NUMERICAL FUNCTIONAL ANALYSIS AND OPTIMIZATION
2020, VOL. 41, NO. 1, 16–112
https://doi.org/10.1080/01630563.2018.1490756

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# Solution Theory, Variational Formulations, and Functional a Posteriori Error Estimates for General First Order Systems with Applications to Electro-Magneto-Statics and More

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**ABSTRACT**

We prove a comprehensive solution theory using tools from functional analysis, show corresponding variational formulations, and present functional a posteriori error estimates for general linear first order systems of type

$$\begin{aligned} \mathrm{A}_2 x &= f, \\ \mathrm{A}_1^* x &= g, \end{aligned}$$

for two densely defined and closed (possibly unbounded) linear operators $\mathrm{A}_1$ and $\mathrm{A}_2$ having the complex property $\mathrm{A}_2\mathrm{A}_1 = 0$. As a prototypical application we will discuss the system of electro-magneto statics in 3D with mixed tangential and normal boundary conditions

$$\begin{aligned} \operatorname{rot} E &= F, \\ -\operatorname{div} \varepsilon E &= g. \end{aligned}$$

Our theory covers a lot more applications in 2D, 3D, and ND, such as general differential forms and all kind of systems arising, e.g., in general relativity, biharmonic problems, Stokes equations, or linear elasticity, to mention just a few, for example

$$\begin{aligned} \mathrm{d}E &= F, & \operatorname{Rot}_{\mathbb{S}} M &= F, & \operatorname{Div}_{\mathbb{T}} T &= F, & \operatorname{Rot}\operatorname{Rot}_{\mathbb{S}}^{\top} S &= F, \\ -\delta\varepsilon E &= G, & \operatorname{div}\operatorname{Div}_{\mathbb{S}}\varepsilon M &= G, & \operatorname{sym}\operatorname{Rot}_{\mathbb{T}}\varepsilon T &= G, & -\operatorname{Div}_{\mathbb{S}}\varepsilon S &= G, \end{aligned}$$

all with possibly mixed boundary conditions of generalized tangential and normal type. Second order systems of types

$$\begin{aligned} \mathrm{A}_2^*\mathrm{A}_2 x &= f, & \mathrm{A}_2^*\mathrm{A}_2 x &= f, \\ \mathrm{A}_1^* x &= g, & \mathrm{A}_1\mathrm{A}_1^* x &= g \end{aligned}$$

will be considered as well using the same techniques.

**ARTICLE HISTORY**
Received 14 June 2018
Revised 15 June 2018
Accepted 15 June 2018

**KEYWORDS**
Electro-magneto statics; functional a posteriori error estimates; general first order systems; mixed boundary conditions

**1991 MATHEMATICS SUBJECT CLASSIFICATION**
35F05; 35J46; 47A05; 47A50; 65N15; 78A25; 78A30

## 1. Introduction

Throughout this article we assume the following: For $\ell \in \mathbb{Z}$ let $\mathrm{H}_\ell$ be Hilbert spaces. Moreover, let

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$$\mathrm{A}_\ell : D(\mathrm{A}_\ell) \subset H_\ell \to H_{\ell+1}$$

be densely defined and closed (possibly unbounded) linear operators. In applications, often almost all operators $\mathrm{A}_\ell$ will be zero, i.e., only finitely many $\mathrm{A}_\ell$ are different from zero, typically $\mathrm{A}_0, \mathrm{A}_1, \mathrm{A}_2, \mathrm{A}_3, \mathrm{A}_4$ in 3D PDE applications or $\mathrm{A}_0, \mathrm{A}_1, \ldots, \mathrm{A}_N, \mathrm{A}_{N+1}$ in ND PDE applications. Here, $D(\mathrm{A})$ denotes the domain of definition of a linear operator A and we introduce by $N(\mathrm{A})$ and $R(\mathrm{A})$ its kernel and range, respectively. Inner product, norm, orthogonality, orthogonal sum and difference of (or in) an Hilbert space $H$ will be denoted by $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathrm{H}}, |\,\cdot\,|_{\mathrm{H}}, \perp_{\mathrm{H}}$, and $\oplus_{\mathrm{H}}, \ominus_{\mathrm{H}}$, respectively. We note that $D(\mathrm{A})$, equipped with the graph inner product, is a Hilbert space itself. Moreover, we assume that the operators $\mathrm{A}_\ell$ satisfy the sequence or complex property, this is for all $\ell$

$$R(\mathrm{A}_\ell) \subset N(\mathrm{A}_{\ell+1}) \tag{1.1}$$

or equivalently $\mathrm{A}_{\ell+1}\mathrm{A}_\ell \subset 0$. Then the (Hilbert space) adjoint operators

$$\mathrm{A}_\ell^* : D(\mathrm{A}_\ell^*) \subset H_{\ell+1} \to H_\ell$$

defined by the relation

$$\forall\ x \in D(\mathrm{A}_\ell) \quad \forall\ y \in D(\mathrm{A}_\ell^*) \qquad \langle \mathrm{A}_\ell x, y\rangle_{H_{\ell+1}} = \langle x, \mathrm{A}_\ell^* y\rangle_{H_\ell}$$

satisfy also the sequence or complex property, i.e., for all $\ell$

$$R(\mathrm{A}_{\ell+1}^*) \subset N(\mathrm{A}_\ell^*) \tag{1.2}$$

or equivalently $\mathrm{A}_\ell^*\mathrm{A}_{\ell+1}^* \subset 0$. We note $\mathrm{A}_\ell^{**} = \overline{\mathrm{A}_\ell} = \mathrm{A}_\ell$, i.e., $(\mathrm{A}_\ell, \mathrm{A}_\ell^*)$ are dual pairs. The complex

$$\cdots \xrightarrow{\mathrm{A}_{\ell-1}} D(\mathrm{A}_\ell) \xrightarrow{\mathrm{A}_\ell} D(\mathrm{A}_{\ell+1}) \xrightarrow{\mathrm{A}_{\ell+1}} \cdots \tag{1.3}$$

is called closed, if all ranges $R(\mathrm{A}_\ell)$ are closed, and called exact, if $R(\mathrm{A}_\ell) = N(\mathrm{A}_{\ell+1})$ holds for all $\ell$. By the closed range theorem, (1.3) is closed resp. exact, if and only if the adjoint complex

$$\cdots \xleftarrow{\mathrm{A}_{\ell-1}^*} D(\mathrm{A}_{\ell-1}^*) \xleftarrow{\mathrm{A}_\ell^*} D(\mathrm{A}_\ell^*) \xleftarrow{\mathrm{A}_{\ell+1}^*} \cdots \tag{1.4}$$

is closed resp. exact. For all $\ell$ and by the projection theorem the Helmholtz type decompositions

$$\begin{aligned} H_\ell &= N(\mathrm{A}_\ell) \oplus_{H_\ell} \overline{R(\mathrm{A}_\ell^*)}, & N(\mathrm{A}_\ell) &= R(\mathrm{A}_\ell^*)^{\perp_{H_\ell}}, \\ H_\ell &= \overline{R(\mathrm{A}_{\ell-1})} \oplus_{H_\ell} N(\mathrm{A}_{\ell-1}^*), & N(\mathrm{A}_{\ell-1}^*) &= R(\mathrm{A}_{\ell-1})^{\perp_{H_\ell}} \end{aligned}$$

hold. Moreover, the complex properties (1.1) and (1.2) show

$$N(\mathrm{A}_\ell) = \overline{R(\mathrm{A}_{\ell-1})} \oplus_{H_\ell} K_\ell, \quad N(\mathrm{A}_{\ell-1}^*) = K_\ell \oplus_{H_\ell} \overline{R(\mathrm{A}_\ell^*)},$$

where we introduce the cohomology groups

$$K_\ell := N(\mathrm{A}_\ell) \cap N(\mathrm{A}^*_{\ell-1}).$$

Therefore, we obtain the refined Helmholtz type decompositions

$$H_\ell = \overline{R(\mathrm{A}_{\ell-1})} \oplus_{H_\ell} K_\ell \oplus_{H_\ell} \overline{R(\mathrm{A}^*_\ell)}.$$

Note that, if $\mathrm{A}_\ell$ has closed range then by the closed range theorem and the projection theorem

$$R(\mathrm{A}_\ell) = N(\mathrm{A}^*_\ell)^{\perp_{H_{\ell+1}}}, \quad R(\mathrm{A}^*_\ell) = N(\mathrm{A}_\ell)^{\perp_{H_\ell}}.$$

Finally, we define for all $\ell$ the domains of definition for our mixed problems

$$D_\ell := D(\mathrm{A}_\ell) \cap D(\mathrm{A}^*_{\ell-1}).$$

### 1.1. Aims and main results

The central aim of this article is to prove functional a posteriori error estimates in the spirit of Sergey Repin, see, e.g., [1–5], for the linear system

$$\begin{aligned} \mathrm{A}_2 x &= f, \\ \mathrm{A}^*_1 x &= g, \\ \pi_2 x &= k \end{aligned} \tag{1.5}$$

with

$$x \in D_2 = D(\mathrm{A}_2) \cap D(\mathrm{A}^*_1),$$

where $\pi_2 : H_2 \to K_2 = N(\mathrm{A}_2) \cap N(\mathrm{A}^*_1)$ denotes the orthonormal projector onto the cohomology group or kernel $K_2$. We recall the complex property $\mathrm{A}_2\mathrm{A}_1 = 0$, and hence also $\mathrm{A}^*_1\mathrm{A}^*_2 = 0$. Obviously, $f \in R(\mathrm{A}_2), g \in R(\mathrm{A}^*_1)$, and $k \in K_2$ are necessary for solvability of (1.5) and there exists at most one solution to (1.5). A proper solution theory for (1.5), i.e., existence of a solution of (1.5) depending continuously on the data, will be given in the next section. The main result for this is Theorem 3.3 and reads as follows:

**Theorem I** (Theorem 3.3) *Let $R(\mathrm{A}_1)$ and $R(\mathrm{A}_2)$ be closed. Then (1.5) is uniquely solvable in $D_2$, if and only if $f \in R(\mathrm{A}_2), g \in R(\mathrm{A}^*_1)$, and $k \in K_2$. The solution $x \in D_2$ depends linearly and continuously on the data, i.e., $|x|_{H_2} \leq c_2|f|_{H_3} + c_1|g|_{H_1} + |k|_{H_2}$.*

**Remark 2** (Lemma 2.1, Lemma 2.3, (2.4))

(i) *By the closed range theorem, $R(\mathrm{A}_1)$ resp. $R(\mathrm{A}_2)$ is closed, if and only if $R(\mathrm{A}^*_1)$ resp. $R(\mathrm{A}^*_2)$ is closed. Moreover, $R(\mathrm{A}_1)$ and $R(\mathrm{A}_2)$ are closed, if, e.g., $D_2 \hookrightarrow H_2$ is compact, see Lemma 2.3, in which case $K_2$ is also finite dimensional, see General Assumption 3.1 and Remark 3.2.*

(ii) *By the closed graph theorem the following assertions are equivalent:*
- *The range* $R(\mathrm{A}_1)$ *is closed in* $H_2$.
- *There exists* $0 < c < \infty$ *such that for all* $\phi \in D(\mathrm{A}_1) \cap N(\mathrm{A}_1)^{\perp_{H_1}}$ *it holds* $|\phi|_{H_1} \leq c|\mathrm{A}_1\phi|_{H_2}$.
- *The inverse* $\mathcal{A}_1^{-1} : R(\mathrm{A}_1) \to D(\mathrm{A}_1) \cap N(\mathrm{A}_1)^{\perp_{H_1}}$ *is continuous, where* $\mathcal{A}_1$ *is the corresponding reduced operator of* $\mathrm{A}_1$, *i.e., the restriction of* $\mathrm{A}_1$ *to* $D(\mathrm{A}_1) \cap N(\mathrm{A}_1)^{\perp_{H_1}}$.

(iii) *If* $R(\mathrm{A}_1)$ *is closed, then* $c_1$ *is defined as the best possible constant in (ii) and hence equals the norm of the inverse* $\mathcal{A}_1^{-1}$ *regarded as operator from* $R(\mathrm{A}_1)$ *to* $N(\mathrm{A}_1)^{\perp_{H_1}}$. *Moreover,* $c_1$ *is also given by the Rayleigh quotient*

$$\inf_{0 \neq \phi \in D(\mathrm{A}_1) \cap N(\mathrm{A}_1)^{\perp_{H_1}}} \frac{|\mathrm{A}_1\phi|^2_{H_2}}{|\phi|^2_{H_1}} = \frac{1}{c_1^2} = \lambda_1,$$

*which defines the smallest positive eigenvalue* $\lambda_1$ *of the selfadjoint operator*[1] $\mathrm{A}_1^*\mathrm{A}_1$.

(iv) *Similar results and definitions as in (ii) and (iii) hold for the constant* $c_2$ *provided that* $R(\mathrm{A}_2)$ *is closed.*

(v) *The unique solution* $x \in D_2$ *in Theorem I is simply given by* $x = \mathcal{A}_2^{-1}f + (\mathcal{A}_1^*)^{-1}g + k$.

Although the solution theory is based on pure functional analysis and operator theory, we shall give a few variational (multiple) saddle point formulations as well propose methods for computing the exact solution $x \in D_2$. These formulations are not only alternatives to prove Theorem I, but also suggestions for possible numerical methods in future applications, and will be discussed extensively, see, e.g., Theorem 3.5, Theorem 3.10, Theorem 3.12, Theorem 3.14, and Theorem 3.17. One of these results reads as follows:

**Theorem III** (Theorem 3.12) *Let* $R(\mathrm{A}_1)$ *and* $R(\mathrm{A}_2)$ *be closed. Moreover, let* $f \in R(\mathrm{A}_2)$ *and* $g \in R(\mathrm{A}_1^*)$. *The unique solution* $x \in D_2$ *in Theorem I can be found by the following two variational double saddle point formulations:*

(i) *There exists a unique triple* $(\hat{x}, z, h) \in D(\mathrm{A}_2) \times (D(\mathrm{A}_1) \cap R(\mathrm{A}_1^*)) \times K_2$, *such that for all triples* $(\xi, \varphi, \kappa) \in D(\mathrm{A}_2) \times (D(\mathrm{A}_1) \cap R(\mathrm{A}_1^*)) \times K_2$

$$\begin{aligned} \langle \mathrm{A}_2\hat{x}, \mathrm{A}_2\xi\rangle_{H_3} + \langle \mathrm{A}_1 z, \xi\rangle_{H_2} + \langle h, \xi\rangle_{H_2} &= \langle f, \mathrm{A}_2\xi\rangle_{H_3}, \\ \langle \hat{x}, \mathrm{A}_1\varphi\rangle_{H_2} &= \langle g, \varphi\rangle_{H_1}, \\ \langle \hat{x}, \kappa\rangle_{H_2} &= \langle k, \kappa\rangle_{H_2}. \end{aligned}$$

[1] Thus $\lambda_1$ is also the smallest positive eigenvalue of the selfadjoint operator $\mathrm{A}_1\mathrm{A}_1^*$.

*It holds* $z=0$ *and* $h=0$ *as well as* $\mathrm{A}_2\hat{x}=f$ *and* $\pi_2\hat{x}=k$*. Moreover, the variational formulation holds for all* $\varphi\in D(\mathrm{A}_1)$*, and thus* $\hat{x}\in D(\mathrm{A}_1^*)$ *with* $\mathrm{A}_1^*\hat{x}=g$*. Finally,* $\hat{x}=x$ *from Theorem I.*

(ii) *There exists a unique triple* $(\hat{x},y,h)\in D(\mathrm{A}_1^*)\times(D(\mathrm{A}_2^*)\cap R(\mathrm{A}_2))\times K_2$*, such that for all triples* $(\zeta,\phi,\kappa)\in D(\mathrm{A}_1^*)\times(D(\mathrm{A}_2^*)\cap R(\mathrm{A}_2))\times K_2$

$$\begin{aligned}\langle \mathrm{A}_1^*\hat{x},\mathrm{A}_1^*\zeta\rangle_{H_1}+\langle \mathrm{A}_2^*y,\zeta\rangle_{H_2}+\langle h,\zeta\rangle_{H_2}&=\langle g,\mathrm{A}_1^*\zeta\rangle_{H_1},\\ \langle \hat{x},\mathrm{A}_2^*\phi\rangle_{H_2}&=\langle f,\phi\rangle_{H_3},\\ \langle \hat{x},\kappa\rangle_{H_2}&=\langle k,\kappa\rangle_{H_2}.\end{aligned}$$

*It holds* $y=0$ *and* $h=0$ *as well as* $\mathrm{A}_1^*\hat{x}=g$ *and* $\pi_2\hat{x}=k$*. The variational formulation holds for all* $\phi\in D(\mathrm{A}_2^*)$*, and thus* $\hat{x}\in D(\mathrm{A}_2)$ *with* $\mathrm{A}_2\hat{x}=f$*. Finally,* $\hat{x}=x$ *from Theorem I.*

Theorem III (i) resp. (ii) is a weak formulation of

$$\mathrm{A}_2^*\mathrm{A}_2\hat{x}+\mathrm{A}_1z+h=\mathrm{A}_2^*f,\qquad \mathrm{A}_1^*\hat{x}=g,\qquad \pi_2\hat{x}=k,$$

resp.

$$\mathrm{A}_1\mathrm{A}_1^*\hat{x}+\mathrm{A}_2^*y+h=\mathrm{A}_1g,\qquad \mathrm{A}_2\hat{x}=f,\qquad \pi_2\hat{x}=k,$$

i.e., in formal matrix notation

$$\begin{bmatrix}\mathrm{A}_2^*\mathrm{A}_2 & \mathrm{A}_1 & \iota_{K_2}\\ \mathrm{A}_1^* & 0 & 0\\ \pi_2=\iota_{K_2}^* & 0 & 0\end{bmatrix}\begin{bmatrix}\hat{x}\\ z\\ h\end{bmatrix}=\begin{bmatrix}\mathrm{A}_2^*f\\ g\\ k\end{bmatrix},\qquad \begin{bmatrix}\mathrm{A}_1\mathrm{A}_1^* & \mathrm{A}_2^* & \iota_{K_2}\\ \mathrm{A}_2 & 0 & 0\\ \pi_2=\iota_{K_2}^* & 0 & 0\end{bmatrix}\begin{bmatrix}\hat{x}\\ y\\ h\end{bmatrix}=\begin{bmatrix}\mathrm{A}_1g\\ f\\ k\end{bmatrix},$$

respectively, where $\iota_{K_2}$ is the canonical embedding of $K_2$ into $H_2$. Note $z=0$, $h=0$ resp. $y=0$, $h=0$. Often the additional condition $z\in R(\mathrm{A}_1^*)$ resp. $y\in R(\mathrm{A}_2)$ is unpleasant, especially for possible numerical applications, and hence the saddle point idea has to be repeated until $D(\mathcal{A}_\ell)=D(\mathrm{A}_\ell)$, i.e., $R(\mathrm{A}_\ell^*)=H_\ell$ resp. $D(\mathcal{A}_\ell^*)=D(\mathrm{A}_\ell^*)$, i.e., $R(\mathrm{A}_\ell)=H_{\ell+1}$ holds for some $\ell$. In 3 D we typically have only the operators $\mathrm{A}_0,\mathrm{A}_1,\mathrm{A}_2,\mathrm{A}_3,\mathrm{A}_4$ with adjoints $\mathrm{A}_0^*,\mathrm{A}_1^*,\mathrm{A}_2^*,\mathrm{A}_3^*,\mathrm{A}_4^*$ forming the Hilbert complexes and it holds $R(\mathrm{A}_0^*)=H_0$ and $R(\mathrm{A}_4)=H_5$. Hence the biggest system in 3 D arising for $\mathrm{A}_2$ resp. $\mathrm{A}_1^*$ as leading operator to compute $\hat{x}=x$ is

$$\begin{bmatrix}\mathrm{A}_2^*\mathrm{A}_2 & \mathrm{A}_1 & 0 & \iota_{K_2} & 0\\ \mathrm{A}_1^* & 0 & \mathrm{A}_0 & 0 & \iota_{K_1}\\ 0 & \mathrm{A}_0^* & 0 & 0 & 0\\ \pi_2=\iota_{K_2}^* & 0 & 0 & 0 & 0\\ 0 & \pi_1=\iota_{K_1}^* & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}\hat{x}\\ z\\ u\\ h_2\\ h_1\end{bmatrix}=\begin{bmatrix}\mathrm{A}_2^*f\\ g\\ 0\\ k\\ 0\end{bmatrix}$$

resp.

$$\begin{bmatrix} A_1A_1^* & A_2^* & 0 & 0 & \iota_{K_2} & 0 & 0 \\ A_2 & 0 & A_3^* & 0 & 0 & \iota_{K_3} & 0 \\ 0 & A_3 & 0 & A_4^* & 0 & 0 & \iota_{K_4} \\ 0 & 0 & A_4 & 0 & 0 & 0 & 0 \\ \pi_2 = \iota_{K_2}^* & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \pi_3 = \iota_{K_3}^* & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \pi_4 = \iota_{K_4}^* & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{x} \\ y \\ v \\ w \\ h_2 \\ h_3 \\ h_4 \end{bmatrix} = \begin{bmatrix} A_1 g \\ f \\ 0 \\ 0 \\ k \\ 0 \\ 0 \end{bmatrix}.$$

Note $z=0$, $u=0$, $h_2=0, h_1=0$ resp. $y=0$, $v=0$, $w=0$, $h_2=0, h_3=0, h_4=0$.

**Remark 4** *Particularly interesting cases are those for which $R(A_1^*)$ in Theorem III (i) or $R(A_2)$ in Theorem III (ii) already have finite co-dimension, i.e., in the best cases $R(A_1^*) = H_1$ or $R(A_2) = H_3$, i.e., $N(A_1) = \{0\}$ or $N(A_2^*) = \{0\}$. Fortunately, these situations are typical in many applications, as we will see at the end of the introduction or in more detail in the Application Section 5. Indeed, typically $N(A_1) = \{0\}$ or at least* $\dim N(A_1) < \infty$ *and* $N(A_3^*) = \{0\}$ *or at least* $\dim N(A_3^*) < \infty$.

Let $\widetilde{x} \in H_2$ and let us consider $\widetilde{x}$ as a possibly (very) nonconforming[2] "approximation" for the exact solution

$$x \in D_2 = D(A_2) \cap D(A_1^*)$$

of (1.5). Proving functional a posteriori error estimates, also called a posteriori error estimates of functional type, for the linear problem (1.5) means, that we will present two-sided estimates for the error

$$e := x - \widetilde{x} \in H_2$$

with the following properties:

① There exist two functionals $\mathcal{M}_{\mp}$, a lower and an upper bound, such that

$$\forall\ z_i, y_j \qquad \mathcal{M}_-(z_1, \ldots, z_I; \widetilde{x}, f, g, k) \le |e|_{H_2} \le \mathcal{M}_+(y_1, \ldots, y_J; \widetilde{x}, f, g, k), \tag{1.6}$$

were the $z_i$ and the $y_j$ belong to some suitable Hilbert spaces. The functionals $\mathcal{M}_{\mp}$ are guaranteed lower and upper bounds for the norm of the error $|e|_{H_2}$ and explicitly computable as long as at least upper bounds for the natural Friedrichs/Poincaré type constants $c_1$ and $c_2$ for the operators $A_1$ and $A_2$ are known[3]. The bounds $\mathcal{M}_{\mp}$ do not depend on the

[2] A conforming "approximation" $\widetilde{x}$ would belong to $D_2$.

[3] Just needed for the upper bound $\mathcal{M}_+$.

possibly and generally unknown exact solution $x$, but only on the data, the approximation $\widetilde{x}$, and the "free" vectors $z_i$, $y_j$.
② The lower and upper bound $\mathcal{M}_{\mp}$ are sharp, i.e.,

$$\max_{z_1,\ldots,z_I} \mathcal{M}_{-}(z_1,\ldots,z_I;\widetilde{x},f,g,k) = |e|_{H_2} = \min_{y_1,\ldots,y_J} \mathcal{M}_{+}(y_1,\ldots,y_J;\widetilde{x},f,g,k). \quad (1.7)$$

③ The minimization over $z_i$ and $y_j$ is "simple," typically a minimization of quadratic functionals.
④ The bounds $\mathcal{M}_{\mp}$ are general in the sense that they do not depend on any specific numerical method which might be used in some possible application.

Concerning the error estimates the main result of this contribution is Corollary 4.6, which summarizes Theorem 4.1, Theorem 4.5, and the corresponding corollaries and reads as follows:

**Theorem V** (Corollary 4.6) *Let $R(A_1)$ and $R(A_2)$ be closed. Moreover, let $x \in D_2$ be the exact solution of (1.5) and let $\widetilde{x} \in H_2$, regarded as nonconforming approximation of $x$. Then the error $e := x - \widetilde{x}$ decomposes orthogonally, i.e.,*

$$e = e_{A_1} + e_{K_2} + e_{A_2^*} \in R(A_1) \oplus_{H_2} K_2 \oplus_{H_2} R(A_2^*),$$
$$|e|_{H_2}^2 = |e_{A_1}|_{H_2}^2 + |e_{K_2}|_{H_2}^2 + |e_{A_2^*}|_{H_2}^2,$$

*and the following a posteriori error estimates for the respective error parts hold:*

(i) *The projection $e_{A_1} \in R(A_1)$ satisfies*

$$\max_{\varphi \in D(A_1)} \mathcal{M}_{-,A_1}(\varphi;\widetilde{x},g) = |e_{A_1}|_{H_2}^2 = \min_{\zeta \in D(A_1^*)} \mathcal{M}_{+,A_1}^2(\zeta;\widetilde{x},g),$$

$$\mathcal{M}_{-,A_1}(\varphi;\widetilde{x},g) := 2\langle g,\varphi\rangle_{H_1} - \langle 2\widetilde{x} + A_1\varphi, A_1\varphi\rangle_{H_2},$$
$$\mathcal{M}_{+,A_1}(\zeta;\widetilde{x},g) := c_1|A_1^*\zeta - g|_{H_1} + |\zeta - \widetilde{x}|_{H_2}.$$

*The maximum is attained at any $\hat{\varphi} \in D(A_1)$ with $A_1\hat{\varphi} = e_{A_1}$ and $\hat{\zeta} := e_{A_1} + \widetilde{x} \in D(A_1^*)$ gives the minimum. It holds $A_1^*\hat{\zeta} = A_1^*x = g$.*

(ii) The projection $e_{A_2^*} \in R(A_2^*)$ satisfies

$$\max_{\phi \in D(A_2^*)} \mathcal{M}_{-,A_2^*}(\phi;\widetilde{x},f) = |e_{A_2^*}|_{H_2}^2 = \min_{\xi \in D(A_2)} \mathcal{M}_{+,A_2^*}^2(\xi;\widetilde{x},f),$$

$$\mathcal{M}_{-,A_2^*}(\phi;\widetilde{x},f) := 2\langle f,\phi\rangle_{H_3} - \langle 2\widetilde{x} + A_2^*\phi, A_2^*\phi\rangle_{H_2},$$
$$\mathcal{M}_{+,A_2^*}(\xi;\widetilde{x},f) := c_2|A_2\xi - f|_{H_3} + |\xi - \widetilde{x}|_{H_2}.$$

*The maximum is attained at any* $\hat{\phi} \in D(\mathrm{A}_2^*)$ *with* $\mathrm{A}_2^*\hat{\phi} = e_{\mathrm{A}_2^*}$ *and* $\hat{\xi} := e_{\mathrm{A}_2^*} + \widetilde{x} \in D(\mathrm{A}_2)$ *gives the minimum. It holds* $\mathrm{A}_2\hat{\xi} = \mathrm{A}_2 x = f$.

(iii) The projection $e_{K_2} = \pi_2 e = k - \pi_2\widetilde{x} \in K_2$ satisfies

$$\max_{\theta\in K_2} \mathcal{M}_{-,K_2}(\theta;\widetilde{x},k) = |e_{K_2}|^2_{H_2} = \min_{\substack{\varphi\in D(\mathrm{A}_1),\\ \phi\in D(\mathrm{A}_2^*)}} \mathcal{M}^2_{+,K_2}(\varphi,\phi;\widetilde{x},k)$$

$$\mathcal{M}_{-,K_2}(\theta;\widetilde{x},k) := \langle 2(k-\widetilde{x})-\theta,\theta\rangle_{H_2},$$

$$\mathcal{M}_{+,K_2}(\varphi,\phi;\widetilde{x},k) := |k-\widetilde{x}+\mathrm{A}_1\varphi+\mathrm{A}_2^*\phi|_{H_2}.$$

*The maximum is attained at* $\hat{\theta} := e_{K_2} \in K_2$ *and the minimum at any pair* $(\hat{\varphi},\hat{\phi}) \in D(\mathrm{A}_1)\times D(\mathrm{A}_2^*)$ *with* $\mathrm{A}_1\hat{\varphi} + \mathrm{A}_2^*\hat{\phi} = (1-\pi_2)\widetilde{x}$.

**Remark VI** (Corollary 4.6 continued, Section 4.3)

(i) *In applications, often* $\widetilde{x} := k + \widetilde{x}_\perp$ *holds with some* $\widetilde{x}_\perp \in K_2^{\perp_{H_2}}$. *In this case* $e_{K_2} = 0$ *and in Theorem V (i) and Theorem V (ii)* $\widetilde{x}$ *can be replaced by* $\widetilde{x}_\perp$ . *Moreover,* $\hat{\zeta}_\perp := e_{\mathrm{A}_1} + \widetilde{x}_\perp \in D(\mathrm{A}_1^*)$ *and* $\hat{\xi}_\perp := e_{\mathrm{A}_2^*} + \widetilde{x}_\perp \in D(\mathrm{A}_2)$ *holds for the attaining minima.*

(ii) *Differentiating the lower bound* $\mathcal{M}_{-,\mathrm{A}_1}(\varphi;\widetilde{x},g)$ *with respect to* $\varphi$ *shows that a possible maximizer* $\hat{\varphi} \in D(\mathrm{A}_1)$ *of the maximum in Theorem V (i) solves the variational formulation*

$$\forall\ \varphi \in D(\mathrm{A}_1) \qquad \langle \mathrm{A}_1\hat{\varphi}, \mathrm{A}_1\varphi\rangle_{H_2} = \langle g,\varphi\rangle_{H_1} - \langle \widetilde{x}, \mathrm{A}_1\varphi\rangle_{H_2}, \qquad (1.8)$$

*which implies* $\mathrm{A}_1\hat{\varphi} + \widetilde{x} \in D(\mathrm{A}_1^*)$ *with* $\mathrm{A}_1^*(\mathrm{A}_1\hat{\varphi}+\widetilde{x}) = g$ *and presents a weak formulation*[4] *of*

$$\mathrm{A}_1^*\mathrm{A}_1\hat{\varphi} = g - \mathrm{A}_1^*\widetilde{x} = \mathrm{A}_1^* e = \mathrm{A}_1^* e_{\mathrm{A}_1}.$$

*By Remark II (ii) A*$_1$ *is strictly positive over* $D(\mathrm{A}_1)\cap N(\mathrm{A}_1)^{\perp_{H_1}}$ *and hence (1.8) admits a unique solution* $\hat{\varphi} \in D(\mathrm{A}_1)\cap N(\mathrm{A}_1)^{\perp_{H_1}}$. *A particularly simple case is again given if N(A*$_1$*) is finite dimensional or even* $N(\mathrm{A}_1) = \{0\}$, *which occurs in many applications.*

(ii') *On the other hand, considering the minimum in Theorem V (i) we can roughly estimate the upper bound by, e.g.,*

$$\mathcal{M}^2_{+,\mathrm{A}_1}(\zeta;\widetilde{x},g) \le 2c_1^2|\mathrm{A}_1^*\zeta - g|^2_{H_1} + 2|\zeta-\widetilde{x}|^2_{H_2}.$$

*Differentiating the right hand side with respect to* $\zeta$ *shows that the minimizer* $\hat{\zeta} \in D(\mathrm{A}_1^*)$ *solves the variational formulation*

$$\forall\ \zeta \in D(\mathrm{A}_1^*) \qquad c_1^2\langle \mathrm{A}_1^*\hat{\zeta}, \mathrm{A}_1^*\zeta\rangle_{H_1} + \langle\hat{\zeta},\zeta\rangle_{H_2} = c_1^2\langle g, \mathrm{A}_1^*\zeta\rangle_{H_1} + \langle\widetilde{x},\zeta\rangle_{H_2}, \qquad (1.9)$$

[4]Thus $\mathrm{A}_1\hat{\varphi} - e_{\mathrm{A}_1} \in N(\mathrm{A}_1^*)\cap R(\mathrm{A}_1) = N(\mathrm{A}_1^*)\cap N(\mathrm{A}_1^*)^{\perp_{H_2}} = \{0\}$.

*which implies* $A_1^*\hat{\zeta} - g \in D(A_1)$ *and* $c_1^2 A_1(A_1^*\hat{\zeta} - g) = (\widetilde{x} - \hat{\zeta})$, *presents a weak formulation of*

$$c_1^2 A_1 A_1^* \hat{\zeta} + \hat{\zeta} = c_1^2 A_1 g + \widetilde{x}.$$

*Unique solvability of (1.9) in* $D(A_1^*)$ *is trivial as the variational formulation reproduces a graph inner product of* $D(A_1^*)$. *An optimized minimization process using a more careful estimate is explained in some detail in Section 4.3.*

(iii) *Similar arguments and formulations hold for Theorem V (ii) and (iii) as well.*

We shall also present a full theory, in particular functional a posteriori error estimates, for linear second order systems such as

$$\begin{aligned} A_2^* A_2 x &= f, \\ A_1^* x &= g, \\ \pi_2 x &= k \end{aligned} \tag{1.10}$$

with $x \in D_2$ such that $A_2 x \in D(A_2^*)$, i.e., $x \in D(A_1^*) \cap D(A_2^* A_2)$. This will follow immediately by the theory developed for the first order system (1.5), since the solution pair

$$(x, y) \in \big(D(A_2) \cap D(A_1^*)\big) \times \big(D(A_3) \cap D(A_2^*)\big)$$

defined by $y := A_2 x \in D(A_2^*) \cap R(A_2)$ solves the system of first order systems

$$\begin{aligned} A_2 x &= y, & A_3 y &= 0, \\ A_1^* x &= g, & A_2^* y &= f, \\ \pi_2 x &= k, & \pi_3 y &= 0. \end{aligned}$$

Analogously, we can treat problems such as

$$\begin{aligned} A_2^* A_2 x &= f, \\ A_1 A_1^* x &= g, \\ \pi_2 x &= k \end{aligned} \tag{1.11}$$

as well, which are strongly related to the generalized Hodge-Helmholtz decomposition of $f + g + k \in H_2$.

### 1.2. Applications

Our main applications will be the linear first order systems of electro-magneto statics as well as related second order rotrot systems and, as a very simple example, the Laplacian, see Section 5, especially Theorem 5.12. In this article, we only discuss homogeneous boundary conditions, noting that the canonical extension to inhomogeneous boundary conditions is straight forward. As we shall give a detailed description of more applications fitting our general theory for the linear systems (1.5), (1.10), (1.11) and for the general complexes (1.3), (1.4) in Section 5.3, we just indicate a few applications by listing some interesting and important underlying complexes arising in, e.g., general electro-magneto statics, for differential forms on Riemannian manifolds, in problems of linear elasticity, Stokes equations, biharmonic theory, general relativity, rot rot rot rot-operators, to mention just a few examples. Although all these systems are allowed to have mixed generalized tangential and normal boundary conditions and inhomogeneous and anisotropic material properties, see Section 5.3, we will just present the cases of full boundary conditions and homogeneous and isotropic material parameters here in this introductory part. For this let $\Omega \subset \mathbb{R}^3$ or $\Omega \subset \mathbb{R}^N, N \geq 2$, be a bounded weak Lipschitz domain.

- electro-magnetics

$$\{0\} \xrightarrow{\mathrm{A}_0=\iota_{\{0\}}} \mathsf{H}^1_\Gamma(\Omega) \xrightarrow{\mathrm{A}_1=\mathrm{grad}_\Gamma} \mathsf{R}_\Gamma(\Omega) \xrightarrow{\mathrm{A}_2=\mathrm{rot}_\Gamma} \mathsf{D}_\Gamma(\Omega) \xrightarrow{\mathrm{A}_3=\mathrm{div}_\Gamma} \mathsf{L}^2(\Omega) \xrightarrow{\mathrm{A}_4=\pi_{\mathbb{R}}} \mathbb{R}$$

$$\{0\} \xleftarrow{\mathrm{A}_0^*=\pi_{\{0\}}} \mathsf{L}^2(\Omega) \xleftarrow{\mathrm{A}_1^*=-\,\mathrm{div}} \mathsf{D}(\Omega) \xleftarrow{\mathrm{A}_2^*=\mathrm{rot}} \mathsf{R}(\Omega) \xleftarrow{\mathrm{A}_3^*=-\,\mathrm{grad}} \mathsf{H}^1(\Omega) \xleftarrow{\mathrm{A}_4^*=\iota_{\mathbb{R}}} \mathbb{R}$$

A typical system for a vector field $E$ is

$$\mathrm{rot}_\Gamma E = F, \qquad -\,\mathrm{div} E = g.$$

This system is well understood, see, e.g., the pioneering work of Norbert Weck [6] and Rainer Picard [7–9]. See also [10].

- generalized electro-magnetics (differential forms)

$$\{0\} \xrightarrow{\mathrm{A}_0=\iota_{\{0\}}} \mathsf{D}^0_\Gamma(\Omega) \xrightarrow{\mathrm{A}_1=\mathrm{d}_\Gamma} \cdots \xrightarrow{\mathrm{A}_{q-1}=\mathrm{d}_\Gamma} \mathsf{D}^{q-1}_\Gamma(\Omega) \xrightarrow{\mathrm{A}_q=\mathrm{d}_\Gamma} \mathsf{D}^q_\Gamma(\Omega) \xrightarrow{\mathrm{A}_{q+1}=\mathrm{d}_\Gamma} \cdots \xrightarrow{\mathrm{A}_N=\mathrm{d}_\Gamma} \mathsf{L}^{2,N}(\Omega) \xrightarrow{\mathrm{A}_{N+1}=\pi_{\mathbb{R}}} \mathbb{R}$$

$$\{0\} \xleftarrow{\mathrm{A}_0^*=\pi_{\{0\}}} \mathsf{L}^{2,0}(\Omega) \xleftarrow{\mathrm{A}_1^*=-\,\delta} \cdots \xleftarrow{\mathrm{A}_{q-1}^*=-\,\delta} \Delta^{q-1}(\Omega) \xleftarrow{\mathrm{A}_q^*=-\,\delta} \Delta^q(\Omega) \xleftarrow{\mathrm{A}_{q+1}^*=-\,\delta} \cdots \xleftarrow{\mathrm{A}_N^*=-\,\delta} \mathsf{D}^N(\Omega) \xleftarrow{\mathrm{A}_{N+1}^*=\iota_{\mathbb{R}}} \mathbb{R}$$

A typical system for a differential form $E$ is

$$\mathrm{d}_\Gamma E = F, \qquad -\,\delta E = G.$$

This system is well understood as well, see, e.g., [7–9, 11, 12].

- biharmonic problems, Stokes problems, and general relativity

$$\{0\} \xrightarrow{\mathrm{A}_0=\iota_{\{0\}}} \mathsf{H}^2_\Gamma(\Omega) \xrightarrow{\mathrm{A}_1=\mathrm{Grad}\,\mathrm{grad}_\Gamma} \mathsf{R}_\Gamma(\Omega;\mathbb{S}) \xrightarrow{\mathrm{A}_2=\mathrm{Rot}_{\mathbb{S},\Gamma}} \mathsf{D}_\Gamma(\Omega;\mathbb{T}) \xrightarrow{\mathrm{A}_3=\mathrm{Div}_{\mathbb{T},\Gamma}} \mathsf{L}^2(\Omega) \xrightarrow{\mathrm{A}_4=\pi_{\mathsf{RT}}} \mathsf{RT}$$

$$\{0\} \xleftarrow{\mathrm{A}_0^*=\pi_{\{0\}}} \mathsf{L}^2(\Omega) \xleftarrow{\mathrm{A}_1^*=\mathrm{div}\,\mathrm{Div}_{\mathbb{S}}} \mathsf{DD}(\Omega;\mathbb{S}) \xleftarrow{\mathrm{A}_2^*=\mathrm{sym}\,\mathrm{Rot}_{\mathbb{T}}} \mathsf{R}_{\mathrm{sym}}(\Omega;\mathbb{T}) \xleftarrow{\mathrm{A}_3^*=-\,\mathrm{dev}\,\mathrm{Grad}} \mathsf{H}^1(\Omega) \xleftarrow{\mathrm{A}_4^*=\iota_{\mathsf{RT}}} \mathsf{RT}$$

A typical system for a symmetric tensor field $S$ resp. a deviatoric (trace free) tensor field $T$ is

$$\operatorname{Rot}_{\mathbb{S},\Gamma} S = F, \quad \operatorname{div}\operatorname{Div}_{\mathbb{S}} S = g \qquad \text{resp.} \qquad \operatorname{Div}_{\mathbb{T},\Gamma}\mathrm{T} = \mathrm{F}, \quad \operatorname{sym}\operatorname{Rot}_{\mathbb{T}}\mathrm{T} = \mathrm{G}.$$

- linear elasticity

$$\{0\} \xrightarrow{\mathrm{A}_0=\iota_{\{0\}}} \mathsf{H}^1_\Gamma(\Omega) \xrightarrow{\mathrm{A}_1=\operatorname{sym}\operatorname{Grad}_\Gamma} \mathsf{RR}^\top_\Gamma(\Omega;\mathbb{S}) \xrightarrow{\mathrm{A}_2=\operatorname{RotRot}^\top_{\mathbb{S},\Gamma}} \mathsf{D}_\Gamma(\Omega;\mathbb{S}) \xrightarrow{\mathrm{A}_3=\operatorname{Div}_{\mathbb{S},\Gamma}} \mathsf{L}^2(\Omega) \xrightarrow{\mathrm{A}_4=\pi_{\mathsf{RM}}} \mathsf{RM}$$

$$\{0\} \xleftarrow{\mathrm{A}_0^*=\pi_{\{0\}}} \mathsf{L}^2(\Omega) \xleftarrow{\mathrm{A}_1^*=-\operatorname{Div}_{\mathbb{S}}} \mathsf{D}(\Omega;\mathbb{S}) \xleftarrow{\mathrm{A}_2^*=\operatorname{RotRot}^\top_{\mathbb{S}}} \mathsf{RR}^\top(\Omega;\mathbb{S}) \xleftarrow{\mathrm{A}_3^*=-\operatorname{sym}\operatorname{Grad}} \mathsf{H}^1(\Omega) \xleftarrow{\mathrm{A}_4^*=\iota_{\mathsf{RM}}} \mathsf{RM}$$

A typical system for a symmetric tensor field $S$ is

$$\operatorname{RotRot}^\top_{\mathbb{S},\Gamma} S = F, \qquad -\operatorname{Div}_{\mathbb{S}} S = G.$$

Here we denote the rigid motions and the global Raviart–Thomas fields of $\Omega$ by

$$\begin{aligned} RM &:= \{P|_\Omega \ : \ P(x) = Qx + b, \ Q \in \mathbb{R}^{3\times 3} \text{ skew} - \text{symmetric}, \ b \in \mathbb{R}^3\}, \\ RT &:= \{P|_\Omega \ : \ P(x) = a\ x + b, \ a \in \mathbb{R}, \ b \in \mathbb{R}^3\}. \end{aligned}$$

## 2. Functional analysis tool box

Let $\ell \in \mathbb{Z}$. By the projection theorem the Helmholtz type decompositions

$$H_\ell = N(\mathrm{A}_\ell) \oplus_{H_\ell} \overline{R(\mathrm{A}_\ell^*)}, H_{\ell+1} = N(\mathrm{A}_\ell^*) \oplus_{H_{\ell+1}} \overline{R(\mathrm{A}_\ell)} \tag{2.1}$$

hold and define in a natural way the reduced operators

$$\begin{aligned} \mathcal{A}_\ell &:= \mathrm{A}_\ell|_{\overline{R(\mathrm{A}_\ell^*)}} : D(\mathcal{A}_\ell) \subset \overline{R(\mathrm{A}_\ell^*)} \to \overline{R(\mathrm{A}_\ell)}, \\ D(\mathcal{A}_\ell) &:= D(\mathrm{A}_\ell) \cap \overline{R(\mathrm{A}_\ell^*)} = D(\mathrm{A}_\ell) \cap N(\mathrm{A}_\ell)^{\perp_{H_\ell}}, \\ \mathcal{A}_\ell^* &:= \mathrm{A}_\ell^*|_{\overline{R(\mathrm{A}_\ell)}} : D(\mathcal{A}_\ell^*) \subset \overline{R(\mathrm{A}_\ell)} \to \overline{R(\mathrm{A}_\ell^*)}, \\ D(\mathcal{A}_\ell^*) &:= D(\mathrm{A}_\ell^*) \cap \overline{R(\mathrm{A}_\ell)} = D(\mathrm{A}_\ell^*) \cap N(\mathrm{A}_\ell^*)^{\perp_{H_{\ell+1}}}, \end{aligned}$$

which are also densely defined and closed linear operators. We note that $\mathcal{A}_\ell$ and $\mathcal{A}_\ell^*$ are indeed adjoint to each other, i.e., $(\mathcal{A}_\ell, \mathcal{A}_\ell^*)$ is a dual pair as well. Now the inverse operators

$$\mathcal{A}_\ell^{-1} : R(\mathrm{A}_\ell) \to D(\mathcal{A}_\ell), \qquad (\mathcal{A}_\ell^*)^{-1} : R(\mathrm{A}_\ell^*) \to D(\mathcal{A}_\ell^*)$$

exist, since $\mathcal{A}_\ell$ and $\mathcal{A}_\ell^*$ are injective by definition, and they are bijective, as, e.g., for $x \in D(\mathcal{A}_\ell)$ and $y := \mathrm{A}_\ell x \in R(\mathrm{A}_\ell)$ we get $\mathcal{A}_\ell^{-1} y = x$ by the injectivity of $\mathcal{A}_\ell$. Furthermore, by the Helmholtz type decompositions (2.1) we have

$$D(\mathrm{A}_\ell) = N(\mathrm{A}_\ell) \oplus_{H_\ell} D(\mathcal{A}_\ell), \qquad D(\mathrm{A}_\ell^*) = N(\mathrm{A}_\ell^*) \oplus_{H_\ell} D(\mathcal{A}_\ell^*) \tag{2.2}$$

and thus we obtain for the ranges

$$R(\mathrm{A}_\ell) = R(\mathcal{A}_\ell), \qquad R(\mathrm{A}_\ell^*) = R(\mathcal{A}_\ell^*). \tag{2.3}$$

By the closed range and closed graph theorem we get immediately the following lemma.

**Lemma 2.1.** *The following assertions are equivalent:*

(i) $\exists\ c_\ell \in (0,\infty)\ \forall\ x \in D(\mathcal{A}_\ell) \qquad |x|_{H_\ell} \le c_\ell |\mathrm{A}_\ell x|_{H_{\ell+1}}$

(i') $\exists\ c^*_\ell \in (0,\infty)\ \forall\ y \in D(\mathcal{A}^*_\ell) \qquad |y|_{H_{\ell+1}} \le c^*_\ell |\mathrm{A}^*_\ell y|_{H_\ell}$

(ii) $R(\mathrm{A}_\ell) = R(\mathcal{A}_\ell)$ *is closed in* $H_{\ell+1}$.

(ii') $R(\mathrm{A}^*_\ell) = R(\mathcal{A}^*_\ell)$ *is closed in* $\mathrm{H}_\ell$.

(iii) $\mathcal{A}_\ell^{-1} : R(\mathrm{A}_\ell) \to D(\mathcal{A}_\ell)$ *is continuous and bijective with norm bounded by* $(1+c_\ell^2)^{1/2}$.

(iii') $(\mathcal{A}^*_\ell)^{-1} : R(\mathrm{A}^*_\ell) \to D(\mathcal{A}^*_\ell)$ *is continuous and bijective with norm bounded by* $(1+c^{*2}_\ell)^{1/2}$.

*Proof.* Note that by the closed range theorem (ii) $\iff$ (ii$^*$) holds. Hence, by symmetry it is sufficient to show (i) $\iff$ (ii) $\iff$ (iii).

(i) $\Rightarrow$ (ii) Pick a sequence $(y_n) \subset R(\mathrm{A}_\ell)$ converging to $y \in H_{\ell+1}$ in $H_{\ell+1}$. By (2.3) there exists a sequence $(x_n) \subset D(\mathcal{A}_\ell)$ with $y_n = \mathrm{A}_\ell x_n$. (i) implies that $(x_n)$ is a Cauchy sequence in $\mathrm{H}_\ell$ and hence there exists some $x \in H_\ell$ with $x_n \to x$ in $\mathrm{H}_\ell$. As $\mathrm{A}_\ell$ is closed, we get $x \in D(\mathrm{A}_\ell)$ and $\mathrm{A}_\ell x = y \in R(\mathrm{A}_\ell)$.

(ii) $\Rightarrow$ (iii) Note that $\mathcal{A}_\ell^{-1} : R(\mathrm{A}_\ell) \to D(\mathcal{A}_\ell)$ is a densely defined and closed linear operator. By (ii), $R(\mathrm{A}\ _\ell)$ is closed and hence itself a Hilbert space. By the closed graph theorem $\mathcal{A}_\ell^{-1}$ is continuous.

(iii) $\Rightarrow$ (i) For $x \in D(\mathcal{A}_\ell)$ let $y := \mathrm{A}_\ell x \in R(\mathrm{A}_\ell)$. Then $x = \mathcal{A}_\ell^{-1} y$ as $\mathcal{A}_\ell$ is injective.[5] Therefore,

$$|x|_{H_\ell} = |\mathcal{A}_\ell^{-1} y|_{H_\ell} \le |\mathcal{A}_\ell^{-1}|_{R(\mathrm{A}_\ell),R(\mathrm{A}^*_\ell)} |y|_{H_{\ell+1}} = c_\ell |\mathrm{A}_\ell x|_{H_{\ell+1}}$$

with $c_\ell := |\mathcal{A}_\ell^{-1}|_{R(\mathrm{A}_\ell),R(\mathrm{A}^*_\ell)}$.

If (i) holds we have for $y \in R(\mathrm{A}_\ell)$ and $x := \mathcal{A}_\ell^{-1} y \in D(\mathcal{A}_\ell)$

$$|\mathcal{A}_\ell^{-1} y|_{H_\ell} \le c_\ell |\mathrm{A}_\ell x|_{H_{\ell+1}} = c_\ell |y|_{H_{\ell+1}}$$

and hence

$$|\mathcal{A}_\ell^{-1}|_{R(\mathrm{A}_\ell),R(\mathrm{A}^*_\ell)} = \sup_{0 \ne y \in R(\mathrm{A}_\ell)} \frac{|\mathcal{A}_\ell^{-1} y|_{H_\ell}}{|y|_{H_{\ell+1}}} \le c_\ell,$$

$$|\mathcal{A}_\ell^{-1}|^2_{R(\mathrm{A}_\ell),D(\mathcal{A}_\ell)} = \sup_{0 \ne y \in R(\mathrm{A}_\ell)} \frac{|\mathcal{A}_\ell^{-1} y|^2_{D(\mathrm{A}_\ell)}}{|y|^2_{H_{\ell+1}}} = \sup_{0 \ne y \in R(\mathrm{A}_\ell)} \frac{|\mathcal{A}_\ell^{-1} y|^2_{H_\ell} + |y|^2_{H_{\ell+1}}}{|y|^2_{H_{\ell+1}}} \le c_\ell^2 + 1,$$

finishing the proof. □

[5]It holds $\mathrm{A}_\ell(x - \mathcal{A}_\ell^{-1} y) = 0$ and thus $x = \mathcal{A}_\ell^{-1} y$.

From now on we assume that we always choose the best Friedrichs/ Poincaré type constants $c_\ell, c_\ell^*$, if they exist in $(0,\infty)$, i.e., $c_\ell$ and $c_\ell^*$ are given by the Rayleigh quotients

$$\frac{1}{c_\ell} := \inf_{0\neq x\in D(\mathcal{A}_\ell)} \frac{|\mathrm{A}_\ell x|_{H_{\ell+1}}}{|x|_{H_\ell}}, \quad \frac{1}{c_\ell^*} := \inf_{0\neq y\in D(\mathcal{A}_\ell^*)} \frac{|\mathrm{A}_\ell^* y|_{H_\ell}}{|y|_{H_{\ell+1}}}.$$

Moreover, we see

$$c_\ell = \sup_{0\neq x\in D(\mathcal{A}_\ell)} \frac{|x|_{H_\ell}}{|\mathrm{A}_\ell x|_{H_{\ell+1}}} = \sup_{0\neq y\in R(\mathrm{A}_\ell)} \frac{|\mathcal{A}_\ell^{-1} y|_{H_\ell}}{|y|_{H_{\ell+1}}} = |\mathcal{A}_\ell^{-1}|_{R(\mathrm{A}_\ell),R(\mathrm{A}_\ell^*)}, \tag{2.4}$$

as $0\neq x\in D(\mathcal{A}_\ell)$ implies $0\neq \mathrm{A}_\ell x$ and for $y := \mathrm{A}_\ell x$ with $x\in D(\mathcal{A}_\ell)$ we have $\mathcal{A}_\ell^{-1} y = x$, both by the injectivity of $\mathcal{A}_\ell$. Analogously, we get

$$c_\ell^* = \sup_{0\neq y\in D(\mathcal{A}_\ell^*)} \frac{|y|_{H_{\ell+1}}}{|\mathrm{A}_\ell^* y|_{H_\ell}} = \sup_{0\neq x\in R(\mathrm{A}_\ell^*)} \frac{|(\mathcal{A}_\ell^*)^{-1} x|_{H_{\ell+1}}}{|x|_{H_\ell}} = |(\mathcal{A}_\ell^*)^{-1}|_{R(\mathrm{A}_\ell^*),R(\mathrm{A}_\ell)}. \tag{2.5}$$

**Lemma 2.2.** *Assume that $c_\ell \in (0,\infty)$ or $c_\ell^* \in (0,\infty)$ exists. Then $c_\ell = c_\ell^*$.*
We note that also in the case $c_\ell = \infty$ or $c_\ell^* = \infty$ we have $c_\ell = c_\ell^* = \infty$.

*Proof.* Let, e.g., $c_\ell^*$ exist in $(0,\infty)$. By Lemma 2.1 also $c_\ell$ exists in $(0,\infty)$ and the ranges $R(\mathrm{A}_\ell) = R(\mathcal{A}_\ell)$ and $R(\mathrm{A}_\ell^*) = R(\mathcal{A}_\ell^*)$ are closed. Then for $x\in D(\mathcal{A}_\ell) = D(\mathrm{A}_\ell)\cap R(\mathrm{A}_\ell^*)$ there is $y\in D(\mathcal{A}_\ell^*)$ with $x = \mathrm{A}_\ell^* y$. More precisely, $y := (\mathcal{A}_\ell^*)^{-1} x \in D(\mathcal{A}_\ell^*)$ is uniquely determined and we have $|y|_{H_{\ell+1}} \leq c_\ell^* |\mathrm{A}_\ell^* y|_{H_\ell}$. But then

$$|x|_{H_\ell}^2 = \langle x, \mathrm{A}_\ell^* y\rangle_{H_\ell} = \langle \mathrm{A}_\ell x, y\rangle_{H_{\ell+1}} \leq |\mathrm{A}_\ell x|_{H_{\ell+1}} |y|_{H_{\ell+1}} \leq c_\ell^* |\mathrm{A}_\ell x|_{H_{\ell+1}} |\mathrm{A}_\ell^* y|_{H_\ell},$$

yielding $|x|_{H_\ell} \leq c_\ell^* |\mathrm{A}_\ell x|_{H_{\ell+1}}$. Therefore, $c_\ell \leq c_\ell^*$ and by symmetry we obtain $c_\ell = c_\ell^*$. □

A standard indirect argument shows the following lemma.

**Lemma 2.3.** *Let $D(\mathcal{A}_\ell) = D(\mathrm{A}_\ell)\cap \overline{R(\mathrm{A}_\ell^*)} \hookrightarrow H_\ell$ be compact. Then the assertions of Lemma 2.1 and Lemma 2.2 hold. Moreover, the inverse operators*

$$\mathcal{A}_\ell^{-1} : R(\mathrm{A}_\ell) \to R(\mathrm{A}_\ell^*), \quad (\mathcal{A}_\ell^*)^{-1} : R(\mathrm{A}_\ell^*) \to R(\mathrm{A}_\ell)$$

*are compact with norms*

$$|\mathcal{A}_\ell^{-1}|_{R(\mathrm{A}_\ell),R(\mathrm{A}_\ell^*)} = |(\mathcal{A}_\ell^*)^{-1}|_{R(\mathrm{A}_\ell^*),R(\mathrm{A}_\ell)} = c_\ell.$$

*Proof.* If, e.g., Lemma 2.1 (i) was wrong, there exists a sequence $(x_n) \subset D(\mathcal{A}_\ell)$ with $|x_n|_{H_\ell} = 1$ and $\mathrm{A}_\ell x_n \to 0$. As $(x_n)$ is bounded in $D(\mathcal{A}_\ell)$ we can extract a subsequence, again denoted by $(x_n)$, with $x_n \to x \in H_\ell$ in $\mathrm{H}_\ell$.

Since $\mathrm{A}_\ell$ is closed, we have $x \in D(\mathrm{A}_\ell)$ and $\mathrm{A}_\ell x = 0$. Hence $x \in N(\mathrm{A}_\ell)$. On the other hand, $(x_n) \subset D(\mathcal{A}_\ell) \subset \overline{R(\mathrm{A}_\ell^*)} = N(\mathrm{A}_\ell)^\perp$ implies $x \in N(\mathrm{A}_\ell)^\perp$. Thus $x = 0$, in contradiction to $1 = |x_n|_{H_\ell} \to |x|_{H_\ell} = 0$. □

**Lemma 2.4.** *The embedding $D(\mathcal{A}_\ell) \hookrightarrow H_\ell$ is compact, if and only if the embedding $D(\mathcal{A}_\ell^*) \hookrightarrow H_{\ell+1}$ is compact. In this case all assertions of Lemma 2.1 and Lemma 2.2 are valid.*

*Proof.* By symmetry it is enough to show one direction. Let, e.g., the embedding $D(\mathcal{A}_\ell) \hookrightarrow H_\ell$ be compact. By Lemma 2.1 and Lemma 2.3, especially $R(\mathrm{A}_\ell) = R(\mathcal{A}_\ell)$ and $R(\mathrm{A}_\ell^*) = R(\mathcal{A}_\ell^*)$ are closed. Let $(y_n) \subset D(\mathcal{A}_\ell^*) = D(\mathrm{A}_\ell^*) \cap R(\mathrm{A}_\ell)$ be a $D(\mathrm{A}_\ell^*)$-bounded sequence. We pick a sequence $(x_n) \subset D(\mathcal{A}_\ell)$ with $y_n = \mathrm{A}_\ell x_n$, i.e., $x_n = \mathcal{A}_\ell^{-1} y_n$. As $\mathcal{A}_\ell^{-1} : R(\mathrm{A}_\ell) \to D(\mathcal{A}_\ell)$ is continuous, $(x_n)$ is bounded in $D(\mathcal{A}_\ell)$ and thus contains a subsequence, again denoted by $(x_n)$, converging in $\mathrm{H}_\ell$ to some $x \in H_\ell$. Now

$$\begin{aligned} |y_n - y_m|_{H_{\ell+1}}^2 &= \langle y_n - y_m, \mathrm{A}_\ell (x_n - x_m) \rangle_{H_{\ell+1}} \\ &= \langle \mathrm{A}_\ell^* (y_n - y_m), x_n - x_m \rangle_{H_\ell} \le c \; |x_n - x_m|_{H_\ell} \end{aligned}$$

as $(y_n)$ is $D(\mathrm{A}_\ell^*)$-bounded. Finally, we see that $(y_n)$ is a Cauchy sequence in $H_{\ell+1}$. □

Let us summarize:

**Corollary 2.5.** *Let $R(\mathrm{A}_\ell)$ be closed. Then*

$$\frac{1}{c_\ell} = \inf_{0 \neq x \in D(\mathcal{A}_\ell)} \frac{|\mathrm{A}_\ell x|_{H_{\ell+1}}}{|x|_{H_\ell}} = \inf_{y \in D(\mathcal{A}_\ell^*)} \frac{|\mathrm{A}_\ell^* y|_{H_\ell}}{|y|_{H_{\ell+1}}}$$

*exists in $(0, \infty)$. Furthermore:*

(i) *The Poincaré type estimates*

$$\forall \; x \in D(\mathcal{A}_\ell) \qquad |x|_{H_\ell} \le c_\ell |\mathrm{A}_\ell x|_{H_{\ell+1}},$$
$$\forall \; y \in D(\mathcal{A}_\ell^*) \qquad |y|_{H_{\ell+1}} \le c_\ell |\mathrm{A}_\ell^* y|_{H_\ell}$$

*hold.*

(ii) *The ranges $R(\mathrm{A}_\ell) = R(\mathcal{A}_\ell)$ and $R(\mathrm{A}_\ell^*) = R(\mathcal{A}_\ell^*)$ are closed. Moreover, $D(\mathcal{A}_\ell) = D(\mathrm{A}_\ell) \cap R(\mathrm{A}_\ell^*)$ and $D(\mathcal{A}_\ell^*) = D(\mathrm{A}_\ell^*) \cap R(\mathrm{A}_\ell)$ with*

$$\mathcal{A}_\ell : D(\mathcal{A}_\ell) \subset R(\mathrm{A}_\ell^*) \to R(\mathrm{A}_\ell), \quad \mathcal{A}_\ell^* : D(\mathcal{A}_\ell^*) \subset R(\mathrm{A}_\ell) \to R(\mathrm{A}_\ell^*).$$

(iii) *The Helmholtz type decompositions*

$$H_\ell = N(\mathrm{A}_\ell) \oplus_{H_\ell} R(\mathrm{A}_\ell^*), \qquad H_{\ell+1} = N(\mathrm{A}_\ell^*) \oplus_{H_{\ell+1}} R(\mathrm{A}_\ell),$$
$$D(\mathrm{A}_\ell) = N(\mathrm{A}_\ell) \oplus_{H_\ell} D(\mathcal{A}_\ell), \quad D(\mathrm{A}_\ell^*) = N(\mathrm{A}_\ell^*) \oplus_{H_{\ell+1}} D(\mathcal{A}_\ell^*)$$

*hold.*

iii. *The inverse operators*

$$\mathcal{A}_\ell^{-1}: R(\mathrm{A}_\ell) \to D(\mathcal{A}_\ell), \quad (\mathcal{A}_\ell^*)^{-1}: R(\mathrm{A}_\ell^*) \to D(\mathcal{A}_\ell^*)$$

*are continuous and bijective with norms* $|\mathcal{A}_\ell^{-1}|_{R(\mathrm{A}_\ell),D(\mathcal{A}_\ell)} = |(\mathcal{A}_\ell^*)^{-1}|_{R(\mathrm{A}_\ell^*),D(\mathcal{A}_\ell^*)} = (1+c_\ell^2)^{1/2}$ *and* $|\mathcal{A}_\ell^{-1}|_{R(\mathrm{A}_\ell),R(\mathrm{A}_\ell^*)} = |(\mathcal{A}_\ell^*)^{-1}|_{R(\mathrm{A}_\ell^*),R(\mathrm{A}_\ell)} = c_\ell$.

**Corollary 2.6.** *Let $D(\mathcal{A}_\ell) \hookrightarrow H_\ell$ be compact. Then $R(\mathrm{A}_\ell)$ is closed and the assertions of Corollary 2.5 hold. Moreover, the inverse operators*

$$\mathcal{A}_\ell^{-1}: R(\mathrm{A}_\ell) \to R(\mathrm{A}_\ell^*), \quad (\mathcal{A}_\ell^*)^{-1}: R(\mathrm{A}_\ell^*) \to R(\mathrm{A}_\ell)$$

*are compact.*

So far, we did not use the complex property (1.1). Hence, Lemma 2.1, Lemma 2.2, Lemma 2.3, Lemma 2.4, and Corollary 2.5, Corollary 2.6 hold without the complex property (1.1). Now the complex property (1.1) enters the theory. Recall the Helmholtz type decompositions (2.1) in the form

$$H_\ell = N(\mathrm{A}_\ell) \oplus_{H_\ell} \overline{R(\mathrm{A}_\ell^*)} = \overline{R(\mathrm{A}_{\ell-1})} \oplus_{H_\ell} N(\mathrm{A}_{\ell-1}^*)$$

hold. Then the complex properties (1.1) and (1.2) yield

$$N(\mathrm{A}_\ell) = \overline{R(\mathrm{A}_{\ell-1})} \oplus_{H_\ell} K_\ell, \quad N(\mathrm{A}_{\ell-1}^*) = K_\ell \oplus_{H_\ell} \overline{R(\mathrm{A}_\ell^*)}, \quad K_\ell = N(\mathrm{A}_\ell) \cap N(\mathrm{A}_{\ell-1}^*).$$

Therefore, we get the refined Helmholtz type decomposition

$$H_\ell = \overline{R(\mathrm{A}_{\ell-1})} \oplus_{H_\ell} K_\ell \oplus_{H_\ell} \overline{R(\mathrm{A}_\ell^*)}. \tag{2.6}$$

**Lemma 2.7.** *The refined Helmholtz type decompositions*

$$\begin{aligned}
H_\ell &= \overline{R(\mathrm{A}_{\ell-1})} \oplus_{H_\ell} K_\ell \oplus_{H_\ell} \overline{R(\mathrm{A}_\ell^*)}, & K_\ell &= N(\mathrm{A}_\ell) \cap N(\mathrm{A}_{\ell-1}^*),\\
N(\mathrm{A}_\ell) &= \overline{R(\mathrm{A}_{\ell-1})} \oplus_{H_\ell} K_\ell, & N(\mathrm{A}_{\ell-1}^*) &= K_\ell \oplus_{H_\ell} \overline{R(\mathrm{A}_\ell^*)},\\
\overline{R(\mathcal{A}_{\ell-1})} = \overline{R(\mathrm{A}_{\ell-1})} &= N(\mathrm{A}_\ell) \ominus_{H_\ell} K_\ell, & \overline{R(\mathcal{A}_\ell^*)} = \overline{R(\mathrm{A}_\ell^*)} &= N(\mathrm{A}_{\ell-1}^*) \ominus_{H_\ell} K_\ell,\\
D(\mathrm{A}_\ell) &= \overline{R(\mathrm{A}_{\ell-1})} \oplus_{H_\ell} K_\ell \oplus_{H_\ell} D(\mathcal{A}_\ell), & D(\mathrm{A}_{\ell-1}^*) &= D(\mathcal{A}_{\ell-1}^*) \oplus_{H_\ell} K_\ell \oplus_{H_\ell} \overline{R(\mathrm{A}_\ell^*)},\\
D_\ell &= D(\mathcal{A}_{\ell-1}^*) \oplus_{H_\ell} K_\ell \oplus_{H_\ell} D(\mathcal{A}_\ell), & D_\ell &= D(\mathrm{A}_\ell) \cap D(\mathrm{A}_{\ell-1}^*)
\end{aligned}$$

*hold. If the range $R(\mathrm{A}_{\ell-1})$ or $R(\mathrm{A}_\ell)$ is closed, the respective closure bars can be dropped and the assertions of Corollary 2.5 are valid. Especially, if $R(\mathrm{A}_{\ell-1})$ and $R(\mathrm{A}_\ell)$ are closed, the assertions of Corollary 2.5 and the refined Helmholtz type decompositions*

$$\begin{aligned}
H_\ell &= R(\mathrm{A}_{\ell-1}) \oplus_{H_\ell} K_\ell \oplus_{H_\ell} R(\mathrm{A}_\ell^*), & K_\ell &= N(\mathrm{A}_\ell) \cap N(\mathrm{A}_{\ell-1}^*),\\
N(\mathrm{A}_\ell) &= R(\mathrm{A}_{\ell-1}) \oplus_{H_\ell} K_\ell, & N(\mathrm{A}_{\ell-1}^*) &= K_\ell \oplus_{H_\ell} R(\mathrm{A}_\ell^*),\\
R(\mathcal{A}_{\ell-1}) = R(\mathrm{A}_{\ell-1}) &= N(\mathrm{A}_\ell) \ominus_{H_\ell} K_\ell, & R(\mathcal{A}_\ell^*) = R(\mathrm{A}_\ell^*) &= N(\mathrm{A}_{\ell-1}^*) \ominus_{H_\ell} K_\ell,\\
D(\mathrm{A}_\ell) &= R(\mathrm{A}_{\ell-1}) \oplus_{H_\ell} K_\ell \oplus_{H_\ell} D(\mathcal{A}_\ell), & D(\mathrm{A}_{\ell-1}^*) &= D(\mathcal{A}_{\ell-1}^*) \oplus_{H_\ell} K_\ell \oplus_{H_\ell} R(\mathrm{A}_\ell^*),\\
D_\ell &= D(\mathcal{A}_{\ell-1}^*) \oplus_{H_\ell} K_\ell \oplus_{H_\ell} D(\mathcal{A}_\ell), & D_\ell &= D(\mathrm{A}_\ell) \cap D(\mathrm{A}_{\ell-1}^*)
\end{aligned}$$

*hold.*

Observe that

$$\begin{aligned} D(\mathcal{A}_\ell) &= D(\mathrm{A}_\ell) \cap \overline{R(\mathrm{A}_\ell^*)} \subset D(\mathrm{A}_\ell) \cap N(\mathrm{A}_{\ell-1}^*) \subset D(\mathrm{A}_\ell) \cap D(\mathrm{A}_{\ell-1}^*) = D_\ell, \\ D(\mathcal{A}_{\ell-1}^*) &= D(\mathrm{A}_{\ell-1}^*) \cap \overline{R(\mathrm{A}_{\ell-1})} \subset D(\mathrm{A}_{\ell-1}^*) \cap N(\mathrm{A}_\ell) \subset D(\mathrm{A}_{\ell-1}^*) \cap D(\mathrm{A}_\ell) = D_\ell. \end{aligned} \tag{2.7}$$

**Lemma 2.8.** *The embeddings $D(\mathcal{A}_\ell) \hookrightarrow H_\ell, D(\mathcal{A}_{\ell-1}) \hookrightarrow H_{\ell-1}$, and $K_\ell \hookrightarrow H_\ell$ are compact, if and only if the embedding $D_\ell \hookrightarrow H_\ell$ is compact. In this case, $K_\ell$ has finite dimension.*

*Proof.* Note that, by Lemma 2.4, $D(\mathcal{A}_{\ell-1}) \hookrightarrow H_{\ell-1}$ is compact, if and only if $D(\mathcal{A}_{\ell-1}^*) \hookrightarrow H_\ell$ is compact.

$\Rightarrow$ Let $(x_n) \subset D_\ell$ be a $D_\ell$-bounded sequence. By the refined Helmholtz type decomposition of Lemma 2.7 we decompose

$$x_n = a_n^* + k_n + a_n \in D(\mathcal{A}_{\ell-1}^*) \oplus_{H_\ell} K_\ell \oplus_{H_\ell} D(\mathcal{A}_\ell).$$

with $\mathrm{A}_\ell x_n = \mathrm{A}_\ell a_n$ and $\mathrm{A}_{\ell-1}^* x_n = \mathrm{A}_{\ell-1}^* a_n^*$. Hence $(a_n)$ is bounded in $D(\mathcal{A}_\ell)$ and $(a_n^*)$ is bounded in $D(\mathcal{A}_{\ell-1}^*)$ and we can extract $\mathrm{H}_\ell$-converging subsequences of $(a_n), (a_n^*)$, and $(k_n)$.

$\Leftarrow$: If $D_\ell \hookrightarrow H_\ell$ is compact, so is $K_\ell \hookrightarrow H_\ell$. Moreover, by (2.7)

$$D(\mathcal{A}_\ell) \subset D_\ell \hookrightarrow H_\ell, \quad D(\mathcal{A}_{\ell-1}^*) \subset D_\ell \hookrightarrow H_\ell.$$

Finally, if $K_\ell \hookrightarrow H_\ell$ is compact, the unit ball in $K_\ell$ is compact, showing that $K_\ell$ has finite dimension. □

**Lemma 2.8.** implies immediately the following result.

**Corollary 2.9.** *Let $D_\ell \hookrightarrow H_\ell$ be compact. Then $R(\mathrm{A}_{\ell-1})$ and $R(\mathrm{A}_\ell)$ are closed, and, besides the assertions of Corollary 2.6, the refined Helmholtz type decompositions of Lemma 2.7 hold and the cohomology group $K_\ell$ is finite dimensional.*

**Remark 2.10.** *Under the assumption that the embedding $D_\ell \hookrightarrow H_\ell$ is compact, all the assertions of this section hold. Especially, the (short) complex*

$$D(\mathrm{A}_{\ell\text{-}1}) \xrightarrow{\mathrm{A}_{\ell\text{-}1}} D(\mathrm{A}_\ell) \xrightarrow{\mathrm{A}_\ell} \mathrm{H}_{\ell+1}$$

*together with its adjoint complex*

$$\mathrm{H}_{\ell\text{-}1} \xleftarrow{\mathrm{A}_{\ell\text{-}1}^*} D(\mathrm{A}_{\ell\text{-}1}^*) \xleftarrow{\mathrm{A}_\ell^*} D(\mathrm{A}_\ell^*)$$

*is closed. These complexes are even exact, if additionally $K_\ell = \{0\}$.*

Defining and recalling the orthonormal projectors

$$\pi_{A_{\ell-1}} := \pi_{\overline{R(A_{\ell-1})}} : H_\ell \to \overline{R(A_{\ell-1})}, \quad \pi_{A_\ell^*} := \pi_{\overline{R(A_\ell^*)}} : H_\ell \to \overline{R(A_\ell^*)}, \quad \pi_\ell : H_\ell \to K_\ell, \tag{2.8}$$

we have $\pi_\ell = 1 - \pi_{A_{\ell-1}} - \pi_{A_\ell^*}$ as well as

$$\begin{aligned} \pi_{A_{\ell-1}} H_\ell &= \pi_{A_{\ell-1}} D(A_\ell) = \pi_{A_{\ell-1}} N(A_\ell) = \overline{R(A_{\ell-1})} = \overline{R(\mathcal{A}_{\ell-1})}, \\ \pi_{A_\ell^*} H_\ell &= \pi_{A_\ell^*} D(A_{\ell-1}^*) = \pi_{A_\ell^*} N(A_{\ell-1}^*) = \overline{R(A_\ell^*)} = \overline{R(\mathcal{A}_\ell^*)} \end{aligned}$$

and

$$\pi_{A_{\ell-1}} D(A_{\ell-1}^*) = \pi_{A_{\ell-1}} D_\ell = D(\mathcal{A}_{\ell-1}^*), \quad \pi_{A_\ell^*} D(A_\ell) = \pi_{A_\ell^*} D_\ell = D(\mathcal{A}_\ell).$$

Moreover

$$\begin{array}{lll} \forall \xi \in D(A_{\ell-1}^*) & \pi_{A_{\ell-1}} \xi \in D(\mathcal{A}_{\ell-1}^*) & \wedge \quad A_{\ell-1}^* \pi_{A_{\ell-1}} \xi = A_{\ell-1}^* \xi, \\ \forall \zeta \in D(A_\ell) & \pi_{A_\ell^*} \zeta \in D(\mathcal{A}_\ell) & \wedge \quad A_\ell \pi_{A_\ell^*} \zeta = A_\ell \zeta. \end{array}$$

We also introduce the orthogonal projectors onto the kernels

$$\pi_{N(A_{\ell-1}^*)} := 1 - \pi_{A_{\ell-1}} : H_\ell \to N(A_{\ell-1}^*), \quad \pi_{N(A_\ell)} := 1 - \pi_{A_\ell^*} : H_\ell \to N(A_\ell).$$

## 3. Solution theory and variational formulations

From now on and throughout this article we suppose the following.

**General Assumption 3.1.** $R(A_1)$ and $R(A_2)$ are closed and $K_2$ is finite dimensional.

Remark 3.2. *The General Assumption 3.1 is satisfied, if, e.g., $D_2 \hookrightarrow H_2$ is compact. The finite dimension of the cohomology group $K_2$ may be dropped.*

### *3.1. First order systems*

We recall the linear first order system (1.5) from the introduction: Find $x \in D_2 = D(A_2) \cap D(A_1^*)$ such that

$$\begin{aligned} A_2 x &= f, \\ A_1^* x &= g, \\ \pi_2 x &= k. \end{aligned} \tag{3.1}$$

Theorem 3.3. (3.1) *is uniquely solvable in $D_2$, if and only if $f \in R(A_2), g \in R(A_1^*)$, and $k \in K_2$. The unique solution $x \in D_2$ is given by*

$$\begin{aligned}
x &:= x_f + x_g + k \in D(\mathcal{A}_2) \oplus_{H_2} D(\mathcal{A}_1^*) \oplus_{H_2} K_2 = D_2, \\
x_f &:= \mathcal{A}_2^{-1} f \in D(\mathcal{A}_2) = D(\mathcal{A}_2) \cap D_2, \\
x_g &:= (\mathcal{A}_1^*)^{-1} g \in D(\mathcal{A}_1^*) = D(\mathcal{A}_1^*) \cap D_2
\end{aligned}$$

*and depends continuously on the data, i.e.,* $|x|_{H_2} \leq c_2 |f|_{H_3} + c_1 |g|_{H_1} + |k|_{H_2}$, *as*

$$|x_f|_{H_2} \leq c_2 |f|_{H_3}, \quad |x_g|_{H_2} \leq c_1 |g|_{H_1}.$$

*It holds*

$$\pi_{\mathrm{A}_2^*} x = x_f, \quad \pi_{\mathrm{A}_1} x = x_g, \quad \pi_2 x = k, \quad |x|_{H_2}^2 = |x_f|_{H_2}^2 + |x_g|_{H_2}^2 + |k|_{H_2}^2.$$

*Proof.* As pointed out in the introduction, we just need to show existence. We use the results of Section 2. Let $f \in R(\mathrm{A}_2), g \in R(\mathrm{A}_1^*), k \in K_2$ and define $x$, $x_f$, and $x_g$ according to the theorem. For the orthogonality we refer to Lemma 2.7. Moreover, $x_f$, $x_g$, and $k$ solve the linear systems

$$\begin{aligned}
\mathrm{A}_2 x_f &= f, & \mathrm{A}_2 x_g &= 0, & \mathrm{A}_2 k &= 0, \\
\mathrm{A}_1^* x_f &= 0, & \mathrm{A}_1^* x_g &= g, & \mathrm{A}_1^* k &= 0, \\
\pi_2 x_f &= 0, & \pi_2 x_g &= 0, & \pi_2 k &= k.
\end{aligned}$$

Thus $x$ solves (3.1) and we have by Corollary 2.5 $|x_f|_{H_2} \leq c_2 |f|_{H_3}$ and $|x_g|_{H_2} \leq c_1 |g|_{H_1}$, which completes the proof of the solution theory. □

**Remark 3.4.** *By orthogonality and with* $\mathrm{A}_2 x = \mathrm{A}_2 x_f = f$ *and* $\mathrm{A}_1^* x = \mathrm{A}_1^* x_g = g$ *we even have*

$$\begin{aligned}
|x|_{H_2}^2 &= |x_f|_{H_2}^2 + |x_g|_{H_2}^2 + |k|_{H_2}^2 \leq c_2^2 |f|_{H_3}^2 + c_1^2 |g|_{H_1}^2 + |k|_{H_2}^2, \\
|x|_{D_2}^2 &= |x_f|_{H_2}^2 + |f|_{H_3}^2 + |x_g|_{H_2}^2 + |g|_{H_1}^2 + |k|_{H_2}^2 \leq (1 + c_2^2) |f|_{H_3}^2 + (1 + c_1^2) |g|_{H_1}^2 + |k|_{H_2}^2.
\end{aligned}$$

### 3.1.1. Variational formulations

Recall the partial solutions

$$\begin{aligned}
x_f &:= \mathcal{A}_2^{-1} f & &\in D(\mathcal{A}_2) = D(\mathrm{A}_2) \cap R(\mathrm{A}_2^*) = D(\mathrm{A}_2) \cap N(\mathrm{A}_1^*) \cap K_2^{\perp_{H_2}}, \\
x_g &:= (\mathcal{A}_1^*)^{-1} g & &\in D(\mathcal{A}_1^*) = D(\mathrm{A}_1^*) \cap R(\mathrm{A}_1) = D(\mathrm{A}_1^*) \cap N(\mathrm{A}_2) \cap K_2^{\perp_{H_2}}.
\end{aligned} \tag{3.2}$$

There are at least two obvious ways to get variational formulations for finding each of the partial solutions $x_f$ and $x_g$. Looking at $x_f \in D(\mathcal{A}_2)$ the first idea is to multiply the equation $\mathrm{A}_2 x_f = f$ by $\mathrm{A}_2 \xi$ with some $\xi \in D(\mathcal{A}_2)$ leading to

$$\forall\ \xi \in D(\mathcal{A}_2) \qquad \langle \mathrm{A}_2 x_f, \mathrm{A}_2 \xi \rangle_{H_3} = \langle f, \mathrm{A}_2 \xi \rangle_{H_3},$$

which is a weak formulation of the second order equation

$$\mathrm{A}_2^*\mathrm{A}_2 x_f = \mathrm{A}_2^* f,$$

more precisely of $\mathrm{A}_2^*(\mathrm{A}_2 x_f - f) = 0$. While the latter choice was straight forward to find $x_f$ itself, the next choice searches for a potential $y_f$, e.g., $y_f := (\mathcal{A}_2^*)^{-1} x_f \in D(\mathcal{A}_2^*)$, of

$$x_f = \mathrm{A}_2^* y_f \in D(\mathcal{A}_2) = D(\mathrm{A}_2) \cap R(\mathrm{A}_2^*) = D(\mathrm{A}_2) \cap R(\mathcal{A}_2^*),$$

see Remark 3.4. Multiplying by $\mathrm{A}_2^*\phi$ with some $\phi \in D(\mathcal{A}_2^*)$ gives

$$\forall\ \phi \in D(\mathcal{A}_2^*) \qquad \langle \mathrm{A}_2^* y_f, \mathrm{A}_2^*\phi \rangle_{H_2} = \langle x_f, \mathrm{A}_2^*\phi \rangle_{H_2} = \langle \mathrm{A}_2 x_f, \phi \rangle_{H_3} = \langle f, \phi \rangle_{H_3},$$

which is a weak formulation of the second order equation

$$\mathrm{A}_2\mathrm{A}_2^* y_f = f.$$

Similar ideas apply to find corresponding variational formulations for $x_g$ as well.

**Theorem 3.5.** *The partial solutions $x_f$ and $x_g$ in Theorem 3.3 can be found by the following four variational formulations:*

(i) *There exists a unique $\widetilde{x}_f \in D(\mathcal{A}_2)$ such that*

$$\forall\ \xi \in D(\mathcal{A}_2) \qquad \langle \mathrm{A}_2 \widetilde{x}_f, \mathrm{A}_2 \xi \rangle_{H_3} = \langle f, \mathrm{A}_2 \xi \rangle_{H_3}. \tag{3.3}$$

(3.3) *is even satisfied for all $\xi \in D(\mathrm{A}_2)$. Moreover, $\mathrm{A}_2 \widetilde{x}_f = f$ holds if and only if $f \in R(\mathrm{A}_2)$. In this case $\widetilde{x}_f = x_f$.*

(i') *There exists a unique potential $y_f \in D(\mathcal{A}_2^*)$ such that*

$$\forall\ \phi \in D(\mathcal{A}_2^*) \qquad \langle \mathrm{A}_2^* y_f, \mathrm{A}_2^* \phi \rangle_{H_2} = \langle f, \phi \rangle_{H_3}. \tag{3.4}$$

(3.4) *even holds for all $\phi \in D(\mathrm{A}_2^*)$ if and only if $f \in R(\mathrm{A}_2)$. In this case we have*

$$\mathrm{A}_2^* y_f \in D(\mathrm{A}_2) \cap R(\mathrm{A}_2^*) = D(\mathcal{A}_2)$$

*with $\mathrm{A}_2\mathrm{A}_2^* y_f = f$ and hence $\mathrm{A}_2^* y_f = x_f$.*

(ii) *There exists a unique $\widetilde{x}_g \in D(\mathcal{A}_1^*)$ such that*

$$\forall\ \zeta \in D(\mathcal{A}_1^*) \qquad \langle \mathrm{A}_1^* \widetilde{x}_g, \mathrm{A}_1^* \zeta \rangle_{H_1} = \langle g, \mathrm{A}_1^* \zeta \rangle_{H_1}. \tag{3.5}$$

(3.5) *is even satisfied for all $\zeta \in D(\mathrm{A}_1^*)$. Moreover, $\mathrm{A}_1^* \widetilde{x}_g = g$ holds if and only if $g \in R(\mathrm{A}_1^*)$. In this case $\widetilde{x}_g = x_g$.*

(ii') *There exists a unique potential* $z_g \in D(\mathcal{A}_1)$ *such that*

$$\forall\ \varphi \in D(\mathcal{A}_1) \qquad \langle \mathrm{A}_1 z_g, \mathrm{A}_1 \varphi \rangle_{H_2} = \langle g, \varphi \rangle_{H_1}. \tag{3.6}$$

(3.6) *even holds for all* $\varphi \in D(\mathrm{A}_1)$ *if and only if* $g \in R(\mathrm{A}_1^*)$. *In this case we have*

$$\mathrm{A}_1 z_g \in D\big(\mathrm{A}_1^*\big) \cap R(\mathrm{A}_1) = D\big(\mathcal{A}_1^*\big)$$

*with* $\mathrm{A}_1^* \mathrm{A}_1 z_g = g$ *and thus* $\mathrm{A}_1 z_g = x_g$

*Proof.* Equation (3.3) is strictly positive (or coercive) over $D(\mathcal{A}_2)$ by the Friedrichs/Poincaré type estimates of Corollary 2.5 (i) and hence a unique $\widetilde{x}_f \in D(\mathcal{A}_2)$ exists by Riesz' representation theorem (or Lax–Milgram's lemma) solving (3.3). By (2.3), i.e., $R(\mathcal{A}_2) = R(\mathrm{A}_2)$, (3.3) holds for all $\xi \in D(\mathrm{A}_2)$. Hence

$$\forall\ \xi \in D(\mathrm{A}_2) \qquad \langle \mathrm{A}_2 \widetilde{x}_f - f, \mathrm{A}_2 \xi \rangle_{H_3} = 0,$$

yielding $\mathrm{A}_2 \widetilde{x}_f - f \in R(\mathrm{A}_2)^{\perp_{H_3}}$. Thus, if $f \in R(\mathrm{A}_2)$, we see $\mathrm{A}_2 \widetilde{x}_f - f \in R(\mathrm{A}_2) \cap R(\mathrm{A}_2)^{\perp_{H_3}} = \{0\}$, i.e., $\mathrm{A}_2 \widetilde{x}_f = f$. As $\widetilde{x}_f \in D(\mathcal{A}_2)$ conclude $\widetilde{x}_f = x_f$ by the injectivity of $\mathcal{A}_2$, which completes the proof of (i).

Equation (3.4) is strictly positive over $D(\mathcal{A}_2^*)$ by Corollary 2.5 (i) and thus a unique $y_f \in D(\mathcal{A}_2^*)$ exists by Riesz' representation theorem solving (3.4). Using Corollary 2.5 (iii) or Lemma 2.7 we can split any $\phi \in D(\mathrm{A}_2^*) = N(\mathrm{A}_2^*) \oplus_{H_3} D(\mathcal{A}_2^*)$ into $\phi = \phi_N + \phi_R$ (null space and range) with $\phi_N \in N(\mathrm{A}_2^*)$, $\phi_R \in D(\mathcal{A}_2^*)$, and $\mathrm{A}_2^* \phi = \mathrm{A}_2^* \phi_R$. Let $f \in R(\mathrm{A}_2)$. Utilizing (3.4) for $\phi_R$ and orthogonality, i.e., $f \in R(\mathrm{A}_2) = N(\mathrm{A}_2^*)^{\perp_{H_3}}$, we get

$$\langle \mathrm{A}_2^* y_f, \mathrm{A}_2^* \phi \rangle_{H_2} = \langle \mathrm{A}_2^* y_f, \mathrm{A}_2^* \phi_R \rangle_{H_2} = \langle f, \phi_R \rangle_{H_3} = \langle f, \phi \rangle_{H_3}.$$

Therefore, (3.4) holds for all $\phi \in D(\mathrm{A}_2^*)$. On the other hand, if (3.4) holds for all $\phi \in D(\mathrm{A}_2^*)$, then $\mathrm{A}_2^* y_f \in D(\mathrm{A}_2)$ with $\mathrm{A}_2 \mathrm{A}_2^* y_f = f$. Hence[6] $f \in R(\mathrm{A}_2)$. Therefore, if $f$ belongs to $R(\mathrm{A}_2)$, we obtain $\mathrm{A}_2^* y_f \in D(\mathrm{A}_2) \cap R(\mathrm{A}_2^*) = D(\mathcal{A}_2)$ with $\mathrm{A}_2 \mathrm{A}_2^* y_f = f$ and hence $\mathrm{A}_2^* y_f = x_f$, again by the injectivity of $\mathcal{A}_2$.

Analogously, we prove (ii) and (ii'). □

[6]Another proof is the following: Pick $\phi \in N(\mathrm{A}_2^*)$ and get by (3.4) directly $\langle f, \phi \rangle_{H_3} = 0$. Thus $f \in N(\mathrm{A}_2^*)^{\perp_{H_3}} = R(\mathrm{A}_2)$.

**Remark 3.6.** *Note that*

$$x_f = \mathcal{A}_2^{-1} f \in D(\mathcal{A}_2), \qquad x_g = (\mathcal{A}_1^*)^{-1} g \in D(\mathcal{A}_1^*),$$
$$y_f = (\mathcal{A}_2^*)^{-1} x_f = (\mathcal{A}_2^*)^{-1} \mathcal{A}_2^{-1} f \in D(\mathcal{A}_2^*), \quad z_g = \mathcal{A}_1^{-1} x_g = \mathcal{A}_1^{-1} (\mathcal{A}_1^*)^{-1} g \in D(\mathcal{A}_1)$$

*hold with* $\mathrm{A}_2\mathrm{A}_2^* y_f = f$ *and* $\mathrm{A}_1^*\mathrm{A}_1 z_g = g$. *Hence* $x_f$, $x_g$, $k$, *and* $y_f$, $z_g$ *solve the first resp. second order systems*

$$\begin{array}{lllllll} \mathrm{A}_2 x_f = f, & \mathrm{A}_2 x_g = 0, & \mathrm{A}_2 k = 0, & \mathrm{A}_2^* y_f = x_f, & \mathrm{A}_2\mathrm{A}_2^* y_f = f, & \mathrm{A}_1 z_g = x_g, & \mathrm{A}_1^*\mathrm{A}_1 z_g = g, \\ \mathrm{A}_1^* x_f = 0, & \mathrm{A}_1^* x_g = g, & \mathrm{A}_1^* k = 0, & \mathrm{A}_3 y_f = 0, & \mathrm{A}_3 y_f = 0, & \mathrm{A}_0^* z_g = 0, & \mathrm{A}_0^* z_g = 0, \\ \pi_2 x_f = 0, & \pi_2 x_g = 0, & \pi_2 k = k, & \pi_3 y_f = 0, & \pi_3 y_f = 0, & \pi_1 z_g = 0, & \pi_1 z_g = 0. \end{array}$$

*Moreover:*

(i) *Equation (3.3) is a weak formulation of*

$$\mathrm{A}_2^*\mathrm{A}_2 \widetilde{x}_f = \mathrm{A}_2^* f, \quad \mathrm{A}_1^* \widetilde{x}_f = 0, \quad \pi_2 \widetilde{x}_f = 0,$$

*i.e., in formal matrix notation*

$$\begin{bmatrix} \mathrm{A}_2^*\mathrm{A}_2 \\ \mathrm{A}_1^* \\ \pi_2 \end{bmatrix} \begin{bmatrix} \widetilde{x}_f \end{bmatrix} = \begin{bmatrix} \mathrm{A}_2^* f \\ 0 \\ 0 \end{bmatrix}.$$

(i') Equation (3.4) *is a weak formulation of*

$$\mathrm{A}_2\mathrm{A}_2^* y_f = f, \quad \mathrm{A}_3 y_f = 0, \quad \pi_3 y_f = 0,$$

*i.e., in formal matrix notation*

$$\begin{bmatrix} \mathrm{A}_2\mathrm{A}_2^* \\ \mathrm{A}_3 \\ \pi_3 \end{bmatrix} \begin{bmatrix} y_f \end{bmatrix} = \begin{bmatrix} f \\ 0 \\ 0 \end{bmatrix}.$$

(ii) *Equation (3.5) is a weak formulation of*

$$\mathrm{A}_1\mathrm{A}_1^* \widetilde{x}_g = \mathrm{A}_1 g, \quad \mathrm{A}_2 \widetilde{x}_g = 0, \quad \pi_2 \widetilde{x}_g = 0,$$

*i.e., in formal matrix notation*

$$\begin{bmatrix} \mathrm{A}_1\mathrm{A}_1^* \\ \mathrm{A}_2 \\ \pi_2 \end{bmatrix} \begin{bmatrix} \widetilde{x}_g \end{bmatrix} = \begin{bmatrix} \mathrm{A}_1 g \\ 0 \\ 0 \end{bmatrix}.$$

(ii') *Equation (3.6) is a weak formulation of*

$$\mathrm{A}_1^*\mathrm{A}_1 z_g = g, \quad \mathrm{A}_0^* z_g = 0, \quad \pi_1 z_g = 0,$$

*i.e., in formal matrix notation*

$$\begin{bmatrix} A_1^* A_1 \\ A_0^* \\ \pi_1 \end{bmatrix} \begin{bmatrix} z_g \end{bmatrix} = \begin{bmatrix} g \\ 0 \\ 0 \end{bmatrix}.$$

We also emphasize that the variational formulations (3.3)–(3.6) have a saddle point structure. We have already seen that, provided $f \in R(A_2)$ and $g \in R(A_1^*)$, the formulations (3.3)–(3.6) are equivalent to the following four problems: Find $\widetilde{x}_f \in D(\mathcal{A}_2), y_f \in D(\mathcal{A}_2^*)$, $\widetilde{x}_g \in D(\mathcal{A}_1^*)$, and $z_g \in D(\mathcal{A}_1)$, such that

$$\forall\ \xi \in D(A_2) \quad \langle A_2 \widetilde{x}_f, A_2 \xi \rangle_{H_3} = \langle f, A_2 \xi \rangle_{H_3}, \tag{3.7}$$

$$\forall\ \phi \in D(A_2^*) \quad \langle A_2^* y_f, A_2^* \phi \rangle_{H_2} = \langle f, \phi \rangle_{H_3}, \tag{3.8}$$

$$\forall\ \zeta \in D(A_1^*) \quad \langle A_1^* \widetilde{x}_g, A_1^* \zeta \rangle_{H_1} = \langle g, A_1^* \zeta \rangle_{H_1}, \tag{3.9}$$

$$\forall\ \varphi \in D(A_1) \quad \langle A_1 z_g, A_1 \varphi \rangle_{H_2} = \langle g, \varphi \rangle_{H_1}. \tag{3.10}$$

Note that in the end one needs only two out of these four formulations for computing

$$x_f = \widetilde{x}_f = A_2^* y_f, \quad x_g = \widetilde{x}_g = A_1 z_g.$$

Moreover,

$$\begin{aligned}
\widetilde{x}_f \in D(\mathcal{A}_2) = D(A_2) \cap R(A_2^*) \quad &\iff \quad \widetilde{x}_f \in D(A_2) \ \wedge \ \widetilde{x}_f \in R(A_2^*) = N(A_2)^{\perp_{H_2}}, \\
y_f \in D(\mathcal{A}_2^*) = D(A_2^*) \cap R(A_2) \quad &\iff \quad y_f \in D(A_2^*) \ \wedge \ y_f \in R(A_2) = N(A_2^*)^{\perp_{H_3}}, \\
\widetilde{x}_g \in D(\mathcal{A}_1^*) = D(A_1^*) \cap R(A_1) \quad &\iff \quad \widetilde{x}_g \in D(A_1^*) \ \wedge \ \widetilde{x}_g \in R(A_1) = N(A_1^*)^{\perp_{H_2}}, \\
z_g \in D(\mathcal{A}_1) = D(A_1) \cap R(A_1^*) \quad &\iff \quad z_g \in D(A_1) \ \wedge \ z_g \in R(A_1^*) = N(A_1)^{\perp_{H_1}}.
\end{aligned}$$

Therefore, the variational formulations (3.7)–(3.10) are equivalent to the following four saddle point problems: Find $\widetilde{x}_f \in D(A_2), y_f \in D(A_2^*)$, $\widetilde{x}_g \in D(A_1^*)$, and $z_g \in D(A_1)$, such that

$$\forall\ \xi \in D(A_2) \quad \langle A_2 \widetilde{x}_f, A_2 \xi \rangle_{H_3} = \langle f, A_2 \xi \rangle_{H_3} \quad \wedge \quad \forall\ \kappa \in N(A_2) \quad \langle \widetilde{x}_f, \kappa \rangle_{H_2} = 0, \tag{3.11}$$

$$\forall\ \phi \in D(A_2^*) \quad \langle A_2^* y_f, A_2^* \phi \rangle_{H_2} = \langle f, \phi \rangle_{H_3} \quad \wedge \quad \forall\ \theta \in N(A_2^*) \quad \langle y_f, \theta \rangle_{H_3} = 0, \tag{3.12}$$

$$\forall\ \zeta \in D(A_1^*) \quad \langle A_1^* \widetilde{x}_g, A_1^* \zeta \rangle_{H_1} = \langle g, A_1^* \zeta \rangle_{H_1} \quad \wedge \quad \forall\ \lambda \in N(A_1^*) \quad \langle \widetilde{x}_g, \lambda \rangle_{H_2} = 0, \tag{3.13}$$

$$\forall\ \varphi \in D(A_1) \quad \langle A_1 z_g, A_1 \varphi \rangle_{H_2} = \langle g, \varphi \rangle_{H_1} \quad \wedge \quad \forall\ \psi \in N(A_1) \quad \langle z_g, \psi \rangle_{H_1} = 0. \tag{3.14}$$

Let us additionally assume that $R(A_0)$ and $R(A_3)$ are closed. Using Lemma 2.7, i.e.,

$$N(\mathrm{A}_1) = R(\mathrm{A}_0) \oplus_{H_1} K_1, \quad N(\mathrm{A}_1^*) = R(\mathrm{A}_2^*) \oplus_{H_2} K_2, \tag{3.15}$$

$$N(\mathrm{A}_2) = R(\mathrm{A}_1) \oplus_{H_2} K_2, \quad N(\mathrm{A}_2^*) = R(\mathrm{A}_3^*) \oplus_{H_3} K_3, \tag{3.16}$$

the systems (3.11)–(3.14) may be further refined to the following four double saddle point formulations: Find $\widetilde{x}_f \in D(\mathrm{A}_2), y_f \in D(\mathrm{A}_2^*)$, $\widetilde{x}_g \in D(\mathrm{A}_1^*)$, and $z_g \in D(\mathrm{A}_1)$, such that

$$\forall\ \xi \in D(\mathrm{A}_2) \quad \langle \mathrm{A}_2\widetilde{x}_f, \mathrm{A}_2\xi\rangle_{H_3} = \langle f, \mathrm{A}_2\xi\rangle_{H_3} \quad \wedge \quad \forall\ \alpha \in D(\mathrm{A}_1) \quad \langle \widetilde{x}_f, \mathrm{A}_1\alpha\rangle_{H_2} = 0$$
$$\wedge \quad \forall\ \kappa \in K_2 \quad \langle \widetilde{x}_f, \kappa\rangle_{H_2} = 0, \tag{3.17}$$

$$\forall\ \phi \in D(\mathrm{A}_2^*) \quad \langle \mathrm{A}_2^* y_f, \mathrm{A}_2^*\phi\rangle_{H_2} = \langle f, \phi\rangle_{H_3} \quad \wedge \quad \forall\ \beta \in D(\mathrm{A}_3^*) \quad \langle y_f, \mathrm{A}_3^*\beta\rangle_{H_3} = 0$$
$$\wedge \quad \forall\ \theta \in K_3 \quad \langle y_f, \theta\rangle_{H_3} = 0, \tag{3.18}$$

$$\forall\ \zeta \in D(\mathrm{A}_1^*) \quad \langle \mathrm{A}_1^*\widetilde{x}_g, \mathrm{A}_1^*\zeta\rangle_{H_1} = \langle g, \mathrm{A}_1^*\zeta\rangle_{H_1} \quad \wedge \quad \forall\ \gamma \in D(\mathrm{A}_2^*) \quad \langle \widetilde{x}_g, \mathrm{A}_2^*\gamma\rangle_{H_2} = 0$$
$$\wedge \quad \forall\ \lambda \in K_2 \quad \langle \widetilde{x}_g, \lambda\rangle_{H_2} = 0, \tag{3.19}$$

$$\forall\ \varphi \in D(\mathrm{A}_1) \quad \langle \mathrm{A}_1 z_g, \mathrm{A}_1\varphi\rangle_{H_2} = \langle g, \varphi\rangle_{H_1} \quad \wedge \quad \forall\ \delta \in D(\mathrm{A}_0) \quad \langle z_g, \mathrm{A}_0\delta\rangle_{H_1} = 0$$
$$\wedge \quad \forall\ \psi \in K_1 \quad \langle z_g, \psi\rangle_{H_1} = 0. \tag{3.20}$$

**Remark 3.7.** *For possible numerical purposes and applications let us mention a few observations:*

(i) *Using the variational formulation (3.11) or (3.17) corresponding to $x_f = \widetilde{x}_f \in D(\mathrm{A}_2)$ for finding a numerical (discrete) approximation $x_{f,h}$ of $x_f$ proposes a $D(A_2)$-conforming method in some finite dimensional (discrete) subspace $D_h(\mathrm{A}_2)$ of $D(A_2)$ giving also a $D(A_2)$-conforming discrete solution $x_{f,h} \in D_h(\mathrm{A}_2) \subset D(\mathrm{A}_2)$.*

(ii) *Using the variational formulation (3.12) or (3.18) corresponding to $x_f = \mathrm{A}_2^* y_f \in R(\mathrm{A}_2^*)$ for finding a discrete approximation $x_{f,h} = \mathrm{A}_2^* y_{f,h}$ of $x_f$ proposes a $D(\mathrm{A}_2^*)$-conforming method in some discrete subspace $D_h(\mathrm{A}_2^*)$ of $D(\mathrm{A}_2^*)$ giving a $D(\mathrm{A}_2^*)$-conforming discrete potential $y_{f,h} \in D_h(\mathrm{A}_2^*) \subset D(\mathrm{A}_2^*)$, but yielding a $D(\mathrm{A}_1^*)$-conforming solution as*

$$x_{f,h} = \mathrm{A}_2^* y_{f,h} \in R(\mathrm{A}_2^*) = N(\mathrm{A}_1^*) \cap K_2^{\perp_{H_2}} \subset D(\mathrm{A}_1^*).$$

(ii') *A possible discrete solution $x_{f,h} = \mathrm{A}_2^* y_{f,h}$ from (ii) satisfies automatically the side conditions*

$$\mathrm{A}_1^* x_{f,h} = 0, \quad \pi_2 x_{f,h} = 0,$$

*i.e., even on the discrete level there is no error in the side conditions. The other option from (i) yields a discrete solution* $x_{f,h}$*, which most probably has got errors in the side conditions.*

(iii) *Similar observations hold for (3.13) or (3.19) with* $D(\mathrm{A}_1^*)$*-conforming discrete solutions and (3.14) or (3.20) with* $D(A_1)$*- resp.* $D(A_2)$*-conforming discrete solutions.*

**Remark 3.8.** *The finite dimensionality of* $K_2$ *may be dropped. Then all assertions of Theorem 3.3 and Theorem 3.5 and all variational and saddle point formulations remain valid. Note that* $R(A_1)$ *and* $R(A_2)$ *are closed, if* $D(\mathcal{A}_1) \hookrightarrow H_1$ *and* $D(\mathcal{A}_2) \hookrightarrow H_2$ *are compact. Moreover, by Lemma 2.8* $D(\mathcal{A}_1) \hookrightarrow H_1$ *and* $D(\mathcal{A}_2) \hookrightarrow H_2$ *are compact and* $K_2$ *is finite dimensional if and only if* $D_2 \hookrightarrow H_2$ *is compact.*

### *3.1.2. Trivial cohomology groups*

The double saddle point formulations (3.17)–(3.20) can be simplified if some assumptions on the cohomology groups are imposed. For this, let additionally to our General Assumption 3.1 the two ranges $R(\mathrm{A}_0)$ and $R(\mathrm{A}_3)$ be closed as well and let the cohomology groups $K_1$ and $K_3$ be trivial. Thus all ranges $R(\mathrm{A}_0), R(\mathrm{A}_1), R(\mathrm{A}_2)$, and $R(\mathrm{A}_3)$ are assumed to be closed and we have

$$K_1 = \{0\}, \quad K_3 = \{0\}.$$

Recalling (3.15) and (3.16) we see

$$N(\mathrm{A}_1) = R(\mathrm{A}_0), \quad N(\mathrm{A}_2^*) = R(\mathrm{A}_3^*).$$

If we now focus on the two double saddle point problems (3.18) and (3.20) we get the following simplified versions: Find $y_f \in D(\mathrm{A}_2^*)$ and $z_g \in D(\mathrm{A}_1)$, such that

$$\forall\ \phi \in D(\mathrm{A}_2^*) \quad \langle \mathrm{A}_2^* y_f, \mathrm{A}_2^* \phi\rangle_{H_2} = \langle f, \phi\rangle_{H_3} \quad \wedge \quad \forall\ \beta \in D(\mathrm{A}_3^*) \quad \langle y_f, \mathrm{A}_3^* \beta\rangle_{H_3} = 0, \tag{3.21}$$

$$\forall\ \varphi \in D(\mathrm{A}_1) \quad \langle \mathrm{A}_1 z_g, \mathrm{A}_1 \varphi\rangle_{H_2} = \langle g, \varphi\rangle_{H_1} \quad \wedge \quad \forall\ \delta \in D(\mathrm{A}_0) \quad \langle z_g, \mathrm{A}_0 \delta\rangle_{H_1} = 0. \tag{3.22}$$

Let us consider the following modified system: Find

$$(y_f, v_f) \in D(\mathrm{A}_2^*) \times D(\mathcal{A}_3^*), \quad (z_g, w_g) \in D(\mathrm{A}_1) \times D(\mathcal{A}_0),$$

such that

$$\forall\ (\phi,\beta)\in D(\mathrm{A}_2^*)\times D(\mathcal{A}_3^*)\ \langle \mathrm{A}_2^*y_f,\mathrm{A}_2^*\phi\rangle_{H_2}+\langle \mathrm{A}_3^*v_f,\phi\rangle_{H_3}=\langle f,\phi\rangle_{H_3}\ \wedge\ \langle y_f,\mathrm{A}_3^*\beta\rangle_{H_3}=0, \tag{3.23}$$

$$\forall\ (\varphi,\delta)\in D(\mathrm{A}_1)\times D(\mathcal{A}_0)\ \langle \mathrm{A}_1z_g,\mathrm{A}_1\varphi\rangle_{H_2}+\langle \mathrm{A}_0w_g,\varphi\rangle_{H_1}=\langle g,\varphi\rangle_{H_1}\ \wedge\ \langle z_g,\mathrm{A}_0\delta\rangle_{H_1}=0. \tag{3.24}$$

The unique solutions $y_f$, $z_g$ of (3.21)–(3.22) yield solutions $(y_f,0),(z_g,0)$ of (3.23)–(3.24). On the other hand, for any solutions $(y_f,v_f),(z_g,w_g)$ of (3.23) and (3.24) we get $\mathrm{A}_3^*v_f=0$ and $\mathrm{A}_0w_g=0$ by testing with $\phi:=\mathrm{A}_3^*v_f\in R(\mathrm{A}_3^*)=N(\mathrm{A}_2^*)\subset D(\mathrm{A}_2^*)$ and $\varphi:=\mathrm{A}_0w_g\in R(\mathrm{A}_0)=N(\mathrm{A}_1)\subset D(\mathrm{A}_1)$ since $f\in R(\mathrm{A}_2)\bot_{H_3}N(\mathrm{A}_2^*)$ and $g\in R(\mathrm{A}_1^*)\bot_{H_1}N(\mathrm{A}_1)$, respectively. Hence, as $v_f\in D(\mathcal{A}_3^*)$ and $w_g\in D(\mathcal{A}_0)$ we see $v_f=0$ and $w_g=0$. Thus, $y_f$, $z_g$ are the unique solutions of (3.21) and (3.22). The latter arguments show that (3.21) and (3.22) and (3.23) and (3.24) are equivalent and both are uniquely solvable.

**Remark 3.9.** *The saddle point formulations (3.23) and (3.24) are also accessible by the standard inf-sup-theory, which is widely used in the numerical community. For this, let us note that the bilinear forms $\langle \mathrm{A}_2^*\,\cdot\,,\mathrm{A}_2^*\,\cdot\,\rangle_{H_2}$ and $\langle \mathrm{A}_1\,\cdot\,,\mathrm{A}_1\,\cdot\,\rangle_{H_2}$ are strictly positive (coercive) over the respective kernels given by each of the second forms, which are by assumption*

$$N(\mathrm{A}_3)=\big(N(\mathrm{A}_3)^\bot\big)^\bot=R(\mathrm{A}_3^*)^\bot=N(\mathrm{A}_2^*)^\bot=R(\mathrm{A}_2),$$
$$N(\mathrm{A}_0^*)=\big(N(\mathrm{A}_0^*)^\bot\big)^\bot=R(\mathrm{A}_0)^\bot=N(\mathrm{A}_1)^\bot=R(\mathrm{A}_1^*),$$

*i.e., over $D(\mathcal{A}_2^*)$ and $D(\mathcal{A}_1)$. Moreover, the inf-sup-conditions are satisfied. For this, we compute by choosing $\phi:=\mathrm{A}_3^*\beta\in R(\mathrm{A}_3^*)=N(\mathrm{A}_2^*)$ and $\varphi:=\mathrm{A}_0\delta\in R(\mathrm{A}_0)=N(\mathrm{A}_1)$ for some given $0\neq\beta\in D(\mathcal{A}_3^*)$ and $0\neq\delta\in D(\mathcal{A}_0)$*

$$\frac{|\mathrm{A}_3^*\beta|_{H_3}}{|\beta|_{D(\mathrm{A}_3^*)}}\le\sup_{0\neq\phi\in D(\mathrm{A}_2^*)}\frac{\langle \mathrm{A}_3^*\beta,\phi\rangle_{H_3}}{|\beta|_{D(\mathrm{A}_3^*)}|\phi|_{D(\mathrm{A}_2^*)}}\le\frac{|\mathrm{A}_3^*\beta|_{H_3}}{|\beta|_{D(\mathrm{A}_3^*)}}\le 1,$$
$$\frac{|\mathrm{A}_0\delta|_{H_1}}{|\delta|_{D(\mathrm{A}_0)}}\le\sup_{0\neq\varphi\in D(\mathrm{A}_1)}\frac{\langle \mathrm{A}_0\delta,\varphi\rangle_{H_1}}{|\delta|_{D(\mathrm{A}_0)}|\varphi|_{D(\mathrm{A}_1)}}\le\frac{|\mathrm{A}_0\delta|_{H_1}}{|\delta|_{D(\mathrm{A}_0)}}\le 1,$$

*which shows that actually equality holds. Thus, the inf-sup-conditions follow*[7]

[7]Note that by (2.4), (2.5), Lemma 2.4, and Corollary 2.5 (iv)

$$\inf_{0\neq\beta\in D(\mathcal{A}_3^*)}\frac{|\mathrm{A}_3^*\beta|_{H_3}^2}{|\beta|_{D(\mathrm{A}_3^*)}^2}=\Bigg(\sup_{0\neq\beta\in D(\mathcal{A}_3^*)}\frac{|\beta|_{H_4}^2+|\mathrm{A}_3^*\beta|_{H_3}^2}{|\mathrm{A}_3^*\beta|_{H_3}^2}\Bigg)^{-1}=\Bigg(1+\sup_{0\neq\beta\in D(\mathcal{A}_3^*)}\frac{|\beta|_{H_4}^2}{|\mathrm{A}_3^*\beta|_{H_3}^2}\Bigg)^{-1}=\frac{1}{1+c_3^2}=|\mathcal{A}_3^{-1}|_{R(\mathrm{A}_3),D(\mathcal{A}_3)}^{-2},$$
$$\inf_{0\neq\delta\in D(\mathcal{A}_0)}\frac{|\mathrm{A}_0\delta|_{H_1}^2}{|\delta|_{D(\mathrm{A}_0)}^2}=\Bigg(\sup_{0\neq\delta\in D(\mathcal{A}_0)}\frac{|\delta|_{H_0}^2+|\mathrm{A}_0\delta|_{H_1}^2}{|\mathrm{A}_0\delta|_{H_1}^2}\Bigg)^{-1}=\Bigg(1+\sup_{0\neq\delta\in D(\mathcal{A}_0)}\frac{|\delta|_{H_0}^2}{|\mathrm{A}_0\delta|_{H_1}^2}\Bigg)^{-1}=\frac{1}{c_0^2+1}=|\mathcal{A}_0^{-1}|_{R(\mathrm{A}_0),D(\mathcal{A}_0)}^{-2}$$

hold.

$$1 \geq \inf_{0 \neq \beta \in D(\mathcal{A}_3^*)} \sup_{0 \neq \phi \in D(\mathrm{A}_2^*)} \frac{\langle \mathrm{A}_3^* \beta, \phi \rangle_{H_3}}{|\beta|_{D(\mathrm{A}_3^*)} |\phi|_{D(\mathrm{A}_2^*)}} = \inf_{0 \neq \beta \in D(\mathcal{A}_3^*)} \frac{|\mathrm{A}_3^* \beta|_{H_3}}{|\beta|_{D(\mathrm{A}_3^*)}} = \left(c_3^2 + 1\right)^{-1/2}$$
$$= |(\mathcal{A}_3^*)^{-1}|^{-1}_{R(\mathrm{A}_3^*), D(\mathcal{A}_3^*)},$$
$$1 \geq \inf_{0 \neq \delta \in D(\mathcal{A}_0)} \sup_{0 \neq \varphi \in D(\mathrm{A}_1)} \frac{\langle \mathrm{A}_0 \delta, \varphi \rangle_{H_1}}{|\delta|_{D(\mathrm{A}_0)} |\varphi|_{D(\mathrm{A}_1)}} = \inf_{0 \neq \delta \in D(\mathcal{A}_0)} \frac{|\mathrm{A}_0 \delta|_{H_1}}{|\delta|_{D(\mathrm{A}_0)}} = \left(c_0^2 + 1\right)^{-1/2}$$
$$= |\mathcal{A}_0^{-1}|^{-1}_{R(\mathrm{A}_0), D(\mathcal{A}_0)},$$

which are actually nothing else than the boundedness of the norms of the respective inverse operators $|(\mathcal{A}_3^*)^{-1}|_{R(\mathrm{A}_3^*), D(\mathcal{A}_3^*)} = |\mathcal{A}_3^{-1}|_{R(\mathrm{A}_3), D(\mathcal{A}_3)}$ and $|\mathcal{A}_0^{-1}|_{R(\mathrm{A}_0), D(\mathcal{A}_0)} = |(\mathcal{A}_0^*)^{-1}|_{R(\mathrm{A}_0^*), D(\mathcal{A}_0^*)}$, i.e., the boundedness of the respective inverse operators $\mathcal{A}_3^{-1}, (\mathcal{A}_3^*)^{-1}, \mathcal{A}_0^{-1}, (\mathcal{A}_0^*)^{-1}$, itself.

Now, if $D(\mathcal{A}_3^*)$ and $D(\mathcal{A}_0)$ are still not suitable and provided that the respective cohomology groups are trivial, we can repeat the procedure to obtain additional saddle point formulations for $v_f$ and $w_g$. Note that (3.23) and (3.24) is equivalent to find $(y_f, v_f, z_g, w_g) \in D(\mathrm{A}_2^*) \times D(\mathcal{A}_3^*) \times D(\mathrm{A}_1) \times D(\mathcal{A}_0)$, such that for all $(\phi, \beta, \varphi, \delta) \in D(\mathrm{A}_2^*) \times D(\mathcal{A}_3^*) \times D(\mathrm{A}_1) \times D(\mathcal{A}_0)$

$$\langle \mathrm{A}_2^* y_f, \mathrm{A}_2^* \phi \rangle_{H_2} + \langle \mathrm{A}_3^* v_f, \phi \rangle_{H_3} + \langle y_f, \mathrm{A}_3^* \beta \rangle_{H_3} + \langle \mathrm{A}_1 z_g, \mathrm{A}_1 \varphi \rangle_{H_2} + \langle \mathrm{A}_0 w_g, \varphi \rangle_{H_1}$$
$$+ \langle z_g, \mathrm{A}_0 \delta \rangle_{H_1} = \langle f, \phi \rangle_{H_3} + \langle g, \varphi \rangle_{H_1}. \tag{3.25}$$

### 3.1.3. More variational formulations

Another idea is to compute the two partial solutions $x_f$ and $x_g$ from Theorem 3.3 together in just one variational formulation for the sum $x_f + x_g$. For this, let $f \in R(\mathrm{A}_2)$ and $g \in R(\mathrm{A}_1^*)$. Recall that $x_f \in D(\mathcal{A}_2)$ and $x_g \in D(\mathcal{A}_1^*)$ are given by the variational formulations in Theorem 3.5 (i) and (ii), i.e.,

$$\forall\ \xi \in D(\mathrm{A}_2) \quad \langle \mathrm{A}_2 x_f, \mathrm{A}_2 \xi \rangle_{H_3} = \langle f, \mathrm{A}_2 \xi \rangle_{H_3}, \tag{3.26}$$
$$\forall\ \zeta \in D(\mathrm{A}_1^*) \quad \langle \mathrm{A}_1^* x_g, \mathrm{A}_1^* \zeta \rangle_{H_1} = \langle g, \mathrm{A}_1^* \zeta \rangle_{H_1}, \tag{3.27}$$

respectively, compare also to the variational formulations (3.17) and (3.19). As $\mathrm{A}_1^* x_f = \mathrm{A}_1^* k = 0$ and $\mathrm{A}_2 x_g = \mathrm{A}_2 k = 0$, these latter two formulations hold for $x = x_f + x_g + k$ as well, i.e.,

$$\forall\ \xi \in D(\mathrm{A}_2) \quad \langle \mathrm{A}_2 x, \mathrm{A}_2 \xi \rangle_{H_3} = \langle f, \mathrm{A}_2 \xi \rangle_{H_3}, \tag{3.28}$$
$$\forall\ \zeta \in D(\mathrm{A}_1^*) \quad \langle \mathrm{A}_1^* x, \mathrm{A}_1^* \zeta \rangle_{H_1} = \langle g, \mathrm{A}_1^* \zeta \rangle_{H_1}. \tag{3.29}$$

The first option is to use (3.28) together with a weak version of $\mathrm{A}_1^* x = g$, i.e.,

$$\forall\ \varphi \in D(\mathrm{A}_1) \qquad \langle g, \varphi \rangle_{H_1} = \langle \mathrm{A}_1^* x, \varphi \rangle_{H_1} = \langle x, \mathrm{A}_1 \varphi \rangle_{H_2}.$$

The second option is to use (3.29) together with a weak version of $A_2 x = f$, i.e.,

$$\forall\ \phi \in D(A_2^*) \qquad \langle f, \phi\rangle_{H_3} = \langle A_2 x, \phi\rangle_{H_3} = \langle x, A_2^*\phi\rangle_{H_2}.$$

For simplicity, let us assume that the cohomology group $K_2$ is trivial.

**Theorem 3.10.** *Let $K_2 = \{0\}$. The unique solution $x = x_f + x_g \in D_2$ in Theorem 3.3 can be found by the following two variational saddle point formulations:*

(i) *There exists a unique pair $(\widetilde{x}, z) \in D(A_2) \times D(\mathcal{A}_1)$ such that*

$$\forall\ (\xi, \varphi) \in D(A_2) \times D(\mathcal{A}_1) \quad \langle A_2\widetilde{x}, A_2\xi\rangle_{H_3} + \langle A_1 z, \xi\rangle_{H_2} = \langle f, A_2\xi\rangle_{H_3}, \tag{3.30}$$

$$\langle \widetilde{x}, A_1\varphi\rangle_{H_2} = \langle g, \varphi\rangle_{H_1}. \tag{3.31}$$

*It holds $z = 0$ as well as*

$$\forall\ (\xi, \varphi) \in D(A_2) \times D(\mathcal{A}_1) \quad \langle A_2\widetilde{x}, A_2\xi\rangle_{H_3} = \langle f, A_2\xi\rangle_{H_3}, \tag{3.32}$$

$$\langle \widetilde{x}, A_1\varphi\rangle_{H_2} = \langle g, \varphi\rangle_{H_1}. \tag{3.33}$$

*Moreover, $A_2\widetilde{x} = f$ if and only if $f \in R(A_2)$. (3.31), (3.33) hold for all $\varphi \in D(A_1)$ if and only if $g \in R(A_1^*)$ if and only if $\widetilde{x} \in D(A_1^*)$ and $A_1^*\widetilde{x} = g$. In this case, i.e., $f \in R(A_2)$ and $g \in R(A_1^*)$, we have $\widetilde{x} = x$ from Theorem 3.3.*

(ii). *There exists a unique pair $(\hat{x}, y) \in D(A_1^*) \times D(\mathcal{A}_2^*)$ such that*

$$\forall\ (\zeta, \phi) \in D(A_1^*) \times D(\mathcal{A}_2^*) \quad \langle A_1^*\hat{x}, A_1^*\zeta\rangle_{H_1} + \langle A_2^* y, \zeta\rangle_{H_2} = \langle g, A_1^*\zeta\rangle_{H_1}, \tag{3.34}$$

$$\langle \hat{x}, A_2^*\phi\rangle_{H_2} = \langle f, \phi\rangle_{H_3}. \tag{3.35}$$

*It holds $y = 0$ as well as*

$$\forall\ (\zeta, \phi) \in D(A_1^*) \times D(\mathcal{A}_2^*) \quad \langle A_1^*\hat{x}, A_1^*\zeta\rangle_{H_1} = \langle g, A_1^*\zeta\rangle_{H_1}, \tag{3.36}$$

$$\langle \hat{x}, A_2^*\phi\rangle_{H_2} = \langle f, \phi\rangle_{H_3}. \tag{3.37}$$

*Moreover, $A_1^*\hat{x} = g$ if and only if $g \in R(A_1^*)$. (3.35), (3.37) hold for all $\phi \in D(A_2^*)$ if and only if $f \in R(A_2)$ if and only if $\hat{x} \in D(A_2)$ and $A_2\hat{x} = f$. In this case, i.e., $f \in R(A_2)$ and $g \in R(A_1^*)$, we have $\hat{x} = x$ from Theorem 3.3.*

*Proof.* We prove unique solvability by standard saddle point theory. By Corollary 2.5 (i) the principal part of (3.30) is strictly positive over the kernel of (3.31), which is

$$D(A_2) \cap N(A_1^*) = D(A_2) \cap R(A_2^*) = D(\mathcal{A}_2),$$

as $K_2 = \{0\}$. Moreover, we have for $0 \neq \varphi \in D(\mathcal{A}_1)$

$$\frac{|\mathrm{A}_1\varphi|_{H_2}}{|\varphi|_{D(\mathrm{A}_1)}} \leq \sup_{0\neq\xi\in D(\mathrm{A}_2)} \frac{\langle \mathrm{A}_1\varphi, \xi\rangle_{H_2}}{|\varphi|_{D(\mathrm{A}_1)}|\xi|_{D(\mathrm{A}_2)}} \leq \frac{|\mathrm{A}_1\varphi|_{H_2}}{|\varphi|_{D(\mathrm{A}_1)}} \leq 1$$

by choosing $\xi := \mathrm{A}_1\varphi \in R(\mathrm{A}_1) = N(\mathrm{A}_2)$, which shows that actually equality holds. Hence

$$\begin{aligned} 1 &\geq \inf_{0\neq\varphi\in D(\mathcal{A}_1)} \sup_{0\neq\xi\in D(\mathrm{A}_2)} \frac{\langle \mathrm{A}_1\varphi, \xi\rangle_{H_2}}{|\varphi|_{D(\mathrm{A}_1)}|\xi|_{D(\mathrm{A}_2)}} = \inf_{0\neq\varphi\in D(\mathcal{A}_1)} \frac{|\mathrm{A}_1\varphi|_{H_2}}{|\varphi|_{D(\mathrm{A}_1)}} \\ &\geq \left(c_1^2+1\right)^{-1/2} = |\mathcal{A}_1^{-1}|^{-1}_{R(\mathrm{A}_1),D(\mathcal{A}_1)}, \end{aligned}$$

which shows that the inf-sup-condition is satisfied. Therefore, (3.30) and (3.31) admits a unique solution. Picking $\xi := \mathrm{A}_1 z \in R(\mathrm{A}_1) = N(\mathrm{A}_2)$ in (3.30) yields $|\mathrm{A}_1 z|^2_{H_2} = 0$ and hence $z = 0$ as $z \in D(\mathcal{A}_1)$. Since $\mathrm{A}_1 z = 0$ even (3.32) and (3.33) are valid. By (3.32) we see $\mathrm{A}_2\tilde{x} - f \in R(\mathrm{A}_2)^{\perp_{H_3}}$, showing $\mathrm{A}_2\tilde{x} = f$ if and only if $f \in R(\mathrm{A}_2)$. Using the orthonormal projector $\pi_{\mathrm{A}_1^*}$ and by (3.33) we see for all $\varphi \in D(\mathrm{A}_1)$ as $\pi_{\mathrm{A}_1^*}\varphi \in D(\mathcal{A}_1)$

$$\langle \tilde{x}, \mathrm{A}_1\varphi\rangle_{H_2} = \langle \tilde{x}, \mathrm{A}_1\pi_{\mathrm{A}_1^*}\varphi\rangle_{H_2} = \langle g, \pi_{\mathrm{A}_1^*}\varphi\rangle_{H_1} = \langle \pi_{\mathrm{A}_1^*} g, \varphi\rangle_{H_1} = \langle g, \varphi\rangle_{H_1},$$

if $g \in R(\mathrm{A}_1^*)$. On the other hand, if (3.33) holds for all $\varphi \in D(\mathrm{A}_1)$, then $\tilde{x} \in D(\mathrm{A}_1^*)$ with $\mathrm{A}_1^*\tilde{x} = g$, especially $g \in R(\mathrm{A}_1^*)$. Therefore, if $f \in R(\mathrm{A}_2)$ and $g \in R(\mathrm{A}_1^*)$, we have $\tilde{x} \in D(\mathrm{A}_2) \cap D(\mathrm{A}_1^*) = D_2$ with $\mathrm{A}_2\tilde{x} = f$ and $\mathrm{A}_1^*\tilde{x} = g$, finally showing $\tilde{x} = x$ by the unique solvability of (3.1) from Theorem 3.3. Analogously we prove (ii). □

**Remark 3.11.** *Let us note the following:*

(i) (3.30) and (3.31) *is a weak formulation of*

$$\mathrm{A}_2^*\mathrm{A}_2\tilde{x} + \mathrm{A}_1 z = \mathrm{A}_2^* f, \quad \mathrm{A}_1^*\tilde{x} = g,$$

*i.e., in formal matrix notation*

$$\begin{bmatrix} \mathrm{A}_2^*\mathrm{A}_2 & \mathrm{A}_1 \\ \mathrm{A}_1^* & 0 \end{bmatrix} \begin{bmatrix} \tilde{x} \\ z \end{bmatrix} = \begin{bmatrix} \mathrm{A}_2^* f \\ g \end{bmatrix}.$$

*Note $z = 0$.*

(ii) (3.34) and (3.35) *is a weak formulation of*

$$\mathrm{A}_1\mathrm{A}_1^*\hat{x} + \mathrm{A}_2^* y = \mathrm{A}_1 g, \quad \mathrm{A}_2\hat{x} = f,$$

*i.e., in formal matrix notation*

$$\begin{bmatrix} \mathrm{A}_1\mathrm{A}_1^* & \mathrm{A}_2^* \\ \mathrm{A}_2 & 0 \end{bmatrix} \begin{bmatrix} \hat{x} \\ y \end{bmatrix} = \begin{bmatrix} \mathrm{A}_1 g \\ f \end{bmatrix}.$$

*Note $y = 0$.*

The restriction $K_2 = \{0\}$ can easily be removed from Theorem 3.10 leading to double saddle point formulations as in (3.17)–(3.20).

**Theorem 3.12.** *The unique solution $x = x_f + x_g + k \in D_2$ in Theorem 3.3 can be found by the following two variational double saddle point formulations:*

(i) *There exists a unique tripple $(\widetilde{x}, z, h) \in D(\mathrm{A}_2) \times D(\mathcal{A}_1) \times K_2$ such that*

$$\begin{aligned}\forall\ (\xi, \varphi, \kappa) \in D(\mathrm{A}_2) \times D(\mathcal{A}_1) \times K_2 \quad \langle \mathrm{A}_2\widetilde{x}, \mathrm{A}_2\xi\rangle_{H_3} + \langle \mathrm{A}_1 z, \xi\rangle_{H_2} + \langle h, \xi\rangle_{H_2} &= \langle f, \mathrm{A}_2\xi\rangle_{H_3},\\ \langle \widetilde{x}, \mathrm{A}_1\varphi\rangle_{H_2} &= \langle g, \varphi\rangle_{H_1},\\ \langle \widetilde{x}, \kappa\rangle_{H_2} &= \langle k, \kappa\rangle_{H_2}.\end{aligned} \tag{3.38}$$

*It holds $z = 0$ and $h = 0$ as well as*

$$\begin{aligned}\forall\ (\xi, \varphi, \kappa) \in D(\mathrm{A}_2) \times D(\mathcal{A}_1) \times K_2 \quad \langle \mathrm{A}_2\widetilde{x}, \mathrm{A}_2\xi\rangle_{H_3} &= \langle f, \mathrm{A}_2\xi\rangle_{H_3},\\ \langle \widetilde{x}, \mathrm{A}_1\varphi\rangle_{H_2} &= \langle g, \varphi\rangle_{H_1},\\ \langle \widetilde{x}, \kappa\rangle_{H_2} &= \langle k, \kappa\rangle_{H_2}.\end{aligned} \tag{3.39}$$

*Moreover, $\mathrm{A}_2\widetilde{x} = f$ if and only if $f \in R(\mathrm{A}_2)$. The second equations of (3.38), (3.39) hold for all $\varphi \in D(\mathrm{A}_1)$ if and only if $g \in R(\mathrm{A}_1^*)$ if and only if $\widetilde{x} \in D(\mathrm{A}_1^*)$ and $\mathrm{A}_1^*\widetilde{x} = g$. Furthermore, $\pi_2\widetilde{x} = k$. In this case, i.e., $f \in R(\mathrm{A}_2)$ and $g \in R(\mathrm{A}_1^*)$, we have $\widetilde{x} = x$ from Theorem 3.3.*

(ii) *There exists a unique triple $(\hat{x}, y, h) \in D(\mathrm{A}_1^*) \times D(\mathcal{A}_2^*) \times K_2$ such that*

$$\begin{aligned}\forall\ (\zeta, \phi, \kappa) \in D(\mathrm{A}_1^*) \times D(\mathcal{A}_2^*) \times K_2 \quad \langle \mathrm{A}_1^*\hat{x}, \mathrm{A}_1^*\zeta\rangle_{H_1} + \langle \mathrm{A}_2^* y, \zeta\rangle_{H_2} + \langle h, \zeta\rangle_{H_2} &= \langle g, \mathrm{A}_1^*\zeta\rangle_{H_1},\\ \langle \hat{x}, \mathrm{A}_2^*\phi\rangle_{H_2} &= \langle f, \phi\rangle_{H_3},\\ \langle \hat{x}, \kappa\rangle_{H_2} &= \langle k, \kappa\rangle_{H_2}.\end{aligned} \tag{3.40}$$

*It holds $y = 0$ and $h = 0$ as well as*

$$\begin{aligned}\forall\ (\zeta, \phi, \kappa) \in D(\mathrm{A}_1^*) \times D(\mathcal{A}_2^*) \times K_2 \quad \langle \mathrm{A}_1^*\hat{x}, \mathrm{A}_1^*\zeta\rangle_{H_1} &= \langle g, \mathrm{A}_1^*\zeta\rangle_{H_1},\\ \langle \hat{x}, \mathrm{A}_2^*\phi\rangle_{H_2} &= \langle f, \phi\rangle_{H_3},\\ \langle \hat{x}, \kappa\rangle_{H_2} &= \langle k, \kappa\rangle_{H_2}.\end{aligned} \tag{3.41}$$

*Moreover, $\mathrm{A}_1^*\hat{x} = g$ if and only if $g \in R(\mathrm{A}_1^*)$. The second equations of (3.40), (3.41) hold for all $\phi \in D(\mathrm{A}_2^*)$ if and only if $f \in R(\mathrm{A}_2)$ if and only if $\hat{x} \in D(\mathrm{A}_2)$ and $\mathrm{A}_2\hat{x} = f$. Furthermore, $\pi_2\hat{x} = k$. In this case, i.e., $f \in R(\mathrm{A}_2)$ and $g \in R(\mathrm{A}_1^*)$, we have $\hat{x} = x$ from Theorem 3.3.*

*Proof.* Again we prove unique solvability by standard (double) saddle point theory. The kernels of the operators encoded in the last two equations of (3.38) are $N(\mathrm{A}_1^*)$ and $K_2^{\perp_{H_2}}$. Hence by Corollary 2.5 (i) the principal part of the first equation in (3.38) is strictly positive over the latter kernels, i.e., over

$$D(\mathrm{A}_2) \cap N(\mathrm{A}_1^*) \cap K_2^{\perp_{H_2}} = D(\mathrm{A}_2) \cap R(\mathrm{A}_2^*) = D(\mathcal{A}_2).$$

Moreover, we have for $\varphi \in D(\mathcal{A}_1)$ and $\kappa \in K_2$ with $(\varphi, \kappa) \neq 0$

$$\frac{\left(|\mathrm{A}_1\varphi|_{H_2}^2 + |\kappa|_{H_2}^2\right)^{1/2}}{\left(|\varphi|_{D(\mathrm{A}_1)}^2 + |\kappa|_{H_2}^2\right)^{1/2}} \leq \sup_{0 \neq \xi \in D(\mathrm{A}_2)} \frac{\langle \mathrm{A}_1\varphi, \xi\rangle_{H_2} + \langle \kappa, \xi\rangle_{H_2}}{\left(|\varphi|_{D(\mathrm{A}_1)}^2 + |\kappa|_{H_2}^2\right)^{1/2} |\xi|_{D(\mathrm{A}_2)}}$$
$$\leq \frac{|\mathrm{A}_1\varphi|_{H_2} + |\kappa|_{H_2}}{\left(|\varphi|_{D(\mathrm{A}_1)}^2 + |\kappa|_{H_2}^2\right)^{1/2}} \leq \sqrt{2}$$

by choosing $\xi := \mathrm{A}_1\varphi + \kappa \in R(\mathrm{A}_1) \oplus K_2 = N(\mathrm{A}_2)$. Hence by Corollary 2.5 (i)

$$\sqrt{2} \geq \inf_{\substack{\kappa \in K_2^{\varphi \in D(\mathcal{A}_1)} \\ (\varphi, \kappa) \neq 0}} \sup_{0 \neq \xi \in D(\mathrm{A}_2)} \frac{\langle \mathrm{A}_1\varphi, \xi\rangle_{H_2} + \langle \kappa, \xi\rangle_{H_2}}{\left(|\varphi|_{D(\mathrm{A}_1)}^2 + |\kappa|_{H_2}^2\right)^{1/2} |\xi|_{D(\mathrm{A}_2)}}$$

$$\geq \inf_{\substack{\kappa \in K_2^{\varphi \in D(\mathcal{A}_1)} \\ (\varphi, \kappa) \neq 0}} \frac{\left(|\mathrm{A}_1\varphi|_{H_2}^2 + |\kappa|_{H_2}^2\right)^{1/2}}{\left(|\varphi|_{D(\mathrm{A}_1)}^2 + |\kappa|_{H_2}^2\right)^{1/2}} \geq \inf_{\substack{\kappa \in K_2^{\varphi \in D(\mathcal{A}_1)} \\ (\varphi, \kappa) \neq 0}} \frac{\left(|\mathrm{A}_1\varphi|_{H_2}^2 + |\kappa|_{H_2}^2\right)^{1/2}}{\left((c_1^2 + 1)|\mathrm{A}_1\varphi|_{H_2}^2 + |\kappa|_{H_2}^2\right)^{1/2}}$$

$$\geq \left(c_1^2 + 1\right)^{-1/2} = |\mathcal{A}_1^{-1}|_{R(\mathrm{A}_1), D(\mathcal{A}_1)}^{-1},$$

which shows that the inf-sup-condition is satisfied. Therefore, (3.38) admits a unique solution. Picking[8] $\xi := \mathrm{A}_1 z \in R(\mathrm{A}_1) = N(\mathrm{A}_2) \cap K_2^{\perp_{H_2}}$ in (3.38) yields $|\mathrm{A}_1 z|_{H_2}^2 = 0$ and hence $z = 0$ as $z \in D(\mathcal{A}_1)$. Choosing $\xi := h \in K_2 = N(\mathrm{A}_2) \cap R(\mathrm{A}_1)^{\perp_{H_2}}$ in (3.38) shows $|h|_{H_2}^2 = 0$. Since $\mathrm{A}_1 z = h = 0$ even (3.39) is valid. By the first equation of (3.39) we see $\mathrm{A}_2\tilde{x} - f \in R(\mathrm{A}_2)^{\perp_{H_3}}$, showing $\mathrm{A}_2\tilde{x} = f$ if and only if $f \in R(\mathrm{A}_2)$. Using the orthonormal projector $\pi_{\mathrm{A}_1^*}$ and by the second equation of (3.39) we get for all $\varphi \in D(\mathrm{A}_1)$ as $\pi_{\mathrm{A}_1^*}\varphi \in D(\mathcal{A}_1)$

$$\langle \tilde{x}, \mathrm{A}_1\varphi\rangle_{H_2} = \langle \tilde{x}, \mathrm{A}_1\pi_{\mathrm{A}_1^*}\varphi\rangle_{H_2} = \langle g, \pi_{\mathrm{A}_1^*}\varphi\rangle_{H_1} = \langle \pi_{\mathrm{A}_1^*} g, \varphi\rangle_{H_1} = \langle g, \varphi\rangle_{H_1},$$

if $g \in R(\mathrm{A}_1^*)$. On the other hand, if the second equation of (3.39) holds for all $\varphi \in D(\mathrm{A}_1)$, then $\tilde{x} \in D(\mathrm{A}_1^*)$ with $\mathrm{A}_1^*\tilde{x} = g$, especially $g \in R(\mathrm{A}_1^*)$. Therefore, if $f \in R(\mathrm{A}_2)$ and $g \in R(\mathrm{A}_1^*)$, we have $\tilde{x} \in D(\mathrm{A}_2) \cap D(\mathrm{A}_1^*) = D_2$ with $\mathrm{A}_2\tilde{x} = f$ and $\mathrm{A}_1^*\tilde{x} = g$. The third equation of (3.39) implies for all $\kappa \in K_2$

[8] We can test directly by $\xi := \mathrm{A}_1 z + h \in R(\mathrm{A}_1) + K_2 = N(\mathrm{A}_2)$ in (3.38) as well, since orthogonality shows immediately $0 = \langle \mathrm{A}_1 z, \xi\rangle_{H_2} + \langle h, \xi\rangle_{H_2} = |\mathrm{A}_1 z|_{H_2}^2 + |h|_{H_2}^2$.

$$0 = \langle \widetilde{x} - k, \kappa \rangle_{H_2} = \langle \widetilde{x} - k, \pi_2 \kappa \rangle_{H_2} = \langle \pi_2 \widetilde{x} - k, \kappa \rangle_{H_2},$$

i.e., $\pi_2 \widetilde{x} = k$. Therefore, $\widetilde{x} = x$ by the unique solvability of (3.1) from Theorem 3.3, which completes the proof of (i). Analogously we prove (ii). □

**Remark 3.13.** *Let us note the following:*

(i) *Using the saddle point formulation in Theorem 3.10 (i) or Theorem 3.12 (i) for finding a numerical approximation $x_h$ of $x$ provides a $D(A_2)$-conforming approximation $x_h \in D(A_2)$ of (3.1), whereas using the saddle point formulation in Theorem 3.10 (ii) or Theorem 3.12 (ii) for finding a numerical approximation $x_h$ of $x$ provides a $D(A_1^*)$-conforming approximation $x_h \in D(A_1^*)$ of (3.1).*

(ii) *The variational formulations in Theorem 3.10 (i), (ii) or Theorem 3.12 (i), (ii) are exactly those from (3.17) and (3.19) for the special right hand sides $g = 0$, $f = 0$, and $k = 0$, respectively.*

(iii) Equation (3.38) *is a weak formulation of*

$$A_2^* A_2 \widetilde{x} + A_1 z + h = A_2^* f, \quad A_1^* \widetilde{x} = g, \quad \pi_2 \widetilde{x} = k,$$

*i.e., in formal matrix notation*

$$\begin{bmatrix} A_2^* A_2 & A_1 & \iota_{K_2} \\ A_1^* & 0 & 0 \\ \pi_2 = \iota_{K_2}^* & 0 & 0 \end{bmatrix} \begin{bmatrix} \widetilde{x} \\ z \\ h \end{bmatrix} = \begin{bmatrix} A_2^* f \\ g \\ k \end{bmatrix},$$

*where $\iota_{K_2}$ is the canonical embedding of $K_2$ into $H_2$. Note $z = 0$ and $h = 0$.*

(iii') Equation (3.40) *is a weak formulation of*

$$A_1 A_1^* \hat{x} + A_2^* y + h = A_1 g, \quad A_2 \hat{x} = f, \quad \pi_2 \hat{x} = k,$$

*i.e., in formal matrix notation*

$$\begin{bmatrix} A_1 A_1^* & A_2^* & \iota_{K_2} \\ A_2 & 0 & 0 \\ \pi_2 = \iota_{K_2}^* & 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{x} \\ y \\ h \end{bmatrix} = \begin{bmatrix} A_1 g \\ f \\ k \end{bmatrix}.$$

*Note $y = 0$ and $h = 0$.*

Finally, we present double saddle point variational formulations for finding the partial solutions in (3.17)–(3.20) as well.

**Theorem 3.14.** *Let additionally $R(A_0)$ and $R(A_3)$ be closed. The partial solutions $x_f = \widetilde{x}_f \in D(\mathcal{A}_2), x_g = \widetilde{x}_g \in D(\mathcal{A}_1^*)$, and their potentials $y_f \in D(\mathcal{A}_2^*)$, $z_g \in D(\mathcal{A}_1)$ from Theorem 3.3, Theorem 3.5, (3.7)–(3.10), (3.11)–(3.14), and*

*(3.17)–(3.20) can be found by the following four variational double saddle point formulations:*

(i) *There exists a unique triple* $(\widetilde{x}_f, u, h) \in D(\mathrm{A}_2) \times D(\mathcal{A}_1) \times K_2$ *such that*

$$\begin{aligned} &\forall\ (\xi, \varphi, \kappa) \in D(\mathrm{A}_2) \times D(\mathcal{A}_1) \times K_2 \\ &\langle \mathrm{A}_2\widetilde{x}_f, \mathrm{A}_2\xi\rangle_{H_3} + \langle \mathrm{A}_1 u, \xi\rangle_{H_2} + \langle h, \xi\rangle_{H_2} = \langle f, \mathrm{A}_2\xi\rangle_{H_3}, \\ &\langle \widetilde{x}_f, \mathrm{A}_1\varphi\rangle_{H_2} = 0, \\ &\langle \widetilde{x}_f, \kappa\rangle_{H_2} = 0. \end{aligned} \tag{3.42}$$

*It holds $u=0$ and $h=0$ as well as (3.17). Moreover, $\mathrm{A}_2\widetilde{x}_f = f$ if and only if $f \in R(\mathrm{A}_2)$. The second equation of (3.42) holds for all $\varphi \in D(\mathrm{A}_1)$ and thus $\widetilde{x}_f \in N(\mathrm{A}_1^*)$. Furthermore, $\pi_2\widetilde{x}_f = 0$. Finally, if $f \in R(\mathrm{A}_2)$, we have $\widetilde{x}_f = x_f$ from Theorem 3.3, see Theorem 3.5 (i).*

(i') *There exists a unique triple* $(y_f, v, h) \in D(\mathrm{A}_2^*) \times D(\mathcal{A}_3^*) \times K_3$ *such that*

$$\begin{aligned} &\forall\ (\phi, \theta, \kappa) \in D(\mathrm{A}_2^*) \times D\big(\mathcal{A}_3^*\big) \times K_3 \\ &\langle \mathrm{A}_2^* y_f, \mathrm{A}_2^*\phi\rangle_{H_2} + \langle \mathrm{A}_3^* v, \phi\rangle_{H_3} + \langle h, \phi\rangle_{H_3} = \langle f, \phi\rangle_{H_3}, \\ &\langle y_f, \mathrm{A}_3^*\theta\rangle_{H_3} = 0, \\ &\langle y_f, \kappa\rangle_{H_3} = 0. \end{aligned} \tag{3.43}$$

*It holds $v=0$ if and only if $f \perp_{H_3} R(\mathrm{A}_3^*)$ if and only if*[9] *$f \in N(\mathrm{A}_3)$. $h=0$ if and only if*[10] *$f \perp_{H_3} K_3$. Thus $v=0$ and $h=0$ if and only if $f \in N(\mathrm{A}_3) \cap K_3^{\perp_{H_3}} = R(\mathrm{A}_2)$. Furthermore, (3.18) holds. Moreover, $\mathrm{A}_2^* y_f \in D(\mathrm{A}_2)$ and $\mathrm{A}_2\mathrm{A}_2^* y_f = f$ if and only if $f \in R(\mathrm{A}_2)$. The second equation of (3.43) holds for all $\theta \in D(\mathrm{A}_3^*)$ and hence $y_f \in N(\mathrm{A}_3)$. Furthermore, $\pi_3 y_f = 0$. Finally, if $f \in R(\mathrm{A}_2)$, we have $\mathrm{A}_2^* y_f = x_f$ from Theorem 3.3, see Theorem 3.5 (i').*

(ii) *There exists a unique triple* $(\widetilde{x}_g, p, h) \in D(\mathrm{A}_1^*) \times D(\mathcal{A}_2^*) \times K_2$ *such that*

$$\begin{aligned} &\forall\ (\zeta, \phi, \kappa) \in D(\mathrm{A}_1^*) \times D\big(\mathcal{A}_2^*\big) \times K_2 \\ &\langle \mathrm{A}_1^*\widetilde{x}_g, \mathrm{A}_1^*\zeta\rangle_{H_1} + \langle \mathrm{A}_2^* p, \zeta\rangle_{H_2} + \langle h, \zeta\rangle_{H_2} = \langle g, \mathrm{A}_1^*\zeta\rangle_{H_1}, \\ &\langle \widetilde{x}_g, \mathrm{A}_2^*\phi\rangle_{H_2} = 0, \\ &\langle \widetilde{x}_g, \kappa\rangle_{H_2} = 0. \end{aligned} \tag{3.44}$$

*It holds $p=0$ and $h=0$ as well as (3.19). Moreover, $\mathrm{A}_1^*\widetilde{x}_g = g$ if and only if $g \in R(\mathrm{A}_1^*)$. The second equation of (3.44) holds for all $\phi \in D(\mathrm{A}_2^*)$ and thus $\widetilde{x}_g \in N(\mathrm{A}_2)$. Furthermore, $\pi_2\widetilde{x}_g = 0$. Finally, if $g \in R(\mathrm{A}_1^*)$, we have $\widetilde{x}_g = x_g$ from Theorem 3.3, see Theorem 3.5 (ii).*

[9] $v=0$ implies $f - h = \mathrm{A}_2\mathrm{A}_2^* y_f \in R(\mathrm{A}_2) \subset N(\mathrm{A}_3)$ and hence $f \in N(\mathrm{A}_3)$.

[10] $h=0$ implies $f - \mathrm{A}_3^* v = \mathrm{A}_2\mathrm{A}_2^* y_f \in R(\mathrm{A}_2) \perp_{H_3} K_3$ and hence $f \perp_{H_3} K_3$.

(ii') There exists a unique triple $(z_g, q, h) \in D(\mathrm{A}_1) \times D(\mathcal{A}_0) \times K_1$ such that

$$\begin{aligned} &\forall\ (\varphi, \vartheta, \kappa) \in D(\mathrm{A}_1) \times D(\mathcal{A}_0) \times K_1 \\ &\langle \mathrm{A}_1 z_g, \mathrm{A}_1 \varphi\rangle_{H_2} + \langle \mathrm{A}_0 q, \varphi\rangle_{H_1} + \langle h, \varphi\rangle_{H_1} = \langle g, \varphi\rangle_{H_1}, \\ &\langle z_g, \mathrm{A}_0 \vartheta\rangle_{H_1} = 0, \\ &\langle z_g, \kappa\rangle_{H_1} = 0. \end{aligned} \tag{3.45}$$

It holds $q = 0$ if and only if $g \perp_{H_1} R(\mathrm{A}_0)$ if and only if[11] $g \in N(\mathrm{A}_0^*)$. $h = 0$ if and only if[12] $g \perp_{H_1} K_1$. Thus $q = 0$ and $h = 0$ if and only if $g \in N(\mathrm{A}_0^*) \cap K_1^{\perp_{H_1}} = R(\mathrm{A}_1^*)$. Furthermore, (3.20) holds. Moreover, $\mathrm{A}_1 z_g \in D(\mathrm{A}_1^*)$ and $\mathrm{A}_1^* \mathrm{A}_1 z_g = g$ if and only if $g \in R(\mathrm{A}_1^*)$. The second equation of (3.45) holds for all $\vartheta \in D(\mathrm{A}_0)$ and hence $z_g \in N(\mathrm{A}_0^*)$. Furthermore, $\pi_1 z_g = 0$. Finally, if $g \in R(\mathrm{A}_1^*)$, we have $\mathrm{A}_1 z_g = x_g$ from Theorem 3.3, see Theorem 3.5 (ii').

*Proof.* The proof follows closely the lines of the proof of Theorem 3.12. □

**Remark 3.15.** *Again we have formal matrix representations:*

(i) *Equation (3.42) is a weak formulation of*

$$\mathrm{A}_2^* \mathrm{A}_2 \widetilde{x}_f + \mathrm{A}_1 u + h = \mathrm{A}_2^* f, \quad \mathrm{A}_1^* \widetilde{x}_f = 0, \quad \pi_2 \widetilde{x}_f = 0,$$

*i.e., in formal matrix notation*

$$\begin{bmatrix} \mathrm{A}_2^* \mathrm{A}_2 & \mathrm{A}_1 & \iota_{K_2} \\ \mathrm{A}_1^* & 0 & 0 \\ \pi_2 = \iota_{K_2}^* & 0 & 0 \end{bmatrix} \begin{bmatrix} \widetilde{x}_f \\ u \\ h \end{bmatrix} = \begin{bmatrix} \mathrm{A}_2^* f \\ 0 \\ 0 \end{bmatrix}.$$

*Note $u = 0$ and $h = 0$.*

(i') *Equation (3.43) is a weak formulation of*

$$\mathrm{A}_2 \mathrm{A}_2^* y_f + \mathrm{A}_3^* v + h = f, \quad \mathrm{A}_3 y_f = 0, \quad \pi_3 y_f = 0,$$

*i.e., in formal matrix notation*

$$\begin{bmatrix} \mathrm{A}_2 \mathrm{A}_2^* & \mathrm{A}_3^* & \iota_{K_3} \\ \mathrm{A}_3 & 0 & 0 \\ \pi_3 = \iota_{K_3}^* & 0 & 0 \end{bmatrix} \begin{bmatrix} y_f \\ v \\ h \end{bmatrix} = \begin{bmatrix} f \\ 0 \\ 0 \end{bmatrix}.$$

*Note $v = 0$ and $h = 0$.*

[11] $q = 0$ implies $g - h = \mathrm{A}_1^* \mathrm{A}_1 z_g \in R(\mathrm{A}_1^*) \subset N(\mathrm{A}_0^*)$ and hence $g \in N(\mathrm{A}_0^*)$.

[12] $h = 0$ implies $g - \mathrm{A}_0 q = \mathrm{A}_1^* \mathrm{A}_1 z_g \in R(\mathrm{A}_1^*) \perp_{H_1} K_1$ and hence $g \perp_{H_1} K_1$.

(ii) *Equation (3.44) is a weak formulation of*

$$\mathrm{A}_1\mathrm{A}_1^*\widetilde{x}_g + \mathrm{A}_2^* p + h = \mathrm{A}_1 g, \quad \mathrm{A}_2\widetilde{x}_g = 0, \quad \pi_2\widetilde{x}_g = 0,$$

*i.e., in formal matrix notation*

$$\begin{bmatrix} \mathrm{A}_1\mathrm{A}_1^* & \mathrm{A}_2^* & \iota_{K_2} \\ \mathrm{A}_2 & 0 & 0 \\ \pi_2 = \iota_{K_2}^* & 0 & 0 \end{bmatrix} \begin{bmatrix} \widetilde{x}_g \\ p \\ h \end{bmatrix} = \begin{bmatrix} \mathrm{A}_1 g \\ 0 \\ 0 \end{bmatrix}.$$

*Note* $p=0$ *and* $h=0$.

(ii') *Equation (3.45) is a weak formulation of*

$$\mathrm{A}_1^*\mathrm{A}_1 z_g + \mathrm{A}_0 q + h = g, \quad \mathrm{A}_0^* z_g = 0, \quad \pi_1 z_g = 0,$$

*i.e., in formal matrix notation*

$$\begin{bmatrix} \mathrm{A}_1^*\mathrm{A}_1 & \mathrm{A}_0 & \iota_{K_1} \\ \mathrm{A}_0^* & 0 & 0 \\ \pi_1 = \iota_{K_1}^* & 0 & 0 \end{bmatrix} \begin{bmatrix} z_g \\ q \\ h \end{bmatrix} = \begin{bmatrix} g \\ 0 \\ 0 \end{bmatrix}.$$

*Note* $q=0$ *and* $h=0$.

### *3.1.4. Even more variational formulations*

In our variational formulations still the unpleasant spaces $D(\mathcal{A}_\ell)$ and $D(\mathcal{A}_\ell^*)$ occur in the side conditions, see, e.g., Theorem 3.12, where

$$z \in D(\mathcal{A}_1) = D(\mathrm{A}_1) \cap R(\mathrm{A}_1^*), \quad y \in D(\mathcal{A}_2^*) = D(\mathrm{A}_2^*) \cap R(\mathrm{A}_2).$$

We can even go one step further and remove these restrictions just by applying the same ideas as before. E.g., in Theorem 3.12

$$\begin{aligned} z \in R(\mathrm{A}_1^*) &= N(\mathrm{A}_0^*) \cap K_1^{\perp_{H_1}} = R(\mathrm{A}_0)^{\perp_{H_1}} \cap K_1^{\perp_{H_1}}, \\ y \in R(\mathrm{A}_2) &= N(\mathrm{A}_3) \cap K_3^{\perp_{H_3}} = R(\mathrm{A}_3^*)^{\perp_{H_3}} \cap K_3^{\perp_{H_3}} \end{aligned}$$

can easily be formulated as additional side conditions. Of course, this procedure can be prolongated ad infinitum depending on the length of the underlying complex.

**Remark 3.16.** *In 3D applications the cohomology groups* $K_0$, $K_1$ *and* $K_4$, $K_5$ *are typically already trivial, see, e.g., the applications section 5.1. Also the kernels* $N(A_0)$ *and* $N(A_4)$ *are always trivial. Moreover, the kernels* $N(A_1)$ *and* $N(\mathrm{A}_3^*)$ *are typically trivial or at least finite dimensional. The same applies to the orthogonal complements of the kernels* $N(\mathrm{A}_0^*)$ *and* $N(A_4)$*. In particular,* $D(\mathcal{A}_1) = D(\mathrm{A}_1)$ *resp.* $D(\mathcal{A}_3^*) = D(\mathrm{A}_3^*)$ *or at least*

$$D(\mathcal{A}_1) = D(\mathrm{A}_1) \cap R(\mathrm{A}_1^*) = D(\mathrm{A}_1) \cap N(\mathrm{A}_1)^{\perp_{H_1}},$$
$$D(\mathcal{A}_3^*) = D(\mathrm{A}_3^*) \cap R(\mathrm{A}_3) = D(\mathrm{A}_3^*) \cap N(\mathrm{A}_3^*)^{\perp_{H_4}},$$

*respectively, with $N(A_1)$ resp. $N(\mathrm{A}_3^*)$ being finite dimensional. We always have $D(\mathcal{A}_0) = D(\mathrm{A}_0)$ and $D(\mathcal{A}_4^*) = D(\mathrm{A}_4^*)$.*

For example Theorem 3.12 can be modified as follows:

**Theorem 3.17.** *Let additionally $R(\mathrm{A}_0), R(\mathrm{A}_3)$, and $R(A_4)$ be closed. Moreover, let $f \in R(\mathrm{A}_2)$ and $g \in R(\mathrm{A}_1^*)$. The unique solution $x = x_f + x_g + k \in D_2$ in Theorem 3.3 can be found by the following three variational quadruple resp. sextuple saddle point formulations:*

(i) *There exists a unique five tuple $(\widetilde{x}, z, u, h_2, h_1) \in D(\mathrm{A}_2) \times D(\mathrm{A}_1) \times D(\mathcal{A}_0) \times K_2 \times K_1$ such that for all $(\xi, \varphi, \vartheta, \kappa, \lambda) \in D(\mathrm{A}_2) \times D(\mathrm{A}_1) \times D(\mathcal{A}_0) \times K_2 \times K_1$*

$$\begin{aligned} \langle \mathrm{A}_2\widetilde{x}, \mathrm{A}_2\xi\rangle_{H_3} + \langle \mathrm{A}_1 z, \xi\rangle_{H_2} + \langle h_2, \xi\rangle_{H_2} &= \langle f, \mathrm{A}_2\xi\rangle_{H_3}, \\ \langle \widetilde{x}, \mathrm{A}_1\varphi\rangle_{H_2} + \langle \mathrm{A}_0 u, \varphi\rangle_{H_1} + \langle h_1, \varphi\rangle_{H_1} &= \langle g, \varphi\rangle_{H_1}, \\ \langle z, \mathrm{A}_0\vartheta\rangle_{H_1} &= 0, \\ \langle \widetilde{x}, \kappa\rangle_{H_2} &= \langle k, \kappa\rangle_{H_2}, \\ \langle z, \lambda\rangle_{H_1} &= 0. \end{aligned} \tag{3.46}$$

*The third equation of (3.46) is valid for all $\vartheta \in D(\mathrm{A}_0)$. It holds $z = 0$ and $h_2 = 0$ as well as $u = 0$ and $h_1 = 0$. Moreover, $\mathrm{A}_2\widetilde{x} = f$ and $\widetilde{x} \in D(\mathrm{A}_1^*)$ with $\mathrm{A}_1^*\widetilde{x} = g$ as well as $\pi_2\widetilde{x} = k$. Finally, $\widetilde{x} = x$ from Theorem 3.3.*

(ii) *There exists a unique five tuple $(\hat{x}, y, v, h_2, h_3) \in D(\mathrm{A}_1^*) \times D(\mathrm{A}_2^*) \times D(\mathcal{A}_3^*) \times K_2 \times K_3$ such that for all $(\zeta, \phi, \theta, \kappa, \lambda) \in D(\mathrm{A}_1^*) \times D(\mathrm{A}_2^*) \times D(\mathcal{A}_3^*) \times K_2 \times K_3$*

$$\begin{aligned} \langle \mathrm{A}_1^*\hat{x}, \mathrm{A}_1^*\zeta\rangle_{H_1} + \langle \mathrm{A}_2^* y, \zeta\rangle_{H_2} + \langle h_2, \zeta\rangle_{H_2} &= \langle g, \mathrm{A}_1^*\zeta\rangle_{H_1}, \\ \langle \hat{x}, \mathrm{A}_2^*\phi\rangle_{H_2} + \langle \mathrm{A}_3^* v, \phi\rangle_{H_3} + \langle h_3, \phi\rangle_{H_3} &= \langle f, \phi\rangle_{H_3}, \\ \langle y, \mathrm{A}_3^*\theta\rangle_{H_3} &= 0, \\ \langle \hat{x}, \kappa\rangle_{H_2} &= \langle k, \kappa\rangle_{H_2}, \\ \langle y, \lambda\rangle_{H_3} &= 0. \end{aligned} \tag{3.47}$$

*The third equation of (3.47) is valid for all $\theta \in D(\mathrm{A}_3^*)$. It holds $y = 0$ and $h_2 = 0$ as well as $v = 0$ and $h_3 = 0$. Moreover, $\mathrm{A}_1^*\hat{x} = g$ and $\hat{x} \in D(\mathrm{A}_2)$ with $\mathrm{A}_2\hat{x} = f$ as well as $\pi_2\hat{x} = k$. Finally, $\hat{x} = x$ from Theorem 3.3.*

(ii') *There exists $(\hat{x}, y, v, w, h_2, h_3, h_4) \in D(\mathrm{A}_1^*) \times D(\mathrm{A}_2^*) \times D(\mathrm{A}_3^*) \times D(\mathcal{A}_4^*) \times K_2 \times K_3 \times K_4$, a unique seven tuple, such that for all $(\zeta, \phi, \theta, \sigma, \kappa, \lambda, \nu) \in D(\mathrm{A}_1^*) \times D(\mathrm{A}_2^*) \times D(\mathrm{A}_3^*) \times D(\mathcal{A}_4^*) \times K_2 \times K_3 \times K_4$*

$$\begin{aligned}\langle \mathrm{A}_1^*\hat{x}, \mathrm{A}_1^*\zeta\rangle_{H_1} + \langle \mathrm{A}_2^* y, \zeta\rangle_{H_2} + \langle h_2, \zeta\rangle_{H_2} &= \langle g, \mathrm{A}_1^*\zeta\rangle_{H_1},\\ \langle \hat{x}, \mathrm{A}_2^*\phi\rangle_{H_2} + \langle \mathrm{A}_3^* v, \phi\rangle_{H_3} + \langle h_3, \phi\rangle_{H_3} &= \langle f, \phi\rangle_{H_3},\\ \langle y, \mathrm{A}_3^*\theta\rangle_{H_3} + \langle \mathrm{A}_4^* w, \theta\rangle_{H_4} + \langle h_4, \theta\rangle_{H_4} &= 0,\\ \langle v, \mathrm{A}_4^*\sigma\rangle_{H_4} &= 0,\\ \langle \hat{x}, \kappa\rangle_{H_2} &= \langle k, \kappa\rangle_{H_2},\\ \langle y, \lambda\rangle_{H_3} &= 0,\\ \langle v, \nu\rangle_{H_4} &= 0.\end{aligned} \tag{3.48}$$

*The fourth equation of (3.48) is valid for all $\sigma \in D(\mathrm{A}_4^*)$. It holds $y=0$, $h_2=0$ and $v=0$, $h_3=0$ as well as $w=0$, $h_4=0$. Moreover, $\mathrm{A}_1^*\hat{x} = g$ and $\hat{x} \in D(\mathrm{A}_2)$ with $\mathrm{A}_2\hat{x} = f$ as well as $\pi_2\hat{x} = k$. Finally, $\hat{x} = x$ from Theorem 3.3.*

**Theorem 3.14** can be extended in the same way.

**Remark 3.18.** *For (ii') recall $R(\mathrm{A}_3) = N(\mathrm{A}_4) \cap K_4^{\perp_{H_4}} = R(\mathrm{A}_4^*)^{\perp_{H_4}} \cap K_4^{\perp_{H_4}}$. Let us also note that generally the solution and test spaces look like*

$$\begin{aligned}&D(\mathrm{A}_\ell) \times D(\mathrm{A}_{\ell-1}) \times \cdots \times D(\mathrm{A}_{\ell-n+1}) \times D(\mathcal{A}_{\ell-n}) \times K_\ell \times K_{\ell-1} \times \cdots \times K_{\ell-n+1},\\ &D(\mathrm{A}_\ell^*) \times D(\mathrm{A}_{\ell+1}^*) \times \cdots \times D(\mathrm{A}_{\ell+n-1}^*) \times D(\mathcal{A}_{\ell+n}^*) \times K_{\ell+1} \times K_{\ell+2} \times \cdots \times K_{\ell+n}.\end{aligned}$$

*Moreover:*

(i) *Equation (3.46) is a weak formulation of*

$$\mathrm{A}_2^*\mathrm{A}_2\tilde{x} + \mathrm{A}_1 z + h_2 = \mathrm{A}_2^* f, \quad \mathrm{A}_1^*\tilde{x} + \mathrm{A}_0 u + h_1 = g,$$
$$\mathrm{A}_0^* z = 0, \quad \pi_2\tilde{x} = k, \quad \pi_1 z = 0,$$

*i.e., in formal matrix notation*

$$\begin{bmatrix} \mathrm{A}_2^*\mathrm{A}_2 & \mathrm{A}_1 & 0 & \iota_{K_2} & 0\\ \mathrm{A}_1^* & 0 & \mathrm{A}_0 & 0 & \iota_{K_1}\\ 0 & \mathrm{A}_0^* & 0 & 0 & 0\\ \pi_2 = \iota_{K_2}^* & 0 & 0 & 0 & 0\\ 0 & \pi_1 = \iota_{K_1}^* & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{x}\\ z\\ u\\ h_2\\ h_1 \end{bmatrix} = \begin{bmatrix} \mathrm{A}_2^* f\\ g\\ 0\\ k\\ 0 \end{bmatrix}.$$

*Note $z=0$, $u=0$ and $h_2 = 0, h_1 = 0$.*

(ii) *Equation (3.47) is a weak formulation of*

$$\mathrm{A}_1\mathrm{A}_1^*\hat{x} + \mathrm{A}_2^* y + h_2 = \mathrm{A}_1 g, \quad \mathrm{A}_2\hat{x} + \mathrm{A}_3^* v + h_3 = f,$$
$$\mathrm{A}_3 y = 0, \quad \pi_2\hat{x} = k, \quad \pi_3 y = 0,$$

*i.e., in formal matrix notation*

$$\begin{bmatrix} A_1A_1^* & A_2^* & 0 & \iota_{K_2} & 0 \\ A_2 & 0 & A_3^* & 0 & \iota_{K_3} \\ 0 & A_3 & 0 & 0 & 0 \\ \pi_2 = \iota_{K_2}^* & 0 & 0 & 0 & 0 \\ 0 & \pi_3 = \iota_{K_3}^* & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{x} \\ y \\ v \\ h_2 \\ h_3 \end{bmatrix} = \begin{bmatrix} A_1 g \\ f \\ 0 \\ k \\ 0 \end{bmatrix}.$$

*Note* $y = 0$*,* $v = 0$ *and* $h_2 = 0, h_3 = 0$.

(ii') *Equation (3.48) is a weak formulation of*

$$A_1A_1^*\hat{x} + A_2^*y + h_2 = A_1 g, \quad A_2\hat{x} + A_3^*v + h_3 = f,$$
$$A_3 y + A_4^* w + h_4 = 0, \quad A_4 v = 0,$$

*and* $\pi_2\hat{x} = k, \pi_3 y = 0, \pi_4 v = 0$*, i.e., in formal matrix notation*

$$\begin{bmatrix} A_1A_1^* & A_2^* & 0 & 0 & \iota_{K_2} & 0 & 0 \\ A_2 & 0 & A_3^* & 0 & 0 & \iota_{K_3} & 0 \\ 0 & A_3 & 0 & A_4^* & 0 & 0 & \iota_{K_4} \\ 0 & 0 & A_4 & 0 & 0 & 0 & 0 \\ \pi_2 = \iota_{K_2}^* & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \pi_3 = \iota_{K_3}^* & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \pi_4 = \iota_{K_4}^* & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{x} \\ y \\ v \\ w \\ h_2 \\ h_3 \\ h_4 \end{bmatrix} = \begin{bmatrix} A_1 g \\ f \\ 0 \\ 0 \\ k \\ 0 \\ 0 \end{bmatrix}.$$

*Note* $y = 0$*,* $v = 0$*,* $w = 0$ *and* $h_2 = 0, h_3 = 0, h_4 = 0$.

### 3.2. Second order systems

We recall the linear second order system (1.10), i.e., find[13]

$$\begin{aligned} x \in \widetilde{D}_2 &:= \{\xi \in D_2 \,:\, A_2\xi \in D(A_2^*)\} \\ &= \{\xi \in D(A_2) \cap D(A_1^*) \,:\, A_2\xi \in D(A_2^*)\} \\ &= D(A_1^*) \cap D(A_2^*A_2) \end{aligned}$$

such that

$$\begin{aligned} A_2^*A_2 x &= f, \\ A_1^* x &= g, \\ \pi_2 x &= k. \end{aligned} \tag{3.49}$$

**Theorem 3.19.** *Equation (3.49) is uniquely solvable in* $\widetilde{D}_2$*, if and only if* $f \in R(A_2^*), g \in R(A_1^*)$*, and* $k \in K_2$*. The unique solution* $x \in \widetilde{D}_2$ *is given by*

[13]We generally define $\widetilde{D}_\ell\{\xi \in D_\ell : A_\ell\xi \in D(A^*{}_\ell)\} = D(A^*{}_{\ell-1}) \cap D(A_\ell A^*{}_\ell)$ for $\ell = 1, \ldots, 3$.

$$x := x_f + x_g + k \in \big(D(\mathcal{A}_2)\oplus_{H_2}D(\mathcal{A}_1^*)\oplus_{H_2}K_2\big)\cap\widetilde{D}_2 = \widetilde{D}_2,$$
$$x_f := \mathcal{A}_2^{-1}\big(\mathcal{A}_2^*\big)^{-1}f \in D\big(\mathcal{A}_2^*\mathcal{A}_2\big) = D(\mathcal{A}_2)\cap\widetilde{D}_2,$$
$$x_g := \big(\mathcal{A}_1^*\big)^{-1}g \in D\big(\mathcal{A}_1^*\big) = D\big(\mathcal{A}_1^*\big)\cap\widetilde{D}_2$$

*and depends continuously on the data, i.e.,* $|x|_{H_2} \le c_2^2|f|_{H_2} + c_1|g|_{H_1} + |k|_{H_2}$, *as*

$$|x_f|_{H_2} \le c_2^2|f|_{H_2}, \quad |x_g|_{H_2} \le c_1|g|_{H_1}.$$

*It holds*

$$\pi_{\mathrm{A}_2^*}x = x_f, \quad \pi_{\mathrm{A}_1}x = x_g, \quad \pi_2 x = k, \quad |x|_{H_2}^2 = |x_f|_{H_2}^2 + |x_g|_{H_2}^2 + |k|_{H_2}^2.$$

*Proof.* The necessary conditions are clear. To show uniqueness, let $x \in \widetilde{D}_2$ solve

$$\mathrm{A}_2^*\mathrm{A}_2 x = 0, \quad \mathrm{A}_1^* x = 0, \quad \pi_2 x = 0.$$

Hence $x \in N(\mathrm{A}_1^*)\cap K_2^{\perp_{H_2}}$ and also $x \in N(\mathrm{A}_2)$ as $\mathrm{A}_2 x \in D(\mathrm{A}_2^*)$ and

$$|\mathrm{A}_2 x|_{H_3}^2 = \langle x, \mathrm{A}_2^*\mathrm{A}_2 x\rangle_{H_2} = 0,$$

yielding $x \in K_2 \cap K_2^{\perp_{H_2}} = \{0\}$. To prove existence, let $f \in R(\mathrm{A}_2^*), g \in R(\mathrm{A}_1^*), k \in K_2$ and define $x$, $x_f$, and $x_g$ according to the theorem. Again the orthogonality follows directly by Lemma 2.7. Moreover, $x_f$, $x_g$, and $k$ solve the linear systems

$$\begin{aligned} \mathrm{A}_2^*\mathrm{A}_2 x_f &= f, & \mathrm{A}_2 x_g &= 0, & \mathrm{A}_2 k &= 0,\\ \mathrm{A}_1^* x_f &= 0, & \mathrm{A}_1^* x_g &= g, & \mathrm{A}_1^* k &= 0,\\ \pi_2 x_f &= 0, & \pi_2 x_g &= 0, & \pi_2 k &= k. \end{aligned}$$

Thus $x$ solves (3.49) and we have by Corollary 2.5 $|x_f|_{H_2} \le c_2|\mathrm{A}_2 x_f|_{H_3} \le c_2^2|f|_{H_2}$ and $|x_g|_{H_2} \le c_1|g|_{H_1}$, completing the proof of the solution theory. □

**Remark 3.20.** *By orthogonality and with* $\mathrm{A}_2 x = (\mathcal{A}_2^*)^{-1}f$, $\mathrm{A}_2^*\mathrm{A}_2 x = f$, *and* $\mathrm{A}_1^* x = g$ *we even have*

$$\begin{aligned} |x|_{H_2}^2 &= |x_f|_{H_2}^2 + |x_g|_{H_2}^2 + |k|_{H_2}^2 \le c_2^4|f|_{H_2}^2 + c_1^2|g|_{H_1}^2 + |k|_{H_2}^2,\\ |x|_{\widetilde{D}_2}^2 &= |x_f|_{H_2}^2 + |\mathrm{A}_2 x|_{H_3}^2 + |f|_{H_2}^2 + |x_g|_{H_2}^2 + |g|_{H_1}^2 + |k|_{H_2}^2\\ &\le \big(1 + c_2^2 + c_2^4\big)|f|_{H_3}^2 + \big(1 + c_1^2\big)|g|_{H_1}^2 + |k|_{H_2}^2. \end{aligned}$$

**Remark 3.21.** *Since the second order system (3.49) decomposes into the two first order systems of shape (1.5) resp. (3.1), i.e.,*

$$\begin{aligned} \mathrm{A}_2 x &= y, & \mathrm{A}_3 y &= 0,\\ \mathrm{A}_1^* x &= g, & \mathrm{A}_2^* y &= f,\\ \pi_2 x &= k, & \pi_3 y &= 0 \end{aligned}$$

*for the pair* $(x, y) \in D_2 \times D_3$ *with* $y := \mathrm{A}_2 x \in D(\mathrm{A}_2^*) \cap R(\mathrm{A}_2) = D(\mathcal{A}_2^*)$, *the solution theory follows directly by Theorem 3.3 as well. One just has to solve and set*

$$
\begin{aligned}
y &:= (\mathcal{A}_2^*)^{-1} f \in D(\mathcal{A}_2^*) \subset R(\mathrm{A}_2), \\
x &:= \mathcal{A}_2^{-1} y + (\mathcal{A}_1^*)^{-1} g + k \in \big(D(\mathcal{A}_2) \oplus_{H_2} D(\mathcal{A}_1^*) \oplus_{H_2} K_2\big) \cap \widetilde{D}_2 = \widetilde{D}_2.
\end{aligned}
$$

### 3.2.1. Variational formulations

We note

$$
\begin{aligned}
D(\mathcal{A}_2^*\mathcal{A}_2) &= D(\mathrm{A}_2^*\mathcal{A}_2) = D(\mathrm{A}_2^*\mathrm{A}_2) \cap D(\mathcal{A}_2) = D(\mathrm{A}_2^*\mathrm{A}_2) \cap R(\mathrm{A}_2^*) \\
&= D(\mathrm{A}_2^*\mathrm{A}_2) \cap N(\mathrm{A}_1^*) \cap K_2^{\perp_{H_2}} \\
&= \widetilde{D}_2 \cap D(\mathcal{A}_2) = \widetilde{D}_2 \cap R(\mathrm{A}_2^*) = \widetilde{D}_2 \cap N(\mathrm{A}_1^*) \cap K_2^{\perp_{H_2}}, \qquad (3.50) \\
D(\mathcal{A}_1^*) &= D(\mathrm{A}_1^*) \cap R(\mathrm{A}_1) = D(\mathrm{A}_1^*) \cap N(\mathrm{A}_2) \cap K_2^{\perp_{H_2}} \\
&= \widetilde{D}_2 \cap D(\mathcal{A}_1^*) = \widetilde{D}_2 \cap R(\mathcal{A}_1) = \widetilde{D}_2 \cap N(\mathrm{A}_2) \cap K_2^{\perp_{H_2}}
\end{aligned}
$$

and recall

$$
\begin{aligned}
x_f &= \mathcal{A}_2^{-1}(\mathcal{A}_2^*)^{-1} f \in D(\mathcal{A}_2^*\mathcal{A}_2) = D(\mathrm{A}_2^*\mathrm{A}_2) \cap R(\mathrm{A}_2^*) = D(\mathrm{A}_2^*\mathrm{A}_2) \cap N(\mathrm{A}_1^*) \cap K_2^{\perp_{H_2}}, \\
x_g &= (\mathcal{A}_1^*)^{-1} g \in D(\mathcal{A}_1^*) = D(\mathrm{A}_1^*) \cap R(\mathrm{A}_1) = D(\mathrm{A}_1^*) \cap N(\mathrm{A}_2) \cap K_2^{\perp_{H_2}}.
\end{aligned}
$$

As in the corresponding section for the first order systems, there are several options for variational formulations for finding each of the partial solutions $x_f$ and $x_g$, which all make sense from a functional analytical point of view. Looking at Remark 3.21 it is clear that all variational formulations proposed for the first order systems from the earlier sections are applicable here as well. Especially for $x_g$ we do not observe anything new. On the other hand, for the second order system related to $x_f$ we can do as follows: The first option is to multiply the equation $\mathrm{A}_2^*\mathrm{A}_2 x_f = f$ by $\mathrm{A}_2^*\mathrm{A}_2\phi$ with some $\phi \in D(\mathcal{A}_2^*\mathcal{A}_2)$ giving the variational formulation

$$
\forall\ \phi \in D(\mathcal{A}_2^*\mathcal{A}_2) \qquad \langle \mathrm{A}_2^*\mathrm{A}_2 x_f, \mathrm{A}_2^*\mathrm{A}_2\phi\rangle_{H_2} = \langle f, \mathrm{A}_2^*\mathrm{A}_2\phi\rangle_{H_2},
$$

which is a weak formulation of the fourth order equation

$$
(\mathrm{A}_2^*\mathrm{A}_2)^2 x_f = \mathrm{A}_2^*\mathrm{A}_2 f,
$$

more precisely of $\mathrm{A}_2^*\mathrm{A}_2(\mathrm{A}_2^*\mathrm{A}_2 x_f - f) = 0$. Perhaps a more convenient choice is to multiply $\mathrm{A}_2^*\mathrm{A}_2 x_f = f$ by some $\xi \in D(\mathcal{A}_2)$ giving the variational formulation

$$
\forall\ \xi \in D(\mathcal{A}_2) \qquad \langle \mathrm{A}_2 x_f, \mathrm{A}_2\xi\rangle_{H_3} = \langle f, \xi\rangle_{H_2},
$$

which is a weak formulation of the second order equation

$$\mathrm{A}_2^*\mathrm{A}_2 x_f = f.$$

The latter choices are finding straight forward $x_f$ itself. As a third option, we propose a formulation to find a potential $y_f$ for $x_f$. For this we go for a second order potential $y_f$ with $\mathrm{A}_2^*\mathrm{A}_2 y_f = x_f$, e.g., $y_f := \mathcal{A}_2^{-1}(\mathcal{A}_2^*)^{-1} x_f \in D(\mathcal{A}_2^*\mathcal{A}_2)$, of

$$x_f = \mathrm{A}_2^*\mathrm{A}_2 y_f \in D\big(\mathcal{A}_2^*\mathcal{A}_2\big) = D\big(\mathrm{A}_2^*\mathrm{A}_2\big) \cap R\big(\mathrm{A}_2^*\big) = D\big(\mathrm{A}_2^*\mathrm{A}_2\big) \cap R\big(\mathcal{A}_2^*\big).$$

Multiplying by $\mathrm{A}_2^*\mathrm{A}_2\tau$ with some $\tau \in D(\mathcal{A}_2^*\mathcal{A}_2)$ gives

$$\forall\ \tau \in D\big(\mathcal{A}_2^*\mathcal{A}_2\big) \quad \langle \mathrm{A}_2^*\mathrm{A}_2 y_f, \mathrm{A}_2^*\mathrm{A}_2\tau\rangle_{H_2} = \langle x_f, \mathrm{A}_2^*\mathrm{A}_2\tau\rangle_{H_2} = \langle \mathrm{A}_2^*\mathrm{A}_2 x_f, \tau\rangle_{H_2} = \langle f, \tau\rangle_{H_2},$$

which is a weak formulation of the fourth order equation

$$\big(\mathrm{A}_2^*\mathrm{A}_2\big)^2 y_f = f.$$

**Theorem 3.22.** *The partial solutions $x_f$ and $x_g$ in Theorem 3.19 can be found by the following variational formulations:*

(i) *There exists a unique $\hat{x}_f \in D(\mathcal{A}_2^*\mathcal{A}_2)$, such that*

$$\forall\ \phi \in D\big(\mathcal{A}_2^*\mathcal{A}_2\big) \qquad \langle \mathrm{A}_2^*\mathrm{A}_2\hat{x}_f, \mathrm{A}_2^*\mathrm{A}_2\phi\rangle_{H_2} = \langle f, \mathrm{A}_2^*\mathrm{A}_2\phi\rangle_{H_2}. \tag{3.51}$$

*Equation (3.51) even holds for all $\phi \in D(\mathrm{A}_2^*\mathrm{A}_2)$. Moreover, $\mathrm{A}_2^*\mathrm{A}_2\hat{x}_f = f$ if and only if $f \in R(\mathrm{A}_2^*)$. In this case $\hat{x}_f = x_f$.*

(i') *There exists a unique $\widetilde{x}_f \in D(\mathcal{A}_2)$, such that*

$$\forall\ \xi \in D(\mathcal{A}_2) \qquad \langle \mathrm{A}_2\widetilde{x}_f, \mathrm{A}_2\xi\rangle_{H_3} = \langle f, \xi\rangle_{H_2}. \tag{3.52}$$

*Equation (3.52) even holds for all $\xi \in D(\mathrm{A}_2)$ if and only if $f \in R(\mathrm{A}_2^*)$. In this case we have*

$$\mathrm{A}_2\widetilde{x}_f \in D\big(\mathrm{A}_2^*\big) \cap R(\mathrm{A}_2) = D\big(\mathcal{A}_2^*\big)$$

*with $\mathrm{A}_2^*\mathrm{A}_2\widetilde{x}_f = f$ and thus $\widetilde{x}_f = x_f$.*

(i'') *There exists a unique potential $y_f \in D(\mathcal{A}_2^*\mathcal{A}_2)$, such that*

$$\forall\ \tau \in D\big(\mathcal{A}_2^*\mathcal{A}_2\big) \qquad \langle \mathrm{A}_2^*\mathrm{A}_2 y_f, \mathrm{A}_2^*\mathrm{A}_2\tau\rangle_{H_2} = \langle f, \tau\rangle_{H_2}. \tag{3.53}$$

*Equation (3.53) even holds for all $\tau \in D(\mathrm{A}_2^*\mathrm{A}_2)$ if and only if $f \in R(\mathrm{A}_2^*)$. In this case we have*

$$\mathrm{A}_2^*\mathrm{A}_2 y_f \in D\big(\mathcal{A}_2^*\mathcal{A}_2\big)$$

*with $(\mathrm{A}_2^*\mathrm{A}_2)^2 y_f = f$ and hence $\mathrm{A}_2^*\mathrm{A}_2 y_f = x_f$.*

(ii) *There exists a unique* $\widetilde{x}_g \in D(\mathcal{A}_1^*)$ *such that*

$$\forall\ \zeta \in D(\mathcal{A}_1^*) \qquad \langle \mathrm{A}_1^* \widetilde{x}_g, \mathrm{A}_1^* \zeta\rangle_{H_1} = \langle g, \mathrm{A}_1^* \zeta\rangle_{H_1}. \tag{3.54}$$

*Equation (3.54) is even satisfied for all* $\zeta \in D(\mathrm{A}_1^*)$. *Moreover,* $\mathrm{A}_1^* \widetilde{x}_g = g$ *holds if and only if* $g \in R(\mathrm{A}_1^*)$. *In this case* $\widetilde{x}_g = x_g$.

(ii') *There exists a unique potential* $z_g \in D(\mathcal{A}_1)$, *such that*

$$\forall\ \varphi \in D(\mathcal{A}_1) \qquad \langle \mathrm{A}_1 z_g, \mathrm{A}_1 \varphi\rangle_{H_2} = \langle g, \varphi\rangle_{H_1}. \tag{3.55}$$

*Equation (3.55) even holds for all* $\varphi \in D(\mathrm{A}_1)$ *if and only if* $g \in R(\mathrm{A}_1^*)$. *In this case we have*

$$\mathrm{A}_1 z_g \in D(\mathrm{A}_1^*) \cap R(\mathrm{A}_1) = D(\mathcal{A}_1^*)$$

*with* $\mathrm{A}_1^* \mathrm{A}_1 z_g = g$ and hence $\mathrm{A}_1 z_g = x_g$

*Proof.* To show (i), let $\phi \in D(\mathcal{A}_2^* \mathcal{A}_2)$. Then $\mathrm{A}_2 \phi$ belongs to $D(\mathcal{A}_2^*)$ and by Corollary 2.5 (i) we see

$$|\mathrm{A}_2^* \mathrm{A}_2 \phi|_{H_2} \geq \frac{1}{c_2} |\mathrm{A}_2 \phi|_{H_3} \geq \frac{1}{c_2^2} |\phi|_{H_2}.$$

Hence, the bilinear form in (3.51) is strictly positive over $D(\mathcal{A}_2^* \mathcal{A}_2)$ and thus Riesz' representation theorem yields the unique solvability of (3.51). From $D(\mathrm{A}_2) = N(\mathrm{A}_2) \oplus_{H_2} D(\mathcal{A}_2)$, see Corollary 2.5 (iii) or Lemma 2.7, and (3.50) we get

$$D(\mathrm{A}_2^* \mathrm{A}_2) = N(\mathrm{A}_2) \oplus_{H_2} D(\mathrm{A}_2^* \mathcal{A}_2) = N(\mathrm{A}_2) \oplus_{H_2} D(\mathcal{A}_2^* \mathcal{A}_2).$$

Therefore $R(\mathrm{A}_2^* \mathrm{A}_2) = R(\mathcal{A}_2^* \mathcal{A}_2)$ and thus (3.51) holds for all $\phi \in D(\mathrm{A}_2^* \mathrm{A}_2)$ as well. Let $\psi \in D(\mathrm{A}_2^*)$ and decompose it according to Corollary 2.5 (iii) or Lemma 2.7 into $\psi = \psi_N + \psi_R \in D(\mathrm{A}_2^*) = N(\mathrm{A}_2^*) \oplus_{H_3} D(\mathcal{A}_2^*)$ (null space and range) and, as $D(\mathcal{A}_2^*) = D(\mathrm{A}_2^*) \cap R(\mathrm{A}_2) = D(\mathcal{A}_2^*) \cap R(\mathcal{A}_2)$, further into[14]

$$\psi = \psi_N + \mathrm{A}_2 \phi_R \in D(\mathrm{A}_2^*) = N(\mathrm{A}_2^*) \oplus_{H_3} \big(D(\mathcal{A}_2^*) \cap R(\mathcal{A}_2)\big), \quad \phi_R \in D(\mathcal{A}_2^* \mathcal{A}_2).$$

Utilizing the latter decomposition and (3.51) we obtain for all $\psi \in D(\mathrm{A}_2^*)$

$$\langle \mathrm{A}_2^* \mathrm{A}_2 \hat{x}_f, \mathrm{A}_2^* \psi\rangle_{H_2} = \langle \mathrm{A}_2^* \mathrm{A}_2 \hat{x}_f, \mathrm{A}_2^* \mathrm{A}_2 \phi_R\rangle_{H_2} = \langle f, \mathrm{A}_2^* \mathrm{A}_2 \phi_R\rangle_{H_2} = \langle f, \mathrm{A}_2^* \psi\rangle_{H_2},$$

[14] Here it would be enough to decompose
$\psi = \psi_N + \mathrm{A}_2 \phi_R \in D(\mathrm{A}_2^*) = N(\mathrm{A}_2^*) \oplus_{H_3} (D(\mathrm{A}_2^*) \cap R(\mathrm{A}_2)), \quad \phi_R \in D(\mathrm{A}_2^* \mathrm{A}_2)$.

which shows $\mathrm{A}_2^*\mathrm{A}_2\hat{x}_f - f \in N(\mathrm{A}_2) = R(\mathrm{A}_2^*)^{\perp_{H_2}}$. Thus, $\mathrm{A}_2^*\mathrm{A}_2\hat{x}_f - f = 0$, if and only if $f \in R(\mathrm{A}_2^*)$. In this case we have $\mathrm{A}_2^*\mathrm{A}_2(\hat{x}_f - x_f) = 0$ and the injectivity of $\mathcal{A}_2^*$ and $\mathcal{A}_2$ shows $\hat{x}_f = x_f$, which finishes the proof of (i).

The left hand side of (3.52) is strictly positive over $D(\mathcal{A}_2)$ and thus Riesz' representation theorem yields the unique solvability of (3.52). Let us recall that the orthonormal projector $\pi_{\mathrm{A}_2^*}$ onto $R(\mathrm{A}_2^*)$ satisfies $\mathrm{A}_2\pi_{\mathrm{A}_2^*}\xi = \mathrm{A}_2\xi$ and $\pi_{\mathrm{A}_2^*}\xi \in D(\mathcal{A}_2)$ for $\xi \in D(\mathrm{A}_2)$ and $\pi_{\mathrm{A}_2^*}f = f$ for $f \in R(\mathrm{A}_2^*)$. Therefore, if $f \in R(\mathrm{A}_2^*)$, then (3.52) yields for $\xi \in D(\mathrm{A}_2)$

$$\langle \mathrm{A}_2\widetilde{x}_f, \mathrm{A}_2\xi\rangle_{H_3} = \langle \mathrm{A}_2\widetilde{x}_f, \mathrm{A}_2\pi_{\mathrm{A}_2^*}\xi\rangle_{H_3} = \langle f, \pi_{\mathrm{A}_2^*}\xi\rangle_{H_2} = \langle \pi_{\mathrm{A}_2^*}f, \xi\rangle_{H_2} = \langle f, \xi\rangle_{H_2},$$

i.e., (3.52) holds for $\xi \in D(\mathrm{A}_2)$. On the other hand, if (3.52) holds for $\xi \in D(\mathrm{A}_2)$, then $\mathrm{A}_2\widetilde{x}_f \in D(\mathrm{A}_2^*)$ and $\mathrm{A}_2^*\mathrm{A}_2\widetilde{x}_f = f$, especially[15] $f \in R(\mathrm{A}_2^*)$. As in this case $\widetilde{x}_f \in D(\mathcal{A}_2)$ and $\mathrm{A}_2\widetilde{x}_f \in D(\mathcal{A}_2^*)$ with $\mathrm{A}_2^*\mathrm{A}_2\widetilde{x}_f = f$, we get $\widetilde{x}_f = x_f$ by the injectivity of $\mathcal{A}_2^*$ and $\mathcal{A}_2$, which shows (i').

In (i") the unique solvability follows as in (i). Let $f \in R(\mathrm{A}_2^*)$. Using the same arguments with the same projector $\pi_{\mathrm{A}_2^*}$ as in (i') we obtain by (3.53) for all $\tau \in D(\mathrm{A}_2^*\mathrm{A}_2)$

$$\begin{aligned}\langle \mathrm{A}_2^*\mathrm{A}_2 y_f, \mathrm{A}_2^*\mathrm{A}_2\tau\rangle_{H_2} &= \langle \mathrm{A}_2^*\mathrm{A}_2 y_f, \mathrm{A}_2^*\mathrm{A}_2\pi_{\mathrm{A}_2^*}\tau\rangle_{H_2} = \langle f, \pi_{\mathrm{A}_2^*}\tau\rangle_{H_2} \\ &= \langle \pi_{\mathrm{A}_2^*}f, \tau\rangle_{H_2} = \langle f, \tau\rangle_{H_2},\end{aligned}$$

as $\pi_{\mathrm{A}_2^*}\tau \in D(\mathrm{A}_2^*\mathcal{A}_2) = D(\mathcal{A}_2^*\mathcal{A}_2)$ by (3.50). Thus (3.53) holds for all $\tau \in D(\mathrm{A}_2^*\mathrm{A}_2)$. On the other hand, if (3.53) holds for all $\tau \in D(\mathrm{A}_2^*\mathrm{A}_2)$, then we obtain $\langle f, \tau\rangle_{H_2} = 0$ for all $\tau \in N(\mathrm{A}_2)$, showing $f \in N(\mathrm{A}_2)^{\perp_{H_2}} = R(\mathrm{A}_2^*)$. Now, in this case of $f \in R(\mathrm{A}_2^*) = R(\mathcal{A}_2^*)$, we define $h := (\mathcal{A}_2^*)^{-1}f \in D(\mathcal{A}_2^*)$ and observe with $\mathrm{A}_2^*h = f$ that by (3.53) for all $\tau \in D(\mathrm{A}_2^*\mathrm{A}_2)$

$$\langle \mathrm{A}_2^*\mathrm{A}_2 y_f, \mathrm{A}_2^*\mathrm{A}_2\tau\rangle_{H_2} = \langle f, \tau\rangle_{H_2} = \langle h, \mathrm{A}_2\tau\rangle_{H_3} = \langle h, \pi_{\mathrm{A}_2}\mathrm{A}_2\tau\rangle_{H_3}. \tag{3.56}$$

As in the proof of (i), let $\psi \in D(\mathrm{A}_2^*)$ and let it be decomposed into

$$\psi = \psi_N + \mathrm{A}_2\tau \in D\big(\mathrm{A}_2^*\big) = N\big(\mathrm{A}_2^*\big) \oplus_{H_3} \big(D\big(\mathrm{A}_2^*\big) \cap R(\mathrm{A}_2)\big), \quad \tau \in D\big(\mathrm{A}_2^*\mathrm{A}_2\big).$$

Using (3.56) and the latter decomposition we see for all $\psi \in D(\mathrm{A}_2^*)$

$$\begin{aligned}\langle \mathrm{A}_2^*\mathrm{A}_2 y_f, \mathrm{A}_2^*\psi\rangle_{H_2} &= \langle \mathrm{A}_2^*\mathrm{A}_2 y_f, \mathrm{A}_2^*\mathrm{A}_2\tau\rangle_{H_2} = \langle h, \pi_{\mathrm{A}_2}\mathrm{A}_2\tau\rangle_{H_3} \\ &= \langle h, \pi_{\mathrm{A}_2}\psi\rangle_{H_3} = \langle h, \psi\rangle_{H_3},\end{aligned}$$

since $h \in D(\mathcal{A}_2^*) \subset R(\mathrm{A}_2)$. Thus $\mathrm{A}_2^*\mathrm{A}_2 y_f \in D(\mathrm{A}_2)$ and $\mathrm{A}_2\mathrm{A}_2^*\mathrm{A}_2 y_f = h \in D(\mathcal{A}_2^*)$, showing

$$\mathrm{A}_2^*\mathrm{A}_2\mathrm{A}_2^*\mathrm{A}_2 y_f = \mathrm{A}_2^*h = f.$$

[15]Another proof is the following: Pick $\xi \in N(\mathrm{A}_2)$ and get by (3.52) directly $\langle f, \xi\rangle_{H_2} = 0$. Thus $f \in N(\mathrm{A}_2)^{\perp_{H_2}} = R(\mathrm{A}_2^*)$.

Since $(\mathrm{A}_2^*\mathrm{A}_2)^2 y_f = (\mathcal{A}_2^*\mathcal{A}_2)^2 y_f$ we get $\mathcal{A}_2^*\mathcal{A}_2(\mathcal{A}_2^*\mathcal{A}_2 y_f - x_f) = 0$ and injectivity yields $\mathcal{A}_2^*\mathcal{A}_2 y_f = x_f$.

(ii) and (ii') are clear from Theorem 3.5 (ii), (ii'). □

**Remark 3.23.** *Note that*

$$\hat{x}_f = \tilde{x}_f = x_f = \mathcal{A}_2^{-1}(\mathcal{A}_2^*)^{-1} f \in D(\mathcal{A}_2^*\mathcal{A}_2), \quad \tilde{x}_g = x_g = (\mathcal{A}_1^*)^{-1} g \in D(\mathcal{A}_1^*),$$
$$y_f = \mathcal{A}_2^{-1}(\mathcal{A}_2^*)^{-1} x_f = \Big(\mathcal{A}_2^{-1}(\mathcal{A}_2^*)^{-1}\Big)^2 f \in D\Big((\mathcal{A}_2^*\mathcal{A}_2)^2\Big),$$
$$z_g = \mathcal{A}_1^{-1} x_g = \mathcal{A}_1^{-1}(\mathcal{A}_1^*)^{-1} g \in D(\mathcal{A}_1^*\mathcal{A}_1)$$

*holds with* $\mathrm{A}_2^*\mathrm{A}_2 x_f = f, \mathrm{A}_2^*\mathrm{A}_2 y_f = x_f$ *and* $\mathrm{A}_1^* x_g = g, \mathrm{A}_1 z_g = x_g$. *Hence* $x_f$, $x_g$, *and* $y_f$, $z_g$ *solve the first resp. second order systems*

$$\begin{aligned} &\mathrm{A}_2^*\mathrm{A}_2 x_f = f, && \mathrm{A}_2 x_g = 0, && \mathrm{A}_2^*\mathrm{A}_2 y_f = x_f, && (\mathrm{A}_2^*\mathrm{A}_2)^2 y_f = f, && \mathrm{A}_1 z_g = x_g, && \mathrm{A}_1^*\mathrm{A}_1 z_g = g,\\ &\mathrm{A}_1^* x_f = 0, && \mathrm{A}_1^* x_g = g, && \mathrm{A}_1^* y_f = 0, && \mathrm{A}_1^* y_f = 0, && \mathrm{A}_0^* z_g = 0, && \mathrm{A}_0^* z_g = 0,\\ &\pi_2 x_f = 0, && \pi_2 x_g = 0, && \pi_2 y_f = 0, && \pi_2 y_f = 0, && \pi_1 z_g = 0, && \pi_1 z_g = 0. \end{aligned}$$

*Moreover:*

(i) *Equation (3.51) is a weak formulation of*

$$(\mathrm{A}_2^*\mathrm{A}_2)^2 \hat{x}_f = \mathrm{A}_2^*\mathrm{A}_2 f, \quad \mathrm{A}_1^*\hat{x}_f = 0, \quad \pi_2\hat{x}_f = 0,$$

*i.e., in formal matrix notation*

$$\begin{bmatrix} (\mathrm{A}_2^*\mathrm{A}_2)^2 \\ \mathrm{A}_1^* \\ \pi_2 \end{bmatrix} \big[\hat{x}_f\big] = \begin{bmatrix} \mathrm{A}_2^*\mathrm{A}_2 f \\ 0 \\ 0 \end{bmatrix}.$$

(i') *Equation (3.52) is a weak formulation of*

$$\mathrm{A}_2^*\mathrm{A}_2\tilde{x}_f = f, \quad \mathrm{A}_1^*\tilde{x}_f = 0, \quad \pi_2\tilde{x}_f = 0,$$

*i.e., in formal matrix notation*

$$\begin{bmatrix} \mathrm{A}_2^*\mathrm{A}_2 \\ \mathrm{A}_1^* \\ \pi_2 \end{bmatrix} \big[\tilde{x}_f\big] = \begin{bmatrix} f \\ 0 \\ 0 \end{bmatrix}.$$

(i'') *Equation (3.53) is a weak formulation of*

$$(\mathrm{A}_2\mathrm{A}_2^*)^2 y_f = f, \quad \mathrm{A}_1^* y_f = 0, \quad \pi_2 y_f = 0,$$

*i.e., in formal matrix notation*

$$\begin{bmatrix} (A_2A_2^*)^2 \\ A_1^* \\ \pi_2 \end{bmatrix} \begin{bmatrix} y_f \end{bmatrix} = \begin{bmatrix} f \\ 0 \\ 0 \end{bmatrix}.$$

(ii) *Equation (3.54) is a weak formulation of*

$$A_1A_1^*\widetilde{x}_g = A_1 g, \quad A_2\widetilde{x}_g = 0, \quad \pi_2\widetilde{x}_g = 0,$$

*i.e., in formal matrix notation*

$$\begin{bmatrix} A_1A_1^* \\ A_2 \\ \pi_2 \end{bmatrix} \begin{bmatrix} \widetilde{x}_g \end{bmatrix} = \begin{bmatrix} A_1 g \\ 0 \\ 0 \end{bmatrix}.$$

(ii') *Equation (3.55) is a weak formulation of*

$$A_1^*A_1 z_g = g, \quad A_0^* z_g = 0, \quad \pi_1 z_g = 0,$$

*i.e., in formal matrix notation*

$$\begin{bmatrix} A_1^*A_1 \\ A_0^* \\ \pi_1 \end{bmatrix} \begin{bmatrix} z_g \end{bmatrix} = \begin{bmatrix} g \\ 0 \\ 0 \end{bmatrix}.$$

As before we emphasize that the variational formulations (3.51)–(3.55) have again saddle point structure. Provided $f \in R(A_2^*)$ and $g \in R(A_1^*)$ the formulations (3.51)–(3.55) are equivalent to the following five problems: Find

$$\begin{aligned}
\hat{x}_f, y_f \in D\big(\mathcal{A}_2^*\mathcal{A}_2\big) = D\big(A_2^*\mathcal{A}_2\big) = D\big(A_2^*A_2\big) \cap R\big(A_2^*\big) = D\big(A_2^*A_2\big) \cap N(A_2)^{\perp_{H_2}}, \\
\widetilde{x}_f \in D(\mathcal{A}_2) = D(A_2) \cap R\big(A_2^*\big) = D(A_2) \cap N(A_2)^{\perp_{H_2}}, \\
\widetilde{x}_g \in D\big(\mathcal{A}_1^*\big) = D\big(A_1^*\big) \cap R(A_1) = D\big(A_1^*\big) \cap N\big(A_1^*\big)^{\perp_{H_2}}, \\
z_g \in D(\mathcal{A}_1) = D(A_1) \cap R\big(A_1^*\big) = D(A_1) \cap N(A_1)^{\perp_{H_1}},
\end{aligned}$$

such that

$$\forall\ \phi \in D\big(A_2^*A_2\big) \quad \langle A_2^*A_2\hat{x}_f, A_2^*A_2\phi\rangle_{H_2} = \langle f, A_2^*A_2\phi\rangle_{H_2}, \tag{3.57}$$

$$\forall\ \xi \in D(A_2) \quad \langle A_2\widetilde{x}_f, A_2\xi\rangle_{H_3} = \langle f, \xi\rangle_{H_2}, \tag{3.58}$$

$$\forall\ \tau \in D\big(A_2^*A_2\big) \quad \langle A_2^*A_2 y_f, A_2^*A_2\tau\rangle_{H_2} = \langle f, \tau\rangle_{H_2}, \tag{3.59}$$

$$\forall\ \zeta \in D\big(A_1^*\big) \quad \langle A_1^*\widetilde{x}_g, A_1^*\zeta\rangle_{H_1} = \langle g, A_1^*\zeta\rangle_{H_1}, \tag{3.60}$$

$$\forall\ \varphi \in D(A_1) \quad \langle A_1 z_g, A_1\varphi\rangle_{H_2} = \langle g, \varphi\rangle_{H_1}. \tag{3.61}$$

Similar to the first order case, the variational formulations (3.57)–(3.61) are equivalent to the following five saddle point problems: Find $\hat{x}_f, y_f \in$

$D(\mathrm{A}_2^*\mathrm{A}_2), \widetilde{x}_f \in D(\mathrm{A}_2), \widetilde{x}_g \in D(\mathrm{A}_1^*), z_g \in D(\mathrm{A}_1)$, such that

$$\begin{array}{llcll}
\forall\ \phi \in D(\mathrm{A}_2^*\mathrm{A}_2) & \langle \mathrm{A}_2^*\mathrm{A}_2\hat{x}_f, \mathrm{A}_2^*\mathrm{A}_2\phi\rangle_{H_2} = \langle f, \mathrm{A}_2^*\mathrm{A}_2\phi\rangle_{H_2} & \wedge & \forall\ \theta \in N(\mathrm{A}_2) & \langle \hat{x}_f, \theta\rangle_{H_2} = 0, \\
\forall\ \xi \in D(\mathrm{A}_2) & \langle \mathrm{A}_2\widetilde{x}_f, \mathrm{A}_2\xi\rangle_{H_3} = \langle f, \xi\rangle_{H_2} & \wedge & \forall\ \kappa \in N(\mathrm{A}_2) & \langle \widetilde{x}_f, \kappa\rangle_{H_2} = 0, \\
\forall\ \tau \in D(\mathrm{A}_2^*\mathrm{A}_2) & \langle \mathrm{A}_2^*\mathrm{A}_2 y_f, \mathrm{A}_2^*\mathrm{A}_2\tau\rangle_{H_2} = \langle f, \tau\rangle_{H_2} & \wedge & \forall\ \sigma \in N(\mathrm{A}_2) & \langle y_f, \sigma\rangle_{H_2} = 0, \\
\forall\ \zeta \in D(\mathrm{A}_1^*) & \langle \mathrm{A}_1^*\widetilde{x}_g, \mathrm{A}_1^*\zeta\rangle_{H_1} = \langle g, \mathrm{A}_1^*\zeta\rangle_{H_1} & \wedge & \forall\ \lambda \in N(\mathrm{A}_1^*) & \langle \widetilde{x}_g, \lambda\rangle_{H_2} = 0, \\
\forall\ \varphi \in D(\mathrm{A}_1) & \langle \mathrm{A}_1 z_g, \mathrm{A}_1\varphi\rangle_{H_2} = \langle g, \varphi\rangle_{H_1} & \wedge & \forall\ \psi \in N(\mathrm{A}_1) & \langle z_g, \psi\rangle_{H_1} = 0.
\end{array}$$

At this point, we have followed the corresponding section for the first order problems up to (3.11)–(3.14). We emphasize that all considerations after (3.11)–(3.14) can be repeated here, giving similar saddle point formulations for the second order problem as well. As an example we present a corresponding result to Theorem 3.10 for finding the solution $x = x_f + x_g$ in just one variational saddle point formulation. For this, let us pick, e.g., the two formulations (3.58) and (3.60) together with the (very) weak versions of $\mathrm{A}_1^* x = g$ resp. $\mathrm{A}_2^*\mathrm{A}_2 x = f$.

**Theorem 3.24.** *Let $K_2 = \{0\}$. The unique solution $x = x_f + x_g \in \widetilde{D}_2$ in Theorem 3.19 can be found by the following two variational saddle point formulations:*

(i) *There exists a unique pair $(\widetilde{x}, z) \in D(\mathrm{A}_2) \times D(\mathcal{A}_1)$ such that*

$$\forall\ (\xi, \varphi) \in D(\mathrm{A}_2) \times D(\mathcal{A}_1) \langle \mathrm{A}_2\widetilde{x}, \mathrm{A}_2\xi\rangle_{H_3} + \langle \mathrm{A}_1 z, \xi\rangle_{H_2} = \langle f, \xi\rangle_{H_2}, \tag{3.62}$$

$$\langle \widetilde{x}, \mathrm{A}_1\varphi\rangle_{H_2} = \langle g, \varphi\rangle_{H_1}. \tag{3.63}$$

*It holds $z = 0$, if and only if $f \in R(\mathrm{A}_2^*)$, if and only if $\mathrm{A}_2\widetilde{x} \in D(\mathrm{A}_2^*)$ with $\mathrm{A}_2^*\mathrm{A}_2\widetilde{x} = f$. In this case*

$$\forall\ (\xi, \varphi) \in D(\mathrm{A}_2) \times D(\mathcal{A}_1) \langle \mathrm{A}_2\widetilde{x}, \mathrm{A}_2\xi\rangle_{H_3} = \langle f, \xi\rangle_{H_2}, \tag{3.64}$$

$$\langle \widetilde{x}, \mathrm{A}_1\varphi\rangle_{H_2} = \langle g, \varphi\rangle_{H_1}. \tag{3.65}$$

*Equations (3.63), (3.65) hold for all $\varphi \in D(\mathrm{A}_1)$ if and only if $g \in R(\mathrm{A}_1^*)$ if and only if $\widetilde{x} \in D(\mathrm{A}_1^*)$ with $\mathrm{A}_1^*\widetilde{x} = g$. Moreover, if $f \in R(\mathrm{A}_2^*)$ and $g \in R(\mathrm{A}_1^*)$, we have $\widetilde{x} = x$ from Theorem 3.19.*

(ii) *There exists a unique pair $(\hat{x}, y) \in D(\mathrm{A}_1^*) \times D(\mathrm{A}_2^*\mathcal{A}_2)$ such that*

$$\forall\ (\zeta, \phi) \in D(\mathrm{A}_1^*) \times D(\mathrm{A}_2^*\mathcal{A}_2) \quad \langle \mathrm{A}_1^*\hat{x}, \mathrm{A}_1^*\zeta\rangle_{H_1} + \langle \mathrm{A}_2^*\mathrm{A}_2 y, \zeta\rangle_{H_2} = \langle g, \mathrm{A}_1^*\zeta\rangle_{H_1}, \tag{3.66}$$

$$\langle \hat{x}, \mathrm{A}_2^*\mathrm{A}_2\phi\rangle_{H_2} = \langle f, \phi\rangle_{H_2}. \tag{3.67}$$

*It holds $y = 0$ as well as*

$$\forall\ (\zeta, \phi) \in D(\mathrm{A}_1^*) \times D(\mathrm{A}_2^*\mathcal{A}_2) \quad \langle \mathrm{A}_1^*\hat{x}, \mathrm{A}_1^*\zeta\rangle_{H_1} = \langle g, \mathrm{A}_1^*\zeta\rangle_{H_1}, \tag{3.68}$$

$$\langle \hat{x}, \mathrm{A}_2^*\mathrm{A}_2\phi\rangle_{H_2} = \langle f, \phi\rangle_{H_2}. \tag{3.69}$$

*Moreover,* $A_1^*\hat{x} = g$ *if and only if* $g \in R(A_1^*)$*. (3.67), (3.69) hold for all* $\phi \in D(A_2^*A_2)$ *if and only if* $f \in R(A_2^*)$ *if and only if* $\hat{x} \in D(A_2^*A_2)$ *with* $A_2^*A_2\hat{x} = f$*. In this case, i.e.,* $f \in R(A_2^*)$ *and* $g \in R(A_1^*)$*, we have* $\hat{x} = x$ *from Theorem 3.19.*

*Proof.* Unique solvability of (i) follows again by standard saddle point theory as in Theorem 3.10 (i). Inserting $\xi := A_1 z \in R(A_1) = N(A_2) = R(A_2^*)^{\perp_{H_2}}$ in (3.62) yields $|A_1 z|_{H_2}^2 = \langle f, A_1 z\rangle_{H_2}$ and hence $A_1 z = 0$, even $z = 0$ as $z \in D(\mathcal{A}_1)$, if $f \in R(A_2^*)$. On the other hand, if $A_1 z = 0$ then (3.62) shows $\langle f, \xi\rangle_{H_2} = 0$ for all $\xi \in N(A_2)$, i.e., $f \in N(A_2)^{\perp_{H_2}} = R(A_2^*)$. Moreover, if $f \in R(A_2^*)$, then (3.64) and (3.65) hold. Especially (3.64) yields $A_2\widetilde{x} \in D(A_2^*)$ and $A_2^*A_2\widetilde{x} = f$. The assertions related to $g$ follow as in the proof of Theorem 3.10 (i). Theorem 3.19 yields $\widetilde{x} = x$, which completes the proof of (i).

For (ii), we pick $\psi \in D(A_2^*)$ and decompose it as in the proof of Theorem 3.22 (i) into

$$\psi = \psi_N + A_2\phi_R \in D(A_2^*) = N(A_2^*) \oplus_{H_3} \big(D(\mathcal{A}_2^*) \cap R(\mathcal{A}_2)\big), \quad \phi_R \in D(\mathcal{A}_2^*\mathcal{A}_2).$$

If $f = 0$, then using the latter decomposition we see for all $\psi \in D(A_2^*)$

$$\langle \hat{x}, A_2^*\psi\rangle_{H_2} = \langle \hat{x}, A_2^*A_2\phi_R\rangle_{H_2} = 0,$$

which holds if and only if $\hat{x} \in N(A_2)$. Thus the kernel of (3.67) equals $N(A_2)$. By Corollary 2.5 (i) the principal part of (3.66) is strictly positive over the kernel of (3.67), which is

$$D(A_1^*) \cap N(A_2) = D(A_1^*) \cap R(A_1) = R(\mathcal{A}_1)$$

as we just have derived and since $K_2 = \{0\}$. Moreover, we have for $0 \neq \phi \in D(A_2^*\mathcal{A}_2)$

$$\frac{|A_2^*A_2\phi|_{H_2}}{|\phi|_{D(A_2^*\mathcal{A}_2)}} \leq \sup_{0 \neq \zeta \in D(A_1^*)} \frac{\langle A_2^*A_2\phi, \zeta\rangle_{H_2}}{|\phi|_{D(A_2^*\mathcal{A}_2)}|\zeta|_{D(A_1^*)}} \leq \frac{|A_2^*A_2\phi|_{H_2}}{|\phi|_{D(A_2^*\mathcal{A}_2)}} \leq 1$$

by choosing[16] $\zeta := A_2^*A_2\phi \in R(A_2^*\mathcal{A}_2) = R(A_2^*) = N(A_1^*)$, which shows that actually equality holds. Hence

$$\begin{aligned} 1 &\geq \inf_{0 \neq \phi \in D(A_2^*\mathcal{A}_2)} \sup_{0 \neq \zeta \in D(A_1^*)} \frac{\langle A_2^*A_2\phi, \zeta\rangle_{H_2}}{|\phi|_{D(A_2^*\mathcal{A}_2)}|\zeta|_{D(A_1^*)}} \\ &= \inf_{0 \neq \phi \in D(A_2^*\mathcal{A}_2)} \frac{|A_2^*A_2\phi|_{H_2}}{|\phi|_{D(\mathcal{A}_2^*\mathcal{A}_2)}} \geq \left(c_2^4 + c_2^2 + 1\right)^{-1/2} = |\mathcal{A}_2^{-1}(\mathcal{A}_2^*)^{-1}|^{-1}_{R(A_2^*), D(\mathcal{A}_2^*\mathcal{A}_2)}, \end{aligned}$$

[16]Indeed we can easily see $R(A_2^*\mathcal{A}_2) = R(A_2^*)$, since $R(A_2^*\mathcal{A}_2) \subset R(A_2^*)$ holds and for $\zeta \in R(A_2^*) = R(\mathcal{A}_2^*)$ there is $\phi := \mathcal{A}_2^{-1}(\mathcal{A}_2^*)^{-1}\zeta \in D(\mathcal{A}_2^*\mathcal{A}_2)$ with $\zeta = A_2^*A_2\phi \in R(\mathcal{A}_2^*\mathcal{A}_2) = R(A_2^*\mathcal{A}_2)$.

which shows that the inf-sup-condition is satisfied. Therefore, (3.66) and (3.67) admits a unique solution by the saddle point theory. Picking $\zeta := \mathrm{A}_2^*\mathrm{A}_2 y \in R(\mathrm{A}_2^*) = N(\mathrm{A}_1^*)$ in (3.66) yields $|\mathrm{A}_2^*\mathrm{A}_2 y|_{H_2}^2 = 0$ and hence $y = 0$ as $y \in D(\mathrm{A}_2^*\mathcal{A}_2) = D(\mathcal{A}_2^*\mathcal{A}_2)$. Since $\mathrm{A}_2^*\mathrm{A}_2 y = 0$ even (3.68) and (3.69) are valid. By (3.68) we see $\mathrm{A}_1^*\hat{x} - g \in R(\mathrm{A}_1^*)^{\perp_{H_1}}$, showing $\mathrm{A}_1^*\hat{x} = g$ if and only if $g \in R(\mathrm{A}_1^*)$. Using the orthonormal projector $\pi_{\mathrm{A}_2^*}$ and by (3.69) we see for all $\phi \in D(\mathrm{A}_2^*\mathrm{A}_2)$ as $\pi_{\mathrm{A}_2^*}\phi \in D(\mathrm{A}_2^*\mathcal{A}_2) = D(\mathcal{A}_2^*\mathcal{A}_2)$

$$\langle \hat{x}, \mathrm{A}_2^*\mathrm{A}_2\phi\rangle_{H_2} = \langle \hat{x}, \mathrm{A}_2^*\mathrm{A}_2\pi_{\mathrm{A}_2^*}\phi\rangle_{H_2} = \langle f, \pi_{\mathrm{A}_2^*}\phi\rangle_{H_2} = \langle \pi_{\mathrm{A}_2^*}f, \phi\rangle_{H_2} = \langle f, \phi\rangle_{H_2},$$

if $f \in R(\mathrm{A}_2^*)$. On the other hand, if (3.69) holds for all $\phi \in D(\mathrm{A}_2^*\mathrm{A}_2)$, then $\langle f, \phi\rangle_{H_2} = 0$ for all $\phi \in N(\mathrm{A}_2)$ and hence $f \in N(\mathrm{A}_2)^{\perp_{H_2}} = R(\mathrm{A}_2^*)$. Now, following the proof of Theorem 3.22 (i”), let $f \in R(\mathrm{A}_2^*) = R(\mathcal{A}_2^*)$ as well as define $h := (\mathcal{A}_2^*)^{-1} f \in D(\mathcal{A}_2^*)$ and observe with $\mathrm{A}_2^* h = f$ that by (3.69) for all $\phi \in D(\mathrm{A}_2^*\mathrm{A}_2)$

$$\langle \hat{x}, \mathrm{A}_2^*\mathrm{A}_2\phi\rangle_{H_2} = \langle f, \phi\rangle_{H_2} = \langle \mathrm{A}_2^* h, \phi\rangle_{H_2} = \langle h, \mathrm{A}_2\phi\rangle_{H_3} = \langle h, \pi_{\mathrm{A}_2}\mathrm{A}_2\phi\rangle_{H_3}. \tag{3.70}$$

As before, let $\psi \in D(\mathrm{A}_2^*)$ and let it be decomposed into

$$\psi = \psi_N + \mathrm{A}_2\phi \in D(\mathrm{A}_2^*) = N(\mathrm{A}_2^*) \oplus_{H_3} \big(D(\mathrm{A}_2^*) \cap R(\mathrm{A}_2)\big), \quad \phi \in D(\mathrm{A}_2^*\mathrm{A}_2).$$

Using (3.70) and the latter decomposition we see for all $\psi \in D(\mathrm{A}_2^*)$

$$\langle \hat{x}, \mathrm{A}_2^*\psi\rangle_{H_2} = \langle \hat{x}, \mathrm{A}_2^*\mathrm{A}_2\phi\rangle_{H_2} = \langle h, \pi_{\mathrm{A}_2}\mathrm{A}_2\phi\rangle_{H_3} = \langle h, \pi_{\mathrm{A}_2}\psi\rangle_{H_3} = \langle h, \psi\rangle_{H_3},$$

since $h \in D(\mathcal{A}_2^*) \subset R(\mathrm{A}_2)$. Thus $\hat{x} \in D(\mathrm{A}_2)$ and $\mathrm{A}_2\hat{x} = h \in D(\mathcal{A}_2^*)$, showing $\hat{x} \in D(\mathrm{A}_2^*\mathrm{A}_2)$ with

$$\mathrm{A}_2^*\mathrm{A}_2\hat{x} = \mathrm{A}_2^* h = f.$$

Finally, if $f \in R(\mathrm{A}_2^*)$ and $g \in R(\mathrm{A}_1^*)$, we have $\hat{x} \in D(\mathrm{A}_2^*\mathrm{A}_2) \cap D(\mathrm{A}_1^*) = \widetilde{D}_2$ with $\mathrm{A}_2^*\mathrm{A}_2\hat{x} = f$ and $\mathrm{A}_1^*\hat{x} = g$ and thus $\hat{x} = x$ by Theorem 3.19, completing the proof. □

**Remark 3.25.** *Let us note the following:*

(i) *Equations (3.62) and (3.63) is a weak formulation of*

$$\mathrm{A}_2^*\mathrm{A}_2\widetilde{x} + \mathrm{A}_1 z = f, \quad \mathrm{A}_1^*\widetilde{x} = g,$$

*i.e., in formal matrix notation*

$$\begin{bmatrix} \mathrm{A}_2^*\mathrm{A}_2 & \mathrm{A}_1 \\ \mathrm{A}_1^* & 0 \end{bmatrix} \begin{bmatrix} \widetilde{x} \\ z \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix}.$$

*Note $z = 0$.*

(ii) *Equations (3.66) and (3.67) is a weak formulation of*

$$\mathrm{A}_1\mathrm{A}_1^*\hat{x} + \mathrm{A}_2^*\mathrm{A}_2 y = \mathrm{A}_1 g, \quad \mathrm{A}_2^*\mathrm{A}_2\hat{x} = f,$$

*i.e., in formal matrix notation*

$$\begin{bmatrix} \mathrm{A}_1\mathrm{A}_1^* & \mathrm{A}_2^*\mathrm{A}_2 \\ \mathrm{A}_2^*\mathrm{A}_2 & 0 \end{bmatrix} \begin{bmatrix} \hat{x} \\ y \end{bmatrix} = \begin{bmatrix} \mathrm{A}_1 g \\ f \end{bmatrix}.$$

*Note $y = 0$.*

A corresponding result to Theorem 3.12 can be formulated as well, skipping the assumption $K_2 = \{0\}$ in Theorem 3.24.

**Theorem 3.26.** *The unique solution $x = x_f + x_g + k \in \widetilde{D}_2$ in Theorem 3.19 can be found by the following two variational double saddle point formulations:*

(i) *There exists a unique triple $(\widetilde{x}, z, h) \in D(\mathrm{A}_2) \times D(\mathcal{A}_1) \times K_2$ such that*

$$\begin{aligned} &\forall\ (\xi, \varphi, \kappa) \in D(\mathrm{A}_2) \times D(\mathcal{A}_1) \times K_2 \\ &\langle \mathrm{A}_2\widetilde{x}, \mathrm{A}_2\xi\rangle_{H_3} + \langle \mathrm{A}_1 z, \xi\rangle_{H_2} + \langle h, \xi\rangle_{H_2} = \langle f, \xi\rangle_{H_2}, \\ &\langle \widetilde{x}, \mathrm{A}_1\varphi\rangle_{H_2} = \langle g, \varphi\rangle_{H_1}, \\ &\langle \widetilde{x}, \kappa\rangle_{H_2} = \langle k, \kappa\rangle_{H_2}. \end{aligned} \tag{3.71}$$

*It holds $z = 0$ and $h = 0$, if and only if $f \in R(\mathrm{A}_2^*)$, if and only if $\mathrm{A}_2\widetilde{x} \in D(\mathrm{A}_2^*)$ with $\mathrm{A}_2^*\mathrm{A}_2\widetilde{x} = f$. In this case*

$$\forall\ (\xi, \varphi, \kappa) \in D(\mathrm{A}_2) \times D(\mathcal{A}_1) \times K_2 \quad \begin{aligned} \langle \mathrm{A}_2\widetilde{x}, \mathrm{A}_2\xi\rangle_{H_3} &= \langle f, \xi\rangle_{H_2}, \\ \langle \widetilde{x}, \mathrm{A}_1\varphi\rangle_{H_2} &= \langle g, \varphi\rangle_{H_1}, \\ \langle \widetilde{x}, \kappa\rangle_{H_2} &= \langle k, \kappa\rangle_{H_2}. \end{aligned} \tag{3.72}$$

(3.71), (3.72) hold for all $\varphi \in D(\mathrm{A}_1)$ if and only if $g \in R(\mathrm{A}_1^*)$ if and only if $\widetilde{x} \in D(\mathrm{A}_1^*)$ with $\mathrm{A}_1^*\widetilde{x} = g$. Furthermore, $\pi_2\widetilde{x} = k$. Moreover, if $f \in R(\mathrm{A}_2^*)$ and $g \in R(\mathrm{A}_1^*)$, we have $\widetilde{x} = x$ from Theorem 3.19.

(ii) *There exists a unique triple $(\hat{x}, y, h) \in D(\mathrm{A}_1^*) \times D(\mathrm{A}_2^*\mathcal{A}_2) \times K_2$ such that*

$$\begin{aligned} &\forall\ (\zeta, \phi, \kappa) \in D(\mathrm{A}_1^*) \times D(\mathrm{A}_2^*\mathcal{A}_2) \times K_2 \\ &\langle \mathrm{A}_1^*\hat{x}, \mathrm{A}_1^*\zeta\rangle_{H_1} + \langle \mathrm{A}_2^*\mathrm{A}_2 y, \zeta\rangle_{H_2} + \langle h, \zeta\rangle_{H_2} = \langle g, \mathrm{A}_1^*\zeta\rangle_{H_1}, \\ &\langle \hat{x}, \mathrm{A}_2^*\mathrm{A}_2\phi\rangle_{H_2} = \langle f, \phi\rangle_{H_2}, \\ &\langle \hat{x}, \kappa\rangle_{H_2} = \langle k, \kappa\rangle_{H_2}. \end{aligned} \tag{3.73}$$

*It holds $y = 0$ and $h = 0$ as well as*

$$\forall\ (\zeta, \phi, \kappa) \in D(\mathrm{A}_1^*) \times D(\mathrm{A}_2^*\mathcal{A}_2) \times K_2 \quad \begin{aligned} \langle \mathrm{A}_1^*\hat{x}, \mathrm{A}_1^*\zeta\rangle_{H_1} &= \langle g, \mathrm{A}_1^*\zeta\rangle_{H_1}, \\ \langle \hat{x}, \mathrm{A}_2^*\mathrm{A}_2\phi\rangle_{H_2} &= \langle f, \phi\rangle_{H_2}, \\ \langle \hat{x}, \kappa\rangle_{H_2} &= \langle k, \kappa\rangle_{H_2}. \end{aligned} \tag{3.74}$$

*Moreover,* $\mathrm{A}_1^*\hat{x}=g$ *if and only if* $g\in R(\mathrm{A}_1^*)$. *(3.73), (3.74) hold for all* $\phi\in D(\mathrm{A}_2^*\mathrm{A}_2)$ *if and only if* $f\in R(\mathrm{A}_2^*)$ *if and only if* $\hat{x}\in D(\mathrm{A}_2^*\mathrm{A}_2)$ *with* $\mathrm{A}_2^*\mathrm{A}_2\hat{x}=f$. *Furthermore,* $\pi_2\hat{x}=k$. *In this case, i.e.,* $f\in R(\mathrm{A}_2^*)$ *and* $g\in R(\mathrm{A}_1^*)$, *we have* $\hat{x}=x$ *from Theorem 3.19.*

**Remark 3.27.** *Let us note the following:*

(i) *Literally, Remark 3.13 (i) holds here as well.*

(ii) *Equation (3.71) is a weak formulation of*

$$\mathrm{A}_2^*\mathrm{A}_2\widetilde{x}+\mathrm{A}_1 z+h=f,\quad \mathrm{A}_1^*\widetilde{x}=g,\quad \pi_2\widetilde{x}=k,$$

*i.e., in formal matrix notation*

$$\begin{bmatrix}\mathrm{A}_2^*\mathrm{A}_2 & \mathrm{A}_1 & \iota_{K_2}\\ \mathrm{A}_1^* & 0 & 0\\ \pi_2=\iota_{K_2}^* & 0 & 0\end{bmatrix}\begin{bmatrix}\widetilde{x}\\ z\\ h\end{bmatrix}=\begin{bmatrix}f\\ g\\ k\end{bmatrix}.$$

*Note* $z=0$ *and* $h=0$.

(ii') *Equation (3.73) is a weak formulation of*

$$\mathrm{A}_1\mathrm{A}_1^*\hat{x}+\mathrm{A}_2^*\mathrm{A}_2 y+h=\mathrm{A}_1 g,\quad \mathrm{A}_2^*\mathrm{A}_2\hat{x}=f,\quad \pi_2\hat{x}=k,$$

*i.e., in formal matrix notation*

$$\begin{bmatrix}\mathrm{A}_1\mathrm{A}_1^* & \mathrm{A}_2^*\mathrm{A}_2 & \iota_{K_2}\\ \mathrm{A}_2^*\mathrm{A}_2 & 0 & 0\\ \pi_2=\iota_{K_2}^* & 0 & 0\end{bmatrix}\begin{bmatrix}\hat{x}\\ y\\ h\end{bmatrix}=\begin{bmatrix}\mathrm{A}_1 g\\ f\\ k\end{bmatrix}.$$

*Note* $y=0$ *and* $h=0$.

For the partial solutions and potentials in Theorem 3.19 and Theorem 3.22 a corresponding result to Theorem 3.14 can be proved as well. It reads as follows:

**Theorem 3.28.** *Let additionally* $R(A_0)$ *be closed. The partial solutions* $x_f=\hat{x}_f=\widetilde{x}_f\in D(\mathcal{A}_2^*\mathcal{A}_2)$, $x_g=\widetilde{x}_g\in D(\mathcal{A}_1^*)$, *and their potentials* $y_f\in D(\mathcal{A}_2^*\mathcal{A}_2)$, $z_g\in D(\mathcal{A}_1)$ *from Theorem 3.19, Theorem 3.22, and (3.57)–(3.61) can be found by the following six variational double saddle point formulations:*

(i) *There exists a unique triple* $(\hat{x}_f,w,h)\in D(\mathrm{A}_2^*\mathrm{A}_2)\times D(\mathcal{A}_1)\times K_2$ *such that*

$$\begin{aligned}&\forall\ (\psi,\varphi,\kappa)\in D(\mathrm{A}_2^*\mathrm{A}_2)\times D(\mathcal{A}_1)\times K_2\\ &\langle \mathrm{A}_2^*\mathrm{A}_2\hat{x}_f,\mathrm{A}_2^*\mathrm{A}_2\psi\rangle_{H_2}+\langle \mathrm{A}_1 w,\psi\rangle_{H_2}+\langle h,\psi\rangle_{H_2}=\langle f,\mathrm{A}_2^*\mathrm{A}_2\psi\rangle_{H_2},\\ &\langle\hat{x}_f,\mathrm{A}_1\varphi\rangle_{H_2}=0,\\ &\langle\hat{x}_f,\kappa\rangle_{H_2}=0.\end{aligned}\tag{3.75}$$

*It holds* $w=0$ *and* $h=0$*. Moreover,* $\mathrm{A}_2\hat{x}_f \in D(\mathrm{A}_2^*)$ *and* $\mathrm{A}_2^*\mathrm{A}_2\hat{x}_f = f$ *if and only if* $f \in R(\mathrm{A}_2^*)$*. The second equation of (3.75) holds for all* $\varphi \in D(\mathrm{A}_1)$ *and thus* $\hat{x}_f \in N(\mathrm{A}_1^*)$*. Furthermore,* $\pi_2\hat{x}_f = 0$*. Finally, if* $f \in R(\mathrm{A}_2^*)$*, we have* $\hat{x}_f = x_f$ *from Theorem 3.19, see Theorem 3.22 (i).*

(i') *There exists a unique triple* $(\widetilde{x}_f, u, h) \in D(\mathrm{A}_2) \times D(\mathcal{A}_1) \times K_2$ *such that*

$$\begin{aligned} \forall\ (\xi, \varphi, \kappa) \in D(\mathrm{A}_2) \times D(\mathcal{A}_1) \times K_2 & \\ \langle \mathrm{A}_2\widetilde{x}_f, \mathrm{A}_2\xi\rangle_{H_3} + \langle \mathrm{A}_1 u, \xi\rangle_{H_2} + \langle h, \xi\rangle_{H_2} &= \langle f, \xi\rangle_{H_2}, \\ \langle \widetilde{x}_f, \mathrm{A}_1\varphi\rangle_{H_2} &= 0, \\ \langle \widetilde{x}_f, \kappa\rangle_{H_2} &= 0. \end{aligned} \tag{3.76}$$

*It holds* $u=0$ *if and only if* $f\perp_{H_2}R(\mathrm{A}_1)$ *if and only if* $f \in N(\mathrm{A}_1^*)$*.* $h=0$ *if and only if* $f\perp_{H_2}K_2$*. Thus* $u=0$ *and* $h=0$ *if and only if* $f \in N(\mathrm{A}_1^*) \cap K_2^{\perp_{H_2}} = R(\mathrm{A}_2^*)$*. Moreover,* $\mathrm{A}_2\hat{x}_f \in D(\mathrm{A}_2^*)$ *and* $\mathrm{A}_2^*\mathrm{A}_2\widetilde{x}_f = f$ *if and only if* $f \in R(\mathrm{A}_2^*)$*. The second equation of (3.76) holds for all* $\varphi \in D(\mathrm{A}_1)$ *and hence* $\widetilde{x}_f \in N(\mathrm{A}_1^*)$*. Furthermore,* $\pi_2\widetilde{x}_f = 0$*. Finally, if* $f \in R(\mathrm{A}_2^*)$*, we have* $\widetilde{x}_f = x_f$ *from Theorem 3.19, see Theorem 3.22 (i').*

(i'') *There exists a unique triple* $(y_f, v, h) \in D(\mathrm{A}_2^*\mathrm{A}_2) \times D(\mathcal{A}_1) \times K_2$ *such that*

$$\begin{aligned} \forall\ (\psi, \varphi, \kappa) \in D(\mathrm{A}_2^*\mathrm{A}_2) \times D(\mathcal{A}_1) \times K_2 & \\ \langle \mathrm{A}_2^*\mathrm{A}_2 y_f, \mathrm{A}_2^*\mathrm{A}_2\psi\rangle_{H_2} + \langle \mathrm{A}_1 v, \psi\rangle_{H_2} + \langle h, \psi\rangle_{H_2} &= \langle f, \psi\rangle_{H_2}, \\ \langle y_f, \mathrm{A}_1\varphi\rangle_{H_2} &= 0, \\ \langle y_f, \kappa\rangle_{H_2} &= 0. \end{aligned} \tag{3.77}$$

*It holds* $v=0$ *if and only if* $f\perp_{H_2}R(\mathrm{A}_1)$ *if and only if* $f \in N(\mathrm{A}_1^*)$*.* $h=0$ *if and only if* $f\perp_{H_2}K_2$*. Thus* $v=0$ *and* $h=0$ *if and only if* $f \in N(\mathrm{A}_1^*) \cap K_2^{\perp_{H_2}} = R(\mathrm{A}_2^*)$*. Moreover,* $\mathrm{A}_2^*\mathrm{A}_2 y_f \in D(\mathrm{A}_2^*\mathrm{A}_2)$ *and* $(\mathrm{A}_2^*\mathrm{A}_2)^2 y_f = f$ *if and only if* $f \in R(\mathrm{A}_2^*)$*. The second equation of (3.77) holds for all* $\varphi \in D(\mathrm{A}_1)$ *and thus* $y_f \in N(\mathrm{A}_1^*)$*. Furthermore,* $\pi_2 y_f = 0$*. Finally, if* $f \in R(\mathrm{A}_2^*)$*, we have* $\mathrm{A}_2^*\mathrm{A}_2 y_f = x_f$ *from Theorem 3.19, see Theorem 3.22 (i'').*

(ii) *There exists a unique triple* $(\widetilde{x}_g, p, h) \in D(\mathrm{A}_1^*) \times D(\mathcal{A}_2^*) \times K_2$ *such that*

$$\begin{aligned} \forall\ (\zeta, \phi, \kappa) \in D(\mathrm{A}_1^*) \times D\big(\mathcal{A}_2^*\big) \times K_2 & \\ \langle \mathrm{A}_1^*\widetilde{x}_g, \mathrm{A}_1^*\zeta\rangle_{H_1} + \langle \mathrm{A}_2^* p, \zeta\rangle_{H_2} + \langle h, \zeta\rangle_{H_2} &= \langle g, \mathrm{A}_1^*\zeta\rangle_{H_1}, \\ \langle \widetilde{x}_g, \mathrm{A}_2^*\phi\rangle_{H_2} &= 0, \\ \langle \widetilde{x}_g, \kappa\rangle_{H_2} &= 0. \end{aligned} \tag{3.78}$$

*It holds* $p=0$ *and* $h=0$*. Moreover,* $\mathrm{A}_1^*\widetilde{x}_g = g$ *if and only if* $g \in R(\mathrm{A}_1^*)$*. The second equation of (3.78) holds for all* $\phi \in D(\mathrm{A}_2^*)$ *and thus* $\widetilde{x}_g \in N(\mathrm{A}_2)$*.*

*Furthermore,* $\pi_2 \widetilde{x}_g = 0$ *. Finally, if* $g \in R(\mathrm{A}_1^*)$ *, we have* $\widetilde{x}_g = x_g$ *from Theorem 3.19, see Theorem 3.22 (ii).*

(ii') *There exists a unique triple* $(\hat{x}_g, q, h) \in D(\mathrm{A}_1^*) \times D(\mathrm{A}_2^* \mathcal{A}_2) \times K_2$ *such that*

$$\begin{aligned} \forall\ (\zeta, \psi, \kappa) \in D(\mathrm{A}_1^*) \times D(\mathrm{A}_2^* \mathcal{A}_2) \times K_2 \\ \langle \mathrm{A}_1^* \hat{x}_g, \mathrm{A}_1^* \zeta \rangle_{H_1} + \langle \mathrm{A}_2^* \mathrm{A}_2 q, \zeta \rangle_{H_2} + \langle h, \zeta \rangle_{H_2} &= \langle g, \mathrm{A}_1^* \zeta \rangle_{H_1}, \\ \langle \hat{x}_g, \mathrm{A}_2^* \mathrm{A}_2 \psi \rangle_{H_2} &= 0, \\ \langle \hat{x}_g, \kappa \rangle_{H_2} &= 0. \end{aligned} \tag{3.79}$$

*It holds* $q=0$ *and* $h=0$*. Moreover,* $\mathrm{A}_1^* \hat{x}_g = g$ *if and only if* $g \in R(\mathrm{A}_1^*)$*. The second equation of (3.79) holds for all* $\psi \in D(\mathrm{A}_2^* \mathrm{A}_2)$ *and thus* $\hat{x}_g \in N(\mathrm{A}_2)$ *as* $\hat{x}_g \perp_{H_2} R(\mathrm{A}_2^* \mathrm{A}_2) = R(\mathrm{A}_2^*)$*. Furthermore,* $\pi_2 \hat{x}_g = 0$*. Finally, if* $g \in R(\mathrm{A}_1^*)$*, we have* $\hat{x}_g = x_g$ *from Theorem 3.19, see Theorem 3.22 (ii).*

(ii'') *There exists a unique triple* $(z_g, r, h) \in D(\mathrm{A}_1) \times D(\mathcal{A}_0) \times K_1$ *such that*

$$\begin{aligned} \forall\ (\varphi, \vartheta, \kappa) \in D(\mathrm{A}_1) \times D(\mathcal{A}_0) \times K_1 \\ \langle \mathrm{A}_1 z_g, \mathrm{A}_1 \varphi \rangle_{H_2} + \langle \mathrm{A}_0 r, \varphi \rangle_{H_1} + \langle h, \varphi \rangle_{H_1} &= \langle g, \varphi \rangle_{H_1}, \\ \langle z_g, \mathrm{A}_0 \vartheta \rangle_{H_1} &= 0, \\ \langle z_g, \kappa \rangle_{H_1} &= 0. \end{aligned} \tag{3.80}$$

*It holds* $r=0$ *if and only if* $g \perp_{H_1} R(\mathrm{A}_0)$ *if and only if* $g \in N(\mathrm{A}_0^*)$*.* $h=0$ *if and only if* $g \perp_{H_1} K_1$*. Thus* $r=0$ *and* $h=0$ *if and only if* $g \in N(\mathrm{A}_0^*) \cap K_1^{\perp_{H_1}} = R(\mathrm{A}_1^*)$*. Moreover,* $\mathrm{A}_1 z_g \in D(\mathrm{A}_1^*)$ *and* $\mathrm{A}_1^* \mathrm{A}_1 z_g = g$ *if and only if* $g \in R(\mathrm{A}_1^*)$*. The second equation of (3.80) holds for all* $\vartheta \in D(\mathrm{A}_0)$ *and hence* $z_g \in N(\mathrm{A}_0^*)$*. Furthermore,* $\pi_1 z_g = 0$*. Finally, if* $g \in R(\mathrm{A}_1^*)$*, we have* $\mathrm{A}_1 z_g = x_g$ *from Theorem 3.19, see Theorem 3.22 (ii').*

*Proof.* The proof utilizes the same techniques as used before. □

**Remark 3.29.** *The formulations in Theorem 3.28 (i') resp. Theorem 3.28 (ii') are the same as in Theorem 3.26 (i) resp. Theorem 3.26 (ii) except of the right hand sides. We note that* $\widetilde{x} = x$ *can also be found by the formulation presented in Theorem 3.28 (i).*

**Remark 3.30.** *Again we have formal matrix representations:*

(i) *Equation (3.75) is a weak formulation of*

$$\left(\mathrm{A}_2^* \mathrm{A}_2\right)^2 \hat{x}_f + \mathrm{A}_1 w + h = \mathrm{A}_2^* \mathrm{A}_2 f, \quad \mathrm{A}_1^* \hat{x}_f = 0, \quad \pi_2 \hat{x}_f = 0,$$

*i.e., in formal matrix notation*

$$\begin{bmatrix} (A_2^*A_2)^2 & A_1 & \imath_{K_2} \\ A_1^* & 0 & 0 \\ \pi_2 = \imath_{K_2}^* & 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{x}_f \\ w \\ h \end{bmatrix} = \begin{bmatrix} A_2^*A_2 f \\ 0 \\ 0 \end{bmatrix}.$$

*Note $w = 0$ and $h = 0$.*

(i') *Equation (3.76) is a weak formulation of*

$$A_2^*A_2\widetilde{x}_f + A_1 u + h = f, \quad A_1^*\widetilde{x}_f = 0, \quad \pi_2\widetilde{x}_f = 0,$$

*i.e., in formal matrix notation*

$$\begin{bmatrix} A_2^*A_2 & A_1 & \imath_{K_2} \\ A_1^* & 0 & 0 \\ \pi_2 = \imath_{K_2}^* & 0 & 0 \end{bmatrix} \begin{bmatrix} \widetilde{x}_f \\ u \\ h \end{bmatrix} = \begin{bmatrix} f \\ 0 \\ 0 \end{bmatrix}.$$

*Note $u = 0$ and $h = 0$.*

(i'') *Equation (3.77) is a weak formulation of*

$$\left(A_2A_2^*\right)^2 y_f + A_1 v + h = f, \quad A_1^* y_f = 0, \quad \pi_2 y_f = 0,$$

*i.e., in formal matrix notation*

$$\begin{bmatrix} (A_2A_2^*)^2 & A_1 & \imath_{K_2} \\ A_1^* & 0 & 0 \\ \pi_2 = \imath_{K_2}^* & 0 & 0 \end{bmatrix} \begin{bmatrix} y_f \\ v \\ h \end{bmatrix} = \begin{bmatrix} f \\ 0 \\ 0 \end{bmatrix}.$$

*Note $v = 0$ and $h = 0$.*

(ii) *Equation (3.78) is a weak formulation of*

$$A_1A_1^*\widetilde{x}_g + A_2^* p + h = A_1 g, \quad A_2\widetilde{x}_g = 0, \quad \pi_2\widetilde{x}_g = 0,$$

*i.e., in formal matrix notation*

$$\begin{bmatrix} A_1A_1^* & A_2^* & \imath_{K_2} \\ A_2 & 0 & 0 \\ \pi_2 = \imath_{K_2}^* & 0 & 0 \end{bmatrix} \begin{bmatrix} \widetilde{x}_g \\ p \\ h \end{bmatrix} = \begin{bmatrix} A_1 g \\ 0 \\ 0 \end{bmatrix}.$$

*Note $p = 0$ and $h = 0$.*

(ii') *Equation (3.79) is a weak formulation of*

$$A_1A_1^*\hat{x}_g + A_2^*A_2 q + h = A_1 g, \quad A_2^*A_2\hat{x}_g = 0, \quad \pi_2\hat{x}_g = 0,$$

*i.e., in formal matrix notation*

$$\begin{bmatrix} A_1A_1^* & A_2^*A_2 & \iota_{K_2} \\ A_2^*A_2 & 0 & 0 \\ \pi_2 = \iota_{K_2}^* & 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{x}_g \\ q \\ h \end{bmatrix} = \begin{bmatrix} A_1 g \\ 0 \\ 0 \end{bmatrix}.$$

*Note $q=0$ and $h=0$.*

(ii") *Equation (3.80) is a weak formulation of*

$$A_1^*A_1 z_g + A_0 r + h = g, \quad A_0^* z_g = 0, \quad \pi_1 z_g = 0,$$

*i.e., in formal matrix notation*

$$\begin{bmatrix} A_1^*A_1 & A_0 & \iota_{K_1} \\ A_0^* & 0 & 0 \\ \pi_1 = \iota_{K_1}^* & 0 & 0 \end{bmatrix} \begin{bmatrix} z_g \\ r \\ h \end{bmatrix} = \begin{bmatrix} g \\ 0 \\ 0 \end{bmatrix}.$$

*Note $r=0$ and $h=0$.*

There is also an analogon for the quadruple saddle point formulations presented in Theorem 3.17. Let us recall from Theorem 3.26 $z \in D(\mathcal{A}_1)$ and $y \in D(A_2^*\mathcal{A}_2)$, i.e.,

$$\begin{aligned} z &\in R(A_1^*) = N\big(A_0^*\big) \cap K_1^{\perp_{H_1}} = R(A_0)^{\perp_{H_1}} \cap K_1^{\perp_{H_1}}, \\ y &\in R(A_2^*) = N(A_1^*) \cap K_2^{\perp_{H_2}} = R(A_1)^{\perp_{H_2}} \cap K_2^{\perp_{H_2}}. \end{aligned}$$

**Theorem 3.31.** *Let additionally $R(A_0)$ be closed. Moreover, let $f \in R(A_2^*)$ and $g \in R(A_1^*)$. The unique solution $x = x_f + x_g + k \in \widetilde{D}_2$ in Theorem 3.19 can be found by the following two variational quadruple saddle point formulations:*

(i) *There exists a unique five tuple $(\widetilde{x}, z, u, h_2, h_1) \in D(A_2) \times D(A_1) \times D(\mathcal{A}_0) \times K_2 \times K_1$ such that for all $(\xi, \varphi, \vartheta, \kappa, \lambda) \in D(A_2) \times D(A_1) \times D(\mathcal{A}_0) \times K_2 \times K_1$*

$$\begin{aligned} \langle A_2\widetilde{x}, A_2\xi\rangle_{H_3} + \langle A_1 z, \xi\rangle_{H_2} + \langle h_2, \xi\rangle_{H_2} &= \langle f, \xi\rangle_{H_2}, \\ \langle \widetilde{x}, A_1\varphi\rangle_{H_2} + \langle A_0 u, \varphi\rangle_{H_1} + \langle h_1, \varphi\rangle_{H_1} &= \langle g, \varphi\rangle_{H_1}, \\ \langle z, A_0\vartheta\rangle_{H_1} &= 0, \\ \langle \widetilde{x}, \kappa\rangle_{H_2} &= \langle k, \kappa\rangle_{H_2}, \\ \langle z, \lambda\rangle_{H_1} &= 0. \end{aligned} \tag{3.81}$$

*The third equation of (3.81) is valid for all $\vartheta \in D(A_0)$. It holds $z=0$ and $h_2=0$ as well as $u=0$ and $h_1=0$. Moreover, $A_2\widetilde{x} \in D(A_2^*)$ with $A_2^*A_2\widetilde{x} = f$ and $\widetilde{x} \in D(A_1^*)$ with $A_1^*\widetilde{x} = g$ as well as $\pi_2\widetilde{x} = k$. Finally, $\widetilde{x} = x$ from Theorem 3.19.*

(ii) *There exists a unique five tuple* $(\hat{x}, y, v, h_2, \hat{h}_2) \in D(A_1^*) \times D(A_2^* A_2) \times D(\mathcal{A}_1) \times K_2 \times K_2$ *such that for all* $(\zeta, \phi, \psi, \kappa, \lambda) \in D(A_1^*) \times D(A_2^* A_2) \times D(\mathcal{A}_1) \times K_2 \times K_2$

$$\begin{aligned}
\langle A_1^* \hat{x}, A_1^* \zeta \rangle_{H_1} + \langle A_2^* A_2 y, \zeta \rangle_{H_2} + \langle h_2, \zeta \rangle_{H_2} &= \langle g, A_1^* \zeta \rangle_{H_1}, \\
\langle \hat{x}, A_2^* A_2 \phi \rangle_{H_2} + \langle A_1 v, \phi \rangle_{H_2} + \langle \hat{h}_2, \phi \rangle_{H_2} &= \langle f, \phi \rangle_{H_2}, \\
\langle y, A_1 \psi \rangle_{H_2} &= 0, \\
\langle \hat{x}, \kappa \rangle_{H_2} &= \langle k, \kappa \rangle_{H_2}, \\
\langle y, \lambda \rangle_{H_2} &= 0.
\end{aligned} \tag{3.82}$$

*The third equation of (3.82) is valid for all* $\psi \in D(A_1)$. *It holds* $y = 0$ *and* $h_2 = 0$ *as well as* $v = 0$ *and* $\hat{h}_2 = 0$. *Moreover,* $A_1^* \hat{x} = g$ *and* $\hat{x} \in D(A_2^* A_2)$ *with* $A_2^* A_2 \hat{x} = f$ *as well as* $\pi_2 \hat{x} = k$. *Finally,* $\hat{x} = x$ *from Theorem 3.19.*

(ii') *There is* $(\hat{x}, y, v, u, h_2, \hat{h}_2, h_1) \in D(A_1^*) \times D(A_2^* A_2) \times D(A_1) \times D(\mathcal{A}_0) \times K_2 \times K_2 \times K_1$ , *a unique seven tuple, such that for all* $(\zeta, \phi, \psi, \vartheta, \kappa, \lambda, \nu) \in D(A_1^*) \times D(A_2^* A_2) \times D(A_1) \times D(\mathcal{A}_0) \times K_2 \times K_2 \times K_1$

$$\begin{aligned}
\langle A_1^* \hat{x}, A_1^* \zeta \rangle_{H_1} + \langle A_2^* A_2 y, \zeta \rangle_{H_2} + \langle h_2, \zeta \rangle_{H_2} &= \langle g, A_1^* \zeta \rangle_{H_1}, \\
\langle \hat{x}, A_2^* A_2 \phi \rangle_{H_2} + \langle A_1 v, \phi \rangle_{H_2} + \langle \hat{h}_2, \phi \rangle_{H_2} &= \langle f, \phi \rangle_{H_2}, \\
\langle y, A_1 \psi \rangle_{H_2} + \langle A_0 u, \psi \rangle_{H_1} + \langle h_1, \psi \rangle_{H_1} &= 0, \\
\langle v, A_0 \vartheta \rangle_{H_1} &= 0, \\
\langle \hat{x}, \kappa \rangle_{H_2} &= \langle k, \kappa \rangle_{H_2}, \\
\langle y, \lambda \rangle_{H_2} &= 0, \\
\langle v, \nu \rangle_{H_1} &= 0.
\end{aligned} \tag{3.83}$$

*The fourth equation of (3.83) is valid for all* $\vartheta \in D(A_0)$. *It holds* $y = 0$, $h_2 = 0$ *and* $v = 0$, $\hat{h}_2 = 0$ *as well as* $u = 0$ *and* $h_1 = 0$. *Moreover,* $A_1^* \hat{x} = g$ *and* $\hat{x} \in D(A_2^* A_2)$ *with* $A_2^* A_2 \hat{x} = f$ *as well as* $\pi_2 \hat{x} = k$. *Finally,* $\hat{x} = x$ *from Theorem 3.19.*

**Theorem 3.28.** can be extended in the same way.

**Remark 3.32.** *Let us note that generally the solution and test spaces look like*

$$\begin{aligned}
&D(A_\ell) \times D(A_{\ell-1}) \times \cdots \times D(A_{\ell-n+1}) \times D(\mathcal{A}_{\ell-n}) \times K_\ell \times K_{\ell-1} \times \cdots \times K_{\ell-n+1}, \\
&D(A_\ell^*) \times D(A_{\ell+1}^* A_{\ell+1}) \times D(A_\ell) \times D(A_{\ell-1}) \times \cdots \\
&\quad \cdots \times D(A_{\ell-n+1}) \times D(\mathcal{A}_{\ell-n}) \times K_{\ell+1} \times K_{\ell+1} \times K_\ell \times K_{\ell-1} \times \cdots \times K_{\ell-n+1}.
\end{aligned}$$

*Moreover:*

(i) *Equation (3.81) is a weak formulation of*

$$A_2^* A_2 \widetilde{x} + A_1 z + h_2 = f, \quad A_1^* \widetilde{x} + A_0 u + h_1 = g, \quad A_0^* z = 0,$$
$$\pi_2 \widetilde{x} = k, \quad \pi_1 z = 0,$$

*i.e., in formal matrix notation*

$$\begin{bmatrix} A_2^*A_2 & A_1 & 0 & \iota_{K_2} & 0 \\ A_1^* & 0 & A_0 & 0 & \iota_{K_1} \\ 0 & A_0^* & 0 & 0 & 0 \\ \pi_2 = \iota_{K_2}^* & 0 & 0 & 0 & 0 \\ 0 & \pi_1 = \iota_{K_1}^* & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \widetilde{x} \\ z \\ u \\ h_2 \\ h_1 \end{bmatrix} = \begin{bmatrix} f \\ g \\ 0 \\ k \\ 0 \end{bmatrix}.$$

*Note* $z=0$, $u=0$ *and* $h_2 = 0, h_1 = 0$.

(ii) *Equation (3.82) is a weak formulation of*

$$A_1A_1^*\hat{x} + A_2^*A_2y + h_2 = A_1g, \quad A_2^*A_2\hat{x} + A_1v + \hat{h}_2 = f,$$
$$A_1^*y = 0, \quad \pi_2\hat{x} = k, \quad \pi_2 y = 0,$$

*i.e., in formal matrix notation*

$$\begin{bmatrix} A_1A_1^* & A_2^*A_2 & 0 & \iota_{K_2} & 0 \\ A_2^*A_2 & 0 & A_1 & 0 & \iota_{K_2} \\ 0 & A_1^* & 0 & 0 & 0 \\ \pi_2 = \iota_{K_2}^* & 0 & 0 & 0 & 0 \\ 0 & \pi_2 = \iota_{K_2}^* & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{x} \\ y \\ v \\ h_2 \\ \hat{h}_2 \end{bmatrix} = \begin{bmatrix} A_1g \\ f \\ 0 \\ k \\ 0 \end{bmatrix}.$$

*Note* $y=0$, $v=0$ *and* $h_2 = 0, \hat{h}_2 = 0$.

(ii') *Equation (3.83) is a weak formulation of*

$$A_1A_1^*\hat{x} + A_2^*A_2y + h_2 = A_1g, \quad A_2^*A_2\hat{x} + A_1v + \hat{h}_2 = f,$$
$$A_1^*y + A_0u + h_1 = 0, \quad A_0^*v = 0,$$

*and* $\pi_2\hat{x} = k, \pi_2 y = 0, \pi_1 v = 0$, *i.e., in formal matrix notation*

$$\begin{bmatrix} A_1A_1^* & A_2^*A_2 & 0 & 0 & \iota_{K_2} & 0 & 0 \\ A_2^*A_2 & 0 & A_1 & 0 & 0 & \iota_{K_2} & 0 \\ 0 & A_1^* & 0 & A_0 & 0 & 0 & \iota_{K_1} \\ 0 & 0 & A_0^* & 0 & 0 & 0 & 0 \\ \pi_2 = \iota_{K_2}^* & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \pi_2 = \iota_{K_2}^* & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \pi_1 = \iota_{K_1}^* & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{x} \\ y \\ v \\ u \\ h_2 \\ \hat{h}_2 \\ h_1 \end{bmatrix} = \begin{bmatrix} A_1g \\ f \\ 0 \\ 0 \\ k \\ 0 \\ 0 \end{bmatrix}.$$

*Note* $y=0$, $v=0$, $u=0$ *and* $h_2 = 0, \hat{h}_2 = 0, h_1 = 0$.

## 4. Functional a posteriori error estimates

Having establishes a solution theory including suitable variational formulations, we now turn to the so-called functional a posteriori error estimates. Note that General Assumption 3.1 is supposed to hold.

### 4.1. First order systems

Let $x \in D_2$ be the exact solution of (3.1) and $\tilde{x} \in H_2$, which may be considered as a nonconforming approximation of $x$. Utilizing the notations from Theorem 3.3 we define and decompose the error

$$\begin{aligned} H_2 \ni e &:= x - \tilde{x} = e_{\mathrm{A}_1} + e_{K_2} + e_{\mathrm{A}_2^*} \in R(\mathrm{A}_1) \oplus_{H_2} K_2 \oplus_{H_2} R(\mathrm{A}_2^*), \\ e_{\mathrm{A}_1} &:= \pi_{\mathrm{A}_1} e = x_g - \pi_{\mathrm{A}_1} \tilde{x} \in R(\mathrm{A}_1), \\ e_{\mathrm{A}_2^*} &:= \pi_{\mathrm{A}_2^*} e = x_f - \pi_{\mathrm{A}_2^*} \tilde{x} \in R(\mathrm{A}_2^*), \\ e_{K_2} &:= \pi_2 e = k - \pi_2 \tilde{x} \in K_2 \end{aligned} \tag{4.1}$$

using the Helmholtz type decompositions of Lemma 2.7. By orthogonality it holds

$$|e|_{H_2}^2 = |e_{\mathrm{A}_1}|_{H_2}^2 + |e_{K_2}|_{H_2}^2 + |e_{\mathrm{A}_2^*}|_{H_2}^2. \tag{4.2}$$

#### 4.1.1. Upper bounds

Testing (4.1) with $\mathrm{A}_1\varphi$ for $\varphi \in D(\mathcal{A}_1)$ we get for all $\zeta \in D(\mathrm{A}_1^*)$ by orthogonality and Corollary 2.5 (i)

$$\begin{aligned} \langle e_{\mathrm{A}_1}, \mathrm{A}_1\varphi\rangle_{H_2} &= \langle e, \mathrm{A}_1\varphi\rangle_{H_2} = \langle \mathrm{A}_1^* x, \varphi\rangle_{H_1} - \langle \tilde{x} - \zeta + \zeta, \mathrm{A}_1\varphi\rangle_{H_2} \\ &= \langle g - \mathrm{A}_1^*\zeta, \varphi\rangle_{H_1} - \langle \pi_{\mathrm{A}_1}(\tilde{x} - \zeta), \mathrm{A}_1\varphi\rangle_{H_2} \\ &\leq |g - \mathrm{A}_1^*\zeta|_{H_1} |\varphi|_{H_1} + |\pi_{\mathrm{A}_1}(\tilde{x} - \zeta)|_{H_2} |\mathrm{A}_1\varphi|_{H_2} \\ &\leq \big(c_1 |g - \mathrm{A}_1^*\zeta|_{H_1} + |\pi_{\mathrm{A}_1}(\tilde{x} - \zeta)|_{H_2}\big) |\mathrm{A}_1\varphi|_{H_2}. \end{aligned} \tag{4.3}$$

As $e_{\mathrm{A}_1} \in R(\mathrm{A}_1) = R(\mathcal{A}_1)$, we have $e_{\mathrm{A}_1} = \mathrm{A}_1\varphi_e$ with $\varphi_e := \mathcal{A}_1^{-1} e_{\mathrm{A}_1} \in D(\mathcal{A}_1)$. Choosing $\varphi := \varphi_e$ in (4.3) we obtain

$$\forall\ \zeta \in D(\mathrm{A}_1^*) \qquad |e_{\mathrm{A}_1}|_{H_2} \leq c_1 |g - \mathrm{A}_1^*\zeta|_{H_1} + |\pi_{\mathrm{A}_1}(\tilde{x} - \zeta)|_{H_2} \leq c_1 |g - \mathrm{A}_1^*\zeta|_{H_1} + |\tilde{x} - \zeta|_{H_2}. \tag{4.4}$$

Analogously, testing with $\mathrm{A}_2^*\phi$ for $\phi \in D(\mathcal{A}_2^*)$ we get for all $\xi \in D(\mathrm{A}_2)$ by orthogonality and Corollary 2.5 (i)

$$\begin{aligned} \langle e_{\mathrm{A}_2^*}, \mathrm{A}_2^*\phi\rangle_{H_2} &= \langle e, \mathrm{A}_2^*\phi\rangle_{H_2} = \langle \mathrm{A}_2 x, \phi\rangle_{H_3} - \langle \tilde{x} - \xi + \xi, \mathrm{A}_2^*\phi\rangle_{H_2} \\ &= \langle f - \mathrm{A}_2\xi, \phi\rangle_{H_3} - \langle \pi_{\mathrm{A}_2^*}(\tilde{x} - \xi), \mathrm{A}_2^*\phi\rangle_{H_2} \\ &\leq |f - \mathrm{A}_2\xi|_{H_3} |\phi|_{H_3} + |\pi_{\mathrm{A}_2^*}(\tilde{x} - \xi)|_{H_2} |\mathrm{A}_2^*\phi|_{H_2} \\ &\leq \big(c_2 |f - \mathrm{A}_2\xi|_{H_3} + |\pi_{\mathrm{A}_2^*}(\tilde{x} - \xi)|_{H_2}\big) |\mathrm{A}_2^*\phi|_{H_2}. \end{aligned} \tag{4.5}$$

As $e_{\mathrm{A}_2^*} \in R(\mathrm{A}_2^*) = R(\mathcal{A}_2^*)$, we have $e_{\mathrm{A}_2^*} = \mathrm{A}_2^*\phi_e$ with $\phi_e := (\mathcal{A}_2^*)^{-1} e_{\mathrm{A}_2^*} \in D(\mathcal{A}_2^*)$. Choosing $\phi := \phi_e$ in (4.5) we obtain

$$\forall\ \xi\in D(\mathrm{A}_2)\qquad |e_{\mathrm{A}_2^*}|_{H_2}\le c_2|f-\mathrm{A}_2\xi|_{H_3}+|\pi_{\mathrm{A}_2^*}(\tilde{x}-\xi)|_{H_2}\le c_2|f-\mathrm{A}_2\xi|_{H_3}+|\tilde{x}-\xi|_{H_2}. \tag{4.6}$$

Finally, for all $\varphi\in D(\mathrm{A}_1)$ and all $\phi\in D(\mathrm{A}_2^*)$ we get by orthogonality

$$|e_{K_2}|_{H_2}^2=\langle e_{K_2},k-\pi_2\tilde{x}+\mathrm{A}_1\varphi+\mathrm{A}_2^*\phi\rangle_{H_2}=\langle e_{K_2},k-\tilde{x}+\mathrm{A}_1\varphi+\mathrm{A}_2^*\phi\rangle_{H_2} \tag{4.7}$$

and thus

$$\forall\ \varphi\in D(\mathrm{A}_1)\quad\forall\ \phi\in D\big(\mathrm{A}_2^*\big)\qquad |e_{K_2}|_{H_2}\le|k-\tilde{x}+\mathrm{A}_1\varphi+\mathrm{A}_2^*\phi|_{H_2}. \tag{4.8}$$

Let us summarize:

**Theorem 4.1.** *Let $x\in D_2$ be the exact solution of (3.1) and $\tilde{x}\in H_2$. Then the following estimates hold for the error $e=x-\tilde{x}$ defined in (4.1):*

(i) *The error decomposes according to (4.1) and (4.2), i.e.,*

$$e=e_{\mathrm{A}_1}+e_{K_2}+e_{\mathrm{A}_2^*}\in R(\mathrm{A}_1)\oplus_{H_2}K_2\oplus_{H_2}R\big(\mathrm{A}_2^*\big),$$
$$|e|_{H_2}^2=|e_{\mathrm{A}_1}|_{H_2}^2+|e_{K_2}|_{H_2}^2+|e_{\mathrm{A}_2^*}|_{H_2}^2.$$

(ii) *The projection $e_{\mathrm{A}_1}=\pi_{\mathrm{A}_1}e=x_g-\pi_{\mathrm{A}_1}\tilde{x}\in R(\mathrm{A}_1)$ satisfies*

$$|e_{\mathrm{A}_1}|_{H_2}=\min_{\zeta\in D\left(\mathrm{A}_1^*\right)}\big(c_1|\mathrm{A}_1^*\zeta-g|_{H_1}+|\zeta-\tilde{x}|_{H_2}\big)$$

*and the minimum is attained at*

$$\hat{\zeta}:=e_{\mathrm{A}_1}+\tilde{x}=\pi_{\mathrm{A}_1}e+\tilde{x}=-(1-\pi_{\mathrm{A}_1})e+x=-\pi_{N\left(\mathrm{A}_1^*\right)}e+x\in D\big(\mathrm{A}_1^*\big)$$

*since $\mathrm{A}_1^*\hat{\zeta}=\mathrm{A}_1^*x=g$.*

(iii) *The projection $e_{\mathrm{A}_2^*}=\pi_{\mathrm{A}_2^*}e=x_f-\pi_{\mathrm{A}_2^*}\tilde{x}\in R(\mathrm{A}_2^*)$ satisfies*

$$|e_{\mathrm{A}_2^*}|_{H_2}=\min_{\xi\in D(\mathrm{A}_2)}\big(c_2|\mathrm{A}_2\xi-f|_{H_3}+|\xi-\tilde{x}|_{H_2}\big)$$

*and the minimum is attained at*

$$\hat{\xi}:=e_{\mathrm{A}_2^*}+\tilde{x}=\pi_{\mathrm{A}_2^*}e+\tilde{x}=-(1-\pi_{\mathrm{A}_2^*})e+x=-\pi_{N(\mathrm{A}_2)}e+x\in D(\mathrm{A}_2)$$

*since $\mathrm{A}_2\hat{\xi}=\mathrm{A}_2x=f$.*

(iv) *The projection $e_{K_2}=\pi_2e=k-\pi_2\tilde{x}\in K_2$ satisfies*

$$|e_{K_2}|_{H_2}=\min_{\varphi\in D(\mathrm{A}_1)}\ \min_{\phi\in D\left(\mathrm{A}_2^*\right)}|k-\tilde{x}+\mathrm{A}_1\varphi+\mathrm{A}_2^*\phi|_{H_2}$$

*and the minimum is attained at any $\hat{\varphi}\in D(\mathrm{A}_1)$ and $\hat{\phi}\in D(\mathrm{A}_2^*)$ solving $\mathrm{A}_1\hat{\varphi}=\pi_{\mathrm{A}_1}\tilde{x}$ and $\mathrm{A}_2^*\hat{\phi}=\pi_{\mathrm{A}_2^*}\tilde{x}$ since $(\pi_{\mathrm{A}_1}+\pi_{\mathrm{A}_2^*})\tilde{x}=(1-\pi_2)\tilde{x}$, especially at*

$$\hat{\varphi} := \mathcal{A}_1^{-1}\pi_{A_1}\widetilde{x} \in D(\mathcal{A}_1), \quad \hat{\phi} := \left(\mathcal{A}_2^*\right)^{-1}\pi_{A_2^*}\widetilde{x} \in D\left(\mathcal{A}_2^*\right).$$

For conforming approximations we get:

**Corollary 4.2.** *Let the assumptions of Theorem 4.1 be satisfied.*

(i) *If $\widetilde{x} \in D(A_1^*)$, then $e \in D(A_1^*)$ and hence $e_{A_1} = \pi_{A_1}e \in D(\mathcal{A}_1^*)$ with $A_1^* e_{A_1} = A_1^* e$ and*

$$|e_{A_1}|_{H_2} \le c_1 |A_1^*\widetilde{x} - g|_{H_1} = c_1 |A_1^* e|_{H_1}$$

*by setting $\zeta := \widetilde{x}$, which also follows directly by the Friedrichs/Poincaré type estimate.*

(ii) *If $\widetilde{x} \in D(A_2)$, then $e \in D(A_2)$ and hence $e_{A_2^*} = \pi_{A_2^*}e \in D(\mathcal{A}_2)$ with $A_2 e_{A_2^*} = A_2 e$ and*

$$|e_{A_2^*}|_{H_2} \le c_2 |A_2\widetilde{x} - f|_{H_3} = c_2 |A_2 e|_{H_3}$$

*by setting $\xi := \widetilde{x}$, which also follows directly by the Friedrichs/Poincaré type estimate.*

(iii) *If $\widetilde{x} \in D_2$, then $e \in D_2$ and*

$$\begin{aligned} |e|_{D_2}^2 &= |e_{A_1}|_{H_2}^2 + |e_{K_2}|_{H_2}^2 + |e_{A_2^*}|_{H_2}^2 + |A_2 e|_{H_3}^2 + |A_1^* e|_{H_1}^2 \\ &\le |e_{K_2}|_{H_2}^2 + \left(1 + c_2^2\right)|A_2 e|_{H_3}^2 + \left(1 + c_1^2\right)|A_1^* e|_{H_1}^2 \end{aligned}$$

*with*

$$e_{K_2} = k - \pi_2\widetilde{x}, \quad A_2 e = f - A_2\widetilde{x}, \quad A_1^* e = g - A_1^*\widetilde{x},$$

*which again also follows immediately by the Friedrichs/Poincaré type estimates.*

**Remark 4.3.** *Corollary 4.2 (iii) shows, that for very conforming $\widetilde{x} \in D_2$ the weighted least squares functional*

$$\mathcal{F}(\widetilde{x}) := |k - \pi_2\widetilde{x}|_{H_2}^2 + \left(1 + c_2^2\right)|A_2\widetilde{x} - f|_{H_3}^2 + \left(1 + c_1^2\right)|A_1^*\widetilde{x} - g|_{H_1}^2$$

*is equivalent to the conforming error, i.e.,*

$$|e|_{D_2}^2 \le \mathcal{F}(\widetilde{x}) \le \left(1 + \max\{c_1, c_2\}^2\right)|e|_{D_2}^2.$$

Recalling the variational resp. saddle point formulations (3.8)–(3.10) resp. (3.12)–(3.14) and that the partial solutions are given by

$$x_f = A_2^* y_f \in D(\mathcal{A}_2), \quad x_g = A_1 z_g \in D\left(\mathcal{A}_1^*\right),$$

a possible numerical method, using these variational formulations in some finite dimensional subspaces to find $\widetilde{y}_f \in D(\mathrm{A}_2^*)$ and $\widetilde{z}_g \in D(\mathrm{A}_1)$, such as the finite element method, will always ensure

$$\widetilde{x}_f := \mathrm{A}_2^*\widetilde{y}_f \in R(\mathrm{A}_2^*) = N(\mathrm{A}_2)^{\perp_{H_2}} \subset N(\mathrm{A}_1^*),$$
$$\widetilde{x}_g := \mathrm{A}_1\widetilde{z}_g \in R(\mathrm{A}_1) = N(\mathrm{A}_1^*)^{\perp_{H_2}} \subset N(\mathrm{A}_2)$$

and thus

$$\widetilde{x}_\perp := \widetilde{x}_f + \widetilde{x}_g \in R(\mathrm{A}_2^*) \oplus_{H_2} R(\mathrm{A}_1) = K_2^{\perp_{H_2}},$$

but maybe not $\widetilde{x}_f \in D(\mathrm{A}_2)$ or $\widetilde{x}_g \in D(\mathrm{A}_1^*)$. Therefore, a reasonable assumption for our nonconforming approximations is

$$\widetilde{x} = \widetilde{x}_\perp + k, \quad \widetilde{x}_\perp \in K_2^{\perp_{H_2}},$$

with $e_{K_2} = \pi_2 e = \pi_2(x - \widetilde{x}) = -\pi_2\widetilde{x}_\perp = 0$.

**Corollary 4.4.** *Let $x \in D_2$ be the exact solution of (3.1) and $\widetilde{x} := k + \widetilde{x}_\perp$ with some $\widetilde{x}_\perp \in K_2^{\perp_{H_2}}$. Then for the error $e$ defined in (4.1) it holds:*

(i) *According to (4.1) and (4.2) the error decomposes, i.e.,*

$$e = x - \widetilde{x} = x_f + x_g - \widetilde{x}_\perp = e_{\mathrm{A}_1} + e_{\mathrm{A}_2^*} \in R(\mathrm{A}_1) \oplus_{H_2} R(\mathrm{A}_2^*) = K_2^{\perp_{H_2}}, \quad e_{K_2} = 0,$$

*and $|e|_{H_2}^2 = |e_{\mathrm{A}_1}|_{H_2}^2 + |e_{\mathrm{A}_2^*}|_{H_2}^2$. Hence there is no error in the "kernel" part.*

(ii) *The projection $e_{\mathrm{A}_1} = \pi_{\mathrm{A}_1} e = x_g - \pi_{\mathrm{A}_1}\widetilde{x} = x_g - \pi_{\mathrm{A}_1}\widetilde{x}_\perp \in R(\mathrm{A}_1)$ satisfies*

$$\begin{aligned} |e_{\mathrm{A}_1}|_{H_2} &= \min_{\zeta \in D(\mathrm{A}_1^*)} \left(c_1|\mathrm{A}_1^*\zeta - g|_{H_1} + |\zeta - \widetilde{x}|_{H_2}\right) \\ &= \min_{\zeta \in D(\mathrm{A}_1^*)} \left(c_1|\mathrm{A}_1^*\zeta - g|_{H_1} + |\zeta - \widetilde{x}_\perp|_{H_2}\right) \end{aligned}$$

*(exchanging $\zeta$ by $\zeta + k$) and the minima are attained at*

$$\hat{\zeta} := e_{\mathrm{A}_1} + \widetilde{x} = \pi_{\mathrm{A}_1} e + \widetilde{x} = -(1 - \pi_{\mathrm{A}_1})e + x = -\pi_{N(\mathrm{A}_1^*)} e + x \in D(\mathrm{A}_1^*),$$
$$\hat{\zeta}_\perp := e_{\mathrm{A}_1} + \widetilde{x}_\perp = \pi_{\mathrm{A}_1} e + \widetilde{x}_\perp = -(1 - \pi_{\mathrm{A}_1})e + x - k = -\pi_{N(\mathrm{A}_1^*)} e + x - k \in D(\mathrm{A}_1^*)$$

*since $\mathrm{A}_1^*\hat{\zeta}_\perp = \mathrm{A}_1^*\hat{\zeta} = \mathrm{A}_1^* x = g$.*

(iii) *The projection $e_{\mathrm{A}_2^*} = \pi_{\mathrm{A}_2^*} e = x_f - \pi_{\mathrm{A}_2^*}\widetilde{x} = x_f - \pi_{\mathrm{A}_2^*}\widetilde{x}_\perp \in R(\mathrm{A}_2^*)$ satisfies*

$$\begin{aligned} |e_{\mathrm{A}_2^*}|_{H_2} &= \min_{\xi \in D(\mathrm{A}_2)} \left(c_2|\mathrm{A}_2\xi - f|_{H_3} + |\xi - \widetilde{x}|_{H_2}\right) \\ &= \min_{\xi \in D(\mathrm{A}_2)} \left(c_2|\mathrm{A}_2\xi - f|_{H_3} + |\xi - \widetilde{x}_\perp|_{H_2}\right) \end{aligned}$$

*(exchanging $\xi$ by $\xi + k$) and the minima are attained at*

$$\hat{\xi} := e_{A_2^*} + \widetilde{x} = \pi_{A_2^*} e + \widetilde{x} = -(1 - \pi_{A_2^*})e + x = -\pi_{N(A_2)} e + x \in D(A_2),$$
$$\hat{\xi}_\perp := e_{A_2^*} + \widetilde{x}_\perp = \pi_{A_2^*} e + \widetilde{x}_\perp = -(1 - \pi_{A_2^*})e + x - k = -\pi_{N(A_2)} e + x - k \in D(A_2)$$

*since* $A_2 \hat{\xi}_\perp = A_2 \hat{\xi} = A_2 x = f$.

### 4.1.2. Lower bounds

In any Hilbert space $H$ we have

$$\forall\ \hat{h} \in H \qquad |\hat{h}|_H^2 = \max_{h \in H} \left( 2\langle \hat{h}, h \rangle_H - |h|_H^2 \right) \tag{4.9}$$

and the maximum is attained at $\hat{h}$. We recall (4.1) and (4.2), especially

$$|e|_{H_2}^2 = |e_{A_1}|_{H_2}^2 + |e_{K_2}|_{H_2}^2 + |e_{A_2^*}|_{H_2}^2.$$

Using (4.9) for $H = R(A_1)$ and orthogonality we get

$$\begin{aligned} |e_{A_1}|_{H_2}^2 &= \max_{\varphi \in D(A_1)} \left( 2\langle e_{A_1}, A_1 \varphi \rangle_{H_2} - |A_1 \varphi|_{H_2}^2 \right) \\ &= \max_{\varphi \in D(A_1)} \left( 2\langle e, A_1 \varphi \rangle_{H_2} - |A_1 \varphi|_{H_2}^2 \right) \\ &= \max_{\varphi \in D(A_1)} \left( 2\langle g, \varphi \rangle_{H_1} - \langle 2\widetilde{x} + A_1 \varphi, A_1 \varphi \rangle_{H_2} \right) \end{aligned}$$

and the maximum is attained at any $\hat{\varphi} \in D(A_1)$ with $A_1 \hat{\varphi} = e_{A_1}$. Analogously for $H = R(A_2^*)$

$$|e_{A_2^*}|_{H_2}^2 = \max_{\phi \in D(A_2^*)} \left( 2\langle f, \phi \rangle_{H_3} - \langle 2\widetilde{x} + A_2^* \phi, A_2^* \phi \rangle_{H_2} \right)$$

and the maximum is attained at any $\hat{\phi} \in D(A_2^*)$ with $A_2^* \hat{\phi} = e_{A_2^*}$. Finally for $H = K_2$ and by orthogonality

$$|e_{K_2}|_{H_2}^2 = \max_{\theta \in K_2} \left( 2\langle e_{K_2}, \theta \rangle_{H_2} - |\theta|_{H_2}^2 \right) = \max_{\theta \in K_2} \langle 2(k - \widetilde{x}) - \theta, \theta \rangle_{H_2}$$

and the maximum is attained at $\hat{\theta} = e_{K_2}$.

**Theorem 4.5.** *Let $x \in D_2$ be the exact solution of (3.1) and $\widetilde{x} \in H_2$. Then the following estimates hold for the error $e = x - \widetilde{x}$ defined in (4.1):*

(i) *The error decomposes according to (4.1) and (4.2), i.e.,*

$$e = e_{A_1} + e_{K_2} + e_{A_2^*} \in R(A_1) \oplus_{H_2} K_2 \oplus_{H_2} R(A_2^*),$$
$$|e|_{H_2}^2 = |e_{A_1}|_{H_2}^2 + |e_{K_2}|_{H_2}^2 + |e_{A_2^*}|_{H_2}^2.$$

(ii) *The projection* $e_{A_1} = \pi_{A_1} e = x_g - \pi_{A_1}\tilde{x} \in R(A_1)$ *satisfies*

$$|e_{A_1}|^2_{H_2} = \max_{\varphi \in D(A_1)} \left( 2\langle g, \varphi\rangle_{H_1} - \langle 2\tilde{x} + A_1\varphi, A_1\varphi\rangle_{H_2} \right)$$

*and the maximum is attained at any* $\hat{\varphi} \in D(A_1)$ *with* $A_1\hat{\varphi} = e_{A_1}$, *e.g., at* $\hat{\varphi} := \mathcal{A}_1^{-1} e_{A_1} \in D(\mathcal{A}_1)$.

(iii) *The projection* $e_{A_2^*} = \pi_{A_2^*} e = x_f - \pi_{A_2^*}\tilde{x} \in R(A_2^*)$ *satisfies*

$$|e_{A_2^*}|^2_{H_2} = \max_{\phi \in D\left(A_2^*\right)} \left( 2\langle f, \phi\rangle_{H_3} - \langle 2\tilde{x} + A_2^*\phi, A_2^*\phi\rangle_{H_2} \right)$$

*and the maximum is attained at any* $\hat{\phi} \in D(A_2^*)$ *with* $A_2^*\hat{\phi} = e_{A_2^*}$, *e.g.,* $\hat{\phi} := (\mathcal{A}_2^*)^{-1} e_{A_2^*} \in D(\mathcal{A}_2^*)$.

(iv) *The projection* $e_{K_2} = \pi_2 e = k - \pi_2\tilde{x} \in K_2$ *satisfies*

$$|e_{K_2}|^2_{H_2} = \max_{\theta \in K_2} \langle 2(k - \tilde{x}) - \theta, \theta\rangle_{H_2}$$

*and the maximum is attained at* $\hat{\theta} := e_{K_2} \in K_2$.

*If* $\tilde{x} := k + \tilde{x}_\perp$ *with some* $\tilde{x}_\perp \in K_2^{\perp_{H_2}}$, *see Corollary 4.4, then* $e_{K_2} = 0$, *and in (ii) and (iii)* $\tilde{x}$ *can be replaced by* $\tilde{x}_\perp$ *as* $k \perp_{H_2} R(A_1) \oplus_{H_2} R(A_2^*)$.

### 4.1.3. Two-sided bounds

We summarize our results from the latter sections.

**Corollary 4.6.** *Let* $x \in D_2$ *be the exact solution of (3.1) and* $\tilde{x} \in H_2$. *Then the following estimates hold for the error* $e = x - \tilde{x}$ *defined in (4.1):*

(i) *The error decomposes according to (4.1) and (4.2), i.e.,*

$$e = e_{A_1} + e_{K_2} + e_{A_2^*} \in R(A_1) \oplus_{H_2} K_2 \oplus_{H_2} R\left(A_2^*\right),$$
$$|e|^2_{H_2} = |e_{A_1}|^2_{H_2} + |e_{K_2}|^2_{H_2} + |e_{A_2^*}|^2_{H_2}.$$

(ii) *The projection* $e_{A_1} = \pi_{A_1} e = x_g - \pi_{A_1}\tilde{x} \in R(A_1)$ *satisfies*

$$\begin{aligned} |e_{A_1}|^2_{H_2} &= \min_{\zeta \in D\left(A_1^*\right)} \left( c_1 |A_1^*\zeta - g|_{H_1} + |\zeta - \tilde{x}|_{H_2} \right)^2 \\ &= \max_{\varphi \in D(A_1)} \left( 2\langle g, \varphi\rangle_{H_1} - \langle 2\tilde{x} + A_1\varphi, A_1\varphi\rangle_{H_2} \right) \end{aligned}$$

*and the minimum resp. maximum is attained at*

$$\hat{\zeta} := e_{A_1} + \tilde{x} \in D\left(A_1^*\right), \quad \hat{\varphi} := \mathcal{A}_1^{-1} e_{A_1} \in D(\mathcal{A}_1)$$

*with* $A_1^*\hat{\zeta} = A_1^* x = g$, *and at any* $\hat{\varphi} \in D(A_1)$ *with* $A_1\hat{\varphi} = e_{A_1}$.

(iii) *The projection* $e_{A_2^*} = \pi_{A_2^*} e = x_f - \pi_{A_2^*}\tilde{x} \in R(A_2^*)$ *satisfies*

$$|e_{A_2^*}|_{H_2}^2 = \min_{\xi \in D(A_2)} \left(c_2|A_2\xi - f|_{H_3} + |\xi - \tilde{x}|_{H_2}\right)^2$$
$$= \max_{\phi \in D\left(A_2^*\right)} \left(2\langle f, \phi\rangle_{H_3} - \langle 2\tilde{x} + A_2^*\phi, A_2^*\phi\rangle_{H_2}\right)$$

*and the minimum resp. maximum is attained at*

$$\hat{\xi} := e_{A_2^*} + \tilde{x} \in D(A_2), \quad \hat{\phi} := \left(\mathcal{A}_2^*\right)^{-1} e_{A_2^*} \in D\left(\mathcal{A}_2^*\right)$$

*with* $A_2\hat{\xi} = A_2 x = f$, *and at any* $\hat{\phi} \in D(A_2^*)$ *with* $A_2^*\hat{\phi} = e_{A_2^*}$.

(iv) *The projection* $e_{K_2} = \pi_2 e = k - \pi_2\tilde{x} \in K_2$ *satisfies*

$$|e_{K_2}|_{H_2}^2 = \min_{\varphi \in D(A_1)} \min_{\phi \in D\left(A_2^*\right)} |k - \tilde{x} + A_1\varphi + A_2^*\phi|_{H_2}^2$$
$$= \max_{\theta \in K_2} \langle 2(k - \tilde{x}) - \theta, \theta\rangle_{H_2}$$

*and the minimum resp. maximum is attained at*

$$\hat{\varphi} := \mathcal{A}_1^{-1}\pi_{A_1}\tilde{x} \in D(\mathcal{A}_1), \quad \hat{\phi} := \left(\mathcal{A}_2^*\right)^{-1}\pi_{A_2^*}\tilde{x} \in D\left(\mathcal{A}_2^*\right), \quad \hat{\theta} := e_{K_2} \in K_2,$$

*and at any* $\hat{\varphi} \in D(A_1)$ *and* $\hat{\phi} \in D(A_2^*)$ *with* $A_1\hat{\varphi} = \pi_{A_1}\tilde{x}$ *and* $A_2^*\hat{\phi} = \pi_{A_2^*}\tilde{x}$.

*If* $\tilde{x} := k + \tilde{x}_\perp$ *with some* $\tilde{x}_\perp \in K_2^{\perp_{H_2}}$, *see Corollary 4.4, then* $e_{K_2} = 0$, *and in (ii) and (iii)* $\tilde{x}$ *can be replaced by* $\tilde{x}_\perp$. *In this case, for the attaining minima it holds*

$$\hat{\zeta}_\perp := e_{A_1} + \tilde{x}_\perp \in D\left(A_1^*\right), \quad \hat{\xi}_\perp := e_{A_2^*} + \tilde{x}_\perp \in D(A_2).$$

### 4.2. Second order systems

Let $x \in \tilde{D}_2$ be the exact solution of (3.49). Recalling Remark 3.21 we introduce the additional quantity $y := A_2 x \in D(\mathcal{A}_2^*)$. Then (3.49) decomposes into two first order systems of shape (1.5) resp. (3.1), i.e.,

$$\begin{aligned} A_2 x &= y, & A_3 y &= 0, \\ A_1^* x &= g, & A_2^* y &= f, \\ \pi_2 x &= k, & \pi_3 y &= 0 \end{aligned}$$

for the pair $(x, y) \in D_2 \times D_3$. Hence, we can immediately apply our results for the first order systems. Let $\tilde{x} \in H_2$ and $\tilde{y} \in H_3$, which may be considered as nonconforming approximations of $x$ and $y$, respectively. Utilizing the notations from Theorem 3.19 we define and decompose the errors

$$
\begin{aligned}
H_2 \ni e &:= x - \widetilde{x} = e_{A_1} + e_{K_2} + e_{A_2^*} \in R(A_1) \oplus_{H_2} K_2 \oplus_{H_2} R\left(A_2^*\right), \\
H_3 \ni h &:= y - \widetilde{y} = h_{A_2} + h_{K_3} + h_{A_3^*} \in R(A_2) \oplus_{H_3} K_3 \oplus_{H_3} R\left(A_3^*\right), \\
e_{A_1} &:= \pi_{A_1} e = x_g - \pi_{A_1}\widetilde{x} \in R(A_1), \quad h_{A_2} := \pi_{A_2} h = y - \pi_{A_2}\widetilde{y} \in R(A_2), \\
e_{A_2^*} &:= \pi_{A_2^*} e = x_y - \pi_{A_2^*}\widetilde{x} \in R\left(A_2^*\right), \quad h_{A_3^*} := \pi_{A_3^*} h = -\pi_{A_3^*}\widetilde{y} \in R\left(A_3^*\right), \\
e_{K_2} &:= \pi_2 e = k - \pi_2 \widetilde{x} \in K_2, \qquad h_{K_3} := \pi_3 e = -\pi_3 \widetilde{y} \in K_3
\end{aligned}
\tag{4.10}
$$

using the Helmholtz type decompositions of Lemma 2.7 and noting $\pi_{A_2} y = y$ as $y \in R(A_2)$. By orthogonality it holds

$$
|e|_{H_2}^2 = |e_{A_1}|_{H_2}^2 + |e_{K_2}|_{H_2}^2 + |e_{A_2^*}|_{H_2}^2, \quad |h|_{H_3}^2 = |h_{A_2}|_{H_3}^2 + |h_{K_3}|_{H_3}^2 + |h_{A_3^*}|_{H_3}^2. \tag{4.11}
$$

Therefore, the results of the latter section can be applied to $e_{A_1}, e_{K_2}, e_{A_2^*}, h_{A_2}, h_{K_3}, h_{A_3^*}$. Especially, by Corollary 4.6 we obtain

$$
|e_{A_1}|_{H_2}^2 = \min_{\zeta \in D\left(A_1^*\right)} \left(c_1 |A_1^* \zeta - g|_{H_1} + |\zeta - \widetilde{x}|_{H_2}\right)^2 = \max_{\varphi \in D(A_1)} \left(2\langle g, \varphi\rangle_{H_1} - \langle 2\widetilde{x} + A_1\varphi, A_1\varphi\rangle_{H_2}\right) \tag{4.12}
$$

and the minimum resp. maximum is attained at $\hat{\zeta} = e_{A_1} + \widetilde{x} \in D(A_1^*)$ and $\hat{\varphi} = \mathcal{A}_1^{-1} e_{A_1} \in D(\mathcal{A}_1)$ with $A_1^* \hat{\zeta} = A_1^* x = g$,

$$
|e_{A_2^*}|_{H_2}^2 = \min_{\xi \in D(A_2)} \left(c_2 |A_2 \xi - y|_{H_3} + |\xi - \widetilde{x}|_{H_2}\right)^2 = \max_{\phi \in D\left(A_2^*\right)} \left(2\langle y, \phi\rangle_{H_3} - \langle 2\widetilde{x} + A_2^*\phi, A_2^*\phi\rangle_{H_2}\right) \tag{4.13}
$$

and the minimum resp. maximum is attained at $\hat{\xi} = e_{A_2^*} + \widetilde{x} \in D(A_2)$ and $\hat{\phi} = (\mathcal{A}_2^*)^{-1} e_{A_2^*} \in D(\mathcal{A}_2^*)$ with $A_2 \hat{\xi} = A_2 x = y$,

$$
|e_{K_2}|_{H_2}^2 = \min_{\varphi \in D(A_1)} \min_{\phi \in D\left(A_2^*\right)} |k - \widetilde{x} + A_1\varphi + A_2^*\phi|_{H_2}^2 = \max_{\theta \in K_2} \langle 2(k - \widetilde{x}) - \theta, \theta\rangle_{H_2} \tag{4.14}
$$

and the minimum resp. maximum is attained at $\hat{\varphi} = \mathcal{A}_1^{-1} \pi_{A_1} \widetilde{x} \in D(\mathcal{A}_1)$, $\hat{\phi} = (\mathcal{A}_2^*)^{-1} \pi_{A_2^*} \widetilde{x} \in D(\mathcal{A}_2^*)$, and $\hat{\theta} = e_{K_2} \in K_2$ with $A_1 \hat{\varphi} + A_2^* \hat{\phi} = (\pi_{A_1} + \pi_{A_2^*})\widetilde{x} = (1 - \pi_2)\widetilde{x}$. If $\widetilde{x} = k + \widetilde{x}_\perp$ with some $\widetilde{x}_\perp \in K_2^{\perp_{H_2}}$, then $e_{K_2} = 0$, and $\widetilde{x}$ can be replaced by $\widetilde{x}_\perp$. If the General Assumption 3.1 holds also for $A_3$, i.e., $R(A_3)$ is closed and (not neccessarily) $K_3$ is finite dimensional, we get the corresponding results for $h_{A_2}, h_{K_3}, h_{A_3^*}$ as well. Replacing $A_1$ by $A_2$ and $A_2$ by $A_3$, Corollary 4.6 yields

$$
|h_{A_2}|_{H_3}^2 = \min_{\zeta \in D\left(A_2^*\right)} \left(c_2 |A_2^* \zeta - f|_{H_2} + |\zeta - \widetilde{y}|_{H_3}\right)^2 = \max_{\varphi \in D(A_2)} \left(2\langle f, \varphi\rangle_{H_2} - \langle 2\widetilde{y} + A_2\varphi, A_2\varphi\rangle_{H_3}\right) \tag{4.15}
$$

and the minimum resp. maximum is attained at $\hat{\zeta} = h_{A_2} + \widetilde{y} \in D(A_2^*)$ and $\hat{\varphi} = \mathcal{A}_2^{-1} h_{A_2} \in D(\mathcal{A}_2)$ with $A_2^* \hat{\zeta} = A_2^* y = f$,

$$|h_{A_3^*}|_{H_3}^2 = \min_{\xi \in D(A_3)} \left(c_3|A_3\xi|_{H_4} + |\xi - \widetilde{y}|_{H_3}\right)^2 = \max_{\phi \in D\left(A_3^*\right)} \left(-\langle 2\widetilde{y} + A_3^*\phi, A_3^*\phi\rangle_{H_3}\right) \tag{4.16}$$

and the minimum resp. maximum is attained at $\hat{\xi} = h_{A_3^*} + \widetilde{y} \in D(A_3)$ and $\hat{\phi} = (\mathcal{A}_3^*)^{-1} h_{A_3^*} \in D(\mathcal{A}_3^*)$ with $A_3\hat{\xi} = A_3 y = 0$, i.e., $\hat{\xi} \in N(A_3)$,

$$|h_{K_3}|_{H_3}^2 = \min_{\varphi \in D(A_2)} \min_{\phi \in D\left(A_3^*\right)} |-\widetilde{y} + A_2\varphi + A_3^*\phi|_{H_3}^2 = \max_{\theta \in K_3} \left(-\langle 2\widetilde{y} + \theta, \theta\rangle_{H_3}\right) \tag{4.17}$$

and the minimum resp. maximum is attained at $\hat{\varphi} = \mathcal{A}_2^{-1}\pi_{A_2}\widetilde{y} \in D(\mathcal{A}_2)$, $\hat{\phi} = (\mathcal{A}_3^*)^{-1}\pi_{A_3^*}\widetilde{y} \in D(\mathcal{A}_3^*)$, and $\hat{\theta} = h_{K_3} \in K_3$ with $A_2\hat{\varphi} + A_3^*\hat{\phi} = (\pi_{A_2} + \pi_{A_3^*})\widetilde{y} = (1 - \pi_3)\widetilde{y}$. If $\widetilde{y} = \widetilde{y}_\perp \in K_3^{\perp_{H_3}}$, then $h_{K_3} = 0$, and $\widetilde{y}$ can be replaced by $\widetilde{y}_\perp$. The upper bound for $|h_{A_3^*}|_{H_3}$ in (4.16) equals

$$|h_{A_3^*}|_{H_3} = \min_{\xi \in N(A_3)} |\xi - \widetilde{y}|_{H_3} = |\hat{\xi} - \widetilde{y}|_{H_3}, \quad \hat{\xi} = h_{A_3^*} + \widetilde{y} \in N(A_3),$$

and so the constant $c_3$ does not play a role. In (4.13) the unknown exact solution $y$ still appears in the upper and in the lower bound. The term $A_2\xi - y \in R(A_2)$ of the upper bound in (4.13) can be handled as an error $h_\xi = y - \widetilde{y}_\xi$ with $\widetilde{y}_\xi = A_2\xi$. As $h_\xi = \pi_{A_2} h_\xi = h_{\xi, A_2}$ we get by (4.15)

$$|A_2\xi - y|_{H_3} = |h_\xi|_{H_3} = \min_{\zeta \in D\left(A_2^*\right)} \left(c_2|A_2^*\zeta - f|_{H_2} + |\zeta - A_2\xi|_{H_3}\right).$$

Another option to compute an upper bound in (4.13) is the following one: As $y \in D(\mathcal{A}_2^*)$ we observe $A_2\xi - y \in D(\mathcal{A}_2^*)$ if $\xi \in D(A_2^* A_2)$. The minimum in (4.13) is attained at $\hat{\xi} = e_{A_2^*} + \widetilde{x} \in D(A_2)$ with $A_2\hat{\xi} = A_2 x = y$. Since $\hat{\xi} \in D(A_2^* A_2)$ and $A_2^* A_2 \hat{\xi} = A_2^* y = f$ we obtain

$$|e_{A_2^*}|_{H_2} = \min_{\xi \in D\left(A_2^* A_2\right)} \left(c_2|A_2\xi - y|_{H_3} + |\xi - \widetilde{x}|_{H_2}\right) = \min_{\xi \in D\left(A_2^* A_2\right)} \left(c_2^2|A_2^* A_2\xi - f|_{H_2} + |\xi - \widetilde{x}|_{H_2}\right),$$

where the latter equality follows by the Friedrichs/Poincaré inequality. To get a lower bound for $|e_{A_2^*}|_{H_2}^2$ in (4.13) we observe $e_{A_2^*} \in R(A_2^*) = R(A_2^* A_2)$ and derive

$$\begin{aligned} |e_{A_2^*}|_{H_2}^2 &= \max_{\phi \in D\left(A_2^* A_2\right)} \left(2\langle e_{A_2^*}, A_2^* A_2\phi\rangle_{H_2} - |A_2^* A_2\phi|_{H_2}^2\right) \\ &= \max_{\phi \in D\left(A_2^* A_2\right)} \left(2\langle e, A_2^* A_2\phi\rangle_{H_2} - |A_2^* A_2\phi|_{H_2}^2\right) \\ &= \max_{\phi \in D\left(A_2^* A_2\right)} \left(2\langle f, \phi\rangle_{H_2} - \langle 2\widetilde{x} + A_2^* A_2\phi, A_2^* A_2\phi\rangle_{H_2}\right). \end{aligned}$$

We summarize the two sided bounds:

**Theorem 4.7.** *Additionally to the General Assumption 3.1, suppose that $R(A_3)$ is closed. Let $x \in \widetilde{D}_2$ be the exact solution of (3.49), $y := A_2 x$, and let*

*$(\widetilde{x},\widetilde{y}) \in H_2 \times H_3$. Then the following estimates hold for the errors $e = x - \widetilde{x}$ and $h = y - \widetilde{y}$ defined in (4.10):*

(i) *The errors decompose, i.e.,*

$$\begin{aligned} e &= e_{\mathrm{A}_1} + e_{K_2} + e_{\mathrm{A}_2^*} \in R(\mathrm{A}_1) \oplus_{H_2} K_2 \oplus_{H_2} R\big(\mathrm{A}_2^*\big), \\ |e|_{H_2}^2 &= |e_{\mathrm{A}_1}|_{H_2}^2 + |e_{K_2}|_{H_2}^2 + |e_{\mathrm{A}_2^*}|_{H_2}^2, \\ h &= h_{\mathrm{A}_2} + h_{K_3} + h_{\mathrm{A}_3^*} \in R(\mathrm{A}_2) \oplus_{H_3} K_3 \oplus_{H_3} R\big(\mathrm{A}_3^*\big), \\ |h|_{H_3}^2 &= |h_{\mathrm{A}_2}|_{H_3}^2 + |h_{K_3}|_{H_3}^2 + |h_{\mathrm{A}_3^*}|_{H_3}^2. \end{aligned}$$

(ii) *The projection $e_{\mathrm{A}_1} = \pi_{\mathrm{A}_1} e = x_g - \pi_{\mathrm{A}_1}\widetilde{x} \in R(\mathrm{A}_1)$ satisfies*

$$\begin{aligned} |e_{\mathrm{A}_1}|_{H_2}^2 &= \min_{\zeta \in D\left(\mathrm{A}_1^*\right)} \big(c_1 |\mathrm{A}_1^*\zeta - g|_{H_1} + |\zeta - \widetilde{x}|_{H_2}\big)^2 \\ &= \max_{\varphi \in D(\mathrm{A}_1)} \big(2\langle g, \varphi\rangle_{H_1} - \langle 2\widetilde{x} + \mathrm{A}_1\varphi, \mathrm{A}_1\varphi\rangle_{H_2}\big) \end{aligned}$$

*and the minimum resp. maximum is attained at*

$$\hat{\zeta} := e_{\mathrm{A}_1} + \widetilde{x} \in D\big(\mathrm{A}_1^*\big), \quad \hat{\varphi} := \mathcal{A}_1^{-1} e_{\mathrm{A}_1} \in D(\mathcal{A}_1)$$

*with $\mathrm{A}_1^*\hat{\zeta} = \mathrm{A}_1^* x = g$.*

(iii) *The projection $e_{\mathrm{A}_2^*} = \pi_{\mathrm{A}_2^*} e = x_y - \pi_{\mathrm{A}_2^*}\widetilde{x} \in R(\mathrm{A}_2^*)$ satisfies*

$$\begin{aligned} |e_{\mathrm{A}_2^*}|_{H_2}^2 &= \min_{\xi \in D(\mathrm{A}_2)} \min_{\zeta \in D\left(\mathrm{A}_2^*\right)} \Big(c_2^2 |\mathrm{A}_2^*\zeta - f|_{H_2} + c_2|\zeta - \mathrm{A}_2\xi|_{H_3} + |\xi - \widetilde{x}|_{H_2}\Big)^2 \\ &= \min_{\xi \in D\left(\mathrm{A}_2^*\mathrm{A}_2\right)} \Big(c_2^2 |\mathrm{A}_2^*\mathrm{A}_2\xi - f|_{H_2} + |\xi - \widetilde{x}|_{H_2}\Big)^2 \\ &= \max_{\phi \in D\left(\mathrm{A}_2^*\mathrm{A}_2\right)} \big(2\langle f, \phi\rangle_{H_2} - \langle 2\widetilde{x} + \mathrm{A}_2^*\mathrm{A}_2\phi, \mathrm{A}_2^*\mathrm{A}_2\phi\rangle_{H_2}\big) \end{aligned}$$

*and the minima resp. maximum are attained at*

$$\begin{aligned} &\hat{\xi} := e_{\mathrm{A}_2^*} + \widetilde{x} \in D\big(\mathrm{A}_2^*\mathrm{A}_2\big), \quad \hat{\zeta} := h_{\xi} + \mathrm{A}_2\xi = y \in D\big(\mathrm{A}_2^*\big), \\ &\hat{\phi} := \mathcal{A}_2^{-1}\big(\mathcal{A}_2^*\big)^{-1} e_{\mathrm{A}_2^*} \in D\big(\mathcal{A}_2^*\mathcal{A}_2\big) \end{aligned}$$

*with $\mathrm{A}_2\hat{\xi} = \mathrm{A}_2 x = y$ and $\mathrm{A}_2^*\mathrm{A}_2\hat{\xi} = \mathrm{A}_2^* y = f$ as well as $\mathrm{A}_2^*\hat{\zeta} = \mathrm{A}_2^* y = f$.*

(iv) *The projection $e_{K_2} = \pi_2 e = k - \pi_2\widetilde{x} \in K_2$ satisfies*

$$\begin{aligned} |e_{K_2}|_{H_2}^2 &= \min_{\varphi \in D(\mathrm{A}_1)} \min_{\phi \in D\left(\mathrm{A}_2^*\right)} |k - \widetilde{x} + \mathrm{A}_1\varphi + \mathrm{A}_2^*\phi|_{H_2}^2 \\ &= \max_{\theta \in K_2} \langle 2(k - \widetilde{x}) - \theta, \theta\rangle_{H_2} \end{aligned}$$

*and the minimum resp. maximum is attained at*

$$\hat{\varphi} := \mathcal{A}_1^{-1}\pi_{A_1}\widetilde{x} \in D(\mathcal{A}_1), \quad \hat{\phi} := \left(\mathcal{A}_2^*\right)^{-1}\pi_{A_2^*}\widetilde{x} \in D\left(\mathcal{A}_2^*\right), \quad \hat{\theta} := e_{K_2} \in K_2$$

*with* $A_1\hat{\varphi} + A_2^*\hat{\phi} = (\pi_{A_1} + \pi_{A_2^*})\widetilde{x} = (1 - \pi_2)\widetilde{x}$.

(v) *The projection* $h_{A_2} = \pi_{A_2}h = y - \pi_{A_2}\widetilde{y} \in R(A_2)$ *satisfies*

$$|h_{A_2}|_{H_3}^2 = \min_{\zeta \in D\left(A_2^*\right)} \left(c_2|A_2^*\zeta - f|_{H_2} + |\zeta - \widetilde{y}|_{H_3}\right)^2$$
$$= \max_{\varphi \in D(A_2)} \left(2\langle f, \varphi\rangle_{H_2} - \langle 2\widetilde{y} + A_2\varphi, A_2\varphi\rangle_{H_3}\right)$$

*and the minimum resp. maximum is attained at*

$$\hat{\zeta} := h_{A_2} + \widetilde{y} \in D\left(A_2^*\right), \quad \hat{\varphi} := \mathcal{A}_2^{-1}h_{A_2} \in D(\mathcal{A}_2)$$

*with* $A_2^*\hat{\zeta} = A_2^*y = f$.

(vi) *The projection* $h_{A_3^*} = \pi_{A_3^*}h = -\pi_{A_3^*}\widetilde{y} \in R(A_3^*)$ *satisfies*

$$|h_{A_3^*}|_{H_3}^2 = \min_{\xi \in D(A_3)} \left(c_3|A_3\xi|_{H_4} + |\xi - \widetilde{y}|_{H_3}\right)^2 = \min_{\xi \in N(A_3)} |\xi - \widetilde{y}|_{H_3}^2$$
$$= \max_{\phi \in D\left(A_3^*\right)} \left(-\langle 2\widetilde{y} + A_3^*\phi, A_3^*\phi\rangle_{H_3}\right)$$

*and the minimum resp. maximum is attained at*

$$\hat{\xi} := h_{A_3^*} + \widetilde{y} \in N(A_3), \quad \hat{\phi} := \left(\mathcal{A}_3^*\right)^{-1}h_{A_3^*} \in D\left(\mathcal{A}_3^*\right)$$

*with* $A_3\hat{\xi} = A_3y = 0$.

(vii) *The projection* $h_{K_3} = \pi_3 e = -\pi_3\widetilde{y} \in K_3$ *satisfies*

$$|h_{K_3}|_{H_3}^2 = \min_{\varphi \in D(A_2)} \min_{\phi \in D\left(A_3^*\right)} |-\widetilde{y} + A_2\varphi + A_3^*\phi|_{H_3}^2$$
$$= \max_{\theta \in K_3} \left(-\langle 2\widetilde{y} + \theta, \theta\rangle_{H_3}\right)$$

*and the minimum resp. maximum is attained at*

$$\hat{\varphi} := \mathcal{A}_2^{-1}\pi_{A_2}\widetilde{y} \in D(\mathcal{A}_2), \quad \hat{\phi} := \left(\mathcal{A}_3^*\right)^{-1}\pi_{A_3^*}\widetilde{y} \in D\left(\mathcal{A}_3^*\right), \quad \hat{\theta} := h_{K_3} \in K_3$$

*with* $A_2\hat{\varphi} + A_3^*\hat{\phi} = (\pi_{A_2} + \pi_{A_3^*})\widetilde{y} = (1 - \pi_3)\widetilde{y}$
*If* $\widetilde{x} = k + \widetilde{x}_\perp$ *with some* $\widetilde{x}_\perp \in K_2^{\perp_{H_2}}$, *then* $e_{K_2} = 0$, *and in (ii) and (iii)* $\widetilde{x}$ *can be replaced by* $\widetilde{x}_\perp$. *If* $\widetilde{y} = \widetilde{y}_\perp \in K_3^{\perp_{H_3}}$, *then* $h_{K_3} = 0$, *and in (v) and (vi)* $\widetilde{y}$ *can be replaced by* $\widetilde{y}_\perp$.

**Remark 4.8.** *A reasonable assumption provided by standard numerical methods is* $\widetilde{y} \in R(\mathrm{A}_2)$*. Hence it often holds* $h_{\mathrm{A}_3^*} = h_{K_3} = 0$.

### 4.3. Computing the error functionals

We propose suitable ways to compute the most important error functionals in Theorem 4.1, Corollary 4.4, and Corollary 4.6. For example, let us focus on Corollary 4.6 (ii), i.e., for $\widetilde{x} \in H_2$ on the error estimates

$$\max_{\varphi \in D(\mathrm{A}_1)} \left(2\langle g, \varphi\rangle_{H_1} - \langle 2\widetilde{x} + \mathrm{A}_1\varphi, \mathrm{A}_1\varphi\rangle_{H_2}\right) = |e_{\mathrm{A}_1}|_{H_2}^2 = \min_{\zeta \in D\left(\mathrm{A}_1^*\right)} \left(c_1|\mathrm{A}_1^*\zeta - g|_{H_1} + |\zeta - \widetilde{x}|_{H_2}\right)^2. \tag{4.18}$$

Before proceeding, let us note that instead of computing the maximum resp. minimum of the lower resp. upper bound we can simply und cheaply choose any $\varphi \in D(\mathrm{A}_1)$ and any $\zeta \in D(\mathrm{A}_1^*)$ given by any method or guess and we obtain the guaranteed error bounds

$$2\langle g, \varphi\rangle_{H_1} - \langle 2\widetilde{x} + \mathrm{A}_1\varphi, \mathrm{A}_1\varphi\rangle_{H_2} \leq |e_{\mathrm{A}_1}|_{H_2}^2 \leq \left(c_1|\mathrm{A}_1^*\zeta - g|_{H_1} + |\zeta - \widetilde{x}|_{H_2}\right)^2.$$

#### 4.3.1. Lower bounds

Considering the maximum on the left hand side of (4.18) we differentiate the lower bound $\Phi(\varphi) := 2\langle g, \varphi\rangle_{H_1} - \langle 2\widetilde{x} + \mathrm{A}_1\varphi, \mathrm{A}_1\varphi\rangle_{H_2}$ with respect to $\varphi$. Hence a maximizer $\hat{\varphi} \in D(\mathrm{A}_1)$ solves the variational formulation

$$\forall\ \varphi \in D(\mathrm{A}_1) \qquad 0 = -\frac{1}{2}\Phi'(\hat{\varphi})\varphi = \langle \mathrm{A}_1\hat{\varphi}, \mathrm{A}_1\varphi\rangle_{H_2} + \langle \widetilde{x}, \mathrm{A}_1\varphi\rangle_{H_2} - \langle g, \varphi\rangle_{H_1}, \tag{4.19}$$

which implies $\mathrm{A}_1\hat{\varphi} + \widetilde{x} \in D(\mathrm{A}_1^*)$ with

$$\mathrm{A}_1^*(\mathrm{A}_1\hat{\varphi} + \widetilde{x}) = g = \mathrm{A}_1^*x$$

and presents a weak formulation of $\mathrm{A}_1^*\mathrm{A}_1\hat{\varphi} = g - \mathrm{A}_1^*\widetilde{x} = \mathrm{A}_1^*e = \mathrm{A}_1^*e_{\mathrm{A}_1}$. By

$$0 = \mathrm{A}_1^*(\mathrm{A}_1\hat{\varphi} + \widetilde{x} - x) = \mathrm{A}_1^*(\mathrm{A}_1\hat{\varphi} - e) = \mathrm{A}_1^*(\mathrm{A}_1\hat{\varphi} - e_{\mathrm{A}_1})$$

we observe $\mathrm{A}_1\hat{\varphi} - e_{\mathrm{A}_1} \in N(\mathrm{A}_1^*) \cap R(\mathrm{A}_1) = N(\mathrm{A}_1^*) \cap N(\mathrm{A}_1^*)^{\perp_{H_2}} = \{0\}$, i.e., $\hat{\varphi}$ solves $\mathrm{A}_1\hat{\varphi} = e_{\mathrm{A}_1}$, see Corollary 4.6 (ii). As $\mathrm{A}_1$ is strictly positive over $D(\mathcal{A}_1) = D(\mathrm{A}_1) \cap R(\mathrm{A}_1^*) = D(\mathrm{A}_1) \cap N(\mathrm{A}_1)^{\perp_{H_1}}$, (4.19) admits a unique solution $\hat{\varphi} \in D(\mathcal{A}_1)$. A particularly simple case is given if $N(\mathrm{A}_1)$ is finite dimensional or even trivial, which occurs in many applications. Otherwise one has to work with the saddle point or double saddle point formulations as we have discussed earlier. The previous considerations show that the unique maximizer $\hat{\varphi} \in D(\mathcal{A}_1)$ is given by

$$\hat{\varphi} = \mathcal{A}_1^{-1} e_{A_1},$$

which is already written down in Corollary 4.6 (ii). Moroever, we finally note

$$\hat{\varphi} = \mathcal{A}_1^{-1} e_{A_1} = \mathcal{A}_1^{-1} \pi_{A_1} e = \mathcal{A}_1^{-1} \pi_{A_1}(x - \tilde{x}) = \mathcal{A}_1^{-1}(x_g - \pi_{A_1}\tilde{x}) = \mathcal{A}_1^{-1}\Big((\mathcal{A}_1^*)^{-1} g - \pi_{A_1}\tilde{x}\Big).$$

If $\tilde{x} \in D(A_1^*)$ then $\pi_{A_1}\tilde{x} \in D(\mathcal{A}_1^*)$ with $A_1^*\pi_{A_1}\tilde{x} = A_1^*\tilde{x}$ and $\hat{\varphi} = \mathcal{A}_1^{-1}(\mathcal{A}_1^*)^{-1}(g - A_1^*\tilde{x})$.

**Remark 4.9.** *The maximum in (4.18) is attained at any $\hat{\varphi} \in D(A_1)$ with $A_1\hat{\varphi} = e_{A_1}$, especially at $\hat{\varphi} = \mathcal{A}_1^{-1} e_{A_1} \in D(\mathcal{A}_1)$. $\hat{\varphi} \in D(A_1)$ can be found by the variational formulation*

$$\forall\ \varphi \in D(A_1) \qquad \langle A_1\hat{\varphi}, A_1\varphi\rangle_{H_2} = \langle g, \varphi\rangle_{H_1} - \langle \tilde{x}, A_1\varphi\rangle_{H_2},$$

*which is coercive (positive) over $D(\mathcal{A}_1)$.*

### *4.3.2. Upper bounds*

For the minimum on the right hand side of (4.18) we can roughly estimate the upper bound by $\Psi(\zeta) := 2c_1^2|A_1^*\zeta - g|_{H_1}^2 + 2|\zeta - \tilde{x}|_{H_2}^2$. Differentiating $\Psi$ shows that the minimizer $\check{\zeta} \in D(A_1^*)$ of $\min_{\zeta \in D(A_1^*)} \Psi(\zeta)$ solves the variational formulation

$$\forall\ \zeta \in D(A_1^*) \qquad 0 = \frac{1}{4}\Psi'\big(\check{\zeta}\big)\zeta \begin{aligned}[t] &= c_1^2\langle A_1^*\check{\zeta} - g, A_1^*\zeta\rangle_{H_1} + \langle \check{\zeta} - \tilde{x}, \zeta\rangle_{H_2} \\ &= c_1^2\langle A_1^*\check{\zeta}, A_1^*\zeta\rangle_{H_1} + \langle \check{\zeta}, \zeta\rangle_{H_2} - c_1^2\langle g, A_1^*\zeta\rangle_{H_1} - \langle \tilde{x}, \zeta\rangle_{H_2}, \end{aligned} \tag{4.20}$$

which implies $A_1^*\check{\zeta} - g \in D(A_1)$ and $c_1^2 A_1(A_1^*\check{\zeta} - g) = (\tilde{x} - \check{\zeta})$, and presents a weak formulation of

$$c_1^2 A_1 A_1^*\check{\zeta} + \check{\zeta} = c_1^2 A_1 g + \tilde{x}.$$

Unique solvability of (4.20) in $D(A_1^*)$ is trivial, as the variational formulation reproduces a graph inner product of $D(A_1^*)$, and we have $\check{\zeta} = (c_1^2 A_1 A_1^* + 1)^{-1}(c_1^2 A_1 g + \tilde{x})$. Moreover, as $g \in R(A_1^*)$ it even holds $A_1^*\check{\zeta} - g \in D(\mathcal{A}_1)$ and hence by the Friedrichs/Poincaré estimate, the equation for $\check{\zeta}$, and inserting $\zeta = \check{\zeta}$ into Corollary 4.6 (ii)

$$|e_{A_1}|_{H_2} \le c_1|A_1^*\check{\zeta} - g|_{H_1} + |\check{\zeta} - \tilde{x}|_{H_2} \le c_1^2|A_1\big(A_1^*\check{\zeta} - g\big)|_{H_1} + |\check{\zeta} - \tilde{x}|_{H_2} = 2\begin{cases} |\check{\zeta} - \tilde{x}|_{H_2}, \\ c_1^2|A_1\big(A_1^*\check{\zeta} - g\big)|_{H_1}. \end{cases} \tag{4.21}$$

This rough minimization process can be improved by using a bit more careful estimate for the square term in (4.18). For this we observe for all $\zeta \in D(\mathrm{A}_1^*)$ and all $t > 0$

$$|e_{\mathrm{A}_1}|_{H_2}^2 \leq \left(1 + t^{-1}\right) c_1^2 \; |\mathrm{A}_1^* \zeta - g|_{H_1}^2 + (1+t)|\zeta - \widetilde{x}|_{H_2}^2 =: \Psi(\widetilde{x}; \zeta, t)$$

and obtain by choosing $\zeta = \hat{\zeta} = e_{\mathrm{A}_1} + \widetilde{x} \in D(\mathrm{A}_1^*)$ from Theorem 4.1, Corollary 4.4 or Corollary 4.6

$$|e_{\mathrm{A}_2^*}|_{H_2}^2 \leq \inf_{t \in (0,\infty)} \inf_{\zeta \in D\left(\mathrm{A}_1^*\right)} \Psi(\widetilde{x}; \zeta, t) \leq \inf_{t \in (0,\infty)} \Psi\left(\widetilde{x}; \hat{\zeta}, t\right) = \inf_{t \in (0,\infty)} (1+t)|e_{\mathrm{A}_1}|_{H_2}^2 = |e_{\mathrm{A}_1}|_{H_2}^2.$$

Thus

$$|e_{\mathrm{A}_1}|_{H_2}^2 = \min_{\zeta \in D\left(\mathrm{A}_1^*\right)^{t \in [0,\infty]},} \Psi(\widetilde{x}; \zeta, t) = \min_{\zeta \in D\left(\mathrm{A}_1^*\right)^{t \in [0,\infty]},} \left(\left(1 + t^{-1}\right) c_1^2 \; |\mathrm{A}_1^* \zeta - g|_{H_1}^2 + (1+t)|\zeta - \widetilde{x}|_{H_2}^2\right), \tag{4.22}$$

and the minimum is attained at $(t, \zeta) = (0, \hat{\zeta})$. For fixed $\zeta \in D(\mathrm{A}_1^*)$ the minimal $t_\zeta \in [0, \infty]$ is given by

$$t_\zeta = \begin{cases} c_1 \dfrac{|\mathrm{A}_1^* \zeta - g|_{H_1}}{|\zeta - \widetilde{x}|_{H_2}} & , \text{ if } \zeta \neq \widetilde{x}, \\ \infty & , \text{ if } \zeta = \widetilde{x}. \end{cases}$$

We note that the case $t_\zeta = \infty$ can only happen if $\widetilde{x} \in D(\mathrm{A}_1^*)$. In any case, inserting $t_\zeta$ into (4.22) we get back the right hand side of (4.18), i.e.,

$$|e_{\mathrm{A}_2^*}|_{H_2}^2 \leq \min_{\zeta \in D\left(\mathrm{A}_1^*\right)} \left(c_1 |\mathrm{A}_1^* \zeta - g|_{H_1} + |\zeta - \widetilde{x}|_{H_2}\right)^2 = |e_{\mathrm{A}_1}|_{H_2}^2.$$

On the other hand, for fixed $0 < t < \infty$ the minimization of $\Psi_t(\zeta) := \Psi(\widetilde{x}; \zeta, t)$ over $\zeta \in D(\mathrm{A}_1^*)$ is equivalent to find $\zeta_t \in D(\mathrm{A}_1^*)$, such that

$$\forall \; \zeta \in D\left(\mathrm{A}_1^*\right) \qquad \frac{t}{2c_1^2(1+t)} \Psi_t'(\zeta_t)(\zeta) = \langle \mathrm{A}_1^* \zeta_t - g, \mathrm{A}_1^* \zeta \rangle_{H_1} + \frac{t}{c_1^2} \langle \zeta_t - \widetilde{x}, \zeta \rangle_{H_2} = 0. \tag{4.23}$$

Especially $\mathrm{A}_1^* \zeta_t - g \in D(\mathrm{A}_1)$ with

$$\mathrm{A}_1\left(\mathrm{A}_1^* \zeta_t - g\right) = \frac{t}{c_1^2} (\widetilde{x} - \zeta_t) \in R(\mathrm{A}_1) \tag{4.24}$$

and hence (4.23) is a standard weak formulation of the coercive problem (in formally strong form) $(\mathrm{A}_1 \mathrm{A}_1^* + \frac{t}{c_1^2})\zeta_t = \mathrm{A}_1 g + \frac{t}{c_1^2} \widetilde{x}$, i.e.,

$$\forall\ \zeta\in D(\mathrm{A}_1^*)\qquad \langle \mathrm{A}_1^*\zeta_t,\mathrm{A}_1^*\zeta\rangle_{H_1}+\frac{t}{c_1^2}\langle\zeta_t,\zeta\rangle_{H_2}=\langle g,\mathrm{A}_1^*\zeta\rangle_{H_1}+\frac{t}{c_1^2}\langle\widetilde{x},\zeta\rangle_{H_2}. \tag{4.25}$$

Moreover, as $g\in R(\mathrm{A}_1^*)$ we even have $\mathrm{A}_1^*\zeta_t-g\in D(\mathcal{A}_1)$ and the strong form holds rigorously if $g\in D(\mathrm{A}_1)$. Furthermore, inserting $\zeta_t$ into (4.22) and using the Friedrichs/Poincaré type estimate shows

$$\begin{aligned}|e_{\mathrm{A}_1}|_{H_2}^2&\le\min_{t\in[0,\infty]}((1+t^{-1})\ c_1^2\ |\mathrm{A}_1^*\zeta_t-g|_{H_1}^2+(1+t)|\zeta_t-\widetilde{x}|_{H_2}^2)\\&\le\min_{t\in[0,\infty]}((1+t^{-1})\ c_1^4\ |\mathrm{A}_1(\mathrm{A}_1^*\zeta_t-g)|_{H_2}^2+(1+t)|\zeta_t-\widetilde{x}|_{H_2}^2)\\&=\begin{cases}\min_{t\in[0,\infty]}\ (1+t)^2|\zeta_t-\widetilde{x}|_{H_2}^2,\\ \min_{t\in[0,\infty]}\ (1+t^{-1})^2\ c_1^4\ |\mathrm{A}_1(\mathrm{A}_1^*\zeta_t-g)|_{H_2}^2,\end{cases}\end{aligned}$$

compare to (4.21). Hence the overestimation by the factor 2 is removed as long as $t$ is close to 0 or $\infty$. A suitable algorithm for computing a good pair $(t,\zeta)$ for approximately minimizing (4.22) is the following:

**Algorithm 4.10.** *Computing a minimizer* $(t,\zeta)$ *in (4.22), i.e., an upper bound for* $|e_{\mathrm{A}_1}|_{H_2}$*:*

- initialization: Set $n:=0$. Pick $\zeta_0\in D(\mathrm{A}_1^*)$ with $\zeta_0\neq\widetilde{x}$.
- loop: Set $n:=n+1$. Compute $t_n=c_1\frac{|\mathrm{A}_1^*\zeta_{n-1}-g|_{H_1}}{|\zeta_{n-1}-\widetilde{x}|_{H_2}}$ and then $\zeta_n\in D(\mathrm{A}_1^*)$ by solving

$$\forall\ \zeta\in D(\mathrm{A}_1^*)\qquad c_1^2\langle\mathrm{A}_1^*\zeta_n,\mathrm{A}_1^*\zeta\rangle_{H_1}+t_n\langle\zeta_n,\zeta\rangle_{H_2}=c_1^2\langle g,\mathrm{A}_1^*\zeta\rangle_{H_1}+t_n\langle\widetilde{x},\zeta\rangle_{H_2}.$$

Compute $\Psi_{\mathrm{A}_1^*}(\widetilde{x};\zeta_n,t_n):=(1+t_n^{-1})\ c_1^2\ |\mathrm{A}_1^*\zeta_n-g|_{H_1}^2+(1+t_n)|\zeta_n-\widetilde{x}|_{H_2}^2.$

- stop if $\Psi_{\mathrm{A}_1^*}(\widetilde{x};\zeta_n,t_n)-\Psi_{\mathrm{A}_1^*}(\widetilde{x};\zeta_{n-1},t_{n-1})$ is small.

**Remark 4.11.** (4.25) *shows for* $\zeta=\zeta_t$

$$\begin{aligned}c_1^2|\mathrm{A}_1^*\zeta_t|_{H_1}^2+t|\zeta_t|_{H_2}^2&=c_1^2\langle g,\mathrm{A}_1^*\zeta_t\rangle_{H_1}+t\langle\widetilde{x},\zeta_t\rangle_{H_2}\\&\le\left(c_1^2|g|_{H_1}^2+t|\widetilde{x}|_{H_2}^2\right)^{1/2}\left(c_1^2|\mathrm{A}_1^*\zeta_t|_{H_1}^2+t|\zeta_t|_{H_2}^2\right)^{1/2}\end{aligned}$$

*and thus*

$$c_1^2|\mathrm{A}_1^*\zeta_t|_{H_1}^2+t|\zeta_t|_{H_2}^2\le c_1^2|g|_{H_1}^2+t|\widetilde{x}|_{H_2}^2.$$

*By* (4.24) *and since* $\mathrm{A}_1^*\zeta_t-g\in D(\mathcal{A}_1)$ *we get*

$$\mathrm{A}_1^*\zeta_t-g=\frac{t}{c_1^2}\mathcal{A}_1^{-1}(\widetilde{x}-\zeta_t)$$

*and hence*

$$|\mathrm{A}_1^*\zeta_t - g|_{H_1} \leq c\ t^{1/2}\Big(|g|_{H_1} + t^{1/2}|\widetilde{x}|_{H_2}\Big)$$

*with $c>0$ independent of $t$ and $\zeta_t$. Let us assume $t_n \to 0$ in Algorithm 4.10. Then by the latter considerations $(\mathrm{A}_1^*\zeta_n)$ and $(t_n^{1/2}\zeta_n)$ are bounded and $\mathrm{A}_1^*\zeta_n \to g$ with the minimal rate $t_n^{1/2}$. Moreover, the projected sequence $(\pi_{\mathrm{A}_1}\zeta_n) \subset D(\mathcal{A}_1^*)$ is bounded in $D(\mathrm{A}_1^*)$ by $\mathrm{A}_1^*\pi_{\mathrm{A}_1}\zeta_n = \mathrm{A}_1^*\zeta_n$ and the Friedrichs/Poincaré estimate $|\pi_{\mathrm{A}_1}\zeta_n|_{H_2} \leq c_1|\mathrm{A}_1^*\pi_{\mathrm{A}_1}\zeta_n|_{H_1}$. If $D(\mathcal{A}_1^*) \hookrightarrow H_2$ is compact, then we can extract a subsequence, again denoted by $(t_n)$, such that $\pi_{\mathrm{A}_1}\zeta_n \to \zeta$ in $H_2$. Thus $\zeta \in D(\mathcal{A}_1^*)$ and $\mathrm{A}_1^*\zeta = g$ as $\mathcal{A}_1^*$ is closed, which shows $\zeta = (\mathcal{A}_1^*)^{-1}g = x_g = \pi_{\mathrm{A}_1}x$, see Theorem 3.3. As the limit $x_g$ is unique, even the whole sequence $\pi_{\mathrm{A}_1}\zeta_n$ converges to $x_g$. For the other part $(1-\pi_{\mathrm{A}_1})\zeta_n \subset N(\mathrm{A}_1^*)$ we apply the projector $1-\pi_{\mathrm{A}_1}$ to (4.24) and obtain $(1-\pi_{\mathrm{A}_1})(\widetilde{x}-\zeta_n)=0$, i.e., $(1-\pi_{\mathrm{A}_1})\zeta_n = (1-\pi_{\mathrm{A}_1})\widetilde{x}$ is constant. Hence*

$$\zeta_n = \pi_{\mathrm{A}_1}\zeta_n + (1-\pi_{\mathrm{A}_1})\zeta_n \to \pi_{\mathrm{A}_1}x + (1-\pi_{\mathrm{A}_1})\widetilde{x} = e_{\mathrm{A}_1} + \widetilde{x} = \hat{\zeta},$$

where $\hat{\zeta} \in D(\mathrm{A}_1^*)$ is the unique minimizer from Corollary 4.6 (ii). Finally, Algorithm 4.10 defines a sequence $(\zeta_n)$ converging in $D(\mathrm{A}_1^*)$ to $\hat{\zeta}$ provided that $D(\mathcal{A}_1^*) \hookrightarrow H_2$ is compact and $t_n \to 0$.

## 5. Applications

### 5.1. Prototype first order system: Electro-magneto statics

As a prototypical example for a first order system we will discuss the system of electro-magneto statics with mixed boundary conditions. Let $\Omega \subset \mathbb{R}^3$ be a bounded weak Lipschitz domain, see [13, Definition 2.3], and let $\Gamma := \partial\Omega$ denote its boundary (Lipschitz manifold), which is supposed to be decomposed into two relatively open weak Lipschitz subdomains (Lipschitz submanifolds) $\Gamma_t$ and $\Gamma_n := \Gamma \setminus \overline{\Gamma_t}$ see [13, Definition 2.5]. Let us consider the linear first order system (in classical strong formulation) for a vector field $E : \Omega \to \mathbb{R}^3$

$$\begin{aligned} \operatorname{rot} E &= F \quad \text{in } \Omega, & n \times E &= 0 \quad \text{at } \Gamma_t, \\ -\operatorname{div} \varepsilon E &= g \quad \text{in } \Omega, & n \cdot \varepsilon E &= 0 \quad \text{at } \Gamma_n, \\ \pi_{\mathcal{H}} E &= K \quad \text{in } \Omega. \end{aligned} \tag{5.1}$$

Here, $\varepsilon : \Omega \to \mathbb{R}^{3\times3}$ is a symmetric, uniformly positive definite $L^\infty$-matrix field and $n$ denotes the outer unit normal at $\Gamma$. Let us put $\mu := \varepsilon^{-1}$. The usual Lebesgue and Sobolev (Hilbert) spaces will be denoted by $L^2(\Omega), H^\ell(\Omega), \ell \in \mathbb{N}_0$, and (in the distributional sense) we introduce

$$R(\Omega) := \{E \in L^2(\Omega) \ : \ \mathrm{rot}E \in L^2(\Omega)\},$$
$$D(\Omega) := \{E \in L^2(\Omega) \ : \ \mathrm{div}E \in L^2(\Omega)\}.$$

Let us also define

$$L^2_\perp(\Omega) := L^2(\Omega) \cap \mathbb{R}^{\perp_{L^2(\Omega)}}, \quad H^1_\perp(\Omega) := H_1(\Omega) \cap L^2_\perp(\Omega).$$

With the test functions or test vector fields

$$C^\infty_{\Gamma_t}(\Omega) := \left\{\varphi|_\Omega \ : \ \varphi \in C^\infty(\mathbb{R}^3), \ \mathrm{supp}\varphi \text{ compact in } \mathbb{R}^3, \ \mathrm{dist}(\mathrm{supp}\varphi, \Gamma_t) > 0\right\},$$
$$C^\infty_\emptyset(\Omega) = C^\infty(\bar{\Omega}),$$

we define as closures of test functions resp. test fields

$$H^1_{\Gamma_t}(\Omega) := \overline{C^\infty_{\Gamma_t}(\Omega)}^{H_1(\Omega)}, \quad R_{\Gamma_t}(\Omega) := \overline{C^\infty_{\Gamma_t}(\Omega)}^{R(\Omega)}, \quad D_{\Gamma_n}(\Omega) := \overline{C^\infty_{\Gamma_n}(\Omega)}^{D(\Omega)},$$

generalizing homogeneous scalar, tangential, and normal traces on $\Gamma_t$ and $\Gamma_n$, respectively. Moreover, we introduce the closed subspaces

$$R_0(\Omega) := \{E \in R(\Omega) \ : \ \mathrm{rot}E = 0\}, \quad D_0(\Omega) := \{E \in D(\Omega) \ : \ \mathrm{div}E = 0\},$$
$$R_{\Gamma_t,0}(\Omega) := R_{\Gamma_t}(\Omega) \cap R_0(\Omega), \qquad D_{\Gamma_n,0}(\Omega) := D_{\Gamma_n}(\Omega) \cap D_0(\Omega),$$

and the Dirichlet–Neumann fields including the corresponding orthonormal projector

$$\mathcal{H}_{t,n,\varepsilon}(\Omega) := R_{\Gamma_t,0}(\Omega) \cap \mu \ D_{\Gamma_n,0}(\Omega), \quad \pi_{\mathcal{H}} : L^2_\varepsilon(\Omega) \to \mathcal{H}_{t,n,\varepsilon}(\Omega).$$

Here, $L^2_\varepsilon(\Omega)$ denotes $L^2(\Omega)$ equipped with the inner product $\langle \ \cdot \ , \ \cdot \ \rangle_{L^2_\varepsilon(\Omega)} := \langle \varepsilon \ \cdot \ , \ \cdot \ \rangle_{L^2(\Omega)}$. Let $H_1 := L^2(\Omega), H_4 := L^2(\Omega)$ (both scalar valued) and $H_2 := L^2_\varepsilon(\Omega), H_3 := L^2(\Omega)$ (both vector valued) as well as

$$\mathrm{A}_1 := \mathrm{grad}_{\Gamma_t} : D(\mathrm{A}_1) := H^1_{\Gamma_t}(\Omega) \subset L^2(\Omega) \to L^2_\varepsilon(\Omega),$$
$$\mathrm{A}_2 := \mathrm{rot}_{\Gamma_t} \ : D(\mathrm{A}_2) := R_{\Gamma_t}(\Omega) \subset L^2_\varepsilon(\Omega) \to L^2(\Omega),$$
$$\mathrm{A}_3 := \mathrm{div}_{\Gamma_t} \ : D(\mathrm{A}_3) := D_{\Gamma_t}(\Omega) \subset L^2(\Omega) \to L^2(\Omega).$$

In [13] it has been shown that the adjoints are

$$\mathrm{A}_1^* = \mathrm{grad}^*_{\Gamma_t} = -\mathrm{div}_{\Gamma_n}\varepsilon : D(\mathrm{A}_1^*) = \mu \ D_{\Gamma_n}(\Omega) \subset L^2_\varepsilon(\Omega) \to L^2(\Omega),$$
$$\mathrm{A}_2^* = \mathrm{rot}^*_{\Gamma_t} = \mu\mathrm{rot}_{\Gamma_n} \quad : D(\mathrm{A}_2^*) = R_{\Gamma_n}(\Omega) \subset L^2(\Omega) \to L^2_\varepsilon(\Omega),$$
$$\mathrm{A}_3^* = \mathrm{div}^*_{\Gamma_t} = -\mathrm{grad}_{\Gamma_n} \quad : D(\mathrm{A}_3^*) = H^1_{\Gamma_n}(\Omega) \subset L^2(\Omega) \to L^2(\Omega).$$

As $\mathrm{A}_1, \mathrm{A}_2, \mathrm{A}_3$ define the well-known de Rham complex, see, e.g., [13, Lemma 2.2], so do their adjoints, i.e., for[17] $\emptyset \neq \Gamma_t \neq \Gamma$

$$\{0\} \xrightarrow{\mathrm{A}_0=\iota_{\{0\}}} \mathsf{H}^1_{\Gamma_t}(\Omega) \xrightarrow{\mathrm{A}_1=\mathrm{grad}_{\Gamma_t}} \mathsf{R}_{\Gamma_t}(\Omega) \xrightarrow{\mathrm{A}_2=\mathrm{rot}_{\Gamma_t}} \mathsf{D}_{\Gamma_t}(\Omega) \xrightarrow{\mathrm{A}_3=\mathrm{div}_{\Gamma_t}} \mathsf{L}^2(\Omega) \xrightarrow{\mathrm{A}_4=\pi_{\{0\}}} \{0\},$$

$$\{0\} \xleftarrow{\mathrm{A}_0^*=\pi_{\{0\}}} \mathsf{L}^2(\Omega) \xleftarrow{\mathrm{A}_1^*=-\,\mathrm{div}_{\Gamma_n}\,\varepsilon} \mu\,\mathsf{D}_{\Gamma_n}(\Omega) \xleftarrow{\mathrm{A}_2^*=\mu\,\mathrm{rot}_{\Gamma_n}} \mathsf{R}_{\Gamma_n}(\Omega) \xleftarrow{\mathrm{A}_3^*=-\,\mathrm{grad}_{\Gamma_n}} \mathsf{H}^1_{\Gamma_n}(\Omega) \xleftarrow{\mathrm{A}_4^*=\iota_{\{0\}}} \{0\},$$

where we have introduced the additional canonical embedding and projection operators $\mathrm{A}_0, \mathrm{A}_0^*, \mathrm{A}_4, \mathrm{A}_4^*$ by

$$\mathrm{A}_0 = \begin{cases} \iota_{\{0\}} : H_0 = \{0\} & , \text{ if } \Gamma_t \neq \emptyset \\ \iota_{\mathbb{R}} : H_0 = \mathbb{R} & , \text{ if } \Gamma_t = \emptyset \end{cases} \to L^2(\Omega),$$
$$\mathrm{A}_4 = \begin{cases} \pi_{\{0\}} & , \text{ if } \Gamma_t \neq \Gamma \\ \pi_{\mathbb{R}} & , \text{ if } \Gamma_t = \Gamma \end{cases} : L^2(\Omega) \to \begin{cases} \{0\} & , \text{ if } \Gamma_t \neq \Gamma \\ \mathbb{R} & , \text{ if } \Gamma_t = \Gamma \end{cases},$$
$$\mathrm{A}_4^* = \begin{cases} \iota_{\{0\}} : H_5 = \{0\} & , \text{ if } \Gamma_t \neq \Gamma \\ \iota_{\mathbb{R}} : H_5 = \mathbb{R} & , \text{ if } \Gamma_t = \Gamma \end{cases} \to L^2(\Omega),$$
$$\mathrm{A}_0^* = \begin{cases} \pi_{\{0\}} & , \text{ if } \Gamma_t \neq \emptyset \\ \pi_{\mathbb{R}} & , \text{ if } \Gamma_t = \emptyset \end{cases} : L^2(\Omega) \to \begin{cases} \{0\} & , \text{ if } \Gamma_t \neq \emptyset \\ \mathbb{R} & , \text{ if } \Gamma_t = \emptyset \end{cases}.$$

For the kernels we have

$$N(\mathrm{A}_0) = \{0\} \qquad N(\mathrm{A}_0^*) = \begin{cases} L^2(\Omega) & , \text{ if } \Gamma_t \neq \emptyset, \\ L^2_\perp(\Omega) & , \text{ if } \Gamma_t = \emptyset, \end{cases},$$
$$N(\mathrm{A}_1) = \begin{cases} \{0\} & , \text{ if } \Gamma_t \neq \emptyset, \\ \mathbb{R} & , \text{ if } \Gamma_t = \emptyset, \end{cases} \qquad N(\mathrm{A}_1^*) = \mu\ D_{\Gamma_n,0}(\Omega),$$
$$N(\mathrm{A}_2) = R_{\Gamma_t,0}(\Omega), \qquad N(\mathrm{A}_2^*) = R_{\Gamma_n,0}(\Omega),$$
$$N(\mathrm{A}_3) = D_{\Gamma_t,0}(\Omega), \qquad N(\mathrm{A}_3^*) = \begin{cases} \{0\} & , \text{ if } \Gamma_t \neq \Gamma, \\ \mathbb{R} & , \text{ if } \Gamma_t = \Gamma, \end{cases},$$
$$N(\mathrm{A}_4) = \begin{cases} L^2(\Omega) & , \text{ if } \Gamma_t \neq \Gamma, \\ L^2_\perp(\Omega) & , \text{ if } \Gamma_t = \Gamma, \end{cases}, \qquad N(\mathrm{A}_4^*) = \{0\}$$

and for the cohomology groups

[17]For $\Gamma_t = \emptyset$ we have

$$\mathbb{R} \xrightarrow{\mathrm{A}_0=\iota_{\mathbb{R}}} \mathsf{H}^1(\Omega) \xrightarrow{\mathrm{A}_1=\mathrm{grad}} \mathsf{R}(\Omega) \xrightarrow{\mathrm{A}_2=\mathrm{rot}} \mathsf{D}(\Omega) \xrightarrow{\mathrm{A}_3=\mathrm{div}} \mathsf{L}^2(\Omega) \xrightarrow{\mathrm{A}_4=\pi_{\{0\}}} \{0\},$$
$$\mathbb{R} \xleftarrow{\mathrm{A}_0^*=\pi_{\mathbb{R}}} \mathsf{L}^2(\Omega) \xleftarrow{\mathrm{A}_1^*=-\,\mathrm{div}_\Gamma\,\varepsilon} \mu\,\mathsf{D}_\Gamma(\Omega) \xleftarrow{\mathrm{A}_2^*=\mu\,\mathrm{rot}_\Gamma} \mathsf{R}_\Gamma(\Omega) \xleftarrow{\mathrm{A}_3^*=-\,\mathrm{grad}_\Gamma} \mathsf{H}^1_\Gamma(\Omega) \xleftarrow{\mathrm{A}_4^*=\iota_{\{0\}}} \{0\},$$

which also shows the case $\Gamma_t = \Gamma$ by interchanging $\Gamma_t$ and $\Gamma_n$ and shifting $\varepsilon$. More precisely, for $\Gamma_t = \Gamma$ it holds

$$\{0\} \xrightarrow{\mathrm{A}_0=\iota_{\{0\}}} \mathsf{H}^1_\Gamma(\Omega) \xrightarrow{\mathrm{A}_1=\mathrm{grad}_\Gamma} \mathsf{R}_\Gamma(\Omega) \xrightarrow{\mathrm{A}_2=\mathrm{rot}_\Gamma} \mathsf{D}_\Gamma(\Omega) \xrightarrow{\mathrm{A}_3=\mathrm{div}_\Gamma} \mathsf{L}^2(\Omega) \xrightarrow{\mathrm{A}_4=\pi_{\mathbb{R}}} \mathbb{R},$$
$$\{0\} \xleftarrow{\mathrm{A}_0^*=\pi_{\{0\}}} \mathsf{L}^2(\Omega) \xleftarrow{\mathrm{A}_1^*=-\,\mathrm{div}\,\varepsilon} \mu\,\mathsf{D}(\Omega) \xleftarrow{\mathrm{A}_2^*=\mu\,\mathrm{rot}} \mathsf{R}(\Omega) \xleftarrow{\mathrm{A}_3^*=-\,\mathrm{grad}} \mathsf{H}^1(\Omega) \xleftarrow{\mathrm{A}_4^*=\iota_{\mathbb{R}}} \mathbb{R}.$$

$$\begin{aligned}
K_0 &= N(\mathrm{A}_0) = \{0\},\\
K_1 &= N(\mathrm{A}_1) \cap N(\mathrm{A}_0^*) = \{0\},\\
K_2 &= N(\mathrm{A}_2) \cap N(\mathrm{A}_1^*) = R_{\Gamma_t,0}(\Omega) \cap \mu\ D_{\Gamma_n,0}(\Omega) = \mathcal{H}_{t,n,\varepsilon}(\Omega),\\
K_3 &= N(\mathrm{A}_3) \cap N(\mathrm{A}_2^*) = D_{\Gamma_t,0}(\Omega) \cap R_{\Gamma_n,0}(\Omega) =: \mathcal{H}_{n,t}(\Omega),\\
K_4 &= N(\mathrm{A}_4) \cap N(\mathrm{A}_3^*) = \{0\},\\
K_5 &= N(\mathrm{A}_4^*) = \{0\}.
\end{aligned}$$

Using the latter operators $\mathrm{A}_2 = \mathrm{rot}_{\Gamma_t}$ and $\mathrm{A}_1^* = -\mathrm{div}_{\Gamma_n}\varepsilon$, the linear first order system (5.1) (in weak formulation) has the form of (1.5) resp. (3.1), i.e., find a vector field

$$E \in D_2 = D(\mathrm{A}_2) \cap D(\mathrm{A}_1^*) = R_{\Gamma_t}(\Omega) \cap \mu\ D_{\Gamma_n}(\Omega),$$

such that

$$\begin{aligned}
\mathrm{rot}_{\Gamma_t} E &= F,\\
-\mathrm{div}_{\Gamma_n}\varepsilon E &= g, \qquad (5.2)\\
\pi_{\mathcal{H}} E &= K,
\end{aligned}$$

where $K_2 = \mathcal{H}_{t,n,\varepsilon}(\Omega)$. In [13, Theorem 5.1] the embedding $D_2, \to H_2$, i.e.,

$$R_{\Gamma_t}(\Omega) \cap \mu\ D_{\Gamma_n}(\Omega), \to L^2_\varepsilon(\Omega),$$

was shown to be compact. Hence also the embedding $D_3 = D(\mathrm{A}_3) \cap D(\mathrm{A}_2^*), \to H_3$, i.e.,

$$D_{\Gamma_t}(\Omega) \cap R_{\Gamma_n}(\Omega), \to L^2(\Omega),$$

is compact. Thus, by the results of the functional analysis toolbox Section 2, all occurring ranges are closed, certain Helmholtz type decompositions hold, corresponding Friedrichs/Poincaré type estimates are valid, and the respective inverse operators are continuous resp. compact. Especially, the reduced operators are

$$\begin{aligned}
\mathcal{A}_1 &:= \widetilde{\mathrm{grad}}_{\Gamma_t} : D(\mathcal{A}_1) = H^1_{\Gamma_t}(\Omega) \cap L^2(\Omega) \subset L^2(\Omega) \to \mathrm{grad}\ H^1_{\Gamma_t}(\Omega),\\
\mathcal{A}_2 &:= \widetilde{\mathrm{rot}}_{\Gamma_t} : D(\mathcal{A}_2) = R_{\Gamma_t}(\Omega) \cap \mu\,\mathrm{rot} R_{\Gamma_n}(\Omega) \subset \mu\,\mathrm{rot} R_{\Gamma_n}(\Omega) \to \mathrm{rot} R_{\Gamma_t}(\Omega),\\
\mathcal{A}_3 &:= \widetilde{\mathrm{div}}_{\Gamma_t} : D(\mathcal{A}_3) = D_{\Gamma_t}(\Omega) \cap \mathrm{grad} H^1_{\Gamma_n}(\Omega) \subset \mathrm{grad} H^1_{\Gamma_n}(\Omega) \to L^2(\Omega),
\end{aligned}$$

where $\mathrm{grad} H^1_{\Gamma_t}(\Omega)$ and $\mu \mathrm{rot} R_{\Gamma_n}(\Omega)$ have to be understood as closed subspaces of $L^2_\varepsilon(\Omega)$, and $L^2(\Omega)$ has to be replaced by $L^2_\perp(\Omega)$ in $\mathcal{A}_1$, if $\Gamma_t = \emptyset$, and in $\mathcal{A}_3$, if $\Gamma_t = \Gamma$, with adjoints

$$\begin{aligned}
\mathcal{A}_1^* &= \widetilde{\mathrm{grad}}_{\Gamma_t}^* = -\widetilde{\mathrm{div}}_{\Gamma_n}\varepsilon : D(\mathcal{A}_1^*) = \mu\ D_{\Gamma_n}(\Omega) \cap \mathrm{grad}\ H^1_{\Gamma_t}(\Omega) \subset \mathrm{grad}\ H^1_{\Gamma_t}(\Omega) \to L^2(\Omega),\\
\mathcal{A}_2^* &= \widetilde{\mathrm{rot}}_{\Gamma_t}^* = \mu\ \widetilde{\mathrm{rot}}_{\Gamma_n} : D(\mathcal{A}_2^*) = R_{\Gamma_n}(\Omega) \cap \mathrm{rot}\ R_{\Gamma_t}(\Omega) \subset \mathrm{rot} R_{\Gamma_t}(\Omega) \to \mu\mathrm{rot}\ R_{\Gamma_n}(\Omega),\\
\mathcal{A}_3^* &= \widetilde{\mathrm{div}}_{\Gamma_t}^* = -\widetilde{\mathrm{grad}}_{\Gamma_n} : D(\mathcal{A}_3^*) = H^1_{\Gamma_n}(\Omega) \cap L^2(\Omega) \subset L^2(\Omega) \to \mathrm{grad} H^1_{\Gamma_n}(\Omega),
\end{aligned}$$

where $L^2(\Omega)$ has to be replaced by $L^2_\perp(\Omega)$ in $\mathcal{A}_1^*$, if $\Gamma_t = \emptyset$, and in $\mathcal{A}_3^*$, if

$\Gamma_t = \Gamma$. Note that the reduced operators possess bounded resp. compact inverse operators. For the ranges we have

$$\begin{aligned}
R(\mathrm{A}_1) = R(\mathcal{A}_1) \subset N(\mathrm{A}_2), \text{i.e.,} \quad & \operatorname{grad} H^1_{\Gamma_t}(\Omega) = \operatorname{grad}\left(H^1_{\Gamma_t}(\Omega) \cap L^2(\Omega)\right) \subset R_{\Gamma_t,0}(\Omega), \\
R(\mathrm{A}_2) = R(\mathcal{A}_2) \subset N(\mathrm{A}_3), \text{i.e.,} \quad & \operatorname{rot} R_{\Gamma_t}(\Omega) = \operatorname{rot}\left(R_{\Gamma_t}(\Omega) \cap \mu \operatorname{rot} R_{\Gamma_n}(\Omega)\right) \subset D_{\Gamma_t,0}(\Omega), \\
R(\mathrm{A}_3) = R(\mathcal{A}_3), \text{i.e.,} \quad & \operatorname{div} D_{\Gamma_t}(\Omega) = \operatorname{div}\left(D_{\Gamma_t}(\Omega) \cap \operatorname{grad} H^1_{\Gamma_n}(\Omega)\right), \\
R(\mathrm{A}_1^*) = R(\mathcal{A}_1^*), \text{i.e.,} \quad & \operatorname{div} D_{\Gamma_n}(\Omega) = \operatorname{div}\left(D_{\Gamma_n}(\Omega) \cap \varepsilon \operatorname{grad} H^1_{\Gamma_t}(\Omega)\right), \\
R(\mathrm{A}_2^*) = R(\mathcal{A}_2^*) \subset N(\mathrm{A}_1^*), \text{i.e.,} \quad & \mu \operatorname{rot} R_{\Gamma_n}(\Omega) = \mu \operatorname{rot}\left(R_{\Gamma_n}(\Omega) \cap \operatorname{rot} R_{\Gamma_t}(\Omega)\right) \subset \mu\, D_{\Gamma_n,0}(\Omega), \\
R(\mathrm{A}_3^*) = R(\mathcal{A}_3^*) \subset N(\mathrm{A}_2^*), \text{i.e.,} \quad & \operatorname{grad} H^1_{\Gamma_n}(\Omega) = \operatorname{grad}\left(H^1_{\Gamma_n}(\Omega) \cap L^2(\Omega)\right) \subset R_{\Gamma_n,0}(\Omega),
\end{aligned}$$

where $L^2(\Omega)$ has to be replaced by $L^2_\perp(\Omega)$ for $\Gamma_t = \emptyset$ resp. $\Gamma_t = \Gamma$. Note that the assertions of $R(\mathrm{A}_3), R(\mathrm{A}_2^*), R(\mathrm{A}_3^*)$ are already included in those of $R(\mathrm{A}_1), R(\mathrm{A}_2), R(\mathrm{A}_1^*)$ by interchanging $\Gamma_t$ and $\Gamma_n$, and setting $\varepsilon := \mathrm{id}$. Furthermore, the following Friedrichs/Poincaré type estimates hold:

$$\begin{aligned}
\forall\ u \in D(\mathcal{A}_1) = H^1_{\Gamma_t}(\Omega) \cap L^2(\Omega) \qquad & |u|_{L^2(\Omega)} \leq c_{fp}\ |\operatorname{grad} u|_{L^2_\varepsilon(\Omega)}, \\
\forall\ E \in D(\mathcal{A}_1^*) = \mu\ D_{\Gamma_n}(\Omega) \cap \operatorname{grad}\ H^1_{\Gamma_t}(\Omega), \qquad & |E|_{L^2_\varepsilon(\Omega)} \leq c_{fp}\ |\operatorname{div} \varepsilon E|_{L^2(\Omega)}, \\
\forall\ E \in D(\mathcal{A}_2) = R_{\Gamma_t}(\Omega) \cap \mu \operatorname{rot}\ R_{\Gamma_n}(\Omega), \qquad & |E|_{L^2_\varepsilon(\Omega)} \leq c_m\ |\operatorname{rot} E|_{L^2(\Omega)}, \\
\forall\ E \in D(\mathcal{A}_2^*) = R_{\Gamma_n}(\Omega) \cap \operatorname{rot}\ R_{\Gamma_t}(\Omega), \qquad & |E|_{L^2(\Omega)} \leq c_m\ |\operatorname{rot} E|_{L^2_\mu}, \\
\forall\ E \in D(\mathcal{A}_3) = D_{\Gamma_t}(\Omega) \cap \operatorname{grad}\ H^1_{\Gamma_n}(\Omega), \qquad & |E|_{L^2(\Omega)} \leq \widetilde{c}_{fp}\ |\operatorname{div} E|_{L^2(\Omega)}, \\
\forall\ u \in D(\mathcal{A}_3^*) = H^1_{\Gamma_n}(\Omega) \cap L^2(\Omega) \qquad & |u|_{L^2(\Omega)} \leq \widetilde{c}_{fp}\ |\operatorname{grad} u|_{L^2(\Omega)},
\end{aligned}$$

where the Friedrichs/Poincaré and Maxwell constants $c_{fp}, c_m, \widetilde{c}_{fp}$, are given by the respective Raleigh quotients, and $L^2(\Omega)$ has to be replaced by $L^2_\perp(\Omega)$ for $\Gamma_t = \emptyset$ resp. $\Gamma_t = \Gamma$. Again note that the latter two assertions are already included in the first two inequalities by interchanging $\Gamma_t$ and $\Gamma_n$ and setting $\varepsilon := \mathrm{id}$.

**Remark 5.1.** *Let the Friedrichs and the Poincaré constants $c_f, c_p$ as well as upper and lower bounds for the matrix field $\varepsilon$ be defined by*

$$\frac{1}{c_p} := \inf_{0 \neq \varphi \in H^1_\perp(\Omega)} \frac{|\operatorname{grad} \varphi|_{L^2(\Omega)}}{|\varphi|_{L^2(\Omega)}}, \qquad \frac{1}{\overline{\varepsilon}} := \inf_{0 \neq \Phi \in L^2(\Omega)} \frac{|\Phi|_{L^2(\Omega)}}{|\Phi|_{L^2_\varepsilon(\Omega)}},$$
$$\frac{1}{c_f} := \inf_{0 \neq \varphi \in H^1_\Gamma(\Omega)} \frac{|\operatorname{grad} \varphi|_{L^2(\Omega)}}{|\varphi|_{L^2(\Omega)}}, \qquad \frac{1}{\underline{\varepsilon}} := \inf_{0 \neq \Phi \in L^2(\Omega)} \frac{|\Phi|_{L^2_\varepsilon(\Omega)}}{|\Phi|_{L^2(\Omega)}}.$$

*In [14], see also [15, 16], the following has been proved for bounded and convex $\Omega$:*

(i) *If $\Gamma_t = \emptyset$ or $\Gamma_t = \Gamma$, then $c_m \leq \overline{\varepsilon}\ c_p \leq \overline{\varepsilon} \operatorname{diam} \Omega / \pi$.*
(ii) *If $\Gamma_t = \emptyset$ we have $c_p / \overline{\varepsilon} \leq c_{fp} \leq c_p$ and $\widetilde{c}_{fp} = c_f < c_p$.*
(iii) *If $\Gamma_t = \Gamma$ it holds $c_f / \overline{\varepsilon} \leq c_{fp} \leq c_f$ and $c_f < c_p = \widetilde{c}_{fp}$.*

Finally, the following Helmholtz decompositions hold:

$$H_1 = L^2(\Omega) = \begin{cases} \{0\} & , \text{ if } \Gamma_t \neq \emptyset, \\ \mathbb{R} & , \text{ if } \Gamma_t = \emptyset, \end{cases} \oplus_{L^2(\Omega)} \begin{cases} L^2(\Omega) & , \text{ if } \Gamma_t \neq \emptyset, \\ L^2_\perp(\Omega) & , \text{ if } \Gamma_t = \emptyset, \end{cases} \qquad (H_1 = R(A_0)\oplus_{H_1} N(A_0^*))$$

$$= \begin{cases} \{0\} & , \text{ if } \Gamma_t \neq \emptyset, \\ \mathbb{R} & , \text{ if } \Gamma_t = \emptyset, \end{cases} \square \oplus_{L^2(\Omega)} \operatorname{div} D_{\Gamma_n}(\Omega), \qquad (H_1 = N(A_1)\oplus_{H_1} R(A_1^*))$$

$$H_2 = L^2_\varepsilon(\Omega) = \operatorname{grad} H^1_{\Gamma_t}(\Omega)\oplus_{L^2_\varepsilon(\Omega)} \mu \,\square\, D_{\Gamma_n,0}(\Omega) \qquad (H_2 = R(A_1)\oplus_{H_2} N(A_1^*))$$
$$= R_{\Gamma_t,0}(\Omega)\oplus_{L^2_\varepsilon(\Omega)} \mu \operatorname{rot} R_{\Gamma_n}(\Omega) \qquad (H_2 = N(A_2)\oplus_{H_2} R(A_2^*))$$
$$= \operatorname{grad} H^1_{\Gamma_t}(\Omega)\oplus_{L^2_\varepsilon(\Omega)} \mathcal{H}_{t,n,\varepsilon}(\Omega)\oplus_{L^2_\varepsilon(\Omega)} \mu \operatorname{rot} R_{\Gamma_n}(\Omega), \quad (H_2 = R(A_1)\oplus_{H_2} K_2 \oplus_{H_2} R(A_1^*))$$

$$H_3 = L^2(\Omega) = \operatorname{grad} H^1_{\Gamma_n}(\Omega)\oplus_{L^2(\Omega)} D_{\Gamma_t,0}(\Omega) \qquad (H_3 = R(A_3^*)\oplus_{H_3} N(A_3))$$
$$= R_{\Gamma_n,0}(\Omega)\oplus_{L^2(\Omega)} \operatorname{rot} R_{\Gamma_t}(\Omega) \qquad (H_3 = N(A_2^*)\oplus_{H_3} R(A_2))$$
$$= \operatorname{grad} H^1_{\Gamma_n}(\Omega)\oplus_{L^2(\Omega)} \mathcal{H}_{n,t}(\Omega)\oplus_{L^2(\Omega)} \operatorname{rot} R_{\Gamma_t}(\Omega), \quad (H_3 = R(A_3^*)\oplus_{H_3} K_3 \oplus_{H_3} R(A_2))$$

$$H_4 = L^2(\Omega) = \begin{cases} \{0\} & , \text{ if } \Gamma_t \neq \Gamma, \\ \mathbb{R} & , \text{ if } \Gamma_t = \Gamma, \end{cases} \oplus_{L^2(\Omega)} \begin{cases} L^2(\Omega) & , \text{ if } \Gamma_t \neq \Gamma, \\ L^2_\perp(\Omega) & , \text{ if } \Gamma_t = \Gamma, \end{cases} \qquad (H_4 = R(A_4^*)\oplus_{H_4} N(A_4))$$
$$= \begin{cases} \{0\} & , \text{ if } \Gamma_t \neq \Gamma, \\ \mathbb{R} & , \text{ if } \Gamma_t = \Gamma, \end{cases} \oplus_{L^2(\Omega)} \operatorname{div} D_{\Gamma_t}(\Omega). \qquad (H_4 = N(A_3^*)\oplus_{H_4} R(A_3))$$

The latter two decompositions are already given by the first two ones by interchanging $\Gamma_t$ and $\Gamma_n$ and setting $\varepsilon := \mathrm{id}$. Especially, it holds

$$\operatorname{grad} H^1_{\Gamma_t}(\Omega) = R_{\Gamma_t,0}(\Omega)\ominus_{L^2_\varepsilon(\Omega)} \mathcal{H}_{t,n,\varepsilon}(\Omega), \quad \mu \operatorname{rot} R_{\Gamma_n}(\Omega) = \mu\, D_{\Gamma_n,0}(\Omega)\ominus_{L^2_\varepsilon(\Omega)} \mathcal{H}_{t,n,\varepsilon}(\Omega),$$
$$\operatorname{grad} H^1_{\Gamma_n}(\Omega) = R_{\Gamma_n,0}(\Omega)\ominus_{L^2(\Omega)} \mathcal{H}_{n,t}(\Omega), \quad \operatorname{rot} R_{\Gamma_t}(\Omega) = D_{\Gamma_t,0}(\Omega)\ominus_{L^2(\Omega)} \mathcal{H}_{n,t}(\Omega).$$

If $\Gamma_t = \Gamma$ and $\Gamma$ is connected, then the Dirichlet fields are trivial, i.e.,

$$\mathcal{H}_{t,n,\varepsilon}(\Omega) = R_{\Gamma,0}(\Omega) \cap \mu\, D_0(\Omega) = \{0\}.$$

If $\Gamma_t = \emptyset$ and $\Omega$ is simply connected, then the Neumann fields are trivial, i.e.,

$$\mathcal{H}_{t,n,\varepsilon}(\Omega) = R_0(\Omega) \cap \mu\, D_{\Gamma,0}(\Omega) = \{0\}.$$

Now we can apply the general results of Section 3 and Section 4.

**Theorem 5.2** (Theorem 3.3). *(5.1) resp. (5.2) is uniquely solvable, if and only if*

$$F \in \operatorname{rot} R_{\Gamma_t}(\Omega) = D_{\Gamma_t,0}(\Omega)\ominus_{L^2(\Omega)} \mathcal{H}_{n,t}(\Omega), \quad g \in L^2(\Omega), \quad K \in \mathcal{H}_{t,n,\varepsilon}(\Omega),$$

*where $L^2(\Omega)$ has to be replaced by $L^2_\perp(\Omega)$ if $\Gamma_t = \emptyset$. The unique solution $E \in R_{\Gamma_t}(\Omega) \cap \mu\, D_{\Gamma_n}(\Omega)$ is given by*

$$E := E_F + E_g + K \in \big(R_{\Gamma_t}(\Omega) \cap \mu \operatorname{rot} R_{\Gamma_n}(\Omega)\big)\oplus_{L^2_\varepsilon(\Omega)} \Big(\mu\, D_{\Gamma_n}(\Omega) \cap \operatorname{grad} H^1_{\Gamma_t}(\Omega)\Big)\oplus_{L^2_\varepsilon(\Omega)} \mathcal{H}_{t,n,\varepsilon}(\Omega)$$
$$= R_{\Gamma_t}(\Omega) \cap \mu\, D_{\Gamma_n}(\Omega),$$
$$E_F := \big(\widetilde{\operatorname{rot}}_{\Gamma_t}\big)^{-1} F \in R_{\Gamma_t}(\Omega) \cap \mu \operatorname{rot} R_{\Gamma_n}(\Omega) = R_{\Gamma_t}(\Omega) \cap \mu\, D_{\Gamma_n,0}(\Omega) \cap \mathcal{H}_{t,n,\varepsilon}(\Omega)^{\perp_{L^2_\varepsilon(\Omega)}},$$
$$E_g := -\Big(\widetilde{\operatorname{div}}_{\Gamma_n}\varepsilon\Big)^{-1} g \in \mu\, D_{\Gamma_n}(\Omega) \cap \operatorname{grad} H^1_{\Gamma_t}(\Omega) = \mu\, D_{\Gamma_n}(\Omega) \cap R_{\Gamma_t,0}(\Omega) \cap \mathcal{H}_{t,n,\varepsilon}(\Omega)^{\perp_{L^2_\varepsilon(\Omega)}}$$

*and depends continuously on the data, i.e.,* $|E|_{L^2_\varepsilon(\Omega)} \leq c_m\, |F|_{L^2(\Omega)} +$

$c_{fp}\ |g|_{L^2(\Omega)} + |K|_{L^2(\Omega)}$, *as*

$$|E_F|_{L^2_\varepsilon(\Omega)} \le c_m\ |F|_{L^2(\Omega)}, \quad |E_g|_{L^2_\varepsilon(\Omega)} \le c_{fp}\ |g|_{L^2(\Omega)}.$$

*Moreover,* $|E|^2_{L^2_\varepsilon(\Omega)} = |E_F|^2_{L^2_\varepsilon(\Omega)} + |E_g|^2_{L^2_\varepsilon(\Omega)} + |K|^2_{L^2_\varepsilon(\Omega)}$.
The partial solutions $E_F$ and $E_g$, solving

$$\begin{aligned} \operatorname{rot}_{\Gamma_t} E_F &= F, & \operatorname{rot}_{\Gamma_t} E_g &= 0, \\ -\operatorname{div}_{\Gamma_n} \varepsilon E_F &= 0, & -\operatorname{div}_{\Gamma_n} \varepsilon E_g &= g, \\ \pi_{\mathcal{H}} E_F &= 0, & \pi_{\mathcal{H}} E_g &= 0, \end{aligned}$$

can be found and computed by the following four variational formulations:

**Theorem 5.3** (Theorem 3.5). *The partial solutions $E_F$ and $E_g$ in Theorem 5.2 can be found by the following four variational formulations:*

(i) *There exists a unique $\widetilde{E}_F \in R_{\Gamma_t}(\Omega) \cap \mu \operatorname{rot} R_{\Gamma_n}(\Omega)$ such that*

$$\forall\ \Phi \in R_{\Gamma_t}(\Omega) \cap \mu \operatorname{rot}\ R_{\Gamma_n}(\Omega) \qquad \langle \operatorname{rot}\widetilde{E}_F, \operatorname{rot}\Phi\rangle_{L^2(\Omega)} = \langle F, \operatorname{rot}\Phi\rangle_{L^2(\Omega)}. \tag{5.3}$$

*Equation (5.3) is even satisfied for all $\Phi \in R_{\Gamma_t}(\Omega)$. Moreover, the equation* $\operatorname{rot}\widetilde{E}_F = F$ *holds if and only if $F \in \operatorname{rot} R_{\Gamma_t}(\Omega)$. In this case $\widetilde{E}_F = E_F$.*

(i') *There exists a unique potential $H_F \in R_{\Gamma_n}(\Omega) \cap \operatorname{rot}\ R_{\Gamma_t}(\Omega)$ such that*

$$\forall\ \Psi \in R_{\Gamma_n}(\Omega) \cap \operatorname{rot}\ R_{\Gamma_t}(\Omega) \qquad \langle \mu \operatorname{rot}\ H_F, \operatorname{rot}\Psi\rangle_{L^2(\Omega)} = \langle F, \Psi\rangle_{L^2(\Omega)}. \tag{5.4}$$

*Equation (5.4) even holds for all $\Psi \in R_{\Gamma_n}(\Omega)$ if and only if $F \in \operatorname{rot} R_{\Gamma_t}(\Omega)$. In this case we have*

$$\mu \operatorname{rot} H_F \in R_{\Gamma_t}(\Omega) \cap \mu \operatorname{rot} R_{\Gamma_n}(\Omega)$$

*with* $\operatorname{rot}\mu\operatorname{rot} H_F = F$ *and hence* $\mu \operatorname{rot} H_F = E_F$

(ii) *Let $\Gamma_t \neq \emptyset$. There is a unique $\widetilde{E}_g \in \mu\ D_{\Gamma_n}(\Omega) \cap \operatorname{grad} H^1_{\Gamma_t}(\Omega)$ such that*

$$\forall\ \Theta \in \mu\ D_{\Gamma_n}(\Omega) \cap \operatorname{grad} H^1_{\Gamma_t}(\Omega) \qquad \langle \operatorname{div} \varepsilon\widetilde{E}_g, \operatorname{div} \varepsilon\Theta\rangle_{L^2(\Omega)} = -\langle g, \operatorname{div} \varepsilon\Theta\rangle_{L^2(\Omega)}. \tag{5.5}$$

*Equation (5.5) is even satisfied for all $\Theta \in \mu\ D_{\Gamma_n}(\Omega)$. Moreover, $-\operatorname{div} \varepsilon\widetilde{E}_g = g$ and $\widetilde{E}_g = E_g$. In the case $\Gamma_t = \emptyset$ the condition $g \in L^2_\perp(\Omega)$ has to be added, i.e., $-\operatorname{div} \varepsilon\widetilde{E}_g = g$ if and only if $g \in L^2_\perp(\Omega)$ and in this case $\widetilde{E}_g = E_g$.*

(ii') *Let $\Gamma_t \neq \emptyset$. There exists a unique potential $u_g \in H^1_{\Gamma_t}(\Omega)$ such that*

$$\forall\ \psi \in H^1_{\Gamma_t}(\Omega) \qquad \langle \varepsilon \operatorname{grad} u_g, \operatorname{grad} \psi\rangle_{L^2(\Omega)} = \langle g, \psi\rangle_{L^2(\Omega)}. \tag{5.6}$$

*It holds*
$$\operatorname{grad} u_g \in \mu\ D_{\Gamma_n}(\Omega) \cap \operatorname{grad} H^1_{\Gamma_t}(\Omega)$$
*with* $-\operatorname{div} \varepsilon \operatorname{grad} u_g = g$ *and thus* $\operatorname{grad} u_g = E_g$. *In the case* $\Gamma_t = \emptyset$ *we replace* $H^1_{\Gamma_t}(\Omega)$ *by* $H^1_\perp(\Omega)$. *Then (5.6) even holds for all* $\psi \in H_1(\Omega)$ *if and only if* $g \in L^2_\perp(\Omega)$. *In this case the other assertions hold as stated before.*

**Remark 5.4** (Remark 3.6). *Let us note the following:*

(i) *It holds* $\operatorname{grad} H^1_{\Gamma_t}(\Omega) = R_{\Gamma_t,0}(\Omega) \cap \mathcal{H}_{t,n,\varepsilon}(\Omega)^{\perp_{L^2_\varepsilon(\Omega)}}$ *and*
$$\mu \operatorname{rot}\ R_{\Gamma_n}(\Omega) = \mu\ D_{\Gamma_n,0}(\Omega) \cap \mathcal{H}_{t,n,\varepsilon}(\Omega)^{\perp_{L^2_\varepsilon(\Omega)}},$$
$$\operatorname{rot}\ R_{\Gamma_t}(\Omega) = D_{\Gamma_t,0}(\Omega) \cap \mathcal{H}_{n,t}(\Omega)^{\perp_{L^2(\Omega)}}.$$

(ii) *We have*
$$E_F = \left(\widetilde{\operatorname{rot}}_{\Gamma_t}\right)^{-1} F \in D\left(\widetilde{\operatorname{rot}}_{\Gamma_t}\right) = R_{\Gamma_t}(\Omega) \cap \mu \operatorname{rot}\ R_{\Gamma_n}(\Omega),$$
$$H_F = \left(\mu\ \widetilde{\operatorname{rot}}_{\Gamma_n}\right)^{-1} E_F = \left(\mu\ \widetilde{\operatorname{rot}}_{\Gamma_n}\right)^{-1} \left(\widetilde{\operatorname{rot}}_{\Gamma_t}\right)^{-1} F \in D\left(\widetilde{\operatorname{rot}}_{\Gamma_t} \mu\ \widetilde{\operatorname{rot}}_{\Gamma_n}\right) \subset R_{\Gamma_n}(\Omega) \cap \operatorname{rot} R_{\Gamma_t}(\Omega),$$
$$E_g = -\left(\widetilde{\operatorname{div}}_{\Gamma_n} \varepsilon\right)^{-1} g \in D\left(\widetilde{\operatorname{div}}_{\Gamma_n} \varepsilon\right) = \mu\ D_{\Gamma_n}(\Omega) \cap \operatorname{grad} H^1_{\Gamma_t}(\Omega),$$
$$u_g = \left(\widetilde{\operatorname{grad}}_{\Gamma_t}\right)^{-1} E_g = -\left(\widetilde{\operatorname{grad}}_{\Gamma_t}\right)^{-1} \left(\widetilde{\operatorname{div}}_{\Gamma_n} \varepsilon\right)^{-1} g \in D\left(\widetilde{\operatorname{div}}_{\Gamma_n} \varepsilon\ \widetilde{\operatorname{grad}}_{\Gamma_t}\right) \subset H^1_{\Gamma_t}(\Omega),$$

*and these vector fields and functions solve*
$$\begin{array}{rrrr}
\operatorname{rot}_{\Gamma_t} E_F = F, & \operatorname{rot}_{\Gamma_t} \mu \operatorname{rot}_{\Gamma_n} H_F = F, & -\operatorname{div}_{\Gamma_n} \varepsilon E_g = g, & -\operatorname{div}_{\Gamma_n} \varepsilon \operatorname{grad}_{\Gamma_t} u_g = g, \\
-\operatorname{div}_{\Gamma_n} \varepsilon E_F = 0, & \operatorname{div}_{\Gamma_t} H_F = 0, & \operatorname{rot}_{\Gamma_t} E_g = 0, & \pi_{\{0\}/\mathbb{R}} u_g = 0, \\
\pi_{\mathcal{H}} E_F = 0, & \pi_{\widetilde{\mathcal{H}}} H_F = 0, & \pi_{\mathcal{H}} E_g = 0, &
\end{array}$$

*where* $\pi_{\widetilde{\mathcal{H}}} : L^2(\Omega) \to \mathcal{H}_{n,t}(\Omega)$ *is the Neumann–Dirichlet orthonormal projector and* $\pi_{\{0\}/\mathbb{R}}$ *denotes* $\pi_{\{0\}}$ *or* $\pi_{\mathbb{R}}$ *if* $\Gamma_t = \emptyset$. *Mooreover, (5.3)–(5.6) are weak formulations of*
$$\begin{array}{rrr}
\mu \operatorname{rot}_{\Gamma_n} \operatorname{rot}_{\Gamma_t} \widetilde{E}_F = \mu \operatorname{rot}_{\Gamma_n} F, & -\operatorname{div}_{\Gamma_n} \varepsilon \widetilde{E}_F = 0, & \pi_{\mathcal{H}} \widetilde{E}_F = 0, \\
\operatorname{rot}_{\Gamma_t} \mu \operatorname{rot}_{\Gamma_n} H_F = F, & \operatorname{div}_{\Gamma_t} H_F = 0, & \pi_{\widetilde{\mathcal{H}}} H_F = 0, \\
-\operatorname{grad}_{\Gamma_t} \operatorname{div}_{\Gamma_n} \varepsilon \widetilde{E}_g = \operatorname{grad}_{\Gamma_t} g, & \operatorname{rot}_{\Gamma_t} \widetilde{E}_g = 0, & \pi_{\mathcal{H}} \widetilde{E}_g = 0, \\
-\operatorname{div}_{\Gamma_n} \varepsilon \operatorname{grad}_{\Gamma_t} u_g = g, & \pi_{\{0\}/\mathbb{R}} u_g = 0, &
\end{array}$$

i.e., in formal matrix notation
$$\begin{bmatrix} \mu \operatorname{rot}_{\Gamma_n} \operatorname{rot}_{\Gamma_t} \\ -\operatorname{div}_{\Gamma_n} \varepsilon \\ \pi_{\mathcal{H}} \end{bmatrix} \begin{bmatrix} \widetilde{E}_F \end{bmatrix} = \begin{bmatrix} \mu \operatorname{rot}_{\Gamma_n} F \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} \operatorname{rot}_{\Gamma_t} \mu \operatorname{rot}_{\Gamma_n} \\ \operatorname{div}_{\Gamma_t} \\ \pi_{\widetilde{\mathcal{H}}} \end{bmatrix} \begin{bmatrix} H_F \end{bmatrix} = \begin{bmatrix} F \\ 0 \\ 0 \end{bmatrix},$$
$$\begin{bmatrix} -\operatorname{grad}_{\Gamma_t} \operatorname{div}_{\Gamma_n} \varepsilon \\ \operatorname{rot}_{\Gamma_t} \\ \pi_{\mathcal{H}} \end{bmatrix} \begin{bmatrix} \widetilde{E}_g \end{bmatrix} = \begin{bmatrix} \operatorname{grad}_{\Gamma_t} g \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} -\operatorname{div}_{\Gamma_n} \varepsilon \operatorname{grad}_{\Gamma_t} \\ \pi_{\{0\}/\mathbb{R}} \end{bmatrix} \begin{bmatrix} u_g \end{bmatrix} = \begin{bmatrix} g \\ 0 \end{bmatrix}.$$

**Remark 5.5** (Remark 3.7). *Let us note the following, especially for possible numerical purposes and applications.*

(i) *Using the variational formulation in Theorem 5.3 (i) corresponding to $E_F = \widetilde{E}_F \in R_{\Gamma_t}(\Omega)$ for finding a numerical (discrete) approximation $E_{F,h}$ of $E_F$ proposes a $R_{\Gamma_t}(\Omega)$-conforming method in some finite dimensional (discrete) subspace $R_{\Gamma_t,h}(\Omega)$ of $R_{\Gamma_t}(\Omega)$ giving also a $R_{\Gamma_t}(\Omega)$-conforming discrete solution $E_{F,h} \in R_{\Gamma_t,h}(\Omega) \subset R_{\Gamma_t}(\Omega)$.*

(i') *Utilizing the variational formulation in Theorem 5.3 (i') for $E_F = \mu\mathrm{rot}H_F \in \mu\mathrm{rot}R_{\Gamma_n}(\Omega)$ to find a discrete approximation $E_{F,h} = \mu\mathrm{rot}H_{F,h}$ of $E_F$ proposes a $R_{\Gamma_n}(\Omega)$-conforming method in some discrete subspace $R_{\Gamma_n,h}(\Omega)$ of $R_{\Gamma_n}(\Omega)$ giving then a $R_{\Gamma_n}(\Omega)$ -conforming discrete potential $H_{F,h} \in R_{\Gamma_n,h}(\Omega) \subset R_{\Gamma_n}(\Omega)$, but yielding a $\mu\ D_{\Gamma_n}(\Omega)$-conforming solution as*

$$E_{F,h} = \mu\mathrm{rot}H_{F,h} \in \mu\,\mathrm{rot}\ R_{\Gamma_n}(\Omega) = \mu\ D_{\Gamma_n,0}(\Omega) \cap \mathcal{H}_{t,n,\varepsilon}(\Omega)^{\perp_{L^2_\varepsilon(\Omega)}} \subset \mu\ D_{\Gamma_n}(\Omega).$$

(ii) *Using the variational formulation in Theorem 5.3 (ii) corresponding to $E_g = \widetilde{E}_g \in \mu\ D_{\Gamma_n}(\Omega)$ for finding a discrete approximation $E_{g,h}$ of $E_g$ proposes a $\mu\ D_{\Gamma_n}(\Omega)$ -conforming method in some discrete subspace $\mu\ D_{\Gamma_n,h}(\Omega)$ of $\mu\ D_{\Gamma_n}(\Omega)$ giving also a $\mu\ D_{\Gamma_n}(\Omega)$-conforming discrete solution $E_{g,h} \in \mu\ D_{\Gamma_n,h}(\Omega) \subset \mu\ D_{\Gamma_n}(\Omega)$.*

(ii') *Utilizing the variational formulation in Theorem 5.3 (ii') for $E_g = \mathrm{grad}\, u_g \in \mathrm{grad}H^1_{\Gamma_t}(\Omega)$ to find a discrete approximation $E_{g,h} = \mathrm{grad}\, u_{g,h}$ of $E_g$ proposes a $H^1_{\Gamma_t}(\Omega)$-conforming method in some discrete subspace $H^1_{\Gamma_t,h}(\Omega)$ of $H^1_{\Gamma_t}(\Omega)$ giving then a $H^1_{\Gamma_t}(\Omega)$-conforming discrete potential $u_{g,h} \in H^1_{\Gamma_t,h}(\Omega) \subset H^1_{\Gamma_t}(\Omega)$, but yielding a $R_{\Gamma_t}(\Omega)$-conforming solution as*

$$E_{g,h} = \mathrm{grad}\ u_{g,h} \in \mathrm{grad}\ H^1_{\Gamma_t}(\Omega) = R_{\Gamma_t,0}(\Omega) \cap \mathcal{H}_{t,n,\varepsilon}(\Omega)^{\perp_{L^2_\varepsilon(\Omega)}} \subset R_{\Gamma_t}(\Omega).$$

(iii) *A possible discrete solution $E_{F,h} = \mu\mathrm{rot}H_{F,h}$ from (ii') satisfies automatically the side conditions*

$$-\,\mathrm{div}_{\Gamma_n}\varepsilon E_{F,h} = 0, \quad \pi_{\mathcal{H}}E_{F,h} = 0,$$

*i.e., even on the discrete level there is no error in the side conditions. The other option from (ii) yields a discrete solution $E_{F,h}$, which most probably has got errors in the side conditions.*

(iii') *A possible discrete solution $E_{g,h} = \mathrm{grad}\, u_{g,h}$ from (iii') satisfies automatically the side conditions*

$$\mathrm{rot}_{\Gamma_t}E_{g,h} = 0, \quad \pi_{\mathcal{H}}E_{g,h} = 0,$$

*i.e., even on the discrete level there is no error in the side conditions. The other option from (iii) yields a discrete solution $E_{g,h}$, which most probably has got errors in the side conditions.*

**Theorem 5.6** (Theorem 3.12). *The unique solution $E = E_F + E_g + K \in R_{\Gamma_t}(\Omega) \cap \mu\ D_{\Gamma_n}(\Omega)$ in Theorem 5.2 can be found by the following two variational double saddle point formulations:*

(i) *Let $\Gamma_t \neq \emptyset$. There exists a unique tripple $(\widetilde{E}, u, H) \in R_{\Gamma_t}(\Omega) \times H^1_{\Gamma_t}(\Omega) \times \mathcal{H}_{t,n,\varepsilon}(\Omega)$ such that for all $(\Phi, \varphi, \Theta) \in R_{\Gamma_t}(\Omega) \times H^1_{\Gamma_t}(\Omega) \times \mathcal{H}_{t,n,\varepsilon}(\Omega)$*

$$\begin{aligned} \langle \text{rot}\ \widetilde{E}, \text{rot}\Phi\rangle_{L^2(\Omega)} + \langle \varepsilon \text{grad}\ u, \Phi\rangle_{L^2(\Omega)} + \langle \varepsilon H, \Phi\rangle_{L^2(\Omega)} &= \langle F, \text{rot}\ \Phi\rangle_{L^2(\Omega)}, \\ \langle \varepsilon\widetilde{E}, \text{grad}\ \varphi\rangle_{L^2(\Omega)} &= \langle g, \varphi\rangle_{L^2(\Omega)}, \\ \langle \varepsilon\widetilde{E}, \Theta\rangle_{L^2(\Omega)} &= \langle \varepsilon K, \Theta\rangle_{L^2(\Omega)}. \end{aligned} \tag{5.7}$$

*It holds $u = 0$ and $H = 0$. $\text{rot}\widetilde{E} = F$ if and only if $F \in \text{rot}R_{\Gamma_t}(\Omega)$. Moreover, $\varepsilon\widetilde{E} \in D_{\Gamma_n}(\Omega)$ and $-\operatorname{div}\varepsilon\widetilde{E} = g$ as well as $\pi_{\mathcal{H}}\widetilde{E} = K$. In this case, i.e., $F \in \text{rot}R_{\Gamma_t}(\Omega)$, we have $\widetilde{E} = E$ from Theorem 5.2. If $\Gamma_t = \emptyset$, we have to replace $H^1_{\Gamma_t}(\Omega)$ by $H^1_{\perp}(\Omega)$. Then (5.7) even holds for all $\varphi \in H_1(\Omega)$ if and only if $g \in L^2_{\perp}(\Omega)$ if and only if $\varepsilon\widetilde{E} \in D_{\Gamma}(\Omega)$ and $-\operatorname{div}\varepsilon\widetilde{E} = g$. Furthermore, $\pi_{\mathcal{H}}\widetilde{E} = K$. In this case, i.e., $F \in \text{rot}R(\Omega)$ and $g \in L^2_{\perp}(\Omega)$, we have $\widetilde{E} = E$ from Theorem 5.2.*

(ii) *Let $\Gamma_t \neq \emptyset$. There exists a unique tripple $(\hat{E}, U, H) \in \mu\ D_{\Gamma_n}(\Omega) \times (R_{\Gamma_n}(\Omega) \cap \text{rot}R_{\Gamma_t}(\Omega)) \times \mathcal{H}_{t,n,\varepsilon}(\Omega)$ such that for all $(\Psi, \Phi, \Theta) \in \mu\ D_{\Gamma_n}(\Omega) \times (R_{\Gamma_n}(\Omega) \cap \text{rot}R_{\Gamma_t}(\Omega)) \times \mathcal{H}_{t,n,\varepsilon}(\Omega)$*

$$\begin{aligned} \langle \text{div}\ \varepsilon\hat{E}, \text{div}\ \varepsilon\Psi\rangle_{L^2(\Omega)} + \langle \text{rot}\ U, \Psi\rangle_{L^2(\Omega)} + \langle \varepsilon H, \Psi\rangle_{L^2(\Omega)} &= -\langle g\rangle \text{div}\ \varepsilon\Psi\rangle_{L^2(\Omega)}, \\ \langle \hat{E}, \text{rot}\ \Phi\rangle_{L^2(\Omega)} &= \langle F, \Phi\rangle_{L^2(\Omega)}, \\ \langle \varepsilon\hat{E}, \Theta\rangle_{L^2(\Omega)} &= \langle \varepsilon K, \Theta\rangle_{L^2(\Omega)}. \end{aligned} \tag{5.8}$$

*It holds $U = 0$ and $H = 0$ as well as $-\operatorname{div}\varepsilon\hat{E} = g$. (5.8) holds for all $\Phi \in R_{\Gamma_n}(\Omega)$ if and only if $F \in \text{rot}R_{\Gamma_t}(\Omega)$ if and only if $\hat{E} \in R_{\Gamma_t}(\Omega)$ with $\text{rot}\hat{E} = F$. Moreover, $\pi_{\mathcal{H}}\hat{E} = K$. In this case, i.e., $F \in \text{rot}R_{\Gamma_t}(\Omega)$, we have $\hat{E} = E$ from Theorem 5.2. If $\Gamma_t = \emptyset$, the condition $g \in L^2_{\perp}(\Omega)$ has to be added, i.e., $-\operatorname{div}\varepsilon\hat{E} = g$ if and only if $g \in L^2_{\perp}(\Omega)$. In this case, i.e., $F \in \text{rot}R(\Omega)$ and $g \in L^2_{\perp}(\Omega)$, we have $\widetilde{E} = E$ from Theorem 5.2.*

**Remark 5.7** (Remark 3.13). *Let us note the following:*

(i) *Using the saddle point formulation in Theorem 5.6 (i) for finding a numerical approximation $E_h$ of $E$ provides a $R_{\Gamma_t}(\Omega)$-conforming approximation $E_h \in R_{\Gamma_t}(\Omega)$ of (5.1) or (5.2), whereas using the saddle point formulation in Theorem 5.6 (ii) for finding a numerical approximation $E_h$ of $E$ provides a $\mu\ D_{\Gamma_n}(\Omega)$-conforming approximation $E_h \in \mu\ D_{\Gamma_n}(\Omega)$ of of (5.1) or (5.2).*

(ii) *Related variational formulations to those presented in Theorem 5.6 have recently been announced and proposed in [1].*

(iii) *(5.7) and (5.8) are weak formulations of*

$$\mu\,\mathrm{rot}_{\Gamma_n}\mathrm{rot}_{\Gamma_t}\widetilde{E} + \mathrm{grad}_{\Gamma_t} u + H = \mu\,\mathrm{rot}_{\Gamma_n} F, \qquad -\mathrm{div}_{\Gamma_n}\varepsilon\widetilde{E} = g, \qquad \pi_{\mathcal{H}}\widetilde{E} = K,$$
$$-\mathrm{grad}_{\Gamma_t}\mathrm{div}_{\Gamma_n}\varepsilon\hat{E} + \mu\,\mathrm{rot}_{\Gamma_n} U + H = \mathrm{grad}_{\Gamma_t} g, \qquad \mathrm{rot}_{\Gamma_t}\hat{E} = F, \qquad \pi_{\mathcal{H}}\hat{E} = K,$$

*i.e., in formal matrix notation*

$$\begin{bmatrix} \mu\,\mathrm{rot}_{\Gamma_n}\mathrm{rot}_{\Gamma_t} & \mathrm{grad}_{\Gamma_t} & \iota_{\mathcal{H}} \\ -\mathrm{div}_{\Gamma_n}\varepsilon & 0 & 0 \\ \pi_{\mathcal{H}} & 0 & 0 \end{bmatrix} \begin{bmatrix} \widetilde{E} \\ u \\ H \end{bmatrix} = \begin{bmatrix} \mu\,\mathrm{rot}_{\Gamma_n} F \\ g \\ K \end{bmatrix},$$
$$\begin{bmatrix} -\mathrm{grad}_{\Gamma_t}\mathrm{div}_{\Gamma_n}\varepsilon & \mu\,\mathrm{rot}_{\Gamma_n} & \iota_{\mathcal{H}} \\ \mathrm{rot}_{\Gamma_t} & 0 & 0 \\ \pi_{\mathcal{H}} & 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{E} \\ U \\ H \end{bmatrix} = \begin{bmatrix} \mathrm{grad}_{\Gamma_t} g \\ F \\ K \end{bmatrix}.$$

**Theorem 5.8** (Theorem 3.14). *The partial solution vector fields* $E_F = \widetilde{E}_F \in R_{\Gamma_t}(\Omega) \cap \mu\mathrm{rot}R_{\Gamma_n}(\Omega)$ *and* $E_g = \widetilde{E}_g \in \mu\ D_{\Gamma_n}(\Omega) \cap \mathrm{grad}H^1_{\Gamma_t}(\Omega)$ *together with their potentials* $H_F \in R_{\Gamma_n}(\Omega) \cap \mathrm{rot}R_{\Gamma_t}(\Omega), u_g \in H^1_{\Gamma_t}(\Omega)$ *resp.* $u_g \in H^1_{\perp}(\Omega)$ *from Theorem 5.2 and Theorem 5.3 can be found by the following four variational double saddle point formulations:*

(i) *Let* $\Gamma_t \neq \emptyset$ *. There exists a unique tripple* $(\widetilde{E}_F, u, H) \in R_{\Gamma_t}(\Omega) \times H^1_{\Gamma_t}(\Omega) \times \mathcal{H}_{t,n,\varepsilon}(\Omega)$ *such that for all* $(\Phi, \varphi, \Theta) \in R_{\Gamma_t}(\Omega) \times H^1_{\Gamma_t}(\Omega) \times \mathcal{H}_{t,n,\varepsilon}(\Omega)$

$$\begin{aligned} \langle \mathrm{rot}\widetilde{E}_F, \mathrm{rot}\Phi\rangle_{L^2(\Omega)} + \langle \varepsilon\,\mathrm{grad}\,u, \Phi\rangle_{L^2(\Omega)} + \langle \varepsilon H, \Phi\rangle_{L^2(\Omega)} &= \langle F, \mathrm{rot}\ \Phi\rangle_{L^2(\Omega)}, \\ \langle \varepsilon\widetilde{E}_F, \mathrm{grad}\varphi\rangle_{L^2(\Omega)} &= 0, \\ \langle \varepsilon\widetilde{E}_F, \Theta\rangle_{L^2(\Omega)} &= 0. \end{aligned} \tag{5.9}$$

*It holds* $u = 0$ *and* $H = 0$. $\mathrm{rot}\widetilde{E}_F = F$ *if and only if* $F \in \mathrm{rot}R_{\Gamma_t}(\Omega)$. *Moreover,* $\varepsilon\widetilde{E}_F \in D_{\Gamma_n,0}(\Omega)$ *and* $\pi_{\mathcal{H}}\widetilde{E}_F = 0$. *Hence, if* $F \in \mathrm{rot}R_{\Gamma_t}(\Omega)$, *we have* $\widetilde{E}_F = E_F$ *from Theorem 5.2, see Theorem 5.3 (i). If* $\Gamma_t = \emptyset$*, we have to replace* $H^1_{\Gamma_t}(\Omega)$ *by* $H^1_{\perp}(\Omega)$. *Then (5.9) even holds for all* $\varphi \in H_1(\Omega)$ *and thus* $\varepsilon\widetilde{E}_F \in D_{\Gamma,0}(\Omega)$. *Furthermore,* $\pi_{\mathcal{H}}\widetilde{E}_F = 0$. *Again, if* $F \in \mathrm{rot}R(\Omega)$, *we have* $\widetilde{E}_F = E_F$ *from Theorem 5.2.*

(i') *Let* $\Gamma_t \neq \Gamma$ *. There exists a unique tripple* $(H_F, v, H) \in R_{\Gamma_n}(\Omega) \times H^1_{\Gamma_n}(\Omega) \times \mathcal{H}_{n,t}(\Omega)$ *such that for all* $(\Psi, \psi, \Theta) \in R_{\Gamma_n}(\Omega) \times H^1_{\Gamma_n}(\Omega) \times \mathcal{H}_{n,t}(\Omega)$

$$\begin{aligned}\langle \mu\, \mathrm{rot} H_F, \mathrm{rot}\Psi\rangle_{L^2(\Omega)} - \langle \mathrm{grad} v, \Psi\rangle_{L^2(\Omega)} + \langle H, \Psi\rangle_{L^2(\Omega)} &= \langle F, \Psi\rangle_{L^2(\Omega)},\\ \langle H_F, \mathrm{grad}\ \psi\rangle_{L^2(\Omega)} &= 0,\\ \langle H_F, \Theta\rangle_{L^2(\Omega)} &= 0.\end{aligned} \tag{5.10}$$

*It holds $v=0$ if and only if $F \perp \mathrm{grad} H^1_{\Gamma_n}(\Omega)$ if and only if $F \in D_{\Gamma_t,0}(\Omega)$. $H=0$ if and only if $F \perp \mathcal{H}_{n,t}(\Omega)$. Thus $v=0$ and $H=0$ if and only if $F \in D_{\Gamma_t,0}(\Omega) \cap \mathcal{H}_{n,t}(\Omega)^{\perp_{L^2(\Omega)}} = \mathrm{rot} R_{\Gamma_t}(\Omega)$. Moreover, $\mu \mathrm{rot} H_F \in R_{\Gamma_t}(\Omega)$ and $\mathrm{rot}\mu\mathrm{rot} H_F = F$ if and only if $F \in \mathrm{rot} R_{\Gamma_t}(\Omega)$. Furthermore, $H_F \in D_{\Gamma_t,0}(\Omega)$ and $\pi_{\widetilde{\mathcal{H}}} H_F = 0$. Hence, if $F \in \mathrm{rot} R_{\Gamma_t}(\Omega)$. we have $\mu \mathrm{rot} H_F = E_F$ from Theorem 5.2, see Theorem 5.3 (i'). If $\Gamma_t = \Gamma$, we have to replace $H^1_{\Gamma_n}(\Omega)$ by $H^1_\perp(\Omega)$. Then (5.10) even holds for all $\psi \in H_1(\Omega)$ and thus $H_F \in D_{\Gamma,0}(\Omega)$. Furthermore, $\pi_{\widetilde{\mathcal{H}}} H_F = 0$. Again, if $F \in \mathrm{rot} R_\Gamma(\Omega)$, we have $\mu \mathrm{rot} H_F = E_F$ from Theorem 5.2.*

(ii) *Let $\Gamma_t \neq \emptyset$. There exists a unique tripple $(\widetilde{E}_g, U, H) \in \mu\ D_{\Gamma_n}(\Omega) \times (R_{\Gamma_n}(\Omega) \cap \mathrm{rot} R_{\Gamma_t}(\Omega)) \times \mathcal{H}_{t,n,\varepsilon}(\Omega)$ such that for all $(\Psi, \Phi, \Theta) \in \mu\ D_{\Gamma_n}(\Omega) \times (R_{\Gamma_n}(\Omega) \cap \mathrm{rot} R_{\Gamma_t}(\Omega)) \times \mathcal{H}_{t,n,\varepsilon}(\Omega)$*

$$\begin{aligned}\langle \mathrm{div}\, \varepsilon\widetilde{E}_g, \mathrm{div}\, \varepsilon\Psi\rangle_{L^2(\Omega)} + \langle \mathrm{rot} U, \Psi\rangle_{L^2(\Omega)} + \langle \varepsilon H, \Psi\rangle_{L^2(\Omega)} &= -\langle g, \mathrm{div}\, \varepsilon\Psi\rangle_{L^2(\Omega)},\\ \langle \widetilde{E}_g, \mathrm{rot}\Phi\rangle_{L^2(\Omega)} &= 0,\\ \langle \varepsilon\widetilde{E}_g, \Theta\rangle_{L^2(\Omega)} &= 0.\end{aligned} \tag{5.11}$$

*It holds $U=0$ and $H=0$ as well as $-\mathrm{div}\, \varepsilon\widetilde{E}_g = g$. (5.11) holds for all $\Phi \in R_{\Gamma_n}(\Omega)$ and hence $\widetilde{E}_g \in R_{\Gamma_t,0}(\Omega)$. Moreover, $\pi_{\mathcal{H}}\widetilde{E}_g = 0$. Finally, we have $\widetilde{E}_g = E_g$ from Theorem 5.2, see Theorem 5.3 (ii). If $\Gamma_t = \emptyset$, the condition $g \in L^2_\perp(\Omega)$ has to be added, i.e., $-\mathrm{div}\, \varepsilon\widetilde{E}_g = g$ if and only if $g \in L^2_\perp(\Omega)$. Again, (5.11) holds for all $\Phi \in R_\Gamma(\Omega)$, $\widetilde{E}_g \in R_0(\Omega)$, and $\pi_{\mathcal{H}}\widetilde{E}_g = 0$. Finally, if $g \in L^2_\perp(\Omega)$, we have $\widetilde{E}_g = E_g$ from Theorem 5.2.*

(ii') *For $\Gamma_t \neq \emptyset$ see Theorem 5.3 (ii'). Let $\Gamma_t = \emptyset$. There exists a unique pair $(u_g, r) \in H_1(\Omega) \times \mathbb{R}$ such that*

$$\forall\ (\psi, \varrho) \in H_1(\Omega) \times \mathbb{R} \qquad \begin{aligned}\langle \varepsilon \mathrm{grad}\, u_g, \mathrm{grad}\, \psi\rangle_{L^2(\Omega)} + \langle \iota_{\mathbb{R}} r, \psi\rangle_{L^2(\Omega)} &= \langle g, \psi\rangle_{L^2(\Omega)},\\ \langle u_g, \iota_{\mathbb{R}}\varrho\rangle_{L^2(\Omega)} &= 0.\end{aligned}$$

*It holds $r=0$ if and only if $g \perp_{L^2(\Omega)} \iota_{\mathbb{R}}\mathbb{R} = \mathbb{R}$ if and only if $g \in L^2_\perp(\Omega) = \mathrm{div} D_\Gamma(\Omega)$. Moreover, $\mathrm{grad}\, u_g \in \mu\ D_\Gamma(\Omega)$ with $-\mathrm{div}\, \varepsilon \mathrm{grad}\, u_g = g$ if and only if $g \in L^2_\perp(\Omega)$. The second equation of (5.12) shows $u_g \in L^2_\perp(\Omega)$, i.e., $u_g \in H^1_\perp(\Omega)$. Finally, if $g \in L^2_\perp(\Omega)$, we have $\mathrm{grad}\, u_g = E_g$ from Theorem 5.2, see Theorem 5.3 (ii').*

**Remark 5.9** (Remark 3.15). (5.9)–(5.12) *are weak formulations of*

$$\begin{aligned}
\mu\,\mathrm{rot}_{\Gamma_n}\mathrm{rot}_{\Gamma_t}\widetilde{E}_F + \mathrm{grad}_{\Gamma_t} u + H &= \mu\,\mathrm{rot}_{\Gamma_n} F, & -\mathrm{div}_{\Gamma_n}\varepsilon\widetilde{E}_F &= 0, & \pi_{\mathcal{H}}\widetilde{E}_F &= 0,\\
\mathrm{rot}_{\Gamma_t}\mu\,\mathrm{rot}_{\Gamma_n} H_F - \mathrm{grad}_{\Gamma_n} v + H &= F, & \mathrm{div}_{\Gamma_t} H_F &= 0, & \pi_{\widetilde{\mathcal{H}}} H_F &= 0,\\
-\mathrm{grad}_{\Gamma_t}\mathrm{div}_{\Gamma_n}\varepsilon\widetilde{E}_g + \mu\,\mathrm{rot}_{\Gamma_n} U + H &= \mathrm{grad}_{\Gamma_t} g, & \mathrm{rot}_{\Gamma_t}\widetilde{E}_g &= 0, & \pi_{\mathcal{H}}\widetilde{E}_g &= 0,\\
-\mathrm{div}_{\Gamma}\varepsilon\mathrm{grad}_{\emptyset} u_g + \iota_{\mathbb{R}} r &= g, & \pi_{\mathbb{R}} u_g &= 0,
\end{aligned}$$

*i.e., in formal matrix notation*

$$\begin{bmatrix} \mu\,\mathrm{rot}_{\Gamma_n}\mathrm{rot}_{\Gamma_t} & \mathrm{grad}_{\Gamma_t} & \iota_{\mathcal{H}} \\ -\mathrm{div}_{\Gamma_n}\varepsilon & 0 & 0 \\ \pi_{\mathcal{H}} & 0 & 0 \end{bmatrix} \begin{bmatrix} \widetilde{E}_F \\ u \\ H \end{bmatrix} = \begin{bmatrix} \mu\,\mathrm{rot}_{\Gamma_n} F \\ 0 \\ 0 \end{bmatrix},$$

$$\begin{bmatrix} \mathrm{rot}_{\Gamma_t}\mu\,\mathrm{rot}_{\Gamma_n} & -\mathrm{grad}_{\Gamma_t} & \iota_{\widetilde{\mathcal{H}}} \\ \mathrm{div}_{\Gamma_t} & 0 & 0 \\ \pi_{\widetilde{\mathcal{H}}} & 0 & 0 \end{bmatrix} \begin{bmatrix} H_F \\ v \\ H \end{bmatrix} = \begin{bmatrix} F \\ 0 \\ 0 \end{bmatrix},$$

$$\begin{bmatrix} -\mathrm{grad}_{\Gamma_t}\mathrm{div}_{\Gamma_n}\varepsilon & \mu\,\mathrm{rot}_{\Gamma_n} & \iota_{\mathcal{H}} \\ \mathrm{rot}_{\Gamma_t} & 0 & 0 \\ \pi_{\mathcal{H}} & 0 & 0 \end{bmatrix} \begin{bmatrix} \widetilde{E}_g \\ U \\ H \end{bmatrix} = \begin{bmatrix} \mathrm{grad}_{\Gamma_t} g \\ 0 \\ 0 \end{bmatrix},$$

$$\begin{bmatrix} -\mathrm{div}\,\Gamma\,\varepsilon\,\mathrm{grad}_{\emptyset}\iota_{\mathbb{R}}\pi_{\mathbb{R}}0 \end{bmatrix} \begin{bmatrix} u_g \\ r \end{bmatrix} = \begin{bmatrix} g \\ 0 \end{bmatrix}.$$

**Theorem 5.10** (Theorem 3.17). *Let* $F \in \mathrm{rot}R_{\Gamma_t}(\Omega)$ *and* $g \in L^2(\Omega)$. *If* $\Gamma_t = \emptyset$, *let* $g \in L^2_{\perp}(\Omega)$. *The unique solution* $E = E_F + E_g + K \in R_{\Gamma_t}(\Omega) \cap \mu\ D_{\Gamma_n}(\Omega)$ *in Theorem 5.2 can be found by the following three variational multiple saddle point formulations:*

(i) *For* $\Gamma_t \neq \emptyset$ *see Theorem 5.6 (i). Let* $\Gamma_t = \emptyset$. *There is* $(\widetilde{E}, u, r, H) \in R(\Omega) \times H_1(\Omega) \times \mathbb{R} \times \mathcal{H}_{t,n,\varepsilon}(\Omega)$, *a unique quadruple, such that for all* $(\Phi, \varphi, б, \Theta) \in R(\Omega) \times H_1(\Omega) \times \mathbb{R} \times \mathcal{H}_{t,n,\varepsilon}(\Omega)$

$$\begin{aligned}
\langle \mathrm{rot}\widetilde{E}, \mathrm{rot}\Phi\rangle_{L^2(\Omega)} + \langle \varepsilon\mathrm{grad}\,u, \Phi\rangle_{L^2(\Omega)} + \langle \varepsilon H, \Phi\rangle_{L^2(\Omega)} &= \langle F, \mathrm{rot}\Phi\rangle_{L^2(\Omega)},\\
\langle \varepsilon\widetilde{E}, \mathrm{grad}\,\varphi\rangle_{L^2(\Omega)} + \langle \iota_{\mathbb{R}} r, \varphi\rangle_{L^2(\Omega)} &= \langle g, \varphi\rangle_{L^2(\Omega)},\\
\langle u, \iota_{\mathbb{R}}\varrho\rangle_{L^2(\Omega)} &= 0,\\
\langle \varepsilon\widetilde{E}, \Theta\rangle_{L^2(\Omega)} &= \langle \varepsilon K, \Theta\rangle_{L^2(\Omega)}.
\end{aligned} \tag{5.13}$$

*It holds* $u = 0$, $H = 0$, *and* $r = 0$. *Moreover,* $\mathrm{rot}\widetilde{E} = F$ *and* $\varepsilon\widetilde{E} \in D_{\Gamma}(\Omega)$ *with* $-\mathrm{div}\,\varepsilon\widetilde{E} = g$ *as well as* $\pi_{\mathcal{H}}\widetilde{E} = K$. *Finally,* $\widetilde{E} = E$ *from Theorem 5.2.*

(ii) Let $\Gamma_t \neq \Gamma$. There is $(\hat{E}, U, v, H, \widetilde{H}) \in \mu\ D_{\Gamma_n}(\Omega) \times R_{\Gamma_n}(\Omega) \times H^1_{\Gamma_n}(\Omega) \times \mathcal{H}_{t,n,\varepsilon}(\Omega) \times \mathcal{H}_{n,t}(\Omega)$ , a unique five tuple, such that for all

$$
\begin{aligned}
(\Psi, \Phi, \psi, \Theta, \widetilde{\Theta}) \in \mu\ D_{\Gamma_n}(\Omega) \times R_{\Gamma_n}(\Omega) \times H^1_{\Gamma_n}(\Omega) \times \mathcal{H}_{t,n,\varepsilon}(\Omega) \times \mathcal{H}_{n,t}(\Omega) \\
\langle \operatorname{div} \varepsilon\hat{E}, \operatorname{div} \varepsilon\Psi\rangle_{L^2(\Omega)} + \langle \operatorname{rot} U, \Psi\rangle_{L^2(\Omega)} + \langle \varepsilon H, \Psi\rangle_{L^2(\Omega)} &= -\langle g, \operatorname{div} \varepsilon\Psi\rangle_{L^2(\Omega)}, \\
\langle \hat{E}, \operatorname{rot}\Phi\rangle_{L^2(\Omega)} - \langle \operatorname{grad} v, \Phi\rangle_{L^2(\Omega)} + \langle \widetilde{H}, \Phi\rangle_{L^2(\Omega)} &= \langle F, \Phi\rangle_{L^2(\Omega)}, \\
-\langle U, \operatorname{grad} \psi\rangle_{L^2(\Omega)} &= 0, \\
\langle \varepsilon\hat{E}, \Theta\rangle_{L^2(\Omega)} &= \langle \varepsilon K, \Theta\rangle_{L^2(\Omega)}, \\
\langle U, \widetilde{\Theta}\rangle_{L^2(\Omega)} &= 0.
\end{aligned}
\tag{5.14}
$$

*It holds $U = 0$, $H = 0$ and $v = 0$, $\widetilde{H} = 0$. Moreover, $-\operatorname{div} \varepsilon\hat{E} = g$ and $\hat{E} \in R_{\Gamma_t}(\Omega)$ with $\operatorname{rot}\hat{E} = F$ as well as $\pi_{\mathcal{H}}\hat{E} = K$. Finally, $\hat{E} = E$ from Theorem 5.2. If $\Gamma_t = \Gamma$, we have to replace $H^1_{\Gamma_n}(\Omega)$ by $H^1_{\perp}(\Omega)$ and the assertions hold as before.*

(ii') *Let $\Gamma_t = \Gamma$. There is $(\hat{E}, U, v, r, H, \widetilde{H}) \in \mu\ D(\Omega) \times R(\Omega) \times H_1(\Omega) \times \mathbb{R} \times \mathcal{H}_{t,n,\varepsilon}(\Omega) \times \mathcal{H}_{n,t}(\Omega)$. a unique six tuple, such that for all $(\Psi, \Phi, \psi, б, \Theta, \widetilde{\Theta}) \in \mu\ D(\Omega) \times R(\Omega) \times H_1(\Omega) \times \mathbb{R} \times \mathcal{H}_{t,n,\varepsilon}(\Omega) \times \mathcal{H}_{n,t}(\Omega)$*

$$
\begin{aligned}
\langle \operatorname{div} \varepsilon\hat{E}, \operatorname{div} \varepsilon\Psi\rangle_{L^2(\Omega)} + \langle \operatorname{rot} U, \Psi\rangle_{L^2(\Omega)} + \langle \varepsilon H, \Psi\rangle_{L^2(\Omega)} &= -\langle g, \operatorname{div} \varepsilon\Psi\rangle_{L^2(\Omega)}, \\
\langle \hat{E}, \operatorname{rot}\Phi\rangle_{L^2(\Omega)} - \langle \operatorname{grad} v, \Phi\rangle_{L^2(\Omega)} + \langle \widetilde{H}, \Phi\rangle_{L^2(\Omega)} &= \langle F, \Phi\rangle_{L^2(\Omega)}, \\
-\langle U, \operatorname{grad} \psi\rangle_{L^2(\Omega)} + \langle \iota_{\mathbb{R}} r, \psi\rangle_{L^2(\Omega)} &= 0, \\
\langle v, \iota_{\mathbb{R}} \varrho\rangle_{L^2(\Omega)} &= 0, \\
\langle \varepsilon\hat{E}, \Theta\rangle_{L^2(\Omega)} &= \langle \varepsilon K, \Theta\rangle_{L^2(\Omega)}, \\
\langle U, \widetilde{\Theta}\rangle_{L^2(\Omega)} &= 0.
\end{aligned}
\tag{5.14}
$$

*It holds $U = 0$, $H = 0$ and $v = 0$, $\widetilde{H} = 0$ as well as $r = 0$. Moreover, $-\operatorname{div} \varepsilon\hat{E} = g$ and $\hat{E} \in R_{\Gamma}(\Omega)$ with $\operatorname{rot}\hat{E} = F$ as well as $\pi_{\mathcal{H}}\hat{E} = K$. Finally, $\hat{E} = E$ from Theorem 5.2.*

**Theorem 5.8** can be extended in the same way.

**Remark 5.11** (Remark 3.18). (5.13)–(5.15) *are weak formulations of*

$$
\begin{aligned}
&\mu \operatorname{rot}_{\Gamma} \operatorname{rot} \widetilde{E} + \operatorname{grad} u + H = \mu \operatorname{rot}_{\Gamma} F, && -\operatorname{div}_{\Gamma} \varepsilon \widetilde{E} + \iota_{\mathbb{R}} r = g, && \pi_{\mathbb{R}} u = 0, \\
&-\operatorname{grad}_{\Gamma_t} \operatorname{div}_{\Gamma_n} \varepsilon \hat{E} + \mu \operatorname{rot}_{\Gamma_n} U + H = \operatorname{grad}_{\Gamma_t} g, && \operatorname{rot}_{\Gamma_t} \hat{E} - \operatorname{grad}_{\Gamma_n} v + \widetilde{H} = F, && \operatorname{div}_{\Gamma_t} U = 0, \\
&-\operatorname{grad}_{\Gamma} \operatorname{div} \varepsilon \hat{E} + \mu \operatorname{rot} U + H = \operatorname{grad}_{\Gamma} g, && \operatorname{rot}_{\Gamma} \hat{E} - \operatorname{grad} v + \widetilde{H} = F, && \operatorname{div}_{\Gamma} U + \iota_{\mathbb{R}} r = 0, \quad \pi_{\mathbb{R}} v = 0,
\end{aligned}
$$

*and $\pi_{\mathcal{H}}\widetilde{E} = K$ as well as $\pi_{\mathcal{H}}\hat{E} = K, \pi_{\widetilde{\mathcal{H}}} U = 0$, resp. $\pi_{\mathcal{H}}\hat{E} = K, \pi_{\widetilde{\mathcal{H}}} U = 0$, i.e., in formal matrix notation*

$$\begin{bmatrix} \mu\,\mathrm{rot}_\Gamma\mathrm{rot} & \mathrm{grad} & 0 & \iota_{\mathcal{H}} \\ -\,\mathrm{div}_\Gamma\varepsilon & 0 & \iota_{\mathbb{R}} & 0 \\ 0 & \pi_{\mathbb{R}} & 0 & 0 \\ \pi_{\mathcal{H}} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \widetilde{E} \\ u \\ r \\ H \end{bmatrix} = \begin{bmatrix} \mu\,\mathrm{rot}_\Gamma F \\ g \\ 0 \\ K \end{bmatrix},$$

$$\begin{bmatrix} -\,\mathrm{grad}_{\Gamma_t}\mathrm{div}_{\Gamma_n}\varepsilon & \mu\,\mathrm{rot}_{\Gamma_n} & 0 & \iota_{\mathcal{H}} & 0 \\ \mathrm{rot}_{\Gamma_t} & 0 & -\,\mathrm{grad}_{\Gamma_n} & 0 & \iota_{\widetilde{\mathcal{H}}} \\ 0 & \mathrm{div}_{\Gamma_t} & 0 & 0 & 0 \\ \pi_{\mathcal{H}} & 0 & 0 & 0 & 0 \\ 0 & \pi_{\widetilde{\mathcal{H}}} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{E} \\ U \\ v \\ H \\ \widetilde{H} \end{bmatrix} = \begin{bmatrix} \mathrm{grad}_{\Gamma_t} g \\ F \\ 0 \\ K \\ 0 \end{bmatrix},$$

$$\begin{bmatrix} -\,\mathrm{grad}_\Gamma\mathrm{div}\,\varepsilon & \mu\mathrm{rot} & 0 & 0 & \iota_{\mathcal{H}} & 0 \\ \mathrm{rot}_\Gamma & 0 & -\,\mathrm{grad} & 0 & 0 & \iota_{\widetilde{\mathcal{H}}} \\ 0 & \mathrm{div}_\Gamma & 0 & \iota_{\mathbb{R}} & 0 & 0 \\ 0 & 0 & \pi_{\mathbb{R}} & 0 & 0 & 0 \\ \pi_{\mathcal{H}} & 0 & 0 & 0 & 0 & 0 \\ 0 & \pi_{\widetilde{\mathcal{H}}} & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{E} \\ U \\ v \\ r \\ H \\ \widetilde{H} \end{bmatrix} = \begin{bmatrix} \mathrm{grad}_\Gamma g \\ F \\ 0 \\ 0 \\ K \\ 0 \end{bmatrix}.$$

We can apply the main functional a posteriori error estimate Corollary 4.6 to (5.1) resp. (5.2).

**Theorem 5.12.** *Let $E \in R_{\Gamma_t}(\Omega) \cap \mu\; D_{\Gamma_n}(\Omega)$ be the exact solution of (5.1) resp. (5.2) and $\widetilde{E} \in L^2_\varepsilon(\Omega)$. Then the following estimates hold for the error $e = E - \widetilde{E}$ defined in (4.1):*

(i) *The error decomposes, i.e., $e = e_{\mathrm{grad}} + e_{\mathcal{H}} + e_{\mathrm{rot}} \in \mathrm{grad}\; H^1_{\Gamma_t}(\Omega) \oplus_{L^2_\varepsilon(\Omega)} \mathcal{H}_{t,n,\varepsilon}(\Omega) \oplus_{L^2_\varepsilon(\Omega)} \mu\,\mathrm{rot}\; R_{\Gamma_n}(\Omega)$ and*

$$|e|^2_{L^2_\varepsilon(\Omega)} = |e_{\mathrm{grad}}|^2_{L^2_\varepsilon(\Omega)} + |e_{\mathcal{H}}|^2_{L^2_\varepsilon(\Omega)} + |e_{\mathrm{rot}}|^2_{L^2_\varepsilon(\Omega)}.$$

(ii) *The projection $e_{\mathrm{grad}} = \pi_{\mathrm{grad}} e = E_g - \pi_{\mathrm{grad}}\widetilde{E} \in \mathrm{grad} H^1_{\Gamma_t}(\Omega)$ satisfies*

$$\begin{aligned} |e_{\mathrm{grad}}|^2_{L^2_\varepsilon(\Omega)} &= \min_{\Phi \in \mu D_{\Gamma_n}(\Omega)} \left( c_{fp}|\mathrm{div}\,\varepsilon\;\Phi + g|_{L^2(\Omega)} + |\Phi - \widetilde{E}|_{L^2_\varepsilon(\Omega)} \right)^2 \\ &= \max_{\varphi \in H^1_{\Gamma_t}(\Omega)} \left( 2\langle g, \varphi\rangle_{L^2(\Omega)} - \langle 2\widetilde{E} + \mathrm{grad}\;\varphi, \varepsilon\mathrm{grad}\;\varphi\rangle_{L^2(\Omega)} \right) \end{aligned}$$

*and the minimum resp. maximum is attained at*

$$\hat{\Phi} := e_{\mathrm{grad}} + \widetilde{E} \in \mu\; D_{\Gamma_n}(\Omega), \qquad \hat{\varphi} := \left(\widetilde{\mathrm{grad}_{\Gamma_t}}\right)^{-1} e_{\mathrm{grad}} \in H^1_{\Gamma_t}(\Omega)$$

*with $-\,\mathrm{div}\,\varepsilon\;\hat{\Phi} = -\,\mathrm{div}\,\varepsilon\; E = g$, where $H^1_{\Gamma_t}(\Omega)$ has to be replaced by $H^1_\perp(\Omega)$, if $\Gamma_t = \emptyset$. In the latter case $\hat{\varphi}$ is unique only up to a constant.*

(iii) *The projection* $e_{\mathrm{rot}} = \pi_{\mathrm{rot}} e = E_F - \pi_{\mathrm{rot}} \widetilde{E} \in \mu \mathrm{rot} R_{\Gamma_n}(\Omega)$ *satisfies*

$$|e_{\mathrm{rot}}|^2_{L^2_\varepsilon(\Omega)} = \min_{\Phi \in R_{\Gamma_t}(\Omega)} \left( c_m |\mathrm{rot}\Phi - F|_{L^2(\Omega)} + |\Phi - \widetilde{E}|_{L^2_\varepsilon(\Omega)} \right)^2$$
$$= \max_{\Psi \in R_{\Gamma_n}(\Omega)} \left( 2\langle F, \Psi \rangle_{L^2(\Omega)} - \langle 2\widetilde{E} + \mu\, \mathrm{rot}\Psi, \mathrm{rot}\Psi \rangle_{L^2(\Omega)} \right)$$

*and the minimum resp. maximum is attained at*

$$\hat{\Phi} := e_{\mathrm{rot}} + \widetilde{E} \in R_{\Gamma_t}(\Omega), \quad \hat{\Psi} := \left(\mu\ \widetilde{\mathrm{rot}}_{\Gamma_n}\right)^{-1} e_{\mathrm{rot}} \in R_{\Gamma_n}(\Omega) \cap \mathrm{rot} R_{\Gamma_t}(\Omega)$$

*with* $\mathrm{rot}\hat{\Phi} = \mathrm{rot}E = F$, *and at any* $\hat{\Psi} \in R_{\Gamma_n}(\Omega)$ *with* $\mu \mathrm{rot}\hat{\Psi} = e_{\mathrm{rot}}$.

(iv) *The projection* $e_{\mathcal{H}} = \pi_{\mathcal{H}} e = H - \pi_{\mathcal{H}} \widetilde{E} \in \mathcal{H}_{t,n,\varepsilon}(\Omega)$ *satisfies*

$$|e_{\mathcal{H}}|^2_{L^2_\varepsilon(\Omega)} = \min_{\varphi \in H^1_{\Gamma_t}(\Omega)} \min_{\Phi \in R_{\Gamma_n}(\Omega)} |H - \widetilde{E} + \mathrm{grad}\,\varphi + \mu \mathrm{rot}\Phi|^2_{L^2_\varepsilon(\Omega)}$$
$$= \max_{\Psi \in \mathcal{H}_{t,n,\varepsilon}(\Omega)} \langle 2(H - \widetilde{E}) - \Psi, \Psi \rangle_{L^2_\varepsilon(\Omega)}$$

*and the minimum resp. maximum is attained at*

$$\hat{\varphi} := \left(\widetilde{\mathrm{grad}}_{\Gamma_t}\right)^{-1} \pi_{\mathrm{grad}} \widetilde{E} \in H^1_{\Gamma_t}(\Omega), \quad \hat{\Phi} := \left(\mu\ \widetilde{\mathrm{rot}}_{\Gamma_n}\right)^{-1} \pi_{\mathrm{rot}} \widetilde{E} \in R_{\Gamma_n}(\Omega) \cap \mathrm{rot} R_{\Gamma_t}(\Omega)$$

*resp.* $\hat{\Psi} := e_{\mathcal{H}} \in \mathcal{H}_{t,n,\varepsilon}(\Omega)$ *with* $\mathrm{grad}\hat{\varphi} + \mu \mathrm{rot}\hat{\phi} = (\pi_{\mathrm{grad}} + \pi_{\mathrm{rot}})\widetilde{E} = (1 - \pi_{\mathcal{H}})\widetilde{E}$, *and at any* $\hat{\Phi} \in R_{\Gamma_n}(\Omega)$ *with* $\mu \mathrm{rot}\hat{\Phi} = \pi_{\mathrm{rot}}\widetilde{E}$, *where* $H^1_{\Gamma_t}(\Omega)$ *has to be replaced by* $H^1_\perp(\Omega)$, *if* $\Gamma_t = \emptyset$. *In the latter case* $\hat{\varphi}$ *is unique only up to a constant.*

If $\widetilde{E} := H + \widetilde{E}_\perp$ with some $\widetilde{E}_\perp \in \mathcal{H}_{t,n,\varepsilon}(\Omega)^{\perp_{L^2_\varepsilon(\Omega)}}$, then $e_{\mathcal{H}} = 0$, and in (ii) and (iii) $\widetilde{E}$ can be replaced by $\widetilde{E}_\perp$. In this case, for the attaining minima it holds

$$\hat{\Phi}_\perp := e_{\mathrm{grad}} + \widetilde{E}_\perp \in \mu\ D_{\Gamma_n}(\Omega), \quad \hat{\Phi}_\perp := e_{\mathrm{rot}} + \widetilde{E}_\perp \in R_{\Gamma_t}(\Omega).$$

**Remark 5.13.** For conforming approximations Corollary 4.2 and Remark 4.3 yield the following:

(i) *If* $\widetilde{E} \in \mu\ D_{\Gamma_n}(\Omega)$, *then* $e \in \mu\ D_{\Gamma_n}(\Omega)$ *and*

$$|e_{\mathrm{grad}}|_{L^2_\varepsilon(\Omega)} \le c_{fp} |\mathrm{div}\,\varepsilon\ \widetilde{E} + g|_{L^2(\Omega)} = c_{fp} |\mathrm{div}\,\varepsilon\ e|_{L^2(\Omega)}.$$

(ii) *If* $\widetilde{E} \in R_{\Gamma_t}(\Omega)$, *then* $e \in R_{\Gamma_t}(\Omega)$ *and*

$$|e_{\mathrm{rot}}|_{L^2_\varepsilon(\Omega)} \le c_m |\mathrm{rot}\widetilde{E} - F|_{L^2(\Omega)} = c_m |\mathrm{rot} e|_{L^2(\Omega)}.$$

(iii) *If* $\widetilde{E} \in R_{\Gamma_t}(\Omega) \cap \mu\ D_{\Gamma_n}(\Omega)$, *then* $e \in R_{\Gamma_t}(\Omega) \cap \mu\ D_{\Gamma_n}(\Omega)$ *and this very conforming error is equivalent to the weighted least squares functional*

$$\mathcal{F}(\widetilde{E}) := |H - \pi_{\mathcal{H}}\widetilde{E}|^2_{L^2_\varepsilon(\Omega)} + (1 + c_m^2)|\mathrm{rot}\widetilde{E} - F|^2_{L^2(\Omega)} + \left(1 + c_{fp}^2\right)|\mathrm{div}\,\varepsilon\,\widetilde{E} + g|^2_{L^2(\Omega)},$$

$$i.e.,\ |e|^2_{R_{\Gamma_t}(\Omega)\cap\mu D_{\Gamma_n}(\Omega)} \le \mathcal{F}(\widetilde{E}) \le (1 + \max\{c_{fp}, c_m\}^2)|e|^2_{R_{\Gamma_t}(\Omega)\cap\mu D_{\Gamma_n}(\Omega)}.$$

## 5.2 Prototype second order systems: Laplacian and rotrot

As prototypical examples for second order systems we will discuss the Laplacian and the rotrot -system, both with mixed boundary conditions. Suppose the assumptions of Section 5.1 are valid and recall the notations. For simplicity and to avoid case studies we assume $\emptyset \neq \Gamma_t \neq \Gamma$.

### 5.2.1. The Laplacian

Suppose $g \in L^2(\Omega)$. Let us consider the linear second order equation (in classical strong formulation) of the perturbed negative Laplacian with mixed boundary conditions for a function $u : \Omega \to \mathbb{R}$

$$-\mathrm{div}\,\varepsilon\,\mathrm{grad}\,u = g \text{ in } \Omega, \quad u = 0 \text{ at } \Gamma_t, \quad n \cdot \varepsilon\,\mathrm{grad}\,u = 0 \text{ at } \Gamma_n. \tag{5.16}$$

The corresponding variational formulation, which is uniquely solvable by Lax-Milgram's lemma, is the following: Find $u \in H^1_{\Gamma_t}(\Omega)$, such that

$$\forall\ \varphi \in H^1_{\Gamma_t}(\Omega) \qquad \langle \mathrm{grad}\,u, \mathrm{grad}\,\varphi\rangle_{L^2_\varepsilon(\Omega)} = \langle g, \varphi\rangle_{L^2(\Omega)}.$$

Then, by definition and the results of [2], we get $\varepsilon\mathrm{grad}\,u \in D_{\Gamma_n}(\Omega)$ with $-\mathrm{div}\,\varepsilon\mathrm{grad}\,u = g$. Hence, by setting

$$E := \mathrm{grad}\ u \in \mu\ D_{\Gamma_n}(\Omega) \cap \mathrm{grad}\ H^1_{\Gamma_t}(\Omega) = \mu\ D_{\Gamma_n}(\Omega) \cap R_{\Gamma_t,0}(\Omega) \cap \mathcal{H}_{t,n,\varepsilon}(\Omega)^{\perp_{L^2_\varepsilon(\Omega)}}$$

we see that the pair $(u, E)$ solves the linear first order system (in classical strong formulation) of electro-magneto statics type with mixed boundary conditions

$$\begin{aligned} \mathrm{grad}\ u = E, \quad \mathrm{rot}\ E &= 0 \quad \text{in } \Omega, \quad u = 0, \quad n \times E = 0 \quad \text{at } \Gamma_t, \\ -\mathrm{div}\,\varepsilon E &= g \quad \text{in } \Omega, \qquad\qquad\quad n \cdot \varepsilon E = 0 \quad \text{at } \Gamma_n, \\ \pi_{\mathcal{H}} E &= 0 \quad \text{in } \Omega. \end{aligned} \tag{5.17}$$

Similar to the latter subsection we define the operators $\mathrm{A}_1, \mathrm{A}_2, \mathrm{A}_3$ and also $\mathrm{A}_0, \mathrm{A}_4$ together with the respective adjoints and reduced operators by the de Rham complexes

$$\{0\} \xrightarrow{\mathrm{A}_0=\iota_{\{0\}}} \mathsf{H}^1_{\Gamma_t}(\Omega) \xrightarrow{\mathrm{A}_1=\mathrm{grad}_{\Gamma_t}} \mathsf{R}_{\Gamma_t}(\Omega) \xrightarrow{\mathrm{A}_2=\mathrm{rot}_{\Gamma_t}} \mathsf{D}_{\Gamma_t}(\Omega) \xrightarrow{\mathrm{A}_3=\mathrm{div}_{\Gamma_t}} \mathsf{L}^2(\Omega) \xrightarrow{\mathrm{A}_4=\pi_{\{0\}}} \{0\},$$

$$\{0\} \xleftarrow{\mathrm{A}_0^*=\pi_{\{0\}}} \mathsf{L}^2(\Omega) \xleftarrow{\mathrm{A}_1^*=-\mathrm{div}_{\Gamma_n}\varepsilon} \mu\,\mathsf{D}_{\Gamma_n}(\Omega) \xleftarrow{\mathrm{A}_2^*=\mu\,\mathrm{rot}_{\Gamma_n}} \mathsf{R}_{\Gamma_n}(\Omega) \xleftarrow{\mathrm{A}_3^*=-\mathrm{grad}_{\Gamma_n}} \mathsf{H}^1_{\Gamma_n}(\Omega) \xleftarrow{\mathrm{A}_4^*=\iota_{\{0\}}} \{0\}.$$

As before, all basic Hilbert spaces are $L^2(\Omega)$ except of $H_2 = L^2_\varepsilon(\Omega)$. Then (5.16) turns to

$$\begin{aligned} \mathrm{A}_1^*\mathrm{A}_1 u &= g, \\ \mathrm{A}_0^* u = \pi_{\{0\}} u &= 0, \\ \pi_1 u = \pi_{\{0\}} u &= 0 \end{aligned}$$

and this system is (again) uniquely solvable by Theorem 3.19 as $g \in L^2(\Omega) = R(\mathrm{A}_1^*)$ with solution $u$ depending continuously on the data. (5.17) reads

$$\begin{aligned} \mathrm{A}_1 u &= \operatorname{grad}_{\Gamma_t} u = E, & \mathrm{A}_2 E &= \operatorname{rot}_{\Gamma_t} E = 0, \\ \mathrm{A}_0^* u &= \pi_{\{0\}} u = 0, & \mathrm{A}_1^* E &= -\operatorname{div}_{\Gamma_n} \varepsilon\ E = g, \\ \pi_1 u &= \pi_{\{0\}} u = 0, & \pi_2 E &= \pi_{\mathcal{H}} E = 0. \end{aligned}$$

We can apply the main functional a posteriori error estimates from Theorem 4.7.

**Theorem 5.14.** *Let $u \in H^1_{\Gamma_t}(\Omega)$ be the exact solution of (5.16), $E := \operatorname{grad} u$, and $(\widetilde{u}, \widetilde{E}) \in L^2(\Omega) \times L^2_\varepsilon(\Omega)$. Then the following estimates hold for the errors $e_u := u - \widetilde{u}$ and $e_E := E - \widetilde{E}$:*

(i) *The error $e_E$ decomposes, i.e.,*

$$e_E = e_{E,\mathrm{grad}} + e_{E,\mathcal{H}} + e_{E,\mathrm{rot}} \in \operatorname{grad}\ H^1_{\Gamma_t}(\Omega) \oplus_{L^2_\varepsilon(\Omega)} \mathcal{H}_{t,n,\varepsilon}(\Omega) \oplus_{L^2_\varepsilon(\Omega)} \mu \operatorname{rot}\ R_{\Gamma_n}(\Omega)$$

*and*

$$|e_E|^2_{L^2_\varepsilon(\Omega)} = |e_{E,\mathrm{grad}}|^2_{L^2_\varepsilon(\Omega)} + |e_{E,\mathcal{H}}|^2_{L^2_\varepsilon(\Omega)} + |e_{E,\mathrm{rot}}|^2_{L^2_\varepsilon(\Omega)}.$$

(ii) $e_u = \pi_{\mathrm{div}} e_u \in \operatorname{div} D_{\Gamma_n}(\Omega) = L^2(\Omega)$ *and*

$$\begin{aligned} |e_u|^2_{L^2(\Omega)} &= \min_{\varphi \in H^1_{\Gamma_t}(\Omega)} \min_{\Phi \in \mu D_{\Gamma_n}(\Omega)} \Big( c_{fp}^2 |\operatorname{div} \varepsilon\ \Phi + g|_{L^2(\Omega)} + c_{fp} |\Phi - \operatorname{grad} \varphi|_{L^2_\varepsilon(\Omega)} + |\varphi - \widetilde{u}|_{L^2(\Omega)} \Big)^2 \\ &= \min_{\substack{\varphi \in H^1_{\Gamma_t}(\Omega), \\ \operatorname{grad} \varphi \in \mu D_{\Gamma_n}(\Omega)}} \Big( c_{fp}^2 |\operatorname{div}\ \varepsilon \operatorname{grad}\ \varphi + g|_{L^2(\Omega)} + |\varphi - \widetilde{u}|_{L^2(\Omega)} \Big)^2 \\ &= \max_{\substack{\phi \in H^1_{\Gamma_t}(\Omega), \\ \operatorname{grad} \phi \in \mu D_{\Gamma_n}(\Omega)}} \Big( 2\langle g, \phi \rangle_{L^2(\Omega)} + \langle 2\widetilde{u} - \operatorname{div} \varepsilon \operatorname{grad} \phi, \operatorname{div} \varepsilon \operatorname{grad} \phi \rangle_{L^2(\Omega)} \Big) \end{aligned}$$

*and the minima resp. maximum are attained at*

$$\begin{aligned} \hat{\varphi} &:= e_u + \widetilde{u} \in H^1_{\Gamma_t}(\Omega), \quad \hat{\Phi} := E \in \mu\ D_{\Gamma_n}(\Omega), \\ \hat{\phi} &:= \Big( \widetilde{\operatorname{grad}}_{\Gamma_t} \Big)^{-1} \Big( -\widetilde{\operatorname{div}}_{\Gamma_n} \varepsilon \Big)^{-1} \in H^1_{\Gamma_t}(\Omega) \end{aligned}$$

*with* $\operatorname{grad} \hat{\varphi}, \operatorname{grad} \hat{\phi} \in \mu\ D_{\Gamma_n}(\Omega)$ *and* $\operatorname{grad} \hat{\varphi} = \operatorname{grad} u = E$ *and* $-\operatorname{div} \varepsilon \operatorname{grad} \hat{\varphi} = -\operatorname{div} \varepsilon\ E = g$ *as well as* $-\operatorname{div} \varepsilon\ \hat{\Phi} = -\operatorname{div} \varepsilon\ E = g$.

(iii) *The projection* $e_{E,\text{grad}} = \pi_{\text{grad}} e_E = E - \pi_{\text{grad}} \widetilde{E} \in \text{grad} H^1_{\Gamma_t}(\Omega)$ *satisfies*

$$|e_{E,\text{grad}}|^2_{L^2_\varepsilon(\Omega)} = \min_{\Phi \in \mu D_{\Gamma_n}(\Omega)} \left( c_{fp} |\operatorname{div} \varepsilon\ \Phi + g|_{L^2(\Omega)} + |\Phi - \widetilde{E}|_{L^2_\varepsilon(\Omega)} \right)^2$$
$$= \max_{\varphi \in H^1_{\Gamma_t}(\Omega)} \left( 2\langle g, \varphi \rangle_{L^2(\Omega)} - \langle 2\widetilde{E} + \text{grad}\,\varphi, \text{grad}\,\varphi \rangle_{L^2_\varepsilon(\Omega)} \right)$$

*and the minimum resp. maximum is attained at*

$$\hat{\Phi} := e_{E,\text{grad}} + \widetilde{E} \in \mu\ D_{\Gamma_n}(\Omega), \quad \hat{\varphi} := \left( \widetilde{\text{grad}}_{\Gamma_t} \right)^{-1} e_{E,\text{grad}} \in H^1_{\Gamma_t}(\Omega)$$

*with* $-\operatorname{div} \varepsilon\ \hat{\Phi} = -\operatorname{div} \varepsilon\ E = g$.

(iv) *The projection* $e_{E,\text{rot}} = \pi_{\text{rot}} e_E = -\pi_{\text{rot}} \widetilde{E} \in \mu \text{rot} R_{\Gamma_n}(\Omega)$ *satisfies*

$$|e_{E,\text{rot}}|^2_{L^2_\varepsilon(\Omega)} = \min_{\Phi \in R_{\Gamma_t}(\Omega)} \left( c_m |\text{rot}\Phi|_{L^2(\Omega)} + |\Phi - \widetilde{E}|_{L^2_\varepsilon(\Omega)} \right)^2 = \min_{\Phi \in R_{\Gamma_t,0}(\Omega)} |\Phi - \widetilde{E}|^2_{L^2_\varepsilon(\Omega)}$$
$$= \max_{\Psi \in R_{\Gamma_n}(\Omega)} \left( -\langle 2\widetilde{E} + \mu\,\text{rot}\Psi, \mu\,\text{rot}\Psi \rangle_{L^2_\varepsilon(\Omega)} \right)$$

*and the minimum resp. maximum is attained at*

$$\hat{\Phi} := e_{E,\text{rot}} + \widetilde{E} \in R_{\Gamma_t,0}(\Omega), \quad \hat{\Psi} := \left( \mu\ \widetilde{\text{rot}}_{\Gamma_n} \right)^{-1} e_{E,\text{rot}} \in R_{\Gamma_n}(\Omega) \cap \text{rot} R_{\Gamma_t}(\Omega)$$

*with* $\text{rot}\hat{\Phi} = \text{rot}E = 0$.

(v) *The projection* $e_{E,\mathcal{H}} = \pi_{\mathcal{H}} e_E = -\pi_{\mathcal{H}} \widetilde{E} \in \mathcal{H}_{t,n,\varepsilon}(\Omega)$ *satisfies*

$$|e_{E,\mathcal{H}}|^2_{L^2_\varepsilon(\Omega)} = \min_{\varphi \in H^1_{\Gamma_t}(\Omega)} \min_{\Phi \in R_{\Gamma_n}(\Omega)} |-\widetilde{E} + \text{grad}\,\varphi + \mu \text{rot}\Phi|^2_{L^2_\varepsilon(\Omega)}$$
$$= \max_{\Psi \in \mathcal{H}_{t,n,\varepsilon}(\Omega)} \left( -\langle 2\widetilde{E} + \Psi, \Psi \rangle_{L^2_\varepsilon(\Omega)} \right)$$

*and the minimum resp. maximum is attained at*

$$\hat{\varphi} := \left( \widetilde{\text{grad}}_{\Gamma_t} \right)^{-1} \pi_{\text{grad}} \widetilde{E} \in H^1_{\Gamma_t}(\Omega), \quad \hat{\Phi} := \left( \mu\ \widetilde{\text{rot}}_{\Gamma_n} \right)^{-1} \pi_{\text{rot}} \widetilde{E} \in R_{\Gamma_n}(\Omega) \cap \text{rot} R_{\Gamma_t}(\Omega)$$

*resp.* $\hat{\Psi} := e_{E,\mathcal{H}} \in \mathcal{H}_{t,n,\varepsilon}(\Omega)$ *with* $\text{grad}\hat{\varphi} + \mu \text{rot}\hat{\phi} = (\pi_{\text{grad}} + \pi_{\text{rot}})\widetilde{E} = (1 - \pi_{\mathcal{H}})\widetilde{E}$.

*If* $\widetilde{E} := \widetilde{E}_\perp$ *with some* $\widetilde{E}_\perp \in \mathcal{H}_{t,n,\varepsilon}(\Omega)^{\perp_{L^2_\varepsilon(\Omega)}}$, *then* $e_{E,\mathcal{H}} = 0$, *and in (iii) and (iv)* $\widetilde{E}$ *can be replaced by* $\widetilde{E}_\perp$. *In this case, for the attaining minima it holds*

$$\hat{\Phi}_\perp := e_{E,\text{grad}} + \widetilde{E}_\perp \in \mu\ D_{\Gamma_n}(\Omega), \quad \hat{\Phi}_\perp := e_{E,\text{rot}} + \widetilde{E}_\perp \in R_{\Gamma_t,0}(\Omega).$$

For conforming approximations $\widetilde{E} \in \text{grad} H^1_{\Gamma_t}(\Omega)$ we have $e_{E,\text{rot}} = e_{E,\mathcal{H}} = 0$ and $e_E = e_{E,\text{grad}}$. Especially, if $\widetilde{u} \in H^1_{\Gamma_t}(\Omega)$ and $\widetilde{E} := \text{grad}\widetilde{u}$ with a conforming approximation $\widetilde{u} \in H^1_{\Gamma_t}(\Omega)$, the estimates of the latter theorem simplify. More precisely, (ii) turns to the following result: If $\widetilde{u} \in H^1_{\Gamma_t}(\Omega)$, then $e_u \in H^1_{\Gamma_t}(\Omega)$ and we can choose, e.g., $\varphi := \widetilde{u}$ yielding, e.g.,

$$|e_u|_{L^2(\Omega)} \leq \min_{\Phi \in \mu D_{\Gamma_n}(\Omega)} \left( c_{fp}^2 |\operatorname{div} \varepsilon\ \Phi + g|_{L^2(\Omega)} + c_{fp} |\Phi - \operatorname{grad}\ \widetilde{u}|_{L^2_\varepsilon(\Omega)} \right),$$

which might not be sharp anymore. Similarly, the results of (iii) read as follows: If $\widetilde{u}$ belongs to $H^1_{\Gamma_t}(\Omega)$, then $\widetilde{E} := \operatorname{grad} \widetilde{u} \in \operatorname{grad} H^1_{\Gamma_t}(\Omega)$ and $\operatorname{grad}(u - \widetilde{u}) = e_E = e_{E,\text{grad}} \in \operatorname{grad} H^1_{\Gamma_t}(\Omega)$ as well as

$$\begin{aligned} |e_E|^2_{L^2_\varepsilon(\Omega)} &= \min_{\Phi \in \mu D_{\Gamma_n}(\Omega)} \left( c_{fp} |\operatorname{div} \varepsilon\ \Phi + g|_{L^2(\Omega)} + |\Phi - \operatorname{grad}\ \widetilde{u}|_{L^2_\varepsilon(\Omega)} \right)^2 \\ &= \max_{\varphi \in H^1_{\Gamma_t}(\Omega)} \left( 2\langle g, \varphi \rangle_{L^2(\Omega)} - \langle \operatorname{grad}(2\widetilde{u} + \varphi), \operatorname{grad}\ \varphi \rangle_{L^2_\varepsilon(\Omega)} \right) \end{aligned}$$

and the minimum resp. maximum is attained at

$$\hat{\Phi} := e_E + \operatorname{grad}\ \widetilde{u} = \operatorname{grad}\ u \in \mu\ D_{\Gamma_n}(\Omega), \quad \hat{\varphi} := \left( \widetilde{\operatorname{grad}_{\Gamma_t}} \right)^{-1} e_E \in H^1_{\Gamma_t}(\Omega)$$

*with* $-\operatorname{div} \varepsilon\ \hat{\Phi} = -\operatorname{div} \varepsilon\ E = g$*. Note that (5.18) are the well known functional a posteriori error estimates for the energy norm associated to the Laplacian, see, e.g., [17].*

### 5.2.2. The *rot rot*-operator

Suppose $F \in \operatorname{rot} R_{\Gamma_t}(\Omega) = D_{\Gamma_t,0}(\Omega) \cap \mathcal{H}_{n,t}(\Omega)^{\perp_{L^2(\Omega)}}$ and $g \in L^2(\Omega)$ as well as $H \in \mathcal{H}_{n,t}(\Omega)$. Let us consider the linear second order equation (in classical strong formulation) of the perturbed rotrot-operator with mixed boundary conditions for a vector field $B : \Omega \to \mathbb{R}^3$

$$\begin{aligned} \operatorname{rot}\ \mu \operatorname{rot} B = F \quad &\text{in } \Omega, \quad n \times B = 0 && \text{at } \Gamma_n, \\ \operatorname{div} \nu B = g \quad &\text{in } \Omega, \quad n \cdot \nu B = 0, \quad n \times \mu \operatorname{rot} B = 0 && \text{at } \Gamma_t, \\ \pi_{\widetilde{\mathcal{H}}} B = H \quad &\text{in } \Omega. \end{aligned} \tag{5.19}$$

Here $\pi_{\widetilde{\mathcal{H}}} : L^2(\Omega) \to \mathcal{H}_{n,t}(\Omega)$ and for simplicity we set $\nu := \text{id}$ for the matrix field $\nu$. The partial solution $B_g$ can be computed by solving a Laplace problem. The corresponding variational formulation, which is uniquely solvable by Lax-Milgram's lemma, to find the partial solution $B_F$ of

$$\begin{aligned} \operatorname{rot} \mu \operatorname{rot} B_F = F \quad &\text{in } \Omega, \quad n \times B_F = 0 && \text{at } \Gamma_n, \\ \operatorname{div} B_F = 0 \quad &\text{in } \Omega, \quad n \cdot B_F = 0, \quad n \times \mu \operatorname{rot} B_F = 0 && \text{at } \Gamma_t, \\ \pi_{\widetilde{\mathcal{H}}} B_F = 0 \quad &\text{in } \Omega, \end{aligned}$$

is the following: Find $B_F \in R_{\Gamma_n}(\Omega) \cap \operatorname{rot} R_{\Gamma_t}(\Omega)$, such that[18]

[18]Note that (5.20) holds for all $\Phi \in R_{\Gamma_n}(\Omega) \cap \operatorname{rot} R_{\Gamma_t}(\Omega)$ if and only if it holds for all $\Phi \in R_{\Gamma_n}(\Omega)$ since $F \in \operatorname{rot} R_{\Gamma_t}(\Omega)$.

$$\forall\ \Phi \in R_{\Gamma_n}(\Omega) \qquad \langle \mathrm{rot} B_F, \mathrm{rot}\Phi \rangle_{L^2_\mu(\Omega)} = \langle F, \Phi \rangle_{L^2(\Omega)}. \tag{5.20}$$

Then, by definition and the results of [2], we get $\mu \mathrm{rot} B_F \in R_{\Gamma_t}(\Omega)$ with $\mathrm{rot}\mu\mathrm{rot} B_F = F$. Hence, by setting

$$E := \mu\mathrm{rot}\ B_F \in R_{\Gamma_t}(\Omega) \cap \mu\mathrm{rot}\ R_{\Gamma_n}(\Omega) = R_{\Gamma_t}(\Omega) \cap \mu\ D_{\Gamma_n,0}(\Omega) \cap \mathcal{H}_{t,n,\varepsilon}(\Omega)^{\perp_{L^2_\varepsilon(\Omega)}}$$

we see that the pair $(B, E)$ solves the linear first order system (in classical strong formulation) of electro-magneto statics type with mixed boundary conditions

$$\begin{array}{llllll} \mu\mathrm{rot}\ B = \mu\mathrm{rot}\ B_F = E, & \mathrm{rot}\ E = F & \text{in } \Omega, & n \times B = 0, & n \cdot \varepsilon E = 0 & \text{at } \Gamma_n, \\ \mathrm{div} B = g, & \mathrm{div}\,\varepsilon E = 0 & \text{in } \Omega, & n \cdot B = 0, & n \times E = 0 & \text{at } \Gamma_t, \\ \pi_{\widetilde{\mathcal{H}}} B = H, & \pi_{\mathcal{H}} E = 0 & \text{in } \Omega. & & & \end{array} \tag{5.21}$$

Let us define operators $\mathrm{T}_1, \mathrm{T}_2, \mathrm{T}_3$ using $\mathrm{A}_1, \mathrm{A}_2, \mathrm{A}_3$ together with the respective adjoints and reduced operators by the complexes

$$\{0\} \xrightarrow{\mathrm{T}_4^* := \iota_{\{0\}}} \mathsf{H}^1_{\Gamma_t}(\Omega) \xrightarrow{\mathrm{T}_3^* := \mathrm{A}_1 = \mathrm{grad}_{\Gamma_t}} \mathsf{R}_{\Gamma_t}(\Omega) \xrightarrow{\mathrm{T}_2^* := \mathrm{A}_2 = \mathrm{rot}_{\Gamma_t}} \mathsf{D}_{\Gamma_t}(\Omega) \xrightarrow{\mathrm{T}_1^* := \mathrm{A}_3 = \mathrm{div}_{\Gamma_t}} \mathsf{L}^2(\Omega) \xrightarrow{\mathrm{T}_0^* := \pi_{\{0\}}} \{0\},$$

$$\{0\} \xrightarrow{\mathrm{T}_4^* := \iota_{\{0\}}} \mathsf{H}^1_{\Gamma_t}(\Omega) \xrightarrow{\mathrm{T}_3^* := \mathrm{A}_1 = \mathrm{grad}_{\Gamma_t}} \mathsf{R}_{\Gamma_t}(\Omega) \xrightarrow{\mathrm{T}_2^* := \mathrm{A}_2 = \mathrm{rot}_{\Gamma_t}} \mathsf{D}_{\Gamma_t}(\Omega) \xrightarrow{\mathrm{T}_1^* := \mathrm{A}_3 = \mathrm{div}_{\Gamma_t}} \mathsf{L}^2(\Omega) \xrightarrow{\mathrm{T}_0^* := \pi_{\{0\}}} \{0\},$$

As before, all basic Hilbert spaces are $L^2(\Omega)$ except of $H_3 = L^2_\varepsilon(\Omega)$, corresponding to the domain of definition of $\mathrm{T}_3$. Then (5.19) turns to

$$\begin{aligned} \mathrm{T}_2^*\mathrm{T}_2 B &= \mathrm{rot}_{\Gamma_t}\mu\,\mathrm{rot}_{\Gamma_n} B = F, \\ \mathrm{T}_1^* B &= \mathrm{div}_{\Gamma_t} B = g, \\ \pi_2 B &= \pi_{\widetilde{\mathcal{H}}} B = H \end{aligned}$$

and this system is uniquely solvable by Theorem 3.19 as $F \in R(\mathrm{T}_2^*), g \in R(\mathrm{T}_1^*)$, and $H \in K_2$ with solution $B$ depending continuously on the data. (5.21) reads

$$\begin{aligned} \mathrm{T}_2 B &= \mu\,\mathrm{rot}_{\Gamma_n} B = E, & \mathrm{T}_3 E &= -\mathrm{div}_{\Gamma_n}\varepsilon\ E = 0, \\ \mathrm{T}_1^* B &= \mathrm{div}_{\Gamma_t} B = g, & \mathrm{T}_2^* E &= \mathrm{rot}_{\Gamma_t} E = F, \\ \pi_2 B &= \pi_{\widetilde{\mathcal{H}}} B = H, & \pi_3 E &= \pi_{\mathcal{H}} E = 0. \end{aligned}$$

Again, we can apply the main functional a posteriori error estimates from Theorem 4.7.

**Theorem 5.15.** *Let $B \in R_{\Gamma_n}(\Omega) \cap D_{\Gamma_t}(\Omega)$ be the exact solution of (5.19), $E := \mu\mathrm{rot} B \in R_{\Gamma_t}(\Omega)$, and $(\widetilde{B}, \widetilde{E}) \in L^2(\Omega) \times L^2_\varepsilon(\Omega)$. Then the following estimates hold for the errors $e_B := B - \widetilde{B}$ and $e_E := E - \widetilde{E}$:*

(i) *The errors $e_B$ and $e_E$ decompose, i.e.,*

$$e_B = e_{B,\mathrm{grad}} + e_{B,\widetilde{\mathcal{H}}} + e_{B,\mathrm{rot}} \in \mathrm{grad}\ H^1_{\Gamma_n}(\Omega) \oplus_{L^2(\Omega)} \mathcal{H}_{n,t}(\Omega) \oplus_{L^2(\Omega)} \mathrm{rot}\ R_{\Gamma_t}(\Omega),$$
$$e_E = e_{E,\mathrm{grad}} + e_{E,\mathcal{H}} + e_{E,\mathrm{rot}} \in \mathrm{grad}\ H^1_{\Gamma_t}(\Omega) \oplus_{L^2_\varepsilon(\Omega)} \mathcal{H}_{t,n,\varepsilon}(\Omega) \oplus_{L^2_\varepsilon(\Omega)} \mu\,\mathrm{rot}\ R_{\Gamma_n}(\Omega)$$

*and*
$$|e_B|^2_{L^2(\Omega)} = |e_{B,\mathrm{grad}}|^2_{L^2(\Omega)} + |e_{B,\widetilde{\mathcal{H}}}|^2_{L^2(\Omega)} + |e_{B,\mathrm{rot}}|^2_{L^2(\Omega)},$$
$$|e_E|^2_{L^2_\varepsilon(\Omega)} = |e_{E,\mathrm{grad}}|^2_{L^2_\varepsilon(\Omega)} + |e_{E,\mathcal{H}}|^2_{L^2_\varepsilon(\Omega)} + |e_{E,\mathrm{rot}}|^2_{L^2_\varepsilon(\Omega)}.$$

(ii) *The projection $e_{B,\mathrm{grad}} = \pi_{\mathrm{grad}} e_B = B_g - \pi_{\mathrm{grad}}\widetilde{B} \in \mathrm{grad} H^1_{\Gamma_n}(\Omega)$ satisfies*

$$|e_{B,\mathrm{grad}}|^2_{L^2(\Omega)} = \min_{\Phi\in D_{\Gamma_t}(\Omega)} \left(\widetilde{c}_{fp}|\mathrm{div}\Phi - g|_{L^2(\Omega)} + |\Phi - \widetilde{B}|_{L^2(\Omega)}\right)^2$$
$$= \max_{\varphi\in H^1_{\Gamma_n}(\Omega)} \left(2\langle g,\varphi\rangle_{L^2(\Omega)} + \langle 2\widetilde{B} - \mathrm{grad}\varphi, \mathrm{grad}\varphi\rangle_{L^2(\Omega)}\right)$$

*and the minimum resp. maximum is attained at*

$$\hat{\Phi} := e_{B,\mathrm{grad}} + \widetilde{B} \in D_{\Gamma_t}(\Omega), \quad \hat{\varphi} := -\left(\widetilde{\mathrm{grad}}_{\Gamma_n}\right)^{-1} e_{B,\mathrm{grad}} \in H^1_{\Gamma_n}(\Omega)$$

*with* $\mathrm{div}\hat{\Phi} = \mathrm{div}B = g$.

(iii) *The projection $e_{B,\mathrm{rot}} = \pi_{\mathrm{rot}} e_B = B_E - \pi_{\mathrm{rot}}\widetilde{B} \in \mathrm{rot}R_{\Gamma_t}(\Omega)$ satisfies*

$$|e_{B,\mathrm{rot}}|^2_{L^2(\Omega)} = \min_{\Psi\in R_{\Gamma_n}(\Omega)} \min_{\Phi\in R_{\Gamma_t}(\Omega)} \left(c^2_m|\mathrm{rot}\Phi - F|_{L^2(\Omega)} + c_m|\Phi - \mu\mathrm{rot}\Psi|_{L^2_\varepsilon(\Omega)} + |\Psi - \widetilde{B}|_{L^2(\Omega)}\right)^2$$
$$= \min_{\mu\mathrm{rot}\Psi\in R_{\Gamma_t}(\Omega)^{\Psi\in R_{\Gamma_n}(\Omega),}} \left(c^2_m|\mathrm{rot}\ \mu\,\mathrm{rot}\ \Psi - F|_{L^2(\Omega)} + |\Psi - \widetilde{B}|_{L^2(\Omega)}\right)^2$$
$$= \max_{\mu\mathrm{rot}\Theta\in R_{\Gamma_t}(\Omega)^{\Theta\in R_{\Gamma_n}(\Omega),}} \left(2\langle F,\Theta\rangle_{L^2(\Omega)} - \langle 2\widetilde{E} + \mathrm{rot}\ \mu\,\mathrm{rot}\ \Theta, \mathrm{rot}\mu\,\mathrm{rot}\Theta\rangle_{L^2(\Omega)}\right)$$

*and the minima resp. maximum is attained at*

$$\hat{\Psi} := e_{B,\mathrm{rot}} + \widetilde{B} \in R_{\Gamma_n}(\Omega), \quad \hat{\Phi} := E \in R_{\Gamma_t}(\Omega),$$

*and* $\hat{\Theta} := (\mu\ \widetilde{\mathrm{rot}}_{\Gamma_n})^{-1}(\widetilde{\mathrm{rot}}_{\Gamma_t})^{-1} e_{B,\mathrm{rot}} \in R_{\Gamma_n}(\Omega) \cap \mathrm{rot}R_{\Gamma_t}(\Omega)$ *with* $\mu\mathrm{rot}\hat{\Psi}$, $\mu\mathrm{rot}\hat{\Theta}, \in R_{\Gamma_t}(\Omega)$ *and* $\mu\mathrm{rot}\hat{\Psi} = \mu\mathrm{rot}B = E$ *and* $\mathrm{rot}\mu\mathrm{rot}\hat{\Psi} = \mathrm{rot}E = F$ *as well as* $\mathrm{rot}\hat{\Phi} = \mathrm{rot}E = F$.

(iv) *The projection $e_{B,\widetilde{\mathcal{H}}} = \pi_{\widetilde{\mathcal{H}}} e_B = H - \pi_{\widetilde{\mathcal{H}}}\widetilde{B} \in \mathcal{H}_{n,t}(\Omega)$ satisfies*

$$|e_{B,\widetilde{\mathcal{H}}}|^2_{L^2(\Omega)} = \min_{\varphi\in H^1_{\Gamma_n}(\Omega)} \min_{\Phi\in R_{\Gamma_t}(\Omega)} |H - \widetilde{B} - \mathrm{grad}\,\varphi + \mathrm{rot}\Phi|^2_{L^2(\Omega)}$$
$$= \max_{\Psi\in\mathcal{H}_{n,t}(\Omega)} \langle 2(H - \widetilde{B}) - \Psi, \Psi\rangle_{L^2(\Omega)}$$

*and the minimum resp. maximum is attained at*

$$\hat{\varphi} := -\left(\widetilde{\mathrm{grad}}_{\Gamma_n}\right)^{-1}\pi_{\mathrm{grad}}\widetilde{B} \in H^1_{\Gamma_n}(\Omega), \quad \hat{\Phi} := \left(\widetilde{\mathrm{rot}}_{\Gamma_t}\right)^{-1}\pi_{\mathrm{rot}}\widetilde{B} \in R_{\Gamma_t}(\Omega) \cap \mu\mathrm{rot}R_{\Gamma_n}(\Omega)$$

*resp.* $\hat{\Psi} := e_{B,\widetilde{\mathcal{H}}} \in \mathcal{H}_{n,t}(\Omega)$ *with* $-\mathrm{grad}\hat{\varphi} + \mathrm{rot}\hat{\phi} = (\pi_{\mathrm{grad}} + \pi_{\mathrm{rot}})\widetilde{B} = (1 - \pi_{\widetilde{\mathcal{H}}})\widetilde{B}.$

(v) *The projection* $e_{E,\mathrm{grad}} = \pi_{\mathrm{grad}} e_E = -\pi_{\mathrm{grad}}\widetilde{E} \in \mathrm{grad}H^1_{\Gamma_t}(\Omega)$ *satisfies*

$$\begin{aligned}|e_{E,\mathrm{grad}}|^2_{L^2_\varepsilon(\Omega)} &= \min_{\Phi\in\mu D_{\Gamma_n}(\Omega)}\left(c_{fp}|\mathrm{div}\,\varepsilon\;\Phi|_{L^2(\Omega)} + |\Phi - \widetilde{E}|_{L^2_\varepsilon(\Omega)}\right)^2 = \min_{\Phi\in\mu D_{\Gamma_n,0}(\Omega)}|\Phi - \widetilde{E}|^2_{L^2_\varepsilon(\Omega)} \\ &= \max_{\varphi\in H^1_{\Gamma_t}(\Omega)}\left(-\langle 2\widetilde{E} + \mathrm{grad}\;\varphi, \mathrm{grad}\;\varphi\rangle_{L^2_\varepsilon(\Omega)}\right)\end{aligned}$$

*and the minimum resp. maximum is attained at*

$$\hat{\Phi} := e_{E,\mathrm{grad}} + \widetilde{E} \in \mu\;D_{\Gamma_n,0}(\Omega), \quad \hat{\varphi} := \left(\widetilde{\mathrm{grad}}_{\Gamma_t}\right)^{-1} e_{E,\mathrm{grad}} \in H^1_{\Gamma_t}(\Omega)$$

with $-\mathrm{div}\,\varepsilon\;\hat{\Phi} = -\mathrm{div}\,\varepsilon\;E = 0.$

(vi) *The projection* $e_{E,\mathrm{rot}} = \pi_{\mathrm{rot}} e_E = E - \pi_{\mathrm{rot}}\widetilde{E} \in \mu\mathrm{rot}R_{\Gamma_n}(\Omega)$ *satisfies*

$$\begin{aligned}|e_{E,\mathrm{rot}}|^2_{L^2_\varepsilon(\Omega)} &= \min_{\Phi\in R_{\Gamma_t}(\Omega)}\left(c_m|\mathrm{rot}\Phi - F|_{L^2(\Omega)} + |\Phi - \widetilde{E}|_{L^2_\varepsilon(\Omega)}\right)^2 \\ &= \max_{\Psi\in R_{\Gamma_n}(\Omega)}\left(2\langle F, \Psi\rangle_{L^2(\Omega)} - \langle 2\widetilde{E} + \mu\mathrm{rot}\Psi, \mu\mathrm{rot}\Psi\rangle_{L^2_\varepsilon(\Omega)}\right)\end{aligned}$$

*and the minimum resp. maximum is attained at*

$$\hat{\Phi} := e_{E,\mathrm{rot}} + \widetilde{E} \in R_{\Gamma_t}(\Omega), \quad \hat{\Psi} := \left(\mu\;\widetilde{\mathrm{rot}}_{\Gamma_n}\right)^{-1} e_{E,\mathrm{rot}} \in R_{\Gamma_n}(\Omega) \cap \mathrm{rot}R_{\Gamma_t}(\Omega)$$

*with* $\mathrm{rot}\hat{\Phi} = \mathrm{rot}E = F.$

(vii) *The projection* $e_{E,\mathcal{H}} = \pi_{\mathcal{H}} e_E = -\pi_{\mathcal{H}}\widetilde{E} \in \mathcal{H}_{t,n,\varepsilon}(\Omega)$ *satisfies*

$$\begin{aligned}|e_{E,\mathcal{H}}|^2_{L^2_\varepsilon(\Omega)} &= \min_{\varphi\in H^1_{\Gamma_t}(\Omega)}\min_{\Phi\in R_{\Gamma_n}(\Omega)}|-\widetilde{E} + \mathrm{grad}\;\varphi + \mu\mathrm{rot}\Phi|^2_{L^2_\varepsilon(\Omega)} \\ &= \max_{\Psi\in\mathcal{H}_{t,n,\varepsilon}(\Omega)}\left(-\langle 2\widetilde{E} + \Psi, \Psi\rangle_{L^2_\varepsilon(\Omega)}\right)\end{aligned}$$

*and the minimum resp. maximum is attained at*

$$\hat{\varphi} := \left(\widetilde{\mathrm{grad}}_{\Gamma_t}\right)^{-1}\pi_{\mathrm{grad}}\widetilde{E} \in H^1_{\Gamma_t}(\Omega), \quad \hat{\Phi} := \left(\mu\;\widetilde{\mathrm{rot}}_{\Gamma_n}\right)^{-1}\pi_{\mathrm{rot}}\widetilde{E} \in R_{\Gamma_n}(\Omega) \cap \mathrm{rot}R_{\Gamma_t}(\Omega)$$

*resp.* $\hat{\Psi} := e_{E,\mathcal{H}} \in \mathcal{H}_{t,n,\varepsilon}(\Omega)$ *with* $\mathrm{grad}\hat{\varphi} + \mu\mathrm{rot}\hat{\phi} = (\pi_{\mathrm{grad}} + \pi_{\mathrm{rot}})\widetilde{E} = (1 - \pi_{\mathcal{H}})\widetilde{E}.$

*If $\widetilde{B} = H + \widetilde{B}_\perp$ with some $\widetilde{B}_\perp \in \mathcal{H}_{n,t}(\Omega)^{\perp_{L^2(\Omega)}}$, then $e_{\widetilde{B},\mathcal{H}} = 0$, and in (ii) and (iii) $\widetilde{B}$ can be replaced by $\widetilde{B}_\perp$. If $\widetilde{E} = \widetilde{E}_\perp$ with some $\widetilde{E}_\perp \in \mathcal{H}_{t,n,\varepsilon}(\Omega)^{\perp_{L^2_\varepsilon(\Omega)}}$, then $e_{E,\mathcal{H}} = 0$, and in (v) and (vi) $\widetilde{E}$ can be replaced by $\widetilde{E}_\perp$.*

A reasonable assumption is, that we have conforming approximations

$$\widetilde{B}_g \in \operatorname{grad} H^1_{\Gamma_n}(\Omega) = R_{\Gamma_n,0}(\Omega) \cap \mathcal{H}_{n,t}(\Omega)^\perp, \quad \widetilde{B}_F \in R_{\Gamma_n}(\Omega)$$

of $B_g \in D_{\Gamma_t}(\Omega) \cap \operatorname{grad} H^1_{\Gamma_n}(\Omega)$ and $B_F \in R_{\Gamma_n}(\Omega) \cap \operatorname{rot} R_{\Gamma_t}(\Omega)$ and hence a conforming approximation

$$\widetilde{E} := \mu \operatorname{rot} \widetilde{B}_F \in \mu \operatorname{rot} R_{\Gamma_n}(\Omega)$$

of $E \in R_{\Gamma_t}(\Omega) \cap \mu \operatorname{rot} R_{\Gamma_n}(\Omega)$, which implies $e_E = e_{E,\mathrm{rot}} \in \mu \operatorname{rot} R_{\Gamma_n}(\Omega)$ and $e_{E,\mathrm{grad}} = e_{E,\mathcal{H}} = 0$ as well as $\widetilde{B} - H = \widetilde{B}_F + \widetilde{B}_g \in R_{\Gamma_n}(\Omega)$ and $e_B \in R_{\Gamma_n}(\Omega)$. In this case the estimates of the latter theorem simplify. More precisely, e.g., (iii) turns to the following result: If $\widetilde{B}_F, \widetilde{B}_g \in R_{\Gamma_n}(\Omega)$, then $\widetilde{B}, e_B \in R_{\Gamma_n}(\Omega)$ and we can choose, e.g., $\Psi := \widetilde{B}$ yielding, e.g.,

$$|e_{B,\mathrm{rot}}|_{L^2(\Omega)} \leq \min_{\Phi \in R_{\Gamma_t}(\Omega)} \left( c_m^2 |\operatorname{rot}\Phi - F|_{L^2(\Omega)} + c_m |\Phi - \mu \operatorname{rot} \widetilde{B}|_{L^2_\varepsilon(\Omega)} \right),$$

which might not be sharp anymore. Similarly, the results of (vi) read as follows: If $\widetilde{B}_F \in R_{\Gamma_n}(\Omega)$, then $\widetilde{E} := \mu \operatorname{rot} \widetilde{B}_F \in \mu \operatorname{rot} R_{\Gamma_n}(\Omega)$ and $\mu \operatorname{rot}(B - \widetilde{B}_F) = e_E = e_{E,\mathrm{rot}} \in \mu \operatorname{rot} R_{\Gamma_n}(\Omega)$ as well as

$$\begin{aligned} |e_E|^2_{L^2_\varepsilon(\Omega)} &= \min\nolimits_{\Phi \in R_{\Gamma_t}(\Omega)} \left( c_m |\operatorname{rot}\Phi - F|_{L^2(\Omega)} + |\Phi - \mu \operatorname{rot} \widetilde{B}_F|_{L^2_\varepsilon(\Omega)} \right)^2 \\ &= \max\nolimits_{\Psi \in R_{\Gamma_n}(\Omega)} \left( 2\langle F, \Psi \rangle_{L^2(\Omega)} - \langle \mu \operatorname{rot}(2\widetilde{B}_F + \Psi), \mu \operatorname{rot} \Psi \rangle_{L^2_\varepsilon(\Omega)} \right) \end{aligned}$$

and the minimum resp. maximum is attained at

$$\hat{\Phi} := e_E + \mu \operatorname{rot} \widetilde{B}_F \in R_{\Gamma_t}(\Omega), \quad \hat{\Psi} := \left( \mu \, \widetilde{\operatorname{rot}}_{\Gamma_n} \right)^{-1} e_E \in R_{\Gamma_n}(\Omega) \cap \operatorname{rot} R_{\Gamma_t}(\Omega)$$

with $\operatorname{rot} \hat{\Phi} = \operatorname{rot} E = F$. Note that (5.22) are in principle the functional a posteriori error estimates for the energy norm associated to the rotrot -operator, which have been proved in [8].

### 5.3. More applications

There are plenty more applications fitting our general theory for the systems (1.5), (1.10), (1.11), i.e.,

$$\begin{aligned} \mathrm{A}_2 x &= f, & \mathrm{A}_2^* \mathrm{A}_2 x &= f, & \mathrm{A}_2^* \mathrm{A}_2 x &= f, \\ \mathrm{A}_1^* x &= g, & \mathrm{A}_1^* x &= g, & \mathrm{A}_1 \mathrm{A}_1^* x &= g, \\ \pi_2 x &= k, & \pi_2 x &= k, & \pi_2 x &= k. \end{aligned}$$

E.g., if we denote the exterior derivative and the co-derivative associated with some Riemannian manifold having compact closure by d and $\delta$, we can discuss problems like

$$\begin{aligned} \mathrm{d}E &= F, & -\delta\mu\mathrm{d}E &= F, & -\delta\mu\mathrm{d}E &= F,\\ -\delta\varepsilon E &= G, & -\delta\varepsilon E &= G, & -\mathrm{d}\delta\varepsilon E &= G,\\ \pi E &= H, & \pi E &= H, & \pi E &= H \end{aligned}$$

for mixed tangential and normal boundary conditions for some differential form $E$. Moreover, problems in linear elasticity, Stokes equations, biharmonic theory, general relativity, rot rot rot rot-operators, to mention just a few examples, fit into our general framework. Note that all these problems feature the underlying complexes (1.3) and (1.4). More precisely, let $\Omega \subset \mathbb{R}^3$ or $\Omega \subset \mathbb{R}^N, N \geq 2$, be a bounded weak Lipschitz domain with weak Lipschitz interface, and, for simplicity, let us just present homogeneous material parameters with $\varepsilon = \mathrm{id}, \mu = \mathrm{id}$ and skip the cohomology projector $\pi$. Then we have the following complexes and linear systems:

- electro-magnetics (as already extensively discussed before)

$$\cdots \xrightarrow{\mathrm{A}_0=\iota\ldots} \mathsf{H}^1_{\Gamma_t}(\Omega) \xrightarrow{\mathrm{A}_1=\mathrm{grad}_{\Gamma_t}} \mathsf{R}_{\Gamma_t}(\Omega) \xrightarrow{\mathrm{A}_2=\mathrm{rot}_{\Gamma_t}} \mathsf{D}_{\Gamma_t}(\Omega) \xrightarrow{\mathrm{A}_3=\mathrm{div}_{\Gamma_t}} \mathsf{L}^2(\Omega) \xrightarrow{\mathrm{A}_4=\pi\ldots} \cdots$$

$$\cdots \xleftarrow{\mathrm{A}_0^*=\pi\ldots} \mathsf{L}^2(\Omega) \xleftarrow{\mathrm{A}_1^*=-\mathrm{div}_{\Gamma_n}} \mathsf{D}_{\Gamma_n}(\Omega) \xleftarrow{\mathrm{A}_2^*=\mathrm{rot}_{\Gamma_n}} \mathsf{R}_{\Gamma_n}(\Omega) \xleftarrow{\mathrm{A}_3^*=-\mathrm{grad}_{\Gamma_n}} \mathsf{H}^1_{\Gamma_n}(\Omega) \xleftarrow{\mathrm{A}_4^*=\iota\ldots} \cdots$$

E.g., we can handle the systems

$$\begin{aligned} \mathrm{rot}_{\Gamma_t} E &= F, & \mathrm{rot}_{\Gamma_n}\mathrm{rot}_{\Gamma_t} E &= F, & -\mathrm{grad}_{\Gamma_n}\mathrm{div}_{\Gamma_t} E &= F,\\ -\mathrm{div}_{\Gamma_n} E &= g, & -\mathrm{div}_{\Gamma_n} E &= g, & \mathrm{rot}_{\Gamma_n} E &= G, \end{aligned}$$

or

$$\begin{aligned} -\mathrm{div}_{\Gamma_n}\mathrm{grad}_{\Gamma_t} u &= f, & \mathrm{rot}_{\Gamma_n}\mathrm{rot}_{\Gamma_t} E &= F,\\ & & -\mathrm{grad}_{\Gamma_t}\mathrm{div}_{\Gamma_n} E &= G. \end{aligned}$$

- generalized electro-magnetics (differential forms)

$$\cdots \xrightarrow{\mathrm{A}_0=\iota\ldots} \mathsf{D}^0_{\Gamma_t}(\Omega) \xrightarrow{\mathrm{A}_1=\mathrm{d}_{\Gamma_t}} \cdots \xrightarrow{\mathrm{A}_{q-1}=\mathrm{d}_{\Gamma_t}} \mathsf{D}^{q-1}_{\Gamma_t}(\Omega) \xrightarrow{\mathrm{A}_q=\mathrm{d}_{\Gamma_t}} \mathsf{D}^q_{\Gamma_t}(\Omega) \xrightarrow{\mathrm{A}_{q+1}=\mathrm{d}_{\Gamma_t}} \cdots \xrightarrow{\mathrm{A}_N=\mathrm{d}_{\Gamma_t}} \mathsf{L}^{2,N}(\Omega) \xrightarrow{\mathrm{A}_{N+1}=\pi\ldots} \cdots$$

$$\cdots \xleftarrow{\mathrm{A}_0^*=\pi\ldots} \mathsf{L}^{2,0}(\Omega) \xleftarrow{\mathrm{A}_1^*=-\delta_{\Gamma_n}} \cdots \xleftarrow{\mathrm{A}_{q-1}^*=-\delta_{\Gamma_n}} \Delta^{q-1}_{\Gamma_n}(\Omega) \xleftarrow{\mathrm{A}_q^*=-\delta_{\Gamma_n}} \Delta^q_{\Gamma_n}(\Omega) \xleftarrow{\mathrm{A}_{q+1}^*=-\delta_{\Gamma_n}} \cdots \xleftarrow{\mathrm{A}_N^*=-\delta_{\Gamma_n}} \mathsf{D}^N_{\Gamma_n}(\Omega) \xleftarrow{\mathrm{A}_{N+1}^*=\iota\ldots} \cdots$$

E.g., we can handle the systems

$$\begin{aligned} \mathrm{d}_{\Gamma_t} E &= F, & -\delta_{\Gamma_n}\mathrm{d}_{\Gamma_t} E &= F, & \mathrm{d}_{\Gamma_t} E &= F, & -\delta_{\Gamma_n}\mathrm{d}_{\Gamma_t} E &= F,\\ -\delta_{\Gamma_n} E &= G, & -\delta_{\Gamma_n} E &= G, & -\mathrm{d}_{\Gamma_t}\delta_{\Gamma_n} E &= G, & -\mathrm{d}_{\Gamma_t}\delta_{\Gamma_n} E &= G. \end{aligned}$$

- biharmonic problems, Stokes problems, and general relativity

$$\cdots \xrightarrow{\mathrm{A}_0=\iota\ldots} \mathsf{H}^2_{\Gamma_t}(\Omega) \xrightarrow{\mathrm{A}_1=\mathrm{Grad}\,\mathrm{grad}_{\Gamma_t}} \mathsf{R}_{\Gamma_t}(\Omega;\mathbb{S}) \xrightarrow{\mathrm{A}_2=\mathrm{Rot}_{\mathbb{S},\Gamma_t}} \mathsf{D}_{\Gamma_t}(\Omega;\mathbb{T}) \xrightarrow{\mathrm{A}_3=\mathrm{Div}_{\mathbb{T},\Gamma_t}} \mathsf{L}^2(\Omega) \xrightarrow{\mathrm{A}_4=\pi\ldots} \cdots$$

$$\cdots \xleftarrow{A_0^*=\pi\ldots} \mathsf{L}^2(\Omega) \xleftarrow{A_1^*=\operatorname{div}\operatorname{Div}_{\mathbb{S},\Gamma_n}} \mathsf{DD}_{\Gamma_n}(\Omega;\mathbb{S}) \xleftarrow{A_2^*=\operatorname{sym}\operatorname{Rot}_{\mathbb{T},\Gamma_n}} \mathsf{R}_{\mathrm{sym},\Gamma_n}(\Omega;\mathbb{T}) \xleftarrow{A_3^*=-\operatorname{dev}\operatorname{Grad}_{\Gamma_n}} \mathsf{H}^1_{\Gamma_n}(\Omega) \xleftarrow{A_4^*=\iota\ldots} \cdots$$

E.g., we can handle the systems

$$\begin{aligned} \operatorname{Rot}_{\mathbb{S},\Gamma_t} S &= F, & \operatorname{Div}_{\mathbb{T},\Gamma_t} T &= F, \\ \operatorname{div}\operatorname{Div}_{\mathbb{S},\Gamma_n} S &= g, & \operatorname{sym}\operatorname{Rot}_{\mathbb{T},\Gamma_n} T &= G, \end{aligned}$$

or

$$\begin{aligned} \operatorname{symRot}_{\mathbb{T},\Gamma_n}\operatorname{Rot}_{\mathbb{S},\Gamma_t} S &= F, & -\operatorname{devGrad}_{\Gamma_n}\operatorname{Div}_{\mathbb{T},\Gamma_t} T &= F, \\ \operatorname{divDiv}_{\mathbb{S},\Gamma_n} S &= g, & \operatorname{symRot}_{\mathbb{T},\Gamma_n} T &= G, \end{aligned}$$

or

$$\begin{aligned} \operatorname{Rot}_{\mathbb{S},\Gamma_t} S &= F, & \operatorname{Div}_{\mathbb{T},\Gamma_t} T &= F, \\ \operatorname{Grad}\ \operatorname{grad}_{\Gamma_t}\operatorname{div}\ \operatorname{Div}_{\mathbb{S},\Gamma_n} S &= G, & \operatorname{Rot}_{\mathbb{S},\Gamma_t}\operatorname{sym}\ \operatorname{Rot}_{\mathbb{T},\Gamma_n} T &= G, \end{aligned}$$

or

$$\operatorname{div}\operatorname{Div}_{\mathbb{S},\Gamma_n}\operatorname{Gradgrad}_{\Gamma_t} u = f, \qquad -\operatorname{Div}_{\mathbb{T},\Gamma_t}\operatorname{devGrad}_{\Gamma_n} E = F.$$

- linear elasticity

$$\cdots \xrightarrow{A_0=\iota\ldots} \mathsf{H}^1_{\Gamma_t}(\Omega) \xrightarrow{A_1=\operatorname{sym}\operatorname{Grad}_{\Gamma_t}} \mathsf{RR}^{\top}_{\Gamma_t}(\Omega;\mathbb{S}) \xrightarrow{A_2=\operatorname{RotRot}^{\top}_{\mathbb{S},\Gamma_t}} \mathsf{D}_{\Gamma_t}(\Omega;\mathbb{S}) \xrightarrow{A_3=\operatorname{Div}_{\mathbb{S},\Gamma_t}} \mathsf{L}^2(\Omega) \xrightarrow{A_4=\pi\ldots} \cdots$$

$$\cdots \xleftarrow{A_0^*=\pi\ldots} \mathsf{L}^2(\Omega) \xleftarrow{A_1^*=-\operatorname{Div}_{\mathbb{S},\Gamma_n}} \mathsf{D}_{\Gamma_n}(\Omega;\mathbb{S}) \xleftarrow{A_2^*=\operatorname{RotRot}^{\top}_{\mathbb{S},\Gamma_n}} \mathsf{RR}^{\top}_{\Gamma_n}(\Omega;\mathbb{S}) \xleftarrow{A_3^*=-\operatorname{sym}\operatorname{Grad}_{\Gamma_n}} \mathsf{H}^1_{\Gamma_n}(\Omega) \xleftarrow{A_4^*=\iota\ldots} \cdots$$

E.g., we can handle the systems

$$\begin{aligned} \operatorname{RotRot}^{\top}_{\mathbb{S},\Gamma_t} S &= F, & \operatorname{RotRot}^{\top}_{\mathbb{S},\Gamma_n}\operatorname{RotRot}^{\top}_{\mathbb{S},\Gamma_t} S &= F, & \operatorname{RotRot}^{\top}_{\mathbb{S},\Gamma_t} S &= F, \\ -\operatorname{Div}_{\mathbb{S},\Gamma_n} S &= G, & -\operatorname{Div}_{\mathbb{S},\Gamma_n} S &= G, & -\operatorname{symGrad}_{\Gamma_t}\operatorname{Div}_{\mathbb{S},\Gamma_n} S &= G, \end{aligned}$$

or

$$\begin{aligned} \operatorname{RotRot}^{\top}_{\mathbb{S},\Gamma_n}\operatorname{RotRot}^{\top}_{\mathbb{S},\Gamma_t} S &= F, & -\operatorname{Div}_{\mathbb{S},\Gamma_n}\operatorname{sym}\ \operatorname{Grad}_{\Gamma_t} E &= G, \\ -\operatorname{sym}\ \operatorname{Grad}_{\Gamma_t}\operatorname{Div}_{\mathbb{S},\Gamma_n} S &= G. & & \end{aligned}$$

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