrm{A}_3^*)} \big(-\langle 2\tilde{y} + \mathrm{A}_3^*\, \phi, \mathrm{A}_3^*\, \phi\rangle_{\mathsf{H}_3}\big) \tag{4.16}$$

and the minimum resp. maximum is attained at $\hat{\xi} = h_{\mathrm{A}_3^*} + \tilde{y} \in D(\mathrm{A}_3)$ and $\hat{\phi} = (\mathcal{A}_3^*)^{-1} h_{\mathrm{A}_3^*} \in D(\mathcal{A}_3^*)$ with $\mathrm{A}_3 \hat{\xi} = \mathrm{A}_3 y = 0$, i.e., $\hat{\xi} \in N(\mathrm{A}_3)$,

$$|h_{K_3}|^2_{\mathsf{H}_3} = \min_{\varphi \in D(\mathrm{A}_2)} \min_{\phi \in D(\mathrm{A}_3^*)} |-\tilde{y} + \mathrm{A}_2 \varphi + \mathrm{A}_3^* \phi|^2_{\mathsf{H}_3} = \max_{\theta \in K_3} \big(-\langle 2\tilde{y} + \theta, \theta \rangle_{\mathsf{H}_3} \big) \tag{4.17}$$

and the minimum resp. maximum is attained at $\hat{\varphi} = (\mathcal{A}_2)^{-1} \pi_{\mathrm{A}_2} \tilde{y} \in D(\mathcal{A}_2)$, $\hat{\phi} = (\mathcal{A}_3^*)^{-1} \pi_{\mathrm{A}_3^*} \tilde{y} \in D(\mathcal{A}_3^*)$, and $\hat{\theta} = h_{K_3} \in K_3$ with $\mathrm{A}_2 \hat{\varphi} + \mathrm{A}_3^* \hat{\phi} = (\pi_{\mathrm{A}_2} + \pi_{\mathrm{A}_3^*})\tilde{y} = (1 - \pi_3)\tilde{y}$. If $\tilde{y} = \tilde{y}_\perp \in K_3^{\perp_{\mathsf{H}_3}}$, then $h_{K_3} = 0$, and $\tilde{y}$ can be replaced by $\tilde{y}_\perp$. The upper bound for $|h_{\mathrm{A}_3^*}|_{\mathsf{H}_3}$ in (4.16) equals

$$|h_{\mathrm{A}_3^*}|_{\mathsf{H}_3} = \min_{\xi \in N(\mathrm{A}_3)} |\xi - \tilde{y}|_{\mathsf{H}_3} = |\hat{\xi} - \tilde{y}|_{\mathsf{H}_3}, \qquad \hat{\xi} = h_{\mathrm{A}_3^*} + \tilde{y} \in N(\mathrm{A}_3),$$

and so the constant c_3 does not play a role. In (4.13) the unknown exact solution y still appears in the upper and in the lower bound. The term $\mathrm{A}_2 \xi - y \in R(\mathrm{A}_2)$ of the upper bound in (4.13) can be handled as an error $h_\xi = y - \tilde{y}_\xi$ with $\tilde{y}_\xi = \mathrm{A}_2 \xi$. As $h_\xi = \pi_{\mathrm{A}_2} h_\xi = h_{\xi, \mathrm{A}_2}$ we get by (4.15)

$$|\mathrm{A}_2 \xi - y|_{\mathsf{H}_3} = |h_\xi|_{\mathsf{H}_3} = \min_{\zeta \in D(\mathrm{A}_2^*)} \big(c_2 |\mathrm{A}_2^* \zeta - f|_{\mathsf{H}_2} + |\zeta - \mathrm{A}_2 \xi|_{\mathsf{H}_3} \big).$$

Another option to compute an upper bound in (4.13) is the following one: As $y \in D(\mathcal{A}_2^*)$ we observe $\mathrm{A}_2 \xi - y \in D(\mathcal{A}_2^*)$ if $\xi \in D(\mathrm{A}_2^* \mathrm{A}_2)$. The minimum in (4.13) is attained at $\hat{\xi} = e_{\mathrm{A}_2^*} + \tilde{x} \in D(\mathrm{A}_2)$ with $\mathrm{A}_2 \hat{\xi} = \mathrm{A}_2 x = y$. Since $\hat{\xi} \in D(\mathrm{A}_2^* \mathrm{A}_2)$ and $\mathrm{A}_2^* \mathrm{A}_2 \hat{\xi} = \mathrm{A}_2^* y = f$ we obtain

$$|e_{\mathrm{A}_2^*}|_{\mathsf{H}_2} = \min_{\xi \in D(\mathrm{A}_2^* \mathrm{A}_2)} \big(c_2 |\mathrm{A}_2 \xi - y|_{\mathsf{H}_3} + |\xi - \tilde{x}|_{\mathsf{H}_2} \big) = \min_{\xi \in D(\mathrm{A}_2^* \mathrm{A}_2)} \big(c_2^2 |\mathrm{A}_2^* \mathrm{A}_2 \xi - f|_{\mathsf{H}_2} + |\xi - \tilde{x}|_{\mathsf{H}_2} \big),$$

where the latter equality follows by the Friedrichs/Poincaré inequality. To get a lower bound for $|e_{\mathrm{A}_2^*}|^2_{\mathsf{H}_2}$ in (4.13) we observe $e_{\mathrm{A}_2^*} \in R(\mathrm{A}_2^*) = R(\mathrm{A}_2^* \mathrm{A}_2)$ and derive

$$\begin{aligned}
|e_{\mathrm{A}_2^*}|^2_{\mathsf{H}_2} &= \max_{\phi \in D(\mathrm{A}_2^* \mathrm{A}_2)} \big(2\langle e_{\mathrm{A}_2^*}, \mathrm{A}_2^* \mathrm{A}_2 \phi \rangle_{\mathsf{H}_2} - |\mathrm{A}_2^* \mathrm{A}_2 \phi|^2_{\mathsf{H}_2} \big) \\
&= \max_{\phi \in D(\mathrm{A}_2^* \mathrm{A}_2)} \big(2\langle e, \mathrm{A}_2^* \mathrm{A}_2 \phi \rangle_{\mathsf{H}_2} - |\mathrm{A}_2^* \mathrm{A}_2 \phi|^2_{\mathsf{H}_2} \big) \\
&= \max_{\phi \in D(\mathrm{A}_2^* \mathrm{A}_2)} \big(2\langle f, \phi \rangle_{\mathsf{H}_2} - \langle 2\tilde{x} + \mathrm{A}_2^* \mathrm{A}_2 \phi, \mathrm{A}_2^* \mathrm{A}_2 \phi \rangle_{\mathsf{H}_2} \big).
\end{aligned}$$

We summarize the two sided bounds:

Theorem 4.7. *Additionally to the General Assumption 3.1, suppose that $R(\mathrm{A}_3)$ is closed. Let $x \in \tilde{D}_2$ be the exact solution of (3.14), $y := \mathrm{A}_2 x$, and let $(\tilde{x}, \tilde{y}) \in \mathsf{H}_2 \times \mathsf{H}_3$. Then the following estimates hold for the errors $e = x - \tilde{x}$ and $h = y - \tilde{y}$ defined in (4.10):*

(i) *The errors decompose, i.e.,*

$$\begin{aligned}
e &= e_{\mathrm{A}_1} + e_{K_2} + e_{\mathrm{A}_2^*} \in R(\mathrm{A}_1) \oplus_{\mathsf{H}_2} K_2 \oplus_{\mathsf{H}_2} R(\mathrm{A}_2^*), & |e|^2_{\mathsf{H}_2} &= |e_{\mathrm{A}_1}|^2_{\mathsf{H}_2} + |e_{K_2}|^2_{\mathsf{H}_2} + |e_{\mathrm{A}_2^*}|^2_{\mathsf{H}_2}, \\
h &= h_{\mathrm{A}_2} + h_{K_3} + h_{\mathrm{A}_3^*} \in R(\mathrm{A}_2) \oplus_{\mathsf{H}_3} K_3 \oplus_{\mathsf{H}_3} R(\mathrm{A}_3^*), & |h|^2_{\mathsf{H}_3} &= |h_{\mathrm{A}_2}|^2_{\mathsf{H}_3} + |h_{K_3}|^2_{\mathsf{H}_3} + |h_{\mathrm{A}_3^*}|^2_{\mathsf{H}_3}.
\end{aligned}$$

(ii) *The projection $e_{\mathrm{A}_1} = \pi_{\mathrm{A}_1} e = x_g - \pi_{\mathrm{A}_1} \tilde{x} \in R(\mathrm{A}_1)$ satisfies*

$$\begin{aligned}
|e_{\mathrm{A}_1}|^2_{\mathsf{H}_2} &= \min_{\zeta \in D(\mathrm{A}_1^*)} \big(c_1 |\mathrm{A}_1^* \zeta - g|_{\mathsf{H}_1} + |\zeta - \tilde{x}|_{\mathsf{H}_2} \big)^2 \\
&= \max_{\varphi \in D(\mathrm{A}_1)} \big(2\langle g, \varphi \rangle_{\mathsf{H}_1} - \langle 2\tilde{x} + \mathrm{A}_1 \varphi, \mathrm{A}_1 \varphi \rangle_{\mathsf{H}_2} \big)
\end{aligned}$$

and the minimum resp. maximum is attained at

$$\hat{\zeta} := e_{\mathrm{A}_1} + \tilde{x} \in D(\mathrm{A}_1^*), \qquad \hat{\varphi} := (\mathcal{A}_1)^{-1} e_{\mathrm{A}_1} \in D(\mathcal{A}_1)$$

with $\mathrm{A}_1^* \hat{\zeta} = \mathrm{A}_1^* x = g$.

(iii) *The projection* $e_{\mathrm{A}_2^*} = \pi_{\mathrm{A}_2^*} e = x_y - \pi_{\mathrm{A}_2^*}\tilde{x} \in R(\mathrm{A}_2^*)$ *satisfies*

$$\begin{aligned}|e_{\mathrm{A}_2^*}|^2_{\mathsf{H}_2} &= \min_{\xi\in D(\mathrm{A}_2)} \min_{\zeta\in D(\mathrm{A}_2^*)} \big(c_2^2|\,\mathrm{A}_2^*\,\zeta - f|_{\mathsf{H}_2} + c_2|\zeta - \mathrm{A}_2\,\xi|_{\mathsf{H}_3} + |\xi - \tilde{x}|_{\mathsf{H}_2}\big)^2\\ &= \min_{\xi\in D(\mathrm{A}_2^*\,\mathrm{A}_2)} \big(c_2^2|\,\mathrm{A}_2^*\,\mathrm{A}_2\,\xi - f|_{\mathsf{H}_2} + |\xi - \tilde{x}|_{\mathsf{H}_2}\big)^2\\ &= \max_{\phi\in D(\mathrm{A}_2^*\,\mathrm{A}_2)} \big(2\langle f,\phi\rangle_{\mathsf{H}_2} - \langle 2\tilde{x} + \mathrm{A}_2^*\,\mathrm{A}_2\,\phi, \mathrm{A}_2^*\,\mathrm{A}_2\,\phi\rangle_{\mathsf{H}_2}\big)\end{aligned}$$

and the minima resp. maximum are attained at

$$\hat{\xi} := e_{\mathrm{A}_2^*} + \tilde{x} \in D(\mathrm{A}_2^*\,\mathrm{A}_2), \quad \hat{\zeta} := h_\xi + \mathrm{A}_2\,\xi = y \in D(\mathrm{A}_2^*), \quad \hat{\phi} := (\mathcal{A}_2)^{-1}(\mathcal{A}_2^*)^{-1} e_{\mathrm{A}_2^*} \in D(\mathcal{A}_2^*\,\mathcal{A}_2)$$

with $\mathrm{A}_2\,\hat{\xi} = \mathrm{A}_2\,x = y$ *and* $\mathrm{A}_2^*\,\mathrm{A}_2\,\hat{\xi} = \mathrm{A}_2^*\,y = f$ *as well as* $\mathrm{A}_2^*\,\hat{\zeta} = \mathrm{A}_2^*\,y = f$.

(iv) *The projection* $e_{K_2} = \pi_2\,e = k - \pi_2\,\tilde{x} \in K_2$ *satisfies*

$$\begin{aligned}|e_{K_2}|^2_{\mathsf{H}_2} &= \min_{\varphi\in D(\mathrm{A}_1)} \min_{\phi\in D(\mathrm{A}_2^*)} |k - \tilde{x} + \mathrm{A}_1\,\varphi + \mathrm{A}_2^*\,\phi|^2_{\mathsf{H}_2}\\ &= \max_{\theta\in K_2} \big\langle 2(k-\tilde{x}) - \theta, \theta\big\rangle_{\mathsf{H}_2}\end{aligned}$$

and the minimum resp. maximum is attained at

$$\hat{\varphi} := (\mathcal{A}_1)^{-1}\pi_{\mathrm{A}_1}\tilde{x} \in D(\mathcal{A}_1), \qquad \hat{\phi} := (\mathcal{A}_2^*)^{-1}\pi_{\mathrm{A}_2^*}\tilde{x} \in D(\mathcal{A}_2^*), \qquad \hat{\theta} := e_{K_2} \in K_2$$

with $\mathrm{A}_1\,\hat{\varphi} + \mathrm{A}_2^*\,\hat{\phi} = (\pi_{\mathrm{A}_1} + \pi_{\mathrm{A}_2^*})\tilde{x} = (1-\pi_2)\tilde{x}$.

(v) *The projection* $h_{\mathrm{A}_2} = \pi_{\mathrm{A}_2} h = y - \pi_{\mathrm{A}_2}\tilde{y} \in R(\mathrm{A}_2)$ *satisfies*

$$\begin{aligned}|h_{\mathrm{A}_2}|^2_{\mathsf{H}_3} &= \min_{\zeta\in D(\mathrm{A}_2^*)} \big(c_2|\,\mathrm{A}_2^*\,\zeta - f|_{\mathsf{H}_2} + |\zeta - \tilde{y}|_{\mathsf{H}_3}\big)^2\\ &= \max_{\varphi\in D(\mathrm{A}_2)} \big(2\langle f,\varphi\rangle_{\mathsf{H}_2} - \langle 2\tilde{y} + \mathrm{A}_2\,\varphi, \mathrm{A}_2\,\varphi\rangle_{\mathsf{H}_3}\big)\end{aligned}$$

and the minimum resp. maximum is attained at

$$\hat{\zeta} := h_{\mathrm{A}_2} + \tilde{y} \in D(\mathrm{A}_2^*), \qquad \hat{\varphi} := (\mathcal{A}_2)^{-1} h_{\mathrm{A}_2} \in D(\mathcal{A}_2)$$

with $\mathrm{A}_2^*\,\hat{\zeta} = \mathrm{A}_2^*\,y = f$.

(vi) *The projection* $h_{\mathrm{A}_3^*} = \pi_{\mathrm{A}_3^*} h = -\pi_{\mathrm{A}_3^*}\tilde{y} \in R(\mathrm{A}_3^*)$ *satisfies*

$$\begin{aligned}|h_{\mathrm{A}_3^*}|^2_{\mathsf{H}_3} &= \min_{\xi\in D(\mathrm{A}_3)} \big(c_3|\,\mathrm{A}_3\,\xi|_{\mathsf{H}_4} + |\xi - \tilde{y}|_{\mathsf{H}_3}\big)^2 = \min_{\xi\in N(\mathrm{A}_3)} |\xi - \tilde{y}|^2_{\mathsf{H}_3}\\ &= \max_{\phi\in D(\mathrm{A}_3^*)} \big(-\langle 2\tilde{y} + \mathrm{A}_3^*\,\phi, \mathrm{A}_3^*\,\phi\rangle_{\mathsf{H}_3}\big)\end{aligned}$$

and the minimum resp. maximum is attained at

$$\hat{\xi} := h_{\mathrm{A}_3^*} + \tilde{y} \in N(\mathrm{A}_3), \qquad \hat{\phi} := (\mathcal{A}_3^*)^{-1} h_{\mathrm{A}_3^*} \in D(\mathcal{A}_3^*)$$

with $\mathrm{A}_3\,\hat{\xi} = \mathrm{A}_3\,y = 0$.

(vii) *The projection* $h_{K_3} = \pi_3\,e = -\,\pi_3\,\tilde{y} \in K_3$ *satisfies*

$$\begin{aligned}|h_{K_3}|^2_{\mathsf{H}_3} &= \min_{\varphi\in D(\mathrm{A}_2)} \min_{\phi\in D(\mathrm{A}_3^*)} |-\tilde{y} + \mathrm{A}_2\,\varphi + \mathrm{A}_3^*\,\phi|^2_{\mathsf{H}_3}\\ &= \max_{\theta\in K_3} \big(-\langle 2\tilde{y} + \theta, \theta\rangle_{\mathsf{H}_3}\big)\end{aligned}$$

and the minimum resp. maximum is attained at

$$\hat{\varphi} := (\mathcal{A}_2)^{-1}\pi_{\mathrm{A}_2}\tilde{y} \in D(\mathcal{A}_2), \qquad \hat{\phi} := (\mathcal{A}_3^*)^{-1}\pi_{\mathrm{A}_3^*}\tilde{y} \in D(\mathcal{A}_3^*), \qquad \hat{\theta} := h_{K_3} \in K_3$$

with $\mathrm{A}_2\,\hat{\varphi} + \mathrm{A}_3^*\,\hat{\phi} = (\pi_{\mathrm{A}_2} + \pi_{\mathrm{A}_3^*})\tilde{y} = (1-\pi_3)\tilde{y}$.

If $\tilde{x} = k + \tilde{x}_\perp$ *with some* $\tilde{x}_\perp \in K_2^{\perp_{\mathsf{H}_2}}$, *then* $e_{K_2} = 0$, *and in (ii) and (iii)* $\tilde{x}$ *can be replaced by* $\tilde{x}_\perp$. *If* $\tilde{y} = \tilde{y}_\perp \in K_3^{\perp_{\mathsf{H}_3}}$, *then* $h_{K_3} = 0$, *and in (v) and (vi)* $\tilde{y}$ *can be replaced by* $\tilde{y}_\perp$.

Remark 4.8. *A reasonable assumption provided by standard numerical methods is $\tilde{y} \in R(\mathrm{A}_2)$. Hence it often holds $h_{\mathrm{A}_3^*} = h_{K_3} = 0$.*

4.3. **Computing the Error Functionals.** We propose a suitable way to compute the most important error functionals in Theorem 4.1, Corollary 4.4, and Corollary 4.6. Let us discuss, e.g.,

$$|e_{\mathrm{A}_2^*}|_{\mathsf{H}_2} = \min_{\xi \in D(\mathrm{A}_2)} \big(c_2 |\, \mathrm{A}_2 \xi - f|_{\mathsf{H}_3} + |\xi - \tilde{x}|_{\mathsf{H}_2} \big), \qquad \tilde{x} \in \mathsf{H}_2 . \tag{4.18}$$

As for all $\xi \in D(\mathrm{A}_2)$ and all $t > 0$

$$|e_{\mathrm{A}_2^*}|^2_{\mathsf{H}_2} \leq (1 + t^{-1})\, c_2^2 \,|\, \mathrm{A}_2 \xi - f|^2_{\mathsf{H}_3} + (1+t)|\xi - \tilde{x}|^2_{\mathsf{H}_2} =: \mathcal{F}(\tilde{x}; \xi, t),$$

we have for $\xi = \hat{\xi}$ from Theorem 4.1, Corollary 4.4 or Corollary 4.6

$$|e_{\mathrm{A}_2^*}|^2_{\mathsf{H}_2} \leq \inf_{t \in (0,\infty)} \inf_{\xi \in D(\mathrm{A}_2)} \mathcal{F}(\tilde{x}; \xi, t) \leq \inf_{t \in (0,\infty)} \mathcal{F}(\tilde{x}; \hat{\xi}, t) = \inf_{t \in (0,\infty)} (1+t)|e_{\mathrm{A}_2^*}|^2_{\mathsf{H}_2} = |e_{\mathrm{A}_2^*}|^2_{\mathsf{H}_2}.$$

Thus

$$|e_{\mathrm{A}_2^*}|^2_{\mathsf{H}_2} = \min_{\substack{t \in [0,\infty], \\ \xi \in D(\mathrm{A}_2)}} \mathcal{F}(\tilde{x}; \xi, t) = \min_{\substack{t \in [0,\infty], \\ \xi \in D(\mathrm{A}_2)}} \big((1 + t^{-1})\, c_2^2 \,|\, \mathrm{A}_2 \xi - f|^2_{\mathsf{H}_3} + (1+t)|\xi - \tilde{x}|^2_{\mathsf{H}_2} \big) \tag{4.19}$$

and the minimum is attained at $(t, \xi) = (0, \hat{\xi})$. For fixed $\xi \in D(\mathrm{A}_2)$ the minimal $t_\xi \in [0, \infty]$ is given by

$$t_\xi = \begin{cases} c_2 \dfrac{|\, \mathrm{A}_2 \xi - f|_{\mathsf{H}_3}}{|\xi - \tilde{x}|_{\mathsf{H}_2}} & \text{, if } \xi \neq \tilde{x}, \\ \infty & \text{, if } \xi = \tilde{x}. \end{cases}$$

We note that the case $t_\xi = \infty$ can only happen if $\tilde{x} \in D(\mathrm{A}_2)$. In any case, inserting t_ξ into (4.19) we get back (4.18), i.e.,

$$|e_{\mathrm{A}_2^*}|^2_{\mathsf{H}_2} \leq \min_{\xi \in D(\mathrm{A}_2)} \big(c_2 |\, \mathrm{A}_2 \xi - f|_{\mathsf{H}_3} + |\xi - \tilde{x}|_{\mathsf{H}_2} \big)^2 = |e_{\mathrm{A}_2^*}|^2_{\mathsf{H}_2}.$$

On the other hand, for fixed $t > 0$ the minimization of $F(\xi) := \mathcal{F}(\tilde{x}; \xi, t)$ over $\xi \in D(\mathrm{A}_2)$ is equivalent to find $\xi_t \in D(\mathrm{A}_2)$, such that

$$\forall\, \xi \in D(\mathrm{A}_2) \qquad \frac{t}{2c_2^2(1+t)} F'(\xi_t)(\xi) = \langle \mathrm{A}_2 \xi_t - f, \mathrm{A}_2 \xi \rangle_{\mathsf{H}_3} + \frac{t}{c_2^2} \langle \xi_t - \tilde{x}, \xi \rangle_{\mathsf{H}_2} = 0. \tag{4.20}$$

Especially $\mathrm{A}_2 \xi_t - f \in D(\mathrm{A}_2^*)$ with $\mathrm{A}_2^*(\mathrm{A}_2 \xi_t - f) = \frac{t}{c_2^2}(\tilde{x} - \xi_t)$ and hence (4.20) is the standard weak formulation of the coercive problem (in formally strong form) $(\mathrm{A}_2^* \mathrm{A}_2 + \frac{t}{c_2^2})\xi_t = \mathrm{A}_2^* f + \frac{t}{c_2^2}\tilde{x}$, i.e.,

$$\forall\, \xi \in D(\mathrm{A}_2) \qquad \langle \mathrm{A}_2 \xi_t, \mathrm{A}_2 \xi \rangle_{\mathsf{H}_3} + \frac{t}{c_2^2} \langle \xi_t, \xi \rangle_{\mathsf{H}_2} = \langle f, \mathrm{A}_2 \xi \rangle_{\mathsf{H}_3} + \frac{t}{c_2^2} \langle \tilde{x}, \xi \rangle_{\mathsf{H}_2}. \tag{4.21}$$

The strong form holds rigorously if $f \in R(\mathrm{A}_2) \cap D(\mathrm{A}_2^*) = D(\mathcal{A}_2^*)$. Moreover, as $f \in R(\mathrm{A}_2)$ we even have

$$\mathrm{A}_2 \xi_t - f \in D(\mathcal{A}_2^*) \quad \text{with} \quad \mathrm{A}_2^*(\mathrm{A}_2 \xi_t - f) = \frac{t}{c_2^2}(\tilde{x} - \xi_t).$$

Inserting ξ_t into (4.19) and using the Friedrichs/Poincaré type estimate shows

$$\begin{aligned} |e_{\mathrm{A}_2^*}|^2_{\mathsf{H}_2} &\leq \min_{t \in [0,\infty]} \big((1 + t^{-1})\, c_2^2 \,|\, \mathrm{A}_2 \xi_t - f|^2_{\mathsf{H}_3} + (1+t)|\xi_t - \tilde{x}|^2_{\mathsf{H}_2} \big) \\ &\leq \min_{t \in [0,\infty]} \big((1 + t^{-1})\, c_2^4 \,\big|\, \mathrm{A}_2^*(\mathrm{A}_2 \xi_t - f)\big|^2_{\mathsf{H}_2} + (1+t)|\xi_t - \tilde{x}|^2_{\mathsf{H}_2} \big) \\ &= \min_{t \in [0,\infty]} (1+t)^2 |\xi_t - \tilde{x}|^2_{\mathsf{H}_2} = |\xi_t - \tilde{x}|^2_{\mathsf{H}_2}. \end{aligned} \tag{4.22}$$

A suitable algorithm for computing a good pair (t, ξ) for approximately minimizing (4.19) is the following:

Algorithm 4.9. *Computing (t, ξ) in (4.19), i.e., an upper bound for $|e_{\mathrm{A}_2^*}|_{\mathsf{H}_2}$:*

- initialization: *Set $n := 0$. Pick $\xi_0 \in D(\mathrm{A}_2)$ with $\xi_0 \neq \tilde{x}$.*

- loop: *Set* $n := n+1$. *Compute* $t_n = c_2 \dfrac{|\operatorname{A}_2 \xi_{n-1} - f|_{\mathsf{H}_3}}{|\xi_{n-1} - \tilde{x}|_{\mathsf{H}_2}}$ *and then* ξ_n *by solving* (4.21), *i.e.,*

$$\forall\, \xi \in D(\operatorname{A}_2) \qquad c_2^2 \langle \operatorname{A}_2 \xi_n, \operatorname{A}_2 \xi\rangle_{\mathsf{H}_3} + t_n \langle \xi_n, \xi\rangle_{\mathsf{H}_2} = c_2^2 \langle f, \operatorname{A}_2 \xi\rangle_{\mathsf{H}_3} + t_n \langle \tilde{x}, \xi\rangle_{\mathsf{H}_2}.$$

  Compute $\mathcal{F}_{\operatorname{A}_2^*}(\tilde{x};\xi_n,t_n) := (1+t_n^{-1})\, c_2^2\, |\operatorname{A}_2 \xi_n - f|_{\mathsf{H}_3}^2 + (1+t_n)|\xi_n - \tilde{x}|_{\mathsf{H}_2}^2$.
- stop *if* $\mathcal{F}_{\operatorname{A}_2^*}(\tilde{x};\xi_n,t_n) - \mathcal{F}_{\operatorname{A}_2^*}(\tilde{x};\xi_{n-1},t_{n-1})$ *is small.*

Similarly we propose the following algorithm:

Algorithm 4.10. *Computing an upper bound for* $|e_{\operatorname{A}_1}|_{\mathsf{H}_2}$*:*

- initialization: *Set* $n := 0$. *Pick* $\zeta_0 \in D(\operatorname{A}_1^*)$ *with* $\zeta_0 \neq \tilde{x}$.
- loop: *Set* $n := n+1$. *Compute* $t_n = c_1 \dfrac{|\operatorname{A}_1^* \zeta_{n-1} - g|_{\mathsf{H}_1}}{|\zeta_{n-1} - \tilde{x}|_{\mathsf{H}_2}}$ *and then* ζ_n *by solving*

$$\forall\, \zeta \in D(\operatorname{A}_1^*) \qquad c_1^2 \langle \operatorname{A}_1^* \zeta_n, \operatorname{A}_1^* \zeta\rangle_{\mathsf{H}_1} + t_n \langle \zeta_n, \zeta\rangle_{\mathsf{H}_2} = c_1^2 \langle g, \operatorname{A}_1^* \zeta\rangle_{\mathsf{H}_1} + t_n \langle \tilde{x}, \zeta\rangle_{\mathsf{H}_2}.$$

  Compute $\mathcal{F}_{\operatorname{A}_1}(\tilde{x};\zeta_n,t_n) := (1+t_n^{-1})\, c_1^2\, |\operatorname{A}_1^* \zeta_n - g|_{\mathsf{H}_1}^2 + (1+t_n)|\zeta_n - \tilde{x}|_{\mathsf{H}_2}^2$.
- stop *if* $\mathcal{F}_{\operatorname{A}_1}(\tilde{x};\zeta_n,t_n) - \mathcal{F}_{\operatorname{A}_1}(\tilde{x};\zeta_{n-1},t_{n-1})$ *is small.*

5. Applications

5.1. **Prototype First Order System: Electro-Magneto Statics.** As a prototypical example for a first order system we will discuss the system of electro-magneto statics with mixed boundary conditions. Let $\Omega \subset \mathbb{R}^3$ be a bounded weak Lipschitz domain, see [1, Definition 2.3], and let $\Gamma := \partial\,\Omega$ denote its boundary (Lipschitz manifold), which is supposed to be decomposed into two relatively open weak Lipschitz subdomains (Lipschitz submanifolds) Γ_{t} and $\Gamma_{\mathsf{n}} := \Gamma \setminus \overline{\Gamma_{\mathsf{t}}}$ see [1, Definition 2.5]. Let us consider the linear first order system (in classical strong formulation) for a vector field $E : \Omega \to \mathbb{R}^3$

$$\begin{aligned} \operatorname{rot} E &= F && \text{in } \Omega, && n \times E = 0 && \text{at } \Gamma_{\mathsf{t}},\\ -\operatorname{div} \varepsilon E &= g && \text{in } \Omega, && n \cdot \varepsilon E = 0 && \text{at } \Gamma_{\mathsf{n}},\\ \pi_{\mathcal{H}} E &= H && \text{in } \Omega. \end{aligned} \tag{5.1}$$

Here, $\varepsilon : \Omega \to \mathbb{R}^{3\times 3}$ is a symmetric, uniformly positive definite L^∞-matrix field and n denotes the outer unit normal at Γ. Let us put $\mu := \varepsilon^{-1}$. The usual Lebesgue and Sobolev (Hilbert) spaces will be denoted by $\mathsf{L}^2(\Omega)$, $\mathsf{H}^\ell(\Omega)$, $\ell \in \mathbb{N}_0$, and (in the distributional sense) we introduce

$$\mathsf{R}(\Omega) := \big\{E \in \mathsf{L}^2(\Omega) \,:\, \operatorname{rot} E \in \mathsf{L}^2(\Omega)\big\}, \qquad \mathsf{D}(\Omega) := \big\{E \in \mathsf{L}^2(\Omega) \,:\, \operatorname{div} E \in \mathsf{L}^2(\Omega)\big\}.$$

With the test functions or test vector fields

$$\mathsf{C}^\infty_{\Gamma_{\mathsf{t}}}(\Omega) := \big\{\varphi|_\Omega \,:\, \varphi \in \mathsf{C}^\infty(\mathbb{R}^3),\ \operatorname{supp}\varphi \text{ compact in } \mathbb{R}^3,\ \operatorname{dist}(\operatorname{supp}\varphi, \Gamma_{\mathsf{t}}) > 0\big\}, \qquad \mathsf{C}^\infty_{\emptyset}(\Omega) = \mathsf{C}^\infty(\overline{\Omega}),$$

we define as closures of test functions resp. test fields

$$\mathsf{H}^1_{\Gamma_{\mathsf{t}}}(\Omega) := \overline{\mathsf{C}^\infty_{\Gamma_{\mathsf{t}}}(\Omega)}^{\mathsf{H}^1(\Omega)}, \qquad \mathsf{R}_{\Gamma_{\mathsf{t}}}(\Omega) := \overline{\mathsf{C}^\infty_{\Gamma_{\mathsf{t}}}(\Omega)}^{\mathsf{R}(\Omega)}, \qquad \mathsf{D}_{\Gamma_{\mathsf{n}}}(\Omega) := \overline{\mathsf{C}^\infty_{\Gamma_{\mathsf{n}}}(\Omega)}^{\mathsf{D}(\Omega)},$$

generalizing homogeneous scalar, tangential, and normal traces on Γ_{t} and Γ_{n}, respectively. Moreover, we introduce the closed subspaces

$$\begin{aligned} \mathsf{R}_0(\Omega) &:= \{E \in \mathsf{R}(\Omega) \,:\, \operatorname{rot} E = 0\}, & \mathsf{D}_0(\Omega) &:= \{E \in \mathsf{D}(\Omega) \,:\, \operatorname{div} E = 0\},\\ \mathsf{R}_{\Gamma_{\mathsf{t}},0}(\Omega) &:= \mathsf{R}_{\Gamma_{\mathsf{t}}}(\Omega) \cap \mathsf{R}_0(\Omega), & \mathsf{D}_{\Gamma_{\mathsf{n}},0}(\Omega) &:= \mathsf{D}_{\Gamma_{\mathsf{n}}}(\Omega) \cap \mathsf{D}_0(\Omega), \end{aligned}$$

and the Dirichlet-Neumann fields including the corresponding orthonormal projector

$$\mathcal{H}_{\mathsf{t},\mathsf{n},\varepsilon}(\Omega) := \mathsf{R}_{\Gamma_{\mathsf{t}},0}(\Omega) \cap \mu\, \mathsf{D}_{\Gamma_{\mathsf{n}},0}(\Omega), \qquad \pi_{\mathcal{H}} : \mathsf{L}^2_\varepsilon(\Omega) \to \mathcal{H}_{\mathsf{t},\mathsf{n},\varepsilon}(\Omega).$$

Here, $\mathsf{L}^2_\varepsilon(\Omega)$ denotes $\mathsf{L}^2(\Omega)$ equipped with the inner product $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathsf{L}^2_\varepsilon(\Omega)} := \langle \varepsilon\,\cdot\,,\,\cdot\,\rangle_{\mathsf{L}^2(\Omega)}$. Let $\mathsf{H}_1 := \mathsf{L}^2(\Omega)$, $\mathsf{H}_4 := \mathsf{L}^2(\Omega)$ (both scalar valued) and $\mathsf{H}_2 := \mathsf{L}^2_\varepsilon(\Omega)$, $\mathsf{H}_3 := \mathsf{L}^2(\Omega)$ (both vector valued) as well as

$$\operatorname{A}_1 := \operatorname{grad}_{\Gamma_{\mathsf{t}}} : D(\operatorname{A}_1) := \mathsf{H}^1_{\Gamma_{\mathsf{t}}}(\Omega) \subset \mathsf{L}^2(\Omega) \to \mathsf{L}^2_\varepsilon(\Omega),$$

$$A_2 := \operatorname{rot}_{\Gamma_t} : D(A_2) := \mathsf{R}_{\Gamma_t}(\Omega) \subset \mathsf{L}^2_\varepsilon(\Omega) \to \mathsf{L}^2(\Omega),$$
$$A_3 := \operatorname{div}_{\Gamma_t} : D(A_3) := \mathsf{D}_{\Gamma_t}(\Omega) \subset \mathsf{L}^2(\Omega) \to \mathsf{L}^2(\Omega).$$

In [1] it has been shown that the adjoints are

$$A_1^* = \operatorname{grad}^*_{\Gamma_t} = -\operatorname{div}_{\Gamma_n} \varepsilon : D(A_1^*) = \mu\, \mathsf{D}_{\Gamma_n}(\Omega) \subset \mathsf{L}^2_\varepsilon(\Omega) \to \mathsf{L}^2(\Omega),$$
$$A_2^* = \operatorname{rot}^*_{\Gamma_t} = \mu \operatorname{rot}_{\Gamma_n} : D(A_2^*) = \mathsf{R}_{\Gamma_n}(\Omega) \subset \mathsf{L}^2(\Omega) \to \mathsf{L}^2_\varepsilon(\Omega),$$
$$A_3^* = \operatorname{div}^*_{\Gamma_t} = -\operatorname{grad}_{\Gamma_n} : D(A_3^*) = \mathsf{H}^1_{\Gamma_n}(\Omega) \subset \mathsf{L}^2(\Omega) \to \mathsf{L}^2(\Omega).$$

For the kernels we have

$$N(A_1) = \begin{cases} \{0\} & \text{, if } \Gamma_t \neq \emptyset, \\ \mathbb{R} & \text{, if } \Gamma_t = \emptyset, \end{cases} \qquad N(A_1^*) = \mu\, \mathsf{D}_{\Gamma_n,0}(\Omega),$$
$$N(A_2) = \mathsf{R}_{\Gamma_t,0}(\Omega), \qquad N(A_2^*) = \mathsf{R}_{\Gamma_n,0}(\Omega),$$
$$N(A_3) = \mathsf{D}_{\Gamma_t,0}(\Omega), \qquad N(A_3^*) = \begin{cases} \{0\} & \text{, if } \Gamma_t \neq \Gamma, \\ \mathbb{R} & \text{, if } \Gamma_t = \Gamma. \end{cases}$$

As A_1, A_2, A_3 define a well known complex, see, e.g., [1, Lemma 2.2], so do their adjoints, i.e., for[vi] $\emptyset \neq \Gamma_t \neq \Gamma$

$$\{0\} \xrightarrow{0} \mathsf{H}^1_{\Gamma_t}(\Omega) \xrightarrow{A_1=\operatorname{grad}_{\Gamma_t}} \mathsf{R}_{\Gamma_t}(\Omega) \xrightarrow{A_2=\operatorname{rot}_{\Gamma_t}} \mathsf{D}_{\Gamma_t}(\Omega) \xrightarrow{A_3=\operatorname{div}_{\Gamma_t}} \mathsf{L}^2(\Omega) \xrightarrow{0} \{0\},$$
$$\{0\} \xleftarrow{0} \mathsf{L}^2(\Omega) \xleftarrow{A_1^*=-\operatorname{div}_{\Gamma_n}\varepsilon} \mu\, \mathsf{D}_{\Gamma_n}(\Omega) \xleftarrow{A_2^*=\mu \operatorname{rot}_{\Gamma_n}} \mathsf{R}_{\Gamma_n}(\Omega) \xleftarrow{A_3^*=-\operatorname{grad}_{\Gamma_n}} \mathsf{H}^1_{\Gamma_n}(\Omega) \xleftarrow{0} \{0\}.$$

Using the latter operators A_2 and A_1^*, the linear first order system (5.1) (in weak formulation) has the form of (1.5) resp. (3.1), i.e., find a vector field

$$E \in D_2 = D(A_2) \cap D(A_1^*) = \mathsf{R}_{\Gamma_t}(\Omega) \cap \mu\, \mathsf{D}_{\Gamma_n}(\Omega),$$

such that

$$\begin{aligned} \operatorname{rot}_{\Gamma_t} E &= F, \\ -\operatorname{div}_{\Gamma_n} \varepsilon E &= g, \\ \pi_{\mathcal{H}} E &= H, \end{aligned} \tag{5.2}$$

where $K_2 = \mathcal{H}_{t,n,\varepsilon}(\Omega)$. In [1, Theorem 5.1] the embedding $D_2 \hookrightarrow \mathsf{H}_2$, i.e.,

$$\mathsf{R}_{\Gamma_t}(\Omega) \cap \mu\, \mathsf{D}_{\Gamma_n}(\Omega) \hookrightarrow \mathsf{L}^2_\varepsilon(\Omega),$$

was shown to be compact. Hence also the embedding $D_3 = D(A_3) \cap D(A_2^*) \hookrightarrow \mathsf{H}_3$, i.e.,

$$\mathsf{D}_{\Gamma_t}(\Omega) \cap \mathsf{R}_{\Gamma_n}(\Omega) \hookrightarrow \mathsf{L}^2(\Omega),$$

is compact. Thus, by the results of the functional analysis toolbox Section 2, all occurring ranges are closed, certain Helmholtz type decompositions hold, corresponding Friedrichs/Poincaré type estimates

[vi] For $\Gamma_t = \emptyset$ we have

$$\mathbb{R} \xrightarrow{\iota_{\mathbb{R}}} \mathsf{H}^1(\Omega) \xrightarrow{A_1=\operatorname{grad}} \mathsf{R}(\Omega) \xrightarrow{A_2=\operatorname{rot}} \mathsf{D}(\Omega) \xrightarrow{A_3=\operatorname{div}} \mathsf{L}^2(\Omega) \xrightarrow{0} \{0\},$$
$$\mathbb{R} \xleftarrow{\pi_{\mathbb{R}}} \mathsf{L}^2(\Omega) \xleftarrow{A_1^*=-\operatorname{div}_\Gamma \varepsilon} \mu\, \mathsf{D}_\Gamma(\Omega) \xleftarrow{A_2^*=\mu \operatorname{rot}_\Gamma} \mathsf{R}_\Gamma(\Omega) \xleftarrow{A_3^*=-\operatorname{grad}_\Gamma} \mathsf{H}^1_\Gamma(\Omega) \xleftarrow{0} \{0\},$$

which also shows the case $\Gamma_t = \Gamma$ by interchanging Γ_t and Γ_n and shifting ε. More precisely, for $\Gamma_t = \Gamma$ it holds

$$\{0\} \xrightarrow{0} \mathsf{H}^1_\Gamma(\Omega) \xrightarrow{A_1=\operatorname{grad}_\Gamma} \mathsf{R}_\Gamma(\Omega) \xrightarrow{A_2=\operatorname{rot}_\Gamma} \mathsf{D}_\Gamma(\Omega) \xrightarrow{A_3=\operatorname{div}_\Gamma} \mathsf{L}^2(\Omega) \xrightarrow{\pi_{\mathbb{R}}} \mathbb{R},$$
$$\{0\} \xleftarrow{0} \mathsf{L}^2(\Omega) \xleftarrow{A_1^*=-\operatorname{div}\varepsilon} \mu\, \mathsf{D}(\Omega) \xleftarrow{A_2^*=\mu \operatorname{rot}} \mathsf{R}(\Omega) \xleftarrow{A_3^*=-\operatorname{grad}} \mathsf{H}^1(\Omega) \xleftarrow{\iota_{\mathbb{R}}} \mathbb{R}.$$

are valid, and the respective inverse operators are continuous resp. compact. Especially, the reduced operators are

$$\begin{aligned}\mathcal{A}_1 &:= \widetilde{\operatorname{grad}}_{\Gamma_t} : D(\mathcal{A}_1) = \mathsf{H}^1_{\Gamma_t}(\Omega) \cap \mathsf{L}^2(\Omega) \subset \mathsf{L}^2(\Omega) \to \operatorname{grad} \mathsf{H}^1_{\Gamma_t}(\Omega),\\ \mathcal{A}_2 &:= \widetilde{\operatorname{rot}}_{\Gamma_t} : D(\mathcal{A}_2) = \mathsf{R}_{\Gamma_t}(\Omega) \cap \mu \operatorname{rot} \mathsf{R}_{\Gamma_n}(\Omega) \subset \mu \operatorname{rot} \mathsf{R}_{\Gamma_n}(\Omega) \to \operatorname{rot} \mathsf{R}_{\Gamma_t}(\Omega),\\ \mathcal{A}_3 &:= \widetilde{\operatorname{div}}_{\Gamma_t} : D(\mathcal{A}_3) = \mathsf{D}_{\Gamma_t}(\Omega) \cap \operatorname{grad} \mathsf{H}^1_{\Gamma_n}(\Omega) \subset \operatorname{grad} \mathsf{H}^1_{\Gamma_n}(\Omega) \to \mathsf{L}^2(\Omega),\end{aligned}$$

where $\operatorname{grad} \mathsf{H}^1_{\Gamma_t}(\Omega)$ and $\mu \operatorname{rot} \mathsf{R}_{\Gamma_n}(\Omega)$ have to be understood as closed subspaces of $\mathsf{L}^2_\varepsilon(\Omega)$, and $\mathsf{L}^2(\Omega)$ has to be replaced by $\mathsf{L}^2_\perp(\Omega) := \mathsf{L}^2(\Omega) \cap \mathbb{R}^{\perp_{\mathsf{L}^2(\Omega)}}$ in $\mathcal{A}_1$, if $\Gamma_t = \emptyset$, and in $\mathcal{A}_3$, if $\Gamma_t = \Gamma$, with adjoints

$$\begin{aligned}\mathcal{A}_1^* &= \widetilde{\operatorname{grad}}_{\Gamma_t}^* = -\widetilde{\operatorname{div}}_{\Gamma_n}\varepsilon : D(\mathcal{A}_1^*) = \mu \mathsf{D}_{\Gamma_n}(\Omega) \cap \operatorname{grad} \mathsf{H}^1_{\Gamma_t}(\Omega) \subset \operatorname{grad} \mathsf{H}^1_{\Gamma_t}(\Omega) \to \mathsf{L}^2(\Omega),\\ \mathcal{A}_2^* &= \widetilde{\operatorname{rot}}_{\Gamma_t}^* = \mu \widetilde{\operatorname{rot}}_{\Gamma_n} : D(\mathcal{A}_2^*) = \mathsf{R}_{\Gamma_n}(\Omega) \cap \operatorname{rot} \mathsf{R}_{\Gamma_t}(\Omega) \subset \operatorname{rot} \mathsf{R}_{\Gamma_t}(\Omega) \to \mu \operatorname{rot} \mathsf{R}_{\Gamma_n}(\Omega),\\ \mathcal{A}_3^* &= \widetilde{\operatorname{div}}_{\Gamma_t}^* = -\widetilde{\operatorname{grad}}_{\Gamma_n} : D(\mathcal{A}_3^*) = \mathsf{H}^1_{\Gamma_n}(\Omega) \cap \mathsf{L}^2(\Omega) \subset \mathsf{L}^2(\Omega) \to \operatorname{grad} \mathsf{H}^1_{\Gamma_n}(\Omega),\end{aligned}$$

where $\mathsf{L}^2(\Omega)$ has to be replaced by $\mathsf{L}^2_\perp(\Omega)$ in $\mathcal{A}_1^*$, if $\Gamma_t = \emptyset$, and in $\mathcal{A}_3^*$, if $\Gamma_t = \Gamma$. Note that the reduced operators possess bounded resp. compact inverse operators. For the ranges we have

$$\begin{aligned}
&R(\mathrm{A}_1) = R(\mathcal{A}_1) \subset N(\mathrm{A}_2), \text{ i.e.,} && \operatorname{grad} \mathsf{H}^1_{\Gamma_t}(\Omega) = \operatorname{grad}\big(\mathsf{H}^1_{\Gamma_t}(\Omega) \cap \mathsf{L}^2(\Omega)\big) \subset \mathsf{R}_{\Gamma_t,0}(\Omega),\\
&R(\mathrm{A}_2) = R(\mathcal{A}_2) \subset N(\mathrm{A}_3), \text{ i.e.,} && \operatorname{rot} \mathsf{R}_{\Gamma_t}(\Omega) = \operatorname{rot}\big(\mathsf{R}_{\Gamma_t}(\Omega) \cap \mu \operatorname{rot} \mathsf{R}_{\Gamma_n}(\Omega)\big) \subset \mathsf{D}_{\Gamma_t,0}(\Omega),\\
&R(\mathrm{A}_3) = R(\mathcal{A}_3), \text{ i.e.,} && \operatorname{div} \mathsf{D}_{\Gamma_t}(\Omega) = \operatorname{div}\big(\mathsf{D}_{\Gamma_t}(\Omega) \cap \operatorname{grad} \mathsf{H}^1_{\Gamma_n}(\Omega)\big),\\
&R(\mathrm{A}_1^*) = R(\mathcal{A}_1^*), \text{ i.e.,} && \operatorname{div} \mathsf{D}_{\Gamma_n}(\Omega) = \operatorname{div}\big(\mathsf{D}_{\Gamma_n}(\Omega) \cap \varepsilon \operatorname{grad} \mathsf{H}^1_{\Gamma_t}(\Omega)\big),\\
&R(\mathrm{A}_2^*) = R(\mathcal{A}_2^*) \subset N(\mathrm{A}_1^*), \text{ i.e.,} && \mu \operatorname{rot} \mathsf{R}_{\Gamma_n}(\Omega) = \mu \operatorname{rot}\big(\mathsf{R}_{\Gamma_n}(\Omega) \cap \operatorname{rot} \mathsf{R}_{\Gamma_t}(\Omega)\big) \subset \mu \mathsf{D}_{\Gamma_n,0}(\Omega),\\
&R(\mathrm{A}_3^*) = R(\mathcal{A}_3^*) \subset N(\mathrm{A}_2^*), \text{ i.e.,} && \operatorname{grad} \mathsf{H}^1_{\Gamma_n}(\Omega) = \operatorname{grad}\big(\mathsf{H}^1_{\Gamma_n}(\Omega) \cap \mathsf{L}^2(\Omega)\big) \subset \mathsf{R}_{\Gamma_n,0}(\Omega),
\end{aligned}$$

where $\mathsf{L}^2(\Omega)$ has to be replaced by $\mathsf{L}^2_\perp(\Omega)$ for $\Gamma_t = \emptyset$ resp. $\Gamma_t = \Gamma$. Note that the assertions of $R(\mathrm{A}_3)$, $R(\mathrm{A}_2^*)$, $R(\mathrm{A}_3^*)$ are already included in those of $R(\mathrm{A}_1)$, $R(\mathrm{A}_2)$, $R(\mathrm{A}_1^*)$ by interchanging Γ_t and Γ_n and setting $\varepsilon := \mathrm{id}$. Furthermore, the following Friedrichs/Poincaré type estimates hold:

$$\begin{aligned}
&\forall\, u \in D(\mathcal{A}_1) = \mathsf{H}^1_{\Gamma_t}(\Omega) \cap \mathsf{L}^2(\Omega) && |u|_{\mathsf{L}^2(\Omega)} \leq c_{\mathsf{fp}}\, |\operatorname{grad} u|_{\mathsf{L}^2_\varepsilon(\Omega)},\\
&\forall\, E \in D(\mathcal{A}_1^*) = \mu \mathsf{D}_{\Gamma_n}(\Omega) \cap \operatorname{grad} \mathsf{H}^1_{\Gamma_t}(\Omega), && |E|_{\mathsf{L}^2_\varepsilon(\Omega)} \leq c_{\mathsf{fp}}\, |\operatorname{div} \varepsilon E|_{\mathsf{L}^2(\Omega)},\\
&\forall\, E \in D(\mathcal{A}_2) = \mathsf{R}_{\Gamma_t}(\Omega) \cap \mu \operatorname{rot} \mathsf{R}_{\Gamma_n}(\Omega), && |E|_{\mathsf{L}^2_\varepsilon(\Omega)} \leq c_{\mathsf{m}}\, |\operatorname{rot} E|_{\mathsf{L}^2(\Omega)},\\
&\forall\, E \in D(\mathcal{A}_2^*) = \mathsf{R}_{\Gamma_n}(\Omega) \cap \operatorname{rot} \mathsf{R}_{\Gamma_t}(\Omega), && |E|_{\mathsf{L}^2(\Omega)} \leq c_{\mathsf{m}}\, |\operatorname{rot} E|_{\mathsf{L}^2_\mu(\Omega)},\\
&\forall\, E \in D(\mathcal{A}_3) = \mathsf{D}_{\Gamma_t}(\Omega) \cap \operatorname{grad} \mathsf{H}^1_{\Gamma_n}(\Omega), && |E|_{\mathsf{L}^2(\Omega)} \leq \tilde{c}_{\mathsf{fp}}\, |\operatorname{div} E|_{\mathsf{L}^2(\Omega)},\\
&\forall\, u \in D(\mathcal{A}_3^*) = \mathsf{H}^1_{\Gamma_n}(\Omega) \cap \mathsf{L}^2(\Omega) && |u|_{\mathsf{L}^2(\Omega)} \leq \tilde{c}_{\mathsf{fp}}\, |\operatorname{grad} u|_{\mathsf{L}^2(\Omega)},
\end{aligned}$$

where the Friedrichs/Poincaré and Maxwell constants c_{fp}, c_{m}, $\tilde{c}_{\mathsf{fp}}$, are given by the respective Raleigh quotients, and $\mathsf{L}^2(\Omega)$ has to be replaced by $\mathsf{L}^2_\perp(\Omega)$ for $\Gamma_t = \emptyset$ resp. $\Gamma_t = \Gamma$. Again note that the latter two assertions are already included in the first two inequalities by interchanging Γ_t and Γ_n and setting $\varepsilon := \mathrm{id}$. Finally, the following Helmholtz decompositions hold:

$$\mathsf{H}_1 = \mathsf{L}^2(\Omega) = \operatorname{div} \mathsf{D}_{\Gamma_n}(\Omega) \oplus_{\mathsf{L}^2(\Omega)} \begin{cases} \{0\} & \text{, if } \Gamma_t \neq \emptyset,\\ \mathbb{R} & \text{, if } \Gamma_t = \emptyset,\end{cases}$$

$$\begin{aligned}\mathsf{H}_2 = \mathsf{L}^2_\varepsilon(\Omega) &= \operatorname{grad} \mathsf{H}^1_{\Gamma_t}(\Omega) \oplus_{\mathsf{L}^2_\varepsilon(\Omega)} \mu \mathsf{D}_{\Gamma_n,0}(\Omega)\\ &= \mathsf{R}_{\Gamma_t,0}(\Omega) \oplus_{\mathsf{L}^2_\varepsilon(\Omega)} \mu \operatorname{rot} \mathsf{R}_{\Gamma_n}(\Omega)\\ &= \operatorname{grad} \mathsf{H}^1_{\Gamma_t}(\Omega) \oplus_{\mathsf{L}^2_\varepsilon(\Omega)} \mathcal{H}_{\mathsf{t},\mathsf{n},\varepsilon}(\Omega) \oplus_{\mathsf{L}^2_\varepsilon(\Omega)} \mu \operatorname{rot} \mathsf{R}_{\Gamma_n}(\Omega),\end{aligned}$$

$$\begin{aligned}
\mathsf{H}_3 = \mathsf{L}^2(\Omega) &= \operatorname{grad} \mathsf{H}^1_{\Gamma_\mathsf{n}}(\Omega) \oplus_{\mathsf{L}^2(\Omega)} \mathsf{D}_{\Gamma_\mathsf{t},0}(\Omega)\\
&= \mathsf{R}_{\Gamma_\mathsf{n},0}(\Omega) \oplus_{\mathsf{L}^2(\Omega)} \operatorname{rot} \mathsf{R}_{\Gamma_\mathsf{t}}(\Omega)\\
&= \operatorname{grad} \mathsf{H}^1_{\Gamma_\mathsf{n}}(\Omega) \oplus_{\mathsf{L}^2(\Omega)} \mathcal{H}_{\mathsf{n},\mathsf{t}}(\Omega) \oplus_{\mathsf{L}^2(\Omega)} \operatorname{rot} \mathsf{R}_{\Gamma_\mathsf{t}}(\Omega), \qquad \mathcal{H}_{\mathsf{n},\mathsf{t}}(\Omega) = \mathsf{R}_{\Gamma_\mathsf{n},0}(\Omega) \cap \mathsf{D}_{\Gamma_\mathsf{t},0}(\Omega),\\
\mathsf{H}_4 = \mathsf{L}^2(\Omega) &= \operatorname{div} \mathsf{D}_{\Gamma_\mathsf{t}}(\Omega) \oplus_{\mathsf{L}^2(\Omega)} \begin{cases} \{0\} & \text{, if } \Gamma_\mathsf{t} \neq \Gamma,\\ \mathbb{R} & \text{, if } \Gamma_\mathsf{t} = \Gamma. \end{cases}
\end{aligned}$$

The latter two decompositions are already given by the first two ones by interchanging Γ_t and Γ_n and setting $\varepsilon := \mathrm{id}$. Especially, it holds

$$\operatorname{grad} \mathsf{H}^1_{\Gamma_\mathsf{t}}(\Omega) = \mathsf{R}_{\Gamma_\mathsf{t},0}(\Omega) \ominus_{\mathsf{L}^2_\varepsilon(\Omega)} \mathcal{H}_{\mathsf{t},\mathsf{n},\varepsilon}(\Omega), \qquad \mu \operatorname{rot} \mathsf{R}_{\Gamma_\mathsf{n}}(\Omega) = \mu\, \mathsf{D}_{\Gamma_\mathsf{n},0}(\Omega) \ominus_{\mathsf{L}^2_\varepsilon(\Omega)} \mathcal{H}_{\mathsf{t},\mathsf{n},\varepsilon}(\Omega),$$

$$\operatorname{grad} \mathsf{H}^1_{\Gamma_\mathsf{n}}(\Omega) = \mathsf{R}_{\Gamma_\mathsf{n},0}(\Omega) \ominus_{\mathsf{L}^2(\Omega)} \mathcal{H}_{\mathsf{n},\mathsf{t}}(\Omega), \qquad \operatorname{rot} \mathsf{R}_{\Gamma_\mathsf{t}}(\Omega) = \mathsf{D}_{\Gamma_\mathsf{t},0}(\Omega) \ominus_{\mathsf{L}^2(\Omega)} \mathcal{H}_{\mathsf{n},\mathsf{t}}(\Omega).$$

If $\Gamma_\mathsf{t} = \Gamma$ and Γ is connected, then the Dirichlet fields are trivial, i.e.,

$$\mathcal{H}_{\mathsf{t},\mathsf{n},\varepsilon}(\Omega) = \mathsf{R}_{\Gamma,0}(\Omega) \cap \mu\, \mathsf{D}_0(\Omega) = \{0\}.$$

If $\Gamma_\mathsf{t} = \emptyset$ and Ω is simply connected, then the Neumann fields are trivial, i.e.,

$$\mathcal{H}_{\mathsf{t},\mathsf{n},\varepsilon}(\Omega) = \mathsf{R}_0(\Omega) \cap \mu\, \mathsf{D}_{\Gamma,0}(\Omega) = \{0\}.$$

Now we can apply the general results of Theorem 3.3 and Corollary 4.6.

Theorem 5.1. (5.1) *resp.* (5.2) *is uniquely solvable, if and only if*

$$F \in \operatorname{rot} \mathsf{R}_{\Gamma_\mathsf{t}}(\Omega) = \mathsf{D}_{\Gamma_\mathsf{t},0}(\Omega) \ominus_{\mathsf{L}^2(\Omega)} \mathcal{H}_{\mathsf{n},\mathsf{t}}(\Omega), \quad g \in \mathsf{L}^2(\Omega), \quad H \in \mathcal{H}_{\mathsf{t},\mathsf{n},\varepsilon}(\Omega),$$

where $\mathsf{L}^2(\Omega)$ *has to be replaced by* $\mathsf{L}^2_\perp(\Omega)$ *if* $\Gamma_\mathsf{t} = \emptyset$. *The unique solution* $E \in \mathsf{R}_{\Gamma_\mathsf{t}}(\Omega) \cap \mu\, \mathsf{D}_{\Gamma_\mathsf{n}}(\Omega)$ *is given by*

$$\begin{aligned}
E := E_F + E_g + H &\in \big(\mathsf{R}_{\Gamma_\mathsf{t}}(\Omega) \cap \mu \operatorname{rot} \mathsf{R}_{\Gamma_\mathsf{n}}(\Omega)\big) \oplus_{\mathsf{L}^2_\varepsilon(\Omega)} \big(\mu\, \mathsf{D}_{\Gamma_\mathsf{n}}(\Omega) \cap \operatorname{grad} \mathsf{H}^1_{\Gamma_\mathsf{t}}(\Omega)\big) \oplus_{\mathsf{L}^2_\varepsilon(\Omega)} \mathcal{H}_{\mathsf{t},\mathsf{n},\varepsilon}(\Omega)\\
&= \mathsf{R}_{\Gamma_\mathsf{t}}(\Omega) \cap \mu\, \mathsf{D}_{\Gamma_\mathsf{n}}(\Omega),\\
E_F := (\widetilde{\operatorname{rot}}_{\Gamma_\mathsf{t}})^{-1} F &\in \mathsf{R}_{\Gamma_\mathsf{t}}(\Omega) \cap \mu \operatorname{rot} \mathsf{R}_{\Gamma_\mathsf{n}}(\Omega) = \mathsf{R}_{\Gamma_\mathsf{t}}(\Omega) \cap \mu\, \mathsf{D}_{\Gamma_\mathsf{n},0}(\Omega) \cap \mathcal{H}_{\mathsf{t},\mathsf{n},\varepsilon}(\Omega)^{\perp_{\mathsf{L}^2_\varepsilon(\Omega)}},\\
E_g := -(\widetilde{\operatorname{div}}_{\Gamma_\mathsf{n}} \varepsilon)^{-1} g &\in \mu\, \mathsf{D}_{\Gamma_\mathsf{n}}(\Omega) \cap \operatorname{grad} \mathsf{H}^1_{\Gamma_\mathsf{t}}(\Omega) = \mu\, \mathsf{D}_{\Gamma_\mathsf{n}}(\Omega) \cap \mathsf{R}_{\Gamma_\mathsf{t},0}(\Omega) \cap \mathcal{H}_{\mathsf{t},\mathsf{n},\varepsilon}(\Omega)^{\perp_{\mathsf{L}^2_\varepsilon(\Omega)}}
\end{aligned}$$

and depends continuously on the data, i.e., $|E|_{\mathsf{L}^2_\varepsilon(\Omega)} \leq c_\mathsf{m} |F|_{\mathsf{L}^2(\Omega)} + c_\mathsf{fp} |g|_{\mathsf{L}^2(\Omega)} + |H|_{\mathsf{L}^2(\Omega)}$, *as*

$$|E_F|_{\mathsf{L}^2_\varepsilon(\Omega)} \leq c_\mathsf{m} |F|_{\mathsf{L}^2(\Omega)}, \qquad |E_g|_{\mathsf{L}^2_\varepsilon(\Omega)} \leq c_\mathsf{fp} |g|_{\mathsf{L}^2(\Omega)}.$$

Moreover, $|E|^2_{\mathsf{L}^2_\varepsilon(\Omega)} = |E_F|^2_{\mathsf{L}^2_\varepsilon(\Omega)} + |E_g|^2_{\mathsf{L}^2_\varepsilon(\Omega)} + |H|^2_{\mathsf{L}^2_\varepsilon(\Omega)}$.

The partial solutions E_F and E_g, solving

$$\begin{aligned}
\operatorname{rot}_{\Gamma_\mathsf{t}} E_F &= F, & \operatorname{rot}_{\Gamma_\mathsf{t}} E_g &= 0,\\
-\operatorname{div}_{\Gamma_\mathsf{n}} \varepsilon E_F &= 0, & -\operatorname{div}_{\Gamma_\mathsf{n}} \varepsilon E_g &= g,\\
\pi_\mathcal{H} E_F &= 0, & \pi_\mathcal{H} E_g &= 0,
\end{aligned}$$

can be found and computed by the following two variational formulations: There exist unique potentials $U_F \in \mathsf{R}_{\Gamma_\mathsf{n}}(\Omega) \cap \operatorname{rot} \mathsf{R}_{\Gamma_\mathsf{t}}(\Omega)$ and $u_g \in \mathsf{H}^1_{\Gamma_\mathsf{t}}(\Omega)$, where $\mathsf{H}^1_{\Gamma_\mathsf{t}}(\Omega)$ has to be replaced by $\mathsf{H}^1(\Omega) \cap \mathsf{L}^2_\perp(\Omega)$, if $\Gamma_\mathsf{t} = \emptyset$, such that

(5.3) $$\forall\, \Phi \in \mathsf{R}_{\Gamma_\mathsf{n}}(\Omega) \qquad \langle \operatorname{rot} U_F, \operatorname{rot} \Phi \rangle_{\mathsf{L}^2_\mu(\Omega)} = \langle F, \Phi \rangle_{\mathsf{L}^2(\Omega)},$$

(5.4) $$\forall\, \varphi \in \mathsf{H}^1_{\Gamma_\mathsf{t}}(\Omega) \qquad \langle \operatorname{grad} u_g, \operatorname{grad} \varphi \rangle_{\mathsf{L}^2_\varepsilon(\Omega)} = \langle g, \varphi \rangle_{\mathsf{L}^2(\Omega)}.$$

It holds

$$\mu \operatorname{rot} U_F = E_F, \qquad \operatorname{grad} u_g = E_g.$$

Moreover, the variational formulation (5.3) is equivalent to the following saddle point problem: Find $U_F \in \mathsf{R}_{\Gamma_\mathsf{n}}(\Omega)$, such that

$$\forall\,\Phi \in \mathsf{R}_{\Gamma_\mathsf{n}}(\Omega) \quad \langle \operatorname{rot} U_F, \operatorname{rot}\Phi\rangle_{\mathsf{L}^2_\mu(\Omega)} = \langle F,\Phi\rangle_{\mathsf{L}^2(\Omega)} \quad \wedge \quad \forall\,\Psi \in \mathsf{R}_{\Gamma_\mathsf{n},0}(\Omega) \quad \langle U_F,\Psi\rangle_{\mathsf{L}^2(\Omega)} = 0. \tag{5.5}$$

As $\mathsf{R}_{\Gamma_\mathsf{n},0}(\Omega) = \operatorname{grad}\mathsf{H}^1_{\Gamma_\mathsf{n}}(\Omega) \oplus_{\mathsf{L}^2(\Omega)} \mathcal{H}_{\mathsf{n},\mathsf{t}}(\Omega)$ we may specify: In the special case

$$\mathcal{H}_{\mathsf{n},\mathsf{t}}(\Omega) = \{0\},$$

the saddle point problem (5.5) is equivalent to: Find $U_F \in \mathsf{R}_{\Gamma_\mathsf{n}}(\Omega)$, such that

$$\forall\,\Phi \in \mathsf{R}_{\Gamma_\mathsf{n}}(\Omega) \quad \langle \operatorname{rot} U_F, \operatorname{rot}\Phi\rangle_{\mathsf{L}^2_\mu(\Omega)} = \langle F,\Phi\rangle_{\mathsf{L}^2(\Omega)} \quad \wedge \quad \forall\,\psi \in \mathsf{H}^1_{\Gamma_\mathsf{n}}(\Omega) \quad \langle U_F,\operatorname{grad}\psi\rangle_{\mathsf{L}^2(\Omega)} = 0. \tag{5.6}$$

Following the procedure leading to (3.11)-(3.12) we observe that (5.6) is equivalent to the following saddle point formulation: Find $(U_F, u_F) \in \mathsf{R}_{\Gamma_\mathsf{n}}(\Omega) \times \mathsf{H}^1_{\Gamma_\mathsf{n}}(\Omega)$, such that for all $(\Phi,\psi) \in \mathsf{R}_{\Gamma_\mathsf{n}}(\Omega) \times \mathsf{H}^1_{\Gamma_\mathsf{n}}(\Omega)$

$$\langle \operatorname{rot} U_F, \operatorname{rot}\Phi\rangle_{\mathsf{L}^2_\mu(\Omega)} + \langle \Phi, \operatorname{grad} u_F\rangle_{\mathsf{L}^2(\Omega)} = \langle F,\Phi\rangle_{\mathsf{L}^2(\Omega)} \qquad \wedge \qquad \langle U_F, \operatorname{grad}\psi\rangle_{\mathsf{L}^2(\Omega)} = 0, \tag{5.7}$$

where $\mathsf{H}^1_{\Gamma_\mathsf{n}}(\Omega)$ has to be replaced by $\mathsf{H}^1(\Omega) \cap \mathsf{L}^2_\perp(\Omega)$, if $\Gamma_\mathsf{t} = \Gamma$. Every solution of (5.7) satisfies $u_F = 0$ and the inf-sup-condition reads

$$\inf_{0\neq\psi\in\mathsf{H}^1_{\Gamma_\mathsf{n}}(\Omega)} \sup_{0\neq\Phi\in\mathsf{R}_{\Gamma_\mathsf{n}}(\Omega)} \frac{\langle\Phi,\operatorname{grad}\psi\rangle_{\mathsf{L}^2(\Omega)}}{|\Phi|_{\mathsf{R}_{\Gamma_\mathsf{n}}(\Omega)}|\psi|_{\mathsf{H}^1_{\Gamma_\mathsf{n}}(\Omega)}} \geq \inf_{0\neq\psi\in\mathsf{H}^1_{\Gamma_\mathsf{n}}(\Omega)} \frac{|\operatorname{grad}\psi|_{\mathsf{L}^2(\Omega)}}{|\psi|_{\mathsf{H}^1_{\Gamma_\mathsf{n}}(\Omega)}} = (\tilde{c}^2_{\mathsf{fp}}+1)^{-1/2}.$$

Note that in $\mathsf{H}^1_{\Gamma_\mathsf{n}}(\Omega)$ resp. $\mathsf{H}^1(\Omega) \cap \mathsf{L}^2_\perp(\Omega)$ we can also use the H^1-half norm $|\cdot|_{\mathsf{H}^1_{\Gamma_\mathsf{n}}(\Omega)} = |\operatorname{grad}\cdot|_{\mathsf{L}^2(\Omega)}$ yielding

$$\inf_{0\neq\psi\in\mathsf{H}^1_{\Gamma_\mathsf{n}}(\Omega)} \sup_{0\neq\Phi\in\mathsf{R}_{\Gamma_\mathsf{n}}(\Omega)} \frac{\langle\Phi,\operatorname{grad}\psi\rangle_{\mathsf{L}^2(\Omega)}}{|\Phi|_{\mathsf{R}_{\Gamma_\mathsf{n}}(\Omega)}|\psi|_{\mathsf{H}^1_{\Gamma_\mathsf{n}}(\Omega)}} \geq 1.$$

Remark 5.2. *We emphasize that in* [5]*, see also* [4, 6]*, the following has been proved: If $\Gamma_\mathsf{t} = \emptyset$ or $\Gamma_\mathsf{t} = \Gamma$, and Ω is convex, then*

$$c_\mathsf{m} \leq \overline{\varepsilon}\, c_\mathsf{p} \leq \overline{\varepsilon}\,\frac{\operatorname{diam}\Omega}{\pi},$$

where the Poincaré constant c_p and $\overline{\varepsilon}$ are given by

$$\frac{1}{c_\mathsf{p}} := \inf_{0\neq\varphi\in\mathsf{H}^1(\Omega)\cap\mathsf{L}^2_\perp(\Omega)} \frac{|\operatorname{grad}\varphi|_{\mathsf{L}^2(\Omega)}}{|\varphi|_{\mathsf{L}^2(\Omega)}}, \qquad \frac{1}{\overline{\varepsilon}} := \inf_{0\neq\Phi\in\mathsf{L}^2(\Omega)} \frac{|\Phi|_{\mathsf{L}^2(\Omega)}}{|\Phi|_{\mathsf{L}^2_\varepsilon(\Omega)}}.$$

Moreover, for $\Gamma_\mathsf{t} = \emptyset$ and convex Ω we have

$$\frac{1}{\overline{\varepsilon}} c_\mathsf{p} \leq c_\mathsf{fp} \leq \underline{\varepsilon} c_\mathsf{p}, \qquad \tilde{c}_\mathsf{fp} = c_\mathsf{f} < c_\mathsf{p},$$

where the Friedrichs constant c_f and $\underline{\varepsilon}$ are given by

$$\frac{1}{c_\mathsf{f}} := \inf_{0\neq\varphi\in\mathsf{H}^1_\Gamma(\Omega)} \frac{|\operatorname{grad}\varphi|_{\mathsf{L}^2(\Omega)}}{|\varphi|_{\mathsf{L}^2(\Omega)}}, \qquad \frac{1}{\underline{\varepsilon}} := \inf_{0\neq\Phi\in\mathsf{L}^2(\Omega)} \frac{|\Phi|_{\mathsf{L}^2_\varepsilon(\Omega)}}{|\Phi|_{\mathsf{L}^2(\Omega)}}.$$

For $\Gamma_\mathsf{t} = \Gamma$ and convex Ω it holds

$$\frac{1}{\overline{\varepsilon}} c_\mathsf{f} \leq c_\mathsf{fp} \leq \underline{\varepsilon} c_\mathsf{f}, \qquad c_\mathsf{f} < c_\mathsf{p} = \tilde{c}_\mathsf{fp}.$$

We can apply the main functional a posteriori error estimate Corollary 4.6 to (5.1) resp. (5.2).

Theorem 5.3. *Let $E \in \mathsf{R}_{\Gamma_\mathsf{t}}(\Omega) \cap \mu\,\mathsf{D}_{\Gamma_\mathsf{n}}(\Omega)$ be the exact solution of* (5.1) *resp.* (5.2) *and $\tilde{E} \in \mathsf{L}^2_\varepsilon(\Omega)$. Then the following estimates hold for the error $e = E - \tilde{E}$ defined in* (4.1)*:*

- **(i)** *The error decomposes, i.e., $e = e_{\mathrm{grad}} + e_{\mathcal{H}} + e_{\mathrm{rot}} \in \operatorname{grad}\mathsf{H}^1_{\Gamma_\mathsf{t}}(\Omega) \oplus_{\mathsf{L}^2_\varepsilon(\Omega)} \mathcal{H}_{\mathsf{t},\mathsf{n},\varepsilon}(\Omega) \oplus_{\mathsf{L}^2_\varepsilon(\Omega)} \mu\operatorname{rot}\mathsf{R}_{\Gamma_\mathsf{n}}(\Omega)$ and*

$$|e|^2_{\mathsf{L}^2_\varepsilon(\Omega)} = |e_{\mathrm{grad}}|^2_{\mathsf{L}^2_\varepsilon(\Omega)} + |e_{\mathcal{H}}|^2_{\mathsf{L}^2_\varepsilon(\Omega)} + |e_{\mathrm{rot}}|^2_{\mathsf{L}^2_\varepsilon(\Omega)}.$$

(ii) *The projection* $e_{\mathrm{grad}} = \pi_{\mathrm{grad}} e = E_g - \pi_{\mathrm{grad}} \tilde{E} \in \operatorname{grad} \mathsf{H}^1_{\Gamma_{\mathsf{t}}}(\Omega)$ *satisfies*

$$|e_{\mathrm{grad}}|^2_{\mathsf{L}^2_\varepsilon(\Omega)} = \min_{\Phi \in \mu\, \mathsf{D}_{\Gamma_{\mathsf{n}}}(\Omega)} \big(c_{\mathsf{fp}} |\operatorname{div} \varepsilon\, \Phi + g|_{\mathsf{L}^2(\Omega)} + |\Phi - \tilde{E}|_{\mathsf{L}^2_\varepsilon(\Omega)} \big)^2$$
$$= \max_{\varphi \in \mathsf{H}^1_{\Gamma_{\mathsf{t}}}(\Omega)} \big(2\langle g, \varphi \rangle_{\mathsf{L}^2(\Omega)} - \langle 2\tilde{E} + \operatorname{grad} \varphi, \operatorname{grad} \varphi \rangle_{\mathsf{L}^2_\varepsilon(\Omega)} \big)$$

and the minimum resp. maximum is attained at

$$\hat{\Phi} := e_{\mathrm{grad}} + \tilde{E} \in \mu\, \mathsf{D}_{\Gamma_{\mathsf{n}}}(\Omega), \qquad \hat{\varphi} := (\widetilde{\operatorname{grad}}_{\Gamma_{\mathsf{t}}})^{-1} e_{\mathrm{grad}} \in \mathsf{H}^1_{\Gamma_{\mathsf{t}}}(\Omega)$$

with $-\operatorname{div} \varepsilon\, \hat{\Phi} = -\operatorname{div} \varepsilon\, E = g$*, where* $\mathsf{H}^1_{\Gamma_{\mathsf{t}}}(\Omega)$ *has to be replaced by* $\mathsf{H}^1(\Omega) \cap \mathsf{L}^2_\perp(\Omega)$*, if* $\Gamma_{\mathsf{t}} = \emptyset$*.*

(iii) *The projection* $e_{\mathrm{rot}} = \pi_{\mathrm{rot}} e = E_F - \pi_{\mathrm{rot}} \tilde{E} \in \mu \operatorname{rot} \mathsf{R}_{\Gamma_{\mathsf{n}}}(\Omega)$ *satisfies*

$$|e_{\mathrm{rot}}|^2_{\mathsf{L}^2_\varepsilon(\Omega)} = \min_{\Phi \in \mathsf{R}_{\Gamma_{\mathsf{t}}}(\Omega)} \big(c_{\mathsf{m}} |\operatorname{rot} \Phi - F|_{\mathsf{L}^2(\Omega)} + |\Phi - \tilde{E}|_{\mathsf{L}^2_\varepsilon(\Omega)} \big)^2$$
$$= \max_{\Psi \in \mathsf{R}_{\Gamma_{\mathsf{n}}}(\Omega)} \big(2\langle F, \Psi \rangle_{\mathsf{L}^2(\Omega)} - \langle 2\tilde{E} + \mu \operatorname{rot} \Psi, \mu \operatorname{rot} \Psi \rangle_{\mathsf{L}^2_\varepsilon(\Omega)} \big)$$

and the minimum resp. maximum is attained at

$$\hat{\Phi} := e_{\mathrm{rot}} + \tilde{E} \in \mathsf{R}_{\Gamma_{\mathsf{t}}}(\Omega), \qquad \hat{\Psi} := (\mu \widetilde{\operatorname{rot}}_{\Gamma_{\mathsf{n}}})^{-1} e_{\mathrm{rot}} \in \mathsf{R}_{\Gamma_{\mathsf{n}}}(\Omega) \cap \operatorname{rot} \mathsf{R}_{\Gamma_{\mathsf{t}}}(\Omega)$$

with $\operatorname{rot} \hat{\Phi} = \operatorname{rot} E = F$*.*

(iv) *The projection* $e_{\mathcal{H}} = \pi_{\mathcal{H}} e = H - \pi_{\mathcal{H}} \tilde{E} \in \mathcal{H}_{\mathsf{t},\mathsf{n},\varepsilon}(\Omega)$ *satisfies*

$$|e_{\mathcal{H}}|^2_{\mathsf{L}^2_\varepsilon(\Omega)} = \min_{\varphi \in \mathsf{H}^1_{\Gamma_{\mathsf{t}}}(\Omega)} \min_{\Phi \in \mathsf{R}_{\Gamma_{\mathsf{n}}}(\Omega)} |H - \tilde{E} + \operatorname{grad} \varphi + \mu \operatorname{rot} \Phi|^2_{\mathsf{L}^2_\varepsilon(\Omega)}$$
$$= \max_{\Psi \in \mathcal{H}_{\mathsf{t},\mathsf{n},\varepsilon}(\Omega)} \big\langle 2(H - \tilde{E}) - \Psi, \Psi \big\rangle_{\mathsf{L}^2_\varepsilon(\Omega)}$$

and the minimum resp. maximum is attained at

$$\hat{\varphi} := (\widetilde{\operatorname{grad}}_{\Gamma_{\mathsf{t}}})^{-1} \pi_{\mathrm{grad}} \tilde{E} \in \mathsf{H}^1_{\Gamma_{\mathsf{t}}}(\Omega), \qquad \hat{\Phi} := (\mu \widetilde{\operatorname{rot}}_{\Gamma_{\mathsf{n}}})^{-1} \pi_{\mathrm{rot}} \tilde{E} \in \mathsf{R}_{\Gamma_{\mathsf{n}}}(\Omega) \cap \operatorname{rot} \mathsf{R}_{\Gamma_{\mathsf{t}}}(\Omega)$$

resp. $\hat{\Psi} := e_{\mathcal{H}} \in \mathcal{H}_{\mathsf{t},\mathsf{n},\varepsilon}(\Omega)$ *with* $\operatorname{grad} \hat{\varphi} + \mu \operatorname{rot} \hat{\phi} = (\pi_{\mathrm{grad}} + \pi_{\mathrm{rot}}) \tilde{E} = (1 - \pi_{\mathcal{H}}) \tilde{E}$*, where* $\mathsf{H}^1_{\Gamma_{\mathsf{t}}}(\Omega)$ *has to be replaced by* $\mathsf{H}^1(\Omega) \cap \mathsf{L}^2_\perp(\Omega)$*, if* $\Gamma_{\mathsf{t}} = \emptyset$*.*

If $\tilde{E} := H + \tilde{E}_\perp$ *with some* $\tilde{E}_\perp \in \mathcal{H}_{\mathsf{t},\mathsf{n},\varepsilon}(\Omega)^{\perp_{\mathsf{L}^2_\varepsilon(\Omega)}}$*, then* $e_{\mathcal{H}} = 0$*, and in (ii) and (iii)* $\tilde{E}$ *can be replaced by* $\tilde{E}_\perp$*. In this case, for the attaining minima it holds*

$$\hat{\Phi}_\perp := e_{\mathrm{grad}} + \tilde{E}_\perp \in \mu\, \mathsf{D}_{\Gamma_{\mathsf{n}}}(\Omega), \qquad \hat{\Phi}_\perp := e_{\mathrm{rot}} + \tilde{E}_\perp \in \mathsf{R}_{\Gamma_{\mathsf{t}}}(\Omega).$$

Remark 5.4. *For conforming approximations Corollary 4.2 and Remark 4.3 yield the following:*

(i) *If* $\tilde{E} \in \mu\, \mathsf{D}_{\Gamma_{\mathsf{n}}}(\Omega)$*, then* $e \in \mu\, \mathsf{D}_{\Gamma_{\mathsf{n}}}(\Omega)$ *and*

$$|e_{\mathrm{grad}}|_{\mathsf{L}^2_\varepsilon(\Omega)} \le c_{\mathsf{fp}} |\operatorname{div} \varepsilon\, \tilde{E} + g|_{\mathsf{L}^2(\Omega)} = c_{\mathsf{fp}} |\operatorname{div} \varepsilon\, e|_{\mathsf{L}^2(\Omega)}.$$

(ii) *If* $\tilde{E} \in \mathsf{R}_{\Gamma_{\mathsf{t}}}(\Omega)$*, then* $e \in \mathsf{R}_{\Gamma_{\mathsf{t}}}(\Omega)$ *and*

$$|e_{\mathrm{rot}}|_{\mathsf{L}^2_\varepsilon(\Omega)} \le c_{\mathsf{m}} |\operatorname{rot} \tilde{E} - F|_{\mathsf{L}^2(\Omega)} = c_{\mathsf{m}} |\operatorname{rot} e|_{\mathsf{L}^2(\Omega)}.$$

(iii) *If* $\tilde{E} \in \mathsf{R}_{\Gamma_{\mathsf{t}}}(\Omega) \cap \mu\, \mathsf{D}_{\Gamma_{\mathsf{n}}}(\Omega)$*, then* $e \in \mathsf{R}_{\Gamma_{\mathsf{t}}}(\Omega) \cap \mu\, \mathsf{D}_{\Gamma_{\mathsf{n}}}(\Omega)$ *and this very conforming error is equivalent to the weighted least squares functional*

$$\mathcal{F}(\tilde{E}) := |H - \pi_{\mathcal{H}} \tilde{E}|^2_{\mathsf{L}^2_\varepsilon(\Omega)} + (1 + c_{\mathsf{m}}^2) |\operatorname{rot} \tilde{E} - F|^2_{\mathsf{L}^2(\Omega)} + (1 + c_{\mathsf{fp}}^2) |\operatorname{div} \varepsilon\, \tilde{E} + g|^2_{\mathsf{L}^2(\Omega)},$$

i.e., $|e|^2_{\mathsf{R}_{\Gamma_{\mathsf{t}}}(\Omega) \cap \mu\, \mathsf{D}_{\Gamma_{\mathsf{n}}}(\Omega)} \le \mathcal{F}(\tilde{E}) \le (1 + \max\{c_{\mathsf{fp}}, c_{\mathsf{m}}\}^2) |e|^2_{\mathsf{R}_{\Gamma_{\mathsf{t}}}(\Omega) \cap \mu\, \mathsf{D}_{\Gamma_{\mathsf{n}}}(\Omega)}$*.*

5.2. **Prototype Second Order Systems: Laplacian and** rot rot**.** As prototypical examples for second order systems we will discuss the Laplacian and the rot rot-system, both with mixed boundary conditions. Suppose the assumptions of Section 5.1 are valid and recall the notations. For simplicity and to avoid case studies we assume $\emptyset \neq \Gamma_{\mathsf{t}} \neq \Gamma$.

5.2.1. *The Laplacian.* Suppose $g \in \mathsf{L}^2(\Omega)$. Let us consider the linear second order equation (in classical strong formulation) of the perturbed negative Laplacian with mixed boundary conditions for a function $u : \Omega \to \mathbb{R}$

$$-\operatorname{div} \varepsilon \operatorname{grad} u = g \text{ in } \Omega, \qquad u = 0 \text{ at } \Gamma_{\mathsf{t}}, \qquad n \cdot \varepsilon \operatorname{grad} u = 0 \text{ at } \Gamma_{\mathsf{n}}. \tag{5.8}$$

The corresponding variational formulation, which is uniquely solvable by Lax-Milgram's lemma, is the following: Find $u \in \mathsf{H}^1_{\Gamma_{\mathsf{t}}}(\Omega)$, such that

$$\forall\, \varphi \in \mathsf{H}^1_{\Gamma_{\mathsf{t}}}(\Omega) \qquad \langle \operatorname{grad} u, \operatorname{grad} \varphi \rangle_{\mathsf{L}^2_\varepsilon(\Omega)} = \langle g, \varphi \rangle_{\mathsf{L}^2(\Omega)}.$$

Then, by definition and the results of [1], we get $\varepsilon \operatorname{grad} u \in \mathsf{D}_{\Gamma_{\mathsf{n}}}(\Omega)$ with $-\operatorname{div} \varepsilon \operatorname{grad} u = g$. Hence, by setting

$$E := \operatorname{grad} u \in \mu\, \mathsf{D}_{\Gamma_{\mathsf{n}}}(\Omega) \cap \operatorname{grad} \mathsf{H}^1_{\Gamma_{\mathsf{t}}}(\Omega) = \mu\, \mathsf{D}_{\Gamma_{\mathsf{n}}}(\Omega) \cap \mathsf{R}_{\Gamma_{\mathsf{t}},0}(\Omega) \cap \mathcal{H}_{\mathsf{t},\mathsf{n},\varepsilon}(\Omega)^{\perp_{\mathsf{L}^2_\varepsilon(\Omega)}}$$

we see that the pair (u, E) solves the linear first order system (in classical strong formulation) of electro-magneto statics type with mixed boundary conditions

$$\begin{aligned} \operatorname{grad} u = E, \qquad \operatorname{rot} E &= 0 \quad \text{in } \Omega, & u = 0, \quad n \times E &= 0 \quad \text{at } \Gamma_{\mathsf{t}}, \\ -\operatorname{div} \varepsilon E &= g \quad \text{in } \Omega, & n \cdot \varepsilon E &= 0 \quad \text{at } \Gamma_{\mathsf{n}}, \\ \pi_{\mathcal{H}} E &= 0 \quad \text{in } \Omega. \end{aligned} \tag{5.9}$$

Similar to the latter subsection we define the operators A_1, A_2, A_3 and also A_0, A_4 together with the respective adjoints and reduced operators by the complexes

$$\{0\} \xrightarrow{\mathrm{A}_0 = 0} \mathsf{H}^1_{\Gamma_{\mathsf{t}}}(\Omega) \xrightarrow{\mathrm{A}_1 = \operatorname{grad}_{\Gamma_{\mathsf{t}}}} \mathsf{R}_{\Gamma_{\mathsf{t}}}(\Omega) \xrightarrow{\mathrm{A}_2 = \operatorname{rot}_{\Gamma_{\mathsf{t}}}} \mathsf{D}_{\Gamma_{\mathsf{t}}}(\Omega) \xrightarrow{\mathrm{A}_3 = \operatorname{div}_{\Gamma_{\mathsf{t}}}} \mathsf{L}^2(\Omega) \xrightarrow{\mathrm{A}_4 = 0} \{0\},$$

$$\{0\} \xleftarrow{\mathrm{A}_0^* = 0} \mathsf{L}^2(\Omega) \xleftarrow{\mathrm{A}_1^* = -\operatorname{div}_{\Gamma_{\mathsf{n}}} \varepsilon} \mu\, \mathsf{D}_{\Gamma_{\mathsf{n}}}(\Omega) \xleftarrow{\mathrm{A}_2^* = \mu \operatorname{rot}_{\Gamma_{\mathsf{n}}}} \mathsf{R}_{\Gamma_{\mathsf{n}}}(\Omega) \xleftarrow{\mathrm{A}_3^* = -\operatorname{grad}_{\Gamma_{\mathsf{n}}}} \mathsf{H}^1_{\Gamma_{\mathsf{n}}}(\Omega) \xleftarrow{\mathrm{A}_4^* = 0} \{0\}.$$

As before, all basic Hilbert spaces are $\mathsf{L}^2(\Omega)$ except of $\mathsf{H}_2 = \mathsf{L}^2_\varepsilon(\Omega)$. Then (5.8) turns to

$$\begin{aligned} \mathrm{A}_1^* \mathrm{A}_1 u &= g, \\ \mathrm{A}_0^* u = 0\, u &= 0, \\ \pi_1 u = \pi_{\{0\}} u &= 0 \end{aligned}$$

and this system is (again) uniquely solvable by Theorem 3.6 as $g \in \mathsf{L}^2(\Omega) = R(\mathrm{A}_1^*)$ with solution u depending continuously on the data. (5.9) reads

$$\begin{aligned} \mathrm{A}_1 u = \operatorname{grad}_{\Gamma_{\mathsf{t}}} u &= E, & \mathrm{A}_2 E = \operatorname{rot}_{\Gamma_{\mathsf{t}}} E &= 0, \\ \mathrm{A}_0^* u = 0\, u &= 0, & \mathrm{A}_1^* E = -\operatorname{div}_{\Gamma_{\mathsf{n}}} \varepsilon E &= g, \\ \pi_1 u = \pi_{\{0\}} u &= 0, & \pi_2 E = \pi_{\mathcal{H}} E &= 0. \end{aligned}$$

We can apply the main functional a posteriori error estimates from Theorem 4.7.

Theorem 5.5. *Let $u \in \mathsf{H}^1_{\Gamma_{\mathsf{t}}}(\Omega)$ be the exact solution of (5.8), $E := \operatorname{grad} u$, and $(\tilde{u}, \tilde{E}) \in \mathsf{L}^2(\Omega) \times \mathsf{L}^2_\varepsilon(\Omega)$. Then the following estimates hold for the errors $e_u := u - \tilde{u}$ and $e_E := E - \tilde{E}$:*

(i) *The error e_E decomposes, i.e.,*

$$e_E = e_{E,\mathrm{grad}} + e_{E,\mathcal{H}} + e_{E,\mathrm{rot}} \in \operatorname{grad} \mathsf{H}^1_{\Gamma_{\mathsf{t}}}(\Omega) \oplus_{\mathsf{L}^2_\varepsilon(\Omega)} \mathcal{H}_{\mathsf{t},\mathsf{n},\varepsilon}(\Omega) \oplus_{\mathsf{L}^2_\varepsilon(\Omega)} \mu \operatorname{rot} \mathsf{R}_{\Gamma_{\mathsf{n}}}(\Omega)$$

and

$$|e_E|^2_{\mathsf{L}^2_\varepsilon(\Omega)} = |e_{E,\mathrm{grad}}|^2_{\mathsf{L}^2_\varepsilon(\Omega)} + |e_{E,\mathcal{H}}|^2_{\mathsf{L}^2_\varepsilon(\Omega)} + |e_{E,\mathrm{rot}}|^2_{\mathsf{L}^2_\varepsilon(\Omega)}.$$

(ii) $e_u=\pi_{\operatorname{div}}e_u\in\operatorname{div}\mathsf{D}_{\Gamma_\mathsf{n}}(\Omega)=\mathsf{L}^2(\Omega)$ *and*

$$\begin{aligned}|e_u|^2_{\mathsf{L}^2(\Omega)}&=\min_{\varphi\in\mathsf{H}^1_{\Gamma_\mathsf{t}}(\Omega)}\min_{\Phi\in\mu\,\mathsf{D}_{\Gamma_\mathsf{n}}(\Omega)}\big(c_{\mathsf{fp}}^2|\operatorname{div}\varepsilon\,\Phi+g|_{\mathsf{L}^2(\Omega)}+c_{\mathsf{fp}}|\Phi-\operatorname{grad}\varphi|_{\mathsf{L}^2_\varepsilon(\Omega)}+|\varphi-\tilde u|_{\mathsf{L}^2(\Omega)}\big)^2\\&=\min_{\substack{\varphi\in\mathsf{H}^1_{\Gamma_\mathsf{t}}(\Omega),\\\operatorname{grad}\varphi\in\mu\,\mathsf{D}_{\Gamma_\mathsf{n}}(\Omega)}}\big(c_{\mathsf{fp}}^2|\operatorname{div}\varepsilon\operatorname{grad}\varphi+g|_{\mathsf{L}^2(\Omega)}+|\varphi-\tilde u|_{\mathsf{L}^2(\Omega)}\big)^2\\&=\max_{\substack{\phi\in\mathsf{H}^1_{\Gamma_\mathsf{t}}(\Omega),\\\operatorname{grad}\phi\in\mu\,\mathsf{D}_{\Gamma_\mathsf{n}}(\Omega)}}\big(2\langle g,\phi\rangle_{\mathsf{L}^2(\Omega)}+\langle 2\tilde u-\operatorname{div}\varepsilon\operatorname{grad}\phi,\operatorname{div}\varepsilon\operatorname{grad}\phi\rangle_{\mathsf{L}^2(\Omega)}\big)\end{aligned}$$

and the minima resp. maximum are attained at

$$\hat\varphi:=e_u+\tilde u\in\mathsf{H}^1_{\Gamma_\mathsf{t}}(\Omega),\qquad\hat\Phi:=E\in\mu\,\mathsf{D}_{\Gamma_\mathsf{n}}(\Omega),\qquad\hat\phi:=(\widetilde{\operatorname{grad}}_{\Gamma_\mathsf{t}})^{-1}(-\widetilde{\operatorname{div}}_{\Gamma_\mathsf{n}}\varepsilon)^{-1}\in\mathsf{H}^1_{\Gamma_\mathsf{t}}(\Omega)$$

with $\operatorname{grad}\hat\varphi,\operatorname{grad}\hat\phi\in\mu\,\mathsf{D}_{\Gamma_\mathsf{n}}(\Omega)$ *and* $\operatorname{grad}\hat\varphi=\operatorname{grad}u=E$ *and* $-\operatorname{div}\varepsilon\operatorname{grad}\hat\varphi=-\operatorname{div}\varepsilon E=g$ *as well as* $-\operatorname{div}\varepsilon\,\hat\Phi=-\operatorname{div}\varepsilon\,E=g$.

(iii) *The projection* $e_{E,\mathrm{grad}}=\pi_{\mathrm{grad}}e_E=E-\pi_{\mathrm{grad}}\tilde E\in\operatorname{grad}\mathsf{H}^1_{\Gamma_\mathsf{t}}(\Omega)$ *satisfies*

$$\begin{aligned}|e_{E,\mathrm{grad}}|^2_{\mathsf{L}^2_\varepsilon(\Omega)}&=\min_{\Phi\in\mu\,\mathsf{D}_{\Gamma_\mathsf{n}}(\Omega)}\big(c_{\mathsf{fp}}|\operatorname{div}\varepsilon\,\Phi+g|_{\mathsf{L}^2(\Omega)}+|\Phi-\tilde E|_{\mathsf{L}^2_\varepsilon(\Omega)}\big)^2\\&=\max_{\varphi\in\mathsf{H}^1_{\Gamma_\mathsf{t}}(\Omega)}\big(2\langle g,\varphi\rangle_{\mathsf{L}^2(\Omega)}-\langle 2\tilde E+\operatorname{grad}\varphi,\operatorname{grad}\varphi\rangle_{\mathsf{L}^2_\varepsilon(\Omega)}\big)\end{aligned}$$

and the minimum resp. maximum is attained at

$$\hat\Phi:=e_{E,\mathrm{grad}}+\tilde E\in\mu\,\mathsf{D}_{\Gamma_\mathsf{n}}(\Omega),\qquad\hat\varphi:=(\widetilde{\operatorname{grad}}_{\Gamma_\mathsf{t}})^{-1}e_{E,\mathrm{grad}}\in\mathsf{H}^1_{\Gamma_\mathsf{t}}(\Omega)$$

with $-\operatorname{div}\varepsilon\,\hat\Phi=-\operatorname{div}\varepsilon\,E=g$.

(iv) *The projection* $e_{E,\mathrm{rot}}=\pi_{\mathrm{rot}}e_E=-\pi_{\mathrm{rot}}\tilde E\in\mu\operatorname{rot}\mathsf{R}_{\Gamma_\mathsf{n}}(\Omega)$ *satisfies*

$$\begin{aligned}|e_{E,\mathrm{rot}}|^2_{\mathsf{L}^2_\varepsilon(\Omega)}&=\min_{\Phi\in\mathsf{R}_{\Gamma_\mathsf{t}}(\Omega)}\big(c_{\mathsf{m}}|\operatorname{rot}\Phi|_{\mathsf{L}^2(\Omega)}+|\Phi-\tilde E|_{\mathsf{L}^2_\varepsilon(\Omega)}\big)^2=\min_{\Phi\in\mathsf{R}_{\Gamma_\mathsf{t},0}(\Omega)}|\Phi-\tilde E|^2_{\mathsf{L}^2_\varepsilon(\Omega)}\\&=\max_{\Psi\in\mathsf{R}_{\Gamma_\mathsf{n}}(\Omega)}\big(-\langle 2\tilde E+\mu\operatorname{rot}\Psi,\mu\operatorname{rot}\Psi\rangle_{\mathsf{L}^2_\varepsilon(\Omega)}\big)\end{aligned}$$

and the minimum resp. maximum is attained at

$$\hat\Phi:=e_{E,\mathrm{rot}}+\tilde E\in\mathsf{R}_{\Gamma_\mathsf{t},0}(\Omega),\qquad\hat\Psi:=(\mu\,\widetilde{\operatorname{rot}}_{\Gamma_\mathsf{n}})^{-1}e_{E,\mathrm{rot}}\in\mathsf{R}_{\Gamma_\mathsf{n}}(\Omega)\cap\operatorname{rot}\mathsf{R}_{\Gamma_\mathsf{t}}(\Omega)$$

with $\operatorname{rot}\hat\Phi=\operatorname{rot}E=0$.

(v) *The projection* $e_{E,\mathcal{H}}=\pi_{\mathcal{H}}e_E=-\pi_{\mathcal{H}}\tilde E\in\mathcal{H}_{\mathsf{t},\mathsf{n},\varepsilon}(\Omega)$ *satisfies*

$$\begin{aligned}|e_{E,\mathcal{H}}|^2_{\mathsf{L}^2_\varepsilon(\Omega)}&=\min_{\varphi\in\mathsf{H}^1_{\Gamma_\mathsf{t}}(\Omega)}\min_{\Phi\in\mathsf{R}_{\Gamma_\mathsf{n}}(\Omega)}|-\tilde E+\operatorname{grad}\varphi+\mu\operatorname{rot}\Phi|^2_{\mathsf{L}^2_\varepsilon(\Omega)}\\&=\max_{\Psi\in\mathcal{H}_{\mathsf{t},\mathsf{n},\varepsilon}(\Omega)}\big(-\big\langle 2\tilde E+\Psi,\Psi\big\rangle_{\mathsf{L}^2_\varepsilon(\Omega)}\big)\end{aligned}$$

and the minimum resp. maximum is attained at

$$\hat\varphi:=(\widetilde{\operatorname{grad}}_{\Gamma_\mathsf{t}})^{-1}\pi_{\mathrm{grad}}\tilde E\in\mathsf{H}^1_{\Gamma_\mathsf{t}}(\Omega),\qquad\hat\Phi:=(\mu\,\widetilde{\operatorname{rot}}_{\Gamma_\mathsf{n}})^{-1}\pi_{\mathrm{rot}}\tilde E\in\mathsf{R}_{\Gamma_\mathsf{n}}(\Omega)\cap\operatorname{rot}\mathsf{R}_{\Gamma_\mathsf{t}}(\Omega)$$

resp. $\hat\Psi:=e_{E,\mathcal{H}}\in\mathcal{H}_{\mathsf{t},\mathsf{n},\varepsilon}(\Omega)$ *with* $\operatorname{grad}\hat\varphi+\mu\operatorname{rot}\hat\phi=(\pi_{\mathrm{grad}}+\pi_{\mathrm{rot}})\tilde E=(1-\pi_{\mathcal{H}})\tilde E$.

If $\tilde E:=\tilde E_\perp$ *with some* $\tilde E_\perp\in\mathcal{H}_{\mathsf{t},\mathsf{n},\varepsilon}(\Omega)^{\perp_{\mathsf{L}^2_\varepsilon(\Omega)}}$, *then* $e_{E,\mathcal{H}}=0$, *and in (iii) and (iv)* $\tilde E$ *can be replaced by* $\tilde E_\perp$. *In this case, for the attaining minima it holds*

$$\hat\Phi_\perp:=e_{E,\mathrm{grad}}+\tilde E_\perp\in\mu\,\mathsf{D}_{\Gamma_\mathsf{n}}(\Omega),\qquad\hat\Phi_\perp:=e_{E,\mathrm{rot}}+\tilde E_\perp\in\mathsf{R}_{\Gamma_\mathsf{t},0}(\Omega).$$

For conforming approximations $\tilde{E} \in \operatorname{grad} \mathsf{H}^1_{\Gamma_t}(\Omega)$ we have $e_{E,\mathrm{rot}} = e_{E,\mathcal{H}} = 0$ and $e_E = e_{E,\mathrm{grad}}$. Especially, if $\tilde{u} \in \mathsf{H}^1_{\Gamma_t}(\Omega)$ and $\tilde{E} := \operatorname{grad} \tilde{u}$ with a conforming approximation $\tilde{u} \in \mathsf{H}^1_{\Gamma_t}(\Omega)$, the estimates of the latter theorem simplify. More precisely, (ii) turns to the following result: If $\tilde{u} \in \mathsf{H}^1_{\Gamma_t}(\Omega)$, then $e_u \in \mathsf{H}^1_{\Gamma_t}(\Omega)$ and we can choose, e.g., $\varphi := \tilde{u}$ yielding, e.g.,

$$|e_u|_{\mathsf{L}^2(\Omega)} \leq \min_{\Phi \in \mu \, \mathsf{D}_{\Gamma_n}(\Omega)} \big(c_{\mathsf{fp}}^2 |\operatorname{div} \varepsilon \, \Phi + g|_{\mathsf{L}^2(\Omega)} + c_{\mathsf{fp}} |\Phi - \operatorname{grad} \tilde{u}|_{\mathsf{L}^2_\varepsilon(\Omega)} \big),$$

which might not be sharp anymore. Similarly, the results of (iii) read as follows: If $\tilde{u}$ belongs to $\mathsf{H}^1_{\Gamma_t}(\Omega)$, then $\tilde{E} := \operatorname{grad} \tilde{u} \in \operatorname{grad} \mathsf{H}^1_{\Gamma_t}(\Omega)$ and $\operatorname{grad}(u - \tilde{u}) = e_E = e_{E,\mathrm{grad}} \in \operatorname{grad} \mathsf{H}^1_{\Gamma_t}(\Omega)$ as well as

$$\begin{aligned} |e_E|^2_{\mathsf{L}^2_\varepsilon(\Omega)} &= \min_{\Phi \in \mu \, \mathsf{D}_{\Gamma_n}(\Omega)} \big(c_{\mathsf{fp}} |\operatorname{div} \varepsilon \, \Phi + g|_{\mathsf{L}^2(\Omega)} + |\Phi - \operatorname{grad} \tilde{u}|_{\mathsf{L}^2_\varepsilon(\Omega)} \big)^2 \\ &= \max_{\varphi \in \mathsf{H}^1_{\Gamma_t}(\Omega)} \big(2\langle g, \varphi \rangle_{\mathsf{L}^2(\Omega)} - \langle \operatorname{grad}(2\tilde{u} + \varphi), \operatorname{grad} \varphi \rangle_{\mathsf{L}^2_\varepsilon(\Omega)} \big) \end{aligned} \tag{5.10}$$

and the minimum resp. maximum is attained at

$$\hat{\Phi} := e_E + \operatorname{grad} \tilde{u} = \operatorname{grad} u \in \mu \, \mathsf{D}_{\Gamma_n}(\Omega), \qquad \hat{\varphi} := (\widetilde{\operatorname{grad}}_{\Gamma_t})^{-1} e_E \in \mathsf{H}^1_{\Gamma_t}(\Omega)$$

with $-\operatorname{div} \varepsilon \, \hat{\Phi} = -\operatorname{div} \varepsilon \, E = g$. Note that (5.10) are the well known functional a posteriori error estimates for the energy norm associated to the Laplacian, see, e.g., [8].

5.2.2. *The* rot rot*-operator.* Suppose $F \in \operatorname{rot} \mathsf{R}_{\Gamma_t}(\Omega) = \mathsf{D}_{\Gamma_t,0}(\Omega) \cap \mathcal{H}_{\mathsf{n},\mathsf{t}}(\Omega)^{\perp_{\mathsf{L}^2(\Omega)}}$ and $g \in \mathsf{L}^2(\Omega)$ as well as $H \in \mathcal{H}_{\mathsf{n},\mathsf{t}}(\Omega)$. Let us consider the linear second order equation (in classical strong formulation) of the perturbed rot rot-operator with mixed boundary conditions for a vector field $B : \Omega \to \mathbb{R}^3$

$$\begin{aligned} \operatorname{rot} \mu \operatorname{rot} B &= F && \text{in } \Omega, && n \times B = 0 && && \text{at } \Gamma_{\mathsf{n}}, \\ \operatorname{div} \nu B &= g && \text{in } \Omega, && n \cdot \nu B = 0, && n \times \mu \operatorname{rot} B = 0 && \text{at } \Gamma_{\mathsf{t}}, \\ \pi_{\tilde{\mathcal{H}}} B &= H && \text{in } \Omega. \end{aligned} \tag{5.11}$$

Here $\pi_{\tilde{\mathcal{H}}} : \mathsf{L}^2(\Omega) \to \mathcal{H}_{\mathsf{n},\mathsf{t}}(\Omega)$ and for simplicity we set $\nu := \mathrm{id}$ for the matrix field ν. The partial solution B_g can be computed by solving a Laplace problem. The corresponding variational formulation, which is uniquely solvable by Lax-Milgram's lemma, to find the partial solution B_F of

$$\begin{aligned} \operatorname{rot} \mu \operatorname{rot} B_F &= F && \text{in } \Omega, && n \times B_F = 0 && && \text{at } \Gamma_{\mathsf{n}}, \\ \operatorname{div} B_F &= 0 && \text{in } \Omega, && n \cdot B_F = 0, && n \times \mu \operatorname{rot} B_F = 0 && \text{at } \Gamma_{\mathsf{t}}, \\ \pi_{\tilde{\mathcal{H}}} B_F &= 0 && \text{in } \Omega, \end{aligned}$$

is the following: Find $B_F \in \mathsf{R}_{\Gamma_n}(\Omega) \cap \operatorname{rot} \mathsf{R}_{\Gamma_t}(\Omega)$, such that[vii]

$$\forall \, \Phi \in \mathsf{R}_{\Gamma_n}(\Omega) \qquad \langle \operatorname{rot} B_F, \operatorname{rot} \Phi \rangle_{\mathsf{L}^2_\mu(\Omega)} = \langle F, \Phi \rangle_{\mathsf{L}^2(\Omega)}. \tag{5.12}$$

Then, by definition and the results of [1], we get $\mu \operatorname{rot} B_F \in \mathsf{R}_{\Gamma_t}(\Omega)$ with $\operatorname{rot} \mu \operatorname{rot} B_F = F$. Hence, by setting

$$E := \mu \operatorname{rot} B_F \in \mathsf{R}_{\Gamma_t}(\Omega) \cap \mu \operatorname{rot} \mathsf{R}_{\Gamma_n}(\Omega) = \mathsf{R}_{\Gamma_t}(\Omega) \cap \mu \, \mathsf{D}_{\Gamma_n,0}(\Omega) \cap \mathcal{H}_{\mathsf{t},\mathsf{n},\varepsilon}(\Omega)^{\perp_{\mathsf{L}^2_\varepsilon(\Omega)}}$$

we see that the pair (B, E) solves the linear first order system (in classical strong formulation) of electro-magneto statics type with mixed boundary conditions

$$\begin{aligned} \mu \operatorname{rot} B &= \mu \operatorname{rot} B_F = E, && \operatorname{rot} E = F && \text{in } \Omega, && n \times B = 0, && n \cdot \varepsilon E = 0 && \text{at } \Gamma_{\mathsf{n}}, \\ \operatorname{div} B &= g, && \operatorname{div} \varepsilon E = 0 && \text{in } \Omega, && n \cdot B = 0, && n \times E = 0 && \text{at } \Gamma_{\mathsf{t}}, \\ \pi_{\tilde{\mathcal{H}}} B &= H, && \pi_{\mathcal{H}} E = 0 && \text{in } \Omega. \end{aligned} \tag{5.13}$$

[vii]Note that (5.12) holds for all $\Phi \in \mathsf{R}_{\Gamma_n}(\Omega) \cap \operatorname{rot} \mathsf{R}_{\Gamma_t}(\Omega)$ if and only if it holds for all $\Phi \in \mathsf{R}_{\Gamma_n}(\Omega)$ since $F \in \operatorname{rot} \mathsf{R}_{\Gamma_t}(\Omega)$.

Let us define operators T_1, T_2, T_3 using A_1, A_2, A_3 together with the respective adjoints and reduced operators by the complexes

$$\{0\} \xrightarrow{0} \mathsf{H}^1_{\Gamma_\mathsf{t}}(\Omega) \xrightarrow{\mathrm{T}_3^*:=\mathrm{A}_1=\operatorname{grad}_{\Gamma_\mathsf{t}}} \mathsf{R}_{\Gamma_\mathsf{t}}(\Omega) \xrightarrow{\mathrm{T}_2^*:=\mathrm{A}_2=\operatorname{rot}_{\Gamma_\mathsf{t}}} \mathsf{D}_{\Gamma_\mathsf{t}}(\Omega) \xrightarrow{\mathrm{T}_1^*:=\mathrm{A}_3=\operatorname{div}_{\Gamma_\mathsf{t}}} \mathsf{L}^2(\Omega) \xrightarrow{0} \{0\},$$

$$\{0\} \xleftarrow{0} \mathsf{L}^2(\Omega) \xleftarrow{\mathrm{T}_3:=\mathrm{A}_1^*=-\operatorname{div}_{\Gamma_\mathsf{n}}\varepsilon} \mu\,\mathsf{D}_{\Gamma_\mathsf{n}}(\Omega) \xleftarrow{\mathrm{T}_2:=\mathrm{A}_2^*=\mu\operatorname{rot}_{\Gamma_\mathsf{n}}} \mathsf{R}_{\Gamma_\mathsf{n}}(\Omega) \xleftarrow{\mathrm{T}_1:=\mathrm{A}_3^*=-\operatorname{grad}_{\Gamma_\mathsf{n}}} \mathsf{H}^1_{\Gamma_\mathsf{n}}(\Omega) \xleftarrow{0} \{0\}.$$

As before, all basic Hilbert spaces are $\mathsf{L}^2(\Omega)$ except of $\mathsf{H}_3 = \mathsf{L}^2_\varepsilon(\Omega)$, corresponding to the domain of definition of T_3. Then (5.11) turns to

$$\begin{aligned}\mathrm{T}_2^*\,\mathrm{T}_2\,B &= \operatorname{rot}_{\Gamma_\mathsf{t}}\mu\operatorname{rot}_{\Gamma_\mathsf{n}}B = F,\\ \mathrm{T}_1^*\,B &= \operatorname{div}_{\Gamma_\mathsf{t}}B = g,\\ \pi_2\,B &= \pi_{\tilde{\mathcal{H}}}B = H\end{aligned}$$

and this system is uniquely solvable by Theorem 3.6 as $F \in R(\mathrm{T}_2^*)$, $g \in R(\mathrm{T}_1^*)$, and $H \in K_2$ with solution B depending continuously on the data. (5.13) reads

$$\begin{aligned}\mathrm{T}_2\,B &= \mu\operatorname{rot}_{\Gamma_\mathsf{n}}B = E, & \mathrm{T}_3\,E &= -\operatorname{div}_{\Gamma_\mathsf{n}}\varepsilon\,E = 0,\\ \mathrm{T}_1^*\,B &= \operatorname{div}_{\Gamma_\mathsf{t}}B = g, & \mathrm{T}_2^*\,E &= \operatorname{rot}_{\Gamma_\mathsf{t}}E = F,\\ \pi_2\,B &= \pi_{\tilde{\mathcal{H}}}B = H, & \pi_3\,E &= \pi_{\mathcal{H}}E = 0.\end{aligned}$$

Again, we can apply the main functional a posteriori error estimates from Theorem 4.7.

Theorem 5.6. *Let $B \in \mathsf{R}_{\Gamma_\mathsf{n}}(\Omega) \cap \mathsf{D}_{\Gamma_\mathsf{t}}(\Omega)$ be the exact solution of (5.11), $E := \mu\operatorname{rot}B \in \mathsf{R}_{\Gamma_\mathsf{t}}(\Omega)$, and $(\tilde{B}, \tilde{E}) \in \mathsf{L}^2(\Omega) \times \mathsf{L}^2_\varepsilon(\Omega)$. Then the following estimates hold for the errors $e_B := B - \tilde{B}$ and $e_E := E - \tilde{E}$:*

(i) *The errors e_B and e_E decompose, i.e.,*

$$\begin{aligned}e_B &= e_{B,\mathrm{grad}} + e_{B,\tilde{\mathcal{H}}} + e_{B,\mathrm{rot}} \in \operatorname{grad}\mathsf{H}^1_{\Gamma_\mathsf{n}}(\Omega) \oplus_{\mathsf{L}^2(\Omega)} \mathcal{H}_{\mathsf{n},\mathsf{t}}(\Omega) \oplus_{\mathsf{L}^2(\Omega)} \operatorname{rot}\mathsf{R}_{\Gamma_\mathsf{t}}(\Omega),\\ e_E &= e_{E,\mathrm{grad}} + e_{E,\mathcal{H}} + e_{E,\mathrm{rot}} \in \operatorname{grad}\mathsf{H}^1_{\Gamma_\mathsf{t}}(\Omega) \oplus_{\mathsf{L}^2_\varepsilon(\Omega)} \mathcal{H}_{\mathsf{t},\mathsf{n},\varepsilon}(\Omega) \oplus_{\mathsf{L}^2_\varepsilon(\Omega)} \mu\operatorname{rot}\mathsf{R}_{\Gamma_\mathsf{n}}(\Omega)\end{aligned}$$

and

$$\begin{aligned}|e_B|^2_{\mathsf{L}^2(\Omega)} &= |e_{B,\mathrm{grad}}|^2_{\mathsf{L}^2(\Omega)} + |e_{B,\tilde{\mathcal{H}}}|^2_{\mathsf{L}^2(\Omega)} + |e_{B,\mathrm{rot}}|^2_{\mathsf{L}^2(\Omega)},\\ |e_E|^2_{\mathsf{L}^2_\varepsilon(\Omega)} &= |e_{E,\mathrm{grad}}|^2_{\mathsf{L}^2_\varepsilon(\Omega)} + |e_{E,\mathcal{H}}|^2_{\mathsf{L}^2_\varepsilon(\Omega)} + |e_{E,\mathrm{rot}}|^2_{\mathsf{L}^2_\varepsilon(\Omega)}.\end{aligned}$$

(ii) *The projection $e_{B,\mathrm{grad}} = \pi_{\mathrm{grad}}e_B = B_g - \pi_{\mathrm{grad}}\tilde{B} \in \operatorname{grad}\mathsf{H}^1_{\Gamma_\mathsf{n}}(\Omega)$ satisfies*

$$\begin{aligned}|e_{B,\mathrm{grad}}|^2_{\mathsf{L}^2(\Omega)} &= \min_{\Phi\in\mathsf{D}_{\Gamma_\mathsf{t}}(\Omega)}\big(\tilde{c}_{\mathsf{fp}}|\operatorname{div}\Phi - g|_{\mathsf{L}^2(\Omega)} + |\Phi - \tilde{B}|_{\mathsf{L}^2(\Omega)}\big)^2\\ &= \max_{\varphi\in\mathsf{H}^1_{\Gamma_\mathsf{n}}(\Omega)}\big(2\langle g,\varphi\rangle_{\mathsf{L}^2(\Omega)} + \langle 2\tilde{B} - \operatorname{grad}\varphi, \operatorname{grad}\varphi\rangle_{\mathsf{L}^2(\Omega)}\big)\end{aligned}$$

and the minimum resp. maximum is attained at

$$\hat{\Phi} := e_{B,\mathrm{grad}} + \tilde{B} \in \mathsf{D}_{\Gamma_\mathsf{t}}(\Omega), \qquad \hat{\varphi} := -(\widetilde{\operatorname{grad}}_{\Gamma_\mathsf{n}})^{-1}e_{B,\mathrm{grad}} \in \mathsf{H}^1_{\Gamma_\mathsf{n}}(\Omega)$$

with $\operatorname{div}\hat{\Phi} = \operatorname{div}B = g$.

(iii) *The projection $e_{B,\mathrm{rot}} = \pi_{\mathrm{rot}}e_B = B_E - \pi_{\mathrm{rot}}\tilde{B} \in \operatorname{rot}\mathsf{R}_{\Gamma_\mathsf{t}}(\Omega)$ satisfies*

$$\begin{aligned}|e_{B,\mathrm{rot}}|^2_{\mathsf{L}^2(\Omega)} &= \min_{\Psi\in\mathsf{R}_{\Gamma_\mathsf{n}}(\Omega)}\min_{\Phi\in\mathsf{R}_{\Gamma_\mathsf{t}}(\Omega)}\big(c_\mathsf{m}^2|\operatorname{rot}\Phi - F|_{\mathsf{L}^2(\Omega)} + c_\mathsf{m}|\Phi - \mu\operatorname{rot}\Psi|_{\mathsf{L}^2_\varepsilon(\Omega)} + |\Psi - \tilde{B}|_{\mathsf{L}^2(\Omega)}\big)^2\\ &= \min_{\substack{\Psi\in\mathsf{R}_{\Gamma_\mathsf{n}}(\Omega),\\ \mu\operatorname{rot}\Psi\in\mathsf{R}_{\Gamma_\mathsf{t}}(\Omega)}}\big(c_\mathsf{m}^2|\operatorname{rot}\mu\operatorname{rot}\Psi - F|_{\mathsf{L}^2(\Omega)} + |\Psi - \tilde{B}|_{\mathsf{L}^2(\Omega)}\big)^2\\ &= \max_{\substack{\Theta\in\mathsf{R}_{\Gamma_\mathsf{n}}(\Omega),\\ \mu\operatorname{rot}\Theta\in\mathsf{R}_{\Gamma_\mathsf{t}}(\Omega)}}\big(2\langle F,\Theta\rangle_{\mathsf{L}^2(\Omega)} - \langle 2\tilde{E} + \operatorname{rot}\mu\operatorname{rot}\Theta, \operatorname{rot}\mu\operatorname{rot}\Theta\rangle_{\mathsf{L}^2(\Omega)}\big)\end{aligned}$$

and the minima resp. maximum is attained at

$$\hat{\Psi} := e_{B,\mathrm{rot}} + \tilde{B} \in \mathsf{R}_{\Gamma_\mathsf{n}}(\Omega), \qquad \hat{\Phi} := E \in \mathsf{R}_{\Gamma_\mathsf{t}}(\Omega),$$

and $\hat{\Theta} := (\mu\,\widetilde{\mathrm{rot}}_{\Gamma_\mathsf{n}})^{-1}(\widetilde{\mathrm{rot}}_{\Gamma_\mathsf{t}})^{-1}e_{B,\mathrm{rot}} \in \mathsf{R}_{\Gamma_\mathsf{n}}(\Omega) \cap \mathrm{rot}\,\mathsf{R}_{\Gamma_\mathsf{t}}(\Omega)$ *with* $\mu\,\mathrm{rot}\,\hat{\Psi}, \mu\,\mathrm{rot}\,\hat{\Theta}, \in \mathsf{R}_{\Gamma_\mathsf{t}}(\Omega)$ *and* $\mu\,\mathrm{rot}\,\hat{\Psi} = \mu\,\mathrm{rot}\,B = E$ *and* $\mathrm{rot}\,\mu\,\mathrm{rot}\,\hat{\Psi} = \mathrm{rot}\,E = F$ *as well as* $\mathrm{rot}\,\hat{\Phi} = \mathrm{rot}\,E = F$.

(iv) *The projection* $e_{B,\tilde{\mathcal{H}}} = \pi_{\tilde{\mathcal{H}}}e_B = H - \pi_{\tilde{\mathcal{H}}}\tilde{B} \in \mathcal{H}_{\mathsf{n},\mathsf{t}}(\Omega)$ *satisfies*

$$\begin{aligned}|e_{B,\tilde{\mathcal{H}}}|^2_{\mathsf{L}^2(\Omega)} &= \min_{\varphi\in\mathsf{H}^1_{\Gamma_\mathsf{n}}(\Omega)}\ \min_{\Phi\in\mathsf{R}_{\Gamma_\mathsf{t}}(\Omega)} |H - \tilde{B} - \mathrm{grad}\,\varphi + \mathrm{rot}\,\Phi|^2_{\mathsf{L}^2(\Omega)}\\ &= \max_{\Psi\in\mathcal{H}_{\mathsf{n},\mathsf{t}}(\Omega)} \big\langle 2(H-\tilde{B}) - \Psi, \Psi\big\rangle_{\mathsf{L}^2(\Omega)}\end{aligned}$$

and the minimum resp. maximum is attained at

$$\hat{\varphi} := -(\widetilde{\mathrm{grad}}_{\Gamma_\mathsf{n}})^{-1}\pi_{\mathrm{grad}}\tilde{B} \in \mathsf{H}^1_{\Gamma_\mathsf{n}}(\Omega), \qquad \hat{\Phi} := (\widetilde{\mathrm{rot}}_{\Gamma_\mathsf{t}})^{-1}\pi_{\mathrm{rot}}\tilde{B} \in \mathsf{R}_{\Gamma_\mathsf{t}}(\Omega) \cap \mu\,\mathrm{rot}\,\mathsf{R}_{\Gamma_\mathsf{n}}(\Omega)$$

resp. $\hat{\Psi} := e_{B,\tilde{\mathcal{H}}} \in \mathcal{H}_{\mathsf{n},\mathsf{t}}(\Omega)$ *with* $-\mathrm{grad}\,\hat{\varphi} + \mathrm{rot}\,\hat{\phi} = (\pi_{\mathrm{grad}} + \pi_{\mathrm{rot}})\tilde{B} = (1 - \pi_{\tilde{\mathcal{H}}})\tilde{B}$.

(v) *The projection* $e_{E,\mathrm{grad}} = \pi_{\mathrm{grad}}e_E = -\pi_{\mathrm{grad}}\tilde{E} \in \mathrm{grad}\,\mathsf{H}^1_{\Gamma_\mathsf{t}}(\Omega)$ *satisfies*

$$\begin{aligned}|e_{E,\mathrm{grad}}|^2_{\mathsf{L}^2_\varepsilon(\Omega)} &= \min_{\Phi\in\mu\,\mathsf{D}_{\Gamma_\mathsf{n}}(\Omega)} \big(c_{\mathsf{fp}}|\,\mathrm{div}\,\varepsilon\,\Phi|_{\mathsf{L}^2(\Omega)} + |\Phi - \tilde{E}|_{\mathsf{L}^2_\varepsilon(\Omega)}\big)^2 = \min_{\Phi\in\mu\,\mathsf{D}_{\Gamma_\mathsf{n},0}(\Omega)} |\Phi - \tilde{E}|^2_{\mathsf{L}^2_\varepsilon(\Omega)}\\ &= \max_{\varphi\in\mathsf{H}^1_{\Gamma_\mathsf{t}}(\Omega)} \big(-\langle 2\tilde{E} + \mathrm{grad}\,\varphi, \mathrm{grad}\,\varphi\rangle_{\mathsf{L}^2_\varepsilon(\Omega)}\big)\end{aligned}$$

and the minimum resp. maximum is attained at

$$\hat{\Phi} := e_{E,\mathrm{grad}} + \tilde{E} \in \mu\,\mathsf{D}_{\Gamma_\mathsf{n},0}(\Omega), \qquad \hat{\varphi} := (\widetilde{\mathrm{grad}}_{\Gamma_\mathsf{t}})^{-1}e_{E,\mathrm{grad}} \in \mathsf{H}^1_{\Gamma_\mathsf{t}}(\Omega)$$

with $-\mathrm{div}\,\varepsilon\,\hat{\Phi} = -\mathrm{div}\,\varepsilon\,E = 0$.

(vi) *The projection* $e_{E,\mathrm{rot}} = \pi_{\mathrm{rot}}e_E = E - \pi_{\mathrm{rot}}\tilde{E} \in \mu\,\mathrm{rot}\,\mathsf{R}_{\Gamma_\mathsf{n}}(\Omega)$ *satisfies*

$$\begin{aligned}|e_{E,\mathrm{rot}}|^2_{\mathsf{L}^2_\varepsilon(\Omega)} &= \min_{\Phi\in\mathsf{R}_{\Gamma_\mathsf{t}}(\Omega)} \big(c_{\mathsf{m}}|\,\mathrm{rot}\,\Phi - F|_{\mathsf{L}^2(\Omega)} + |\Phi - \tilde{E}|_{\mathsf{L}^2_\varepsilon(\Omega)}\big)^2\\ &= \max_{\Psi\in\mathsf{R}_{\Gamma_\mathsf{n}}(\Omega)} \big(2\langle F, \Psi\rangle_{\mathsf{L}^2(\Omega)} - \langle 2\tilde{E} + \mu\,\mathrm{rot}\,\Psi, \mu\,\mathrm{rot}\,\Psi\rangle_{\mathsf{L}^2_\varepsilon(\Omega)}\big)\end{aligned}$$

and the minimum resp. maximum is attained at

$$\hat{\Phi} := e_{E,\mathrm{rot}} + \tilde{E} \in \mathsf{R}_{\Gamma_\mathsf{t}}(\Omega), \qquad \hat{\Psi} := (\mu\,\widetilde{\mathrm{rot}}_{\Gamma_\mathsf{n}})^{-1}e_{E,\mathrm{rot}} \in \mathsf{R}_{\Gamma_\mathsf{n}}(\Omega) \cap \mathrm{rot}\,\mathsf{R}_{\Gamma_\mathsf{t}}(\Omega)$$

with $\mathrm{rot}\,\hat{\Phi} = \mathrm{rot}\,E = F$.

(vii) *The projection* $e_{E,\mathcal{H}} = \pi_{\mathcal{H}}e_E = -\pi_{\mathcal{H}}\tilde{E} \in \mathcal{H}_{\mathsf{t},\mathsf{n},\varepsilon}(\Omega)$ *satisfies*

$$\begin{aligned}|e_{E,\mathcal{H}}|^2_{\mathsf{L}^2_\varepsilon(\Omega)} &= \min_{\varphi\in\mathsf{H}^1_{\Gamma_\mathsf{t}}(\Omega)}\ \min_{\Phi\in\mathsf{R}_{\Gamma_\mathsf{n}}(\Omega)} |-\tilde{E} + \mathrm{grad}\,\varphi + \mu\,\mathrm{rot}\,\Phi|^2_{\mathsf{L}^2_\varepsilon(\Omega)}\\ &= \max_{\Psi\in\mathcal{H}_{\mathsf{t},\mathsf{n},\varepsilon}(\Omega)} \big(-\big\langle 2\tilde{E} + \Psi, \Psi\big\rangle_{\mathsf{L}^2_\varepsilon(\Omega)}\big)\end{aligned}$$

and the minimum resp. maximum is attained at

$$\hat{\varphi} := (\widetilde{\mathrm{grad}}_{\Gamma_\mathsf{t}})^{-1}\pi_{\mathrm{grad}}\tilde{E} \in \mathsf{H}^1_{\Gamma_\mathsf{t}}(\Omega), \qquad \hat{\Phi} := (\mu\,\widetilde{\mathrm{rot}}_{\Gamma_\mathsf{n}})^{-1}\pi_{\mathrm{rot}}\tilde{E} \in \mathsf{R}_{\Gamma_\mathsf{n}}(\Omega) \cap \mathrm{rot}\,\mathsf{R}_{\Gamma_\mathsf{t}}(\Omega)$$

resp. $\hat{\Psi} := e_{E,\mathcal{H}} \in \mathcal{H}_{\mathsf{t},\mathsf{n},\varepsilon}(\Omega)$ *with* $\mathrm{grad}\,\hat{\varphi} + \mu\,\mathrm{rot}\,\hat{\phi} = (\pi_{\mathrm{grad}} + \pi_{\mathrm{rot}})\tilde{E} = (1 - \pi_{\mathcal{H}})\tilde{E}$.

If $\tilde{B} = H + \tilde{B}_\perp$ *with some* $\tilde{B}_\perp \in \mathcal{H}_{\mathsf{n},\mathsf{t}}(\Omega)^{\perp_{\mathsf{L}^2(\Omega)}}$, *then* $e_{B,\tilde{\mathcal{H}}} = 0$, *and in (ii) and (iii)* $\tilde{B}$ *can be replaced by* $\tilde{B}_\perp$. *If* $\tilde{E} = \tilde{E}_\perp$ *with some* $\tilde{E}_\perp \in \mathcal{H}_{\mathsf{t},\mathsf{n},\varepsilon}(\Omega)^{\perp_{\mathsf{L}^2_\varepsilon(\Omega)}}$, *then* $e_{E,\mathcal{H}} = 0$, *and in (v) and (vi)* $\tilde{E}$ *can be replaced by* $\tilde{E}_\perp$.

A reasonable assumption is, that we have conforming approximations

$$\tilde{B}_g \in \operatorname{grad} \mathsf{H}^1_{\Gamma_\mathsf{n}}(\Omega) = \mathsf{R}_{\Gamma_\mathsf{n},0}(\Omega) \cap \mathcal{H}_{\mathsf{n},\mathsf{t}}(\Omega)^\perp, \qquad \tilde{B}_F \in \mathsf{R}_{\Gamma_\mathsf{n}}(\Omega)$$

of $B_g \in \mathsf{D}_{\Gamma_\mathsf{t}}(\Omega) \cap \operatorname{grad} \mathsf{H}^1_{\Gamma_\mathsf{n}}(\Omega)$ and $B_F \in \mathsf{R}_{\Gamma_\mathsf{n}}(\Omega) \cap \operatorname{rot} \mathsf{R}_{\Gamma_\mathsf{t}}(\Omega)$ and hence a conforming approximation

$$\tilde{E} := \mu \operatorname{rot} \tilde{B}_F \in \mu \operatorname{rot} \mathsf{R}_{\Gamma_\mathsf{n}}(\Omega)$$

of $E \in \mathsf{R}_{\Gamma_\mathsf{t}}(\Omega) \cap \mu \operatorname{rot} \mathsf{R}_{\Gamma_\mathsf{n}}(\Omega)$, which implies $e_E = e_{E,\mathrm{rot}} \in \mu \operatorname{rot} \mathsf{R}_{\Gamma_\mathsf{n}}(\Omega)$ and $e_{E,\mathrm{grad}} = e_{E,\mathcal{H}} = 0$ as well as $\tilde{B} - H = \tilde{B}_F + \tilde{B}_g \in \mathsf{R}_{\Gamma_\mathsf{n}}(\Omega)$ and $e_B \in \mathsf{R}_{\Gamma_\mathsf{n}}(\Omega)$. In this case the estimates of the latter theorem simplify. More precisely, e.g., (iii) turns to the following result: If $\tilde{B}_F, \tilde{B}_g \in \mathsf{R}_{\Gamma_\mathsf{n}}(\Omega)$, then $\tilde{B}, e_B \in \mathsf{R}_{\Gamma_\mathsf{n}}(\Omega)$ and we can choose, e.g., $\Psi := \tilde{B}$ yielding, e.g.,

$$|e_{B,\mathrm{rot}}|_{\mathsf{L}^2(\Omega)} \leq \min_{\Phi \in \mathsf{R}_{\Gamma_\mathsf{t}}(\Omega)} \big(c_\mathsf{m}^2 |\operatorname{rot} \Phi - F|_{\mathsf{L}^2(\Omega)} + c_\mathsf{m} |\Phi - \mu \operatorname{rot} \tilde{B}|_{\mathsf{L}^2_\varepsilon(\Omega)} \big),$$

which might not be sharp anymore. Similarly, the results of (vi) read as follows: If $\tilde{B}_F \in \mathsf{R}_{\Gamma_\mathsf{n}}(\Omega)$, then $\tilde{E} := \mu \operatorname{rot} \tilde{B}_F \in \mu \operatorname{rot} \mathsf{R}_{\Gamma_\mathsf{n}}(\Omega)$ and $\mu \operatorname{rot}(B - \tilde{B}_F) = e_E = e_{E,\mathrm{rot}} \in \mu \operatorname{rot} \mathsf{R}_{\Gamma_\mathsf{n}}(\Omega)$ as well as

$$
\begin{aligned}
|e_E|^2_{\mathsf{L}^2_\varepsilon(\Omega)} &= \min_{\Phi \in \mathsf{R}_{\Gamma_\mathsf{t}}(\Omega)} \big(c_\mathsf{m} |\operatorname{rot} \Phi - F|_{\mathsf{L}^2(\Omega)} + |\Phi - \mu \operatorname{rot} \tilde{B}_F|_{\mathsf{L}^2_\varepsilon(\Omega)} \big)^2 \\
&= \max_{\Psi \in \mathsf{R}_{\Gamma_\mathsf{n}}(\Omega)} \big(2\langle F, \Psi \rangle_{\mathsf{L}^2(\Omega)} - \langle \mu \operatorname{rot}(2\tilde{B}_F + \Psi), \mu \operatorname{rot} \Psi \rangle_{\mathsf{L}^2_\varepsilon(\Omega)} \big)
\end{aligned}
\tag{5.14}
$$

and the minimum resp. maximum is attained at

$$\hat{\Phi} := e_E + \mu \operatorname{rot} \tilde{B}_F \in \mathsf{R}_{\Gamma_\mathsf{t}}(\Omega), \qquad \hat{\Psi} := (\mu \widetilde{\operatorname{rot}}_{\Gamma_\mathsf{n}})^{-1} e_E \in \mathsf{R}_{\Gamma_\mathsf{n}}(\Omega) \cap \operatorname{rot} \mathsf{R}_{\Gamma_\mathsf{t}}(\Omega)$$

with $\operatorname{rot} \hat{\Phi} = \operatorname{rot} E = F$. Note that (5.14) are in principle the functional a posteriori error estimates for the energy norm associated to the rot rot-operator, which have been proved in [7].

5.3. **More Applications.** There are a lot more applications. If we denote the exterior derivative and the co-derivative associated with some compact Riemannian manifold by d and δ, we can discuss problems like

$$
\begin{aligned}
\mathrm{d}\, E &= F, & \delta\, \mathrm{d}\, E &= F, & \delta\, \mathrm{d}\, E &= F, \\
\delta\, \varepsilon E &= G, & \delta\, \varepsilon E &= G, & \mathrm{d}\, \delta\, \varepsilon E &= G, \\
\pi E &= H, & \pi E &= H, & \pi E &= H
\end{aligned}
$$

for mixed tangential and normal boundary conditions for some differential form E. Moreover, problems in linear elasticity, Stokes equations, biharmonic theory, rot rot rot rot-operators, ... fit into our general framework. Note that all these problems feature the underlying complexes (1.3)-(1.4), such as

$$\mathsf{H}^1_{\Gamma_\mathsf{t}}(\Omega) \xrightarrow{\mathrm{A}_1 = \operatorname{grad}_{\Gamma_\mathsf{t}}} \mathsf{R}_{\Gamma_\mathsf{t}}(\Omega) \xrightarrow{\mathrm{A}_2 = \operatorname{rot}_{\Gamma_\mathsf{t}}} \mathsf{D}_{\Gamma_\mathsf{t}}(\Omega) \xrightarrow{\mathrm{A}_3 = \operatorname{div}_{\Gamma_\mathsf{t}}} \mathsf{L}^2(\Omega),$$

$$\mathsf{L}^2(\Omega) \xleftarrow{\mathrm{A}_1^* = -\operatorname{div}_{\Gamma_\mathsf{n}}} \mathsf{D}_{\Gamma_\mathsf{n}}(\Omega) \xleftarrow{\mathrm{A}_2^* = \operatorname{rot}_{\Gamma_\mathsf{n}}} \mathsf{R}_{\Gamma_\mathsf{n}}(\Omega) \xleftarrow{\mathrm{A}_3^* = -\operatorname{grad}_{\Gamma_\mathsf{n}}} \mathsf{H}^1_{\Gamma_\mathsf{n}}(\Omega)$$

for electro-magnetics,

$$\mathsf{D}^0_{\Gamma_\mathsf{t}}(\Omega) \xrightarrow{\mathrm{A}_1 = \mathrm{d}_{\Gamma_\mathsf{t}}} \mathsf{D}^1_{\Gamma_\mathsf{t}}(\Omega) \xrightarrow{\mathrm{A}_2 = \mathrm{d}_{\Gamma_\mathsf{t}}} \mathsf{D}^2_{\Gamma_\mathsf{t}}(\Omega) \xrightarrow{\mathrm{A}_3 = \mathrm{d}_{\Gamma_\mathsf{t}}} \mathsf{L}^{2,3}(\Omega),$$

$$\mathsf{L}^{2,0}(\Omega) \xleftarrow{\mathrm{A}_1^* = \delta_{\Gamma_\mathsf{n}}} \Delta^1_{\Gamma_\mathsf{n}}(\Omega) \xleftarrow{\mathrm{A}_2^* = \delta_{\Gamma_\mathsf{n}}} \Delta^2_{\Gamma_\mathsf{n}}(\Omega) \xleftarrow{\mathrm{A}_3^* = \delta_{\Gamma_\mathsf{n}}} \Delta^3_{\Gamma_\mathsf{n}}(\Omega)$$

for generalized electro-magnetics (differential forms),

$$\mathsf{H}^2_{\Gamma_\mathsf{t}}(\Omega) \xrightarrow{\mathrm{A}_1 = \operatorname{Grad} \operatorname{grad}_{\Gamma_\mathsf{t}}} \mathsf{R}_{\Gamma_\mathsf{t}}(\Omega; \mathbb{S}) \xrightarrow{\mathrm{A}_2 = \operatorname{Rot}_{\mathbb{S},\Gamma_\mathsf{t}}} \mathsf{D}_{\Gamma_\mathsf{t}}(\Omega; \mathbb{T}) \xrightarrow{\mathrm{A}_3 = \operatorname{Div}_{\mathbb{T},\Gamma_\mathsf{t}}} \mathsf{L}^2(\Omega),$$

$$\mathsf{L}^2(\Omega) \xleftarrow{\mathrm{A}_1^* = \operatorname{div} \operatorname{Div}_{\mathbb{S},\Gamma_\mathsf{n}}} \mathsf{DD}_{\Gamma_\mathsf{n}}(\Omega; \mathbb{S}) \xleftarrow{\mathrm{A}_2^* = \operatorname{sym} \operatorname{Rot}_{\mathbb{T},\Gamma_\mathsf{n}}} \mathsf{R}_{\mathrm{sym},\Gamma_\mathsf{n}}(\Omega; \mathbb{T}) \xleftarrow{\mathrm{A}_3^* = -\operatorname{dev} \operatorname{Grad}_{\Gamma_\mathsf{n}}} \mathsf{H}^1_{\Gamma_\mathsf{n}}(\Omega)$$

for biharmonic and Stokes problems, and

$$\mathsf{H}^1_{\Gamma_t}(\Omega) \xrightarrow{\mathrm{A}_1=\operatorname{sym}\operatorname{Grad}_{\Gamma_t}} \mathsf{RR}^\top_{\Gamma_t}(\Omega;\mathbb{S}) \xrightarrow{\mathrm{A}_2=\operatorname{RotRot}^\top_{\mathbb{S},\Gamma_t}} \mathsf{D}_{\Gamma_t}(\Omega;\mathbb{S}) \xrightarrow{\mathrm{A}_3=\operatorname{Div}_{\mathbb{S},\Gamma_t}} \mathsf{L}^2(\Omega),$$

$$\mathsf{L}^2(\Omega) \xleftarrow{\mathrm{A}_1^*=-\operatorname{Div}_{\mathbb{S},\Gamma_n}} \mathsf{D}_{\Gamma_n}(\Omega;\mathbb{S}) \xleftarrow{\mathrm{A}_2^*=\operatorname{RotRot}^\top_{\mathbb{S},\Gamma_n}} \mathsf{RR}^\top_{\Gamma_n}(\Omega;\mathbb{S}) \xleftarrow{\mathrm{A}_3^*=-\operatorname{sym}\operatorname{Grad}_{\Gamma_n}} \mathsf{H}^1_{\Gamma_n}(\Omega)$$

for linear elasticity.

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