

A Global div-curl-Lemma for Mixed Boundary Conditions in Weak Lipschitz Domains

Dirk Pauly

Abstract We prove a global version of the so-called div-curl-lemma, a crucial result for compensated compactness and in homogenization theory, for mixed tangential and normal boundary conditions in bounded weak Lipschitz domains in 3D and weak Lipschitz interfaces. The crucial tools and the core of our arguments are the de Rham complex and Weck's selection theorem, the essential compact embedding result for Maxwell's equations.

1 Introduction and Main Results

We shall prove a global (and hence also a local) version of the so-called div-curl-lemma, with mixed tangential and normal boundary conditions for bounded weak Lipschitz domains Ω in 3D, more precisely for admissible pairs (Ω, Γ_t) of a bounded weak Lipschitz domain $\Omega \subset \mathbb{R}^3$ and a part Γ_t of its boundary Γ , see Definition 1 for details.

Theorem 1 (global div-curl-lemma) *Let (Ω, Γ_t) be admissible and let*

- (i) $E_n, E \in D(\text{curl}_{\Gamma_t})$,
- (i') $E_n \rightharpoonup E$ in ${}^t D(\text{curl}_{\Gamma_t})$,
- (ii) $H_n, H \in D(\text{div}_{\Gamma_n})$,
- (ii') $H_n \rightharpoonup H$ in ${}^2 D(\text{div}_{\Gamma_n})$.

Then

- (iii) $\langle E_n, H_n \rangle_{L^2(\Omega)} \rightarrow \langle E, H \rangle_{L^2(\Omega)}$.

Here, we introduce the densely defined and closed linear operators ∇_{Γ_t} , curl_{Γ_t} , div_{Γ_t} with domains of definition³ $D(\nabla_{\Gamma_t})$, $D(\text{curl}_{\Gamma_t})$, $D(\text{div}_{\Gamma_t})$ as closures of the classical differential operators from vector analysis acting on $L^2(\Omega)$ and defined on smooth test functions resp. test vector fields bounded away from the boundary part Γ_t given by

Dirk Pauly
Fakultät für Mathematik, Universität Duisburg-Essen, Campus Essen, Germany
e-mail: dirk.pauly@uni-due.de

¹ In particular, $E_n \rightharpoonup E$ in $L^2(\Omega)$ and $\text{curl } E_n \rightharpoonup \text{curl } E$ in $L^2(\Omega)$.

² In particular, $H_n \rightharpoonup H$ in $L^2(\Omega)$ and $\text{div } H_n \rightharpoonup \text{div } H$ in $L^2(\Omega)$.

³ Various notations like

$$C_{\Gamma_t}^{\infty}(\Omega) := \{\varphi|_{\Omega} : \varphi \in C^{\infty}(\mathbb{R}^3), \text{supp } \varphi \text{ compact, } \text{dist}(\text{supp } \varphi, \Gamma_t) > 0\}.$$

As shown in [3, Theorem 4.5] (weak equals strong in terms of definitions of boundary conditions) their adjoints are given by $-\text{div}_{\Gamma_n}$, curl_{Γ_n} , $-\nabla_{\Gamma_n}$ defined in the same way. Note that these operators are unbounded and that the domains of definition are Hilbert spaces equipped with the respective proper graph inner products.

Corollary 1 (local div-curl-lemma) *Let $\Omega \subset \mathbb{R}^3$ be an open set and let*

- (i) $E_n, E \in D(\text{curl})$,
- (i') $E_n \rightharpoonup E$ in $D(\text{curl})$,
- (ii) $H_n, H \in D(\text{div})$,
- (ii') $H_n \rightharpoonup H$ in $D(\text{div})$.

Then

$$\text{(iii)} \quad \forall \varphi \in C_{\Gamma_t}^{\infty}(\Omega) \quad \langle \varphi E_n, H_n \rangle_{L^2(\Omega)} \rightarrow \langle \varphi E, H \rangle_{L^2(\Omega)}.$$

$C_{\Gamma_t}^{\infty}(\Omega)$ may be replaced by $C_{\Gamma_t}^1(\Omega)$ or even $C_{\Gamma_t}^{0,1}(\Omega)$, the space of Lipschitz continuous functions vanishing in a neighbourhood of Γ . Moreover, the boundedness of (E_n) and (H_n) in local spaces is sufficient for the assertion to hold.

The div-curl-lemma, or compensated compactness, see the original papers by Murat [13] and Tartar [23] or [6, 22], and its variants and extensions have plenty important applications. It is widely used in the theory of homogenization of (nonlinear) partial differential equations, see, e.g., [22]. Moreover, it is crucial in establishing compactness and regularity results for nonlinear partial differential equations such as harmonic maps, see, e.g., [8, 7, 19]. Numerical applications can be found, e.g., in [2]. The div-curl-lemma is further a crucial tool in the homogenization of stochastic partial differential equations, especially with certain random coefficients, see, e.g., the survey [1] and the literature cited therein, e.g., [9].

For an extensive discussion and a historical overview of the div-curl-lemma see [24]. More recent discussions can be found, e.g., in [4, 25]. Recently, in [26, 17] the div-curl-lemma has been proved in a general Hilbert space setting which allows for various applications in mathematical physics. Interesting and new applications to homogenization of partial differential equations can be found in [27].

Let us also mention that the div-curl-lemma is particularly useful to treat homogenization of problems arising in plasticity, see, e.g., a recent preprint on this topic [21], for which the preprint [20] provides the important key div-curl-lemma. Unfortunately, in [20, 21] a $H^1(\Omega)$ -detour is used as the core argument for the proofs. The same detour is utilized in the recent contribution [11] where div-curl-type lemmas

$$\begin{aligned} D(\nabla_{\Gamma_t}) &= H(\nabla_{\Gamma_t}, \Omega) = H_{\Gamma_t}(\nabla, \Omega) = H_{\Gamma_t}^1(\Omega) = \mathring{H}_{\Gamma_t}^1(\Omega), \\ D(\text{curl}_{\Gamma_t}) &= H(\text{curl}_{\Gamma_t}, \Omega) = H_{\Gamma_t}(\text{curl}, \Omega) = R_{\Gamma_t}(\Omega) = \mathring{R}_{\Gamma_t}(\Omega), \\ D(\text{div}_{\Gamma_t}) &= H(\text{div}_{\Gamma_t}, \Omega) = H_{\Gamma_t}(\text{div}, \Omega) = D_{\Gamma_t}(\Omega) = \mathring{D}_{\Gamma_t}(\Omega) \end{aligned}$$

can be found frequently in the literature, where also $\text{curl} = \text{rot}$ is used.

are presented which also allow for inhomogeneous boundary conditions. This unnecessarily high regularity assumption of $H^1(\Omega)$ -fields excludes results like [11, 20, 21] to be applied to important applications which are stated, e.g., in Lipschitz domains.

2 Notations, Preliminaries, and Proofs

Definition 1 (admissible domains) We call a pair (Ω, Γ_t) admissible, if

- (i) $\Omega \subset \mathbb{R}^3$ is a bounded weak Lipschitz domain in the sense of [3, Definition 2.3]
- (ii) with boundary $\Gamma := \partial\Omega$, which is divided into two relatively open weak Lipschitz subsets $\Gamma_t \subset \Gamma$ and its complement $\Gamma_n := \Gamma \setminus \overline{\Gamma_t}$ in the sense of [3, Definition 2.5].

Note that strong Lipschitz domains (locally below a graph of a Lipschitz function) are weak Lipschitz domains (the boundary is a Lipschitz manifold) which holds for the boundary as well as for the interface. The reverse implication is not true due to the failure of the implicit function theorem for Lipschitz mappings. Throughout this paper we shall assume the latter regularity of Ω , and $\Gamma, \Gamma_t, \Gamma_n$.

Recently, in [3], Weck's selection theorem [29], also known as the Maxwell compactness property, has been shown to hold for such bounded weak Lipschitz domains and mixed boundary conditions. More precisely, the following holds:

Lemma 1 (Weck's selection theorem) *Let (Ω, Γ_t) be admissible. Then the embedding*

$$D(\text{curl}_{\Gamma_t}) \cap D(\text{div}_{\Gamma_n}) \hookrightarrow L^2(\Omega)$$

is compact.

For a proof see [3, Theorem 4.7]. A short historical overview of Weck's selection theorem is given in the introduction of [3], see also the original paper [29] and [18, 28, 5, 30, 10, 12] for simpler proofs and generalizations.

Let us emphasize that our assumptions also allow for Rellich's selection theorem, i.e., the embedding

$$D(\nabla_{\Gamma_t}) \hookrightarrow L^2(\Omega) \tag{1}$$

is compact, see, e.g., [3, Theorem 4.8]. By density we have the two rules of integration by parts

$$\forall u \in D(\nabla_{\Gamma_t}) \quad \forall H \in D(\text{div}_{\Gamma_n}) \quad \langle \nabla u, H \rangle_{L^2(\Omega)} = -\langle u, \text{div} H \rangle_{L^2(\Omega)}, \tag{2}$$

$$\forall E \in D(\text{curl}_{\Gamma_t}) \quad \forall H \in D(\text{curl}_{\Gamma_n}) \quad \langle \text{curl} E, H \rangle_{L^2(\Omega)} = \langle E, \text{curl} H \rangle_{L^2(\Omega)}. \tag{3}$$

A direct consequence of Lemma 1 is the compactness of the unit ball in

$$\mathcal{H}(\Omega) := N(\text{curl}_{\Gamma_t}) \cap N(\text{div}_{\Gamma_n}),$$

the space of so-called Dirichlet-Neumann fields. Hence $\mathcal{H}(\Omega)$ is finite-dimensional. Here and in the following we denote the kernels and the ranges of our operators ∇_{Γ_t} , curl_{Γ_t} , div_{Γ_t} by

$$N(\nabla_{\Gamma_t}), \quad N(\text{curl}_{\Gamma_t}), \quad N(\text{div}_{\Gamma_t}), \quad R(\nabla_{\Gamma_t}), \quad R(\text{curl}_{\Gamma_t}), \quad R(\text{div}_{\Gamma_t}).$$

Another immediate consequence of Weck's selection theorem, stated in Lemma 1, using a standard indirect argument, is the so-called Maxwell estimate, i.e., there exists $c_m > 0$ such that for all $E \in D(\text{curl}_{\Gamma_t}) \cap D(\text{div}_{\Gamma_n}) \cap \mathcal{H}(\Omega)^{\perp L^2(\Omega)}$

$$|E|_{L^2(\Omega)} \leq c_m (|\text{curl } E|_{L^2(\Omega)} + |\text{div } E|_{L^2(\Omega)}), \quad (4)$$

see [3, Theorem 5.1]. Recent estimates for the Maxwell constant c_m can be found in [14, 15, 16]. Analogously, Rellich's selection theorem (1) shows the Friedrichs/Poincaré estimate, i.e., there exists $c_{t,p} > 0$ such that for all $u \in D(\nabla_{\Gamma_t})$

$$|u|_{L^2(\Omega)} \leq c_{t,p} |\nabla u|_{L^2(\Omega)}, \quad (5)$$

see [3, Theorem 4.8]. To avoid case studies due to the one-dimensional kernel \mathbb{R} of ∇ when using the Friedrichs/Poincaré estimate in the case $\Gamma_t = \emptyset$, we also define

$$D(\nabla_{\emptyset}) := D(\nabla) \cap \mathbb{R}^{\perp L^2(\Omega)} = \left\{ u \in H^1(\Omega) : \int_{\Omega} u = 0 \right\}.$$

By the projection theorem, applied to our densely defined and closed (unbounded) linear operator

$$\nabla_{\Gamma_t} : D(\nabla_{\Gamma_t}) \subset L^2(\Omega) \longrightarrow L^2(\Omega)$$

with (Hilbert space) adjoint

$$\nabla_{\Gamma_t}^* = -\text{div}_{\Gamma_n} : D(\text{div}_{\Gamma_n}) \subset L^2(\Omega) \longrightarrow L^2(\Omega),$$

where we have used [3, Theorem 4.5] (weak equals strong), we get the simple (orthogonal) Helmholtz decomposition

$$L^2(\Omega) = R(\nabla_{\Gamma_t}) \oplus_{L^2(\Omega)} N(\text{div}_{\Gamma_n}), \quad (6)$$

see [3, Theorem 5.3 or (13)], which immediately implies the orthogonal decomposition

$$D(\text{curl}_{\Gamma_t}) = R(\nabla_{\Gamma_t}) \oplus_{L^2(\Omega)} (D(\text{curl}_{\Gamma_t}) \cap N(\text{div}_{\Gamma_n})) \quad (7)$$

as the complex property $R(\nabla_{\Gamma_t}) \subset N(\text{curl}_{\Gamma_t})$ holds. Here $\oplus_{L^2(\Omega)}$ in the decompositions (6) and (7) denotes the orthogonal sum in the Hilbert space $L^2(\Omega)$. By (5) the range $R(\nabla_{\Gamma_t})$ is closed in $L^2(\Omega)$, see also [3, Lemma 5.2]. Note that we call (6) a simple Helmholtz decomposition, since the refined Helmholtz decomposition

$$L^2(\Omega) = R(\nabla_{\Gamma_t}) \oplus_{L^2(\Omega)} \mathcal{H}(\Omega) \oplus_{L^2(\Omega)} R(\text{curl}_{\Gamma_n})$$

holds as well, see [3, Theorem 5.3], where also $R(\text{curl}_{\Gamma_n})$ is closed in $L^2(\Omega)$ as a consequence of (4), see [3, Lemma 5.2].

Proof of Theorem 1 By (7) we have $D(\text{curl}_{\Gamma_t}) \ni E_n = \nabla u_n + \tilde{E}_n$ with some $u_n \in D(\nabla_{\Gamma_t})$ and $\tilde{E}_n \in D(\text{curl}_{\Gamma_t}) \cap N(\text{div}_{\Gamma_n})$. Then (u_n) is bounded in $H^1(\Omega)$ by orthogonality and the Friedrichs/Poincaré estimate (5). By orthogonality (\tilde{E}_n) is bounded in $D(\text{curl}_{\Gamma_t}) \cap N(\text{div}_{\Gamma_n})$ and $\text{curl } \tilde{E}_n = \text{curl } E_n$. Hence, using Rellich's and Weck's selection theorems there exist $u \in D(\nabla_{\Gamma_t})$ and $\tilde{E} \in D(\text{curl}_{\Gamma_t}) \cap N(\text{div}_{\Gamma_n})$ and we can extract two subsequences, again denoted by (u_n) and (\tilde{E}_n) such that $u_n \rightharpoonup u$ in $D(\nabla_{\Gamma_t})$ and $u_n \rightarrow u$ in $L^2(\Omega)$ as well as $\tilde{E}_n \rightharpoonup \tilde{E}$ in $D(\text{curl}_{\Gamma_t}) \cap N(\text{div}_{\Gamma_n})$ and $\tilde{E}_n \rightarrow \tilde{E}$ in $L^2(\Omega)$. We observe $E = \nabla u + \tilde{E}$, giving the simple Helmholtz decomposition for E . Finally, by (2)

$$\begin{aligned} \langle E_n, H_n \rangle_{L^2(\Omega)} &= \langle \nabla u_n, H_n \rangle_{L^2(\Omega)} + \langle \tilde{E}_n, H_n \rangle_{L^2(\Omega)} \\ &= -\langle u_n, \text{div } H_n \rangle_{L^2(\Omega)} + \langle \tilde{E}_n, H_n \rangle_{L^2(\Omega)} \\ &\rightarrow -\langle u, \text{div } H \rangle_{L^2(\Omega)} + \langle \tilde{E}, H \rangle_{L^2(\Omega)} \\ &= \langle \nabla u, H \rangle_{L^2(\Omega)} + \langle \tilde{E}, H \rangle_{L^2(\Omega)} \\ &= \langle E, H \rangle_{L^2(\Omega)}. \end{aligned}$$

As the limit is unique, the original sequence $(\langle E_n, H_n \rangle_{L^2(\Omega)})$ already converges to the limit $\langle E, H \rangle_{L^2(\Omega)}$. ■

Proof of Corollary 1 Let $\Gamma_t := \Gamma$ and hence $\Gamma_n = \emptyset$. The sequence (φE_n) is bounded in $D(\text{curl}_{\Gamma})$ and (H_n) is bounded in $D(\text{div})$. Theorem 1 shows the assertion. ■

3 Generalizations and the Classical div-curl-Lemma

In [26, 17] more general div-curl-lemmas have been presented. In particular in [17] we can find the following generalization to distributions.

Theorem 2 (alternative global div-curl-lemma) *Let $\Omega \subset \mathbb{R}^3$ be a bounded strong Lipschitz domain with trivial topology (all Betti numbers vanish) and let*

- (i) $E_n, H_n, E, H \in L^2(\Omega)$,
- (i') $E_n \rightharpoonup E$ and $H_n \rightharpoonup H$ in $L^2(\Omega)$.

Moreover, let either

- (ii) $(\widehat{\text{curl}} E_n)$ be relatively compact in $\dot{H}^{-1}(\Omega)$,
- (iii) $(\widehat{\text{div}} H_n)$ be relatively compact in $H^{-1}(\Omega)$,

or

- (ii') $(\widetilde{\text{curl}} E_n)$ be relatively compact in $H^{-1}(\Omega)$,
 (iii') $(\widetilde{\text{div}} H_n)$ be relatively compact in $\dot{H}^{-1}(\Omega)$.

Then

$$(iv) \quad \langle E_n, H_n \rangle_{L^2(\Omega)} \rightarrow \langle E, H \rangle_{L^2(\Omega)}.$$

Here, $H^{-1}(\Omega) := \dot{H}^1(\Omega)'$ and $\dot{H}^{-1}(\Omega) := H^1(\Omega)'$ and the distributional extensions

$$\begin{aligned} \widetilde{\text{curl}} : L^2(\Omega) &\rightarrow H^{-1}(\Omega), & \widetilde{\text{div}} : L^2(\Omega) &\rightarrow H^{-1}(\Omega), \\ \widehat{\text{curl}} : L^2(\Omega) &\rightarrow \dot{H}^{-1}(\Omega), & \widehat{\text{div}} : L^2(\Omega) &\rightarrow \dot{H}^{-1}(\Omega) \end{aligned}$$

of curl and div, respectively, are defined for $E \in L^2(\Omega)$ by

$$\begin{aligned} \widetilde{\text{curl}} E(\Phi) &:= \langle \text{curl } \Phi, E \rangle_{L^2(\Omega)}, & \Phi &\in \dot{H}^1(\Omega), \\ \widehat{\text{curl}} E(\Phi) &:= \langle \text{curl } \Phi, E \rangle_{L^2(\Omega)}, & \Phi &\in H^1(\Omega), \\ \widetilde{\text{div}} E(\varphi) &:= -\langle \nabla \varphi, E \rangle_{L^2(\Omega)}, & \varphi &\in \dot{H}^1(\Omega), \\ \widehat{\text{div}} E(\varphi) &:= -\langle \nabla \varphi, E \rangle_{L^2(\Omega)}, & \varphi &\in H^1(\Omega). \end{aligned}$$

Finally, we emphasize that Theorem 2 provides a global result while the classical div-curl-lemma by Murat [13] and Tartar [23] is a local one. The latter may be formulated as follows:

Theorem 3 (classical div-curl-lemma) *Let $\Omega \subset \mathbb{R}^3$ be an open set and let*

- (i) $E_n, H_n, E, H \in L^2(\Omega)$,
 (i') $E_n \rightharpoonup E$ and $H_n \rightharpoonup H$ in $L^2(\Omega)$,
 (ii) $(\widetilde{\text{curl}} E_n)$ and $(\widetilde{\text{div}} H_n)$ be relatively compact in $H^{-1}(\Omega)$.

Then

$$(iii) \quad \forall \varphi \in C_1^\infty(\Omega) \quad \langle \varphi E_n, H_n \rangle_{L^2(\Omega)} \rightarrow \langle \varphi E, H \rangle_{L^2(\Omega)}.$$

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