Uniqueness of Integrable Solutions to $\nabla\zeta = G\zeta, \zeta|_\Gamma = 0$ for Integrable Tensor-Coefficients G and Applications to Elasticity

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Let $\Omega \subset R^N$ be bounded Lipschitz and $\emptyset \neq \Gamma \subset \partial\Omega$ relatively open. We show that the solution to the linear first order system

$$\nabla\zeta = G\zeta, \quad \zeta|_\Gamma = 0 \tag{1}$$

vanishes if $G \in \mathsf{L}^1(\Omega; \mathbb{R}^{(N\times N)\times N})$ and $\zeta \in \mathsf{W}^{1,1}(\Omega; \mathbb{R}^N)$, (e.g. $\zeta \in L^2$, $G \in L^2$). We prove

$$\|\!|\cdot|\!\| : \mathsf{C}^\infty_\circ(\Omega, \Gamma; \mathbb{R}^3) \to [0, \infty), \quad u \mapsto \|\operatorname{sym}(\nabla u P^{-1})\|_{\mathsf{L}^2(\Omega)}$$

to be a norm if $P \in \mathsf{L}^\infty(\Omega; \mathbb{R}^{3\times 3})$ with $\operatorname{Curl} P \in \mathsf{L}^p(\Omega; \mathbb{R}^{3\times 3})$, $\operatorname{Curl} P^{-1} \in \mathsf{L}^q(\Omega; \mathbb{R}^{3\times 3})$ for some $p, q > 1$ with $1/p + 1/q = 1$ and $\det P \geq c^+ > 0$. We give a new proof for the so called 'in-finitesimal rigid displacement lemma' in curvilinear coordinates: Let $\Phi \in \mathsf{H}^1(\Omega; \mathbb{R}^3)$, $\Omega \subset \mathbb{R}^3$, satisfy $\operatorname{sym}(\nabla\Phi^\top \nabla\Psi) = 0$ for some $\Psi \in \mathsf{W}^{1,\infty}(\Omega; \mathbb{R}^3) \cap \mathsf{H}^2(\Omega; \mathbb{R}^3)$ with $\det \nabla\Psi \geq c^+ > 0$. Then there are $a \in \mathbb{R}^3$ and a constant skew-symmetric matrix $A \in \mathfrak{so}(3)$, such that $\Phi = A\Psi + a$.

1 Introduction

Korn's inequality plays a very important role in linear elasticity. [1–5] This is reason enough to consider the question, whether it also holds for non-constant coefficients:

$$\exists\, c > 0 \quad \forall u \in \mathsf{H}^1_\circ(\Omega, \Gamma; \mathbb{R}^3) \qquad \|\operatorname{sym}(\nabla u P^{-1})\|_{\mathsf{L}^2(\Omega)} \geq c\|u\|_{\mathsf{H}^1(\Omega)}. \tag{2}$$

Quadratic forms of this type arise in applications to geometrically exact models of shells, plates and membranes, in micromorphic and Cosserat type models and in plasticity, [6–9]. And indeed, Korn's inequality with non-constant coefficients does hold, if $P, P^{-1}, \operatorname{Curl} P \in \mathsf{C}^1(\overline{\Omega}; \mathbb{R}^{3\times 3})$ as Neff has shown in [3], see also [10]. It has been proved in [10] to hold for continuous P^{-1}, whereas it can be violated for $P^{-1} \in \mathsf{L}^\infty(\Omega)$ or $P^{-1} \in SO(3)$ a.e. The counterexamples, given by Pompe in [10–12], each use the fact that for such P an expression of the form of

$$\|\!|u|\!\| := \|\operatorname{sym}(\nabla u P^{-1})\|_{\mathsf{L}^2(\Omega)} \tag{3}$$

is not a norm (it has a nontrivial kernel) on the spaces of functions considered. We aim to understand this expression better. By considering the first order system of equations (1) and proving the uniqueness of its solution, we can show that (3) is a norm on $\mathsf{C}^\infty_\circ(\Omega, \Gamma; \mathbb{R}^3) := \{u \in \mathsf{C}^\infty(\overline{\Omega}; \mathbb{R}^3) \,:\, \operatorname{dist}(\operatorname{supp} u, \Gamma) > 0\}$, where $\mathsf{C}^\infty(\overline{\Omega}; \mathbb{R}^3) := \{u|_\Omega \,:\, u \in \mathsf{C}^\infty_\circ(\mathbb{R}^3; \mathbb{R}^3)\}$.

The main part of our proof that $\|\!|\cdot|\!\|$ is a norm consists in obtaining $u = 0$ from $\operatorname{sym}(\nabla u P^{-1}) = 0$. By taking $P = \nabla\Psi$ to be a gradient, we present another proof of the infinitesimal rigid displacement lemma (which is important for linear elasticity in curvilinear coordinates; [13–15]), in dimension $N = 3$ which yields $\Phi = A\Psi + a$ with $A \in \mathfrak{so}(N)$, $a \in \mathbb{R}^N$. We need slightly more regularity than in [13] however. The key tool for obtaining our results is Neff's formula for the Curl of the product of two matrices, the first of which is skew-symmetric (see [3]).

2 Results

Let us first note that by ∇ we denote not only the gradient of a scalar-valued function, but also (as an usual gradient row-wise) the derivative or Jacobian of a vector-field. The Curl of a matrix is to be taken row-wise as usual curl for vector fields.

Theorem 2.1 (Unique Continuation) *Let $\Omega \subset \mathbb{R}^N$, $N \in \mathbb{N}$, be a Lipschitz domain, Γ be a relatively open and non-empty subset of $\partial\Omega$ and $G \in \mathsf{L}^1(\Omega; \mathbb{R}^{(N\times N)\times N})$. If $\zeta \in \mathsf{W}^{1,1}(\Omega; \mathbb{R}^N)$ solves*

$$\nabla\zeta = G\,\zeta, \quad \zeta|_\Gamma = 0,$$

then $\zeta = 0$.

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The differential equation itself cannot guarantee that a weak solution $\zeta \in \mathsf{L}^1(\Omega, \mathbb{R}^N)$ necessarily belongs to $\mathsf{W}^{1,1}(\Omega; \mathbb{R}^N)$. But this can be ensured by requiring higher integrability of G and ζ, since for bounded domains, e.g., the conditions $G \in \mathsf{L}^2(\Omega)$ and $\zeta \in \mathsf{L}^2(\Omega)$ imply $\nabla\zeta \in \mathsf{L}^1(\Omega)$ and hence $\zeta \in \mathsf{W}^{1,1}(\Omega)$; then an application of the theorem ensures that $\zeta = 0$. Thus we obtain the uniqueness of $\mathsf{L}^2(\Omega)$-solutions if the coefficients G are square-integrable. Of course, the same holds if $\zeta \in \mathsf{L}^p(\Omega)$ for arbitrary $p \geq 1$. Then G at least needs to be an $\mathsf{L}^q(\Omega)$-function, where $1/p + 1/q = 1$.

Theorem 2.2 (Norm) *Let $\Omega \subset \mathbb{R}^3$ be a Lipschitz domain, $\emptyset \neq \Gamma \subset \partial\Omega$ be relatively open, $P \in \mathsf{L}^\infty(\Omega; \mathbb{R}^{3\times3})$ with $\det P \geq c^+ > 0$, $\operatorname{Curl} P \in \mathsf{L}^p(\Omega; \mathbb{R}^{3\times3})$, $\operatorname{Curl} P^{-1} \in \mathsf{L}^q(\Omega; \mathbb{R}^{3\times3})$ for some $p, q > 1$ with $1/p + 1/q = 1$. Then*

$$|\!|\!| \cdot |\!|\!| : \mathsf{C}^\infty_\circ(\Omega, \Gamma; \mathbb{R}^3) \to [0, \infty), \quad u \mapsto \| \operatorname{sym}(\nabla u P^{-1}) \|_{\mathsf{L}^2(\Omega)} \tag{4}$$

defines a norm.

Remark 2.3 In the case of $p = q = 2$ and for $P \in \mathrm{SO}(3)$ a.e., $\operatorname{Curl} P^{-1} \in \mathsf{L}^2(\Omega)$ is no additional condition, since then $\operatorname{Curl} P \in \mathsf{L}^2(\Omega) \Leftrightarrow \operatorname{Curl} P^{-1} \in \mathsf{L}^2(\Omega)$. (Note that if $P \in \mathrm{SO}(3)$ a.e., then $P, \operatorname{Curl} P \in \mathsf{L}^p(\Omega)$ is equivalent to $P \in \mathsf{W}^{1,p}(\Omega)$, cf. [16].)

Remark 2.4 Since the norms $|\!|\!| \cdot |\!|\!|$ and $\| \cdot \|_{\mathsf{H}^1(\Omega)}$ are not shown to be equivalent, it is not clear whether the spaces $\mathsf{H}^1_\circ(\Omega, \Gamma) = \overline{\mathsf{C}^\infty_\circ(\Omega, \Gamma)}^{\| \cdot \|_{\mathsf{H}^1(\Omega)}}$ and $\overline{\mathsf{C}^\infty_\circ(\Omega, \Gamma)}^{|\!|\!| \cdot |\!|\!|}$ coincide. However, by [10], these norms are equivalent if $P \in \mathsf{C}^0(\overline{\Omega})$ with $\det P \geq c^+ > 0$.

Theorem 2.5 (Infinitesimal Rigid Displacement Lemma) *Let $\Omega \subset \mathbb{R}^3$ be a Lipschitz domain. Moreover, let $\Phi \in \mathsf{W}^{1,p}(\Omega; \mathbb{R}^3)$ and $\Psi \in \mathsf{W}^{1,\infty}(\Omega; \mathbb{R}^3) \cap \mathsf{W}^{2,q}(\Omega; \mathbb{R}^3)$ with $\det \nabla\Psi \geq c^+ > 0$ a.e. and $p, q > 1$, $1/p + 1/q = 1$. If*

$$\operatorname{sym}(\nabla\Phi^\top \nabla\Psi) = 0$$

then there exist $a \in \mathbb{R}^3$ and a constant skew-symmetric matrix $A \in \mathfrak{so}(3)$, such that $\Phi = A\Psi + a$.

3 Sketch of proofs

An application of Gronwall's inequality yields Theorem 2.1 for Ω being an interval. The case where Ω is a cube, and Γ a face of it can be reduced to this situation, ensuring also the unique continuation property for (1). For a general Lipschitz domain Ω we use a transformation of Γ and a neighborhood thereof onto such a cube. The proofs of Theorem 2.2 and Theorem 2.5 both rely heavily on the formula for the Curl of a product of two matrices, which reads (see [3])

$$\operatorname{Curl}(XY) = \operatorname{mat} L_Y(\operatorname{vec} \nabla \operatorname{axl} X) + X \operatorname{Curl} Y, \quad \det L_Y = -2(\det Y)^3$$

in the special case of a skew-symmetric X. There $\operatorname{mat}\colon \mathbb{R}^9 \to \mathbb{R}^{3\times3}$, $\operatorname{vec}\colon \mathbb{R}^{3\times3} \to \mathbb{R}^9$, $\operatorname{axl}\colon \mathfrak{so}(3) \to \mathbb{R}^3$ and $L\colon \mathbb{R}^{3\times3} \to \mathbb{R}^{9\times9}$ denote usual identification operators. It is used to reduce Theorem 2.2 to (1) where an application of Theorem 2.1 yields the definiteness. This formula is also used to prove the infinitesimal rigid displacement lemma: With a suitable approximation of $A = \nabla\Phi(\nabla\Psi)^{-1}$ it is possible to show that the weak partial derivatives of A vanish.
These results have been announced in [17] and published with details in [18].

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