

Sámar school

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Differential Forms and FEM

Part 1 : Theory

1 Manifolds

1. 1 Topological Spaces

Let (X, d) be a metric space. We call a subset $S \subset X$ "open" iff

$$\forall s \in S \exists r > 0 \quad U(s, r) \subset S ,$$

where $U(s, r) = \{t \in X : d(t, s) < r\}$ with $r > 0$ denotes the open ball of radius r around s . Then we have for the family of all open subsets of X

$\mathcal{O}(X) : \text{power set of } X$

$$\mathcal{O}_+ = \{O \in \mathcal{P}(X) : O \text{ open}\}$$

$$(O_1) \quad \emptyset, X \in \mathcal{O}$$

$$(O_2) \quad O_1, \dots, O_L \in \mathcal{O} \Rightarrow \bigcap_{l=1}^L O_l \in \mathcal{O}$$

$$(O_3) \quad \{O_i\}_{i \in I} \subset \mathcal{O} \Rightarrow \bigcup_{i \in I} O_i \in \mathcal{O}$$

Note: Δ

"finite intersections of open sets are open."

"arbitrary unions of open sets are open."

Definition 1 Let X be a set. A family $\mathcal{O} \subset \mathcal{P}(X)$

satisfying $(O_1), (O_2), (O_3)$ is called "topology" for X and $X = (X, \mathcal{O})$ is called a topological space. We don't call the elements of \mathcal{O} the open subsets of X or shortly open.

Let (X, \mathcal{O}) be a topological space.

Definition 2 $A \in \mathcal{P}(X)$ is called "closed", iff $X \setminus A \in \mathcal{O}$.

We write for the family of all closed subsets of X

$$\mathcal{A}^- = \{A \in \mathcal{P}(X) : X \setminus A \in \mathcal{O}\} .$$

Remark 3 (i) Metric spaces are topological spaces.

(ii) By $(O_1) - (O_3)$ we get immediately

$$(A_1) \quad \emptyset, X \in \mathcal{A}^-$$

$$(A_2) \quad A_1, \dots, A_L \in \mathcal{A}^- \Rightarrow \bigcup_{l=1}^L A_l \in \mathcal{A}^-$$

$$(A_3) \quad \{A_i\}_{i \in I} \subset \mathcal{A}^- \Rightarrow \bigcap_{i \in I} A_i \in \mathcal{A}^-$$

Note: Δ

"finite unions of closed sets are closed"

"arbitrary intersections of closed sets are closed."

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Example 4 Every set becomes by

$\Omega := \mathcal{P}(X)$ (discrete topology) (finest)

$\Omega = \{\emptyset, X\}$ (indiscrete topology) (coarsest)

a topological space.

Definition 5 Let (X, Ω) be a topological space. X "separates" the points of X or is called "Hausdorff", iff

$$\forall x, y \in X, x \neq y \Rightarrow \exists U, V \in \Omega \quad x \in U, y \in V \\ \text{and} \quad U \cap V = \emptyset.$$

Definition 6 Let (X, Ω) be a topological space and $S \subset X$. Then we define

(i) $S^\circ := \text{int } S := \bigcup_{\substack{\Omega \in \Omega \\ \Omega \subset S}} \Omega$, the "interior" or "inner set" of S ;

(ii) $\bar{S} := \text{cls } S := \bigcap_{\substack{A \subset X \\ S \subset A}} A$, the "closure" of S ;

(iii) $\partial S := \bar{S} \cap \overline{S^c}$, the "boundary" of S ;

(iv) $\text{AP}(S) := \{x \in X : \bigwedge_{\substack{x \in \Omega \subset S \\ \Omega \in \Omega}} \Omega \cap S \neq \emptyset\}$, the "accumulation points" of S .

The $\dot{S} := S \setminus \{x\}$ for some fixed $x \in X$.

Definition 7 Let (X, Ω_X) , (Y, Ω_Y) be topological spaces and $f: X \rightarrow Y$ a function. Then we call f "continuous" and write $f \in C^0(X, Y)$, iff

$$f^{-1}(\Omega_Y) \subset \Omega_X.$$

($f^{-1}(S)$: "preimage" of S)

Remark 8 f is continuous iff $f^{-1}(U_X) \subset \Omega_X$.

Proof: $A \in \Omega_Y \Rightarrow \forall A \in \Omega_Y \Rightarrow f^{-1}(Y \setminus A) \in \Omega_X \Rightarrow f^{-1}(A) \in \Omega_X$.
 "
 $\Rightarrow f^{-1}(A) \in \Omega_X$

Note: Preimages of open resp. closed sets are open resp. closed. ?

Definition 9 Let (X, \mathcal{O}) be a topological space. We say $K \subset X$ is "compact", iff every open covering of K , i.e. $\mathcal{C} \subset \mathcal{O}$ with $K \subset \bigcup_{C \in \mathcal{C}} C$, contains a finite subcovering, i.e. $\mathcal{C}_{fin} \subset \mathcal{C}$ with $K \subset \bigcup_{C \in \mathcal{C}_{fin}} C$ and $\#\mathcal{C}_{fin} < \infty$.

Theorem 10 Let $(X, \mathcal{O}_X), (Y, \mathcal{O}_Y)$ be topological spaces and $f \in C^0(X, Y)$ as well as X be compact.

(i) Then $f(X)$ is compact.

(ii) If f is injective, i.e. $f^{-1}(x) = \emptyset$ exists, and Y Hausdorff, then $f^{-1} \in C^0(f(X), X)$.

Proof: (i) $f(X) \subset \bigcup_{i \in I} O_i$, $O_i \in \mathcal{O}_Y$.

$$\Rightarrow X = f^{-1}(f(X)) \subset f^{-1}\left(\bigcup_{i \in I} O_i\right) = \bigcup_{i \in I} f^{-1}(O_i) \in \mathcal{O}_X$$

$$\Rightarrow X = \bigcup_{i \in I_{fin}} f^{-1}(O_i)$$

$$\Rightarrow f(X) = \bigcup_{i \in I_{fin}} f(f(O_i)) \subset \bigcup_{i \in I_{fin}} O_i$$

(ii) Let $A \in \mathcal{O}_Y$. aim: $f^{-1}(A) \in \mathcal{O}_X$. (since $(f^{-1})^{-1} = f$)
 A is compact. $\Rightarrow f(A)$ compact $\stackrel{Y \text{ Hausd.}}{\Rightarrow} f(A)$ closed ■

Theorem 11 Corollary II: Let X be a compact topological space and $f \in C^0(Y, \mathbb{R})$. Then f possesses maximum and minimum.
 Proof: $f(X)$ compact $\Rightarrow f(X)$ bounded and closed. ■

Theorem 12 (Stone-Weierstraß) Let X be a compact topological space and $C^0(X, \mathbb{R})$ be equip with $\|\cdot\|_\infty$, i.e.

$\|f\|_\infty := \max_{x \in X} |f(x)|$. Then any algebra $A \subset C^0(X, \mathbb{R})$, which separates the points of X and possesses the 1, is dense in $C^0(X, \mathbb{R})$.
 I.e. $\overline{A} = C^0(X, \mathbb{R})$ (since $C^0(X, \mathbb{R})$ is already closed)

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Lemma 13 Compact subsets of metric spaces are bounded and closed.

1.2 Differentiable Manifolds

Let (X, \mathcal{O}) denote a topological Hausdorff space satisfying the second axiom of countability, i.e. there exists a countable basis of the topology \mathcal{O} .

Moreover let $N \in \mathbb{N}$.

Definition 14 A " N -dimensional chart" or "chart" for X is a homeomorphism (bijective and C^l -continuous)

$$\varphi: U^{\text{open}} \subset X \rightarrow U^{\text{open}} \subset \mathbb{R}^N .$$

We call $U = U(\varphi)$ a "chart domain". We call X locally Euclidean, if every point of X belongs to a chart domain U .

Remark 15 Inverse charts may be regarded as "classical" "usual" parameterizations known from vector analysis in the calculus course.

Definition 16 For two charts $(U, \varphi), (V, \psi)$ we call for X we call the homeomorphism $\psi \circ \varphi^{-1}$, i.e.

$$\psi \circ \varphi^{-1}: \psi(U \cap V) \subset \mathbb{R}^N \rightarrow \varphi(U \cap V) \subset \mathbb{R}^N ,$$

a "change of charts." If $\psi \circ \varphi^{-1}$ is even C^l with $l \in \mathbb{N}_0 \cup \{\infty\}$ we say that φ and ψ change C^l . If $l = \infty$ we "differentiable" means $l = \infty$.

Definition 17 A family $\mathcal{A} := \{(U_i, \varphi_i) : i \in I\}$ is called atlas for X , if $X = \bigcup_{i \in I} U_i$. If all changes of charts are differentiable we say call it a "differentiable atlas". Then we denote (X, \mathcal{A}) a differentiable N -dimensional manifold. Shortly we call X a manifold.

Remark 18 φ is called "compatible" with an (differentiable) atlas, if φ changes differentiable with all $\psi_i \in \mathcal{A}$.
 By taking all compatible charts (with \mathcal{A}) one ends up with a maximal atlas denoted by \mathcal{B} .

Now we want to define some local properties of maps

$$\varphi: X \rightarrow Y .$$

For this we pick a chart (U, φ) for some point $u \in X$ with $u \in U$ and look at the pull-back $\tilde{\varphi}$

$$\tilde{\varphi} := \varphi \circ \varphi^{-1}: U = \varphi(U) \rightarrow Y$$

If φ has a property locally at x , then we say that φ has a property locally at u , if $\tilde{\varphi}$ has got this property locally at $x = \varphi(u)$. with respect to φ

If the property is even independent of the chart φ , then we say that φ has this property locally at u .
 For example, φ is differentiable in u , if $\tilde{\varphi} \circ \varphi^{-1}$ is differentiable in $x = \varphi(u)$. Of course, if this holds for one chart φ of an differentiable atlas \mathcal{A} , then it holds for every chart ψ of \mathcal{A} , since the changes of charts are differentiable, i.e.

$$\textcircled{C} \quad f \circ \varphi^{-1} = \underbrace{f \circ \tilde{\varphi}}_{\in C^\infty} \circ \underbrace{\varphi^{-1}}_{\in C^\infty} : Y(U, V) \rightarrow \mathbb{R} .$$

Analogously we may proceed if Y also the image manifold is a manifold. But φ should be continuous in this case, since we have to guarantee that $\varphi(u) \in V$ holds.

Remark 19 Let $f: X \rightarrow Y \in C^0(X, Y)$ be a mapping between the two manifolds X, Y . Then there exists for every chart (V, ψ) from around $f(u) = v \in V$ a chart (U, φ) of u with $f(u) \in V$.

Proof: $v \in \psi^{-1}(\{v\}) \subset f^{-1}(V)$ open $\subset X$. Let $(\tilde{U}, \tilde{\varphi})$ be a chart of u . Then $U := \tilde{U} \cap f^{-1}(V)$ is open and $\varphi := \tilde{\varphi}|_U$ maps U to V . □

Definition 20 A continuous mapping between two manifolds X and Y is called differentiable C^l (or differentiable, if $l=\infty$) in $u \in X$, if f is C^l (or differentiable, if $l=\infty$) with respect to charts, i.e.

$$f(u, \varphi_u) \rightarrow f(v, \varphi_v) \quad \tilde{f} := \varphi_0 f \circ \varphi^{-1} |_{\varphi(u)} : \varphi(u) \subset \mathbb{R}^n \rightarrow \varphi(v) \subset \mathbb{R}^m$$

$$\in C^l \text{ (or differentiable, if } l=\infty)$$

$$\Leftrightarrow \forall (u, \varphi_u) \forall (v, \varphi_v) \quad \tilde{f} \dots .$$

f is called C^l (or diff...) , if f is C^l (or diff...) ~~in all~~ in every $u \in X$. We call f a C^l -diffeomorphism ~~if~~ if f ~~is~~ (or diffeomorphism), if f is C^l (or C^∞) and bijective and f as well as f^{-1} are both C^l (or C^∞).

In the following we fix:

X ~~man~~ (diff.) manifold of dimension N , $u, \varphi \in X$

Y (diff) manifold of dimension M , $v, \tilde{\varphi} \in Y$

(U, φ) chart for u, φ_u with $\tilde{x}_i = \varphi_i(u)$, $x = \varphi(u)$

$(V, \tilde{\varphi})$ chart for $v, \tilde{\varphi}_v$ with $y_i = \tilde{\varphi}_v(v)$, $y = \tilde{\varphi}(v)$

$f: X \rightarrow Y$ differentiable, $\varphi(\tilde{u}) = \tilde{v}$, $f(u) = v$

$\tilde{f}: \varphi_0 f \circ \varphi^{-1} |_{\varphi(u)} : \varphi(u) \subset \mathbb{R}^n \rightarrow \varphi(v) \subset \mathbb{R}^m$ (diff.)

Remark 21 (i) X, Y, f manifolds and $f \in C^\infty(X, Y)$, $g \in C^\infty(Y, Z)$

$$\Rightarrow g \circ f \in C^\infty(X, Z)$$

(ii) charts are diffeomorphisms.

Proof: (i) $g \circ g_0 f \circ \varphi^{-1} = \underbrace{g \circ g_0}_{C^\infty} \underbrace{\varphi^{-1} \circ f \circ \varphi}_{C^\infty} |_{\varphi(u)} ((w, z) \text{ chart})$

(ii) φ chart. $\left| \begin{array}{l} \varphi \in C^\infty \Leftrightarrow \text{id} \circ \varphi \circ \varphi^{-1} \in C^\infty \quad (Y = \mathbb{R}^M) \\ \text{chart id} \quad \text{chart} \\ \varphi \circ \varphi^{-1} \text{id} \quad \varphi \circ \varphi^{-1} \text{id} \in C^\infty \\ \text{id} \end{array} \right. \quad \boxed{\square}$

Lemma and Definition #22 The rank

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$$\text{rank}(f, \hat{u}) := \text{rank } \tilde{\Phi}'(\hat{x})$$

of f in $\hat{u} \in X$ is independent of the chosen chart.

Proof: $\tilde{\Phi}' = (\varphi_0 \circ f \circ \varphi^{-1})' = \cancel{\varphi_0 \circ f \circ \tilde{\Phi}^{-1} \circ \tilde{\varphi}_0}$

$$= (\varphi_0 \tilde{\varphi}^{-1} \circ \tilde{\varphi} \circ f \circ \tilde{\Phi}' \circ \tilde{\varphi} \circ \varphi^{-1})'$$

$$= (\varphi_0 \tilde{\varphi}^{-1})' \underbrace{(\tilde{\varphi} \circ f \circ \tilde{\Phi}^{-1})'}_{\stackrel{\text{diffeo}}{=}\hat{f}} (\tilde{\Phi} \circ \varphi^{-1})'$$

$$\stackrel{\text{diffeo}}{=} \hat{f}$$

$$\Rightarrow \text{rank } \tilde{\Phi}' = \text{rank } \hat{f} \quad \blacksquare$$

Theorem #23 (inverse function theorem)

Let $n = N$ and f possess maximal rank N . Then f is a local diffeomorphism at u . Furthermore there are charts (U, u) for u and (V, v) for v with $\tilde{f} = id_{\mathbb{R}^n}$ on $\tilde{\Phi}(U)$.

Proof. $\text{rank } \tilde{f}'(\hat{x}) = N \Leftrightarrow \tilde{f}$ local diffeo at \hat{x} .

inverse mapping theorem in \mathbb{R}^n

$$\Rightarrow \exists \tilde{U} \subset \varphi(U) : \hat{f} := \tilde{\Phi}|_{\tilde{U}} : \tilde{U} \rightarrow \tilde{V} := \tilde{\Phi}(\tilde{U}) \text{ bijective}$$

$$\text{and } \hat{f}, \hat{f}^{-1} \in C^\infty.$$

$$\text{Moreover: } id|_{\tilde{U}} = \hat{f}^{-1} \circ \hat{f} = \underbrace{\hat{f}^{-1} \circ \varphi \circ f \circ \varphi^{-1}}_{= \tilde{\varphi} : \varphi \circ f \circ \varphi^{-1}(\tilde{U}) \rightarrow \tilde{U}} =: \tilde{\varphi} \Rightarrow \tilde{f} = \hat{f}$$

chart for v . \blacksquare

Definition and

Theorem #24 (regular point theorem)

Let N, n . u is called a "regular point" if $\text{rank}(f_u) = n$, otherwise "singular point". At a regular point there exist α charts (U, u) and (V, v) , with such that

$$\tilde{f}(x) = (x_1, \dots, x_n) \quad \forall x \in U$$

holds, i.e. \tilde{f} is the projection onto the first n -components.

$\hat{x} = \varphi(\hat{u})$.

Proof: $\tilde{\varphi}'(\hat{x}) = \left[\begin{array}{c|c} \partial_1 \tilde{\varphi}_1(\hat{x}) \dots \partial_n \tilde{\varphi}_1(\hat{x}), \partial_{n+1} \tilde{\varphi}_1(\hat{x}) \dots \partial_n \tilde{\varphi}_1(\hat{x}) \\ \vdots \\ \partial_1 \tilde{\varphi}_m(\hat{x}) \dots \partial_n \tilde{\varphi}_m(\hat{x}), \partial_{n+1} \tilde{\varphi}_m(\hat{x}) \dots \partial_n \tilde{\varphi}_m(\hat{x}) \end{array} \right] \quad (8)$

$\qquad\qquad\qquad \Rightarrow A$

$\operatorname{rank} \tilde{\varphi}'(\hat{x}) = n \iff$ u.l.o.g. A regular $\Rightarrow \det A \neq 0$.
 $\qquad\qquad\qquad \Rightarrow$ changing variables

Then $B := \begin{bmatrix} \tilde{\varphi}'(\hat{x}) \\ 0 \text{ id}_{\mathbb{R}^{N-n}} \end{bmatrix}$ regular, i.e. $\det B = \det A \neq 0$.

Thus $F: U(u) \subset \mathbb{R}^N \rightarrow \mathbb{R}^N$ is a local diffeomorphism at \hat{x} .
 $x \mapsto \begin{bmatrix} \tilde{\varphi}(x) \\ x_{n+1} \\ \vdots \\ x_N \end{bmatrix}$

$\Rightarrow \exists U' \subset U(u) \quad F|_{U'} \text{ bijective} \wedge F, F^{-1} \in C^\infty,$
 a.e. $F: U' \rightarrow V' := F(U')$ diffeomorphism.

$$\Rightarrow V' \ni y = F \circ \tilde{\varphi}^{-1} \cancel{= F^{-1}(\tilde{\varphi}(x), x_{n+1}, \dots, x_N)}$$

$$= (\tilde{\varphi}(F^{-1}(y)), F_{n+1}^{-1}(y), \dots, F_N^{-1}(y))$$

$$\Rightarrow \forall y \in V': y_m = \tilde{\varphi}_m(F^{-1}(y)) \quad , m = 1, \dots, n$$

$$y_m = F_m^{-1}(y) \quad , m = n+1, \dots, N.$$

$\Rightarrow \tilde{\varphi} \circ F^{-1}$ is the projection onto the first n components.

Furthermore $\tilde{\varphi} = \varphi \circ \underbrace{\tilde{\varphi} \circ F^{-1}}_{=(F \circ \varphi)^{-1}} =: \tilde{\tilde{\varphi}}$

Since $F \circ \varphi$ is a chart for u . $\tilde{\varphi} = \tilde{\tilde{\varphi}}$ \blacksquare
 a.e. $(F \circ \varphi, \tilde{\varphi}'(u'))$

Notation #25

$\mathbb{R}_0^k := \{x \in \mathbb{R}^N : x_{n+1} = \dots = x_N = 0\} = \{(x_0) \in \mathbb{R}^N : x \in \mathbb{R}_0^k\}$
 $\subset \mathbb{R}^N$

and $P_k := \mathbb{R}_0^k \subset \mathbb{R}^N \rightarrow \mathbb{R}^k$ projection $\Rightarrow \mathbb{R}_0^k \cong \mathbb{R}^k$.
 $(x_0) \mapsto x$

$$\boxed{\begin{array}{l} \text{projection } \Rightarrow \mathbb{R}_0^k \cong \mathbb{R}^k \\ \Pi_k : \mathbb{R}^N \rightarrow \mathbb{R}^k \\ x \mapsto (x_1, \dots, x_k) \end{array}}$$

Definition and lemma #26

Submanifold of X of dimension k resp. codimension $n := N-k$, if for every point $u \in \Sigma$ there exists a chart (φ, U) of X for U with

$\varphi : U \subset X \rightarrow \varphi(U) \subset \mathbb{R}^N$ bijective (clear!),

$$\varphi(U \cap \Sigma) = \underline{P_k \circ \varphi}(u) = \varphi(U) \cap \mathbb{R}_0^k \cong P_k \circ \varphi(U).$$

Using the chart $(\varphi_\Sigma, U_\Sigma)$ with $U_\Sigma := U \cap \Sigma$ and $\varphi_\Sigma := P_k \circ \varphi$, i.e. the "flattener", Σ becomes itself a k -dimensional manifold (diff.).

Proof: φ_Σ bijective is clear.

$$\tilde{\varphi}_\Sigma \circ \varphi_\Sigma^{-1} = P_k \circ \underbrace{\tilde{\varphi} \circ \varphi^{-1}}_{C^\infty} \circ P_k^{-1} \in C^\infty$$

$$\text{and } \tilde{\varphi}_\Sigma \circ \varphi_\Sigma^{-1}(x) = P_k \circ \tilde{\varphi} \circ \varphi^{-1}(x_0) =$$

□

Definition and

Theorem #27 (regular value theorem)

$v \in Y$ is called a "singular value" of f , if

$f^{-1}(\{v\})$ contains a singular point, otherwise "regular value". In the latter case $\Sigma := f^{-1}(\{v\})$ is a $(k := N-n)$ -dimensional sub-manifold of X .

Proof:

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$$\forall u \in S_2 \quad \text{rank}(f, u) = n.$$

No k: $u \in S_2 \Leftrightarrow f(u) = v \Leftrightarrow u \circ f(u) = \hat{Y} = O$ (u.l.o.g.)

Theorem 11 \Rightarrow $(4,4), (4,5) \vdash \tilde{\phi} = \pi_{\bar{A}} |_{\psi(u)}$

Define flat for $\hat{u} \in S$

$$\phi : u \rightarrow \mathbb{R}^n$$

$$u \mapsto (\underbrace{\varphi_{n+1}(u), \dots, \varphi_n(u)}_{\in \mathbb{R}^k}, \underbrace{\varphi_0 f(u)}_{\in \mathbb{R}^m})$$

Then:

$$\begin{aligned} u \in S &\Rightarrow \phi(u) = (\varphi_{n+1}(u), \dots, \varphi_n(u), 0) \in \mathbb{R}_0^k \\ u \notin S &\Rightarrow \phi(u) = \dots, \varphi_0(u) \neq 0 \in \mathbb{R}^k \\ &\Rightarrow \phi(u) \notin \mathbb{R}_0^k. \end{aligned}$$

$$\Rightarrow \phi(\cup S) = \phi(U \cap S) = \phi(U) \cap \mathbb{R}_0^k$$

and $\phi \circ \psi^{-1}(x) = (x_{n+1}, \dots, x_n, \frac{\tilde{x}}{x_n}(x)) = (x_{n+1}, x_n, x_{n-1}, x_n)$
 $\Rightarrow \phi \circ \psi^{-1}$ bij. $\Rightarrow \phi$ bij.

amples ~~1~~ 28

(i) \mathbb{R}^n is a differentiable manifold with $\text{id} = (\mathbb{R}^n, \text{id})$.

(ii) Any vector space V of dimension N is naturally a N -dimensional manifold with

$$\varphi: V \rightarrow \mathbb{R}^n$$

$$v = \sum_{i=1}^n v_i e_i \mapsto \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$

where $\{e^u\}_{u=1,\dots,N}$ is a basis for V .

(iii) $g \in C^\infty(\mathbb{R}^N, \mathbb{R}^n)$, $N \geq n$.

$\{\nabla g_{\mu}(x)\}_{\mu=1,\dots,n}$ linear independent

for all $x \in \Omega := g^{-1}(\{\vec{y}\})$, $\vec{y} \in \mathbb{R}^n$.

Then Ω is a $(k := N - n)$ -dimensional sub-manifold of \mathbb{R}^N .

1.3 Construction of manifolds

Theorem 29 Let $n \geq N$ and $f: X \rightarrow Y$ a ~~differentiable homeomorphism~~ with $\text{rank}(f, u) = N$ for all $u \in X$. Then $f(X)$ is a N -dimensional (differentiable) ~~manifold~~ sub-manifold of Y .

Proof: ~~$\forall \tilde{U} \in \mathcal{F}(X)$, $U := f^{-1}(\tilde{U})$, $\tilde{U} = f^{-1}(U)$~~

~~Pick $(\tilde{U}, \tilde{\varphi})$ chart at \tilde{U} . $U := f^{-1}(\tilde{U}) \ni \tilde{u}$.~~

~~w.l.o.g. U is Pick chart $U \subset \tilde{U}$ with $u \in U$.~~

~~Pick chart (U, φ) with $u \in U$ and $U \subset \tilde{U}$. $v = f(u)$~~

~~Since $(U, \varphi), (\tilde{U}, \tilde{\varphi})$ are charts for $f(X)$ and~~

$$\varphi \circ f^{-1} \circ (\tilde{\varphi} \circ \tilde{f}^{-1})^{-1} = \varphi \circ \tilde{f}^{-1} \circ \tilde{\varphi} \circ \tilde{f}^{-1} \in C^\infty$$

\downarrow
 $= \text{id}$

~~$f(X)$ is a differentiable manifold.~~

~~$f(X)$ sub-manifold of Y ?~~

~~$\varphi \circ f(U)$ is a N -dim. manifold in \mathbb{R}^N ,~~
~~because $\varphi \circ f \circ \tilde{f}$ is a parametrisation of $\varphi(\tilde{U})$~~

~~Since $\text{rank}(\tilde{f}, x) = \text{rank}(f, u) = N$.~~

~~$\Rightarrow \phi: V \subset \mathbb{R}^N \rightarrow \mathbb{R}^n$ diffeomorphism~~

~~with $v \in \varphi(f(U)) \Leftrightarrow \phi(v), \dots, \phi_{n+1}(v) = 0$,~~

~~i.e. $\phi(\varphi(f(u))) = \phi(u)$~~

The maps $(\varphi_0 \circ f^{-1})$ are charts for $f(U)$ are charts for $f(X)$ since f is bijective and (12)

$$\varphi_0 \circ f^{-1} \circ (\tilde{\varphi}_0 \circ f^{-1})^{-1} = \underbrace{\varphi_0 \circ f^{-1} \circ f}_{=\text{id}} \circ \tilde{\varphi}_0^{-1} = \varphi_0 \circ \tilde{\varphi}_0^{-1} \in C^\infty.$$

$\Rightarrow f(X)$ N -dim. manifold.

$f(X)$ submanifold of Y ?

Let $v \in f(X) \subset Y$ and (U, v) be a chart for V .
w.l.o.g. $\varphi_0(U) \subset V$.
Then $\tilde{\varphi}$ has rank N ~~exact~~. Thus $\tilde{\varphi}: \varphi_0(U) \rightarrow \varphi_0(f(U))$ is a parametrization for $\varphi_0(f(U))$.

$\Rightarrow \varphi_0(f(U))$ is a N -dim manifold in \mathbb{R}^N .

$\Rightarrow \exists$ diffeom. $\phi: V' \setminus \varphi_0(f(U)) \cap \mathbb{R}^N \rightarrow \mathbb{R}^N$ with
 $y \in \varphi_0(f(U)) \cap V' \Leftrightarrow \phi_{n+1}(y) = \dots = \phi_N(y) = 0 \forall y \in V'$

i.e. $\phi(\varphi_0(f(U)) \cap V') = \phi(\varphi_0(f(U))) \cap \mathbb{R}_0^N$ ~~is a chart~~

$\Rightarrow (\phi \circ \varphi, \tilde{V})$ is chart for $v \in Y$.
and w.l.o.g. $f(U) \subset \tilde{V}$.

i.e. $\phi(\varphi_0(f(U)) \cap V') = \phi(V') \cap \mathbb{R}_0^N$.

Set $\tilde{V} :=$ w.l.o.g. $\varphi(V)$
w.l.o.g. $V' \subset \varphi(V)$. Set $\tilde{V}' := \varphi^{-1}(V') \subset V$.

Then $(\phi \circ \varphi, \tilde{V})$ is a chart for v and

$$\begin{aligned}\phi \circ \varphi(\varphi_0(f(U)) \cap \tilde{V}) &= \phi(\varphi_0(f(U)) \cap V') \\ &= \phi(V') \cap \mathbb{R}_0^N \\ &= \phi \circ \varphi(\tilde{V}') \cap \mathbb{R}_0^N.\end{aligned}$$

~~Thus~~ ~~$\phi \circ \varphi$~~



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Theorem 30 (i) $N = n$. Then $X \times Y$ is a N -dim. manifold.

(ii) $X \times Y$ is a $(n+m)$ -dim. manifold.

Proof:

(i) $u \in X \times Y \Rightarrow$ w.l.o.g. $u \in X \Rightarrow (\varphi, U)$ chart for u .
no new chart changes since X and Y are both open and closed.

(ii) $\phi: U \times V \subset X \times Y \rightarrow \mathbb{R}^n \times \mathbb{R}^m \cong \mathbb{R}^{n+m}$
 $(x, y) \mapsto (\varphi(x), \psi(y))$

are charts for $X \times Y$.

$$\phi^{-1}(x, y) = (\varphi^{-1}(x), \psi^{-1}(y))$$

$$\Rightarrow \phi \circ \phi^{-1} = (\varphi \circ \varphi^{-1}, \psi \circ \psi^{-1}) \in C^\infty$$

\uparrow \uparrow
 C^∞ C^∞



1.4 Exercises

①* Let $X := \mathbb{R}$ and $\mathcal{A}_1 := \{\varphi\}$, $\mathcal{A}_2 := \{g\}$ with $\varphi = \text{id}$ and $g = g(t) = t^3$.

(i) Are (X, \mathcal{A}_1) differentiable manifolds?

(ii) Are φ and g compatible?

(iii) Are (X, \mathcal{A}_1) diffeomorphic?

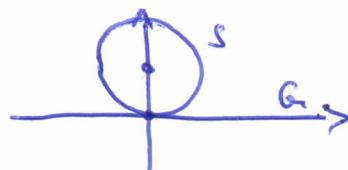
② (i) Is the boundary of a circular cone in \mathbb{R}^3 (with the induced metric) a differentiable manifold?



(ii) Same question for $X := S \cup G_1$;

$$S := \mathbb{S}_{\mathbb{R}^2}(e^2, 1)$$

$$G_1 := \{te^1 : t \in \mathbb{R}\}$$



③ We define in \mathbb{R} an equivalence relation

$$x \sim y \Leftrightarrow x - y \in \mathbb{Z}$$

and the corresponding equivalence classes $[x]$.

Show that

$$\mathbb{R}^{\sim} := \{[x] : x \in \mathbb{R}\}$$

with

$$d([x], [y]) := \min_{u \in \mathbb{Z}} \{ |x - y - u|\}$$

is a metric space. Introduce in \mathbb{R}^{\sim} an atlas, such that $X := (\mathbb{R}^{\sim}, \mathcal{A})$ becomes a differentiable manifold. Show that X is diffeomorphic to

$$S^1 := \{x \in \mathbb{R}^2 : |x| = 1\} = S^1_{\mathbb{R}^2}(0, 1)$$

④ We introduce ~~the~~ on the strip

$$S := \{x \in \mathbb{R}^2 : x_2 \in (-1, 1)\}$$

an equivalence relation by

$$x \sim y \Leftrightarrow \exists u \in \mathbb{Z} : x_1 - y_1 = 2\pi u \wedge x_2 = (-1)^u y_2.$$

Show that ~~S~~ is a metric space and a differentiable manifold, which is diffeomorphic to the "Nöbius strip" $\phi(S)$ with $\phi: S \rightarrow \mathbb{R}^3$ and defined by

$$\phi(t, s) := 2 \begin{bmatrix} \cos t \\ \sin t \\ 0 \end{bmatrix} + s \left(\sin \frac{t}{2} \begin{bmatrix} \cos t \\ \sin t \\ 0 \end{bmatrix} + \cos \frac{t}{2} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right).$$

⑤ * Show that X , the set of all real $(N \times N)$ -matrices is a differentiable manifold of dimension N^2 and that $O := \{A \in X : A \text{ orthogonal}^{\text{normal}}\}$, ~~the other~~ is a $\frac{1}{2}(N-1)N$ -dimensional sub-manifold of X .

Hint: $Y := \{A \in X : X \text{ symmetric}\}$ is a (15)
 $\frac{1}{2}N(N+1)$ -dimensional manifold. Show that
 $i_{\text{cl}}|_X$ is a regular value of Φ

$$\begin{aligned}\Phi: X &\rightarrow Y \\ A &\mapsto A^t A\end{aligned}$$

⑥ Let X be the space of all regular $(N \times N)$ -matrices,
i.e. $X = \{A \in M(N, N) : \det A \neq 0\}$. ($\dim X = N^2$,
since X is a open subset of $M(N, N)$ (Neumann's series!))
Let S be a symmetric matrix.

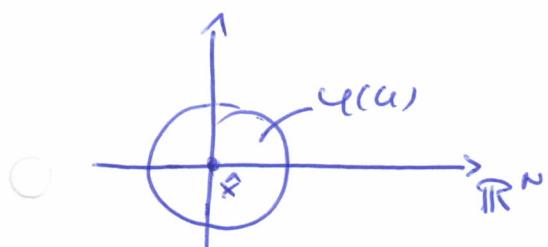
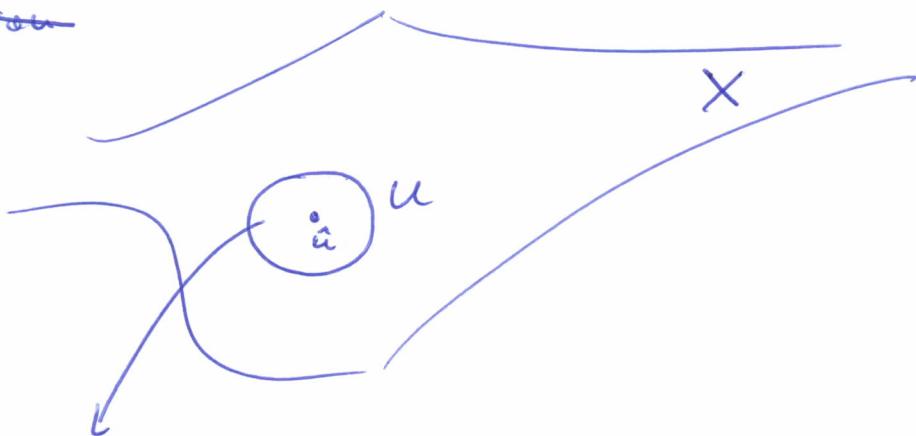
Show that $\mathcal{S} := \{A \in X : \langle SAx, Ax \rangle = \langle Sx, x \rangle \forall x \in \mathbb{R}^N\}$

is a differentiable submanifold of X and
determine $\dim \mathcal{S}$.

2. Tangential Space and Differential

2.1 The Tangential Space

Definition



Definition Let $\hat{u} \in X$.

(i) f is called a "gen" at \hat{u} , if there exists a open neighbourhood U of \hat{u} with $f \in C^{\infty}(U, \mathbb{R})$. We ~~also~~ say that two gens ^{f,g} are equivalent at \hat{u} , if there exists a open neighbourhood of \hat{u} , such that $f|_U = g|_U$.

The equivalence classes of gens become in a natural way a algebra, which we will denote by $C^{\infty}(\hat{u})$.

~~(ii) Let A be a algebra. Then a linear mapping~~

~~is called a "derivation", if it satisfies the product rule~~

$$v(f \cdot g) = f(\hat{u}) \cdot v(g) + g(\hat{u}) \cdot v(f)$$

~~for all~~

(c) A linear mapping $\tau: C^\infty(\hat{U}) \rightarrow \mathbb{R}$ is called a "derivation" at \hat{u} , if τ satisfies the product rule, i.e.

$$\tau(f \cdot g) = f(\hat{u}) \tau(g) + g(\hat{u}) \tau(f) \quad \forall f, g \in C^\infty(\hat{U}).$$

Note: $\tau \in (C^\infty(\hat{U}))'$.

Lemma 2 Let τ be a derivation at \hat{u} . Then

$$(i) \quad \tau(1) = 0$$

$$(ii) \quad \tau(f) = \tau(g) \iff f \sim g$$

$$(iii) \quad \tau(f \cdot g) = 0 \iff f(\hat{u}) = g(\hat{u}) = 0$$

Proof:

$$(i) \quad \tau(1) = \tau(1 \cdot 1) = \tau(1) + \tau(1) \Rightarrow \tau(1) = 0$$

$$(ii) \quad f \sim g \Rightarrow f - g \sim 0 \Rightarrow \tau(f) - \tau(g) = \tau(f - g) = \tau(0) = 0$$

$$(iii) \quad \text{#} \checkmark \quad \blacksquare$$

Example 3

(i) The coordinate functions or coordinates

$$\varphi_i: U \rightarrow \mathbb{R}$$

of a chart (φ_i, U) are forms at $u \in U$, i.e. $\varphi_i \in C^\infty(u)$.

(ii) The "partial derivatives"

$$\partial_i^{\varphi}: C^\infty(u) \rightarrow \mathbb{R}, \quad x = \varphi(u)$$

$$f \mapsto \partial_i^{\varphi}(f \circ \varphi^{-1})(x)$$

are derivations at u . These depend of course on φ .

well defined?: $f, g \in C^\infty(u)$ with $f \sim g$.

$$\Rightarrow f - g \sim 0 \Rightarrow \partial_i^{\varphi}(f - g) = \partial_i^{\varphi}(0 \circ \varphi^{-1})(x) = 0$$

$$= \partial_i^{\varphi} f - \partial_i^{\varphi} g$$

$$\text{Linear?}: \partial_i^{\varphi}(\lambda f + g) = \partial_i^{\varphi}(x f \circ \varphi^{-1} + g \circ \varphi^{-1})(x)$$

$$= \lambda \partial_i^{\varphi} f + \partial_i^{\varphi} g$$

$$\begin{aligned}
 \text{product rule: } \partial_i^\varphi(f \cdot g) &= \partial_i((f \cdot g) \circ \varphi^{-1})(x) \\
 &= \partial_i(\underbrace{f \circ \varphi^{-1}}_{=u})(x) \cdot g(\underbrace{\varphi^{-1}(x)}_{=u}) \\
 &\quad + \underbrace{\partial_i(g \circ \varphi^{-1})(x)}_{\in \mathcal{L}} \cdot f(\underbrace{\varphi^{-1}(x)}_{=u}) \\
 &= \partial_i^\varphi g
 \end{aligned} \tag{18}$$

Definition and Theorem 4

The set

$$T_u(X) := \{ \tau : \text{derivation at } u \}$$

is naturally a N -dimensional vector space, since the special derivations

$$\{ \partial_u^\varphi \}_{u=1, \dots, N}$$

form a basis, called "chart basis", of $T_u(X)$.

$T_u(X)$ is called the "tangential space" at u of X .

Proof: vector space is clear, as well as $\partial_u^\varphi \in T_u(X)$.

$$\text{Since } \partial_u^\varphi(\varphi_m) = \partial_u(\underbrace{\varphi_m \circ \varphi^{-1}}_{\in C^0(u)}) = \partial_u \varphi_m = \delta_{um}$$

The ∂_u^φ are linear independent.

$$(0 = \sum_{u=1}^n \lambda_u \partial_u^\varphi \underset{\varphi_m}{=} 0 \Rightarrow \lambda_m = 0)$$

Note: $\{\partial_u^\varphi\}$ is the dual basis of $C^0(u)^*$ for $\{\varphi_m\} \subset C^0(u)$

$$\Rightarrow \dim T_u(X) \geq N.$$

Let v be in $T_u(X)$.

$$\text{ansatz: } v = \sum_{u=1}^N v_u \partial_u^\varphi$$

$$\Rightarrow v_u = v(\varphi_u)$$

$$\Rightarrow v = \sum_{u=1}^N v(\varphi_u) \partial_u^\varphi \in \text{lin } \{ \partial_u^\varphi \}_{u=1, \dots, N}$$

to show!

Let f be in $C^\infty(\hat{G})$ arbitrary.

(19)

Calculation:

$$\cancel{\tau(f)} = \cancel{\tau(f(\hat{u}) + f)}$$

$$f(u) = f(\hat{u}) + f(u) - f(\hat{u})$$

$$, x = \varphi(u)$$

$$= f(\hat{u}) + \underbrace{f \circ \varphi^{-1}(x) - f \circ \varphi^{-1}(\hat{x})}_{=: H(x)} = H(x)$$

$$\cancel{\tau(t \cdot x)} := \cancel{f \circ \varphi^{-1}}$$

$$\cancel{u(t)} := H(t \cdot x). \Rightarrow u(1) = H(x), u(0) = H(\hat{x}) \quad H: U' \subset \mathbb{R}^n \rightarrow \mathbb{R}$$

$$u(t) = H(t(x - \hat{x}) + \hat{x}) \Rightarrow u(1) = H(x), u(0) = H(\hat{x})$$

$$\begin{aligned} \Rightarrow u(1) - u(0) &= \int_0^1 u'(t) dt = \int_0^1 H'(t(x - \hat{x}) + \hat{x})(x - \hat{x}) dt \\ &= \int_0^1 \partial_u(f \circ \varphi^{-1})(t(x - \hat{x}) + \hat{x})(\varphi_u - \hat{\varphi}_u) dt \\ &= \int_0^1 \partial_u(f \circ \varphi)(t(x - \hat{x}) + \hat{x}) dt (\varphi_u - \hat{\varphi}_u) \\ &\qquad\qquad\qquad =: \alpha_u(x) \\ \cancel{\tau(f(\hat{u}))} &= 0 \end{aligned}$$

$$\Rightarrow \cancel{\tau(f)} = \cancel{\sum_{u=1}^N \varepsilon(u) \alpha_u(x)}$$

$$\Rightarrow f = f(\hat{u}) + \alpha_u \circ \varphi \cdot (\varphi_u - \varphi_u(\hat{u}))$$

$$\Rightarrow \cancel{\tau(f)} = \sum_{u=1}^N \cancel{\varepsilon(u) \alpha_u \circ \varphi \cdot (\varphi_u - \varphi_u(\hat{u}))}$$

Lemma 2(i)

$$\begin{aligned} &= \sum_{u=1}^N \alpha_u \circ \varphi(\hat{u}) \cdot \cancel{\varepsilon(\varphi_u - \varphi_u(\hat{u}))} \\ &\qquad\qquad\qquad = \cancel{\varepsilon(\varphi_u)} - \cancel{\varepsilon(\varphi_u(\hat{u}))} = 0 \end{aligned}$$

$$+ \sum_{u=1}^N \underbrace{(\varphi_u(\hat{u}) - \varphi_u(\hat{u}))}_{=0} \cancel{\varepsilon(\alpha_u \circ \varphi)} = 0$$

$$= \sum_{u=1}^N \cancel{\varepsilon(\varphi_u)} \underbrace{\alpha_u(x)}_{=0}$$

$$= \int_0^1 \partial_u(f \circ \varphi)(\hat{x}) dt = \partial_u(f \circ \varphi^{-1})(\hat{x})$$

$$= \partial_u^F \varphi$$

■

Remark 5 The bases $\{\partial_u^\varphi\}_{u=1,\dots,n}$ of $T_\varphi(X)$ (20)

depend on φ and u . We have the isomorphisms

$$I := I_u^\varphi : T_\varphi(X) \rightarrow \mathbb{R}^n$$
$$\tau = \sum_{u=1}^n t_u \partial_u^\varphi \mapsto \begin{bmatrix} \tau_1 \\ \vdots \\ \tau_n \end{bmatrix}$$

which of course depend on φ and u as well.

Thus $I(\partial_u^\varphi) = e^u$. We have

$$\partial_u^\varphi f = \partial_u(\varphi \circ \varphi^{-1})(x) = \langle e^u, \nabla(\varphi \circ \varphi^{-1})(x) \rangle = \partial_{e^u}(\varphi \circ \varphi^{-1})(x)$$
$$\tau(f) = \sum_{u=1}^n t_u \partial_u(\varphi \circ \varphi^{-1})(x) = \langle \tau, \nabla(\varphi \circ \varphi^{-1})(x) \rangle = \partial_\tau(\varphi \circ \varphi^{-1})(x).$$

Hence ∂_u^φ may be identified with the u -th partial derivative. In the special case $X = \mathbb{R}^n$ and $\varphi = \text{id}$ we have

$$\partial_u^{\text{id}} f = \partial_u f(x),$$
$$\tau(f) = \partial_\tau f(x).$$

Hence ∂_u^{id} may then be identified with the u -th partial derivative ∂_u and the tangential vector τ may be identified with the directional derivative of τ .

$$\partial_u^{\text{id}} \stackrel{?}{=} \partial_u = \langle e^u, \nabla \cdot \rangle \xrightarrow{I} e^u$$
$$\tau \stackrel{?}{=} \partial_\tau = \langle \tau, \nabla \cdot \rangle \xrightarrow{I} \tau$$

Remark 6 "Changing of chart bases".

$$\partial_u^\varphi = \sum_{m=1}^n \lambda_m \partial_m^\varphi \Rightarrow \partial_u^\varphi(\varphi_m) = \lambda_m$$
$$\Rightarrow \partial_m^\varphi = \sum_{u=1}^n \partial_m(\varphi_u) \partial_u^\varphi = \sum_{u=1}^n \partial_m(\varphi_u \circ \varphi^{-1})(x) \partial_u^\varphi$$

2.2 The Differential

Definition 7 Let $\Phi \in C^\infty(X, Y)$, $u \in X$, $\Phi(u) = v$. Then the mapping

$$d\Phi_u : T_u(X) \rightarrow T_v(Y)$$

$$\tau \mapsto d\Phi_u \tau$$

where $d\Phi_u \tau(\Phi) := \tau(f \circ \Phi) \in \mathbb{R}$ for all $f \in C^\infty(v)$, is called the differential of Φ at u .

Theorem 8 The differential $d\Phi_u$ is linear well defined and linear. Choosing $\{\partial_u^k\}_{u=1,\dots,n}$ and $\{\partial_v^l\}_{v=1,\dots,n}$ as bases for $T_u(X)$ and $T_v(Y)$ the differential admits the representation

$$\tilde{\Phi}'(x).$$

Proof:

① $d\Phi_u \tau$ is derivation at v .

$$\begin{aligned} d\Phi_u \tau(\lambda f + g) &= \tau((\lambda f + g) \circ \Phi) = \tau(\lambda f \circ \Phi + g \circ \Phi) \\ &= \lambda \tau(f \circ \Phi) + \tau(g \circ \Phi) \\ &= \lambda d\Phi_u \tau(f) + d\Phi_u \tau(g) \end{aligned}$$

$$\Rightarrow d\Phi_u \tau \in C^\infty(v)'$$

$$\begin{aligned} d\Phi_u \tau(f \cdot g) &= \tau((f \cdot g) \circ \Phi) = \tau(f \circ \Phi \cdot g \circ \Phi) \\ &= f \circ \Phi(u) \tau(g \circ \Phi) + g \circ \Phi(u) \cdot \tau(f \circ \Phi) \\ &= f(v) d\Phi_u \tau(g) + g(v) d\Phi_u \tau(f) \end{aligned}$$

$\Rightarrow d\Phi_u \tau$ satisfies the product rule.

(22)

② dF_u linear, since

$$\begin{aligned} dF_u(\tau + \sigma)(\phi) &= (\tau + \sigma)(f \circ F) \\ &= \tau(f \circ F) + \sigma(f \circ F) \\ &= dF_u \tau(\phi) + dF_u \sigma(\phi) \end{aligned}$$

$$\Rightarrow dF_u(\tau + \sigma) = dF_u \tau + dF_u \sigma$$

$$\begin{aligned} ③ dF_u(\partial_u^\psi)(\phi) &= \partial_u^\psi(f \circ F) = \partial_u(f \circ F \circ \psi^{-1})(x) \\ &= \partial_u(f \circ \psi^{-1} \circ \psi \circ F \circ \psi^{-1})(x), \tilde{\psi} = \psi \circ F \circ \psi^{-1} \\ &= \partial_u(f \circ \psi^{-1}) \underbrace{(\tilde{\psi}(x))}_{=x} \cdot \partial_u \tilde{\psi}(x) \\ &= \partial_u \tilde{\psi}(x) \partial_u^\psi(\phi) \end{aligned}$$

$$\Rightarrow dF_u(\partial_u^\psi) = \partial_u \tilde{\psi}(x) \partial_u^\psi$$

$$\Rightarrow dF_u \left|_{\{\partial_u^\psi\}} \right. = \tilde{\psi}'(x)$$

□

Theorem 9 (chain rule)

$$\tilde{\psi} \in C^\infty(X, Y), \tilde{G}_v \in C^\infty(Y, Z)$$

$$\Rightarrow d(G_v \circ \tilde{\psi})_u = dG_{v,u} \circ dF_u$$

Proof:

$$\begin{aligned} d(G_v \circ \tilde{\psi})_u \tau(\phi) &= \tau(f \circ G_v \circ \tilde{\psi}) \\ &= \underbrace{dF_u \tau}_{=\sigma}(f \circ G_v) \\ &= dG_{v,u}(dF_u \tau)(\phi) \end{aligned}$$

$$\Rightarrow d(G_v \circ \tilde{\psi})_u \tau = dG_{v,u} \circ dF_u \tau$$

$$\Rightarrow d(G_v \circ \tilde{\psi})_u = dG_{v,u} \circ dF_u$$

□

~~Theorem 8.12~~

Note From the previous proof we get

$$dF_u(\partial_u^\varphi) = \partial_u \tilde{F}_u(x) \partial_u^\varphi = \partial_u^\varphi (\varphi_0 F) \partial_u^\varphi$$

$$\Rightarrow I(dF_u(\partial_u^\varphi)) = \begin{bmatrix} \partial_u \tilde{F}_1(x) \\ \vdots \\ \partial_u \tilde{F}_n(x) \end{bmatrix} = \partial_u \tilde{F}(x) \in \mathbb{R}^n$$

$$= \partial_u^\varphi (\varphi_0 F)$$

Especially for $\varphi = \text{id}$ we obtain with $\varphi = \text{id}$

$$I(dF_u(\partial_u^\varphi)) = \partial_u^\varphi F = \partial_u(F \circ \varphi^{-1})(x) \in \mathbb{R}^n.$$

For charts the chart φ we get ($n=N$)

$$I(d\varphi_u(\partial_u^\varphi)) = \underbrace{\partial_u(\varphi_0 \varphi^{-1})}_{=\text{id}}(x) = e^u \in \mathbb{R}^n,$$

$$I(d\varphi_{u_n}(\partial_u^\varphi)) = \underbrace{\partial_u(\varphi_{u_n} \circ \varphi^{-1})}_{=\text{id}_{u_n}}(x) = \delta_{uu_n} \in \mathbb{R}. \quad \textcircled{*}$$

Hence every $f \in C^\infty(X, \mathbb{R})$ defines by $I \circ d\varphi_u$ an element of the dual space $T_u(X)'$. Since $\dim T_u(X)' = N$ and because of $\textcircled{*}$ the coordinates φ_u generate the dual basis of $T_u(X)'$, i.e.

$$T_u(X) = \text{lin} \{ \partial_1^\varphi, \dots, \partial_N^\varphi \},$$

$$T_u(X)' = \text{lin} \{ I \circ d\varphi_{u_1}, \dots, I \circ d\varphi_{u_N} \}.$$

Note: $\omega = \sum_{u=1}^N \omega_u I \circ d\varphi_{u_n} \Rightarrow \omega_u = \omega(\partial_u^\varphi)$

$$\Rightarrow \omega = \sum_{u=1}^N \omega(\partial_u^\varphi) I \circ d\varphi_{u_n} \quad \textcircled{**}$$

Remark: $I \circ d\varphi_{u_n}(\partial_u^\varphi) = \delta_{uu_n} = \partial_u^\varphi(\varphi_{u_n})$

Lemma 40 (dual basis)

(24)

Let V be a vector space and $\{b_1, \dots, b_N\}$ a basis of V . Furthermore let $B_u \in V^*$ with $B_u(b_m) = \delta_{um}$. Then $\dim V' = \text{Liu} \{B_1, \dots, B_N\}$. Particularly $\dim V' = N$.

Proof:

$$\textcircled{1} \quad \sum \lambda_u B_u = 0 \stackrel{b_m}{\Rightarrow} \lambda_m = 0 \Rightarrow B_u \text{ linear independent}$$

$$\textcircled{2} \quad \phi \in V' : \text{ansatz } \hat{\phi} = \sum \phi_u B_u \stackrel{b_m}{\Rightarrow} \phi_u = \phi(b_u) \\ \Rightarrow \hat{\phi} = \sum \phi(b_u) B_u =: \tilde{\phi}$$

~~ϕ and $\tilde{\phi}$ are~~

ϕ equals $\tilde{\phi}$ on the basis $\{b_1, \dots, b_N\} \Rightarrow \phi = \tilde{\phi}$.

$$\Rightarrow \phi = \sum \phi(b_u) B_u$$

$$\Rightarrow \dim V' = N \wedge \text{Liu} \{B_1, \dots, B_N\} = V'$$

\textcircled{3} B_u unique? Of course or

$$\tilde{B}_u = \sum \lambda_u B_u \\ \Rightarrow \delta_{um} = \tilde{B}_u(b_m) = \lambda_m \Rightarrow \tilde{B}_u = B_u.$$

■

Corollary 11 the chart differential $I \circ d\alpha_m$ build

the a basis of $T_u(X)'$, the dual space of $T_u(X)$. Thus for any $F \in C^\infty(X, \mathbb{R})$ the differential dF_u is represented by

$$dF_u = \sum_{u=1}^n \partial_u^F d\alpha_m$$

Proof: $I \circ dF_u = \sum_{u=1}^n \underbrace{I \circ dF_u(\partial_u^F)}_{T_u(X)'} I \circ d\alpha_m$ by $\textcircled{**}$.

$$= \partial_u^F F$$

$$\Rightarrow dF_u = \sum_{u=1}^n \partial_u^F F d\alpha_m$$

■

From now on we often neglect the isomorphism \mathbb{I} and thus identify the objects.  (25)

Remark 12 We note the important formulas in the special case $Y = \mathbb{R}^n$:

$$dF_u(\partial_u^\varphi) = \partial_u^\varphi F \in \mathbb{R}^n, \quad F \in C^\infty(X, \mathbb{R}^n)$$

$$d\psi_{uu}(\partial_u^\varphi) = \partial_u^\varphi \psi_{uu} = \delta_{uu},$$

$$dF_u = \sum_{u=1}^n \partial_u^\varphi F \, d\psi_{uu}, \quad F \in C^\infty(X, \mathbb{R}).$$

Definition 13 Let (ψ, U) a chart for $u \in X$. We define

$$T(U) := \bigcup_{u \in U} T_u(X), \quad T(X) := \bigcup_{u \in X} T_u(X),$$

$$\pi : T(U) \rightarrow U, \quad \text{if } \tau \in T_u(X), \\ \tau \mapsto u$$

$$\phi : T(U) \rightarrow \mathbb{R}^n \times \mathbb{R}^n$$

$$\tau \mapsto (\psi \circ \pi(\tau), I(\tau)).$$

$S \subset T(X)$ is called "open", if for any $\tau \in S$ there exists an open subset W of $\mathbb{R}^{2n} \cong \mathbb{R}^n \times \mathbb{R}^n$, such that

$$\tau \in \pi^{-1}(W) \subset S.$$

Theorem 14 $T(X)$ is a topological Hausdorff space. If ψ, φ are compatible then ϕ, ψ are compatible as well. Equipped with this atlas the "tangential bundle" $T(X)$ becomes a $2N$ -dimensional differentiable manifold.

changing chart?

$$\phi \circ \varphi^{-1}(y, b) = \phi(\tau) = (x, a)$$

$$\text{if } \varphi(u) = y, \varphi(v) = x$$

$I = I^{\varphi}, I^{\tau}$ depend on φ, τ !

$$\cancel{b} = (\cancel{I^{\tau}})^{-1}(\tau) = (\cancel{I^{\varphi}})^{-1} \Rightarrow a = I^{\varphi} \circ (I^{\tau})^{-1}(b)$$

$$\cancel{b} = I^{\tau} \quad \tau = (I^{\tau})^{-1}(b) = \sum_{n=1}^N b_n \partial_n^{\tau}$$

$$= \sum_{m,n=1}^N b_n \partial_n^{\tau}(\varphi_m) \partial_m^{\varphi}$$

$$= \sum_{m=1}^N \underbrace{\left(\sum_{n=1}^N \partial_n^{\tau}(\varphi_m) b_n \right)}_{\partial_m^{\varphi}} \partial_m^{\varphi}$$

$$= \cancel{\sum_{n=1}^N} \partial_n(\varphi_m \circ \varphi^{-1})(y) b_n$$

$$= ((\varphi \circ \varphi^{-1})'(y) b)_m$$

$$\Rightarrow a = (\varphi \circ \varphi^{-1})'(y) b$$

Thus

Remark 15 The change of chart is given by

$$\phi \circ \varphi^{-1}(y, b) = (\varphi \circ \varphi^{-1}(y), (\varphi \circ \varphi^{-1})'(y) b).$$

Definition 16 A mapping

$$\tau: X \rightarrow T(X)$$

$$u \mapsto \tau(u), \quad \tau(u) \in T_u(X)$$

is called a "vector field" on X .

Lemma 17 Every vector field $\tau: X \rightarrow T(X)$ is represented in $u \in U \subset X$ by

$$\tau(u) = \sum_{n=1}^{\infty} \tau_n(u) \partial_u^n, \quad \partial_u^n \in T_u(X).$$

~~mostly~~ Mostly we write just

$$\tau = \sum_{n=1}^{\infty} \tau_n \partial_u^n.$$

We leave $\tau_n(u) = \tau(u)(\varphi_n)$ for $\tau_n: U \rightarrow \mathbb{R}$.

The vector fields build in a natural way a vector space.
(of infinite dimension!) The following statements are equivalent:

C (i) $\tau \in C^\infty(X, T(X))$

(ii) $\bigwedge_{\varphi} \bigwedge_u \tau_n \in C^\infty(U, \mathbb{R})$

In this case we call a vector field differentiable.

Definition 18 For $f \in C^\infty(X, Y)$ we define the differential

$$d\bar{f}: C^\infty(X, T(X)) \rightarrow C^\infty(Y, T(Y))$$
$$\tau \mapsto d\bar{f}(\tau)$$

by $d\bar{f}(\tau)(v) := d\bar{f}_u(\tau(u))$, if $\tau(u) \in T_u(X) \subset T(X)$.

We often mostly work with $d\bar{f}(\tau)$. Then

$$d\bar{f} \in C^\infty(C^\infty(X, T(X)), C^\infty(Y, T(Y)))$$

and $d\bar{f}$ linear. We often write just

$$d\bar{f}: T(X) \rightarrow T(Y)$$

Proof of Lemma 17

(28)

$$\tau \in C^\infty(X, T(x))$$

$$\Leftrightarrow \bigwedge_{\varphi, \psi} \psi \circ \tau \circ \varphi^{-1} \in C^\infty(\varphi(U), \mathbb{R}^{2n})$$

$$\Leftrightarrow \bigwedge_{\varphi, \psi} \psi \left(\sum_{u=1}^n \tau_u \circ \varphi^{-1} \partial_u^\varphi \right) \in C^\infty(\varphi(U), \mathbb{R}^{2n})$$

$$\Leftrightarrow \bigwedge_{\varphi, \psi} \left(\underbrace{\varphi \circ \varphi^{-1}}_{=\text{id}}, \tau_1 \circ \varphi^{-1}, \dots, \tau_n \circ \varphi^{-1} \right) \in C^\infty(\varphi(U), \mathbb{R}^{2n})$$

$$\Leftrightarrow \bigwedge_{\varphi} \bigwedge_{u=1, \dots, n} \tau_u \circ \varphi^{-1} \in C^\infty(\varphi(U), \mathbb{R})$$

$$\Leftrightarrow \bigwedge_{\varphi} \bigwedge_{u=1, \dots, n} \tau_u \in C^\infty(U, \mathbb{R}). \quad \blacksquare$$

Remark 18 (Local representation of ~~the~~ the differential)

$$\varphi \circ dF \circ \tilde{\Phi}^{-1}(x, a), \quad x = \varphi(u), \quad a = \varphi(v)$$

$$= \varphi \circ dF(\tau) = \varphi \left(\sum_{m=1}^n dF(\tau)(\varphi_m) \partial_m^\varphi \right)$$

$$= (\gamma, dF(\tau)(\varphi_1), \dots, dF(\tau)(\varphi_n))$$

$$\begin{aligned} \text{Since } dF(\tau)(\varphi_m) &= \sum_{u=1}^n a_u \underbrace{dF(\partial_u^\varphi)(\varphi_m)}_{= \partial_u \tilde{F}_m(x) \underbrace{\partial_u^\varphi(\varphi_m)}_{= \tau_{lm}}} \\ &= \sum_{u=1}^n \partial_u \tilde{F}_m(x) a_u \\ &= (\tilde{F}'(x) a)_m \end{aligned}$$

We obtain

$$\varphi \circ dF \circ \tilde{\Phi}^{-1}(x, a) = (\tilde{F}(x), \tilde{F}'(x) a).$$

This was clear since \tilde{F}' is the matrix representation of the linear map dF in the bases $\{\partial_u^\varphi\}_{u=1, \dots, n}$, $\{\partial_m^\varphi\}_{m=1, \dots, n}$. And x resp. γ is the vector corresponding to x resp. a .

2.3 Exercises

(29)

①* Let φ, ψ be charts at $u \in X$ with $\varphi_1 = \psi_1$.

$$(i) \quad \partial_x^{\varphi} = \partial_x^{\psi} ?$$

$$(ii) \quad d\varphi_i = d\psi_i ?$$

②* $F: X \rightarrow Y$ constant $\Rightarrow dF = 0$.

\Leftrightarrow holds if X connected.

③ Let φ, ψ be charts at u with $\partial_u^{\varphi} = \partial_u^{\psi}, u=1, n$.

Show that φ and ψ may only differ from each other

by a constant, if $U \cap V$ is connected. Give a

counterexample that this ~~is not~~ is generally wrong
in general if $U \cap V$ is not connected.

④* Show $d\text{id}_u = \text{id}_{T_u(X)}$.

⑤* Let $\varphi \subset X$ be a k -dimensional submanifold of X .
and

$$\begin{matrix} Z & \subset & \varphi & \subset & X \\ & & u & \mapsto & u \end{matrix}$$

the natural embedding. Show:

$$(i) \quad d_Z(\partial_e^{\varphi|_{\varphi}}) = \partial_e^{\psi}, \quad e=1, \dots, L$$

$$(ii) \quad \dim d_Z(\partial_e^{\varphi|_{\varphi}}) \dim d_Z T_u(\varphi) = L$$

$$(iii) \quad \psi|_{\varphi} \text{ constant} \Rightarrow d_Z T_u(\varphi) \subset \text{ker}(d\psi_u)$$

(iv) Let $v \in Y$ be a regular value of ψ and
 $\varphi := \psi^{-1}(\{v\})$. Then

$$d_Z T_u(\varphi) = \text{ker}(d\psi_u).$$

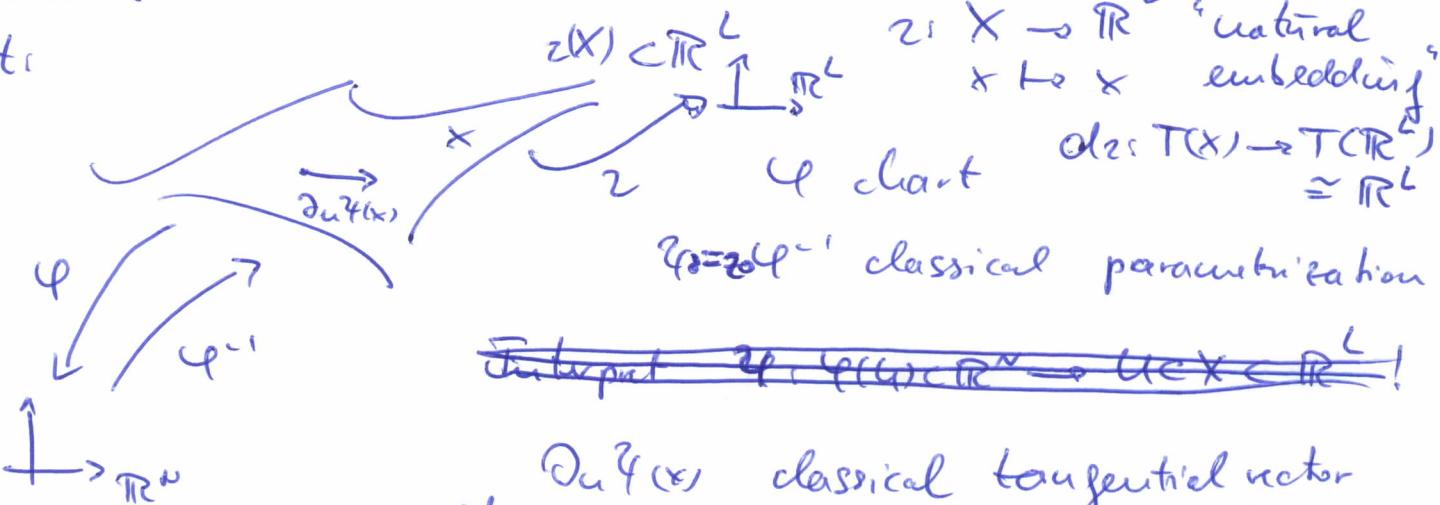
②* In which way may a tangential vector?

(36)

$$\Gamma(X) \ni v = T_u \varphi \stackrel{?}{=} \begin{bmatrix} v_1 \\ \vdots \\ v_N \end{bmatrix} \in \mathbb{R}^N$$

be interpreted as the "usual" or "well known" classical tangential vector in the case of a N -manifold $X \subset \mathbb{R}^L$?

Heute:



$$d\varphi(\partial_u^\varphi) = \partial_u^\varphi (\text{id}_{\mathbb{R}} \circ \varphi) \stackrel{\text{id}}{=} \partial_u^\varphi (\varphi^{-1} \circ \text{id}_{\mathbb{R}}) = \partial_u^\varphi \varphi \circ \partial_u^\varphi \stackrel{\text{id}}{=} \partial_u^\varphi \varphi$$

~~$\partial_u^\varphi \varphi = \partial_u^\varphi (\text{id} \circ \varphi^{-1}) \circ \varphi = \partial_u^\varphi \text{id}_{\mathbb{R}}$~~

$$\Rightarrow d\varphi(\partial_u^\varphi) = \partial_u^\varphi \varphi \in \mathbb{R}^L$$

Note: $\partial_u^\varphi \varphi = \langle e^u, \nabla \varphi(x) \rangle, \quad u=1, \dots, N.$

Especially for $X^{\text{open}} \subset \mathbb{R}^N$, i.e. $N=L$, we have

with $\varphi = \text{id}_{\mathbb{R}^N}$

$$\partial_u \text{id}_{\mathbb{R}^N} = \partial_u \text{id}_{\mathbb{R}^N} \circ \varphi = e^u \in \mathbb{R}^N.$$

③* Let U be an open subset of \mathbb{R}^N .
Let $\varphi = \text{id}$ and $\varphi(r, \theta, \varphi) := \begin{bmatrix} r \cos \theta & r \sin \theta \\ \cos \theta & \sin \theta \end{bmatrix}$, i.e.

φ is the identity chart and φ are polar coordinates.
Express $\partial_u := \partial_u^\varphi = \partial_u^\varphi$ and in terms of $\partial_r^\varphi, \partial_\theta^\varphi, \partial_\varphi^\varphi$ and vice versa. Do the same with dx^u and $dr, d\theta, d\varphi$.

(8)* "velocity vector"

Let $\gamma := (\alpha, \beta)$ be an interval in \mathbb{R} and $f \in C^\infty(\gamma, X)$, i.e. a curve

$$d\gamma \left(\frac{d}{dt} \right) \in T(X)$$

and $I = I \circ \gamma$. Then $\frac{d}{dt} := \partial_u^1 id \in T(\gamma)$.

$$\Gamma \quad d\gamma \left(\frac{d}{dt} \right)(\varphi), \quad \varphi \in C^\infty(X, \mathbb{R}), \quad t \in \gamma, \quad \gamma \circ \varphi = u$$

$$= \frac{d}{dt} (\varphi \circ \gamma) = \frac{d}{dt} (\underbrace{\varphi \circ \gamma \circ \text{id}^{-1}}_{\varphi^{-1} \circ \varphi}(t))$$

$$\approx \frac{d}{dt} \varphi^{-1} \circ \varphi$$

$$= \partial_u (\varphi \circ \gamma^{-1})(x) \cdot (\varphi_u \circ \gamma)'(t)$$

$$= (\varphi_u \circ \gamma)'(t) \partial_u^\varphi \varphi$$

$$\Rightarrow d\gamma \left(\frac{d}{dt} \right) = (\varphi_u \circ \gamma)'(t) \partial_u^\varphi$$

$$\Rightarrow I \circ d\gamma \left(\frac{d}{dt} \right) = (\varphi \circ \gamma)'(t) \in \mathbb{R}^n$$

"pullback" of the classical tangential vector

(i) We call $\dot{\gamma}(t) := d\gamma \left(\frac{d}{dt} \right)$ the "velocity vector" of γ at u . The mapping

$$C^\infty(\gamma, X) \rightarrow T(X)$$

$$\gamma \mapsto \dot{\gamma}(t)$$

is surjective.

Let $u \in X$ and $\tau \in T_u(X)$. Define the curve

$$\gamma: (-\varepsilon, \varepsilon) \rightarrow X$$

$$t \mapsto \varphi^{-1}(t\tau(\varphi_1), \dots, t\tau(\varphi_N)) = \varphi^{-1}\left(t \begin{bmatrix} \tau(\varphi_1) \\ \vdots \\ \tau(\varphi_N) \end{bmatrix}\right)$$

Then $\dot{\gamma}(t) = (t \cdot \tau(\varphi_u))'(t) \partial_u^\varphi = \underbrace{\tau(\varphi_u)}_{= \tau_u} \partial_u^\varphi = \tau$

(iii) By

$$\gamma \sim \gamma' \Leftrightarrow \dot{\gamma}(t) = \dot{\gamma}'(t)$$

we may define equivalence classes ~~is~~ on the space of all C^∞ -curves ~~from~~ on X . Note that we do not need the existence of a tangential space to define this classes, i.e.

$$\gamma \sim \gamma' \Leftrightarrow (\varphi \circ \gamma)'(t) = (\varphi \circ \gamma')'(t)$$

This definition is independent of the chart φ and thus $T_u(X)$ could be also defined as the equivalence classes of these curves.

$$\begin{aligned} \Gamma(\varphi \circ \gamma)' &= (\varphi \circ \varphi^{-1} \circ \varphi \circ \gamma)' = (\varphi \circ \varphi^{-1})' (\varphi \circ \gamma)' \\ &= (\varphi \circ \varphi^{-1})' (\varphi \circ \gamma')' = (\varphi \circ \gamma')' \quad] \end{aligned}$$

3 Differential Forms

(33)

3.1 Multilinear Algebra

In the following let V be a vector space (over \mathbb{R}) of dimension N and $\{e^u\}_{u=1,\dots,n}$ be a basis of V as well as $\{\delta^u\}_{u=1,\dots,n}$ be the dual basis ~~of V'~~ of V' , i.e.

$$\delta^u(e^v) = \delta_{uv} .$$

Definition and Lemma 1 Let ω be a q -linear ($q \in \mathbb{Z}$) mapping on V , i.e.

$$\omega: V^q := \underbrace{V \times \dots \times V}_{q\text{-times}} \rightarrow \mathbb{R} \quad \text{multilinear} .$$

Then \circlearrowleft

- (i) v_1^1, \dots, v_q^q linear ~~dependent~~ $\Rightarrow \omega(v_1^1, \dots, v_q^q) = 0$.
- (ii) $\exists u \neq m \quad v^u = v^m \Rightarrow \omega(v^1, \dots, v^q) = 0$
- (iii) $\forall \pi \in \overline{\Pi}_q \quad \omega(v^{\pi 1}, \dots, v^{\pi q}) = \text{sgn } \pi \omega(v^1, \dots, v^q)$,
where $\overline{\Pi}_q$ is the set of all permutations of $\{1, \dots, q\}$,
i.e. $\pi \in \overline{\Pi}_q \Leftrightarrow \pi: \{1, \dots, q\} \rightarrow \{1, \dots, q\}$ bijection.
- (iv) $\forall u, m \quad \omega(v^1, \dots, v^u, \dots, v^m, \dots, v^q) = -\omega(v^1, \dots, v^m, \dots, v^u, \dots, v^q)$

Such multi-linear mappings are called "alternating"
or "alternating (q)-forms". We shall call them shortly
 q -forms. The space of all q -forms over V will
be denoted by

$$A^q(V)$$

and ~~but~~ builds in a natural way a vector space ..

Proof: (i) \Rightarrow (ii) \vee (iii) \Rightarrow (iv) \vee

~~(ii) \Rightarrow (iii)~~

$$(ii) \Rightarrow (i): \quad v^1 = \sum_{e=2}^q \lambda_e v^e$$

$$\Rightarrow w(v^1, \dots, v^q) = \sum_{e=2}^q \lambda_e \underbrace{w(v^e, v^2, \dots, v^q)}_{\Rightarrow 0} = 0.$$

(iv) \Rightarrow (iii): Any permutation is a product (circ!) of transpositions.

(iv) \Rightarrow (iv): $\emptyset = w(v^1, \dots, v^u + v^m, \dots, v^u + v^m, \dots, v^q)$

$$= w(v^1, \dots, v^u, \dots, v^u, \dots, v^q) \cancel{+} = 0$$

$$+ w(v^1, \dots, v^u, \dots, v^m, \dots, v^q)$$

$$+ w(v^1, \dots, v^m, \dots, v^u, \dots, v^q)$$

$$+ w(v^1, \dots, v^m, \dots, v^m, \dots, v^q) \cancel{+} = 0$$

(iii) \Rightarrow (ii): $w(v^1, \dots, \underset{\substack{\| \\ m}}{v^u}, \dots, \underset{\substack{\| \\ u}}{v^m}, \dots, v^q) = w(v^1, \dots, v^u, \dots, v^u, \dots, v^q)$

$$= -w(v^1, \dots, v^u, \dots, v^u, \dots, v^q)$$

$$\Rightarrow w(v^1, \dots, v^u, \dots, v^u, \dots, v^q) = 0.$$

Convention and

Lemma 2 $A^q(V) = \{0\}$ for $q \in \mathbb{Z} \setminus \{0, \dots, n\}$.

Proof: Lemma 1 (i) $\Rightarrow A^q(V) = \{0\}$ for $q \geq n+1$.

$q \leq -1$ convention.



Remark 3

$$A^0(V) := \mathbb{R} \quad , \quad A^1(V) = V' .$$

Definition and lemma 4 Let U, V, W be vector spaces and $A: U \rightarrow V, B: V \rightarrow W$ be linear mappings. Moreover let

$$A^*: A^q(V) \rightarrow A^q(U)$$

$$w \mapsto A^*w$$

for $q=0$.

where $A^*w(v^1, \dots, v^q) := w(Av^1, \dots, Av^q), q \geq 1, A^* := \text{id}_{\mathbb{R}}$

Then A^* is well defined and linear and $A^*B^* = (BA)^*$ holds. If $q=n$ and $U=V$ we have

$$A^*w = \det A \cdot w$$

Proof: A^* is well defined and linear.

$$(BA)^* \omega(u^1, \dots, u^q) = \omega(BAu^1, \dots, BAu^q)$$

$$= B^* \omega(Au^1, \dots, Au^q) = A^* B^* \omega(u^1, \dots, u^q)$$

□

Definition and Lemma 5 The set

$$\mathbb{I}^{q,N} := \{ I : \{1, \dots, q\} \rightarrow \{1, \dots, N\} : \bigwedge_{u, w} I(u) < I(w) \}$$

contain exactly $\binom{N}{q}$ elements.

Notation: $i_u := I(u)$, $I = (i_1, \dots, i_q)$. "ordered multi-index".

The mapping

$$I^q : A^q(V) \rightarrow \mathbb{R}^{\binom{N}{q}}$$

$$w \mapsto [w_I]_{I \in \mathbb{I}^{q,N}},$$

where $w_I := w(e^{i_1}, \dots, e^{i_q})$, is a Isomorphism.

Thus we may identify $A^q(V)$ with $\mathbb{R}^{\binom{N}{q}}$ via I^q .

In particular we have

$$\dim A^q(V) = \binom{N}{q} = \frac{N!}{q!(N-q)!}$$

w_I are called the components of w .

Proof: $\# \mathbb{I}^{q,N} = \# \{ \text{subsets of } q \text{ elements from } \{1, \dots, N\} \text{ without repetitions} \}$.

I^q linear ✓

I^q injective, clear since $\forall I \quad w_I = 0$ implies $w = 0$. (basis and alternating!)

I^q surjective; Let $[w_I]_{I \in \mathbb{I}^{q,N}} \in \mathbb{R}^{\binom{N}{q}}$. Define

$$w(e^{i_1}, \dots, e^{i_q}) := w_I.$$

ordered!

$$\Rightarrow w \in A^q(V) \text{ and } I^q(w) = [w_I]_{I \in \mathbb{I}^{q,N}}$$

"alternation ~~under~~ by definition".

□

Proof: well defined \forall linear.

$$\begin{aligned} & \cancel{(BA)^* w(e^1, \dots, e^n) = w(BAe^1, \dots, BAe^n)} \\ & = B^* w(Ae^1, \dots, Ae^n) = A^* B^* w(e^1, \dots, e^n) \end{aligned}$$



Remark 6 (i) $A^* = \text{id}_{\mathbb{R}}$, if $q=0$.

(ii) $\text{id}_U^* = \text{id}_{A^q(U)}$, if $U=V$, $A=\text{id}$.

(iii) $A^* w = w \circ A = w A$, if $q=1$.

(iv) $A^* w = \det A w$, if $q=N$, $U=V$.

(v) $\forall a \in \mathbb{R} \exists^* w \in A^N(V) \quad w(e^1, \dots, e^n) = a$.

(vi) $0 \neq w \in A^N(V) \Rightarrow w(e^1, \dots, e^n) \neq 0$.

Proof: (i) ✓, (ii) ✓, (iii) ✓

$$\cancel{\leftrightarrow w(e^1, \dots, e^n) = a}$$

(v) $w \neq 0 \Rightarrow \exists v^1, \dots, v^N : w(v^1, \dots, v^N) = a \neq 0$.

$$v^q = \sum_{i_1=1}^N \lambda_{i_1}^{(q)} e^{i_1} \sum_{i_2=1}^N \dots \sum_{i_N=1}^N \lambda_{i_1}^{(1)} \dots \lambda_{i_N}^{(N)} w(e^{i_1}, \dots, e^{i_N}) = a \neq 0$$

$$= \left(\sum_{i_1=1}^N \dots \sum_{i_N=1}^N \lambda_{i_1}^{(1)} \dots \lambda_{i_N}^{(N)} \cancel{\text{sgn}(\{i_1, \dots, i_N\} \rightarrow \{1, \dots, N\})} \right) \cdot w(e^1, \dots, e^n)$$

$$\Rightarrow w(e^1, \dots, e^n) \neq 0.$$

(i) ~~$w =$~~ $a=0 \Rightarrow w := 0$.

$a \neq 0$. Let $w \neq 0 \Rightarrow w(e^1, \dots, e^n) = \tilde{a} \neq 0$.

$$\Rightarrow \frac{a}{\tilde{a}} w(e^1, \dots, e^n) = a.$$

(iv) $A: V \rightarrow V$, $I: V \rightarrow \mathbb{R}^N$
 $v = v_u e^u \mapsto \begin{bmatrix} v_1 \\ \vdots \\ v_N \end{bmatrix}$

~~usual definition~~ $\Rightarrow \det A = \det(I A I^{-1}) = \det \tilde{A}$

~~$\det A$~~

$V \xrightarrow{A} V$

It is enough to show it on a basis, i.e. on

$\tilde{A} := I A I^{-1}$

$\begin{array}{ccc} I & & I \\ \downarrow & & \downarrow \\ \mathbb{R}^N & \xrightarrow{IAI^{-1}} & \mathbb{R}^N \end{array}$

{ e^u }. See ⊕

$$A^* \omega(e^1, \dots, e^n)$$

$$= \omega(Ae^1, \dots, Ae^n)$$

$$= \omega(\underbrace{I' \tilde{A} I e^1}, \dots, \underbrace{I' \tilde{A} I e^n})$$

$$= \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ i \\ \vdots \\ 0 \end{bmatrix}$$

$$= (I')^* \omega(\tilde{A} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \dots, \tilde{A} \begin{bmatrix} 0 \\ i \\ \vdots \\ 0 \end{bmatrix})$$

$$= (I')^* \omega(\tilde{A}_1, \dots, \tilde{A}_n) \quad , \text{ if } \tilde{A} = (\tilde{A}_1 \dots \tilde{A}_n)$$

Since $(I')^* \omega \in A^\wedge(\mathbb{R}^n)$ and $\dim A^\wedge(\mathbb{R}^n) = \binom{n}{n-1} = 1$

and $\det \in A^\wedge(\mathbb{R}^n)$ we have

$$(I')^* \omega = c \cdot \det$$

$$\Rightarrow A^* \omega(e^1, \dots, e^n) = c \det(\tilde{A}_1, \dots, \tilde{A}_n)$$

$$= c \det \tilde{A} = c \det A .$$

on the other hand we have

$$\omega(e^1, \dots, e^n) = (I')^* \omega(Ie^1, \dots, Ie^n)$$

$$\underbrace{I' I}_{I'^* I} = c \underbrace{\det \left(\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 \\ i \\ \vdots \\ 0 \end{bmatrix} \right)}_{\equiv 1} = c$$

$$\Rightarrow A^* \omega(e^1, \dots, e^n) = \det A \omega(e^1, \dots, e^n)$$

$$\Rightarrow A^* \omega = \det A \cdot \omega$$

abstractive proof of (iv): $\{A\}_{\{e^i\}}^{\{e^j\}} = (a_{ij})_{i,j=1,\dots,N}$

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$$\cancel{A^* \omega(v^1, \dots, v^N)} = \omega(Av^1, \dots, Av^N)$$

$$= \sum_{i_1, \dots, i_N=1}^N \cancel{\omega(v_{i_1}^1, Ae^{i_1}, \dots, v_{i_N}^N, Ae^{i_N})}$$

\Downarrow

$$\sum_{j_1=1}^N a_{j_1, i_1} e^{j_1}$$

$$= \sum_{i_1, \dots, i_N=1}^N \sum_{j_1=1}^N a_{j_1, i_1} \dots a_{j_N, i_N} v_{i_1}^1 \dots v_{i_N}^N \cancel{\omega(e^{j_1}, \dots, e^{j_N})}$$

$$\text{assume } A^* \omega(e^1, \dots, e^N) = \det A \omega(e^1, \dots, e^N). \quad \textcircled{*}$$

Then

$$\begin{aligned} A^* \omega(v^1, \dots, v^N) &= \sum_{i_1, \dots, i_N=1}^N v_{i_1}^1 \dots v_{i_N}^N A^* \omega(e^{i_1}, \dots, e^{i_N}) \\ &= \sum_{i_1, \dots, i_N=1}^N v_{i_1}^1 \dots v_{i_N}^N \det A \omega = \sum_{\pi \in \Pi_N} v_{\pi_1}^1 \dots v_{\pi_N}^N A^* \omega(e^{\pi_1}, \dots, e^{\pi_N}) \\ &= \sum_{\pi \in \Pi_N} \operatorname{sgn} \pi v_{\pi_1}^1 \dots v_{\pi_N}^N \underbrace{A^* \omega(e^1, \dots, e^N)}_{= \det A \omega(e^1, \dots, e^N)} \\ &= \det A \sum_{\pi \in \Pi_N} v_{\pi_1}^1 \dots v_{\pi_N}^N \omega(e^{\pi_1}, \dots, e^{\pi_N}) \\ &= \det A \underbrace{\sum_{i_1, \dots, i_N=1}^N v_{i_1}^1 \dots v_{i_N}^N \omega(e^{i_1}, \dots, e^{i_N})}_{\omega(v^1, \dots, v^N)} \end{aligned}$$

It remains to prove $\textcircled{*}$:

$$\begin{aligned} A^* \omega(e^1, \dots, e^N) &= \omega(Ae^1, \dots, Ae^N) = \sum_{i_1, \dots, i_N=1}^N a_{i_1 1} \dots a_{i_N N} \omega(e^{i_1}, \dots, e^{i_N}) \\ &= \sum_{\pi \in \Pi_N} a_{\pi_1 1} \dots a_{\pi_N N} \omega(e^{\pi_1}, \dots, e^{\pi_N}) \\ &= \sum_{\pi \in \Pi_N} \operatorname{sgn} \pi a_{\pi_1 1} \dots a_{\pi_N N} \underbrace{\omega(e^1, \dots, e^N)}_{\det(a_{ij}) = \det A} \\ &= \det A \omega(e^1, \dots, e^N) \end{aligned}$$

□

Remark 7

$$\dim A^0(V) = 1, \quad \dim A^1(V) = \dim A^{N+1}(V) = N$$

$$\dim A^N(V) = 1$$

$$A^N(V) = \text{Lin} \{ \tilde{\omega} \},$$

where $\tilde{\omega}$ is the determinant-form, i.e. $\tilde{\omega}(e^1, \dots, e^N) = 1$.

Definition and Theorem 8 For all $a, p \in \mathbb{Z}$ there exists a bilinear mapping

$$\wedge: A^q(V) \times A^p(V) \rightarrow A^{q+p}(V),$$

$$\text{such that: } (\omega, \gamma) \mapsto \wedge(\omega, \gamma) := \omega \wedge \gamma$$

$$(i) (\omega \wedge \gamma) \wedge \delta = \omega \wedge (\gamma \wedge \delta) \quad \text{"associativity"}$$

$$(ii) \omega \wedge \gamma = (-)^{q+p} \gamma \wedge \omega \quad \text{"anti-commutativity"}$$

$$(iii) \alpha \wedge \omega = \alpha \cdot \omega, \quad \alpha \in A^0(V) = \mathbb{R}$$

$$(iv) A^*(\omega \wedge \gamma) = (A^*\omega) \wedge (A^*\gamma) \quad \text{"naturality"}$$

for all $A: U \rightarrow V$ linear.

$$(v) \text{ If } V = \mathbb{R}^N \text{ and } \{e^u\}_{u=1, \dots, N} \text{ is the standard basis } e_u^u = \delta_{uu} \text{ then}$$

$$\delta^1 \wedge \dots \wedge \delta^N(e^1, \dots, e^N) = 1 \quad \forall N \geq 1.$$

By (i) - (v) \wedge is uniquely determined.

Proof: Let

$$\pi := \pi_u := \{ \text{permutations of } \{1, \dots, u\} \}, \quad \# \pi_u = u!$$

and define

$$\omega \wedge \gamma (v^1, \dots, v^q, v^{q+1}, \dots, v^{q+p})$$

$$:= \frac{1}{q!p!} \sum_{\pi \in \pi_{q+p}} \underbrace{\text{sgn } \pi \cdot \omega(v^{\pi(1)}, \dots, v^{\pi(q)}) \cdot \gamma(v^{\pi(q+1)}, \dots, v^{\pi(q+p)})}_{\text{multi-linear}}$$

$\omega \wedge \gamma$ multi-linear?

(39)

Look at $(\lambda u^1 + w^1, v^2, \dots, v^{q+p})$ and $\pi(u) = 1$

$$\begin{aligned} \text{Then } \omega(v^{\pi(u)}, \dots, \lambda u^{\pi(u)} + w^{\pi(u)}, \dots, v^{\pi(q)}) &\cdot \gamma(v^{\pi(q+1)}, \dots, v^{\pi(q+p)}) \\ &= \lambda \omega(v^{\pi(1)}, \dots, u^{\pi(u)}, \dots, v^{\pi(q)}) \cdot \gamma(v^{\pi(q+1)}, \dots, v^{\pi(q+p)}) \\ &\quad + \omega(v^{\pi(1)}, \dots, w^{\pi(u)}, \dots, v^{\pi(q)}) \cdot \gamma(v^{\pi(q+1)}, \dots, v^{\pi(q+p)}) \end{aligned}$$

for $\pi(u) = 1$ and $1 \leq u \leq q$

and

$$\begin{aligned} \omega(v^{\pi(1)}, \dots, v^{\pi(q)}) &\cdot \gamma(v^{\pi(q+1)}, \dots, \lambda u^{\pi(u)} + w^{\pi(u)}, \dots, v^{\pi(q+p)}) \\ &= \lambda \omega(v^{\pi(1)}, \dots, v^{\pi(q)}) \cdot \gamma(v^{\pi(q+1)}, \dots, u^{\pi(u)}, \dots, v^{\pi(q+p)}) \\ &\quad + \cancel{\omega}(v^{\pi(1)}, \dots, v^{\pi(q)}) \cdot \gamma(v^{\pi(q+1)}, \dots, w^{\pi(u)}, \dots, v^{\pi(q+p)}) \end{aligned}$$

for $\pi(u) = 1$ and $q+1 \leq u \leq q+p$.

$$\Rightarrow \omega \wedge \gamma(\lambda u^1 + w^1, v^2, \dots, v^{q+p})$$

$$= \lambda \omega \wedge \gamma(u^1, v^2, \dots, v^{q+p}) + \omega \wedge \gamma(w^1, v^2, \dots, v^{q+p})$$

$\omega \wedge \gamma$ alternating?

q.p! $\omega \wedge \gamma(v^1, v^2, v^3, \dots, v^{q+p})$ with $v^2 = v^1$.

$$= \sum_{\pi \in \Pi_{q+p}} \text{sgn} \pi \omega(v^{\pi(1)}, \dots, v^{\pi(q)}) \gamma(v^{\pi(q+1)}, \dots, v^{\pi(q+p)})$$

$\sum_{\pi \in \Pi_{q+p}}$ only over π with $1 \leq \pi^{-1}(1) \leq q$ and $q+1 \leq \pi^{-1}(2) \leq q+p$
 or $1 \leq \pi^{-1}(2) \leq q$ and $q+1 \leq \pi^{-1}(1) \leq q+p$.

$$= \sum_{u=1}^q \sum_{u=q+1}^{q+p} \left[\sum_{\substack{\pi \in \Pi_{q+p} \\ \pi(u)=1, 1 \leq u \leq q \\ \pi(u)=2, q+1 \leq u \leq q+p}} \text{sgn} \pi \omega(v^{\pi(1)}, \dots, v^1, \dots, v^{\pi(q)}) \right. \\ \left. \cdot \gamma(v^{\pi(q+1)}, \dots, \cancel{v^2}, \dots, v^{\pi(q+p)}) \right]$$

$$+ \sum_{\substack{\pi \in \Pi_{q+p} \\ \pi(u)=2, 1 \leq u \leq q \\ \pi(u)=1, q+1 \leq u \leq q+p}} \text{sgn} \pi \omega(v^{\pi(1)}, \dots, v^2, \dots, v^{\pi(q)}) \\ \cdot \gamma(v^{\pi(q+1)}, \dots, \cancel{v^1}, \dots, v^{\pi(q+p)})$$

here interchanging
 ~~$\pi(u)$~~
 ~~$\pi(u)$~~
 and $\pi(u)$!

(40)

$$= \sum_{m=1}^q \sum_{u=q+h}^{q+p} \sum_{\pi \in \tilde{\Pi}_{q+h}^1} \operatorname{sgn} \pi .$$

$$\pi(u) = 1, 1 \leq u \leq q$$

$$\pi(u) = 2, q+h \leq u \leq q+p$$

$$\left(\omega(v^{\pi(1)}, \dots, v^1, \dots, v^{\pi(q)}) \cdot \gamma(v^{\pi(q+h)}, \dots, v^2, \dots, v^{\pi(q+p)}) \right. \\ \left. - \omega(v^{\pi(1)}, \dots, v^1, \dots, v^{\pi(q)}) \cdot \gamma(v^{\pi(q+h)}, \dots, v^2, \dots, v^{\pi(q+p)}) \right)$$

$$= 0 .$$

\Rightarrow 1 well defined.

(i) ~~left for the~~ just calculate

$$(ii) \quad u \quad u$$

$$(iii) \quad \checkmark$$

$$(iv) \quad \checkmark \quad (v) \quad \checkmark$$

□

To show uniqueness we first prove:

Lemma 9 We have by (i), ..., (v) (only!)

$$\delta^{\pi(1)} \wedge \dots \wedge \delta^{\pi(q)} (e^{\pi(1)}, \dots, e^{\pi(q)}) = \begin{cases} \operatorname{sgn}(\pi, \pi'), & \pi \sim \pi' \\ 0, & \text{otherwise} \end{cases}$$

where $\pi \sim \pi' \iff \{\pi(1), \dots, \pi(q)\} = \{\pi'(1), \dots, \pi'(q)\}$

and $\operatorname{sgn}(\pi, \pi') := \operatorname{sgn}(\hat{\pi})$ with $\pi = \pi' \circ \hat{\pi}$.

w.l.o.g.

Proof: w.l.o.g. $\pi' = \text{id}_q$ and $\operatorname{sgn}(\pi) = \operatorname{sgn}(\hat{\pi})$.

So we have to show

$$\delta^{\pi(1)} \wedge \dots \wedge \delta^{\pi(q)} (e^1, \dots, e^q) = \begin{cases} \operatorname{sgn} \pi, & \pi \sim \text{id}_q \\ 0, & \text{otherwise.} \end{cases}$$

1. $\delta^{\circ} \wedge \delta^{\circ} = 0$, since (i)

2. If there exists i, j with $\pi(i) = \pi(j)$ we have

$$\delta^{\pi(1)} \wedge \dots \wedge \overset{(i)}{\delta^{\pi(q)}} \pm \underbrace{(\delta^{\pi(i)} \wedge \delta^{\pi(j)})}_{=0 \text{ because of 1.}} \wedge \delta^{\pi_{-i}} \dots \wedge \delta^{\pi_{-j}} = 0 .$$

3. $\varphi: V \rightarrow V$ natural embedding (41)
 $e^i \mapsto \begin{cases} e^i & , i=1, \dots, q \\ 0 & , i=q+1, \dots, N \end{cases}$ of $\text{Lin}\{e^1, \dots, e^q\}$ in V .

$$\text{Note } z^{*\sigma^i}(e^j) = \sigma^i(z e^j)$$

$$= \begin{cases} \sigma^i(e^j) & , j=1, \dots, q \\ 0 & , j=q+1, \dots, N \end{cases}$$

$$= \begin{cases} \sigma^i(e^j) & , j=1, \dots, q \\ 0 & , j=q+1, \dots, N \end{cases} = \begin{cases} \sigma^i(e^j) & , j=1, \dots, q \\ 0 & , j=q+1, \dots, N \end{cases}$$

Then

$$\begin{aligned} & \sigma^{\pi_1} \wedge \dots \wedge \sigma^{\pi_q}(e^1, \dots, e^q) \\ &= \sigma^{\pi_1} \wedge \dots \wedge \sigma^{\pi_q}(z e^1, \dots, z e^q) \\ &= z^{*\sigma^{\pi_1}} \wedge \dots \wedge z^{*\sigma^{\pi_q}}(e^1, \dots, e^q) \\ &= \begin{cases} \sigma^{\pi_1} \wedge \dots \wedge \sigma^{\pi_q}(e^1, \dots, e^q) & , \{\pi_1, \dots, \pi_q\} \subset \{1, \dots, q\} \\ 0 & , \text{otherwise} \end{cases} \end{aligned}$$

and

$$\sigma^{\pi_1} \wedge \dots \wedge \sigma^{\pi_q}(e^1, \dots, e^q) \neq 0$$

$$\Rightarrow \{\pi_1, \dots, \pi_q\} = \{1, \dots, q\}$$



Finally for $\{\pi_1, \dots, \pi_q\} = \{1, \dots, q\}$

$$\begin{aligned} & \sigma^{\pi_1} \wedge \dots \wedge \sigma^{\pi_q}(e^1, \dots, e^q) \\ &= \underbrace{\text{sgn}(\pi) \sigma^1 \wedge \dots \wedge \sigma^q(e^1, \dots, e^q)}_{=1 \text{ since } \sigma^i(e^i) = \sigma_0} \end{aligned}$$

(*)

= 1 since $\sigma^i(e^i) = \sigma_0$, "dual basis"

Remark 10 $\forall \pi, \pi': \{1, \dots, N\} \rightarrow \{1, \dots, N\}$ (not w.r.t. a permutation)

(i) $\sigma^{\pi_1} \wedge \dots \wedge \sigma^{\pi_q}(e^{\pi_1}, \dots, e^{\pi_q}) = 0$, if $\exists i \neq j: \pi_i = \pi_j$ or $\pi'^i = \pi'^j$
 or $\{\pi_1, \dots, \pi_q\} \neq \{\pi'^1, \dots, \pi'^q\}$

(ii) If $\{\pi_1, \dots, \pi_q\} = \{\pi'^1, \dots, \pi'^q\}$, then

$$\sigma^{\pi_1} \wedge \dots \wedge \sigma^{\pi_q}(e^{\pi_1}, \dots, e^{\pi_q}) = \text{sgn}(\pi, \pi')$$

(*) follows not only by the definition of a but also
by the properties. For this let

(42)

$$A: V \rightarrow \mathbb{R}^N$$

$$e^i \mapsto \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{R}^N$$

Then

$$(A^{-1})^* \delta^c \left(\begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}_{\text{dual}} \right) = \delta^c(e^j) = \delta_{ij}$$

$\Rightarrow (A^{-1})^* \delta^c$ ~~is a dual basis~~ and dual basis in $\mathbb{R}^{N'}$
For the standard basis $\begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} = A e^i$.

We compute

$$\begin{aligned} & \delta^1 \wedge \dots \wedge \delta^N(e^1, \dots, e^N) \\ &= (A^{-1})^* \delta^1 \wedge \dots \wedge (A^{-1})^* \delta^N(A e^1, \dots, A e^N) = 1 \text{ by (v).} \end{aligned}$$

□

Finally to show uniqueness we note that by Lemma ~~9~~ $\delta^1 \wedge \dots \wedge \delta^N$ does not depend on the special operation \wedge but only on the properties (i), ..., (v). Since the δ^I form a basis of $A^q(V)$ the proof is complete.

□

Lemma II For $I \in \mathbb{I}^{q,N}$ let $\delta^I := \delta^{i_1} \wedge \dots \wedge \delta^{i_q}$.

Then $\{\delta^I\}_{I \in \mathbb{I}^{q,N}}$ is a basis of $A^q(V)$ and for
any every $w \in A^q(V)$ we have

$$w = \sum_{I \in \mathbb{I}^{q,N}} w_I \delta^I$$

with $w_I = w(e^{i_1}, \dots, e^{i_q})$ from Lemma 5.

Proof: Let $e^7 := (e^{s_1}, \dots, e^{s_q})$.

(43)

$$\sigma^{s_1} \dots \sigma^{s_q}(e^{s_1}, \dots, e^{s_q}) = \begin{cases} \text{sgn}(\dots) & , \{e^{s_1}, \dots, e^{s_q}\} = \{s_1, \dots, s_q\} \\ 0 & , \text{otherwise} \end{cases}$$

$$\Rightarrow \sigma^I(e^7) = \begin{cases} 1 & , I=7 \\ 0 & , \text{otherwise} \end{cases} = \sigma_I 7$$

I, 7 ordered!

$\Rightarrow \{\sigma^I\}_{I \in \mathbb{I}^{q, N}}$ linear independent

$\Rightarrow \{\sigma^I\}_I$ basis since the dimensions are OK.

$$\sum_I w_I \sigma^I(e^7) = w_7 = \omega(e^7) \quad \checkmark 7$$

$$\Rightarrow \omega = \sum_I w_I \sigma^I$$

■

Lemma and

Definition 12 Let V be a vector space with scalar product

$$\langle \cdot, \cdot \rangle: V \times V \rightarrow \mathbb{R}$$
$$(u, v) \mapsto \langle u, v \rangle$$

Then $b: V \rightarrow V^{\bullet 1}$ is an isomorphism and
 $v \mapsto \langle \cdot, v \rangle$

we denote it's inverse by $\#$: $\# := (b \cdot)^{-1}$.

b : flat, $\#$: sharp (curly)

$\#$ makes the linear mapping $\alpha \mapsto \sharp \alpha$, i.e. a vector $\# \alpha \in V$.

Lemma 13 (i) The mapping

$$\phi: A^q(V)^l \rightarrow A^q(V^l)$$

defined by $\phi(\psi)(w^1, \dots, w^q) := \psi(w^1, \dots, w^q)$ is an isomorphism.

(ii) The mapping $\psi: A^q(V^l) \rightarrow A^q(V)$ defined by

$\psi(\alpha)(v^1, \dots, v^q) := \alpha(bv^1, \dots, bv^q)$ is an isomorphism, if

V is a vector space with scalar product.

Proof: (i) $\dim V = n = \dim A^q(V) \Rightarrow \dim (A^q(V))' = \dim A^q(V)$. (44)

Furthermore ϕ is injective, since $\phi(v) = 0$ implies

$$\phi(v)(w^1, \dots, w^q) = \psi(w^1 \wedge \dots \wedge w^q) = 0 \quad \forall w^1, \dots, w^q \in V.$$

$$\text{Check } v^i \Rightarrow \psi(\delta^I) = 0 \quad \forall I \in \mathbb{I}^{q, n}$$

$$\text{but } \Rightarrow \psi = 0.$$

Thus ϕ is an isomorphism.

$$(ii) \dim A^q(V') = \dim A^q(V).$$

$$\psi(v) = 0$$

$$\Rightarrow \alpha(b_v^1, \dots, b_v^q) = 0 \quad \forall v^1, \dots, v^q \in V.$$

α isomorphism

$$\Rightarrow \alpha(w^1, \dots, w^q) = 0 \quad \forall w^1, \dots, w^q \in V'$$

$$\Rightarrow \alpha = 0$$

$\Rightarrow \alpha$ injective $\Rightarrow \alpha$ isomorphism. □

Theorem 14 Let V be a vector space with scalar product of dimension n . Then

$$\langle \cdot, \cdot \rangle_q : A^q(V) \times A^q(V) \rightarrow \mathbb{R} \\ (\omega, \eta) \mapsto \underbrace{[\phi^{-1} \circ \psi^{-1}(\omega)](\eta)}_{\in A^q(V)'} \in \mathbb{R}$$

Defines a scalar product on $A^q(V)$.

Proof: $\langle \cdot, \cdot \rangle_q$ well defined. \checkmark

$$\langle \omega, \eta \rangle_q = (\phi^{-1} \circ \psi^{-1})(\omega)(\eta) = ?$$

$$\text{AN} \Rightarrow \phi^{-1} \circ \psi^{-1}(\omega) = \omega' \Leftrightarrow \omega = \psi \circ \phi(\omega') \in A^q(V)$$

$$\Rightarrow \omega(v^1, \dots, v^q) = \phi(\omega')(b_{v^1}, \dots, b_{v^q}) = \omega'(b_{v^1} \wedge \dots \wedge b_{v^q}) \quad (*) \\ \forall v^1, \dots, v^q \in V.$$

We note: Let now $\{e^i\}$ be an orthonormal basis of V .

Then $b_{e^i}(e^j) \stackrel{\text{Def}}{=} \langle e^i, e^j \rangle = \delta_{ij} = \delta^i(e^j) \quad \forall i, j$, i.e.

$$b_{e^i} = \delta^i, \quad e^i = \# \delta^i.$$

$$\text{Since } \langle \omega, z \rangle_q = \sum_{I, 7} w_I z_I \underbrace{\phi' \circ \varphi^{-1}(\sigma^I)(\sigma^7)}_{= \langle \sigma^I, \sigma^7 \rangle_q}, \quad (45)$$

it is sufficient to calculate $\langle \sigma^I, \sigma^7 \rangle_q$ for the basis $\{\sigma^I\}_{I \in \mathbb{I}^{q,n}}$ of $A^q(V)$.

$$\begin{aligned} \langle \sigma^I, \sigma^7 \rangle_q &= \phi' \circ \varphi^{-1}(\sigma^I)(\sigma^7) = \phi'(\varphi^{-1}(\sigma^I))(\sigma^{s_1} \dots \sigma^{s_q}) \\ &\stackrel{*}{=} \sigma^I(e^{s_1}, \dots, e^{s_q}) \\ &= \sigma^I(e^7) \quad (= \sigma^{c_1} \dots \sigma^{c_q}(e^{s_1}, \dots, e^{s_q})) \\ &= \delta_{I7} \end{aligned}$$

Now

$\langle \cdot, \cdot \rangle$ scalar product?

$$1. |\omega|_q^2 := \langle \omega, \omega \rangle_q = 0 \Leftrightarrow \sum_{I, 7} w_I z_I \underbrace{\langle \sigma^I, \sigma^7 \rangle_q}_{= \delta_{I7}} = \sum_I w_I^2 = 0$$

$$\Leftrightarrow w_I = 0 \quad \forall I \quad \Leftrightarrow \omega = 0$$

$$\begin{aligned} 2. \langle \omega, z \rangle_q &= \sum_{I, 7} w_I z_I \underbrace{\langle \sigma^I, \sigma^7 \rangle_q}_{= \delta_{I7}} = \sum_I w_I z_I = \sum_I z_I w_I \\ &= \langle z, \omega \rangle_q \end{aligned}$$

3. $\langle \cdot, \cdot \rangle$ bilinear \checkmark . ■

Lemma 15 Let $\{e^i\}_{i=1, \dots, n}$ be an orthonormal basis of V .

Then $b_e^i = \sigma^i$, $e^i = \# \sigma^i$ and

$$\{\sigma^I\}_{I \in \mathbb{I}^{q,n}}$$

is an orthonormal basis of $A^q(V)$. We have

$$\langle \sigma^I, \sigma^7 \rangle_q = \sigma^I(e^7) = \delta_{I7}.$$

Lemma 16 Let $g := (\langle e^i, e^j \rangle)_{i,j=1,\dots,n}$. Then

$$\langle \delta^i, \delta^j \rangle_1 = (g^{-1})_{i,j} \quad , \quad i,j = 1, \dots, n.$$

Proof: Let $\{\tilde{e}^i\}$ be an orthonormal basis of V and $\{\tilde{\delta}^i\}$ the dual basis. $\Rightarrow \{\tilde{\delta}^i\}$ orthonormal basis as well.

$$\text{Let } e^i = \sum_u \alpha_{ui} \tilde{e}^u, \quad \delta^i = \sum_u \beta_{ui} \tilde{\delta}^u.$$

Then

$$\bullet \quad \langle \delta^i, \delta^j \rangle_1 = \underbrace{\sum_{u,m} \beta_{ui} \beta_{mj}}_{=\delta_{um}} \langle \tilde{\delta}^u, \tilde{\delta}^m \rangle_1 = \sum_u \beta_{ui} \beta_{uj}$$

$$= ({}^t \beta \beta)_{i,j} \quad ,$$

$$\bullet \quad g_{ij} = \langle e^i, e^j \rangle = \sum_{u,m} \alpha_{ui} \alpha_{mj} \underbrace{\langle \tilde{e}^u, \tilde{e}^m \rangle}_{=\delta_{um}} = \sum_u \alpha_{ui} \alpha_{uj}$$

$$= ({}^t \alpha \alpha)_{i,j} \quad ,$$

$$\bullet \quad \delta_{ij} = \delta^i(e^j) = \sum_{u,m} \beta_{ui} \alpha_{mj} \underbrace{\tilde{\delta}^u(\tilde{e}^m)}_{=\delta_{um}} = \sum_u \beta_{ui} \alpha_{uj}$$

$$= {}^t \beta \alpha$$

$$\Downarrow \quad \text{Id} = {}^t \beta \alpha, \quad g = {}^t \alpha \alpha, \quad \langle \delta^i, \delta^j \rangle_1 = {}^t \beta \beta$$

$$\Downarrow \quad \langle \delta^i, \delta^j \rangle_1 = ({}^t \beta \beta)_{i,j} = (\alpha^{-1} {}^t \alpha^{-1})_{i,j} = (g^{-1} {}^t \alpha {}^t \alpha^{-1})_{i,j} = (g^{-1})_{i,j} \quad . \quad \blacksquare$$

Corollary 17 If g is positive definite where then

$$\left\{ \frac{1}{\sqrt{g_{uu}}} e^u \right\}_{u=1,\dots,N}, \quad \left\{ \frac{1}{\sqrt{g_{uu}}} \delta^u \right\}_{u=1,\dots,N}$$

are orthonormal bases of V and $V' = A^*(V)$.

Proof:

$$\bullet \quad \langle \frac{1}{\sqrt{g_{uu}}} e^u, \frac{1}{\sqrt{g_{wm}}} e^w \rangle = \frac{1}{\sqrt{g_{uu}}} \frac{1}{\sqrt{g_{wm}}} \underbrace{\langle e^u, e^w \rangle}_{=g_{uw}} = \left(\frac{1}{\sqrt{g_{uu}}} \frac{1}{\sqrt{g_{ww}}} \right)_{i,j} = \delta_{ij}$$

$$\bullet \quad \langle \frac{1}{\sqrt{g_{ui}}} \delta^u, \frac{1}{\sqrt{g_{wj}}} \delta^w \rangle_1 = \frac{1}{\sqrt{g_{ui}}} \frac{1}{\sqrt{g_{wj}}} \underbrace{\langle \delta^u, \delta^w \rangle_1}_{=g_{uw}} = \left(\frac{1}{\sqrt{g_{ui}}} \frac{1}{\sqrt{g_{wj}}} \right)_{i,j} = \delta_{ij} \quad . \quad \blacksquare$$

(47)

Example 18 What is $\langle \omega, \gamma \rangle_1$ for $\omega, \gamma \in A^1(V) = V'$

and general $\{e^i\}, \{\delta^i\}$?

We have

$$\langle \omega, \gamma \rangle_1 = \sum_{i,j=1}^n \omega_i \gamma_j \underbrace{\langle \delta^i, \delta^j \rangle_1}_{= g_{ij}^{-1}} = \langle \bar{\omega}, \bar{g}^{-1} \bar{\gamma} \rangle_{\mathbb{R}^n}.$$

Example 19 Same as Example 18 with q .

$$\begin{aligned} \langle \omega, \gamma \rangle_q &= \sum_{I,7} \omega_I \gamma_7 \langle \delta^I, \delta^7 \rangle_q \\ &= \sum_{I,7} \omega_I \gamma_7 \beta_{I\bar{I}} \beta_{\bar{I}\bar{7}} \underbrace{\langle \delta^{\bar{I}}, \delta^{\bar{7}} \rangle_q}_{= \bar{g}_{\bar{I},\bar{7}}} \\ &= \sum_{\substack{I,7,\bar{I},\bar{7}} \atop {I,7}} \omega_I \gamma_7 \beta_{I\bar{I}} \beta_{\bar{I}\bar{7}} \\ &= \sum_{\substack{\bar{I},\bar{7} \\ I,7}} \omega_I \gamma_7 \underbrace{\beta_{\bar{I}_1 i_1} \cdots \beta_{\bar{I}_q i_q} \beta_{\bar{I}_1 j_1} \cdots \beta_{\bar{I}_q j_q}}_{= \beta_{\bar{I}_1 i_1} \beta_{\bar{I}_2 i_2} \cdots \beta_{\bar{I}_q i_q} \beta_{\bar{I}_1 j_1} \cdots \beta_{\bar{I}_q j_q}} \\ &= (\bar{g}^{-1})_{i_1 i_1} \cdots (\bar{g}^{-1})_{i_q i_q} \\ &= \bar{g}_{i_1 i_1} \cdots \bar{g}_{i_q i_q} = \bar{g}_{I\bar{I}} \\ &= \sum_{I,7} \omega_I \gamma_7 \bar{g}_{I\bar{I}} \end{aligned}$$

Especially we get

$$\langle \delta^I, \delta^7 \rangle_q = \bar{g}_{I\bar{I}} \quad \triangle$$

Note 20 Then $\{g_{I\bar{I}}^{\frac{1}{2}}, \delta^I\}$ is an orthonormal basis of $A^q(V)$.

Definition 21 "Orientation of a vector space" (48)

Two bases $\{e^1, \dots, e^n\}$ and $\{\tilde{e}^1, \dots, \tilde{e}^n\}$ of V are called have the same orientation, if the matrix of the change of bases has got a positive determinant. Then we call $\{e^i\}$ and $\{\tilde{e}^i\}$ equivalent and write $\{e^i\} \sim \{\tilde{e}^i\}$.

Lemma 22 The set of all bases decomposes into two equivalence classes. One contains all basis with ~~each one~~ contains only basis with same orientation. An orientation of V is a choice of one class. The element

Definition 23 An orientation of V is a choice of one equivalence class. The elements of this class are called positive oriented, otherwise negative oriented.

The standard basis in \mathbb{R}^n is called positive oriented.

Definition and Lemma 24

Let V be a N -dimensional vector space with scalar product and orientation. Let $\{e^i\}$ be a positive oriented orthonormal basis of V . Then we call the N -form $\hat{\omega} \in \Lambda^N(V)$ with

$$\hat{\omega}(e^1, \dots, e^n) = +1$$

the "volume form" of V . Then $\hat{\omega}(\tilde{e}^1, \dots, \tilde{e}^n) = 1$ holds for any positive oriented orthonormal basis of V .

Proof. Let $\{\tilde{e}^i\}$ be the other basis and $\{e^i\}$

$$\mathcal{B}: V \rightarrow V \quad b = (b_{ij}) = \mathcal{B}_{\{e^i\}}$$

$$e^i \mapsto \tilde{e}^i$$

with $\det \mathcal{B} > 0$. We have

$$\begin{aligned} \hat{\omega}(\tilde{e}^1, \dots, \tilde{e}^n) &= \hat{\omega}(\mathcal{B}e^1, \dots, \mathcal{B}e^n) = \mathcal{B}^t \hat{\omega}(e^1, \dots, e^n) \\ &\stackrel{\text{Perm}(i^n)}{=} \det \mathcal{B} \hat{\omega}(e^1, \dots, e^n) = \det \mathcal{B}. \end{aligned}$$

Moreover we have

$$\delta_{ij} = \langle \tilde{e}^i, \tilde{e}^j \rangle = \sum_{u,u} b_{ui} b_{uj} \underbrace{\langle e^i, e^u \rangle}_{= \delta_{iu}} = \sum_u b_{ui} b_{uj} = (b^t b)_{ij}$$

$$\Rightarrow \text{Id} = {}^t b b.$$

$$\Rightarrow (\det b)^2 = 1 \Rightarrow \det b = \pm 1.$$

$\Rightarrow \det b = +1$, since \mathcal{B} keeps the orientation. \blacksquare

From now on let V be a N -dimensional vector space with scalar product and orientation.

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Lemma 25 Let $\{e^i\}$ be positive oriented. Then

$$\hat{\omega} = (\det g)^{\frac{1}{2}} \delta^1 \wedge \dots \wedge \delta^N .$$

Proof: Let $\{\tilde{e}^i\}$ be a positive oriented orthonormal basis. Then by \circledast

$$\begin{aligned} \delta^1 \wedge \dots \wedge \delta^N (\tilde{e}^1, \dots, \tilde{e}^N) &= \delta^1 \wedge \dots \wedge \delta^N (\mathcal{B}e^1, \dots, \mathcal{B}e^N) \\ &= \mathcal{B}^*(\delta^1 \wedge \dots \wedge \delta^N)(e^1, \dots, e^N) \\ &= \det \mathcal{B} \underbrace{\delta^1 \wedge \dots \wedge \delta^N(e^1, \dots, e^N)}_{=1} = \det \mathcal{B} > 0 . \end{aligned}$$

By \circledast we compute with $b^{-1} = (b^{-1}_{ij}) = \mathcal{B}^{-1}$

$$g_{ij} = \langle e^i, e^j \rangle = \cancel{\langle \mathcal{B}^{-1}\tilde{e}^i, \mathcal{B}^{-1}\tilde{e}^j \rangle} \underbrace{\equiv b^{-1}}_{=f^{-1}}$$

$$g_{ij} = \langle \tilde{e}^i, \tilde{e}^j \rangle = \sum_{u,m} b_{ui} b_{mj} \underbrace{\langle e^u, e^m \rangle}_{=g_{um}} = {}^t b g b^{-1}_{ij} = g_{um}$$

$$\Rightarrow \text{Id} = {}^t b g b$$

$$\Rightarrow {}^t b^{-1} b = g$$

$$\Rightarrow (\det b^{-1})^2 \cancel{= \det g} \Rightarrow \det \mathcal{B} = \det b = \det g .$$

$$\Rightarrow 1 = (\det b)^2 \det g$$

$$\Rightarrow 0 < \det \mathcal{B} = \det b = (\det g)^{-\frac{1}{2}}$$

$$\text{Finally } (\det g)^{\frac{1}{2}} \delta^1 \wedge \dots \wedge \delta^N (\tilde{e}^1, \dots, \tilde{e}^N) = (\det g)^{\frac{1}{2}} (\det g)^{-\frac{1}{2}} = 1 .$$

Remark 26 Let $\{e^i\}$ be a positive oriented orthonormal basis. Then

$$\hat{\omega} = \delta^1 \wedge \dots \wedge \delta^N .$$

50

Definition 27 For every $q \in \mathbb{N}$ we define

the "Hodge star operator" (or "star operator" or " \star -operator") as the linear operator

$$\star : A^q(V) \rightarrow A^{N-q}(V)$$

defined by $\sigma^I := \text{sgn}(I, I') \sigma^{I'}$ on a positively-oriented

~~the~~ ~~other~~ ~~complement~~ orthogonal basis $\{\phi^c\}_{c=1, \dots, N}$ of $A^*(V)$, where $I^c \in \{1, \dots, N\} \setminus I$ is the ordered complementary index to I and $\text{sgn}(I, I^c)$ is the sign of the permutation mapping (I, I^c) to $(1, \dots, N)$.

Lemma 28 We have like \mathbb{R} an every positively oriented orthonormal basis. $\{e_1, e_2, \dots, e_n\} \subset A^1(V)$

$$\begin{aligned} \text{(c)} \quad * \text{ acts like } \oplus \text{ on every polynomial in } & A^q(V) \\ \text{(d)} \quad \omega_1 * \zeta = \langle \omega_1, \zeta \rangle_q \omega & \end{aligned}$$

$$(ii) \quad * \delta^I = \text{Sgn}(I, I') \delta^{I'} \quad , \text{ if } |I| = \{1, -1, N\} \text{ and} \\ \text{even } I \text{ and } I' \text{ are not ordered.}$$

$$(iii) \quad *1 = \hat{\omega} \quad , \quad *\hat{\omega} = 1$$

$$(cv) \quad ** = (-1)^{q(N-q)} \cancel{\text{_____}}$$

$$(v) \quad \omega \wedge \{ = (-1)^{q(N-q)} \langle \omega, * \{ \rangle_s \hat{\omega} \quad \forall \omega \in A^q(V), \{ \in A^{N-q}(V)$$

$$(vi) \quad \langle *w, *z \rangle_{N-q} = \langle w, z \rangle_q \quad \forall w, z \in A^q(V)$$

(viii) A change of orientation implies a change of the sign of $*$.

$$(viii) \quad \langle w, z \rangle_q = * (w \wedge *z) \text{ positively}$$

Proof: Let $\{e_i\}$ be an oriented orthonormal basis of V .

$$(ii) \quad w_1 * z = \sum_{I=7} \omega_I z_I \xrightarrow{\sigma^I * \sigma^7} \\ = \text{sgn}(7,7') \sigma^I * \sigma^{7'} \stackrel{I=7}{=} \cancel{\sigma^7} \cancel{\sigma^{7'}} \xrightarrow{\text{cancel}} I=7$$

$$= \sum_I \omega_I z_I \underbrace{\operatorname{sgn}(I, I') \sigma^I \wedge \sigma^{I'}}_{\sigma^{I'}_1 \wedge \cdots \wedge \sigma^{I'}_n}.$$

$$= \sum_{I=1}^n u_I z_I$$

$$= \sum_{I, J} w_I z_J \langle \sigma^I, \sigma^J \rangle, \quad \hat{\omega} = \langle \omega, z \rangle, \quad \hat{\omega}$$

$$(iii) * \sigma^I = \operatorname{sgn} I * \sigma^{I_{\text{ord}}} = \operatorname{sgn} I \operatorname{sgn}(I_{\text{ord}}, I'_{\text{ord}}) \sigma^{I'_{\text{ord}}} \quad (51)$$

$$= \underbrace{\operatorname{sgn} I \operatorname{sgn} I'_{\text{ord}}}_{=1} \underbrace{\operatorname{sgn}(I_{\text{ord}}, I'_{\text{ord}})}_{=\operatorname{sgn}(I, I')} \sigma^{I'}$$

$$(iv) *(\sigma^1 \wedge \dots \wedge \sigma^N) = \hat{\omega} \quad ,$$

$$*\hat{\omega} = 1$$

$$(v) ** \sigma^I = \operatorname{sgn}(I, I') * \sigma^{I'} = \underbrace{\operatorname{sgn}(I, I')}_{q-\text{times}} \underbrace{\operatorname{sgn}(I', I)}_{(N-q)-\text{times}} \sigma^I$$

sorting sorting

$$= (-1)^{q(N-q)} \sigma^I \Rightarrow ** = (-1)^{q(N-q)}$$

$$(vi) \omega \wedge \zeta \stackrel{(iv)}{=} (-1)^{q(N-q)} \omega \wedge *(\zeta) = (-1)^{q(N-q)} \langle \omega, * \zeta \rangle_q \hat{\omega}$$

$$\begin{aligned} \langle * \omega, * \zeta \rangle_{N-q} &\stackrel{(iii)}{=} * \langle * \omega, * \zeta \rangle_{N-q} \hat{\omega} \stackrel{(i)}{=} *(*\omega \wedge * \zeta) \\ &= (-1)^{q(N-q)} *(*\omega \wedge \zeta) \\ &= (-1)^{q(N-q)+q(N-q)} *(\zeta \wedge *\omega) \\ &\stackrel{(ii)}{=} * \langle \omega, \zeta \rangle_q \hat{\omega} \stackrel{(iii)}{=} \langle \omega, \zeta \rangle_q \end{aligned}$$

$$(vii) \stackrel{(i)}{=} \omega \wedge * \zeta = \langle \omega, \zeta \rangle_q \hat{\omega} \Rightarrow * \text{ changes sign.}$$

$\begin{matrix} q \\ \text{keeps sign} \end{matrix} \quad \begin{matrix} R \\ \text{changes sign} \end{matrix}$

$$(viii) \langle \omega, \zeta \rangle_q = \langle \omega, \zeta \rangle_q * \hat{\omega} = *(\omega \wedge * \zeta)$$

(v) Let $\{\tilde{\sigma}^i\}$ be another positively oriented orthonormal basis.

$$\text{Then } \tilde{\sigma}^I = \sum_j \tilde{w}_j \tilde{\sigma}^j \text{ with } \tilde{w}_j = \tilde{\sigma}^I(e^j), \quad j \in \mathbb{Z}_{\leq N}^{q, N}$$

$$\begin{aligned} * \tilde{\sigma}^I &= \sum_j \tilde{w}_j * \tilde{\sigma}^j = \sum_j \tilde{w}_j \operatorname{sgn}(j, j') \tilde{\sigma}^{j'} \\ &= \sum_j \tilde{\sigma}^I(e^j) \operatorname{sgn}(j, j') \tilde{\sigma}^{j'} \\ &= \sum_{j' \in \mathbb{Z}_{\leq N}^{q, N}} \tilde{\sigma}^I(e^{j'}) \end{aligned}$$

$$\text{Then by (v)} \quad \tilde{\sigma}^I \wedge * \tilde{\sigma}^7 = \langle \tilde{\sigma}^I, \tilde{\sigma}^7 \rangle_q \hat{\omega} = \sigma_I \hat{\omega}$$

and with $* \tilde{\sigma}^7 = \sum_{k \in \mathbb{I}^{N-9, N}} \omega_k \tilde{\sigma}^k$

(52)

$$\begin{aligned}\tilde{\sigma}^I * \tilde{\sigma}^7 &= \sum_{k \in \mathbb{I}^{N-9, N}} \omega_k \underbrace{\tilde{\sigma}^I * \tilde{\sigma}^k}_{\text{cancel } \tilde{\sigma}^I \text{ and } \tilde{\sigma}^k} \\ &= \cancel{\omega_k} \delta_{I,k} \tilde{\sigma}^I \\ &= \delta_{I',k} \tilde{\sigma}^1 * \dots * \tilde{\sigma}^N \operatorname{sgn}(I, I') \\ &= \omega_{I'} \operatorname{sgn}(I, I') \underbrace{\tilde{\sigma}^1 * \dots * \tilde{\sigma}^N}_{\hat{\omega}}\end{aligned}$$

$\Rightarrow \omega_{I'} = \operatorname{sgn}(I, I') \delta_{I,I'}$, i.e. ~~$I' = k, I = k'$~~

~~$* \tilde{\sigma}^k = \operatorname{sgn}(I, I') \sum_k \operatorname{sgn}(I, I') \tilde{\sigma}^k$~~

i.e.

$$\begin{aligned}\omega_{k'} &= \operatorname{sgn}(k, k') \underbrace{\delta_{k',k}}_{\text{cancel } k' \text{ and } k} \\ &= \delta_{k,k'}\end{aligned}$$

$\Rightarrow * \tilde{\sigma}^7 = \sum_k \operatorname{sgn}(k, k') \delta_{k,k'} \tilde{\sigma}^k = \operatorname{sgn}(7, 7') \tilde{\sigma}^{7'}$ ■

Finally we compute the $*$ -operator for a general basis.

Let $\{\tilde{e}^i\}, \{\tilde{\sigma}^i\}$ be positively oriented orthonormal bases. Then with

$$\sigma^i = \sum_u \beta_{ui} \tilde{\sigma}^u$$

we have

$$\sigma^I = \sum_I \beta_{JI} \tilde{\sigma}^J, \quad \tilde{\sigma}^I = \sum_I \beta_{IJ} \tilde{\sigma}^J$$

and "notation"!

3.2 Multilinear Algebra on ~~Tangent~~ Manifolds,

In this section we just put

$$V := T(X) \text{ , i.e. } V \subset T_u(X)$$

$$e^q := \partial_u^q$$

$$\omega := d\psi_u$$

and translate our results from section 3.1.

Lemma and Definition 29

~~($\forall A^q(X) = A^q T_u(X)$ possesses the basis~~
 ~~$\{\partial_u^1, \dots, \partial_u^q\}$~~
~~with coefficients~~

We define $A_u^q(X) := A^q T_u(X)$ and $A^q(X) := \bigcup_{u \in X} A_u^q(X)$.

Then $A^q(X)$ becomes a differentiable manifold by the charts

$$\phi : A^q(X) \longrightarrow \mathbb{R}^N \times \mathbb{R}^{N \choose q}$$

$$\omega \mapsto (\psi_0 \pi(\omega), I^q(\omega)) \quad \text{if } \omega \in A_u^q(X)$$

(see section 2, Def. 13 and Def. 5), where $\pi(\omega) = u$ and $I^q(\omega) = (\omega_I)_{I \in \mathbb{I}^{q,n}}$ and $\omega_I = \omega(\partial_u^1, \dots, \partial_u^q)$,

A mapping

$$\omega : X \rightarrow A^q(X)$$

$$u \mapsto \omega(u) \quad \text{with } \omega(u) \in A_u^q(X)$$

is called a q -form on X . The set $A^q(X)$ of all q -forms builds in a natural way a vector space.

For $\omega \in A^q(X)$ the following assertions are equivalent:

- (i) ω is differentiable
- (ii) $\bigwedge_I \omega_I \in C^\infty(X, \mathbb{R})$.

In this case we call ω differentiable and write
 $\omega \in C^{\infty, q}(X)$.

Remark 30

$$C^{\infty, 0}(X) = C^\infty(X, \mathbb{R}).$$

Lemma and Definition 31 Let $F \in C^\infty(X, Y)$ and $\omega \in A^q(Y)$. We define the "pullback" of ω by

$$\begin{aligned} F^*\omega: X &\rightarrow A^q(X) \\ u &\mapsto F^*\omega(u) \end{aligned}$$

and $F^*\omega(v^1, \dots, v^N) := \omega(dF_u v^1, \dots, dF_u v^N)$

for $v^1, \dots, v^N \in T_u(X)$. Then we have

$$(i) \quad \omega \in C^{\infty, q}(Y) \Rightarrow F^*\omega \in C^{\infty, q}(X)$$

$$(ii) \quad (\cancel{G \circ F})^* \omega \quad (\cancel{(G \circ F)})^* \omega = F^* G^* \omega$$

for $g \in A^q(Z)$ and $G \in C^\infty(Y, Z)$.

$$\text{Proof: } (i) \quad F^*\omega = (dF)^* \omega$$

$$\begin{aligned} &= (G \circ F)^* \omega = (d(G \circ F))^* \omega = (dG \circ dF)^* \omega \\ &= (dF)^* (dG)^* \omega = F^* G^* \omega \end{aligned}$$

$$(ii) \quad (F^*\omega)_I = F^*\omega(\partial_I^q) = \omega(dF \partial_{i_1}^q, \dots, dF \partial_{i_q}^q)$$

$$= \omega(\partial_{c_1}^q F, \dots, \partial_{c_q}^q F \partial_m^q)$$

$$= \sum_{m_1, \dots, m_q=1}^n \partial_{c_1}^q F_{m_1} \cdots \partial_{c_q}^q F_{m_q} \underbrace{\omega(\partial_{m_1}^q, \dots, \partial_{m_q}^q)}_{= \pm \omega_F}$$

Thus $(F^*\omega)_I \in C^\infty(X, \mathbb{R})$ since

(55)

$\partial_i F_I \in C^\infty(X, \mathbb{R})$ and $\omega_I \in C^\infty(X, \mathbb{R})$.

■

$\Rightarrow F^*\omega \in C^{\infty, q}(X)$.

Lemma and Definition 32 By $w \wedge z(u) := w(u) \wedge z(u)$
~~bilinear mapping~~ we define two ~~the~~ bilinear
mappings

$$\wedge : A^q(X) \times A^p(X) \rightarrow A^{q+p}(X),$$

$$\wedge : C^{\infty, q}(X) \times C^{\infty, p}(X) \rightarrow C^{\infty, q+p}(X),$$

which have the properties of Theorem 8 (Def. of \wedge).

Moreover we have

$$(i) \quad f \wedge w = f \cdot w \quad \forall f \in A^0(X) \text{ resp. } C^{\infty, 0}(X) = C^\infty(X)$$

$$(ii) \quad F^*(w \wedge z) = F^*w \wedge F^*z \quad \forall f \in C^\infty(Y, X)$$

Proof: (i) ✓

$$(ii) \quad F^*(w \wedge z) = (dF)^*(w \wedge z) = (dF)^*w \wedge (dF)^*z \\ = F^*w \wedge F^*z.$$

■

Lemma 33 Since $\{d\varphi_u\}_{u=1,\dots,n} \subset A^1(X)$ is the dual basis
to $\{\partial_u^q\}_{u=1,\dots,n} \subset T(X)$ the set

$$\{d\varphi_I\}_{I \in \mathbb{I}^{q,n}}$$

is a basis of $A^q(X)$ and we have for $w \in A^p(X)$

$$w = \sum_{I \in \mathbb{I}^{q,p}} w_I d\varphi_I,$$

where $w_I = w(\partial_I^q)$.

Remark 34 Note

$$\begin{aligned} d\varphi^I(\partial_{\bar{z}}^q) &= d\varphi_{i_1} \wedge \dots \wedge d\varphi_{i_q} (\partial_{i_1}^{q_1}, \dots, \partial_{i_q}^{q_q}) \\ &= \text{sgn}(I, \tau) \, \delta^I_{\bar{z}} \end{aligned}$$

Definition and Theorem 35 For any q there exists exactly one linear mapping, the "exterior derivative",

$$d : C^{\infty, q}(X) \rightarrow C^{\infty, q+1}(X) ,$$

such that:

- (i) dF is the differential for $F \in C^{\infty, 0}(X) = C^{\infty}(X)$.
- (ii) $d \circ d = 0$ (complex property)
- (iii) $d(w \wedge \bar{z}) = dw \wedge \bar{z} + (-)^q w \wedge d\bar{z}$ if $w \in C^{\infty, q}(X)$, $\bar{z} \in C^{\infty, p}(X)$. (product rule)

Proof: Only for the proof: d differential, D exterior derivative

① $X = U$ chart domain. Let $\omega = \omega_I \, d\varphi_I$

$$= D\omega_I \wedge d\varphi_I + (-)^0 \omega \wedge D d\varphi_I$$

$$\begin{aligned} D d\varphi_I &= D(d\varphi_{i_1} \wedge \dots \wedge d\varphi_{i_q}) \\ &= \overbrace{(D d\varphi_{i_1}) \wedge d\varphi_{i_2} \wedge \dots \wedge d\varphi_{i_q}}^{=0} \\ &\quad + (-)^1 d\varphi_{i_1} \wedge \underbrace{D(d\varphi_{i_2} \wedge \dots \wedge d\varphi_{i_q})}_{=0 \text{ by induction}} \\ &= 0 \end{aligned}$$

$$\Rightarrow D\omega = D\omega_I \wedge d\varphi_I = \omega_I \wedge d\varphi_I$$

$\Rightarrow D$ is unique uniqueness for D .

For the existence of \mathcal{D} we use the definition

(57)

$$\mathcal{D}\omega := d\omega_I \wedge d\varphi_I.$$

Then \mathcal{D} is linear and well defined since

$$\mathcal{D}: C^{\infty, q}(U) \rightarrow C^{\infty, q+1}(U).$$

(i) ✓

$$\begin{aligned}
 \text{(ii)} \quad \mathcal{D}(\omega \wedge \gamma) &= \mathcal{D}(\omega_I \gamma_I \wedge d\varphi_I \wedge d\varphi_I) \\
 &= d(\omega_I \gamma_I) \wedge d\varphi_I \wedge d\varphi_I \\
 &= \cancel{\gamma_I} d\omega_I \wedge d\varphi_I \wedge d\varphi_I \\
 &\quad + \omega_I d\cancel{\gamma_I} \wedge d\varphi_I \wedge d\varphi_I \\
 &= (\underbrace{d\omega_I \wedge d\varphi_I}_{= \mathcal{D}\omega}) \wedge (\cancel{\gamma_I} d\varphi_I) \\
 &\quad = \cancel{\gamma_I} \\
 &+ (-1)^q (\cancel{\omega_I d\varphi_I}) \wedge (\cancel{d\gamma_I \wedge d\varphi_I}) \\
 &= \cancel{\omega} \mathcal{D}\gamma \\
 &= \mathcal{D}\omega \wedge \gamma + (-1)^q \cancel{\omega \wedge \mathcal{D}\gamma}.
 \end{aligned}$$

$$\text{(iii)} \quad \mathcal{D}\mathcal{D}\omega = \mathcal{D}(d\omega_I \wedge d\varphi_I)$$

$$\begin{aligned}
 &= (\underbrace{\mathcal{D}d\omega_I}_{= 0}) \wedge d\varphi_I + (-1)^q d\omega_I \wedge \mathcal{D}(d\varphi_I)
 \end{aligned}$$

(*)

$$\begin{aligned}
 \mathcal{D}\mathcal{D}\omega_I &= \mathcal{D}(\partial_u^\varphi(\omega_I) d\varphi_u) \\
 &= d(\partial_u^\varphi(\omega_I)) \wedge d\varphi_u \quad \cancel{+ \partial_u^\varphi(d\varphi_I) \wedge d\varphi_u} \\
 &= d(\partial_u^\varphi(\omega_I \circ \varphi^{-1})) \wedge d\varphi_u \quad \cancel{+ \partial_u^\varphi(d\varphi_I \circ \varphi^{-1}) \wedge d\varphi_u} \\
 &= \partial_m^\varphi(\partial_u^\varphi(\omega_I \circ \varphi^{-1})) \circ \varphi d\varphi_m \wedge d\varphi_u \\
 &= \partial_m^\varphi(\partial_u^\varphi(\omega_I \circ \varphi^{-1}) \circ \varphi \circ \varphi^{-1}) d\varphi_m \wedge d\varphi_u \\
 &= \partial_m^\varphi \partial_u^\varphi(\omega_I \circ \varphi^{-1}) d\varphi_m \wedge d\varphi_u \\
 &= \sum_{\substack{u, m=1 \\ u \neq m}} \underbrace{\partial_m^\varphi \partial_u^\varphi(\omega_I \circ \varphi^{-1})}_{= \partial_u \partial_m (\text{sym})} \underbrace{d\varphi_m \wedge d\varphi_u}_{= - d\varphi_u \wedge d\varphi_m (\text{skew sym})} = 0
 \end{aligned}$$

$$\Rightarrow \text{DD}\omega = - \underbrace{d\omega_I \wedge (\text{D} d\ell^I)}_{= \text{D}(d\ell_1, \dots, d\ell_q)}$$

Since the product rule holds already ~~too~~ it is enough (by induction) to show $\text{DD}f = 0$ for functions. But this is already clear by ④.

$$\Rightarrow \text{DD}\omega = 0.$$

② general X .

$$D\omega := D_{U,U} \omega, \text{ where } (U, \varphi) \text{ is a chart.}$$

independent of charts?

Uniqueness?

The technical details are left to the audience. □

Remark 36 In local (chart) coordinates we have
for $\omega = \sum_{I \in \mathbb{I}^{q,n}} \omega_I d\ell^I \in C^{\infty,q}(X)$ the representation

$$d\omega = d\omega_I \wedge d\ell^I$$

$$= \sum_{u=1}^n \sum_{I \in \mathbb{I}^{q,n}} (\omega_I)(\partial_u^\varphi) d\ell_u \wedge d\ell^I$$

$$= \sum_{u=1}^n \sum_{I \in \mathbb{I}^{q,n}} \partial_u^\varphi(\omega_I) d\ell^u \wedge d\ell^I.$$

Lemma 37 For $F \in C^\infty(X, Y)$ and $\omega \in C^{\infty,q}(Y)$
we have

$$d(F^*\omega) = F^*(d\omega),$$

i.e. d commutes with F^* , i.e. d is natural.

Note: $F^*\omega \in C^{\infty,q}(X)$.

Proof: For $\omega \in C^{\infty,0}(Y) = \text{C}^\infty(Y)$ we have

(59)

$$d(F^*\omega) = d(\omega \circ F),$$

$$F^*dw = dw(dF) = dw \circ dF,$$

i.e. this is just the chain rule.

Let $\omega \in C^{\infty,0}(Y)$. Then

$$\omega = \sum_I \omega_I d\varphi^I$$

$$d\omega = \sum_I dw_I \wedge d\varphi^I$$

and

$$F^*\omega = \sum_I \omega_I \circ F \underbrace{F^*d\varphi^I}_{= d\varphi_{i_1} \wedge \dots \wedge d\varphi_{i_q}}$$

$$= \sum_I \omega_I \circ F \underbrace{F^*d\varphi_{i_1}}_{= dF^*\varphi_{i_1}} \wedge \dots \wedge \underbrace{F^*d\varphi_{i_q}}_{= dF^*\varphi_{i_q}}$$

$$= \sum_I \omega_I \circ F d(\varphi \circ F)^I$$

$$F^*dw = \sum_I F^*dw_I \wedge F^*d\varphi^I$$

$$= \sum_I d(\omega_I \circ F) \wedge d(\varphi \circ F)^I$$

Now

$$dF^*\omega = \sum_I d(\omega_I \circ F) \wedge d(\varphi \circ F)^I (= F^*dw)$$

$$+ \sum_I \omega_I \circ F d d(\varphi \circ F)^I$$

Now

$$\begin{aligned} d d(\varphi_0 \tilde{F})^I &= d(d(\varphi_0 F)_{i_1, 1} \dots, d(\varphi_0 F)_{i_q, q}) \\ &= 0 \quad \text{by induction, product rule} \\ &\quad \text{and } dd = 0. \end{aligned}$$

Thus

$$d F^* \omega = F^* d \omega$$



3.3 Orientation, star operator, scalar product and co-derivative

We already know orientations for vector spaces V .

Definition 38 An orientation for X is a family of orientations for $T_u(X)$ for all $u \in X$, such that for any $u \in X$ there exists a chart (U, φ) , ~~such that the differential $d\varphi_u: T_u(X) \rightarrow T_{\varphi(u)} \cong \mathbb{R}^n$ maps positively oriented bases of $T_u(X)$~~ which is orientation preserving. A chart (U, φ) is called "orientation preserving", if for any $u \in X$ ~~there exists a chart~~ the differential $d\varphi_u: T_u(X) \rightarrow T_{\varphi(u)} \cong \mathbb{R}^n$ maps positively oriented bases of $T_u(X)$ onto positively oriented bases of \mathbb{R}^n .

X is called orientable, if an orientation for X exists.

Remark 39 The orientations of $T_u(X)$ are locally "constant".

Definition 40: A diffeomorphism $F: X \rightarrow Y$ is called "orientation preserving", if for any $u \in X$ the differential $dF_u: T_u(X) \rightarrow T_{F(u)}(Y)$ is orientation preserving.

Remark 41 $d\varphi(\partial_u^\varphi) = \partial_u^\varphi(\varphi) = e^u$. Thus a chart φ is orientation preserving, iff the basis $\{\partial_u^\varphi\}_{u=1,\dots,n}$ of $T_u(X)$ is positively oriented for every point $u \in U$.

Definition and lemma 42

- (i) X is orientable
- (ii) There exist an atlas \mathcal{U} for X , such that all chart changes are orientation preserving, i.e. such that all Jacobian matrices of the chart changes have positive determinants.
- (iii) There exists $\tilde{\omega} \in \Omega^{n,n}(X)$ with $\tilde{\omega}(u) \neq 0$ for all $u \in X$.

Remark 43 The choice of such a $\tilde{\omega}$ is called ~~orient~~ an orientation of X as well. A N -form ω is then called positive resp. nonnegative, if

$$\omega = f \tilde{\omega} \quad \text{with } f \in A^0(X), \quad f > 0 \text{ resp. } f \geq 0.$$

A basis (e^1, \dots, e^n) of $T_u(X)$ is then called positive, if $\tilde{\omega}(e^1, \dots, e^n) > 0$. A chart (U, φ) is then called positive, if $d\varphi_u$ maps positively oriented bases of $T_u(X)$ onto positively oriented bases of \mathbb{R}^n .

Lemma 44 X is not orientable, if there exist a closed C^∞ -curve $\gamma: [0, 1] \rightarrow X$ and C^∞ -mappings $e^i: [0, 1] \rightarrow T(X)$, such that

- (i) $\bigwedge_{t \in [0, 1]} \{e^1(t), \dots, e^n(t)\}$ is basis of $T_{\gamma(t)}(X)$
- (ii) $\{e^1(0), \dots, e^n(0)\}$ and $\{e^1(1), \dots, e^n(1)\}$ have different orientations.

Remark 45 For example Möbius' strip is not orientable.

From now on let X be an oriented, N -dimensional differentiable manifold \hookrightarrow , such that the tangential spaces $T_u(X)$ possess scalar products $\langle \cdot, \cdot \rangle$, which are differentiable, i.e. $x \mapsto \langle \partial_x^1, \partial_x^2 \rangle \in \mathbb{R}$ are differentiable for all $x \in u$.

Now ~~$d d(\varphi \circ F)^E = d (\det(\varphi \circ F))_{ij_1 \dots j_N} d(\varphi \circ F)_{i_1 i_2}$~~ (62)

$$= 0 \quad \text{by induction, product rule and } dd = 0.$$

Thus

$$dF^* \omega = F^* d\omega.$$

Definition 46 ~~then~~ We call X a Riemannian manifold. ■

Lemma and Definition 47 By $(*\omega)(u) := *(\omega(u))$ we define ~~two~~ two linear mappings

$$*: A^q(X) \rightarrow A^{N-q}(X),$$

$$*: C^\infty, q(X) \rightarrow C^\infty, N-q(X),$$

which possess the properties of Definition 27 and Lemma 28.

Lemma and Definition 48 By $\langle \omega, \gamma \rangle_q(u) := \langle \omega(u), \gamma(u) \rangle_q$

we define two bilinear mappings

$$\langle \cdot, \cdot \rangle: A^q(X) \times A^q(X) \rightarrow A^0(\mathbb{R})$$

$$\langle \cdot, \cdot \rangle: C^\infty, q(X) \times C^\infty, q(X) \rightarrow C^\infty, 0(\mathbb{R}) = C^\infty(\mathbb{R}),$$

which possess the properties of the scalar product from Theorem 14.

~~Definition 49~~ ~~orientable~~

Definition 49 We define the ω -derivative by

$$\delta: C^\infty, q(\mathbb{R}) \rightarrow C^\infty, q-1(\mathbb{R}), \quad \delta := (-1)^{(q-1)N} * d *$$

~~isomorphism~~

Remark 50 $\delta \delta = 0$

3.4 Exercises

① Let $\omega \in A^N V$, $\omega \neq 0$. Show that

$$\phi_\omega : V \rightarrow A^{N-1} V$$

Defined by $\phi_\omega(v)(v^1, \dots, v^{N-1}) := \omega(v, v^1, \dots, v^{N-1})$
is an Isomorphism.

② Let $\{\omega_i\}_{i=1, \dots, N}$ be a basis of $V^* = A^*(V)$.

Moreover let $\{\alpha_j\}_{j=1, \dots, L} \subset A^*(V)$ for $1 \leq j \leq L$

$$(i) \sum_{l=1}^L \alpha_l \wedge \omega_l = 0$$

$$(ii) \exists (\alpha_{lk})_{l,k=1, \dots, L} \text{ sym. matrix} \quad \alpha_k = \sum_{l=1}^L \alpha_{lk} \omega_l.$$

③ Let $\{\omega_e\}_{e=1, \dots, L}$, $\{\alpha_e\}_{e=1, \dots, L} \subset A^*(V) = V^*$.

Show: $\{\omega_1, \dots, \omega_L\}$ linear independent, iff $\omega_1 \wedge \dots \wedge \omega_L \neq 0$,
and that in this case (i)

$$(i) \lim \{\omega_e\}_{e=1, \dots, L} = \lim \{\alpha_e\}_{e=1, \dots, L}$$

$$(ii) \bigvee_{c \in \mathbb{R} \setminus \{0\}} \alpha_1 \wedge \dots \wedge \alpha_L = c \omega_1 \wedge \dots \wedge \omega_L.$$

④ Let $\{\omega_i\}_{i=1, \dots, N} \subset C^{\infty, 1}(X)$ with

$$\forall u \in X \quad \omega_1(u) \wedge \dots \wedge \omega_N(u) \neq 0.$$

Moreover let $1 \leq l \leq N$ and $\alpha_1, \dots, \alpha_L \in C^{\infty, 1}(X)$ with

$$\sum_{e=1}^L \alpha_e \wedge \omega_e = 0.$$

Show: There exist $\alpha_{lk} \in C^{\infty, 0}(X)$, $l, k = 1, \dots, L$,
with $\alpha_{lk} = \alpha_{lk}$ and

$$\alpha_L = \sum_{e=1}^L \alpha_{lk} \omega_e.$$

⑤ We introduce isomorphisms with ($\text{id} = \star$, notation!) ⑥4

$$\phi_0 : C^\infty(\mathbb{R}^3) \rightarrow C^{\infty,0}(\mathbb{R}^3)$$

$$\varphi \mapsto \varphi$$

$$\phi_1 : C^\infty(\mathbb{R}^3)^3 \longrightarrow C^{\infty,1}(\mathbb{R}^3)$$

$$E = E_i e^i = \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix} \longmapsto E_i dx^i$$

$$\phi_2 : C^\infty(\mathbb{R}^3)^3 \longrightarrow C^{\infty,2}(\mathbb{R}^3)$$

$$E \longmapsto E_i \star dx^i$$

$$\phi_3 : C^\infty(\mathbb{R}^3) \longrightarrow C^{\infty,3}(\mathbb{R}^3)$$

$$\varphi \longmapsto \star\varphi = \varphi dx^1 \wedge dx^2 \wedge dx^3$$

Calculate:

$$(i) \quad \phi_1^{-1} d \phi_0(\varphi)$$

$$(ii) \quad \phi_2^{-1} d \phi_1(E)$$

$$(iii) \quad \phi_3^{-1} d \phi_2(E)$$

⑥ Let $w \in C^{\infty,q}(X)$, $\varphi \in C^{\infty,p}(X)$ and $y \in C^{\infty,r}(X)$.

For which q, p, r we have can guarantee

$$(i) \quad d(dw \wedge \varphi + w \wedge dy) = 0 \quad ?$$

$$(ii) \quad d(dw \wedge \varphi \wedge y + w \wedge d\varphi \wedge y + w \wedge \varphi \wedge dy) = 0 \quad ?$$

What do you get for general q, p, r ?

⑦ For $E \in C^\infty(\mathbb{R}^3)$ write

E and dE

in polar coordinates & with

$$\tilde{\varphi}'(r, \theta, \varphi) := r \begin{bmatrix} \cos \theta & \cos \varphi \\ \sin \theta & \cos \varphi \\ \sin \varphi \end{bmatrix}$$

What do you get for classical vector fields?

This gives the representation of grad, curl, div in polar coordinates

⑧ Determine $\alpha, \beta, \gamma \in C^\infty$, such that

$$\{\alpha dr, \beta d\theta, \gamma d\varphi\}$$

becomes an orthonormal basis.

What about 2- and 3-forms?

⑨ What is the natural metric tensor on a submanifold of \mathbb{R}^k ?

⑩ Let $F \in C^\infty(X, Y)$ and $\omega \in C^{0, q}(Y)$.

Calculate $F^*\omega$ in local chart coordinates.

What happens in the special case $X = \mathbb{R}^n$, $Y = \mathbb{R}^m$.

~~Answers~~

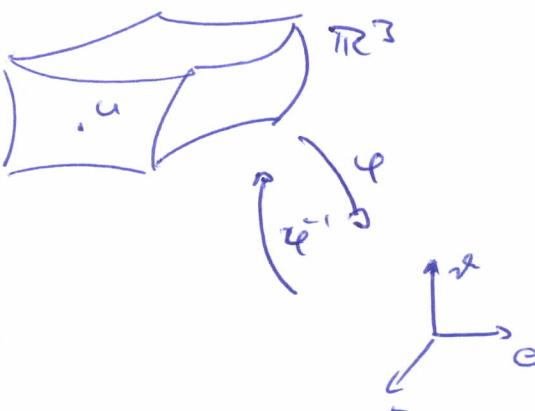
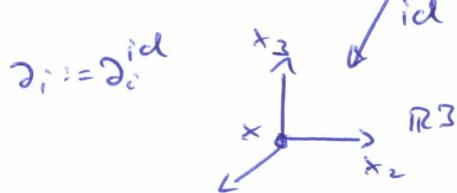
3.5 Solutions

(66)

~~Übungsaufgaben~~

polar coordinates

1. way: chart changing



$$\varphi^{-1}(r, \theta, \varphi) := r \begin{bmatrix} \cos \theta & \sin \theta \\ \sin \theta & \cos \theta \\ \end{bmatrix}$$

$$\partial_r := \partial_r^\varphi$$

$$\partial_\theta := -\dots$$

$$\partial_i := \partial_i^\varphi$$

$$\begin{aligned} dx_u &= dx_u(\partial_r) dr + dx_u(\partial_\theta) d\theta + dx_u(\partial_\varphi) d\varphi \\ &= \partial_r(x_u \circ \varphi^{-1}) dr + \partial_\theta(x_u \circ \varphi^{-1}) d\theta + \partial_\varphi(x_u \circ \varphi^{-1}) d\varphi \\ &= \partial_r \varphi_u^{-1} dr + \partial_\theta \varphi_u^{-1} d\theta + \partial_\varphi \varphi_u^{-1} d\varphi \end{aligned}$$

$$dx = (\varphi^{-1})^{*} \begin{bmatrix} dr \\ d\theta \\ d\varphi \end{bmatrix}$$

$(\varphi^{-1})'$ is easily invertible.

$$\begin{aligned} dx_1 \wedge dx_2 &= dx_1 \wedge dx_2(\partial_r, \partial_\theta) dr \wedge d\theta \\ &\quad + dx_1 \wedge dx_2(\partial_\theta, \partial_\varphi) d\theta \wedge d\varphi \\ &\quad + dx_1 \wedge dx_2(\partial_r, \partial_\varphi) dr \wedge d\varphi \\ &= (dx_1(\partial_r) \cdot dx_2(\partial_\theta) - dx_1(\partial_\theta) \cdot dx_2(\partial_r)) dr \wedge d\theta \\ &\quad + (dx_1(\partial_\theta) \cdot dx_2(\partial_\varphi) - dx_1(\partial_\varphi) \cdot dx_2(\partial_\theta)) d\theta \wedge d\varphi \\ &\quad + (dx_1(\partial_r) \cdot dx_2(\partial_\varphi) - dx_1(\partial_\varphi) \cdot dx_2(\partial_r)) dr \wedge d\varphi \\ &= (\partial_r \varphi_1^{-1} \partial_\theta \varphi_2^{-1} - \partial_\theta \varphi_1^{-1} \partial_r \varphi_2^{-1}) dr \wedge d\theta \\ &\quad + (\partial_\theta \varphi_1^{-1} \cdot \partial_\varphi \varphi_2^{-1} - \partial_\varphi \varphi_1^{-1} \partial_\theta \varphi_2^{-1}) d\theta \wedge d\varphi \\ &\quad + (\partial_r \varphi_1^{-1} \partial_\varphi \varphi_2^{-1} - \partial_\varphi \varphi_1^{-1} \partial_r \varphi_2^{-1}) dr \wedge d\varphi \\ &= (\nabla_{(r, \theta, \varphi)} \varphi_1^{-1} \times \nabla_{(r, \theta, \varphi)} \varphi_2^{-1}) \cdot \begin{bmatrix} d\theta \wedge d\varphi \\ dr \wedge d\varphi \\ dr \wedge d\theta \end{bmatrix} \end{aligned}$$

on the other hand

(67)

$$\begin{aligned}
 dx_1 \wedge dx_2 &= (\partial_r \varphi_i^{-1} dr + \partial_\theta \varphi_i^{-1} d\theta + \partial_\alpha \varphi_i^{-1} d\alpha) \\
 &\wedge (\partial_r \varphi_2^{-1} dr + \partial_\theta \varphi_2^{-1} d\theta + \partial_\alpha \varphi_2^{-1} d\alpha) \\
 &= (\partial_r \varphi_i^{-1} \partial_\theta \varphi_2^{-1} - \partial_\theta \varphi_i^{-1} \partial_r \varphi_2^{-1}) dr \wedge d\theta \\
 &+ (\partial_\theta \varphi_i^{-1} \partial_\alpha \varphi_2^{-1} - \partial_\alpha \varphi_i^{-1} \partial_\theta \varphi_2^{-1}) d\theta \wedge d\alpha \\
 &+ (\partial_\alpha \varphi_i^{-1} \partial_r \varphi_2^{-1} - \partial_r \varphi_i^{-1} \partial_\alpha \varphi_2^{-1}) d\alpha \wedge dr \\
 &= \text{the same}
 \end{aligned}$$

analogously for $dx_2 \wedge dx_3$ and $dx_1 \wedge dx_3$, i.e.

$$dx_i \wedge dx_j = (\nabla_{(r, \theta, \alpha)} \varphi_i^{-1} \times \nabla_{(r, \theta, \alpha)} \varphi_j^{-1}) \cdot \begin{bmatrix} d\alpha \wedge dr \\ dr \wedge d\theta \\ d\theta \wedge d\alpha \end{bmatrix}$$

(guess, it might differ).

$$\Rightarrow \begin{bmatrix} *dx_3 \\ *dx_2 \\ *dx_1 \end{bmatrix} = * \begin{bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{bmatrix} = \begin{bmatrix} (\nabla' \varphi_2^{-1} \times \nabla' \varphi_3^{-1}) \\ (\nabla' \varphi_3^{-1} \times \nabla' \varphi_1^{-1}) \\ (\nabla' \varphi_1^{-1} \times \nabla' \varphi_2^{-1}) \end{bmatrix} \cdot \begin{bmatrix} d\alpha \wedge dr \\ dr \wedge d\theta \\ d\theta \wedge d\alpha \end{bmatrix}$$

where $\nabla' := \nabla_{(r, \theta, \alpha)}$

↑?
Should be
invertible as well!

$$\begin{aligned}
 dx_1 \wedge dx_2 \wedge dx_3 &= dx_1 \wedge dx_2 \wedge dx_3 (\partial_r, \partial_\theta, \partial_\alpha) dr \wedge d\theta \wedge d\alpha \\
 &= (dx_1(\partial_r) \cdot dx_2(\partial_\theta) \cdot dx_3(\partial_\alpha)) \\
 &\quad - dx_1(\partial_\theta) \cdot dx_2(\partial_r) \cdot dx_3(\partial_\alpha) \\
 &\quad - dx_1(\partial_\alpha) \cdot dx_2(\partial_\theta) \cdot dx_3(\partial_r) \\
 &\quad + dx_1(\partial_\theta) \cdot dx_2(\partial_\alpha) \cdot dx_3(\partial_r) \\
 &\quad + dx_1(\partial_\alpha) \cdot dx_2(\partial_r) \cdot dx_3(\partial_\theta) \\
 &\quad - dx_1(\partial_r) \cdot dx_2(\partial_\theta) \cdot dx_3(\partial_\alpha)) dr \wedge d\theta \wedge d\alpha \\
 &= (\partial_r \varphi_i^{-1} \partial_\theta \varphi_2^{-1} \partial_\alpha \varphi_3^{-1} - \partial_\theta \varphi_i^{-1} \partial_r \varphi_2^{-1} \partial_\alpha \varphi_3^{-1} \\
 &\quad - \partial_r \varphi_i^{-1} \partial_\alpha \varphi_2^{-1} \partial_\theta \varphi_3^{-1} + \partial_\theta \varphi_i^{-1} \partial_\alpha \varphi_2^{-1} \partial_r \varphi_3^{-1} \\
 &\quad + \partial_\alpha \varphi_i^{-1} \partial_r \varphi_2^{-1} \partial_\theta \varphi_3^{-1} - \partial_\alpha \varphi_i^{-1} \partial_\theta \varphi_2^{-1} \partial_r \varphi_3^{-1}) \\
 &\quad dr \wedge d\theta \wedge d\alpha
 \end{aligned}$$

(68)

$$= \det(\varphi^{-1})' dr \wedge d\theta \wedge d\varphi$$

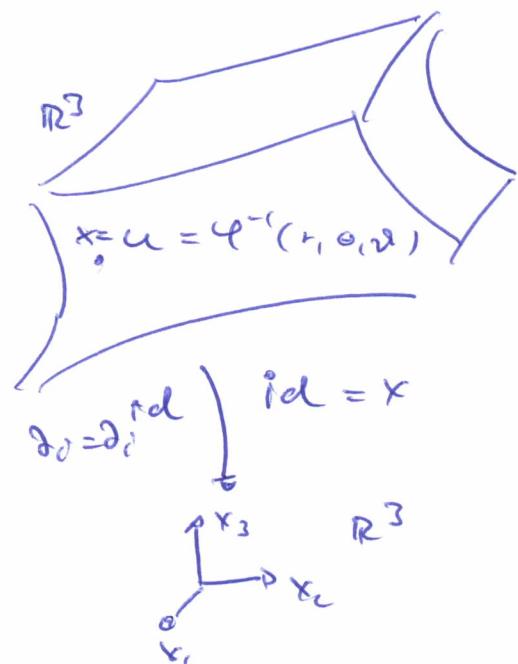
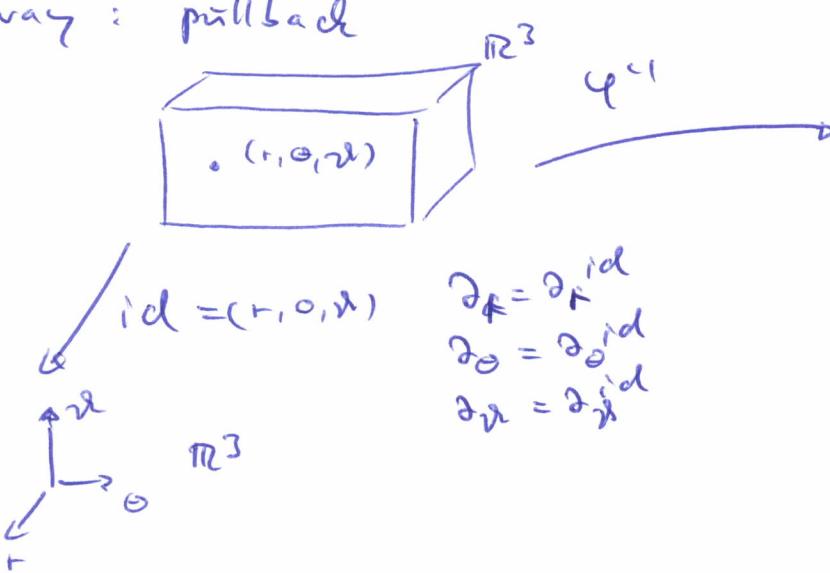
on the other hand

- $dx_1 \wedge dx_2 \wedge dx_3 = (\partial_r \varphi_i^{-1} dr + \partial_\theta \varphi_i^{-1} d\theta + \partial_\varphi \varphi_i^{-1} d\varphi) \wedge (\partial_r \varphi_j^{-1} dr + \partial_\theta \varphi_j^{-1} d\theta + \partial_\varphi \varphi_j^{-1} d\varphi) \wedge (\partial_r \varphi_k^{-1} dr + \partial_\theta \varphi_k^{-1} d\theta + \partial_\varphi \varphi_k^{-1} d\varphi)$

$$= (\partial_r \varphi_i^{-1} \partial_\theta \varphi_j^{-1} \partial_\varphi \varphi_k^{-1} - \partial_r \varphi_i^{-1} \partial_\varphi \varphi_j^{-1} \partial_\theta \varphi_k^{-1} - \partial_\theta \varphi_i^{-1} \partial_r \varphi_j^{-1} \partial_\varphi \varphi_k^{-1} + \partial_\theta \varphi_i^{-1} \partial_\varphi \varphi_j^{-1} \partial_r \varphi_k^{-1} + \partial_\varphi \varphi_i^{-1} \partial_r \varphi_j^{-1} \partial_\theta \varphi_k^{-1} - \partial_\varphi \varphi_i^{-1} \partial_\theta \varphi_j^{-1} \partial_r \varphi_k^{-1}) dr \wedge d\theta \wedge d\varphi$$

= the same

2 way : pullback



- $(\varphi^{-1})^* x = x \circ \varphi^{-1} = \varphi^{-1}$

- $(\varphi^{-1})^* dx_i = dx_u \circ d\varphi^{-1} = d(x_u \circ \varphi^{-1}) = d\varphi_u^{-1}$

$$= d\varphi_u^{-1}(\partial_r) dr + d\varphi_u^{-1}(\partial_\theta) d\theta + d\varphi_u^{-1}(\partial_\varphi) d\varphi$$

$$= \partial_r(\varphi_u^{-1}) dr + \partial_\theta(\varphi_u^{-1}) d\theta + \partial_\varphi(\varphi_u^{-1}) d\varphi$$

$$= \partial_r(\varphi_u^{-1} \circ id) dr + \partial_\theta(\varphi_u^{-1} \circ id) d\theta + \partial_\varphi(\varphi_u^{-1} \circ id) d\varphi$$

$$= \partial_r(\varphi_u^{-1}) dr + \partial_\theta(\varphi_u^{-1}) d\theta + \partial_\varphi(\varphi_u^{-1}) d\varphi$$

$$\Rightarrow \begin{bmatrix} (\varphi^{-1})^* dx_1 \\ (\varphi^{-1})^* dx_2 \\ (\varphi^{-1})^* dx_3 \end{bmatrix} = (\varphi^{-1})' \begin{bmatrix} dr \\ d\theta \\ d\varphi \end{bmatrix}$$

Ok

$$\bullet (\varphi^{-1})^*(dx_1 \wedge dx_2) = \underbrace{(\varphi^{-1})^*dx_1}_{\text{known } 68} \wedge \underbrace{(\varphi^{-1})^*dx_2}_{\text{known}} \quad (69)$$

$$\textcircled{66} \quad \textcircled{67} \quad = \nabla' \varphi_1^{-1} \times \nabla' \varphi_2^{-1} \begin{bmatrix} d\theta \wedge dr \\ dr \wedge d\theta \\ d\theta \wedge d\theta \end{bmatrix}$$

$$\Rightarrow (\varphi^{-1})^* \begin{bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{bmatrix} = \begin{bmatrix} \dots \\ \dots \end{bmatrix} \begin{bmatrix} \dots \\ \dots \end{bmatrix} \quad \textcircled{67}$$

$$\bullet (\varphi^{-1})^*(dx_1 \wedge dx_2 \wedge dx_3) = (\varphi^{-1})^*dx_1 \wedge (\varphi^{-1})^*dx_2 \wedge (\varphi^{-1})^*dx_3$$

$$= (\varphi^{-1})^*d\varphi_1 \text{ known } 68$$

$$= \det(\varphi^{-1})^* dr \wedge d\theta \wedge d\phi \quad \textcircled{68}$$

ON B's $\partial_i := \partial_i^{\text{old}}, \quad \partial_r := \partial_r^{\text{old}}, \dots$

$$\{\partial_i\} \text{ ON B} \Rightarrow \{dx_i\} \text{ ON B}.$$

$$\Rightarrow dx = [e_r \quad r \cos \theta e_\theta \quad r \sin \theta e_\phi] \begin{bmatrix} dr \\ d\theta \\ d\phi \end{bmatrix} = A \begin{bmatrix} dr \\ r \cos \theta d\theta \\ r \sin \theta d\phi \end{bmatrix}$$

with $e_r := \begin{bmatrix} \cos \theta \cos \phi \\ \sin \theta \cos \phi \\ \sin \phi \end{bmatrix}, \quad e_\theta := \begin{bmatrix} -\sin \theta \\ \cos \theta \\ 0 \end{bmatrix}, \quad e_\phi := \begin{bmatrix} -\cos \theta \sin \phi \\ -\sin \theta \sin \phi \\ \cos \phi \end{bmatrix}$

$$\{e_r, e_\theta, e_\phi\} \text{ ON B in } \mathbb{R}^3.$$

$$A := [e_r \ e_\theta \ e_\phi] \text{ ON matrix.} \Rightarrow \tilde{A} = A^* = \begin{bmatrix} {}^t e_r \\ {}^t e_\theta \\ {}^t e_\phi \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} dr \\ r \cos \theta d\theta \\ r \sin \theta d\phi \end{bmatrix} = \tilde{A}^T dx = {}^T A dx = \begin{bmatrix} {}^t e_r \\ {}^t e_\theta \\ {}^t e_\phi \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{bmatrix} \quad \text{ON B}$$

$$\text{Since } \langle \alpha_i dx_i, \alpha_j dx_j \rangle_1 = \underbrace{a_{ui} a_{uj}}_{= \delta_{ui}} \underbrace{\langle dx_u, dx_u \rangle_1}_{= 1} \\ = a_{ui} a_{uj} = a_{iu} a_{uj} = \delta_{ij}$$

where $a_1 = a_r = 1, \quad a_2 = a_\theta = r \cos \theta, \quad a_3 = a_\phi = r$.

$$df = \partial_r f dr + \partial_\theta f d\theta + \partial_\varphi f d\varphi$$

$$= \partial_r f dr + \partial_\theta f d\theta + \partial_\varphi f d\varphi$$

$$= [\partial_r f \quad \frac{\partial_\theta f}{r \cos\varphi} r \cos\theta d\theta \quad \frac{\partial_\varphi f}{r}] \begin{bmatrix} dr \\ r \cos\theta d\theta \\ r d\varphi \end{bmatrix}$$

"grad in
polar
coordinates"

$$\Rightarrow \text{grad}_P f = \begin{bmatrix} \partial_r f \\ \frac{1}{r \cos\varphi} \partial_\theta f \\ \frac{1}{r} \partial_\varphi f \end{bmatrix} = \partial_r f e_r + \frac{1}{r \cos\varphi} \partial_\theta f e_\theta + \frac{1}{r} \partial_\varphi f e_\varphi$$

$$\mathbf{E} \approx E_r e_r + E_\theta e_\theta + E_\varphi e_\varphi$$

$$\mathbf{E} = E_r dr + E_\theta r \cos\theta d\theta + E_\varphi r d\varphi$$

$$d\mathbf{E} = dE_r \wedge dr + d(E_\theta r \cos\theta) \wedge d\theta + d(E_\varphi r) \wedge d\varphi$$

$$= \underbrace{\partial_r E_r dr \wedge dr}_{=0} + \partial_\theta E_r dr \wedge d\theta + \partial_\varphi E_r dr \wedge d\varphi$$

$$+ \partial_r (E_\theta r \cos\theta) dr \wedge d\theta + \partial_\varphi (E_\theta r \cos\theta) dr \wedge d\varphi$$

$$+ \partial_r (E_\varphi r) dr \wedge d\varphi + \partial_\theta (E_\varphi r) dr \wedge d\varphi$$

$$= \partial_\theta E_r dr \wedge d\theta - \partial_\varphi E_r dr \wedge d\varphi$$

$$+ (\partial_r E_\theta r \cos\theta + E_\theta \cos\theta) dr \wedge d\theta$$

$$- \partial_\varphi E_\theta r \cos\theta d\theta \wedge d\varphi + E_\theta \sin\theta d\theta \wedge d\varphi$$

$$+ (\partial_r E_\varphi r + E_\varphi) dr \wedge d\varphi + \partial_\theta E_\varphi dr \wedge d\varphi$$

$$= \underbrace{(\partial_\theta E_r + \partial_r E_\theta r \cos\theta + E_\theta \cos\theta)}_{r \cos\theta} dr \wedge d\theta$$

$$\underbrace{(-\partial_\varphi E_r + \partial_r E_\varphi r + E_\varphi)}_r dr \wedge d\varphi$$

$$+ \underbrace{(E_\theta \sin\theta + r \partial_\theta E_\varphi)}_{r^2 \cos\theta} r^2 \cos\theta d\theta \wedge d\varphi$$

$$= \left(\frac{\partial_\theta E_r}{r \cos\theta} + \partial_r E_\theta + \frac{E_\theta}{r} \right) \xrightarrow{+ r d\theta} = r \times d\theta$$

$$+ \left(-\frac{\partial_\theta E_r}{r} + \partial_r E_\theta + \frac{E_\theta}{r} \right) \xrightarrow{+ (-r \cos\theta) d\theta} = -r \cos\theta \times d\theta$$

$$+ \left(\cancel{\frac{\tan\theta}{r} \frac{E_\theta}{r}} + \frac{\partial_\theta E_\theta}{r \cos\theta} \right) \times dr$$

$$\Rightarrow \text{curl}_p E \approx \left(\frac{\tan\theta}{r} E_\theta + \frac{\partial_\theta E_\theta}{r \cos\theta} \right) \mathbf{e}_r$$

"curl in
polar
coordinates"

$$+ \left(\frac{\partial_\theta E_r}{r} - \partial_r E_\theta - \frac{E_\theta}{r} \right) \mathbf{e}_\theta$$

$$+ \left(\frac{\partial_\theta E_r}{r \cos\theta} + \partial_r E_\theta + \frac{E_\theta}{r} \right) \mathbf{e}_\phi$$

4. Measurability

4.1 Integration on manifolds

Def $\omega \in A^q(X)$ "measurable", if

~~if~~ $\forall u \in X \exists (U, \varphi) \forall I \in \mathbb{I}^{q, \mu} \omega_I \circ \varphi^{-1}$ Lebesgue-measurable in $\varphi(U) \subset \mathbb{R}^n$,
 if $\omega = \sum_I \omega_I d\lambda_I$
 $\Rightarrow \omega_I \circ \varphi^{-1}$ measurable holds for all charts.

Def $U \text{ open } \subset \mathbb{R}^n$, $\omega \in A^n(U)$, $\omega > 0$ measurable

Then

$$\omega = f dx_1 \wedge \dots \wedge dx_n, \quad f: U \rightarrow [0, \infty) \text{ measurable}$$

and we define

$$\int_U \omega := \int_U f d\lambda \in [0, \infty], \quad \lambda \text{ Lebesgue measure in } \mathbb{R}^n$$

Bem $f = f_+ - f_-$



$$f_+ := \max\{f, 0\} \geq 0$$

$$f_- := \max\{-f, 0\} \geq 0$$

$$\int_U \omega := \int_U f_+ d\lambda - \int_U f_- d\lambda \in [-\infty, \infty],$$

$$\text{if } \int_U f_+ d\lambda < \infty \text{ or } \int_U f_- d\lambda < \infty.$$

$$\text{If } \int_U \omega \in \mathbb{R} \text{ we write } \omega \in L^{1, \mathbb{N}}(U).$$

lem $U, V \subset \mathbb{R}^n$, $\# \in \text{C}^0(U, V)$ orientation preserving diffeo. ②

Then for all measurable $w \in A^n(X)$

$$\int_U \#^* w = \int_{V=\#(U)} w , \text{ if one of the integrals exists.}$$

Proof ~~$\#^* w = F^* w(\partial_1, \dots, \partial_n) dx_1 \wedge \dots \wedge dx_n$~~

$$w = f dx_1 \wedge \dots \wedge dx_n = \cancel{w(dF_1, \dots, dF_n)} dx_1 \wedge \dots \wedge dx_n$$

$$\begin{aligned} F^* w &= F^*(f dx_1 \wedge \dots \wedge dx_n) \\ &= \cancel{f \circ F} \cancel{F^* dx_1 \wedge \dots \wedge F^* dx_n} \\ &= \cancel{\det F} = \end{aligned}$$

$$\begin{aligned} &= f \circ F (\det F)^* (dx_1 \wedge \dots \wedge dx_n) \\ &= f \circ F \det(F) dx_1 \wedge \dots \wedge dx_n \\ &= \det F' f \circ F dx_1 \wedge \dots \wedge dx_n \end{aligned}$$

$$F = \tilde{F}$$

$$, \det F' > 0 !$$

$$\Rightarrow \int_U F^* w = \int_U f \circ F \det F' d\lambda$$

$$= \int_V f d\lambda = \int_V w$$



From now on X oriented manifold.

Def and lem (U, φ) orientation preserving - We define for measurable $w \in A^n(U)$, $w \geq 0$

$$\int_U w := \int_{\varphi(U)} (\varphi^{-1})^* w \in [0, \infty]$$

This is independent of the chart φ .

Proof ~~$\int_U (\varphi^{-1})^* w = \int_{\varphi(U)} (\varphi^{-1} \circ \varphi \circ \varphi^{-1})^* w$~~

 ~~$\varphi(U) \overset{\varphi}{\longrightarrow} \varphi(U) \overset{\cancel{(\varphi^{-1} \circ \varphi \circ \varphi^{-1})^*}}{=} (\varphi^{-1})^* (\varphi^{-1} \circ \varphi)^*$~~

lem \mathbb{R}^n

$$\int_{\varphi(U)} (\varphi^{-1})^* w = \int_{\varphi(U)} (\varphi^{-1} \circ \varphi \circ \varphi^{-1})^* w = \int_{\varphi(U)} (\varphi \circ \varphi^{-1})^* (\varphi^{-1})^* w$$

$$= \int_{\varphi(U)} (\varphi^{-1})^* w \stackrel{\text{Def.}}{=} \int_U w$$



Def A $\overset{\text{countable}}{\cup}$ positive atlas, i.e. all charts are positive, (3)

$\Omega_e \Rightarrow X = \bigcup_{e \in \mathbb{N}} U_e$, (U_e, φ_e) chart orientation preserving

Let $\{\psi_e\}_{e \in \mathbb{N}}$ be a $(U_e)_{e \in \mathbb{N}}$ subordined locally finite C^∞ -partition of unity, i.e. $\psi_e \in C^\infty(U_e)$ and

$$\sum_{e \in \mathbb{N}} \psi_e = 1.$$

For measurable $w \in A^*(X)$, what we define

$$\int_X w = \sum_{e \in \mathbb{N}} \int_{\varphi_e(U_e)} (\varphi_e^{-1})^*(\psi_e \cdot w) \in [0, \infty]$$

Independent of (φ_e) and (ψ_e) !

Rem $\int_X w$ for measurable N -forms as in Rem on ①.
 $w \in L^{1, \infty}(X)$.

Rem Choosing a special positive N -form $\hat{\omega}$, we can define integral for $f: X \rightarrow \mathbb{R}$ and X becomes a measure space.

$S \subset X$ measurable $\Leftrightarrow \chi_S \hat{\omega}$ measurable

$$\mu(S) := \int_X \chi_S \hat{\omega} \quad \text{if } S \text{ measurable}$$

$$\int_X f d\mu := \int_X f \cdot \hat{\omega} \quad \text{if } f \text{ measurable and } f \geq 0, \text{ or...}$$

Then the property "measurable" is independent of $\hat{\omega}$.
 $\mu(S)$ and $\int_X f d\mu$ depend of course on the choice of $\hat{\omega}$.

The X, Y orientable oriented manifolds and
 $F \in C^\infty(X, Y)$ orientation preserving diffeo.
 (i.e. $\hat{\omega} \in C^{\infty, N}(Y), \hat{\omega} > 0 \Rightarrow F^*\hat{\omega} \in C^{\infty, N}(X), F^*\hat{\omega} > 0$)
 Then for all measurable $\omega \in A^N(Y)$

$$\int_X F^*\omega = \int_Y \omega ,$$

if one of the integrals exist.

Proof only chart domains.

$$\begin{aligned} \int_U F^*\omega &= \int_{\varphi(U)} (\varphi^{-1})^* F^*\omega = \int_{\varphi(V)} (F \circ \varphi^{-1})^* \omega \\ &= \int_{(\varphi \circ F^{-1})^{-1}(V)} \omega \\ &= \int_V \omega \end{aligned}$$

since $\varphi \circ F^{-1}: V \rightarrow \mathbb{R}^n$ is a chart for Y .

Reeu: $\int_U \omega = ?$

$$\begin{aligned} \textcircled{1} \quad \int_U \omega &= \int_{\varphi(U)} (\varphi^{-1})^* \omega = \int_{\varphi(U)} (\varphi^{-1})^* \omega(\partial_1, \dots, \partial_n) dx_1 \dots dx_n \\ &= \int_{\varphi(U)} (\varphi^{-1})^* \omega(\partial_1, \dots, \partial_n) d\lambda \end{aligned}$$

$$(\varphi^{-1})^* \omega(\partial_1, \dots, \partial_n) = \omega(d\varphi^{-1}(\partial_1), \dots, d\varphi^{-1}(\partial_n))$$

$$\begin{aligned} d\varphi^{-1}(\partial_i) &= \cancel{d\varphi^{-1}(\partial_i)(\varphi_u)} \partial_u^\varphi \\ &= \underbrace{\partial_i \circ (\varphi_u \circ \varphi^{-1})}_{= x_u} \partial_u^\varphi = \partial_{x_u} \partial_u^\varphi = \partial_i^\varphi \end{aligned}$$

$$\Rightarrow \int_U \omega = \int_{\varphi(U)} \omega(\partial_{x_1}^\varphi, \dots, \partial_{x_n}^\varphi) \circ \varphi^{-1} d\lambda$$

$$\textcircled{2} \quad \omega = \omega(\partial_1^q, \dots, \partial_N^q) d\varphi_1 \dots d\varphi_N$$

(5)

$$\Rightarrow \int_U \omega = \int_{\varphi(U)} \omega(\partial_1^q, \dots, \partial_N^q) \underbrace{\circ (\varphi^{-1})^*}_{\varphi^{-1}} d\varphi_1 \dots d\varphi_N$$

$$= (\varphi^{-1})^* d\varphi_1 \dots \wedge (\varphi^{-1})^* d\varphi_N$$

$$= d\varphi_1 d\varphi_2 \wedge \dots \wedge d\varphi_N d\varphi_N$$

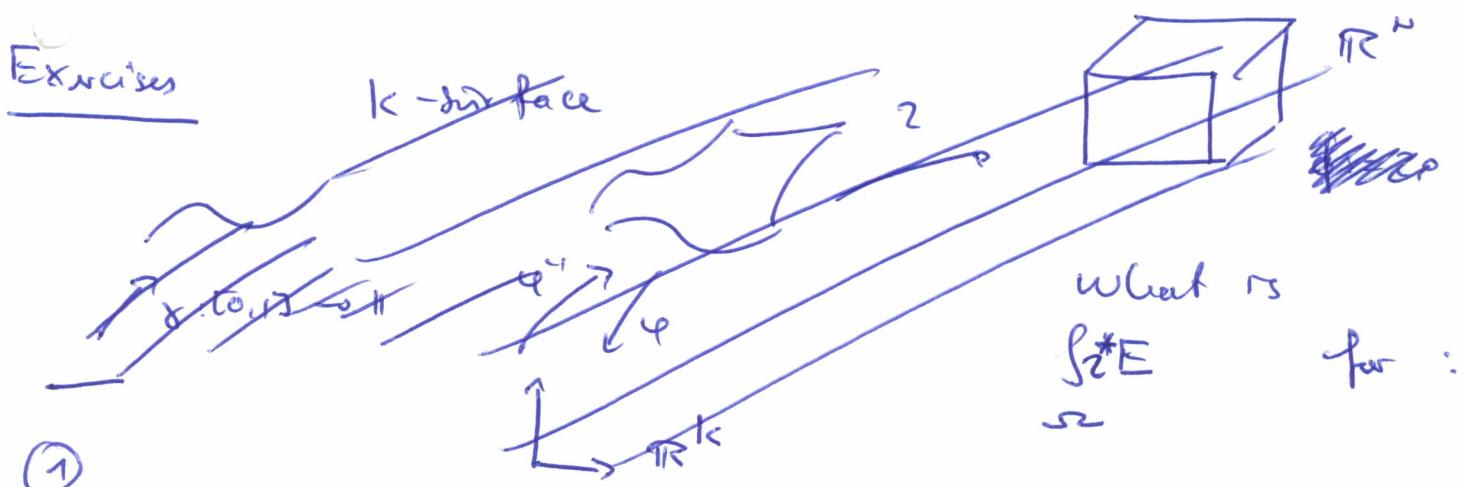
$$= d\underbrace{\varphi_1 \circ \varphi^{-1}}_{=x_1} \wedge \dots \wedge d\underbrace{\varphi_N \circ \varphi^{-1}}_{=x_N}$$

$$= dx_1 \wedge \dots \wedge dx_N$$

$$= \int_{\varphi(U)} \omega(\partial_1^q, \dots, \partial_N^q) \circ \varphi^{-1} d\lambda .$$

Exercises

k -surface



\textcircled{1}

$$\bullet \quad \begin{array}{ccc} \varphi & \rightarrow & \mathbb{R}^3 \\ \overline{\mathbb{R}} & \rightarrow & \mathbb{R}^3 \end{array} \quad E = E_u dx_u$$

$\varphi: [0,1] \rightarrow \mathbb{R}^2$

$$\bullet \quad \begin{array}{ccc} \varphi & \rightarrow & \mathbb{R}^3 \\ \mathbb{R} & \rightarrow & \mathbb{R}^3 \end{array} \quad E = E_u \times dx_u$$

$$\bullet \quad \begin{array}{ccc} \varphi & \rightarrow & \mathbb{R}^3 \\ \mathbb{R}^{-1}, [0,1]^2 & \rightarrow & \mathbb{R}^2 \end{array}$$

$\begin{array}{ccc} \uparrow & & \mathbb{R}^2 \\ \text{[] } & \rightarrow & \mathbb{R}^3 \\ \uparrow & & \mathbb{R}^3 \end{array} \quad E = E dx_1 dx_2 dx_3$

$$\bullet \quad \begin{array}{ccc} \varphi & \rightarrow & \mathbb{R}^3 \\ \mathbb{R}^{-1}, [0,1]^3 & \rightarrow & \mathbb{R}^2 \end{array}$$

Remark $\phi := z \circ \varphi^{-1}$ classical parametrizations

(6)

$$\begin{aligned}
 \text{note } z^* dx_u (\partial_m^q)^{0^{q-1}} &= dx_m dz (\partial_m^q) \circ \phi^{-1} \\
 &= d(x_{u+2}) (\partial_m^q) \circ \phi^{-1} \\
 &\stackrel{?}{=} \partial_m (x_{u+2})^{0^{q-1}} = \partial_m (x_{u+2} \phi^{-1}) \circ \phi \circ \phi^{-1} \\
 &= \partial_m (\phi_u \circ \phi^{-1}) = \partial_m \phi_u
 \end{aligned}$$

classical tangential vector on Σ !

Then

$$\begin{aligned}
 q=1: \quad \int_{\Sigma} z^* E &= \int_{\Sigma} E_{u+2} z^* dx_u = \int_{[0,1]} E_u \cancel{\phi} z^* dx_u (\partial_1^q) dx_1 \\
 &= \cancel{\int_{[0,1]} E_u \cancel{\phi} \cancel{\partial_1} (\partial_u^q)} = \int_{[0,1]} E_u \phi \underbrace{\partial_1 \phi_u}_{=\phi'_u} dx_1 \\
 &= \int_0^1 \langle E_u \phi, \phi' \rangle dx_1 \\
 &= \int_0^1 \langle E_u \phi, t \circ \phi \rangle |\phi'| dt, \quad t \text{ classical} \\
 &\quad \text{tangential} \\
 &\quad \text{unit vector} \\
 &\quad \text{(velocity vector)} \\
 \text{path integral} \\
 \text{over the} \\
 \text{tangential} \\
 \text{component} \\
 \text{of } E &= \int_{\Sigma} \langle E, t \rangle d\omega_1
 \end{aligned}$$

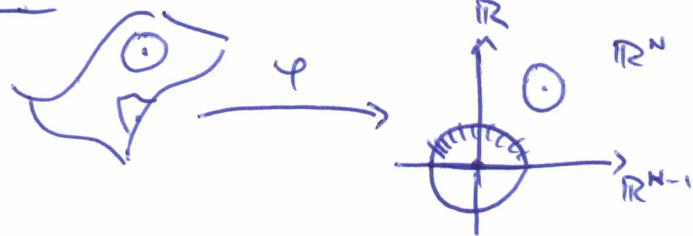
$$\begin{aligned}
 q=2: \quad \int_{\Sigma} z^* E &= \int_{\Sigma} E_{u+2} z^* \underbrace{dx_u}_{= dx_e \wedge dx_\epsilon} = \int_{\Sigma} E_{u+2} z^* dx_e \wedge z^* dx_\epsilon \\
 &= \int_{[0,1]^2} E_{u+2} \phi^{0^{q-1}} z^* dx_e \wedge z^* dx_\epsilon (\partial_1^q, \partial_2^q) \circ \phi^{-1} dx_2 \\
 &= \int_{[0,1]^2} E_u \phi \left(z^* dx_e (\partial_1^q) z^* dx_\epsilon (\partial_2^q) \right. \\
 &\quad \left. - z^* dx_e (\partial_2^q) z^* dx_\epsilon (\partial_1^q) \right) \circ \phi^{-1} dx_2 \\
 &= \int_{[0,1]^2} E_u \phi (\partial_1 \phi_e \partial_2 \phi_\epsilon - \partial_2 \phi_e \partial_1 \phi_\epsilon) dx_2 \\
 &= \int_{[0,1]^2} \langle E_u \phi, \partial_1 \phi \times \partial_2 \phi \rangle dx_2 \\
 &= \int_{[0,1]^2} \langle E_u \phi, u \circ \phi \rangle |\partial_1 \phi \times \partial_2 \phi| dx_2, \quad u \text{ classical normal} \\
 &\quad \text{unit vector (exterior)} \\
 \text{surface integral} \\
 \text{over the} \\
 \text{normal} \\
 \text{component} \\
 \text{of } E &= \int_{\Sigma} \langle E, u \rangle d\omega_2
 \end{aligned}$$

(7)

$$\begin{aligned}
 q=3 : \int_{\Sigma} z^* E &= \int_{\Sigma} E \circ z z^* dx_1 \wedge dx_2 \wedge dx_3 \\
 &= \int_{\Sigma} E \circ z z^* dx_1 \wedge z^* dx_2 \wedge z^* dx_3 \\
 &= \int_{[0,1]^3} E \circ z \circ \phi^{-1}(z^* dx_1 \wedge z^* dx_2 \wedge z^* dx_3) (\partial_1^q, \partial_2^q, \partial_3^q) \det d\lambda_3 \\
 &= \int_{[0,1]^3} E \circ \phi (z^* dx_1(\partial_1^q), z^* dx_2(\partial_2^q), z^* dx_3(\partial_3^q) \\
 &\quad - z^* dx_1(\partial_1^q) z^* dx_2(\partial_3^q), z^* dx_3(\partial_2^q) \\
 &\quad + z^* dx_1(\partial_3^q) z^* dx_2(\partial_1^q), z^* dx_3(\partial_2^q) \\
 &\quad - z^* dx_1(\partial_3^q) z^* dx_2(\partial_2^q), z^* dx_3(\partial_1^q) \\
 &\quad + z^* dx_1(\partial_2^q) z^* dx_2(\partial_3^q), z^* dx_3(\partial_1^q) \\
 &\quad - z^* dx_1(\partial_2^q) z^* dx_2(\partial_1^q), z^* dx_3(\partial_3^q)) \det d\lambda_3 \\
 &= \int_{[0,1]^3} E \circ \phi (\partial_1 \phi_1 \partial_2 \phi_2 \partial_3 \phi_3 - \partial_1 \phi_1 \partial_3 \phi_2 \partial_2 \phi_3 \\
 &\quad + \partial_3 \phi_1 \partial_1 \phi_2 \partial_2 \phi_3 - \partial_3 \phi_1 \partial_2 \phi_2 \partial_1 \phi_3 \\
 &\quad + \partial_2 \phi_1 \partial_3 \phi_2 \partial_1 \phi_3 - \partial_2 \phi_1 \partial_1 \phi_2 \partial_3 \phi_3) \det d\lambda_3 \\
 &= \int_{[0,1]^3} E \circ \phi \underbrace{\det \phi'}_{>0} \det d\lambda_3 \quad \text{since } \phi \text{ is orientation preserving} \\
 &\xrightarrow{\text{classical transformation}} = \int_{\Sigma} E d\lambda_3 \quad (x = \phi(\gamma) \Rightarrow dx = |\det \phi'(\gamma)| d\gamma)
 \end{aligned}$$

4.2 Stokes' Theorem

"manifolds with boundary"



Def "modification of chart"

A N -dimensional chart φ for X is a homeomorphism

$$\varphi: U^{\text{open}} \subset X \rightarrow \varphi(U) \xrightarrow{\sim} \mathbb{R}_+^N, \quad \varphi(U)^{\text{open}} \subset \mathbb{R}^N,$$

where $\mathbb{R}_+^N := \{x \in \mathbb{R}^N : x_N \geq 0\}$.

All the other definitions and assumptions from the previous chapters remain unchanged. A collection of compatible charts is called an atlas, now for the manifold with boundary.

The charts decompose into two classes:

- (i) "inner charts": $\Leftrightarrow \varphi(U) \subset \mathbb{R}_+^N$.
- (ii) "boundary charts": $\Leftrightarrow \varphi(U) \cap \mathbb{R}_+^0 \neq \emptyset$.

~~Atlas~~

Def and lem $u \in X$ is called "inner point" of the manifold with boundary X , if there exists an inner chart $\varphi|_U$, otherwise boundary point. ~~The boundary points~~ We collect the boundary points in the set ∂X , the boundary of X . For all charts φ and all boundary points are $\varphi(u) \in \mathbb{R}_+^0$.

~~$X \setminus \partial X$ is then a manifold (without boundary!) of dimension $\dim(X \setminus \partial X) = \dim X - 1$~~

~~atlas too~~ Then φ must be a boundary chart.

$X \setminus \partial X$ and ∂X equipped with the atlases

$$\begin{aligned} \mathcal{A} &:= \{ \varphi \in \mathcal{A} : \varphi \text{ inner chart} \}, \\ \mathcal{B} &:= \{ \varphi|_{U \cap \partial X} : \varphi \text{ boundary chart} \} \end{aligned}$$

are manifolds (without boundary!) of dimensions

$$\dim(X \setminus \partial X) = \dim X = N$$

$$\dim(\partial X) = \dim X - 1 = N - 1$$

If X is orientable, so are $X/\partial X$ and ∂X .

(9)

Convention If \mathcal{A}_+ positive atlases for X . Then $X/\partial X$ and ∂X may be oriented, such that

$\mathcal{A}^+ \cap \mathcal{A}_+$

$$\mathcal{D}\mathcal{A}_+ = \{\Psi|_{U_n \cap \partial X} : \Psi \in \mathcal{A}_+\}$$

are positive atlases.

Rem The notion "oriented" etc. assumes that there is a tangential space $T_u(X)$. These may be defined by derivations as before. Then even for $u \in \partial X$

$T_u(X)$

is a N -dimensional vector space and $\{\partial_1, \dots, \partial_N\}$ is a (chart) basis. But now we have to define ∂_N as right hand side derivative

$$\partial_N(\varphi) = \lim_{t \rightarrow 0} \frac{1}{t} (\varphi \circ \tilde{\varphi}^{-1}(t e^{\tilde{\varphi}}) - \varphi \circ \underline{\tilde{\varphi}^{-1}(x)}), \quad .$$

Rem Everything will be defined as before. Especially we have

$$A^q(X), \quad \text{Cos}^q(X).$$

LEM $u \in \partial X$ and $\varphi: U \rightarrow \varphi(U) \cap \overline{\mathbb{R}^N_+}$ a boundary chart. Then $\phi: U \cap \partial X \rightarrow \varphi(U) \cap \overline{\mathbb{R}^N_+} = \varphi(U) \cap (\mathbb{R}^{N-1} \times \{0\})$

$$u \mapsto \begin{cases} \varphi'(u) & \text{open} \\ \varphi(u) = x = (x'_1, 0) & \cong x'_1 \end{cases},$$

where $\varphi' = (\varphi_1, \dots, \varphi_{N-1})$, is a chart for $u \in \partial X$ of the manifold ∂X . Let $\{\partial_u^\varphi\}_{u=1,\dots,N} \subset T_u(X)$ and $\{\partial_u^\phi\}_{u=1,\dots,N-1} \subset T_u(\partial X)$.

Then :

$$\begin{aligned} \partial_u^\varphi(\varphi_u) &= \delta_{u,u} & u_{\text{open}} &= 1, \dots, N \\ \partial_u^\phi(\phi_u) &= \delta_{u,u} & u, u &= 1, \dots, N-1 \end{aligned} \quad (1)$$

Now for the natural embedding $z: \partial X \rightarrow X$; we have

$$dz(\partial_u^\varphi) = \partial_u^\varphi, \quad u=1, \dots, N-1$$

and

$$z^* d\varphi_u = \begin{cases} d\varphi_u & , u=1, \dots, N-1 \\ 0 & , u=N \end{cases}.$$

Proof (1) \vee .

$$\begin{aligned} (2): \quad dz(\partial_u^\varphi) &= dz(\partial_u^\varphi)(\varphi_u) \partial_u^\varphi \\ &= \sum_{m=1}^N \partial_m^\varphi(\varphi_u \circ z) \partial_m^\varphi \sum_{m=1}^N \partial_m(\varphi_u \circ z \circ \varphi^{-1}) \partial_m^\varphi \end{aligned}$$

$$\Gamma \varphi_u \circ z \circ \varphi^{-1}(x') = \varphi_u \circ z \circ \underbrace{\varphi^{-1}(x'_{1,0})}_{\mathbb{R}^{N-1}} = \underbrace{\text{id}_u}_{=u}^{\mathbb{R}^N}(x'_{1,0})$$

$$\Rightarrow \varphi_u \circ z \circ \varphi^{-1}: \varphi'(u) \subset \mathbb{R}^{N-1} \rightarrow \mathbb{R}^N, \quad u=1, \dots, N-1$$

$$x' \mapsto x'_u$$

$$\varphi_u \circ z \circ \varphi^{-1}: \varphi'(u) \subset \mathbb{R}^{N-1} \rightarrow \mathbb{R}^N$$

$$x' \mapsto \underbrace{0}_{u}$$

$$= \sum_{m=1}^{N-1} \delta_{um} \partial_m^\varphi = \partial_u^\varphi, \quad u=1, \dots, N-1.$$

(11)

$$z^* d\varphi_u = dz^* \varphi_u = d\varphi_{u+2}$$

$$= \begin{cases} d\varphi_u & , u=1, \dots, N-1 \\ d0=0 & , u=N \end{cases}$$

Since $d\varphi_{u+2} = \begin{cases} d\varphi_u & , u=1, \dots, N-1 \\ 0 & , u=N \end{cases}$

■

Theorem "Stokes' Theorem"

$$\int_X dw = \int_{\partial X} w \quad \forall w \in \overset{\circ}{C}{}^{\infty, N-1}(X) \quad *$$

~~smooth~~

More precisely:

Assumptions:

- (i) X oriented manifold of dimension N with boundary ∂X
- (ii) $z: \partial X \rightarrow X$ embedding
- (iii) $w \in \overset{\circ}{C}{}^{\infty, N-1}(X) = \{ \omega \in C^{\infty}(X) : \text{supp } \omega \subset \text{supp } z \}$, where
 $\overset{\circ}{C}{}^{\infty}(X) := \{ \gamma \in C^{\infty}(X) : \text{supp } \gamma \text{ compact} \}$

Then

$$(iv) \text{ Then } z^* w \in \overset{\circ}{C}{}^{\infty, N-1}(\partial X) \quad \text{and}$$

$$\int_X dw = \int_{\partial X} z^* w .$$

Proof

$$\textcircled{1} \quad X = \mathbb{R}^N. \quad \varphi = \text{id}, \quad d\varphi_u = dx_u$$

$$\varphi = \varphi|_{\mathbb{R}^{N-1}}, \quad d\varphi_u = d\tilde{x}_u$$

$$\omega = \sum_{u=1}^N w_u * dx_u$$

$$dw = \sum_{u,u=1}^N \partial_{uu} w_u dx_u \wedge dx_u \quad (= dw_u \wedge dx_u)$$

$$= \underbrace{\sum_{u=1}^N \partial_u w_u dx_1 \wedge \dots \wedge dx_N}_{= \tilde{\omega}}$$

$$z^+ \omega = \sum_{u=1}^N w_{u,0} z^+ * dx_u$$

$$= \sum_{u=1}^N w_{u,0} z^+ dx_1 \wedge \dots \wedge dx_{u-1} \wedge dx_u + \dots + dx_N$$

$$= \sum_{u=1}^N w_{u,0} z^+ \underbrace{dx_1 \wedge \dots \wedge dx_{u-1}}_{= d\tilde{x}_1} \wedge \underbrace{z^+ dx_u}_{= d\tilde{x}_u} \wedge \underbrace{dx_{u+1} \wedge \dots \wedge dx_N}_{= 0}$$

$$= w_p \circ z^+ d\tilde{x}_1 \wedge \dots \wedge d\tilde{x}_{N-1}, \quad \text{because of the previous lemma.}$$

For $\gamma: \mathbb{R}^{N-1} \times \{0\} \rightarrow \mathbb{R}^N$
 $(x'_1, 0) \mapsto (x'_1, 0)$

$$\Rightarrow z^+ d\text{id}_{\mathbb{R}^N} = d z^+ d\text{id}_{\mathbb{R}^N} = d \text{id}_{\mathbb{R}^N} \circ z^+$$

$$= \begin{cases} d\tilde{x}_u & , u=1, \dots, N-1 \\ 0 & , u=N \end{cases}$$

since $\text{id}_{\mathbb{R}^N} \circ z^+(x'_1, 0) = x'_u, \quad u=1, \dots, N-1$

$$\text{id}_N \circ z^+(x'_1, 0) = 0$$

~~$$z^+ \omega(x) = (-1)^{N+1} w_N(x)$$~~

Now

$$\int_X dw = \int_X \sum_{u=1}^N \partial_{uu} w_u dx_1 \wedge \dots \wedge dx_N$$

$$= \int_{\mathbb{R}^N} \sum_{u=1}^N \partial_{uu} w_u dx_N \quad (\text{Def.})$$

~~Supp is compact
in \mathbb{R}^N \Rightarrow $\int_0^\infty \int_{\text{Supp } \omega} \partial_{uu} w_u dx_N dx_N$ exists~~

Supp ω compact
Fubini/Tonelli \Downarrow

(13)

$$= \sum_{u=1}^N \int_{\mathbb{R}^N} \partial_u w_u d\lambda_N$$

$$= \sum_{u=1}^{N-1} \int_{\mathbb{R}^{N-1}} \underbrace{\int_{\mathbb{R}} \partial_u w_u d\lambda_1(x_1) d\lambda_{N-1}(x_1, \dots, x_{u-1}, x_{u+1}, x_N)}_{= w_u \Big|_{x_u=-\infty}^{x_u=+\infty}} = 0 \text{ since } \text{supp } w_u \text{ compact}$$

$$+ \int_{\mathbb{R}^{N-1}} \underbrace{\int_{-\infty}^{\circ} \partial_N w_N d\lambda_1(x_N) d\lambda_{N-1}(x_1, \dots, x_{N-1})}_{= w_N \Big|_{x_N=-\infty}^{x_N=\circ}} = + w_N(\cdot, \circ)$$

$$= + \int_{\mathbb{R}^{N-1}} w_N(\cdot, \circ) d\lambda_{N-1}$$

On the other hand

$$\begin{aligned} \int_{\partial X} z^\ast \omega &= \int_{\partial X = \mathbb{R}^N} w_N \circ \varphi \, d\tilde{x}_1 \dots d\tilde{x}_{N-1} \\ &= \int_{\mathbb{R}^{N-1}} w_N \circ \varphi \, d\lambda_{N-1} \end{aligned}$$

$$\Gamma \# \varphi'(x') = (x'_{1,0}) \Rightarrow z^\ast \varphi'(x') = (x'_{1,0}) \quad \blacksquare$$

$$= \int_{\mathbb{R}^{N-1}} w_N(x'_1, \circ) d\lambda_{N-1}(x')$$

$$= \int_{\mathbb{R}^{N-1}} w_N(\cdot, \circ) d\lambda_{N-1}$$

$$= \int_X d\omega \quad \square$$

② $x = u$ chart domain and $\text{supp } \omega$ (compact) $\subset U$

(14)

Then

$$\int_X d\omega = \int_U d\omega = \int_{\varphi(u)} (\varphi^{-1})^* d\omega = \int_{\varphi(u)} d(\varphi^{-1})^* \omega$$

$$= \int_{\mathbb{R}^N_+} d(\varphi^{-1})^* \omega \stackrel{(6)}{=} \int_{\partial \mathbb{R}^N_+} z_{\mathbb{R}^N_+}^* (\varphi^{-1})^* \omega$$

~~$$= \int_{\mathbb{R}^N_+} (\varphi^{-1} \circ z_{\mathbb{R}^N_+})^* \omega$$~~

$$= \int_{\mathbb{R}^N_+} z_{\mathbb{R}^N_+}^* (\varphi^{-1})^* (z_{\partial u}^{-1})^* z_{\partial u}^* \omega$$

$$\begin{array}{c} \Gamma \\ z_{\partial u}: \partial u \rightarrow u \\ = z_x: \partial u \rightarrow \end{array}$$

$$= \int_{\mathbb{R}^N_+} (z_{\partial u}^{-1} \circ \varphi^{-1} \circ z_{\mathbb{R}^N_+})^* z_{\partial u}^* \omega$$

$$\begin{aligned} &= \int_{\mathbb{R}^N_+} ((z_{\mathbb{R}^N_+}^{-1} \circ \varphi \circ z_{\partial u})^{-1})^* z_{\partial u}^* \omega \\ &= z_{\mathbb{R}^N_+}^{-1} \circ \varphi \circ z(\partial u) \end{aligned}$$

transformation
formula

$$= \int_{\partial u} z_{\partial u}^* \omega = \int_{\partial X} z_{\partial X}^* \omega.$$

③ general case: $\mathcal{U} = (U_e, \varphi_e)_e$ atlas for X .

(15)

$(\varphi_e)_e$ a $\bigcup_e U_e$ subordinated \mathcal{C}^∞ -partition of unity. Then

$$\begin{aligned}
 \int_X d\omega &= \int_X d\left(\sum_e \varphi_e \omega\right) = \sum_e \int_X d\varphi_e \omega \quad , \text{ supp } \varphi_e \subset U_e \\
 &= \sum_e \int_{U_e} d\varphi_e \omega \stackrel{(2)}{=} \sum_e \int_{\partial U_e} 2^+ \frac{\partial}{\partial \varphi_e} \varphi_e \omega \\
 &= \cancel{\sum_e \int_{\partial X} \varphi_e \circ 2^+ \frac{\partial}{\partial \varphi_e} \omega} = \sum_e \int_{\partial X} \varphi_e \circ 2^+ \frac{\partial}{\partial \varphi_e} \omega \\
 &= \sum_e \int_{\partial X} \frac{\partial}{\partial \varphi_e} 2^+ \frac{\partial}{\partial \varphi_e} \varphi_e \omega = \sum_e \int_{\partial X} \varphi_e \circ 2^+ \frac{\partial}{\partial \varphi_e} \omega \\
 &= \int_{\partial X} \underbrace{\sum_e \varphi_e \circ 2^+ \frac{\partial}{\partial \varphi_e}}_{=1} \cdot 2^+ \frac{\partial}{\partial \varphi_e} \omega = \int_{\partial X} 2^+ \frac{\partial}{\partial \varphi_e} \omega
 \end{aligned}$$



Exercises $\Sigma \subset \mathbb{R}^3$

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$$\text{Stokes: } \oint_{\partial\Sigma} d\omega = \int_{\Sigma} \star \omega \quad \textcircled{*}$$

$$q=1: \omega := \phi_1 E, \quad E \text{ vector field}$$

$$q=2: \omega = \phi_2 E, \quad " \quad " \quad "$$

What is $\textcircled{*}$?

$$1) \quad d\omega = \phi_2 \phi_1^{-1} d\phi_1 E = \phi_2 \operatorname{curl} E, \quad \Sigma \text{ surface}$$

$$\oint_{\partial\Sigma} d\omega = \int_{\Sigma} d_2 \operatorname{curl} E = \int_{\Sigma} \langle \operatorname{curl} E, n \rangle d\sigma_2$$

$$= \int_{\partial\Sigma} \star \omega = \int_{\partial\Sigma} \star d_1 E = \int_{\partial\Sigma} \langle E, t \rangle d\sigma_1$$

"classical
Theorem of
Stokes"

$$2) \quad d\omega = \phi_3 \phi_2^{-1} d\phi_2 E = \phi_3 \operatorname{div} E, \quad \Sigma \text{ volume}$$

$$\int_{\Sigma} d\omega = \int_{\Sigma} d_3 \operatorname{div} E = \int_{\Sigma} \operatorname{div} E d\lambda_3$$

$$= \int_{\partial\Sigma} \star \omega = \int_{\partial\Sigma} \star d_2 E = \int_{\partial\Sigma} \langle E, u \rangle d\sigma_2$$

"classical
Theorem of
Gauß"

4.3 Weak Derivatives

- ① partial integration , tangential trace, normal trace !
- ② $L^{2,q}(\Omega)$!
- ③ $D^q(\Omega)$, $\Delta^q(\Omega)$, $\overset{\circ}{D}{}^q(\Omega)$, $\overset{\circ}{\Delta}{}^q(\Omega)$!
- ④ M selfadjoint ?
- ⑤ Maxwell's equations ?, acoustics ?, Helmholtz ?

Stokes revisited

$$\textcircled{1} \quad X = \mathbb{R}_-^N = \{x \in \mathbb{R}^N : x_1 < 0\}.$$

$$\partial X = \mathbb{R}_0^N = \{x \in \mathbb{R}^N : x_1 = 0\} \stackrel{\text{def}}{=} \mathbb{R}^{N-1}$$

$$\varphi = \text{id}_{\mathbb{R}^N}, \quad \Pi: \mathbb{R}^{N-1} \rightarrow \mathbb{R}^N$$

$$\omega \in \overset{\circ}{C}{}^{\infty, N-1}(X)$$

$$\Rightarrow \omega = \omega_u + d\psi_u$$

$$d\omega = \partial_u \omega_u + d\psi_u$$

$$= \partial_u \omega_u + d\psi_u + d\psi_u$$

$$= \partial_u \omega_u + d\psi_u + d\psi_u$$

$$= \partial_u \omega_u + \hat{\omega}_N$$

$$\text{where } \hat{\omega}_N = d\lambda_1 \wedge \dots \wedge d\lambda_N.$$

with $\Pi(x) = x'$
 $x = (x_1, x')$

$$(\partial_u \omega_u = \omega_u \partial_u \text{id}_{\mathbb{R}})$$

$$\Rightarrow \omega_u = I \circ \partial_u (\partial_u^{-1})$$

~~so that~~ I

$$\partial_u \omega_u (\partial_u^{-1})(\varphi) = \partial_u (\varphi \circ \omega_u)$$

$$= \partial_u (\varphi \circ \omega_u \circ \varphi^{-1}) \circ \varphi$$

$$= \partial_u (\varphi \circ \text{id}_{\mathbb{R}} \circ \text{id}_{\mathbb{R}} \circ \omega_u \circ \varphi^{-1}) \circ \varphi$$

$$= \partial_u (\varphi \circ \text{id}_{\mathbb{R}}) \circ \omega_u \cdot \partial_u (\text{id}_{\mathbb{R}} \circ \omega_u \circ \varphi^{-1}) \circ \varphi$$

$$= \partial_u \varphi \circ \partial_u (\text{id}_{\mathbb{R}} \circ \omega_u)$$

$$\Rightarrow \omega_u = \partial_u \varphi \circ \omega_u$$

$$\int_X d\omega = \int_X \partial_u \omega_u \hat{\omega}_N$$

$$\begin{aligned} & \text{Def. } X \\ &= \int_{\mathbb{R}_-^N} \partial_u \omega_u \circ \varphi^{-1} d\lambda_P \\ & \quad \varphi(X) = \mathbb{R}_-^N \\ & \quad (= X) \end{aligned}$$

$$\begin{aligned} & (\partial_u \omega_u \circ \varphi^{-1}) \\ &= \partial_u (\omega_u \circ \varphi^{-1}) \circ \varphi \circ \varphi^{-1} \\ &= \partial_u \omega_u \end{aligned}$$

$$= \int_{\mathbb{R}_-^N} \partial_u \omega_u d\lambda_P$$

$$\begin{aligned} & \text{comp. supp.} \\ &= \sum_{n=2}^N \int_{\mathbb{R}^{N-1}} \int_{\mathbb{R}} \partial_u \omega_u \frac{d\lambda_n d\lambda_{N-1}}{(\dots, x_n, \dots)} d\lambda_n(x_n) d\lambda_{N-1} \\ & \quad \text{Fubini/Tonelli} \end{aligned}$$

$$= \omega_u(\dots, x_n, \dots) \Big|_{x_n = +\infty} - \Big|_{x_n = -\infty}$$

$$= 0 - 0 \quad (\text{comp. supp.})$$

$$+ \int_{\mathbb{R}^{N-1}} \int_{-\infty}^{\infty} \partial_u \omega_u(x_1, \dots) d\lambda_n(x_n) d\lambda_{N-1}$$

$$= \omega_u(x_1, \dots) \Big|_{x_1 = 0} - \Big|_{x_1 = -\infty}$$

$$= \omega_u(0, \dots) - 0$$