A Korn's inequality for incompatible tensor fields

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Dedicated to Professor Rolf Leis on the occasion of his 80th birthday.

We prove a Korn-type inequality for bounded Lipschitz domains Ω in $\mathbb{R}^3$ and non-symmetric square integrable tensor fields $P : \Omega \to \mathbb{R}^{3\times 3}$ having square integrable rotation $\operatorname{Curl} P : \Omega \to \mathbb{R}^{3\times 3}$. For skew-symmetric P or compatible $P = \nabla v$ our estimate reduces to non-standard variants of Poincaré's or Korn's first inequality, respectively, for which our new estimate can be viewed as a common generalized version.

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with Lipschitz boundary Γ. Moreover, let Γ_t be a relatively open subset of Γ. We prove the following Korn-type inequality: There exists a constant $c > 0$ such that for all $P \in C^\infty(\mathbb{R}^3, \mathbb{R}^{3\times 3})$

$$c \int_\Omega |P|^2 \leq \int_\Omega |\operatorname{sym} P|^2 + \sum_{n=1}^{3} \int_\Omega |\operatorname{curl} P_n|^2 + \sum_{n=1}^{3} \int_{\Gamma_t} |\nu \times P_n|^2.$$

Here, ν denotes the outward unit normal at Γ and the tensor field P consists of the rows P_n^T, where P_n are vector fields in $C^\infty(\mathbb{R}^3, \mathbb{R}^3)$. Of course, this estimate holds true in the appropriate Hilbert space setting. Let $H(\operatorname{Curl}, \Omega)$ be the Hilbert space of all tensor fields having rows in $H(\operatorname{curl}, \Omega)$, the well known Sobolev space generated by curl.

Theorem 0.1 *There exists a constant $c > 0$ such that for all $P \in H(\operatorname{Curl}, \Omega)$ the estimate*

$$c||P||_{L^2(\Omega)} \leq ||\operatorname{sym} P||_{L^2(\Omega)} + \sum_{n=1}^{3} ||\operatorname{curl} P_n||_{L^2(\Omega)} + \sum_{n=1}^{3} ||\tau_t P_n||_{H^{-1/2}(\Gamma_t)}$$

holds true, where τ_t denotes the restricted tangential trace.

Our proofs rely on three essential tools, namely

1. Maxwell's estimate (Poincaré-type estimate for curl and div);
2. Helmholtz' decomposition;
3. Korn's first inequality.

There are immediate consequences of Theorem 0.1.

Corollary 0.2 *There exists a constant $c > 0$ such that*

$$c||P||_{L^2(\Omega)} \leq ||\operatorname{sym} P||_{L^2(\Omega)} + \sum_{n=1}^{3} ||\operatorname{curl} P_n||_{L^2(\Omega)}$$

holds for all $P \in H(\operatorname{Curl}, \Omega)$ with vanishing restricted tangential trace on Γ_t.

Corollary 0.3 *(generalized Korn's first inequality) There exists a constant $c > 0$ such that*

$$c||\nabla v||_{L^2(\Omega)} \leq ||\operatorname{sym} \nabla v||_{L^2(\Omega)}$$

holds for all vector fields $v \in H(\Omega)$ vanishing on Γ_t or having components ∇v_n normal at Γ_t.

Remark 0.4 (generalized Poincaré's inequality) Since vector fields can be identified with skew-symmetric tensor fields and Curl bounds (pointwise) the full gradient of those, we obtain by Theorem 0.1 and Corollary 0.2 new variants of Poincaré's inequality.

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We want to note that similar results hold in $\mathbb{R}^N$ and in the Banach space setting of L^p-spaces.

Application In the theory of extended continuum mechanics we encounter the micromorphic approach. A subvariant of this model can be written in the form of a minimization problem for two fields, i.e., the classical displacement $u : \Omega \subset \mathbb{R}^3 \to \mathbb{R}^3$ and the micromorphic tensor field $P : \Omega \subset \mathbb{R}^3 \to \mathbb{R}^{3\times 3}$. The additional field may be needed in the description of foams and bones [1–4]. The problem is to find the pair (u, P) such that

$$\int_\Omega \mu|\operatorname{sym}(\nabla u - P)|^2 + \frac{\lambda}{2}|\operatorname{tr}(\nabla u - P)|^2 - f\cdot u + h^+\big(\mu|\operatorname{sym} P|^2 + \frac{\lambda}{2}|\operatorname{tr} P|^2\big) + \mu L_c^2|\operatorname{Curl} P|^2 \longrightarrow \min$$

subject to the boundary conditions of place $u|_\Gamma = 0$ and $\nu \times P|_\Gamma = 0$. Here, the problem is driven by the body force f and $L_c > 0$ has dimensions of length, $h^+ > 0$ is a non-dimensional factor and μ, λ are the Lamé-constants of the material. With our estimate at hand one can show that the unique solution satisfies $u \in H^1(\Omega, \mathbb{R}^3)$ and $P \in H(\operatorname{Curl}; \mathbb{R}^{3\times 3})$. In order that the model is invariant with respect to superposed infinitesimal rigid rotations, i.e., $(\nabla u, P) \mapsto (\nabla u + A, P + A)$ for constant $A \in \mathfrak{so}(3)$, the symmetric local contribution $\operatorname{sym} P$ is mandatory.

Provided that $h^+ = 1$, this formulation is a relaxed formulation of linear elasticity, since the stored energy will always be less than the stored energy for the corresponding linear elastic formulation: Just take $P = \nabla u$, where u is the classical solution. This remains true even in the formal limit $L_c \to \infty$.

References

[1] A. Klawonn, P. Neff, O. Rheinbach, and S. Vanis. FETI-DP domain decomposition methods for elasticity with structural changes: P-elasticity. *ESAIM: Math. Mod. Num. Anal.*, 45:563–602, 2011.

[2] P. Neff. *Existence of minimizers for a finite-strain micromorphic elastic solid.* Preprint 2318, http://www3.mathematik.tu-darmstadt.de/fb/mathe/bibliothek/preprints.html, Proc. Roy. Soc. Edinb. A, 136:997–1012, 2006.

[3] P. Neff and S. Forest. *A geometrically exact micromorphic model for elastic metallic foams accounting for affine microstructure. Modelling, existence of minimizers, identification of moduli and computational results.* J. Elasticity, 87:239–276, 2007.

[4] P. Neff, J. Jeong, I. Münch, and H. Ramezani. *Mean field modeling of isotropic random Cauchy elasticity versus microstretch elasticity.* Preprint 2556, http://www3.mathematik.tu-darmstadt.de/fb/mathe/bibliothek/preprints.html, Z. Angew. Math. Phys., 3(60):479–497, 2009.

[5] P. Neff, D. Pauly, and K.-J. Witsch, *An Elementary Extension of Korn's First Inequality to H(Curl) motivated by Gradient Plasticity with Plastic Spin*, submitted (2011), http://arxiv.org/abs/1105.4039; Preprint SM-E-736, Universität Duisburg-Essen, Schriftenreihe der Fakultät für Mathematik, http://www.uni-due.de/mathematik/preprints.shtml.

[6] P. Neff, D. Pauly, and K.-J. Witsch, *Maxwell meets Korn: A New Coercive Inequality for Tensor Fields in* $\mathbb{R}^{N\times N}$ *with Square-Integrable Exterior Derivative*, submitted (2011), http://arxiv.org/abs/1105.5013; Preprint SM-E-737, Universität Duisburg-Essen, Schriftenreihe der Fakultät für Mathematik, http://www.uni-due.de/mathematik/preprints.shtml.

