Mathematical Methods in the Applied Sciences

WILEY

RESEARCH ARTICLE OPEN ACCESS

Shape Derivatives of the Eigenvalues of the De Rham Complex for Lipschitz Deformations and Variable Coefficients: Part I

Pier Domenico Lamberti[1] | Dirk Pauly[2] | Michele Zaccaron[3]

[1]Dipartimento di Tecnica e Gestione dei Sistemi Industriali, University of Padova, Italy | [2]Institut für Analysis, Technische Universität Dresden, Germany | [3]Institut Fresnel, Aix Marseille Univ, CNRS, Centrale Marseille, Marseille, France

Correspondence: Dirk Pauly (dirk.pauly@tu-dresden.de)

Received: 15 July 2025 | **Accepted:** 4 December 2025

Keywords: de Rham complex | eigenvalues | shape derivatives

ABSTRACT

We study eigenvalue problems for the de Rham complex on varying three-dimensional domains. Our analysis includes the Helmholtz equation as well as the Maxwell system with mixed boundary conditions and non-constant coefficients. We provide Hadamard-type formulas for the shape derivatives under weak regularity assumptions on the domain and its perturbations. Our proofs are based on abstract results adapted to varying Hilbert complexes. As a byproduct of our analysis, we give a proof of the celebrated Hellmann–Feynman theorem both for simple and multiple eigenvalues of suitable families of self-adjoint operators in Hilbert space depending on possibly infinite dimensional parameters. This series of papers consists of Parts I and II.

1 | Introduction

The analysis of the dependence of the eigenvalues and eigenfunctions of elliptic operators upon variation of the underlying domain is a classical problem considered in many papers in the literature with applications in approximation, optimization, homogenization, control theory, and mathematical physics. It is impossible to give an account of all contributions in the literature and we refer to the monograph [1] for an introduction to this topic in particular to the method of transplantation used in this paper. Needless to say that the Laplace operator and other second-order partial differential equations have received much more attention than higher order operators and systems, the analysis of which often leads to various technical and theoretical obstructions as well as paradoxes, see for instance [2–6] for polyharmonic operators and to [7, 8] for elliptic systems. From this point of view, the case of the Maxwell system has been investigated even less, cf. [9–14]. In particular, we note that differentiability results and Hadamard-type formulas for shape derivatives are proved in [10, 12, 14, 15] under suitable regularity assumptions on the domains and the corresponding perturbations.

The main aim of the present series of papers is to relax those regularity assumptions gaining one degree of smoothness and to provide a unified approach including both the Helmholz equation and the Maxwell system. This is done by analyzing the corresponding de Rham complex and its domain perturbations. A further contribution of our papers consists in the fact that we consider nonconstant physical parameters such as the electric permittivity ε and the magnetic permeability μ. In particular, we give a rigorous proof of a formula found by Hiromasa Hirakawa in [16], pp. 91–93 which is a Hadamard-type formula for the Maxwell system. Moreover, we consider the general case of mixed Dirichlet–Neumann boundary conditions.

We note that the proof of the Hadamard formulas can be obtained at a formal level by applying the Hellmann–Feynman theorem, a classical result is quantum mechanics that reduces here to a straightforward differentiation of the Rayleigh quotients depending on a parameter (see [17] for a recent discussion on this theorem and references). However, to discuss the dependence of the eigenvalues on infinite dimensional parameters and to consider multiple eigenvalues, we follow the approach developed in [18] and in particular we consider the elementary symmetric functions of the eigenvalues since these functions depend smoothly on the parameters as simple eigenvalues do. The results in [18] concern general families of compact selfadjoint operators in Hilbert space with variable scalar product and are applied in [12] to the Maxwell problem. To do so, the authors of [12] have to consider a penalized problem which requires $\mathsf{C}^{1,1}$ regularity assumptions on the domain perturbations. Here, to consider domain perturbations of class $\mathsf{C}^{0,1}$, we do not penalize the problem but this prevents us from using the results of [18] in a direct way because the operators under consideration are selfadjoint but not compact.[1] Thus, we are forced to give new proofs of abstract theorems concerning families of selfadjoint operators in Hilbert space. As a byproduct of our analysis, we provide a proof of a general version of the Hellmann–Feynman theorem for families of operators suitable for de Rham complexes in Hilbert spaces, see Part II of this paper at hand.

This Part I of the paper series is organized as follows. Section 2 is devoted to notations and preliminaries on the Functional Analysis Toolbox. Section 3 is devoted to the analysis of the eigenvalue problem for the de Rham complex on transplanted domains. In Section 4, we conclude this first part with some formal computations to derive the shape derivatives of the eigenvalues assuming that those are simple and the corresponding eigenvectors are differentiable.

In Part II of this series of papers, we present Hadamard type formulas and related findings obtained by a direct application of the Hellmann–Feynman theorem together with sound proofs of all results.

We conclude this introduction with two subsections where we highlight the main problems under consideration and briefly discuss the approach of domain transplantation used in this paper.

Until stated otherwise, let Ω be a *bounded open set* in $\mathbb{R}^3$ with boundary Γ and let $\lambda_0, \lambda_1 > 0$. Moreover, let $\nu \in \mathsf{L}^\infty(\Omega, \mathbb{R})$ be positive with respect to the $\mathsf{L}^2(\Omega)$-inner product, and let ε and μ be *admissible* symmetric matrix fields, that is, ε and μ belong to $\mathsf{L}^\infty(\Omega, \mathbb{R}^{3\times3}_{\mathrm{sym}})$ and are positive with respect to the $\mathsf{L}^2(\Omega)$-inner product, cf. [19–22]. The required regularity of $\Gamma, \nu, \varepsilon, \mu$ will be specified along the paper.

1.1 | Eigenvalues of the De Rham Complex

We shall consider the Dirichlet Maxwell eigenvalue problem

$$\begin{aligned} \varepsilon^{-1} \operatorname{rot} \mu^{-1} \operatorname{rot} E &= \lambda_1 E \quad && \text{in } \Omega, \\ \mathrm{n} \times E &= 0 && \text{on } \Gamma. \end{aligned} \tag{1}$$

The corresponding Neumann Maxwell eigenvalue problem reads

$$\begin{aligned} \varepsilon^{-1} \operatorname{rot} \mu^{-1} \operatorname{rot} E &= \lambda_1 E \quad && \text{in } \Omega, \\ \mathrm{n} \times \mu^{-1} \operatorname{rot} E &= 0 && \text{on } \Gamma. \end{aligned} \tag{2}$$

Note that any solution of (1) or (2) automatically satisfies $\operatorname{div} \varepsilon E = 0$ in Ω, and that in (1) and (2) we have additional $\mathrm{n} \cdot \operatorname{rot} E = 0$ and $\mathrm{n} \cdot \varepsilon E = 0$ on Γ, respectively.

We investigate also mixed boundary conditions, that is, the Maxwell eigenvalue problem with mixed Dirichlet/Neumann boundary conditions

$$\begin{aligned} \varepsilon^{-1} \operatorname{rot} \mu^{-1} \operatorname{rot} E = \lambda_1 E \quad & \text{in } \Omega, \\ \mathrm{n} \times E = 0 \quad & \text{on } \Gamma_t, \\ \mathrm{n} \times \mu^{-1} \operatorname{rot} E = 0 \quad & \text{on } \Gamma_n, \end{aligned} \tag{3}$$

where Γ is decomposed into two relatively open subsets $\emptyset \subset \Gamma_t \subset \Gamma$ and $\Gamma_n := \Gamma \setminus \overline{\Gamma_t}$. Note that again we have $\operatorname{div} \varepsilon E = 0$ and $\mathrm{n} \cdot \operatorname{rot} E|_{\Gamma_t} = 0$ and $\mathrm{n} \cdot \varepsilon E|_{\Gamma_n} = 0$.

Moreover, we shall discuss the full spectrum of the de Rham complex. Hence, we also investigate the scalar Laplacian and its dual, that is,

$$\begin{aligned} - \nu^{-1} \operatorname{div} \varepsilon \nabla u = \lambda_0 u \quad & \text{in } \Omega, \\ u = 0 \quad & \text{on } \Gamma_t, \\ \mathrm{n} \cdot \varepsilon \nabla u = 0 \quad & \text{on } \Gamma_n, \end{aligned} \tag{4}$$

and

$$\begin{aligned} - \nabla \nu^{-1} \operatorname{div} \varepsilon H = \lambda_0 H \quad & \text{in } \Omega, \\ \nu^{-1} \operatorname{div} \varepsilon H = 0 \quad & \text{on } \Gamma_t, \\ \mathrm{n} \cdot \varepsilon H = 0 \quad & \text{on } \Gamma_n. \end{aligned} \tag{5}$$

As in (3) it holds $\operatorname{rot} H = 0$ and $\mathrm{n} \times H|_{\Gamma_t} = 0$ but only $\int_\Gamma \nu^{-1} \operatorname{div} \varepsilon H = 0$ if $\Gamma_n = \Gamma$.

In view of (3) and (5), we shall also discuss the generalized vector Laplacian

$$\begin{aligned} (\varepsilon^{-1} \operatorname{rot} \mu^{-1} \operatorname{rot} - \nabla \nu^{-1} \operatorname{div} \varepsilon) E = \lambda_{0,1} E \quad & \text{in } \Omega, \qquad \lambda_{0,1} \in \{\lambda_0, \lambda_1\}, \\ \mathrm{n} \times E = 0, \quad \nu^{-1} \operatorname{div} \varepsilon E = 0 \quad & \text{on } \Gamma_t, \\ \mathrm{n} \times \mu^{-1} \operatorname{rot} E = 0, \quad \mathrm{n} \cdot \varepsilon E = 0 \quad & \text{on } \Gamma_n. \end{aligned} \tag{6}$$

Note that for ε, μ, and ν being the identity mappings we have

$$\varepsilon^{-1} \operatorname{rot} \mu^{-1} \operatorname{rot} - \nabla \nu^{-1} \operatorname{div} \varepsilon = \operatorname{rot} \operatorname{rot} - \nabla \operatorname{div} = -\vec{\Delta}.$$

1.2 | Shape Derivatives of Eigenvalues

We intend to study variations of the domain and the boundary conditions by replacing Ω and the boundary parts Γ_t, Γ_n with

$$\Omega_\Phi := \Phi(\Omega), \qquad \Gamma_\Phi := \Phi(\Gamma), \qquad \Gamma_{t,\Phi} := \Phi(\Gamma_t), \qquad \Gamma_{n,\Phi} := \Phi(\Gamma_n),$$

respectively, where

$$\Phi : \Omega \to \Omega_\Phi$$

is a bi-Lipschitz transformation. In particular, for $\ell \in \{0, 1\}$, we are interested in the variations of the eigenvalues

$$0 < \lambda_{\ell,1}(\Phi) \le \lambda_{\ell,2}(\Phi) \le \cdots < \lambda_{\ell,k-1}(\Phi) \le \lambda_{\ell,k}(\Phi) \le \cdots \to \infty$$

in the domain Ω_Φ and their elementary symmetric functions with respect to changing transformations Φ.

For simplicity, assume here that ε, μ are the identity matrices and that $\nu = 1$. Let $\lambda_{0,k}(\Phi)$ and $\lambda_{1,k}(\Phi)$ be eigenvalues with eigenvectors u and E of (4) and (3), respectively. As is well-known these eigenvalues can be written by means of Rayleigh quotients as

$$\lambda_{0,k}(\Phi) = \frac{|\nabla u|^2_{\mathrm{L}^2(\Omega_\Phi)}}{|u|^2_{\mathrm{L}^2(\Omega_\Phi)}}, \qquad \lambda_{1,k}(\Phi) = \frac{|\operatorname{rot} E|^2_{\mathrm{L}^2(\Omega_\Phi)}}{|E|^2_{\mathrm{L}^2(\Omega_\Phi)}}. \tag{7}$$

In particular, assuming that the eigenvectors are normalized in $\mathsf{L}^2(\Omega_\Phi)$ we have

$$\lambda_{0,k}(\Phi) = |\nabla u|^2_{\mathsf{L}^2(\Omega_\Phi)}, \qquad \lambda_{1,k}(\Phi) = |\operatorname{rot} E|^2_{\mathsf{L}^2(\Omega_\Phi)}.$$

Then also the dual eigenvectors

$$H := \lambda_{0,k}^{-1/2}(\Phi)\nabla u, \qquad B := \lambda_{1,k}^{-1/2}(\Phi) \operatorname{rot} E$$

are $\mathsf{L}^2(\Omega_\Phi)$-normalized eigenvectors of (5) and the respective dual of (3). Moreover, we have the dualities

$$u = -\lambda_{0,k}^{-1/2}(\Phi)\operatorname{div} H, \qquad E = \lambda_{1,k}^{-1/2}(\Phi) \operatorname{rot} B,$$
$$\lambda_{0,\Phi} = |\operatorname{div} H|^2_{\mathsf{L}^2(\Omega_\Phi)}, \qquad \lambda_{1,\Phi} = |\operatorname{rot} B|^2_{\mathsf{L}^2(\Omega_\Phi)}.$$

Note that the dual of (3) reads

$$\begin{aligned} \mu^{-1} \operatorname{rot} \varepsilon^{-1} \operatorname{rot} B &= \lambda_1 B && \text{in } \Omega, \\ \mathrm{n} \times \varepsilon^{-1} \operatorname{rot} B &= 0 && \text{on } \Gamma_t, \\ \mathrm{n} \times B &= 0 && \text{on } \Gamma_n, \end{aligned} \tag{8}$$

which incorporates also the conditions $\operatorname{div} \mu B = 0$ and $\mathrm{n} \cdot \mu B|_{\Gamma_t} = 0$ and $\mathrm{n} \cdot \operatorname{rot} B|_{\Gamma_n} = 0$.

In this paper, among other results, we prove Hadamard type formulas for the directional derivatives of the maps $\Phi \mapsto \lambda_{l,k}(\Phi)$ for $l \in \{0, 1\}$. This means that, given a fixed direction $\widetilde{\Psi}$ in the space of Lipschitz transformations Φ, we compute the limit

$$\partial_{\widetilde{\Psi}} \lambda_{l,k}(\Phi) = \lim_{h\to 0} \frac{\lambda_{l,k}(\Phi + h\widetilde{\Psi}) - \lambda_{l,k}(\Phi)}{h}. \tag{9}$$

Recall $\partial_{\widetilde{\Psi}} \lambda_{l,k}(\Phi) = \lambda'_{l,k}(\Phi)\widetilde{\Psi}$. As customary, it is convenient to consider $\widetilde{\Psi}$ as the pull-back of a transformation Ψ defined on Ω_Φ, that is $\widetilde{\Psi} = \Psi \circ \Phi$, and to express the formulas for the derivatives as volume or boundary integrals on Ω_Φ. At a formal level, assuming the differentiability of the eigenvalues and eigenvectors with respect to Φ (which might fail for multiple eigenvalues, cf. Part II), the Hellmann–Feynman theorem allows to obtain the formulas for $\partial_{\widetilde{\Psi}} \lambda_{l,k}(\Phi)$ by differentiating the Rayleigh quotients (7) with respect to Φ (in direction $\widetilde{\Psi}$ and keeping fixed the eigenvectors involved). By doing so, we obtain

$$-\frac{\partial_{\widetilde{\Psi}} \lambda_{0,k}(\Phi)}{\lambda_{0,k}(\Phi)} = \langle (\operatorname{symtr} \nabla\Psi) H, H\rangle_{\mathsf{L}^2(\Omega_\Phi)} + \langle (\operatorname{div} \Psi) u, u\rangle_{\mathsf{L}^2(\Omega_\Phi)}, \tag{10a}$$
$$\frac{\partial_{\widetilde{\Psi}} \lambda_{1,k}(\Phi)}{\lambda_{1,k}(\Phi)} = \langle (\operatorname{symtr} \nabla\Psi) B, B\rangle_{\mathsf{L}^2(\Omega_\Phi)} + \langle (\operatorname{symtr} \nabla\Psi) E, E\rangle_{\mathsf{L}^2(\Omega_\Phi)}, \tag{10b}$$

cf. (31) and Part II, where

$$\operatorname{symtr} \nabla\Psi := 2 \operatorname{sym} \nabla\Psi - \operatorname{tr} \nabla\Psi = \nabla\Psi + (\nabla\Psi)^\top - \operatorname{div} \Psi.$$

Then, under more regularity assumptions on eigenvectors, it is possible to integrate by parts and write these formulas by means of surface integrals as follows:

$$\frac{\partial_{\widetilde{\Psi}} \lambda_{0,k}(\Phi)}{\lambda_{0,k}(\Phi)} = \int_{\Gamma_{n,\Phi}} \big(|H|^2 - |u|^2\big)\Psi \cdot n \, d\sigma - \int_{\Gamma_{t,\Phi}} \big(|H|^2 - |u|^2\big)\Psi \cdot n \, d\sigma, \tag{11a}$$
$$\frac{\partial_{\widetilde{\Psi}} \lambda_{1,k}(\Phi)}{\lambda_{1,k}(\Phi)} = \int_{\Gamma_{n,\Phi}} \big(|B|^2 - |E|^2\big)\Psi \cdot n \, d\sigma - \int_{\Gamma_{t,\Phi}} \big(|B|^2 - |E|^2\big)\Psi \cdot n \, d\sigma, \tag{11b}$$

see Part II. These computations are quite involved.

Note that formula (11a) is well known at least for non-mixed boundary conditions, cf. for example [18, 23], and formula (11b) has been recently proved in [12, 15] for sufficiently regular perturbations. Formula (11b) was found in a heuristic way in [16] with $\Gamma_n = \emptyset$ for arbitrary ε and μ and another interesting equivalent formula was proved in [10], see also [14].

It is important to observe that the left and right derivatives in (9), that is, $h \to 0^{\mp}$, coincide if the eigenvalue under consideration is simple, while they might be different if the eigenvalue is multiple (the difference corresponds to the choice of different eigenvectors in the formulas). This phenomenon is well-known for many eigenvalue problems associated with families of self-adjoint operators depending on some parameters. It is also well-known that for perturbations depending on one scalar parameter, it is possible to apply the Rellich theorem and relabel the eigenvalues to guarantee their differentiability. On the other hand, it was proved in [18] that the elementary symmetric functions of the eigenvalues which bifurcate at a multiple eigenvalue are differentiable no matter whether the parameter involved is one dimensional or not.

2 | Preliminaries

2.1 | Sobolev Spaces and Boundary Conditions

Let $k \in \mathbb{N}_0 \cup \{\infty\}$. We define (for scalar, vector, or tensor fields)

$$\mathsf{C}^k_{\Gamma_t}(\Omega) := \{\psi|_\Omega \ : \ \psi \in \mathsf{C}^k(\mathbb{R}^3),\ \operatorname{supp}\psi \text{ compact},\ \operatorname{dist}(\operatorname{supp}\psi, \Gamma_t) > 0\},$$
$$\mathsf{C}^{0,1}_{\Gamma_t}(\Omega) := \{\psi|_\Omega \ : \ \psi \in \mathsf{C}^{0,1}(\mathbb{R}^3),\ \operatorname{supp}\psi \text{ compact},\ \operatorname{dist}(\operatorname{supp}\psi, \Gamma_t) > 0\}.$$

Recall that Γ_t is a relatively open subset of Γ. Note that $\mathsf{C}^k_{\emptyset}(\Omega)$ and $\mathsf{C}^{0,1}_{\emptyset}(\Omega)$ are often denoted by $\mathsf{C}^k(\overline{\Omega})$ and $\mathsf{C}^{0,1}(\overline{\Omega})$, respectively. With the Lebesgue space $\mathsf{L}^2(\Omega)$ we have the standard Sobolev spaces in the weak sense

$$\begin{aligned}
\mathbf{H}^k(\Omega) &:= \{\psi \in \mathsf{L}^2(\Omega) \ : \ \partial^\alpha\psi \in \mathsf{L}^2(\Omega)\ \forall\, |\alpha| \le k\} \\
&= \{\psi \in \mathsf{L}^2(\Omega) \ : \ \forall\, |\alpha| \le k\ \exists \Psi_\alpha \in \mathsf{L}^2(\Omega)\ \forall\, \theta \in \mathsf{C}^\infty_\Gamma(\Omega) \quad \langle \psi, \partial^\alpha\theta\rangle_{\mathsf{L}^2(\Omega)} = (-1)^{|\alpha|}\langle \psi_\alpha, \theta\rangle_{\mathsf{L}^2(\Omega)}\}, \\
\mathbf{R}(\Omega) &:= \{\Psi \in \mathsf{L}^2(\Omega) \ : \ \operatorname{rot}\Psi \in \mathsf{L}^2(\Omega)\} \\
&= \{\Psi \in \mathsf{L}^2(\Omega) \ : \ \exists \Psi_{\mathrm{rot}} \in \mathsf{L}^2(\Omega)\ \forall\, \Theta \in \mathsf{C}^\infty_\Gamma(\Omega) \quad \langle \Psi, \operatorname{rot}\Theta\rangle_{\mathsf{L}^2(\Omega)} = \langle \Psi_{\mathrm{rot}}, \Theta\rangle_{\mathsf{L}^2(\Omega)}\}, \\
\mathbf{D}(\Omega) &:= \{\Psi \in \mathsf{L}^2(\Omega) \ : \ \operatorname{div}\Psi \in \mathsf{L}^2(\Omega)\} \\
&= \{\Psi \in \mathsf{L}^2(\Omega) \ : \ \exists \psi_{\mathrm{div}} \in \mathsf{L}^2(\Omega)\ \forall\, \theta \in \mathsf{C}^\infty_\Gamma(\Omega) \quad \langle \Psi, \nabla\theta\rangle_{\mathsf{L}^2(\Omega)} = -\langle \psi_{\mathrm{div}}, \theta\rangle_{\mathsf{L}^2(\Omega)}\}.
\end{aligned}$$

Note that $\mathbf{R}(\Omega)$ and $\mathbf{D}(\Omega)$ are the well-known spaces $\mathrm{H}(\mathrm{rot}\,, \Omega)$ and $\mathbf{H}(\mathrm{div}, \Omega)$, respectively. We introduce boundary conditions in the strong sense by

$$\mathsf{H}^k_{\Gamma_t}(\Omega) := \overline{\mathsf{C}^\infty_{\Gamma_t}(\Omega)}^{\mathbf{H}^k(\Omega)}, \qquad \mathsf{R}_{\Gamma_t}(\Omega) := \overline{\mathsf{C}^\infty_{\Gamma_t}(\Omega)}^{\mathbf{R}(\Omega)}, \qquad \mathsf{D}_{\Gamma_t}(\Omega) := \overline{\mathsf{C}^\infty_{\Gamma_t}(\Omega)}^{\mathbf{D}(\Omega)}.$$

By standard Friedrichs' mollification, we observe

$$\mathsf{H}^1_{\Gamma_t}(\Omega) = \overline{\mathsf{C}^{0,1}_{\Gamma_t}(\Omega)}^{\mathbf{H}^1(\Omega)}, \qquad \mathsf{R}_{\Gamma_t}(\Omega) = \overline{\mathsf{C}^{0,1}_{\Gamma_t}(\Omega)}^{\mathbf{R}(\Omega)}, \qquad \mathsf{D}_{\Gamma_t}(\Omega) = \overline{\mathsf{C}^{0,1}_{\Gamma_t}(\Omega)}^{\mathbf{D}(\Omega)}.$$

Also boundary conditions in the weak sense are introduced by

$$\begin{aligned}
\mathbf{H}^k_{\Gamma_t}(\Omega) &:= \left\{\psi \in \mathbf{H}^k(\Omega) \ : \ \forall\, |\alpha| \le k\ \forall\, \theta \in \mathsf{C}^\infty_{\Gamma_n}(\Omega) \quad \langle \psi, \partial^\alpha\theta\rangle_{\mathsf{L}^2(\Omega)} = (-1)^{|\alpha|}\langle \partial^\alpha, \theta\rangle_{\mathsf{L}^2(\Omega)}\right\}, \\
\mathbf{R}_{\Gamma_t}(\Omega) &:= \left\{\Psi \in \mathbf{R}(\Omega) \ : \ \forall\, \Theta \in \mathsf{C}^\infty_{\Gamma_n}(\Omega) \quad \langle \Psi, \operatorname{rot}\Theta\rangle_{\mathsf{L}^2(\Omega)} = \langle \operatorname{rot}\Psi, \Theta\rangle_{\mathsf{L}^2(\Omega)}\right\}, \\
\mathbf{D}_{\Gamma_t}(\Omega) &:= \left\{\Psi \in \mathbf{D}(\Omega) \ : \ \forall\, \theta \in \mathsf{C}^\infty_{\Gamma_n}(\Omega) \quad \langle \Psi, \nabla\theta\rangle_{\mathsf{L}^2(\Omega)} = -\langle \operatorname{div}\Psi, \theta\rangle_{\mathsf{L}^2(\Omega)}\right\}.
\end{aligned}$$

Note that for $\Gamma_t = \emptyset$ we have

$$\mathbf{H}^k_\emptyset(\Omega) = \mathbf{H}^k(\Omega), \qquad \mathbf{R}_\emptyset(\Omega) = \mathbf{R}(\Omega), \qquad \mathbf{D}_\emptyset(\Omega) = \mathbf{D}(\Omega).$$

2.2 | Weak and Strong Boundary Conditions Coincide

For full boundary conditions there is a simple result that "weak equals strong" holds without any additional assumptions in a certain sense (the test fields can be chosen from a possibly larger space). Unfortunately, the proof does not allow for mixed boundary conditions.

Lemma 2.1 (Weak equals strong for full boundary conditions). *It holds*

$$\begin{aligned}
\mathsf{H}^1_\Gamma(\Omega) &= \left\{\psi \in \mathbf{H}^1(\Omega) \ : \ \forall\, \Theta \in \mathbf{D}(\Omega) \quad \langle \psi, \operatorname{div}\Theta\rangle_{\mathsf{L}^2(\Omega)} = -\langle \nabla\psi, \theta\rangle_{\mathsf{L}^2(\Omega)}\right\},\\
\mathsf{R}_\Gamma(\Omega) &= \left\{\Psi \in \mathbf{R}(\Omega) \ : \ \forall\, \Theta \in \mathbf{R}(\Omega) \quad \langle \Psi, \operatorname{rot}\Theta\rangle_{\mathsf{L}^2(\Omega)} = \langle \operatorname{rot}\Psi, \Theta\rangle_{\mathsf{L}^2(\Omega)}\right\},\\
\mathsf{D}_\Gamma(\Omega) &= \left\{\Psi \in \mathbf{D}(\Omega) \ : \ \forall\, \theta \in \mathbf{H}^1(\Omega) \quad \langle \Psi, \nabla\theta\rangle_{\mathsf{L}^2(\Omega)} = -\langle \operatorname{div}\Psi, \theta\rangle_{\mathsf{L}^2(\Omega)}\right\}.
\end{aligned}$$

See the appendix for a proof. For a definition of the segment property used in the following lemma we refer to [24].

Lemma 2.2 (weak equals strong for no boundary conditions/density of smooth fields). *Let Ω have additionally the segment property. Then*

$$\mathsf{H}^k_\emptyset(\Omega) = \mathbf{H}^k(\Omega), \qquad \mathsf{R}_\emptyset(\Omega) = \mathbf{R}(\Omega), \qquad \mathsf{D}_\emptyset(\Omega) = \mathbf{D}(\Omega).$$

In other words, $\mathsf{C}^\infty_\emptyset(\Omega) = \mathsf{C}^\infty(\overline{\Omega})$ is dense in $\mathbf{H}^k(\Omega)$, $\mathbf{R}(\Omega)$, and $\mathbf{D}(\Omega)$, respectively.

Proof. The proof for $\mathsf{H}^1(\Omega)$ can be found, for example, in Agmon [25] or in Wloka [26], Theorem 3.6, and it literally carries over to $\mathsf{R}(\Omega)$ and $\mathsf{D}(\Omega)$ as the mollifiers work similarly for rot and div. The result for $\mathsf{H}^k(\Omega)$ follows by induction. □

In case of Lemma 2.2, we set

$$\mathsf{H}^k(\Omega) := \mathsf{H}^k_\emptyset(\Omega) = \mathbf{H}^k(\Omega), \qquad \mathsf{R}(\Omega) := \mathsf{R}_\emptyset(\Omega) = \mathbf{R}(\Omega), \qquad \mathsf{D}(\Omega) := \mathsf{D}_\emptyset(\Omega) = \mathbf{D}(\Omega).$$

Lemma 2.3 (weak equals strong for full boundary conditions). *Let Ω have additionally the segment property. Then*

$$\mathsf{H}^k_\Gamma(\Omega) = \mathbf{H}^k_\Gamma(\Omega), \qquad \mathsf{R}_\Gamma(\Omega) = \mathbf{R}_\Gamma(\Omega), \qquad \mathsf{D}_\Gamma(\Omega) = \mathbf{D}_\Gamma(\Omega).$$

Proof. This follows by the same technique used in the proof of Lemma 2.1 in combination with the density results from Lemma 2.2, for example, $\mathbf{R}(\Omega) = \mathsf{R}(\Omega) = \mathsf{R}_\emptyset(\Omega) = \overline{\mathsf{C}^\infty_\emptyset(\Omega)}^{\mathsf{R}(\Omega)}$. □

For mixed boundary conditions, that is, $\emptyset \neq \Gamma_t \neq \Gamma$, the question "weak equals strong" is more delicate. The equality can be proved under the assumption that Ω has a Lipschitz boundary in the weak sense and Γ_t has a relative boundary in Γ which is also Lipschitz in the weak sense. In particular, Γ is a Lipschitz manifold of codimension one in $\mathbb{R}^3$ and the relative boundary of Γ_t in Γ is a Lipschitz submanifold of codimension one in Γ. In this case, we say that (Ω, Γ_t) is a weak Lipschitz pair. Recall that usually "Lipschitz in the weak sense" means that the open set can be locally flattened near the boundary by means of a Lipschitz diffeomorphism. This condition is weaker than "Lipschitz in the strong sense" in which case the open set can be locally represented near boundary as a subgraph of a Lipschitz function.

A proof of the following lemma and the precise definition of a weak Lipschitz pair can be found in [19] or [22].

Lemma 2.4 (Weak equals strong for mixed boundary consitions). *Let (Ω, Γ_t) be additionally a weak Lipschitz pair. Then*

$$\mathsf{H}^k_{\Gamma_t}(\Omega) = \mathbf{H}^k_{\Gamma_t}(\Omega), \qquad \mathsf{R}_{\Gamma_t}(\Omega) = \mathbf{R}_{\Gamma_t}(\Omega), \qquad \mathsf{D}_{\Gamma_t}(\Omega) = \mathbf{D}_{\Gamma_t}(\Omega).$$

2.3 | The Transformation Theorem

Let $\Phi \in \mathsf{C}^{0,1}(\mathbb{R}^3, \mathbb{R}^3)$ be such that

$$\Phi : \Omega \to \Phi(\Omega) = \Omega_\Phi$$

is bi-Lipschitz, and regular, that is, $\Phi \in \mathsf{C}^{0,1}(\overline{\Omega}, \overline{\Omega}_\Phi)$ and $\Phi^{-1} \in \mathsf{C}^{0,1}(\overline{\Omega}_\Phi, \overline{\Omega})$ with[2]

$$J_\Phi = \Phi' = (\nabla\Phi)^\top, \qquad \operatorname{ess\,inf} \det J_\Phi > 0.$$

Such regular bi-Lipschitz transformations will be called admissible and we write

$$\Phi \in \mathcal{L}(\Omega).$$

For $\Phi \in \mathcal{L}(\Omega)$ the inverse and adjunct matrix of J_Φ shall be denoted by

$$J_\Phi^{-1}, \qquad \operatorname{adj} J_\Phi := (\det J_\Phi) J_\Phi^{-1},$$

respectively. We denote the composition with Φ by tilde, that is, for any tensor field ψ we define

$$\widetilde{\psi} := \psi \circ \Phi.$$

Moreover, let

$$\Gamma_\Phi := \Phi(\Gamma), \qquad \Gamma_{t,\Phi} := \Phi(\Gamma_t), \qquad \Gamma_{n,\Phi} := \Phi(\Gamma_n).$$

A proof of the following theorem for differential forms can be found in the appendix of [28]. Here we focus on the special case used in this paper.

Theorem 2.1 (Transformation theorem). *Let $u \in \mathsf{H}^1_{\Gamma_{t,\Phi}}(\Omega_\Phi)$, $E \in \mathsf{R}_{\Gamma_{t,\Phi}}(\Omega_\Phi)$, and $H \in \mathsf{D}_{\Gamma_{t,\Phi}}(\Omega_\Phi)$. Then*

$$\tau^0_\Phi u := \widetilde{u} \in \mathsf{H}^1_{\Gamma_t}(\Omega) \quad \text{and} \quad \nabla \tau^0_\Phi u = \tau^1_\Phi \nabla u,$$
$$\tau^1_\Phi E := J_\Phi^\top \widetilde{E} \in \mathsf{R}_{\Gamma_t}(\Omega) \quad \text{and} \quad \operatorname{rot} \tau^1_\Phi E = \tau^2_\Phi \operatorname{rot} E,$$
$$\tau^2_\Phi H := (\operatorname{adj} J_\Phi)\widetilde{H} \in \mathsf{D}_{\Gamma_t}(\Omega) \quad \text{and} \quad \operatorname{div} \tau^2_\Phi H = \tau^3_\Phi \operatorname{div} H.$$

with $\tau^3_\Phi f := (\det J_\Phi)\tau^0_\Phi f = (\det J_\Phi)\widetilde{f} \in \mathsf{L}^2(\Omega)$ for $f \in \mathsf{L}^2(\Omega_\Phi)$. Moreover,

$$\tau^0_\Phi : \mathsf{H}^1_{\Gamma_{t,\Phi}}(\Omega_\Phi) \to \mathsf{H}^1_{\Gamma_t}(\Omega), \qquad \tau^1_\Phi : \mathsf{R}_{\Gamma_{t,\Phi}}(\Omega_\Phi) \to \mathsf{R}_{\Gamma_t}(\Omega),$$
$$\tau^3_\Phi : \mathsf{L}^2(\Omega_\Phi) \to \mathsf{L}^2(\Omega), \qquad \tau^2_\Phi : \mathsf{D}_{\Gamma_{t,\Phi}}(\Omega_\Phi) \to \mathsf{D}_{\Gamma_t}(\Omega)$$

are topological isomorphisms with norms depending on Ω and J_Φ. The inverse operators and the L^2-adjoints, that is, the Hilbert space adjoints of $\tau^q_\Phi : \mathsf{L}^2(\Omega_\Phi) \to \mathsf{L}^2(\Omega)$, are given by

$$(\tau^q_\Phi)^{-1} = \tau^q_{\Phi^{-1}}, \qquad (\tau^0_\Phi)^* = \tau^3_{\Phi^{-1}}, \qquad (\tau^1_\Phi)^* = \tau^2_{\Phi^{-1}}, \qquad (\tau^2_\Phi)^* = \tau^1_{\Phi^{-1}}, \qquad (\tau^3_\Phi)^* = \tau^0_{\Phi^{-1}},$$

respectively.

Proof. If $u \in \mathsf{C}^{0,1}_{\Gamma_{t,\Phi}}(\Omega_\Phi)$ we have by Rademacher's theorem $\widetilde{u} \in \mathsf{C}^{0,1}_{\Gamma_t}(\Omega)$ and the standard chain rule $(\widetilde{u})' = \widetilde{u'}\Phi'$ holds, that is,

$$\nabla\widetilde{u} = \nabla\Phi\widetilde{\nabla u} = J_\Phi^\top \widetilde{\nabla u}. \tag{12}$$

Then we use an approximation argument. For $u \in \mathsf{H}^1_{\Gamma_{t,\Phi}}(\Omega_\Phi)$ we pick a sequence $(u^\ell) \subset \mathsf{C}^{0,1}_{\Gamma_{t,\Phi}}(\Omega_\Phi)$ such that $u^\ell \to E$ in $\mathsf{H}^1_{\Gamma_{t,\Phi}}(\Omega_\Phi)$. Then $\widetilde{u^\ell} \to \widetilde{E}$ and $\widetilde{\nabla u^\ell} \to \widetilde{\nabla u}$ in $\mathsf{L}^2(\Omega)$ by the standard transformation theorem. By (12) we have $\widetilde{u^\ell} \in \mathsf{C}^{0,1}_{\Gamma_t}(\Omega) \subset \mathsf{H}^1_{\Gamma_t}(\Omega)$ with

$$\widetilde{u^\ell} \to \widetilde{u}, \quad \nabla\widetilde{u^\ell} = J_\Phi^\top \widetilde{\nabla u^\ell} \to J_\Phi^\top \widetilde{\nabla u} \qquad \text{in } \mathsf{L}^2(\Omega).$$

Since $\nabla_{\Gamma_t} : \mathsf{H}^1_{\Gamma_t}(\Omega) \subset \mathsf{L}^2(\Omega) \to \mathsf{L}^2(\Omega)$ is closed, we conclude $\widetilde{u} \in \mathsf{H}^1_{\Gamma_t}(\Omega)$ and

$$\nabla\widetilde{u} = J_\Phi^\top \widetilde{\nabla u}. \tag{13}$$

For the classical chain rule in Sobolev spaces see, for example, [18].

Assume $E \in \mathsf{C}^{0,1}_{\Gamma_{t,\Phi}}(\Omega_\Phi)$. Then $\widetilde{E} \in \mathsf{C}^{0,1}_{\Gamma_t}(\Omega)$ and

$$J_\Phi^\top \widetilde{E} = \nabla\Phi\widetilde{E} = [\nabla\Phi_1\ \nabla\Phi_2\ \nabla\Phi_3]\widetilde{E} = \sum_j \widetilde{E}_j \nabla\Phi_j.$$

As $\nabla\Phi_j \in \nabla\mathsf{H}^1(\Omega) \subset \mathsf{R}(\Omega)$ we conclude $J_\Phi^\top \widetilde{E} \in \mathsf{R}(\Omega)$ and

$$\begin{aligned}
\operatorname{rot}(J_\Phi^\top \widetilde{E}) &= \sum_j \nabla\widetilde{E}_j \times \nabla\Phi_j = \sum_j (J_\Phi^\top \widetilde{\nabla E_j}) \times \nabla\Phi_j \\
&= \sum_j \Big([\nabla\Phi_1\ \nabla\Phi_2\ \nabla\Phi_3]\ \widetilde{\nabla E_j} \Big) \times \nabla\Phi_j \\
&= \sum_{j,m} \widetilde{\partial_m E_j} \nabla\Phi_m \times \nabla\Phi_j = \sum_{j<m} (\widetilde{\partial_m E_j} - \widetilde{\partial_j E_m}) \nabla\Phi_m \times \nabla\Phi_j \\
&= [\nabla\Phi_2 \times \nabla\Phi_3 \quad \nabla\Phi_3 \times \nabla\Phi_1 \quad \nabla\Phi_1 \times \nabla\Phi_2]\ \widetilde{\operatorname{rot} E} = (\operatorname{adj} J_\Phi)\ \widetilde{\operatorname{rot} E}.
\end{aligned} \tag{14}$$

Moreover, by a mollification argument it follows that $J_\Phi^\top \widetilde{E} \in \mathsf{R}_{\Gamma_t}(\Omega)$. Again, the general case $E \in \mathsf{R}_{\Gamma_{t,\Phi}}(\Omega_\Phi)$ can be treated by an approximation argument. For this, we consider a sequence $(E^\ell)_{l\in\mathbb{N}} \subset \mathsf{C}^{0,1}_{\Gamma_{t,\Phi}}(\Omega_\Phi)$ such that $E^\ell \to E$ in $\mathsf{R}(\Omega_\Phi)$. Then $\widetilde{E^\ell} \to \widetilde{E}$ and $\widetilde{\operatorname{rot} E^\ell} \to \widetilde{\operatorname{rot} E}$ in $\mathsf{L}^2(\Omega)$. Hence by the previous argument it follows that $J_\Phi^\top \widetilde{E^\ell} \in \mathsf{R}_{\Gamma_t}(\Omega)$ with

$$J_\Phi^\top \widetilde{E^\ell} \to J_\Phi^\top \widetilde{E}, \quad \operatorname{rot}(J_\Phi^\top \widetilde{E^\ell}) = (\operatorname{adj} J_\Phi)\ \widetilde{\operatorname{rot} E^\ell} \to (\operatorname{adj} J_\Phi)\ \widetilde{\operatorname{rot} E} \qquad \text{in } \mathsf{L}^2(\Omega).$$

Since $\operatorname{rot}_{\Gamma_t} : \mathsf{R}_{\Gamma_t}(\Omega) \subset \mathsf{L}^2(\Omega) \to \mathsf{L}^2(\Omega)$ is a closed operator, we conclude $J_\Phi^\top \widetilde{E} \in \mathsf{R}_{\Gamma_t}(\Omega)$ and

$$\operatorname{rot}(J_\Phi^\top \widetilde{E}) = (\operatorname{adj} J_\Phi)\ \widetilde{\operatorname{rot} E}, \tag{15}$$

which completes the proof of the transformation rule for τ_Φ^1.

We now consider the case of τ_Φ^2. Assume $H \in \mathsf{C}^{0,1}_{\Gamma_{t,\Phi}}(\Omega_\Phi)$. Then $\widetilde{H} \in \mathsf{C}^{0,1}_{\Gamma_t}(\Omega)$ and

$$(\operatorname{adj} J_\Phi)\widetilde{H} = [\nabla\Phi_2 \times \nabla\Phi_3 \quad \nabla\Phi_3 \times \nabla\Phi_1 \quad \nabla\Phi_1 \times \nabla\Phi_2]\widetilde{H} = \sum_{(j,m,l)} \widetilde{H}_j \nabla\Phi_m \times \nabla\Phi_l,$$

cf. (14), where the summation is over the three even permutations (j,m,l) of $(1,2,3)$. Since $\nabla\Phi_m \times \nabla\Phi_l = \operatorname{rot}(\Phi_m\nabla\Phi_l) \in \operatorname{rot}\mathsf{R}(\Omega) \subset \mathsf{D}(\Omega)$ we conclude that $(\operatorname{adj} J_\Phi)\widetilde{H} \in \mathsf{D}(\Omega)$ and

$$\begin{aligned}
\operatorname{div}\Big((\operatorname{adj} J_\Phi)\widetilde{H}\Big) &= \sum_{(j,m,l)} \nabla\widetilde{H}_n \cdot (\nabla\Phi_m \times \nabla\Phi_l) = \sum_{(j,m,l)} (J_\Phi^\top \widetilde{\nabla H_j}) \cdot (\nabla\Phi_m \times \nabla\Phi_l) \\
&= \sum_{(j,m,l)} \Big([\nabla\Phi_1\ \nabla\Phi_2\ \nabla\Phi_3]\ \widetilde{\nabla H_j} \Big) \cdot (\nabla\Phi_m \times \nabla\Phi_l) \\
&= \sum_{(j,m,l,k)} \widetilde{\partial_k H_j} \nabla\Phi_k \cdot (\nabla\Phi_m \times \nabla\Phi_l) \\
&\overset{k=j}{=} (\det\nabla\Phi)\ \widetilde{\operatorname{div} H} = (\det J_\Phi)\ \widetilde{\operatorname{div} H}.
\end{aligned} \tag{16}$$

Moreover, by a mollification argument we deduce that $(\operatorname{adj} J_\Phi)\widetilde{H} \in \mathsf{D}_{\Gamma_t}(\Omega)$. The general case $H \in \mathsf{D}_{\Gamma_{t,\Phi}}(\Omega_\Phi)$ can be discussed by an approximation argument as above. Consider a sequence $(H^\ell)_{l\in\mathbb{N}} \subset \mathsf{C}^{0,1}_{\Gamma_{t,\Phi}}(\Omega_\Phi)$ such that $H^\ell \to H$ in $\mathsf{D}(\Omega_\Phi)$. Then $\widetilde{H^\ell} \to \widetilde{H}$ and $\widetilde{\operatorname{div} H^\ell} \to \widetilde{\operatorname{div} H}$ in $\mathsf{L}^2(\Omega)$. Hence by the previous argument we know that $(\operatorname{adj} J_\Phi)\widetilde{H^\ell} \in \mathsf{D}_{\Gamma_t}(\Omega)$ with $(\operatorname{adj} J_\Phi)\widetilde{H^\ell} \to (\operatorname{adj} J_\Phi)\widetilde{H}$ and $\operatorname{div}\Big((\operatorname{adj} J_\Phi)\widetilde{H^\ell}\Big) = (\det J_\Phi)\widetilde{\operatorname{div} H^\ell} \to (\det J_\Phi)\widetilde{\operatorname{div} H}$ in $\mathsf{L}^2(\Omega)$. Since $\operatorname{div}_{\Gamma_t} : \mathsf{D}_{\Gamma_t}(\Omega) \subset \mathsf{L}^2(\Omega) \to \mathsf{L}^2(\Omega)$ is a closed operator, we conclude that $(\operatorname{adj} J_\Phi)\widetilde{H} \in \mathsf{D}_{\Gamma_t}(\Omega)$ and

$$\operatorname{div}\Big((\operatorname{adj} J_\Phi)\widetilde{H}\Big) = (\det J_\Phi)\widetilde{\operatorname{div} H},$$

which completes the proof of the transformation rule for τ^2_Φ.

Concerning the inverse operators and L^2-adjoints we consider, for example, the case $q = 1$ since the other cases can be discussed in a similar way. As

$$\tau^1_{\Phi^{-1}}\tau^1_\Phi E = \tau^1_{\Phi^{-1}} J_\Phi^\top \widetilde{E} = J^\top_{\Phi^{-1}}\Big((J_\Phi^\top \widetilde{E})\circ\Phi^{-1}\Big) = \Big(J_\Phi^{-\top} J_\Phi^\top \widetilde{E}\Big)\circ\Phi^{-1} = E$$

we have $(\tau^1_\Phi)^{-1} = \tau^1_{\Phi^{-1}}$. Moreover, observing that $J_{\Phi^{-1}} = J_\Phi^{-1}\circ\Phi^{-1}$ we get

$$\begin{aligned}
\langle \tau^1_\Phi E, \Psi\rangle_{\mathsf{L}^2(\Omega)} = \langle J_\Phi^\top \widetilde{E}, \Psi\rangle_{\mathsf{L}^2(\Omega)} &= \Big\langle E, (\det J_{\Phi^{-1}})(J_\Phi \Psi)\circ\Phi^{-1}\Big\rangle_{\mathsf{L}^2(\Omega_\Phi)} \\
&= \Big\langle E, (\det J_{\Phi^{-1}}) J^{-1}_{\Phi^{-1}}(\Psi\circ\Phi^{-1})\Big\rangle_{\mathsf{L}^2(\Omega_\Phi)} \\
&= \Big\langle E, (\operatorname{adj} J_{\Phi^{-1}})(\Psi\circ\Phi^{-1})\Big\rangle_{\mathsf{L}^2(\Omega_\Phi)} = \langle E, \tau^2_{\Phi^{-1}}\Psi\rangle_{\mathsf{L}^2(\Omega_\Phi)},
\end{aligned}$$

and hence $(\tau^1_\Phi)^* = \tau^2_{\Phi^{-1}}$. □

Remark 2.1 (Transformation theorem). For the divergence there is also a duality argument leading to the result of Theorem 2.1. For this, let $H \in \mathsf{D}(\Omega_\Phi)$ and pick some $\psi \in \mathsf{C}^{0,1}_\Gamma(\Omega)$. Then $\phi := \psi\circ\Phi^{-1} \in \mathsf{C}^{0,1}_{\Gamma,\Phi}(\Omega_\Phi)$ and $\widetilde{\phi} = \psi$. By the chain rule we compute

$$\begin{aligned}
\langle H, \nabla\phi\rangle_{\mathsf{L}^2(\Omega_\Phi)} &= -\langle \operatorname{div} H, \phi\rangle_{\mathsf{L}^2(\Omega_\Phi)} = -\Big\langle (\det J_\Phi)\widetilde{\operatorname{div} H}, \psi\Big\rangle_{\mathsf{L}^2(\Omega)} \\
= \Big\langle (\det J_\Phi)\widetilde{H}, \widetilde{\nabla\phi}\Big\rangle_{\mathsf{L}^2(\Omega)} &= \Big\langle (\det J_\Phi)\widetilde{H}, J_\Phi^{-\top}\nabla\widetilde{\phi}\Big\rangle_{\mathsf{L}^2(\Omega)} = \Big\langle (\operatorname{adj} J_\Phi)\widetilde{H}, \nabla\psi\Big\rangle_{\mathsf{L}^2(\Omega)}.
\end{aligned}$$

Hence, $(\operatorname{adj} J_\Phi)\widetilde{H} \in \mathsf{D}(\Omega)$ and $\operatorname{div}\Big((\operatorname{adj} J_\Phi)\widetilde{H}\Big) = (\det J_\Phi)\widetilde{\operatorname{div} H}$. Note that this duality argument does not apply for the rot operator.

Corollary 2.1 (Transformation theorem). *Let* $E \in \mathsf{R}_{\Gamma_{t,\Phi}}(\Omega_\Phi) \cap \varepsilon^{-1}\mathsf{D}_{\Gamma_{n,\Phi}}(\Omega_\Phi)$. *Then*

$$\tau^1_\Phi E \in \mathsf{R}_{\Gamma_t}(\Omega) \cap \varepsilon_\Phi^{-1}\mathsf{D}_{\Gamma_n}(\Omega)$$

and it holds

$$\operatorname{rot}\tau^1_\Phi E = \tau^2_\Phi \operatorname{rot} E, \qquad \operatorname{div}\varepsilon_\Phi\tau^1_\Phi E = \tau^3_\Phi \operatorname{div}\varepsilon E, \qquad \varepsilon_\Phi\tau^1_\Phi = \tau^2_\Phi\varepsilon$$

with $\varepsilon_\Phi := \tau^2_\Phi\varepsilon\tau^1_{\Phi^{-1}} = (\det J_\Phi)J_\Phi^{-1}\widetilde{\varepsilon}J_\Phi^{-\top} = (\operatorname{adj} J_\Phi)\widetilde{\varepsilon}J_\Phi^{-\top}$. *Moreover,*

$$\tau^1_\Phi : \mathsf{R}_{\Gamma_{t,\Phi}}(\Omega_\Phi) \cap \varepsilon^{-1}\mathsf{D}_{\Gamma_{n,\Phi}}(\Omega_\Phi) \to \mathsf{R}_{\Gamma_t}(\Omega) \cap \varepsilon_\Phi^{-1}\mathsf{D}_{\Gamma_n}(\Omega)$$

is a topological isomorphism with norm depending on Ω, ε, *and* J_Φ. *The inverse is given by* $\tau^1_{\Phi^{-1}}$.

Proof. Using Theorem 2.1 we compute for $\varepsilon E \in \mathsf{D}_{\Gamma_{n,\Phi}}(\Omega_\Phi)$

$$\tau^3_\Phi \operatorname{div}\varepsilon E = \operatorname{div}\tau^2_\Phi\varepsilon E = \operatorname{div}\tau^2_\Phi\varepsilon\tau^1_{\Phi^{-1}}\tau^1_\Phi E = \operatorname{div}\varepsilon_\Phi\tau^1_\Phi E,$$

with $\varepsilon_\Phi = \tau^2_\Phi\varepsilon\tau^1_{\Phi^{-1}} = (\operatorname{adj} J_\Phi)\widetilde{\varepsilon\tau^1_{\Phi^{-1}}} = (\operatorname{adj} J_\Phi)\widetilde{\varepsilon}J_\Phi^{-\top} = (\det J_\Phi)J_\Phi^{-1}\widetilde{\varepsilon}J_\Phi^{-\top}$. □

Remark 2.2 (Transformation theorem). More explicitly, in Theorem 2.1 and Corollary 2.1 it holds

$$\begin{aligned}
&\forall\, u \in \mathsf{H}^1_{\Gamma_{t,\Phi}}(\Omega_\Phi) && \nabla\widetilde{u} = J_\Phi^\top\widetilde{\nabla u}, \\
&\forall\, E \in \mathsf{R}_{\Gamma_{t,\Phi}}(\Omega_\Phi) && \operatorname{rot}(J_\Phi^\top\widetilde{E}) = (\operatorname{adj} J_\Phi)\ \widetilde{\operatorname{rot} E}, \\
&\forall\, H \in \mathsf{D}_{\Gamma_{t,\Phi}}(\Omega_\Phi) && \operatorname{div}\Big((\operatorname{adj} J_\Phi)\ \widetilde{H}\Big) = (\det J_\Phi)\ \widetilde{\operatorname{div} H}, \\
&\forall\, E \in \varepsilon^{-1}\mathsf{D}_{\Gamma_{n,\Phi}}(\Omega_\Phi) && \operatorname{div}(\varepsilon_\Phi J_\Phi^\top\widetilde{E}) = (\det J_\Phi)\ \widetilde{\operatorname{div}\varepsilon E}.
\end{aligned}$$

Remark 2.3 (Transformation theorem). The transformations τ_Φ^q are just the well-known pullback maps for differential q-forms applied to the corresponding vector proxies, that is, $\tau_\Phi^q = \Phi^*$ on differentials forms F of degree q. Using the exterior derivative d the latter formulas reduce to

$$\mathrm{d}\tau_\Phi^q F = \mathrm{d}\Phi^* F = \Phi^* \mathrm{d}F = \tau_\Phi^{q+1}\mathrm{d}F.$$

2.4 | Functional Analysis Toolbox

We collect and cite some parts from [20–22, 29–31], cf. [31–35], of the so-called functional analysis toolbox (FA-ToolBox).

2.4.1 | Single Operators and Hilbert Space Adjoints

Let $\mathrm{A} : D(\mathrm{A}) \subset \mathsf{H}_0 \to \mathsf{H}_1$ be a densely defined and closed (unbounded[3]) linear operator with domain of definition $D(\mathrm{A})$ on two Hilbert spaces H_0 and H_1. Then the Hilbert space adjoint $\mathrm{A}^* : D(\mathrm{A}^*) \subset \mathsf{H}_1 \to \mathsf{H}_0$ is well defined and characterized by

$$\forall\, x \in D(\mathrm{A}) \quad \forall\, y \in D(\mathrm{A}^*) \qquad \langle \mathrm{A}x, y\rangle_{\mathsf{H}_1} = \langle x, \mathrm{A}^* y\rangle_{\mathsf{H}_0}.$$

The operators A and A^* are both densely defined, closed, and typically unbounded. We call $(\mathrm{A}, \mathrm{A}^*)$ a dual pair as $(\mathrm{A}^*)^* = \overline{\mathrm{A}} = \mathrm{A}$. The projection theorem shows

$$\mathsf{H}_0 = N(\mathrm{A}) \oplus_{\mathsf{H}_0} \overline{R(\mathrm{A}^*)}, \quad \mathsf{H}_1 = N(\mathrm{A}^*) \oplus_{\mathsf{H}_1} \overline{R(\mathrm{A})}, \tag{17}$$

often called Helmholtz/Hodge/Weyl decompositions, where we introduce the notation N for the kernel (or null space) and R for the range of a linear operator. These orthogonal decompositions reduce the operators A and A^*, leading to the injective operators $\mathcal{A} := \mathrm{A}|_{\overline{R(\mathrm{A}^*)}}$ and $\mathcal{A}^* := \mathrm{A}^*|_{\overline{R(\mathrm{A})}}$, that is

$$\mathcal{A} : D(\mathcal{A}) \subset \overline{R(\mathrm{A}^*)} \to \overline{R(\mathrm{A})}, \qquad D(\mathcal{A}) = D(\mathcal{A}) \cap \overline{R(\mathrm{A}^*)},$$
$$\mathcal{A}^* : D(\mathcal{A}^*) \subset \overline{R(\mathrm{A})} \to \overline{R(\mathrm{A}^*)}, \qquad D(\mathcal{A}^*) = D(\mathcal{A}^*) \cap \overline{R(\mathrm{A})},$$

which are again densely defined and closed (unbounded) linear operators. Note that

$$\overline{R(\mathrm{A}^*)} = N(\mathrm{A})^{\perp_{\mathsf{H}_0}}, \quad \overline{R(\mathrm{A})} = N(\mathrm{A}^*)^{\perp_{\mathsf{H}_1}},$$

and that $\mathcal{A}$ and $\mathcal{A}^*$ are indeed adjoint to each other, that is, $(\mathcal{A}, \mathcal{A}^*)$ is a dual pair as well. Then the inverse operators

$$\mathcal{A}^{-1} : R(\mathrm{A}) \to D(\mathcal{A}), \quad (\mathcal{A}^*)^{-1} : R(\mathrm{A}^*) \to D(\mathcal{A}^*),$$

are well defined and bijective, but possibly unbounded. Furthermore, by (17) we have the refined Helmholtz type decompositions

$$D(\mathrm{A}) = N(\mathrm{A}) \oplus_{\mathsf{H}_0} D(\mathcal{A}), \quad D(\mathrm{A}^*) = N(\mathrm{A}^*) \oplus_{\mathsf{H}_1} D(\mathcal{A}^*), \tag{18}$$

and thus we obtain for the ranges

$$R(\mathrm{A}) = R(\mathcal{A}), \quad R(\mathrm{A}^*) = R(\mathcal{A}^*).$$

Note that $D(\mathrm{A})$, $D(\mathcal{A})$ and $D(\mathrm{A}^*)$, $D(\mathcal{A}^*)$ equipped with the respective graph norms are Hilbert spaces.

The following result is a well-known and direct consequence of the closed graph theorem and the closed range theorem.

Lemma 2.5 (fa-toolbox lemma 1). *The following assertions are equivalent*:

i. $\exists\, c_{\mathrm{A}} \in (0, \infty)\ \forall\, x \in D(\mathcal{A})\ |x|_{\mathsf{H}_0} \le c_{\mathrm{A}} |\mathrm{A}x|_{\mathsf{H}_1}$

*i**. $\exists\, c_{\mathrm{A}^*} \in (0, \infty)\ \forall\, y \in D(\mathcal{A}^*)\ |y|_{\mathsf{H}_1} \le c_{\mathrm{A}^*} |\mathrm{A}^* y|_{\mathsf{H}_0}$

ii. $R(\mathrm{A}) = R(\mathcal{A})$ *is closed in* H_1.

*ii**. $R(\mathrm{A}^*) = R(\mathcal{A}^*)$ *is closed in* H_0.

iii. $\mathcal{A}^{-1} : R(\mathrm{A}) \to D(\mathcal{A})$ *is bounded.*

*iii**. $(\mathcal{A}^*)^{-1} : R(\mathrm{A}^*) \to D(\mathcal{A}^*)$ *is bounded.*

iv. $\mathcal{A} : D(\mathcal{A}) \to R(\mathrm{A})$ *is a topological isomorphism.*

*iv**. $\mathcal{A}^* : D(\mathcal{A}^*) \to R(\mathrm{A}^*)$ *is a topological isomorphism.*

The latter inequalities will be called Friedrichs-Poincaré type estimates.

Lemma 2.6 (fa-toolbox lemma 2). *The following assertions are equivalent*:

i. $D(\mathcal{A}) \hookrightarrow \mathsf{H}_0$ *is compact.*

*i**. $D(\mathcal{A}^*) \hookrightarrow \mathsf{H}_1$ *is compact.*

ii. $\mathcal{A}^{-1} : R(\mathrm{A}) \to R(\mathrm{A}^*)$ *is compact.*

*ii**. $(\mathcal{A}^*)^{-1} : R(\mathrm{A}^*) \to R(\mathrm{A})$ *is compact.*

Remark 2.4 (Sufficient assumptions for the first fa-toolbox lemmas).

i. If $R(\mathrm{A})$ is closed, then the assertions of Lemma 2.5 hold.

ii. If $D(\mathcal{A}) \hookrightarrow \mathsf{H}_0$ is compact, then the assertions of Lemma 2.5 (and Lemma 2.6) hold. In particular, the Friedrichs-Poincaré type estimates hold, all ranges are closed and the inverse operators are compact.

2.4.2 | Spectra and Point Spectra

We emphasize that

$$\mathrm{A}^*\,\mathrm{A} \geq 0, \qquad \mathrm{A}\,\mathrm{A}^* \geq 0 \tag{19}$$

are self-adjoint with essentially (except of 0) the same non-negative spectrum. The same holds true for the reduced operators $\mathcal{A}^*\mathcal{A}, \mathcal{A}\,\mathcal{A}^* > 0$. We shall give more details for the point spectrum in the next lemma.

Lemma 2.7 (fa-toolbox lemma 3/eigenvalues). *Let* $D(\mathcal{A}) \hookrightarrow \mathsf{H}_0$ *be compact. Then the operators in (19) are self-adjoint, non-negative, and have pure and discrete point spectra with no accumulation point in* $\mathbb{R}$. *Moreover*,

$$\sigma(\mathcal{A}^*\mathcal{A}) = \sigma(\mathrm{A}^*\,\mathrm{A}) \setminus \{0\} = \sigma(\mathrm{A}\,\mathrm{A}^*) \setminus \{0\} = \sigma(\mathcal{A}\,\mathcal{A}^*) = \{\lambda_k\}_{k\in\mathbb{N}} \subset (0, \infty)$$

with eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_{k-1} \leq \lambda_k \leq \cdots \to \infty$. *Only finitely many eigenvalues coincide, they are repeated according their multiplicity, and it holds*

$$N(\mathrm{A}^*\,\mathrm{A} - \lambda_k) = N(\mathcal{A}^*\mathcal{A} - \lambda_k), \qquad N(\mathrm{A}\,\mathrm{A}^* - \lambda_k) = N(\mathcal{A}\,\mathcal{A}^* - \lambda_k).$$

Remark 2.5 (variational formulations). For any eigenvector x of $\mathrm{A}^*\,\mathrm{A}$ associated with an eigenvalue λ_k we have

$$(\mathrm{A}^*\,\mathrm{A} - \lambda_k)x = 0, \qquad x \in D(\mathrm{A}^*\,\mathrm{A}) \cap R(\mathrm{A}^*) = D(\mathcal{A}^*\mathcal{A}) \subset D(\mathcal{A}),$$

and the variational formulation

$$\forall\, \phi \in D(\mathrm{A}) \qquad \langle \mathrm{A}x, \mathrm{A}\phi \rangle_{\mathsf{H}_1} = \lambda_k \langle x, \phi \rangle_{\mathsf{H}_0},$$

holds. The corresponding results hold for any eigenvector y of $\mathrm{A}\,\mathrm{A}^*$ to λ_k. Note that, for example, $y = \mathrm{A}x$.

For $x \in N(\mathrm{A}^*\mathrm{A} - \lambda_k)$ and $y := \lambda_k^{-1/2}\mathrm{A}x$ we observe that

$$x = \lambda_k^{-1}\mathrm{A}^*\mathrm{A}x = \lambda_k^{-1/2}\mathrm{A}^*y, \qquad \mathrm{A}\mathrm{A}^*y = \lambda_k^{1/2}\mathrm{A}x = \lambda_k y,$$

and

$$|y|_{\mathsf{H}_1}^2 = \lambda_k^{-1}|\mathrm{A}x|_{\mathsf{H}_1}^2 = \lambda_k^{-1}\langle \mathrm{A}^*\mathrm{A}x, x\rangle_{\mathsf{H}_0} = |x|_{\mathsf{H}_0}^2,$$

which shows the following:

Lemma 2.8 (Eigenvectors). *The following statements hold*:

i. *If x is an eigenvector of $\mathrm{A}^*\mathrm{A}$ for the eigenvalue λ_k, then $y := \mathrm{A}x$ is an eigenvector of $\mathrm{A}\mathrm{A}^*$ for the same eigenvalue λ_k.*

i*. *If y is an eigenvector of $\mathrm{A}\mathrm{A}^*$ for the eigenvalue λ_k, then $x := \mathrm{A}^*y$ is an eigenvector of $\mathrm{A}^*\mathrm{A}$ for the same eigenvalue λ_k.*

Lemma 2.9 (Eigenvalues, Friedrichs-Poincaré type constants, and Rayleigh quotients). *The best constants in Lemma 2.5 (i) and (i*) are given by the Rayleigh quotients and equal each other and the inverse of the square root the first positive eigenvalue of $\mathrm{A}^*\mathrm{A}$ and $\mathrm{A}\mathrm{A}^*$, that is,*

$$\lambda_1^{1/2} = \frac{1}{c_{\mathrm{A}}} = \inf_{0\neq x\in D(\mathcal{A})}\frac{|\mathrm{A}x|_{\mathsf{H}_1}}{|x|_{\mathsf{H}_0}} = \inf_{0\neq y\in D(\mathcal{A}^*)}\frac{|\mathrm{A}^*y|_{\mathsf{H}_0}}{|y|_{\mathsf{H}_1}} = \frac{1}{c_{\mathrm{A}^*}}.$$

Note that similar formulas hold for all eigenvalues, that is,

$$\lambda_k^{1/2} = \inf_x \frac{|\mathrm{A}x|_{\mathsf{H}_1}}{|x|_{\mathsf{H}_0}} = \inf_y \frac{|\mathrm{A}^*y|_{\mathsf{H}_0}}{|y|_{\mathsf{H}_1}},$$

where the infima are taken over all $0 \neq x \in D(\mathcal{A})$ and $0 \neq y \in D(\mathcal{A}^*)$ with $x \perp_{\mathsf{H}_0} \bigoplus_{\ell=1}^{k-1} N(\mathrm{A}^*\mathrm{A} - \lambda_\ell)$ and $y \perp_{\mathsf{H}_1} \bigoplus_{\ell=1}^{k-1} N(\mathrm{A}\mathrm{A}^* - \lambda_\ell)$. All infima are minima and are attained at the corresponding eigenvectors, that is, for all k and all eigenvectors $x_k \in N(\mathrm{A}^*\mathrm{A} - \lambda_k)$ and $y_k \in N(\mathrm{A}\mathrm{A}^* - \lambda_k)$ we have

$$\frac{\langle \mathrm{A}^*\mathrm{A}x_k, x_k\rangle_{\mathsf{H}_0}}{|x_k|_{\mathsf{H}_0}^2} = \frac{|\mathrm{A}x_k|_{\mathsf{H}_1}^2}{|x_k|_{\mathsf{H}_0}^2} = \lambda_k = \frac{|\mathrm{A}^*y_k|_{\mathsf{H}_0}^2}{|y_k|_{\mathsf{H}_1}^2} = \frac{\langle \mathrm{A}\mathrm{A}^*y_k, y_k\rangle_{\mathsf{H}_1}}{|y_k|_{\mathsf{H}_1}^2}.$$

2.4.3 | Hilbert Complexes

Now, let

$$\mathrm{A}_0 : D(\mathrm{A}_0) \subset \mathsf{H}_0 \to \mathsf{H}_1, \quad \mathrm{A}_1 : D(\mathrm{A}_1) \subset \mathsf{H}_1 \to \mathsf{H}_2$$

be two densely defined and closed linear operators on three Hilbert spaces H_0, H_1, and H_2 with adjoints

$$\mathrm{A}_0^* : D(\mathrm{A}_0^*) \subset \mathsf{H}_1 \to \mathsf{H}_0, \quad \mathrm{A}_1^* : D(\mathrm{A}_1^*) \subset \mathsf{H}_2 \to \mathsf{H}_1,$$

as well as reduced operators $\mathcal{A}_0$, $\mathcal{A}_0^*$, and $\mathcal{A}_1$, $\mathcal{A}_1^*$. Furthermore, we assume the complex property of A_0 and A_1, that is $\mathrm{A}_1\mathrm{A}_0 = 0$, that is,

$$R(\mathrm{A}_0) \subset N(\mathrm{A}_1), \tag{20}$$

being equivalent to $R(\mathrm{A}_1^*) \subset N(\mathrm{A}_0^*)$. Recall that

$$R(\mathrm{A}_0) = R(\mathcal{A}_0), \quad R(\mathrm{A}_0^*) = R(\mathcal{A}_0^*), \quad R(\mathrm{A}_1) = R(\mathcal{A}_1), \quad R(\mathrm{A}_1^*) = R(\mathcal{A}_1^*).$$

From the Helmholtz type decompositions (17) for $\mathrm{A} = \mathrm{A}_0$ and $\mathrm{A} = \mathrm{A}_1$ we get in particular

$$\mathsf{H}_1 = \overline{R(\mathrm{A}_0)} \oplus_{\mathsf{H}_1} N(\mathrm{A}_0^*), \quad \mathsf{H}_1 = \overline{R(\mathrm{A}_1^*)} \oplus_{\mathsf{H}_1} N(\mathrm{A}_1). \tag{21}$$

Introducing the cohomology group

$$N_{0,1} := N(\mathrm{A}_1) \cap N(\mathrm{A}_0^*),$$

we obtain the refined Helmholtz type decompositions

$$\begin{aligned} N(\mathrm{A}_1) &= \overline{R(\mathrm{A}_0)} \oplus_{\mathsf{H}_1} N_{0,1}, & N(\mathrm{A}_0^*) &= \overline{R(\mathrm{A}_1^*)} \oplus_{\mathsf{H}_1} N_{0,1}, \\ D(\mathrm{A}_1) &= \overline{R(\mathrm{A}_0)} \oplus_{\mathsf{H}_1} \big(D(\mathrm{A}_1) \cap N(\mathrm{A}_0^*)\big), & D(\mathrm{A}_0^*) &= \overline{R(\mathrm{A}_1^*)} \oplus_{\mathsf{H}_1} \big(D(\mathrm{A}_0^*) \cap N(\mathrm{A}_1)\big), \end{aligned} \tag{22}$$

and therefore the Helmholtz type decomposition

$$\mathsf{H}_1 = \overline{R(\mathrm{A}_0)} \oplus_{\mathsf{H}_1} N_{0,1} \oplus_{\mathsf{H}_1} \overline{R(\mathrm{A}_1^*)}, \tag{23}$$

follows. Let us remark that the first line of (22) can also be written as

$$\overline{R(\mathrm{A}_0)} = N(\mathrm{A}_1) \cap N_{0,1}^{\perp_{\mathsf{H}_1}}, \quad \overline{R(\mathrm{A}_1^*)} = N(\mathrm{A}_0^*) \cap N_{0,1}^{\perp_{\mathsf{H}_1}}.$$

Note that (23) can be further refined and specialized, for example, to

$$\begin{aligned} D(\mathrm{A}_1) &= \overline{R(\mathrm{A}_0)} \oplus_{\mathsf{H}_1} N_{0,1} \oplus_{\mathsf{H}_1} D(\mathcal{A}_1), \\ D(\mathrm{A}_0^*) &= D(\mathcal{A}_0^*) \oplus_{\mathsf{H}_1} N_{0,1} \oplus_{\mathsf{H}_1} \overline{R(\mathrm{A}_1^*)}, \\ D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) &= D(\mathcal{A}_0^*) \oplus_{\mathsf{H}_1} N_{0,1} \oplus_{\mathsf{H}_1} D(\mathcal{A}_1). \end{aligned} \tag{24}$$

We observe

$$\begin{aligned} D(\mathcal{A}_1) &= D(\mathrm{A}_1) \cap \overline{R(\mathrm{A}_1^*)} \subset D(\mathrm{A}_1) \cap N(\mathrm{A}_0^*) \subset D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*), \\ D(\mathcal{A}_0^*) &= D(\mathrm{A}_0^*) \cap \overline{R(\mathrm{A}_0)} \subset D(\mathrm{A}_0^*) \cap N(\mathrm{A}_1) \subset D(\mathrm{A}_0^*) \cap D(\mathrm{A}_1), \end{aligned}$$

and using the refined Helmholtz type decompositions (23) and (24) as well as the results of Lemma 2.6 we immediately see:

Lemma 2.10 (fa-toolbox lemma 4/compact embeddings). *The following assertions are equivalent*:

i. $D(\mathcal{A}_0) \hookrightarrow \mathsf{H}_0$, $D(\mathcal{A}_1) \hookrightarrow \mathsf{H}_1$, *and* $N_{0,1} \hookrightarrow \mathsf{H}_1$ *are compact*.

ii. $D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) \hookrightarrow \mathsf{H}_1$ *is compact*.

In this case, the cohomology group $N_{0,1}$ *has finite dimension*.

We summarize:

Lemma 2.11 (fa-toolbox lemma 5). *Let the ranges* $R(\mathrm{A}_0)$ *and* $R(\mathrm{A}_1)$ *be closed. Then* $R(\mathrm{A}_0^*)$ *and* $R(\mathrm{A}_1^*)$ *are also closed, and the corresponding Friedrichs-Poincaré type estimates hold, that is, there exists a positive constant* c *such that*

$$\begin{aligned} &\forall\, z \in D(\mathcal{A}_0) = D(\mathrm{A}_0) \cap R(\mathrm{A}_0^*) && |z|_{\mathsf{H}_0} \le c|\mathrm{A}_0 z|_{\mathsf{H}_1}, \\ &\forall\, x \in D(\mathcal{A}_0^*) = D(\mathrm{A}_0^*) \cap R(\mathrm{A}_0) = D(\mathrm{A}_0^*) \cap N(\mathrm{A}_1) \cap N_{0,1}^{\perp_{\mathsf{H}_1}} && |x|_{\mathsf{H}_1} \le c|\mathrm{A}_0^* x|_{\mathsf{H}_0}, \\ &\forall\, x \in D(\mathcal{A}_1) = D(\mathrm{A}_1) \cap R(\mathrm{A}_1^*) = D(\mathrm{A}_1) \cap N(\mathrm{A}_0^*) \cap N_{0,1}^{\perp_{\mathsf{H}_1}} && |x|_{\mathsf{H}_1} \le c|\mathrm{A}_1 x|_{\mathsf{H}_2}, \\ &\forall\, y \in D(\mathcal{A}_1^*) = D(\mathrm{A}_1^*) \cap R(\mathrm{A}_1) && |y|_{\mathsf{H}_2} \le c|\mathrm{A}_1^* y|_{\mathsf{H}_1}, \end{aligned}$$

and

$$\forall\, x \in D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) \cap N_{0,1}^{\perp_{\mathsf{H}_1}} \qquad |x|_{\mathsf{H}_1} \le c\big(|\mathrm{A}_1 x|_{\mathsf{H}_2} + |\mathrm{A}_0^* x|_{\mathsf{H}_0}\big).$$

Moreover, all Helmholtz type decompositions (21–24) hold with closed ranges, in particular

$$\mathsf{H}_1 = R(\mathrm{A}_0) \oplus_{\mathsf{H}_1} N_{0,1} \oplus_{\mathsf{H}_1} R(\mathrm{A}_1^*).$$

In other words, the primal and dual complex

$$\mathsf{H}_0 \underset{\mathrm{A}_0^*}{\overset{\mathrm{A}_0}{\rightleftarrows}} \mathsf{H}_1 \underset{\mathrm{A}_1^*}{\overset{\mathrm{A}_1}{\rightleftarrows}} \mathsf{H}_2 \tag{25}$$

is a Hilbert complex of closed and densely defined linear operators. We call the complex *closed* if the ranges $R(\mathrm{A}_0)$ and $R(\mathrm{A}_1)$ are closed. The complex is *exact* if $N_{0,1} = \{0\}$. The complex is called *compact*, if the embedding

$$D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) \hookrightarrow \mathsf{H}_1, \tag{26}$$

is compact.

2.4.4 | Generalized Laplacian

Finally, we present some results for the densely defined and closed generalized Laplacian

$$\rho \mathrm{A}_0 \mathrm{A}_0^* + \mathrm{A}_1^* \mathrm{A}_1 : D(\tau \mathrm{A}_0 \mathrm{A}_0^* + \mathrm{A}_1^* \mathrm{A}_1) \subset \mathsf{H}_1 \to \mathsf{H}_1, \qquad \rho > 0,$$

with $D(\rho \mathrm{A}_0 \mathrm{A}_0^* + \mathrm{A}_1^* \mathrm{A}_1) := D(\mathrm{A}_1^* \mathrm{A}_1) \cap D(\mathrm{A}_0 \mathrm{A}_0^*) \subset D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*)$.

Lemma 2.12 (fa-toolbox lemma 6/eigenvalues). *Let $D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) \hookrightarrow \mathsf{H}_1$ be compact. Then $\rho \mathrm{A}_0 \mathrm{A}_0^* + \mathrm{A}_1^* \mathrm{A}_1$ is self-adjoint, non-negative, and has pure and discrete point spectrum with no accumulation point, that is,*

$$\sigma(\rho \mathrm{A}_0 \mathrm{A}_0^* + \mathrm{A}_1^* \mathrm{A}_1) \setminus \{0\} = \big(\rho\sigma(\mathrm{A}_0^* \mathrm{A}_0) \setminus \{0\}\big) \cup \big(\sigma(\mathrm{A}_1^* \mathrm{A}_1) \setminus \{0\}\big) = \rho\{\lambda_{0,k}\}_{k \in \mathbb{N}} \cup \{\lambda_{1,k}\}_{k \in \mathbb{N}},$$

with eigenvalues $0 < \lambda_{\ell,1} \le \lambda_{\ell,2} \le \cdots \le \lambda_{\ell,k-1} \le \lambda_{\ell,k} \le \cdots \to \infty$ of $\mathcal{A}_\ell^* \mathcal{A}_\ell$ for $\ell \in \{0,1\}$. Only finitely many eigenvalues coincide and they are repeated according to their multiplicity. Moreover,

$$\rho \mathrm{A}_0 \mathrm{A}_0^* + \mathrm{A}_1^* \mathrm{A}_1 : D(\rho \mathrm{A}_0 \mathrm{A}_0^* + \mathrm{A}_1^* \mathrm{A}_1) \cap N_{0,1}^{\perp_{\mathsf{H}_1}} \to N_{0,1}^{\perp_{\mathsf{H}_1}},$$

is a topological isomorphism.

Remark 2.6 (Helmholtz decomposition). Let $\rho = 1$. Then $\mathrm{A}_0 \mathrm{A}_0^* + \mathrm{A}_1^* \mathrm{A}_1$ provides the Helmholtz decomposition from Lemma 2.11. To see this, let us denote the orthonormal projector onto the cohomology group $N_{0,1}$ by $\pi_{N_{0,1}} : \mathsf{H}_1 \to N_{0,1}$. Then, for $x \in \mathsf{H}_1$ we have $(1 - \pi_{N_{0,1}})x \in N_{0,1}^{\perp_{\mathsf{H}_1}}$ and

$$\begin{aligned} x &= \pi_{N_{0,1}} x + (1 - \pi_{N_{0,1}})x \\ &= \pi_{N_{0,1}} x + (\mathrm{A}_0 \mathrm{A}_0^* + \mathrm{A}_1^* \mathrm{A}_1)(\mathrm{A}_0 \mathrm{A}_0^* + \mathrm{A}_1^* \mathrm{A}_1)^{-1}(1 - \pi_{N_{0,1}})x \in N_{0,1} \oplus_{\mathsf{H}_1} R(\mathrm{A}_0) \oplus_{\mathsf{H}_1} R(\mathrm{A}_1^*). \end{aligned}$$

2.5 | De Rham Complex

In this subsection, let additionally (Ω, Γ_t) be a weak Lipschitz pair. Let us consider the densely defined and closed (unbounded) linear operators

$$\begin{aligned} \mathrm{A}_0 &:= \nabla_{\Gamma_t} : \mathsf{H}^1_{\Gamma_t}(\Omega) \subset \mathsf{L}^2_\nu(\Omega) \to \mathsf{L}^2_\varepsilon(\Omega), \\ \mathrm{A}_1 &:= \mu^{-1} \operatorname{rot}_{\Gamma_t} : \mathsf{R}_{\Gamma_t}(\Omega) \subset \mathsf{L}^2_\varepsilon(\Omega) \to \mathsf{L}^2_\mu(\Omega), \\ \mathrm{A}_2 &:= \kappa^{-1} \operatorname{div}_{\Gamma_t} \mu : \mu^{-1}\mathsf{D}_{\Gamma_t}(\Omega) \subset \mathsf{L}^2_\mu(\Omega) \to \mathsf{L}^2_\kappa(\Omega), \end{aligned}$$

together with their densely defined and closed (unbounded) adjoints

$$\begin{aligned} \mathrm{A}_0^* &= -\nu^{-1} \operatorname{div}_{\Gamma_n} \varepsilon : \varepsilon^{-1}\mathsf{D}_{\Gamma_n}(\Omega) \subset \mathsf{L}^2_\varepsilon(\Omega) \to \mathsf{L}^2_\nu(\Omega), \\ \mathrm{A}_1^* &= \varepsilon^{-1} \operatorname{rot}_{\Gamma_n} : \mathsf{R}_{\Gamma_n}(\Omega) \subset \mathsf{L}^2_\mu(\Omega) \to \mathsf{L}^2_\varepsilon(\Omega), \\ \mathrm{A}_2^* &= -\nabla_{\Gamma_n} : \mathsf{H}^1_{\Gamma_n}(\Omega) \subset \mathsf{L}^2_\kappa(\Omega) \to \mathsf{L}^2_\mu(\Omega), \end{aligned}$$

where we introduce the weighted Lebesgue space $\mathsf{L}^2_\varepsilon(\Omega)$ as $\mathsf{L}^2(\Omega)$ equipped with the weighted and equivalent inner product

$$\langle\,\cdot\,,\,\cdot\,\rangle_{\mathsf{L}^2_\varepsilon(\Omega)} := \langle\varepsilon\,\cdot\,,\,\cdot\,\rangle_{\mathsf{L}^2(\Omega)},$$

(same for μ, ν, and κ). For the adjoints we refer to [19] and [22] (weak and strong boundary conditions coincide). Recall that $\mathrm{A}_\ell^{**} = \overline{\mathrm{A}_\ell} = \mathrm{A}_\ell$ and that $(\mathrm{A}_\ell, \mathrm{A}_\ell^*)$ are dual pairs.

Remark 2.7. The latter operators satisfy the complex properties $R(\mathrm{A}_\ell) \subset N(\mathrm{A}_{\ell+1})$ and build the well-known de Rham Hilbert complex

$$N(\mathrm{A}_0) \underset{\mathrm{A}_{-1}^*=\pi_{N(\mathrm{A}_0)}}{\overset{\mathrm{A}_{-1}:=\iota_{N(\mathrm{A}_0)}}{\rightleftarrows}} \mathsf{L}^2_\nu(\Omega) \underset{\mathrm{A}_0^*=-\nu^{-1}\operatorname{div}_{\Gamma_n}\varepsilon}{\overset{\mathrm{A}_0=\nabla_{\Gamma_t}}{\rightleftarrows}} \mathsf{L}^2_\varepsilon(\Omega) \underset{\mathrm{A}_1^*=\varepsilon^{-1}\operatorname{rot}_{\Gamma_n}}{\overset{\mathrm{A}_1=\mu^{-1}\operatorname{rot}_{\Gamma_t}}{\rightleftarrows}} \mathsf{L}^2_\mu(\Omega) \underset{\mathrm{A}_2^*=-\nabla_{\Gamma_n}}{\overset{\mathrm{A}_2=\kappa^{-1}\operatorname{div}_{\Gamma_t}\mu}{\rightleftarrows}} \mathsf{L}^2_\kappa(\Omega) \underset{\mathrm{A}_3^*=\iota_{N(\mathrm{A}_2^*)}}{\overset{\mathrm{A}_3:=\pi_{N(\mathrm{A}_2^*)}}{\rightleftarrows}} N(\mathrm{A}_2^*).$$

Here, ι and π denote the canonical embedding and the orthogonal projector, respectively.

Theorem 2.2 (Weck's selection theorem). *The embedding*

$$D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) = \mathsf{R}_{\Gamma_t}(\Omega) \cap \varepsilon^{-1}\mathsf{D}_{\Gamma_n}(\Omega) \hookrightarrow \mathsf{L}^2_\varepsilon(\Omega) = \mathsf{H}_1,$$

is compact.

A proof can be found in [19] and [22].

Remark 2.8 (Weck's selection theorem). Note that by Theorem 2.2 also

$$D(\mathrm{A}_2) \cap D(\mathrm{A}_1^*) = \mu^{-1}\mathsf{D}_{\Gamma_t}(\Omega) \cap \mathsf{R}_{\Gamma_n}(\Omega) \hookrightarrow \mathsf{L}^2_\mu(\Omega) = \mathsf{H}_2,$$

is compact. Moreover, $D(\mathrm{A}_0) = \mathsf{H}^1_{\Gamma_t}(\Omega) \hookrightarrow \mathsf{L}^2_\nu(\Omega) = \mathsf{H}_0$ and $D(\mathrm{A}_2^*) = \mathsf{H}^1_{\Gamma_n}(\Omega) \hookrightarrow \mathsf{L}^2_\kappa(\Omega) = \mathsf{H}_3$ are trivially compact by Rellich's selection theorem. The first compact embedding result for non-smooth domains, that is, for piecewise smooth and globally strong Lipschitz boundaries and full boundary conditions, was given by Weck [36]. First results for strong Lipschitz boundaries and full boundary conditions had been proved in [37, 38]. First results for strong Lipschitz boundaries and mixed boundary conditions can be found in [39, 40].

Theorem 2.2 together with Lemma 2.10 shows that $D(\mathcal{A}_1) \hookrightarrow \mathsf{H}_1$ and $D(\mathcal{A}_0), D(\mathrm{A}_0) \hookrightarrow \mathsf{H}_1$ are compact. Hence by Lemma 2.7 we see that

$$\begin{aligned}
\mathrm{A}_0^*\,\mathrm{A}_0 &= -\nu^{-1}\operatorname{div}_{\Gamma_n}\varepsilon\nabla_{\Gamma_t}, & \mathrm{A}_1^*\,\mathrm{A}_1 &= \varepsilon^{-1}\operatorname{rot}_{\Gamma_n}\mu^{-1}\operatorname{rot}_{\Gamma_t},\\
\mathrm{A}_0\,\mathrm{A}_0^* &= -\nabla_{\Gamma_t}\nu^{-1}\operatorname{div}_{\Gamma_n}\varepsilon, & \mathrm{A}_1\mathrm{A}_1^* &= \mu^{-1}\operatorname{rot}_{\Gamma_t}\varepsilon^{-1}\operatorname{rot}_{\Gamma_n}
\end{aligned} \tag{27}$$

are self-adjoint, non-negative, and have pure and discrete point spectrum with no accumulation point. Moreover, as

$$\sigma(\mathrm{A}_\ell^*\,\mathrm{A}_\ell) \setminus \{0\} = \sigma(\mathcal{A}_\ell^*\,\mathcal{A}_\ell) = \sigma(\mathcal{A}_\ell\,\mathcal{A}_\ell^*) = \sigma(\mathrm{A}_\ell\,\mathrm{A}_\ell^*) \setminus \{0\} \subset (0,\infty), \qquad \ell \in \{0,1\},$$

we get:

Theorem 2.3 (eigenvalues of the de Rham complex). *It holds*

$$\begin{aligned}
\sigma(\mu^{-1}\operatorname{rot}_{\Gamma_t}\varepsilon^{-1}\operatorname{rot}_{\Gamma_n}) \setminus \{0\} = \sigma(\varepsilon^{-1}\operatorname{rot}_{\Gamma_n}\mu^{-1}\operatorname{rot}_{\Gamma_t}) \setminus \{0\} &= \{\lambda_{1,k}\}_{k\in\mathbb{N}} \subset (0,\infty),\\
\sigma(-\nabla_{\Gamma_t}\nu^{-1}\operatorname{div}_{\Gamma_n}\varepsilon) \setminus \{0\} = \sigma(-\nu^{-1}\operatorname{div}_{\Gamma_n}\varepsilon\nabla_{\Gamma_t}) \setminus \{0\} &= \{\lambda_{0,k}\}_{k\in\mathbb{N}} \subset (0,\infty),
\end{aligned}$$

with eigenvalues $0 < \lambda_{\ell,1} \le \lambda_{\ell,2} \le \ldots \le \lambda_{\ell,k-1} \le \lambda_{\ell,k} \le \cdots \to \infty$ *for* $\ell \in \{0,1\}$. *Only finitely many eigenvalues coincide and they are repeated according to their multiplicity.*

For the generalized Laplacian

$$\rho\mathrm{A}_0\,\mathrm{A}_0^* + \mathrm{A}_1^*\,\mathrm{A}_1 = -\rho\nabla_{\Gamma_t}\nu^{-1}\operatorname{div}_{\Gamma_n}\varepsilon + \varepsilon^{-1}\operatorname{rot}_{\Gamma_n}\mu^{-1}\operatorname{rot}_{\Gamma_t}$$

we have the following result:

Theorem 2.4 (eigenvalues of the generalised Laplacian). *$\varepsilon^{-1}\operatorname{rot}_{\Gamma_n}\mu^{-1}\operatorname{rot}_{\Gamma_t}-\rho\nabla_{\Gamma_t}\nu^{-1}\operatorname{div}_{\Gamma_n}\varepsilon$ is self-adjoint, non-negative, and has pure and discrete point spectrum with no accumulation point, that is,*

$$\begin{aligned}&\sigma(-\rho\nabla_{\Gamma_t}\nu^{-1}\operatorname{div}_{\Gamma_n}\varepsilon+\varepsilon^{-1}\operatorname{rot}_{\Gamma_n}\mu^{-1}\operatorname{rot}_{\Gamma_t})\setminus\{0\}\\&\quad=\big(\rho\sigma(-\nabla_{\Gamma_t}\nu^{-1}\operatorname{div}_{\Gamma_n}\varepsilon)\setminus\{0\}\big)\cup\sigma\big(\varepsilon^{-1}\operatorname{rot}_{\Gamma_n}\mu^{-1}\operatorname{rot}_{\Gamma_t})\setminus\{0\}\big)=\rho\{\lambda_{0,k}\}_{k\in\mathbb{N}}\cup\{\lambda_{1,k}\}_{k\in\mathbb{N}}.\end{aligned}$$

Only finitely many eigenvalues coincide and they are repeated according to their multiplicity.

3 | Eigenvalues

Let

$$\Phi:\Omega\to\Omega_\Phi=\Phi(\Omega)$$

be a bi-Lipschitz transformation. We are interested in the eigenvalue problems (1–6), in particular, in the dependence of the eigenvalues and related symmetric functions on the domain Ω, more precisely on the domain Ω_Φ, when Φ is varying. For this, we consider unbounded linear operators of the de Rham complex in $\mathsf{L}^2(\Omega_\Phi)$ together with their Lipschitz transformed relatives in $\mathsf{L}^2(\Omega)$.

From now on, let additionally (Ω,Γ_t) be a weak Lipschitz pair.

3.1 | Operators of the De Rham Complex

Let us define the densely defined and closed (unbounded) linear operators

$$\begin{aligned}\mathrm{A}_{0,\Phi}&:=\nabla_{\Gamma_{t,\Phi}}:\mathsf{H}^1_{\Gamma_{t,\Phi}}(\Omega_\Phi)\subset\mathsf{L}^2_\nu(\Omega_\Phi)\to\mathsf{L}^2_\varepsilon(\Omega_\Phi),\\\mathrm{A}_{1,\Phi}&:=\mu^{-1}\operatorname{rot}_{\Gamma_{t,\Phi}}:\mathsf{R}_{\Gamma_{t,\Phi}}(\Omega_\Phi)\subset\mathsf{L}^2_\varepsilon(\Omega_\Phi)\to\mathsf{L}^2_\mu(\Omega_\Phi),\end{aligned}$$

together with their densely defined and closed (unbounded) adjoints

$$\begin{aligned}\mathrm{A}^*_{0,\Phi}&=-\nu^{-1}\operatorname{div}_{\Gamma_{n,\Phi}}\varepsilon:\varepsilon^{-1}\mathsf{D}_{\Gamma_{n,\Phi}}(\Omega_\Phi)\subset\mathsf{L}^2_\varepsilon(\Omega_\Phi)\to\mathsf{L}^2_\nu(\Omega_\Phi),\\\mathrm{A}^*_{1,\Phi}&=\varepsilon^{-1}\operatorname{rot}_{\Gamma_{n,\Phi}}:\mathsf{R}_{\Gamma_{n,\Phi}}(\Omega_\Phi)\subset\mathsf{L}^2_\mu(\Omega_\Phi)\to\mathsf{L}^2_\varepsilon(\Omega_\Phi).\end{aligned}$$

Recall that $(\mathrm{A}_{\ell,\Phi},\mathrm{A}^*_{\ell,\Phi})$ are dual pairs. Moreover, let

$$\begin{aligned}\mathrm{A}_0&:=\nabla_{\Gamma_t}:\mathsf{H}^1_{\Gamma_t}(\Omega)\subset\mathsf{L}^2_{\nu_\Phi}(\Omega)\to\mathsf{L}^2_{\varepsilon_\Phi}(\Omega), & \varepsilon_\Phi&:=\tau^2_\Phi\varepsilon\tau^1_{\Phi^{-1}}=(\det J_\Phi)J_\Phi^{-1}\widetilde{\varepsilon}J_\Phi^{-\top},\\\mathrm{A}_1&:=\mu_\Phi^{-1}\operatorname{rot}_{\Gamma_t}:\mathsf{R}_{\Gamma_t}(\Omega)\subset\mathsf{L}^2_{\varepsilon_\Phi}(\Omega)\to\mathsf{L}^2_{\mu_\Phi}(\Omega), & \mu_\Phi&:=\tau^2_\Phi\mu\tau^1_{\Phi^{-1}}=(\det J_\Phi)J_\Phi^{-1}\widetilde{\mu}J_\Phi^{-\top},\\\mathrm{A}^*_0&=-\nu_\Phi^{-1}\operatorname{div}_{\Gamma_n}\varepsilon_\Phi:\varepsilon_\Phi^{-1}\mathsf{D}_{\Gamma_n}(\Omega)\subset\mathsf{L}^2_{\varepsilon_\Phi}(\Omega)\to\mathsf{L}^2_{\nu_\Phi}(\Omega), & \nu_\Phi&:=\tau^3_\Phi\nu\tau^0_{\Phi^{-1}}=(\det J_\Phi)\widetilde{\nu},\\\mathrm{A}^*_1&=\varepsilon_\Phi^{-1}\operatorname{rot}_{\Gamma_n}:\mathsf{R}_{\Gamma_n}(\Omega)\subset\mathsf{L}^2_{\mu_\Phi}(\Omega)\to\mathsf{L}^2_{\varepsilon_\Phi}(\Omega),\end{aligned}$$

which are also densely defined and closed (unbounded) linear operators. Again, $(\mathrm{A}_\ell,\mathrm{A}^*_\ell)$ are dual pairs. Note that the inner products in the weighted Lebesgue spaces $\mathsf{L}^2_{\varepsilon_\Phi}(\Omega)$, $\mathsf{L}^2_{\mu_\Phi}(\Omega)$, and $\mathsf{L}^2_{\nu_\Phi}(\Omega)$ read explicitly

$$\begin{aligned}\langle\,\cdot\,,\,\cdot\,\rangle_{\mathsf{L}^2_{\varepsilon_\Phi}(\Omega)}&=\langle\varepsilon_\Phi\,\cdot\,,\,\cdot\,\rangle_{\mathsf{L}^2(\Omega)}=\big\langle(\det J_\Phi)\widetilde{\varepsilon}J_\Phi^{-\top}\,\cdot\,,J_\Phi^{-\top}\,\cdot\,\big\rangle_{\mathsf{L}^2(\Omega)},\\\langle\,\cdot\,,\,\cdot\,\rangle_{\mathsf{L}^2_{\mu_\Phi}(\Omega)}&=\langle\mu_\Phi\,\cdot\,,\,\cdot\,\rangle_{\mathsf{L}^2(\Omega)}=\big\langle(\det J_\Phi)\widetilde{\mu}J_\Phi^{-\top}\,\cdot\,,J_\Phi^{-\top}\,\cdot\,\big\rangle_{\mathsf{L}^2(\Omega)},\\\langle\,\cdot\,,\,\cdot\,\rangle_{\mathsf{L}^2_{\nu_\Phi}(\Omega)}&=\langle\nu_\Phi\,\cdot\,,\,\cdot\,\rangle_{\mathsf{L}^2(\Omega)}=\langle(\det J_\Phi)\widetilde{\nu}\,\cdot\,,\,\cdot\,\rangle_{\mathsf{L}^2(\Omega)},\end{aligned}$$

and that ε_Φ, μ_Φ, and ν_Φ are admissible transformations. Note that, here, the operators $\mathrm{A}_{2,\Phi}$ and A_2 are not needed due to their equivalence to $\mathrm{A}^*_{0,\Phi}$ and A^*_0.

3.2 | Unitary Equivalence and Spectrum

Using the pullbacks τ_Φ^q of $\Phi : \Omega \to \Phi(\Omega) = \Omega_\Phi$ from Theorem 2.1 and Corollary 2.1 and the corresponding inverse pullbacks $(\tau_\Phi^q)^{-1} = \tau_{\Phi^{-1}}^q$ we compute

$$\tau_\Phi^1 \mathrm{A}_{0,\Phi} u = \tau_\Phi^1 \nabla_{\Gamma_{t,\Phi}} u = \nabla_{\Gamma_t} \tau_\Phi^0 u = \mathrm{A}_0 \tau_\Phi^0 u,$$
$$\tau_\Phi^1 \mathrm{A}_{1,\Phi} E = \tau_\Phi^1 \mu^{-1} \operatorname{rot}_{\Gamma_{t,\Phi}} E = \tau_\Phi^1 \mu^{-1} \tau_{\Phi^{-1}}^2 \tau_\Phi^2 \operatorname{rot}_{\Gamma_{t,\Phi}} E = \mu_\Phi^{-1} \operatorname{rot}_{\Gamma_t} \tau_\Phi^1 E = \mathrm{A}_1 \tau_\Phi^1 E,$$
$$\tau_\Phi^0 \mathrm{A}_{0,\Phi}^* H = -\tau_\Phi^0 \nu^{-1} \tau_{\Phi^{-1}}^3 \tau_\Phi^3 \operatorname{div}_{\Gamma_{n,\Phi}} \varepsilon H = -\nu_\Phi^{-1} \operatorname{div}_{\Gamma_n} \tau_\Phi^2 \varepsilon H = -\nu_\Phi^{-1} \operatorname{div}_{\Gamma_n} \varepsilon_\Phi \tau_\Phi^1 H = \mathrm{A}_0^* \tau_\Phi^1 H,$$

and obtain by symmetry the following result.

Lemma 3.1. *It holds*

$$\mathrm{A}_{0,\Phi} = \tau_{\Phi^{-1}}^1 \mathrm{A}_0 \tau_\Phi^0, \qquad \mathrm{A}_{1,\Phi} = \tau_{\Phi^{-1}}^1 \mathrm{A}_1 \tau_\Phi^1,$$
$$\mathrm{A}_{0,\Phi}^* = \tau_{\Phi^{-1}}^0 \mathrm{A}_0^* \tau_\Phi^1, \qquad \mathrm{A}_{1,\Phi}^* = \tau_{\Phi^{-1}}^1 \mathrm{A}_1^* \tau_\Phi^1.$$

Remark 3.1. We emphasize that the adjoints of the pullbacks in Lemma 3.1 are given by

$$(\tau_\Phi^0)^* = \tau_{\Phi^{-1}}^0 = (\tau_\Phi^0)^{-1}, \qquad (\tau_\Phi^1)^* = \tau_{\Phi^{-1}}^1 = (\tau_\Phi^1)^{-1},$$

apparently in contradiction to the adjoints from Theorem 2.1. This is due to the formulations in weighted L^2-spaces. Note that, for example, we have to consider $\tau_\Phi^1 : \mathsf{L}_\varepsilon^2(\Omega_\Phi) \to \mathsf{L}_{\varepsilon_\Phi}^2(\Omega)$, which leads with $\tau_{\Phi^{-1}}^2 \varepsilon_\Phi = \varepsilon \tau_{\Phi^{-1}}^1$ to

$$\langle \tau_\Phi^1 E, \Psi \rangle_{\mathsf{L}_{\varepsilon_\Phi}^2(\Omega)} = \langle \tau_\Phi^1 E, \varepsilon_\Phi \Psi \rangle_{\mathsf{L}^2(\Omega)} = \langle E, \tau_{\Phi^{-1}}^2 \varepsilon_\Phi \Psi \rangle_{\mathsf{L}^2(\Omega_\Phi)} = \langle E, \tau_{\Phi^{-1}}^1 \Psi \rangle_{\mathsf{L}_\varepsilon^2(\Omega_\Phi)},$$

that is, $(\tau_\Phi^1)^* = \tau_{\Phi^{-1}}^1$. Analogously, we treat $\tau_\Phi^0 : \mathsf{L}_\nu^2(\Omega_\Phi) \to \mathsf{L}_{\nu_\Phi}^2(\Omega)$.

Theorem 3.1. $\mathrm{A}_{\ell,\Phi}^* \mathrm{A}_{\ell,\Phi}$ *and* $\mathrm{A}_{\ell,\Phi} \mathrm{A}_{\ell,\Phi}^*$ *are unitarily equivalent to* $\mathrm{A}_\ell^* \mathrm{A}_\ell$ *and* $\mathrm{A}_\ell \mathrm{A}_\ell^*$*, respectively. More precisely,*

$$\mathrm{A}_{0,\Phi}^* \mathrm{A}_{0,\Phi} = \tau_{\Phi^{-1}}^0 \mathrm{A}_0^* \mathrm{A}_0 \tau_\Phi^0, \qquad \mathrm{A}_{1,\Phi}^* \mathrm{A}_{1,\Phi} = \tau_{\Phi^{-1}}^1 \mathrm{A}_1^* \mathrm{A}_1 \tau_\Phi^1,$$
$$\mathrm{A}_{0,\Phi} \mathrm{A}_{0,\Phi}^* = \tau_{\Phi^{-1}}^1 \mathrm{A}_0 \mathrm{A}_0^* \tau_\Phi^1, \qquad \mathrm{A}_{1,\Phi} \mathrm{A}_{1,\Phi}^* = \tau_{\Phi^{-1}}^1 \mathrm{A}_1 \mathrm{A}_1^* \tau_\Phi^1.$$

Moreover, $\rho \mathrm{A}_{0,\Phi} \mathrm{A}_{0,\Phi}^* + \mathrm{A}_{1,\Phi}^* \mathrm{A}_{1,\Phi}$ *and* $\rho \mathrm{A}_0 \mathrm{A}_0^* + \mathrm{A}_1^* \mathrm{A}_1$ *are unitarily equivalent, that is,*

$$\rho \mathrm{A}_{0,\Phi} \mathrm{A}_{0,\Phi}^* + \mathrm{A}_{1,\Phi}^* \mathrm{A}_{1,\Phi} = \tau_{\Phi^{-1}}^1 (\rho \mathrm{A}_0 \mathrm{A}_0^* + \mathrm{A}_1^* \mathrm{A}_1) \tau_\Phi^1.$$

Proof. Apply Lemma 3.1. □

Corollary 3.1. *The positive parts of the spectra of* $\mathrm{A}_{\ell,\Phi}^* \mathrm{A}_{\ell,\Phi}$, $\mathrm{A}_{\ell,\Phi} \mathrm{A}_{\ell,\Phi}^*$, *and* $\mathrm{A}_\ell^* \mathrm{A}_\ell$, $\mathrm{A}_\ell \mathrm{A}_\ell^*$ *coincide. More precisely,*

$$\sigma(\mathrm{A}_{\ell,\Phi}^* \mathrm{A}_{\ell,\Phi}) \setminus \{0\} = \sigma(\mathrm{A}_{\ell,\Phi} \mathrm{A}_{\ell,\Phi}^*) \setminus \{0\} = \sigma(\mathrm{A}_\ell \mathrm{A}_\ell^*) \setminus \{0\} = \sigma(\mathrm{A}_\ell^* \mathrm{A}_\ell) \setminus \{0\}.$$

Moreover, the positive parts of the spectra of $\rho \mathrm{A}_{0,\Phi} \mathrm{A}_{0,\Phi}^* + \mathrm{A}_{1,\Phi}^* \mathrm{A}_{1,\Phi}$, *and* $\rho \mathrm{A}_0 \mathrm{A}_0^* + \mathrm{A}_1^* \mathrm{A}_1$ *coincide, that is,*

$$\sigma(\rho \mathrm{A}_{0,\Phi} \mathrm{A}_{0,\Phi}^* + \mathrm{A}_{1,\Phi}^* \mathrm{A}_{1,\Phi}) \setminus \{0\} = \sigma(\rho \mathrm{A}_0 \mathrm{A}_0^* + \mathrm{A}_1^* \mathrm{A}_1) \setminus \{0\}.$$

Proof. Recall (19) and apply Theorem 3.1. □

Remark 3.2 (Eigenvectors). It holds:

- u is an eigenvector of $\mathrm{A}_{0,\Phi}^* \mathrm{A}_{0,\Phi}$, if and only if $\tau_\Phi^0 u$ is an eigenvector of $\mathrm{A}_0^* \mathrm{A}_0$.

- E is an eigenvector of $\mathrm{A}_{1,\Phi}^*\mathrm{A}_{1,\Phi}$ and $\mathrm{A}_{0,\Phi}\mathrm{A}_{0,\Phi}^*$, respectively, if and only if $\tau_\Phi^1 E$ is an eigenvector of $\mathrm{A}_1^*\mathrm{A}_1$ and $\mathrm{A}_0\mathrm{A}_0^*$, respectively.
- H is an eigenvector of $\mathrm{A}_{1,\Phi}\mathrm{A}_{1,\Phi}^*$, if and only if $\tau_\Phi^1 H$ is an eigenvector of $\mathrm{A}_1\mathrm{A}_1^*$.

Note that by Lemma 3.1 and Remark 3.1 we have, for example, for $F, G \in \mathsf{L}^2_\varepsilon(\Omega_\Phi)$

$$\langle \tau_\Phi^1 F, \tau_\Phi^1 G\rangle_{\mathsf{L}^2_{\varepsilon_\Phi}(\Omega)} = \langle F, \tau_{\Phi^{-1}}^1\tau_\Phi^1 G\rangle_{\mathsf{L}^2_\varepsilon(\Omega_\Phi)} = \langle F, G\rangle_{\mathsf{L}^2_\varepsilon(\Omega_\Phi)},$$

and for $F, G \in D(\mathrm{A}_{1,\Phi})$

$$\begin{aligned}\langle \mathrm{A}_1\tau_\Phi^1 F, \mathrm{A}_1\tau_\Phi^1 G\rangle_{\mathsf{L}^2_{\mu_\Phi}(\Omega)} &= \langle \tau_\Phi^1\mathrm{A}_{1,\Phi}F, \tau_\Phi^1\mathrm{A}_{1,\Phi}G\rangle_{\mathsf{L}^2_{\mu_\Phi}(\Omega)} = \langle \mathrm{A}_{1,\Phi}F, \mathrm{A}_{1,\Phi}G\rangle_{\mathsf{L}^2_\mu(\Omega_\Phi)},\\ \langle \tau_\Phi^1 F, \tau_\Phi^1 G\rangle_{D(\mathrm{A}_1)} &= \langle F, G\rangle_{D(\mathrm{A}_{1,\Phi})}.\end{aligned}$$

For a definition of the inner products $\langle\,\cdot\,,\,\cdot\,\rangle_{D(\mathrm{A}_1)}$ and $\langle\,\cdot\,,\,\cdot\,\rangle_{D(\mathrm{A}_{1,\Phi})}$ see the next remark.

Hence we get:

Remark 3.3 (Isometries and orthonormal bases). The transformations τ_Φ^q are isometries. In particular, orthonormal bases are mapped to orthonormal bases. More precisely:

- $\tau_\Phi^0 : \mathsf{L}^2_\nu(\Omega_\Phi) \to \mathsf{L}^2_{\nu_\Phi}(\Omega)$ and $\tau_\Phi^0 : D(\mathrm{A}_{0,\Phi}) \to D(\mathrm{A}_0)$, the latter

  $$D(\mathrm{A}_{0,\Phi}) = \mathsf{H}^1_{\Gamma_{t,\Phi}}(\Omega_\Phi), \qquad D(\mathrm{A}_0) = \mathsf{H}^1_{\Gamma_t}(\Omega)$$

  equipped with the inner products

  $$\begin{aligned}\langle\,\cdot\,,\,\cdot\,\rangle_{D(\mathrm{A}_{0,\Phi})} &= \langle\,\cdot\,,\,\cdot\,\rangle_{\mathsf{L}^2_\nu(\Omega_\Phi)} + \langle \mathrm{A}_{0,\Phi}\,\cdot\,, \mathrm{A}_{0,\Phi}\,\cdot\,\rangle_{\mathsf{L}^2_\varepsilon(\Omega_\Phi)} = \langle\,\cdot\,,\,\cdot\,\rangle_{\mathsf{L}^2_\nu(\Omega_\Phi)} + \langle \nabla\,\cdot\,, \nabla\,\cdot\,\rangle_{\mathsf{L}^2_\varepsilon(\Omega_\Phi)},\\ \langle\,\cdot\,,\,\cdot\,\rangle_{D(\mathrm{A}_0)} &= \langle\,\cdot\,,\,\cdot\,\rangle_{\mathsf{L}^2_{\nu_\Phi}(\Omega)} + \langle \mathrm{A}_0\,\cdot\,, \mathrm{A}_0\,\cdot\,\rangle_{\mathsf{L}^2_{\varepsilon_\Phi}(\Omega)} = \langle\,\cdot\,,\,\cdot\,\rangle_{\mathsf{L}^2_{\nu_\Phi}(\Omega)} + \langle \nabla\,\cdot\,, \nabla\,\cdot\,\rangle_{\mathsf{L}^2_{\varepsilon_\Phi}(\Omega)},\end{aligned}$$

  are isometries. Hence τ_Φ^0 maps a $\mathsf{L}^2_\nu(\Omega_\Phi)$-orthonormal basis or a $D(\mathrm{A}_{0,\Phi})$-orthonormal basis $\{u_m\}$ to the $\mathsf{L}^2_{\nu_\Phi}(\Omega)$-orthonormal basis or the $D(\mathrm{A}_0)$-orthonormal basis $\{\tau_\Phi^0 u_m\}$, respectively, and vice versa.
- $\tau_\Phi^1 : \mathsf{L}^2_\varepsilon(\Omega_\Phi) \to \mathsf{L}^2_{\varepsilon_\Phi}(\Omega)$, $\tau_\Phi^1 : \mathsf{L}^2_\mu(\Omega_\Phi) \to \mathsf{L}^2_{\mu_\Phi}(\Omega)$ and

  $$\tau_\Phi^1 : D(\mathrm{A}_{1,\Phi}) \to D(\mathrm{A}_1), \quad \tau_\Phi^1 : D(\mathrm{A}_{1,\Phi}^*) \to D(\mathrm{A}_1^*), \quad \tau_\Phi^1 : D(\mathrm{A}_{0,\Phi}^*) \to D(\mathrm{A}_{0,\Phi}^*),$$

  the latter

  $$\begin{aligned}D(\mathrm{A}_{1,\Phi}) &= \mathsf{R}_{\Gamma_{t,\Phi}}(\Omega_\Phi), & D(\mathrm{A}_1) &= \mathsf{R}_{\Gamma_t}(\Omega),\\ D(\mathrm{A}_{1,\Phi}^*) &= \mathsf{R}_{\Gamma_{n,\Phi}}(\Omega_\Phi), & D(\mathrm{A}_1^*) &= \mathsf{R}_{\Gamma_n}(\Omega),\\ D(\mathrm{A}_{0,\Phi}^*) &= \varepsilon^{-1}\mathsf{D}_{\Gamma_{n,\Phi}}(\Omega_\Phi), & D(\mathrm{A}_0^*) &= \varepsilon_\Phi^{-1}\mathsf{D}_{\Gamma_n}(\Omega)\end{aligned}$$

  equipped with the inner products

  $$\begin{aligned}\langle\,\cdot\,,\,\cdot\,\rangle_{D(\mathrm{A}_{1,\Phi})} &= \langle\,\cdot\,,\,\cdot\,\rangle_{\mathsf{L}^2_\varepsilon(\Omega_\Phi)} + \langle \mathrm{A}_{1,\Phi}\,\cdot\,, \mathrm{A}_{1,\Phi}\,\cdot\,\rangle_{\mathsf{L}^2_\mu(\Omega_\Phi)} = \langle\,\cdot\,,\,\cdot\,\rangle_{\mathsf{L}^2_\varepsilon(\Omega_\Phi)} + \langle \operatorname{rot}\,\cdot\,, \operatorname{rot}\,\cdot\,\rangle_{\mathsf{L}^2_{\mu^{-1}}(\Omega_\Phi)},\\ \langle\,\cdot\,,\,\cdot\,\rangle_{D(\mathrm{A}_1)} &= \langle\,\cdot\,,\,\cdot\,\rangle_{\mathsf{L}^2_{\varepsilon_\Phi}(\Omega)} + \langle \mathrm{A}_1\,\cdot\,, \mathrm{A}_1\,\cdot\,\rangle_{\mathsf{L}^2_{\mu_\Phi}(\Omega)} = \langle\,\cdot\,,\,\cdot\,\rangle_{\mathsf{L}^2_{\varepsilon_\Phi}(\Omega)} + \langle \operatorname{rot}\,\cdot\,, \operatorname{rot}\,\cdot\,\rangle_{\mathsf{L}^2_{\mu_\Phi^{-1}}(\Omega)},\\ \langle\,\cdot\,,\,\cdot\,\rangle_{D(\mathrm{A}_{1,\Phi}^*)} &= \langle\,\cdot\,,\,\cdot\,\rangle_{\mathsf{L}^2_\mu(\Omega_\Phi)} + \langle \mathrm{A}_{1,\Phi}^*\,\cdot\,, \mathrm{A}_{1,\Phi}^*\,\cdot\,\rangle_{\mathsf{L}^2_\varepsilon(\Omega_\Phi)} = \langle\,\cdot\,,\,\cdot\,\rangle_{\mathsf{L}^2_\mu(\Omega_\Phi)} + \langle \operatorname{rot}\,\cdot\,, \operatorname{rot}\,\cdot\,\rangle_{\mathsf{L}^2_{\varepsilon^{-1}}(\Omega_\Phi)},\\ \langle\,\cdot\,,\,\cdot\,\rangle_{D(\mathrm{A}_1^*)} &= \langle\,\cdot\,,\,\cdot\,\rangle_{\mathsf{L}^2_{\mu_\Phi}(\Omega)} + \langle \mathrm{A}_1^*\,\cdot\,, \mathrm{A}_1^*\,\cdot\,\rangle_{\mathsf{L}^2_{\varepsilon_\Phi}(\Omega)} = \langle\,\cdot\,,\,\cdot\,\rangle_{\mathsf{L}^2_{\mu_\Phi}(\Omega)} + \langle \operatorname{rot}\,\cdot\,, \operatorname{rot}\,\cdot\,\rangle_{\mathsf{L}^2_{\varepsilon_\Phi^{-1}}(\Omega)},\\ \langle\,\cdot\,,\,\cdot\,\rangle_{D(\mathrm{A}_{0,\Phi}^*)} &= \langle\,\cdot\,,\,\cdot\,\rangle_{\mathsf{L}^2_\varepsilon(\Omega_\Phi)} + \langle \mathrm{A}_{0,\Phi}^*\,\cdot\,, \mathrm{A}_{0,\Phi}^*\,\cdot\,\rangle_{\mathsf{L}^2_\nu(\Omega_\Phi)} = \langle\,\cdot\,,\,\cdot\,\rangle_{\mathsf{L}^2_\varepsilon(\Omega_\Phi)} + \langle \operatorname{div}\varepsilon\,\cdot\,, \operatorname{div}\varepsilon\,\cdot\,\rangle_{\mathsf{L}^2_{\nu^{-1}}(\Omega_\Phi)},\\ \langle\,\cdot\,,\,\cdot\,\rangle_{D(\mathrm{A}_0^*)} &= \langle\,\cdot\,,\,\cdot\,\rangle_{\mathsf{L}^2_{\varepsilon_\Phi}(\Omega)} + \langle \mathrm{A}_0^*\,\cdot\,, \mathrm{A}_0^*\,\cdot\,\rangle_{\mathsf{L}^2_{\nu_\Phi}(\Omega)} = \langle\,\cdot\,,\,\cdot\,\rangle_{\mathsf{L}^2_{\varepsilon_\Phi}(\Omega)} + \langle \operatorname{div}\varepsilon_\Phi\,\cdot\,, \operatorname{div}\varepsilon_\Phi\,\cdot\,\rangle_{\mathsf{L}^2_{\nu_\Phi^{-1}}(\Omega)},\end{aligned}$$

are isometries. Hence τ^1_Φ maps a $\mathsf{L}^2_\varepsilon(\Omega_\Phi)$-orthonormal basis or a $D(\mathrm{A}_{1,\Phi})$-orthonormal basis or a $D(\mathrm{A}^*_{0,\Phi})$-orthonormal basis $\{E_m\}$ to the $\mathsf{L}^2_{\varepsilon_\Phi}(\Omega)$-orthonormal basis or the $D(\mathrm{A}_1)$-orthonormal basis or the $D(\mathrm{A}^*_0)$-orthonormal basis $\{\tau^1_\Phi E_m\}$, respectively, and vice versa. Analogously, τ^1_Φ maps a $\mathsf{L}^2_\mu(\Omega_\Phi)$-orthonormal basis or a $D(\mathrm{A}^*_{1,\Phi})$-orthonormal basis $\{H_m\}$ to the $\mathsf{L}^2_{\mu_\Phi}(\Omega)$-orthonormal basis or the $D(\mathrm{A}^*_1)$-orthonormal basis $\{\tau^1_\Phi H_m\}$, respectively, and vice versa.

3.3 | Point Spectrum

Now and more precisely (3–5) read for the domain Ω_Φ

$$\begin{aligned}
\mathrm{A}^*_{0,\Phi}\mathrm{A}_{0,\Phi}u &= -\nu^{-1}\operatorname{div}_{\Gamma_{n,\Phi}}\varepsilon\nabla_{\Gamma_{t,\Phi}}u = \lambda_0 u && \text{in } \mathsf{L}^2_\nu(\Omega_\Phi),\\
\mathrm{A}^*_{1,\Phi}\mathrm{A}_{1,\Phi}E &= \varepsilon^{-1}\operatorname{rot}_{\Gamma_{n,\Phi}}\mu^{-1}\operatorname{rot}_{\Gamma_{t,\Phi}}E = \lambda_1 E && \text{in } \mathsf{L}^2_\varepsilon(\Omega_\Phi),\\
\mathrm{A}_{0,\Phi}\mathrm{A}^*_{0,\Phi}H &= -\nabla_{\Gamma_{t,\Phi}}\nu^{-1}\operatorname{div}_{\Gamma_{n,\Phi}}\varepsilon H = \lambda_0 H && \text{in } \mathsf{L}^2_\varepsilon(\Omega_\Phi)
\end{aligned} \tag{28}$$

with some eigenvectors

$$\begin{aligned}
u &\in D(\mathrm{A}^*_{0,\Phi}\mathrm{A}_{0,\Phi}) = D(\nu^{-1}\operatorname{div}_{\Gamma_{n,\Phi}}\varepsilon\nabla_{\Gamma_{t,\Phi}}) = \Big\{\psi\in\mathsf{H}^1_{\Gamma_{t,\Phi}}(\Omega_\Phi)\ :\ \varepsilon\nabla\psi\in\mathsf{D}_{\Gamma_{n,\Phi}}(\Omega_\Phi)\Big\},\\
E &\in D(\mathrm{A}^*_{1,\Phi}\mathrm{A}_{1,\Phi}) = D(\varepsilon^{-1}\operatorname{rot}_{\Gamma_{n,\Phi}}\mu^{-1}\operatorname{rot}_{\Gamma_{t,\Phi}}) = \Big\{\Psi\in\mathsf{R}_{\Gamma_{t,\Phi}}(\Omega_\Phi)\ :\ \mu^{-1}\operatorname{rot}\Psi\in\mathsf{R}_{\Gamma_{n,\Phi}}(\Omega_\Phi)\Big\},\\
H &\in D(\mathrm{A}_{0,\Phi}\mathrm{A}^*_{0,\Phi}) = D(\nabla_{\Gamma_{t,\Phi}}\nu^{-1}\operatorname{div}_{\Gamma_{n,\Phi}}\varepsilon) = \Big\{\Psi\in\varepsilon^{-1}\mathsf{D}_{\Gamma_{n,\Phi}}(\Omega_\Phi)\ :\ \nu^{-1}\operatorname{div}\varepsilon\Psi\in\mathsf{H}^1_{\Gamma_{t,\Phi}}(\Omega_\Phi)\Big\}.
\end{aligned}$$

Remark 3.4. Note that for the eigenfields E and H a normal and tangential boundary condition is induced by the complex property since

$$\begin{aligned}
E &\in R(\mathrm{A}^*_{1,\Phi})\subset N(\mathrm{A}^*_{0,\Phi}) = N(\nu^{-1}\operatorname{div}_{\Gamma_{n,\Phi}}\varepsilon) = \Big\{\Psi\in\mathsf{L}^2(\Omega_\Phi)\ :\ \operatorname{div}\varepsilon\Psi=0,\ \mathrm{n}\cdot\varepsilon\Psi|_{\Gamma_{n,\Phi}}=0\Big\},\\
H &\in R(\mathrm{A}_{0,\Phi})\subset N(\mathrm{A}_{1,\Phi}) = N(\mu^{-1}\operatorname{rot}_{\Gamma_{t,\Phi}}) = \Big\{\Psi\in\mathsf{L}^2(\Omega_\Phi)\ :\ \operatorname{rot}\Psi=0,\ \mathrm{n}\times\Psi|_{\Gamma_{t,\Phi}}=0\Big\},
\end{aligned}$$

respectively.

We want to discuss (28) equivalently in Ω using the pullbacks τ^q_Φ and Theorems 2.3, 3.1, and Corollary 3.1.

Theorem 3.2 (eigenvalues of the de Rham complex). *$\mathrm{A}^*_{\ell,\Phi}\mathrm{A}_{\ell,\Phi}$ and $\mathrm{A}^*_\ell\mathrm{A}_\ell$ are unitarily equivalent. The same holds for $\mathrm{A}_{\ell,\Phi}\mathrm{A}^*_{\ell,\Phi}$ and $\mathrm{A}_\ell\mathrm{A}^*_\ell$. All these operators are self-adjoint and non-negative and have pure and discrete point spectrum with no accumulation point. Moreover, the positive parts of the spectra coincide, that is,*

$$\begin{aligned}
&\sigma(\mu^{-1}\operatorname{rot}_{\Gamma_{t,\Phi}}\varepsilon^{-1}\operatorname{rot}_{\Gamma_{n,\Phi}})\setminus\{0\} = \sigma(\varepsilon^{-1}\operatorname{rot}_{\Gamma_{n,\Phi}}\mu^{-1}\operatorname{rot}_{\Gamma_{t,\Phi}})\setminus\{0\}\\
&\quad = \sigma(\mu_\Phi^{-1}\operatorname{rot}_{\Gamma_t}\varepsilon_\Phi^{-1}\operatorname{rot}_{\Gamma_n})\setminus\{0\} = \sigma(\varepsilon_\Phi^{-1}\operatorname{rot}_{\Gamma_n}\mu_\Phi^{-1}\operatorname{rot}_{\Gamma_t})\setminus\{0\} = \{\lambda_{1,\Phi,k}\}_{k\in\mathbb{N}}\subset(0,\infty)
\end{aligned}$$

and

$$\begin{aligned}
&\sigma(-\nabla_{\Gamma_{t,\Phi}}\nu^{-1}\operatorname{div}_{\Gamma_{n,\Phi}}\varepsilon)\setminus\{0\} = \sigma(-\nu^{-1}\operatorname{div}_{\Gamma_{n,\Phi}}\varepsilon\nabla_{\Gamma_{t,\Phi}})\setminus\{0\}\\
&\quad = \sigma(-\nabla_{\Gamma_t}\nu_\Phi^{-1}\operatorname{div}_{\Gamma_n}\varepsilon_\Phi)\setminus\{0\} = \sigma(-\nu_\Phi^{-1}\operatorname{div}_{\Gamma_n}\varepsilon_\Phi\nabla_{\Gamma_t})\setminus\{0\} = \{\lambda_{0,\Phi,k}\}_{k\in\mathbb{N}}\subset(0,\infty)
\end{aligned}$$

with eigenvalues $0<\lambda_{\ell,\Phi,1}\le\lambda_{\ell,\Phi,2}\le\ldots\le\lambda_{\ell,\Phi,k-1}\le\lambda_{\ell,\Phi,k}\le\cdots\to\infty$. Only finitely many eigenvalues coincide and they are repeated according to their multiplicity.

Proof. Theorems 2.3, 3.1, and Corollary 3.1 yield

$$\begin{aligned}
&\sigma(\mathrm{A}^*_{\ell,\Phi}\mathrm{A}_{\ell,\Phi})\setminus\{0\} = \sigma(\mathrm{A}_{\ell,\Phi}\mathrm{A}^*_{\ell,\Phi})\setminus\{0\} = \sigma(\mathcal{A}_{\ell,\Phi}\mathcal{A}^*_{\ell,\Phi}) = \sigma(\mathcal{A}^*_{\ell,\Phi}\mathcal{A}_{\ell,\Phi})\\
&\quad = \sigma(\mathrm{A}^*_\ell\mathrm{A}_\ell)\setminus\{0\} = \sigma(\mathrm{A}_\ell\mathrm{A}^*_\ell)\setminus\{0\} = \sigma(\mathcal{A}_\ell\mathcal{A}^*_\ell) = \sigma(\mathcal{A}^*_\ell\mathcal{A}_\ell) =: \{\lambda_{\ell,\Phi,k}\}_{k\in\mathbb{N}}
\end{aligned}$$

for $\ell\in\{0,1\}$. □

Remark 3.5. Note that by the definition of weak Lipschitz pair it follows directly that Φ maps a weak Lipschitz pair (Ω, Γ_t) to a weak Lipschitz pair $(\Omega_\Phi, \Gamma_{t,\Phi})$.

Theorems 3.1, 3.2 and Remark 3.2, show that eigenvectors u, E, and H in (28) for the domain Ω_Φ and for the eigenvalues λ_0 and λ_1 are mapped to eigenvectors $\tau^0_\Phi u$, $\tau^1_\Phi E$, and $\tau^1_\Phi H$ for the domain Ω for the same eigenvalues and vice versa. More precisely, u, E, and H are eigenvectors in (28), if and only if

$$\begin{aligned}
\mathrm{A}_0^* \mathrm{A}_0 \tau^0_\Phi u &= -\nu_\Phi^{-1} \mathrm{div}_{\Gamma_n} \varepsilon_\Phi \nabla_{\Gamma_t} \tau^0_\Phi u = \lambda_0 \tau^0_\Phi u && \text{in } \mathsf{L}^2_{\nu_\Phi}(\Omega),\\
\mathrm{A}_1^* \mathrm{A}_1 \tau^1_\Phi E &= \varepsilon_\Phi^{-1} \mathrm{rot}_{\Gamma_n} \mu_\Phi^{-1} \mathrm{rot}_{\Gamma_t} \tau^1_\Phi E = \lambda_1 \tau^1_\Phi E && \text{in } \mathsf{L}^2_{\varepsilon_\Phi}(\Omega),\\
\mathrm{A}_0 \mathrm{A}_0^* \tau^1_\Phi H &= -\nabla_{\Gamma_t} \nu_\Phi^{-1} \mathrm{div}_{\Gamma_n} \varepsilon_\Phi \tau^1_\Phi H = \lambda_0 \tau^1_\Phi H && \text{in } \mathsf{L}^2_{\varepsilon_\Phi}(\Omega)
\end{aligned} \tag{29}$$

with eigenvectors

$$\begin{aligned}
\tau^0_\Phi u &\in D(\mathrm{A}_0^* \mathrm{A}_0) = D(\nu_\Phi^{-1} \mathrm{div}_{\Gamma_n} \varepsilon_\Phi \nabla_{\Gamma_t}) = \Big\{ \psi \in \mathsf{H}^1_{\Gamma_t}(\Omega) \ : \ \varepsilon_\Phi \nabla \psi \in \mathsf{D}_{\Gamma_n}(\Omega) \Big\},\\
\tau^1_\Phi E &\in D(\mathrm{A}_1^* \mathrm{A}_1) = D(\varepsilon_\Phi^{-1} \mathrm{rot}_{\Gamma_n} \mu_\Phi^{-1} \mathrm{rot}_{\Gamma_t}) = \big\{ \Psi \in \mathsf{R}_{\Gamma_t}(\Omega) \ : \ \mu_\Phi^{-1} \mathrm{rot}\, \Psi \in \mathsf{R}_{\Gamma_n}(\Omega) \big\},\\
\tau^1_\Phi H &\in D(\mathrm{A}_0 \mathrm{A}_0^*) = D(\nabla_{\Gamma_t} \nu_\Phi^{-1} \mathrm{div}_{\Gamma_n} \varepsilon_\Phi) = \Big\{ \Psi \in \varepsilon_\Phi^{-1} \mathsf{D}_{\Gamma_n}(\Omega) \ : \ \nu_\Phi^{-1} \mathrm{div}\, \varepsilon_\Phi \Psi \in \mathsf{H}^1_{\Gamma_t}(\Omega) \Big\}.
\end{aligned}$$

Remark 3.6. We have

$$\begin{aligned}
u &\in D(\mathrm{A}_{0,\Phi}^* \mathrm{A}_{0,\Phi}) \subset D(\mathrm{A}_{0,\Phi}) = D(\nabla_{\Gamma_{t,\Phi}}) = \mathsf{H}^1_{\Gamma_{t,\Phi}}(\Omega_\Phi),\\
E &\in D(\mathrm{A}_{1,\Phi}^* \mathrm{A}_{1,\Phi}) \cap N(\mathrm{A}_{0,\Phi}^*) \subset D(\mathrm{A}_{1,\Phi}) \cap N(\mathrm{A}_{0,\Phi}^*) = D(\mathrm{rot}_{\Gamma_{t,\Phi}}) \cap N(\mathrm{div}_{\Gamma_{n,\Phi}} \varepsilon)\\
&= \Big\{ \Psi \in \mathsf{R}_{\Gamma_{t,\Phi}}(\Omega_\Phi) \cap \varepsilon^{-1} \mathsf{D}_{\Gamma_{n,\Phi}}(\Omega_\Phi) \ : \ \mathrm{div}\, \varepsilon \Psi = 0 \Big\}\\
&= \Big\{ \Psi \in \mathsf{R}(\Omega_\Phi) \cap \varepsilon^{-1} \mathsf{D}(\Omega_\Phi) \ : \ \mathrm{div}\, \varepsilon \Psi = 0,\ \mathrm{n} \times \Psi|_{\Gamma_{t,\Phi}} = 0,\ \mathrm{n} \cdot \varepsilon \Psi|_{\Gamma_{n,\Phi}} = 0 \Big\},\\
H &\in D(\mathrm{A}_{0,\Phi} \mathrm{A}_{0,\Phi}^*) \cap N(\mathrm{A}_{1,\Phi}) \subset D(\mathrm{A}_{0,\Phi}^*) \cap N(\mathrm{A}_{1,\Phi}) = D(\mathrm{div}_{\Gamma_{n,\Phi}} \varepsilon) \cap N(\mathrm{rot}_{\Gamma_{t,\Phi}})\\
&= \Big\{ \Psi \in \mathsf{R}_{\Gamma_{t,\Phi}}(\Omega_\Phi) \cap \varepsilon^{-1} \mathsf{D}_{\Gamma_{n,\Phi}}(\Omega_\Phi) \ : \ \mathrm{rot}\, \Psi = 0 \Big\}\\
&= \Big\{ \Psi \in \mathsf{R}(\Omega_\Phi) \cap \varepsilon^{-1} \mathsf{D}(\Omega_\Phi) \ : \ \mathrm{rot}\, \Psi = 0,\ \mathrm{n} \times \Psi|_{\Gamma_{t,\Phi}} = 0,\ \mathrm{n} \cdot \varepsilon \Psi|_{\Gamma_{n,\Phi}} = 0 \Big\},
\end{aligned}$$

and for the transformed fields

$$\begin{aligned}
\tau^0_\Phi u &\in D(\mathrm{A}_0^* \mathrm{A}_0) \subset D(\mathrm{A}_0) = D(\nabla_{\Gamma_t}) = \mathsf{H}^1_{\Gamma_t}(\Omega),\\
\tau^1_\Phi E &\in D(\mathrm{A}_1^* \mathrm{A}_1) \cap N(\mathrm{A}_0^*) \subset D(\mathrm{A}_1) \cap N(\mathrm{A}_0^*) = D(\mathrm{rot}_{\Gamma_t}) \cap N(\mathrm{div}_{\Gamma_n} \varepsilon_\Phi)\\
&= \big\{ \Psi \in \mathsf{R}_{\Gamma_t}(\Omega) \cap \varepsilon_\Phi^{-1} \mathsf{D}_{\Gamma_n}(\Omega) \ : \ \mathrm{div}\, \varepsilon_\Phi \Psi = 0 \big\}\\
&= \big\{ \Psi \in \mathsf{R}(\Omega) \cap \varepsilon_\Phi^{-1} \mathsf{D}(\Omega) \ : \ \mathrm{div}\, \varepsilon_\Phi \Psi = 0,\ \mathrm{n} \times \Psi|_{\Gamma_t} = 0,\ \mathrm{n} \cdot \varepsilon_\Phi \Psi|_{\Gamma_n} = 0 \big\},\\
\tau^1_\Phi H &\in D(\mathrm{A}_0 \mathrm{A}_0^*) \cap N(\mathrm{A}_1) \subset D(\mathrm{A}_0^*) \cap N(\mathrm{A}_1) = D(\mathrm{div}_{\Gamma_n} \varepsilon_\Phi) \cap N(\mathrm{rot}_{\Gamma_t})\\
&= \big\{ \Psi \in \mathsf{R}_{\Gamma_t}(\Omega) \cap \varepsilon_\Phi^{-1} \mathsf{D}_{\Gamma_n}(\Omega) \ : \ \mathrm{rot}\, \Psi = 0 \big\}\\
&= \big\{ \Psi \in \mathsf{R}(\Omega) \cap \varepsilon_\Phi^{-1} \mathsf{D}(\Omega) \ : \ \mathrm{rot}\, \Psi = 0,\ \mathrm{n} \times \Psi|_{\Gamma_t} = 0,\ \mathrm{n} \cdot \varepsilon_\Phi \Psi|_{\Gamma_n} = 0 \big\}.
\end{aligned}$$

For corresponding variational formulations, see the appendix.

Theorem 3.3 (eigenvalues of the generalised Laplacian). *The operators* $\rho \mathrm{A}_{0,\Phi} \mathrm{A}_{0,\Phi}^* + \mathrm{A}_{1,\Phi}^* \mathrm{A}_{1,\Phi}$ *and* $\rho \mathrm{A}_0 \mathrm{A}_0^* + \mathrm{A}_1^* \mathrm{A}_1$ *are unitarily equivalent. Moreover, both are self-adjoint and non-negative and have pure and discrete point spectrum with no accumulation point. Moreover, the positive parts of the spectra coincide, that is,*

$$\begin{aligned}
&\sigma(-\rho \nabla_{\Gamma_{t,\Phi}} \nu^{-1} \mathrm{div}_{\Gamma_{n,\Phi}} \varepsilon + \varepsilon^{-1} \mathrm{rot}_{\Gamma_{n,\Phi}} \mu^{-1} \mathrm{rot}_{\Gamma_{t,\Phi}}) \setminus \{0\}\\
&= \sigma(-\rho \nabla_{\Gamma_t} \nu_\Phi^{-1} \mathrm{div}_{\Gamma_n} \varepsilon_\Phi + \varepsilon_\Phi^{-1} \mathrm{rot}_{\Gamma_n} \mu_\Phi^{-1} \mathrm{rot}_{\Gamma_t}) \setminus \{0\}\\
&= \big(\rho \sigma(-\nabla_{\Gamma_t} \nu_\Phi^{-1} \mathrm{div}_{\Gamma_n} \varepsilon_\Phi) \cup \sigma(\varepsilon_\Phi^{-1} \mathrm{rot}_{\Gamma_n} \mu_\Phi^{-1} \mathrm{rot}_{\Gamma_t}) \big) \setminus \{0\} = \rho \{ \lambda_{0,\Phi,k} \}_{k \in \mathbb{N}} \cup \{ \lambda_{1,\Phi,k} \}_{k \in \mathbb{N}}.
\end{aligned}$$

Only finitely many eigenvalues coincide and they are repeated according to their multiplicity.

4 | Conclusion and Outlook

4.1 | Eigenvalues and Rayleigh Quotients

We recall our results on the de Rham eigenvalues which are important for the study of their shape derivatives (variations of the domain and the boundary conditions via the Lipschitz maps $\Phi : \Omega \to \Phi(\Omega) = \Omega_\Phi$) in the second part of this paper. So far we have shown that for bounded weak Lipschitz pairs (Ω, Γ_t) the de Rahm complex has countably many eigenvalues

$$0 < \lambda_{\ell,\Phi,1} \leq \lambda_{\ell,\Phi,2} \leq \cdots \leq \lambda_{\ell,\Phi,k-1} \leq \lambda_{\ell,\Phi,k} \leq \cdots \to \infty, \qquad \ell \in \{0,1\}.$$

For a fixed index k we set

$$\lambda_{0,\Phi} := \lambda_{0,\Phi,k}, \qquad \lambda_{1,\Phi} := \lambda_{1,\Phi,k}.$$

Moreover, by Lemma 2.9 and (28), (29) the eigenvalues are given by the Rayleigh quotients of the eigenfields, this is

$$\begin{aligned}
\lambda_{0,\Phi} &= \frac{|\mathrm{A}_{0,\Phi} u|^2_{\mathrm{L}^2_\varepsilon(\Omega_\Phi)}}{|u|^2_{\mathrm{L}^2_\nu(\Omega_\Phi)}} = \frac{\langle \varepsilon \nabla_{\Gamma_t,\Phi} u, \nabla_{\Gamma_t,\Phi} u \rangle_{\mathrm{L}^2(\Omega_\Phi)}}{\langle \nu u, u \rangle_{\mathrm{L}^2(\Omega_\Phi)}} \\
&= \frac{|\mathrm{A}_0 \tau^0_\Phi u|^2_{\mathrm{L}^2_{\varepsilon_\Phi}(\Omega)}}{|\tau^0_\Phi u|^2_{\mathrm{L}^2_{\nu_\Phi}(\Omega)}} = \frac{\langle \varepsilon_\Phi \nabla_{\Gamma_t} \tau^0_\Phi u, \nabla_{\Gamma_t} \tau^0_\Phi u \rangle_{\mathrm{L}^2(\Omega)}}{\langle \nu_\Phi \tau^0_\Phi u, \tau^0_\Phi u \rangle_{\mathrm{L}^2(\Omega)}} \\
&= \frac{|\mathrm{A}^*_{0,\Phi} H|^2_{\mathrm{L}^2_\nu(\Omega_\Phi)}}{|H|^2_{\mathrm{L}^2_\varepsilon(\Omega_\Phi)}} = \frac{\langle \nu^{-1} \operatorname{div}_{\Gamma_n,\Phi} \varepsilon H, \operatorname{div}_{\Gamma_n,\Phi} \varepsilon H \rangle_{\mathrm{L}^2(\Omega_\Phi)}}{\langle \varepsilon H, H \rangle_{\mathrm{L}^2(\Omega_\Phi)}} \\
&= \frac{|\mathrm{A}^*_0 \tau^1_\Phi H|^2_{\mathrm{L}^2_{\nu_\Phi}(\Omega)}}{|\tau^1_\Phi H|^2_{\mathrm{L}^2_{\varepsilon_\Phi}(\Omega)}} = \frac{\langle \nu_\Phi^{-1} \operatorname{div}_{\Gamma_n} \varepsilon_\Phi \tau^1_\Phi H, \operatorname{div}_{\Gamma_n} \varepsilon_\Phi \tau^1_\Phi H \rangle_{\mathrm{L}^2(\Omega)}}{\langle \varepsilon_\Phi \tau^1_\Phi H, \tau^1_\Phi H \rangle_{\mathrm{L}^2(\Omega)}}, \\
\lambda_{1,\Phi} &= \frac{|\mathrm{A}_{1,\Phi} E|^2_{\mathrm{L}^2_\mu(\Omega_\Phi)}}{|E|^2_{\mathrm{L}^2_\varepsilon(\Omega_\Phi)}} = \frac{\langle \mu^{-1} \operatorname{rot}_{\Gamma_t,\Phi} E, \ \operatorname{rot}_{\Gamma_t,\Phi} E \rangle_{\mathrm{L}^2(\Omega_\Phi)}}{\langle \varepsilon E, E \rangle_{\mathrm{L}^2(\Omega_\Phi)}} \\
&= \frac{|\mathrm{A}_1 \tau^1_\Phi E|^2_{\mathrm{L}^2_{\mu_\Phi}(\Omega)}}{|\tau^1_\Phi E|^2_{\mathrm{L}^2_{\varepsilon_\Phi}(\Omega)}} = \frac{\langle \mu_\Phi^{-1} \operatorname{rot}_{\Gamma_t} \tau^1_\Phi E, \ \operatorname{rot}_{\Gamma_t} \tau^1_\Phi E \rangle_{\mathrm{L}^2(\Omega)}}{\langle \varepsilon_\Phi \tau^1_\Phi E, \tau^1_\Phi E \rangle_{\mathrm{L}^2(\Omega)}},
\end{aligned} \tag{30}$$

with eigenfields u, E, H, and

$$\tau^0_\Phi u = \widetilde{u}, \qquad \tau^1_\Phi E = J_\Phi^\top \widetilde{E}, \qquad \tau^2_\Phi H = (\operatorname{adj} J_\Phi) \widetilde{H},$$

respectively. Note that the eigenvalues $\lambda_{\ell,\Phi,k}$ are depending not only on Φ (shape of the domain) but also on the mixed boundary conditions imposed on Γ_t and Γ_n and on the coefficients ε, μ, and ν, which we do not indicate explicitly in our notations, that is,

$$\lambda_{0,\Phi,k} = \lambda_{0,\Phi,k}(\Omega, \Gamma_t, \varepsilon, \nu), \qquad \lambda_{1,\Phi,k} = \lambda_{1,\Phi,k}(\Omega, \Gamma_t, \varepsilon, \mu).$$

4.2 | Heuristic Shape Derivatives of Eigenvalues

In this final subsection, we want to conclude with formal computations to derive shape derivatives of the eigenvalues assuming[4] that the corresponding (pull-backs of the) eigenvectors are differentiable with respect to Φ. This means we investigate the behavior of the eigenvalues under variations of the domain and the boundary conditions. More precisely, we investigate the differentiable dependence of the eigenvalues of the de Rahm complex if the domain, that is, the mapping Φ, is changing in certain subsets of bi-Lipschitz transformations. Here the space $\mathsf{C}^{0,1}(\overline{\Omega}, \mathbb{R}^3)$ of Lipschitz maps from $\overline{\Omega}$ to $\mathbb{R}^3$ is endowed with its standard norm.

Theorem 4.1. *Let ε, μ, and ν be of class C^1. Let u, E, H be normalized eigenfields such that*

$$|\tau^0_\Phi u|_{\mathrm{L}^2_{\nu_\Phi}(\Omega)} = |u|_{\mathrm{L}^2_\nu(\Omega_\Phi)} = |\tau^1_\Phi E|_{\mathrm{L}^2_{\varepsilon_\Phi}(\Omega)} = |E|_{\mathrm{L}^2_\varepsilon(\Omega_\Phi)} = |\tau^1_\Phi H|_{\mathrm{L}^2_{\varepsilon_\Phi}(\Omega)} = |H|_{\mathrm{L}^2_\varepsilon(\Omega_\Phi)} = 1,$$

and

$$\lambda_{0,\Phi} = |\mathrm{A}_0\tau^0_\Phi u|^2_{\mathsf{L}^2_{\varepsilon_\Phi}(\Omega)} = |\nabla_{\Gamma_t}\tau^0_\Phi u|^2_{\mathsf{L}^2_{\varepsilon_\Phi}(\Omega)} = \langle \varepsilon_\Phi \nabla_{\Gamma_t}\tau^0_\Phi u, \nabla_{\Gamma_t}\tau^0_\Phi u\rangle_{\mathsf{L}^2(\Omega)},$$

$$\lambda_{1,\Phi} = |\mathrm{A}_1\tau^1_\Phi E|^2_{\mathsf{L}^2_{\mu_\Phi}(\Omega)} = |\mu_\Phi^{-1}\operatorname{rot}_{\Gamma_t}\tau^1_\Phi E|^2_{\mathsf{L}^2_{\mu_\Phi}(\Omega)} = \langle \mu_\Phi^{-1}\operatorname{rot}_{\Gamma_t}\tau^1_\Phi E,\ \operatorname{rot}_{\Gamma_t}\tau^1_\Phi E\rangle_{\mathsf{L}^2(\Omega)},$$

$$\lambda_{0,\Phi} = |\mathrm{A}_0^*\tau^1_\Phi H|^2_{\mathsf{L}^2_{\nu_\Phi}(\Omega)} = |\nu_\Phi^{-1}\operatorname{div}_{\Gamma_n}\varepsilon_\Phi\tau^1_\Phi H|^2_{\mathsf{L}^2_{\nu_\Phi}(\Omega)} = \langle \nu_\Phi^{-1}\operatorname{div}_{\Gamma_n}\varepsilon_\Phi\tau^1_\Phi H, \operatorname{div}_{\Gamma_n}\varepsilon_\Phi\tau^1_\Phi H\rangle_{\mathsf{L}^2(\Omega)},$$

cf. Remark 3.3 and (30). Assume that $\tau^0_\Phi u$, $\tau^1_\Phi E$, $\tau^1_\Phi H$ are differentiable with respect to $\Phi \in \mathsf{C}^{0,1}(\overline{\Omega}, \mathbb{R}^3)$. Then the directional derivatives of the eigenvalues with respect to a direction $\widetilde{\Psi} \in \mathsf{C}^{0,1}(\overline{\Omega}, \mathbb{R}^3)$ are given by

$$\begin{aligned}
\partial_{\widetilde{\Psi}}\lambda_{0,\Phi} &= |\mathrm{A}_0\tau^0_\Phi u|^2_{\mathsf{L}^2_{(\partial_{\widetilde{\Psi}}\varepsilon_\Phi)}(\Omega)} - \lambda_{0,\Phi}|\tau^0_\Phi u|^2_{\mathsf{L}^2_{(\partial_{\widetilde{\Psi}}\nu_\Phi)}(\Omega)}\\
&= |\nabla_{\Gamma_t}\tau^0_\Phi u|^2_{\mathsf{L}^2_{(\partial_{\widetilde{\Psi}}\varepsilon_\Phi)}(\Omega)} - \lambda_{0,\Phi}|\tau^0_\Phi u|^2_{\mathsf{L}^2_{(\partial_{\widetilde{\Psi}}\nu_\Phi)}(\Omega)}\\
&= \big\langle (\partial_{\widetilde{\Psi}}\varepsilon_\Phi)\nabla_{\Gamma_t}\tau^0_\Phi u, \nabla_{\Gamma_t}\tau^0_\Phi u\big\rangle_{\mathsf{L}^2(\Omega)} - \lambda_{0,\Phi}\big\langle (\partial_{\widetilde{\Psi}}\nu_\Phi)\tau^0_\Phi u, \tau^0_\Phi u\big\rangle_{\mathsf{L}^2(\Omega)},\\
\partial_{\widetilde{\Psi}}\lambda_{1,\Phi} &= |\mathrm{A}_1\tau^1_\Phi E|^2_{\mathsf{L}^2_{-(\partial_{\widetilde{\Psi}}\mu_\Phi)}(\Omega)} - \lambda_{1,\Phi}|\tau^1_\Phi E|^2_{\mathsf{L}^2_{(\partial_{\widetilde{\Psi}}\varepsilon_\Phi)}(\Omega)}\\
&= |\operatorname{rot}_{\Gamma_t}\tau^1_\Phi E|^2_{\mathsf{L}^2_{(\partial_{\widetilde{\Psi}}\mu_\Phi^{-1})}(\Omega)} - \lambda_{1,\Phi}|\tau^1_\Phi E|^2_{\mathsf{L}^2_{(\partial_{\widetilde{\Psi}}\varepsilon_\Phi)}(\Omega)}\\
&= \big\langle (\partial_{\widetilde{\Psi}}\mu_\Phi^{-1})\operatorname{rot}_{\Gamma_t}\tau^1_\Phi E,\ \operatorname{rot}_{\Gamma_t}\tau^1_\Phi E\big\rangle_{\mathsf{L}^2(\Omega)} - \lambda_{1,\Phi}\big\langle (\partial_{\widetilde{\Psi}}\varepsilon_\Phi)\tau^1_\Phi E, \tau^1_\Phi E\big\rangle_{\mathsf{L}^2(\Omega)},\\
\partial_{\widetilde{\Psi}}\lambda_{0,\Phi} &= |\mathrm{A}_0^*\tau^1_\Phi H|^2_{\mathsf{L}^2_{-(\partial_{\widetilde{\Psi}}\nu_\Phi)}(\Omega)} - \lambda_{0,\Phi}|\tau^1_\Phi H|^2_{\mathsf{L}^2_{-(\partial_{\widetilde{\Psi}}\varepsilon_\Phi)}(\Omega)}\\
&= |\operatorname{div}_{\Gamma_n}\varepsilon_\Phi\tau^1_\Phi H|^2_{\mathsf{L}^2_{(\partial_{\widetilde{\Psi}}\nu_\Phi^{-1})}(\Omega)} + \lambda_{0,\Phi}|\tau^1_\Phi H|^2_{\mathsf{L}^2_{(\partial_{\widetilde{\Psi}}\varepsilon_\Phi)}(\Omega)}\\
&= \big\langle (\partial_{\widetilde{\Psi}}\nu_\Phi^{-1})\operatorname{div}_{\Gamma_n}\varepsilon_\Phi\tau^1_\Phi H, \operatorname{div}_{\Gamma_n}\varepsilon_\Phi\tau^1_\Phi H\big\rangle_{\mathsf{L}^2(\Omega)} + \lambda_{0,\Phi}\big\langle (\partial_{\widetilde{\Psi}}\varepsilon_\Phi)\tau^1_\Phi H, \tau^1_\Phi H\big\rangle_{\mathsf{L}^2(\Omega)}.
\end{aligned}$$

Here, we have formally used the norm notation although the tensor fields $\pm\partial_{\widetilde{\Psi}}(\cdots)^{\pm}_\Phi$ do not necessarily generate proper $\mathsf{L}^2(\Omega)$-inner products. Note that

$$\partial_{\widetilde{\Psi}}\varepsilon_\Phi = -\varepsilon_\Phi(\partial_{\widetilde{\Psi}}\varepsilon_\Phi^{-1})\varepsilon_\Phi, \qquad \partial_{\widetilde{\Psi}}\mu_\Phi = -\mu_\Phi(\partial_{\widetilde{\Psi}}\mu_\Phi^{-1})\mu_\Phi, \qquad \partial_{\widetilde{\Psi}}\nu_\Phi = -\nu_\Phi^2\partial_{\widetilde{\Psi}}\nu_\Phi^{-1},$$

cf. Remark A.2. Furthermore, we understand terms like $\partial_{\widetilde{\Psi}}\varepsilon$ in the sense of

$$\partial_{\widetilde{\Psi}}\varepsilon := [\partial_{\widetilde{\Psi}}\varepsilon_{j,m}].$$

Proof. We elaborate the computations only for $\lambda_{1,\Phi}$ and postpone the calculations of the remaining cases to Appendix A. By (30) and the quotient rule we compute

$$\begin{aligned}
(\partial_{\widetilde{\Psi}}\lambda_{1,\Phi})\langle\varepsilon_\Phi\tau^1_\Phi E, \tau^1_\Phi E\rangle^2_{\mathsf{L}^2(\Omega)} &= \langle\varepsilon_\Phi\tau^1_\Phi E, \tau^1_\Phi E\rangle_{\mathsf{L}^2(\Omega)}\partial_{\widetilde{\Psi}}\langle\mu_\Phi^{-1}\operatorname{rot}_{\Gamma_t}\tau^1_\Phi E,\ \operatorname{rot}_{\Gamma_t}\tau^1_\Phi E\rangle_{\mathsf{L}^2(\Omega)}\\
&\quad - \langle\mu_\Phi^{-1}\operatorname{rot}_{\Gamma_t}\tau^1_\Phi E,\ \operatorname{rot}_{\Gamma_t}\tau^1_\Phi E\rangle_{\mathsf{L}^2(\Omega)}\partial_{\widetilde{\Psi}}\langle\varepsilon_\Phi\tau^1_\Phi E, \tau^1_\Phi E\rangle_{\mathsf{L}^2(\Omega)}\\
&= \langle\varepsilon_\Phi\tau^1_\Phi E, \tau^1_\Phi E\rangle_{\mathsf{L}^2(\Omega)}\Big(\big\langle(\partial_{\widetilde{\Psi}}\mu_\Phi^{-1})\operatorname{rot}_{\Gamma_t}\tau^1_\Phi E,\ \operatorname{rot}_{\Gamma_t}\tau^1_\Phi E\big\rangle_{\mathsf{L}^2(\Omega)}\\
&\quad + 2\Re\langle\mu_\Phi^{-1}\operatorname{rot}_{\Gamma_t}\tau^1_\Phi E,\ \operatorname{rot}_{\Gamma_t}\partial_{\widetilde{\Psi}}\tau^1_\Phi E\rangle_{\mathsf{L}^2(\Omega)}\Big)\\
&\quad - \langle\mu_\Phi^{-1}\operatorname{rot}_{\Gamma_t}\tau^1_\Phi E,\ \operatorname{rot}_{\Gamma_t}\tau^1_\Phi E\rangle_{\mathsf{L}^2(\Omega)}\Big(\big\langle(\partial_{\widetilde{\Psi}}\varepsilon_\Phi)\tau^1_\Phi E, \tau^1_\Phi E\big\rangle_{\mathsf{L}^2(\Omega)}\\
&\quad + 2\Re\langle\varepsilon_\Phi\tau^1_\Phi E, \partial_{\widetilde{\Psi}}\tau^1_\Phi E\rangle_{\mathsf{L}^2(\Omega)}\Big).
\end{aligned}$$

Note that we assume that $\tau^1_\Phi E$ is differentiable and hence $\partial_{\widetilde{\Psi}}\tau^1_\Phi E$ exists. Thus using

$$\langle\mu_\Phi^{-1}\operatorname{rot}_{\Gamma_t}\tau^1_\Phi E,\ \operatorname{rot}_{\Gamma_t}\tau^1_\Phi E\rangle_{\mathsf{L}^2(\Omega)} = \lambda_{1,\Phi}\langle\varepsilon_\Phi\tau^1_\Phi E, \tau^1_\Phi E\rangle_{\mathsf{L}^2(\Omega)},$$

we see

$$
\begin{aligned}
(\partial_{\widetilde{\Psi}}\lambda_{1,\Phi})\langle\varepsilon_\Phi\tau^1_\Phi E,\tau^1_\Phi E\rangle_{\mathsf{L}^2(\Omega)} &= \Big\langle\big(\partial_{\widetilde{\Psi}}\mu_\Phi^{-1}\big)\operatorname{rot}_{\Gamma_t}\tau^1_\Phi E,\ \operatorname{rot}_{\Gamma_t}\tau^1_\Phi E\Big\rangle_{\mathsf{L}^2(\Omega)} \\
&\quad + 2\Re\langle\mu_\Phi^{-1}\operatorname{rot}_{\Gamma_t}\tau^1_\Phi E,\ \operatorname{rot}_{\Gamma_t}\partial_{\widetilde{\Psi}}\tau^1_\Phi E\rangle_{\mathsf{L}^2(\Omega)} \\
&\quad - \lambda_{1,\Phi}\Big(\Big\langle\big(\partial_{\widetilde{\Psi}}\varepsilon_\Phi\big)\tau^1_\Phi E,\tau^1_\Phi E\Big\rangle_{\mathsf{L}^2(\Omega)} + 2\Re\langle\varepsilon_\Phi\tau^1_\Phi E,\partial_{\widetilde{\Psi}}\tau^1_\Phi E\rangle_{\mathsf{L}^2(\Omega)}\Big) \\
&= \Big\langle\big(\partial_{\widetilde{\Psi}}\mu_\Phi^{-1}\big)\operatorname{rot}_{\Gamma_t}\tau^1_\Phi E,\ \operatorname{rot}_{\Gamma_t}\tau^1_\Phi E\Big\rangle_{\mathsf{L}^2(\Omega)} \\
&\quad + 2\Re\langle \mathrm{A}_1\tau^1_\Phi E,\mathrm{A}_1\partial_{\widetilde{\Psi}}\tau^1_\Phi E\rangle_{\mathsf{L}^2_{\mu_\Phi}(\Omega)} \\
&\quad - \lambda_{1,\Phi}\Big(\Big\langle\big(\partial_{\widetilde{\Psi}}\varepsilon_\Phi\big)\tau^1_\Phi E,\tau^1_\Phi E\Big\rangle_{\mathsf{L}^2(\Omega)} + 2\Re\langle\tau^1_\Phi E,\partial_{\widetilde{\Psi}}\tau^1_\Phi E\rangle_{\mathsf{L}^2_{\varepsilon_\Phi}(\Omega)}\Big) \\
&= \Big\langle\big(\partial_{\widetilde{\Psi}}\mu_\Phi^{-1}\big)\operatorname{rot}_{\Gamma_t}\tau^1_\Phi E,\ \operatorname{rot}_{\Gamma_t}\tau^1_\Phi E\Big\rangle_{\mathsf{L}^2(\Omega)} \\
&\quad - \lambda_{1,\Phi}\Big\langle\big(\partial_{\widetilde{\Psi}}\varepsilon_\Phi\big)\tau^1_\Phi E,\tau^1_\Phi E\Big\rangle_{\mathsf{L}^2(\Omega)} \\
&\quad + 2\Re\Bigg\langle\underbrace{(\mathrm{A}_1^*\mathrm{A}_1-\lambda_{1,\Phi})\tau^1_\Phi E}_{=0},\partial_{\widetilde{\Psi}}\tau^1_\Phi E\Bigg\rangle_{\mathsf{L}^2_{\varepsilon_\Phi}(\Omega)} \\
&= \langle\operatorname{rot}_{\Gamma_t}\tau^1_\Phi E,\ \operatorname{rot}_{\Gamma_t}\tau^1_\Phi E\rangle_{\mathsf{L}^2_{(\partial_{\widetilde{\Psi}}\mu_\Phi^{-1})}(\Omega)} - \lambda_{1,\Phi}\langle\tau^1_\Phi E,\tau^1_\Phi E\rangle_{\mathsf{L}^2_{(\partial_{\widetilde{\Psi}}\varepsilon_\Phi)}(\Omega)}.
\end{aligned}
$$

Note that $\partial_{\widetilde{\Psi}}\varepsilon_\Phi = -\varepsilon_\Phi(\partial_{\widetilde{\Psi}}\varepsilon_\Phi^{-1})\varepsilon_\Phi$. For a normalized eigenfield E, we obtain the assertions. □

By Lemma A.1, we have

$$
\begin{aligned}
\partial_{\widetilde{\Psi}}\varepsilon_\Phi &= (\det J_\Phi)J_\Phi^{-1}\Big(\partial_{\widetilde{\Psi}}\widetilde{\varepsilon} + (\widetilde{\operatorname{div}\Psi})\widetilde{\varepsilon} - 2\operatorname{sym}(\widetilde{J_\Psi}\widetilde{\varepsilon})\Big)J_\Phi^{-\top}, \\
\partial_{\widetilde{\Psi}}\varepsilon_\Phi^{-1} &= (\det J_\Phi)^{-1}J_\Phi^{\top}\Big(\partial_{\widetilde{\Psi}}\widetilde{\varepsilon^{-1}} - (\widetilde{\operatorname{div}\Psi})\widetilde{\varepsilon^{-1}} + 2\operatorname{sym}(\widetilde{\varepsilon^{-1}}\widetilde{J_\Psi})\Big)J_\Phi, \\
\partial_{\widetilde{\Psi}}\nu_\Phi &= (\det J_\Phi)\Big(\partial_{\widetilde{\Psi}}\widetilde{\nu} + (\widetilde{\operatorname{div}\Psi})\widetilde{\nu}\Big), \\
\partial_{\widetilde{\Psi}}\nu_\Phi^{-1} &= -(\det J_\Phi)^{-1}\widetilde{\nu}^{-2}\Big(\partial_{\widetilde{\Psi}}\widetilde{\nu} + (\widetilde{\operatorname{div}\Psi})\widetilde{\nu}\Big) = (\det J_\Phi)^{-1}\Big(\partial_{\widetilde{\Psi}}\widetilde{\nu^{-1}} - (\widetilde{\operatorname{div}\Psi})\widetilde{\nu^{-1}}\Big),
\end{aligned}
$$

where the same formulas hold for ε replaced by μ. Recall that by definition $2\operatorname{sym} M = M + M^\top$ for any square matrix M. Note that, for example, for ε, we have

$$\partial_{\widetilde{\Psi}}\widetilde{\varepsilon}_{j,m} = \widetilde{J_{\varepsilon_{j,m}}}\widetilde{\Psi},$$

and hence

$$(\partial_{\widetilde{\Psi}}\widetilde{\varepsilon}_{j,m})\circ\Phi^{-1} = J_{\varepsilon_{j,m}}\Psi := \partial_\Psi\varepsilon_{j,m}.$$

Theorem 4.2. *Let the assumptions of Theorem 4.1 be satisfied. Then*

$$
\begin{aligned}
\partial_{\widetilde{\Psi}}\lambda_{0,\Phi} &= \Big\langle\big(\partial_\Psi\varepsilon + (\operatorname{div}\Psi)\varepsilon - 2\operatorname{sym}(J_\Psi\varepsilon)\big)\nabla_{\Gamma_{t,\Phi}}u,\nabla_{\Gamma_{t,\Phi}}u\Big\rangle_{\mathsf{L}^2(\Omega_\Phi)} \\
&\quad - \lambda_{0,\Phi}\Big\langle\big(\partial_\Psi\nu + (\operatorname{div}\Psi)\nu\big)u,u\Big\rangle_{\mathsf{L}^2(\Omega_\Phi)}, \\
\partial_{\widetilde{\Psi}}\lambda_{1,\Phi} &= \Big\langle\big(\partial_\Psi\mu^{-1} - (\operatorname{div}\Psi)\mu^{-1} + 2\operatorname{sym}(\mu^{-1}J_\Psi)\big)\operatorname{rot}_{\Gamma_{t,\Phi}}E,\ \operatorname{rot}_{\Gamma_{t,\Phi}}E\Big\rangle_{\mathsf{L}^2(\Omega_\Phi)} \\
&\quad - \lambda_{1,\Phi}\Big\langle\big(\partial_\Psi\varepsilon + (\operatorname{div}\Psi)\varepsilon - 2\operatorname{sym}(J_\Psi\varepsilon)\big)E,E\Big\rangle_{\mathsf{L}^2(\Omega_\Phi)}, \\
\partial_{\widetilde{\Psi}}\lambda_{0,\Phi} &= \Big\langle\big(\partial_\Psi\nu^{-1} - (\operatorname{div}\Psi)\nu^{-1}\big)\operatorname{div}_{\Gamma_{n,\Phi}}\varepsilon H,\operatorname{div}_{\Gamma_{n,\Phi}}\varepsilon H\Big\rangle_{\mathsf{L}^2(\Omega_\Phi)} \\
&\quad + \lambda_{0,\Phi}\Big\langle\big(\partial_\Psi\varepsilon + (\operatorname{div}\Psi)\varepsilon - 2\operatorname{sym}(J_\Psi\varepsilon)\big)H,H\Big\rangle_{\mathsf{L}^2(\Omega_\Phi)}.
\end{aligned}
$$

Recalling Lemma 2.8 and with the dual eigenvectors

$$
\begin{aligned}
H^* &:= \lambda_{0,\Phi}^{-1/2} \mathrm{A}_{0,\Phi} u = \lambda_{0,\Phi}^{-1/2} \nabla_{\Gamma_{t,\Phi}} u, & u &= \lambda_{0,\Phi}^{-1/2} \mathrm{A}_{0,\Phi}^* H^* = -\lambda_{0,\Phi}^{-1/2} \nu^{-1} \operatorname{div}_{\Gamma_{n,\Phi}} \varepsilon H^*,\\
u^* &:= \lambda_{0,\Phi}^{-1/2} \mathrm{A}_{0,\Phi}^* H = -\lambda_{0,\Phi}^{-1/2} \nu^{-1} \operatorname{div}_{\Gamma_{n,\Phi}} \varepsilon H, & H &= \lambda_{0,\Phi}^{-1/2} \mathrm{A}_{0,\Phi} u^* = \lambda_{0,\Phi}^{-1/2} \nabla_{\Gamma_{t,\Phi}} u^*,\\
E^* &:= \lambda_{1,\Phi}^{-1/2} \mathrm{A}_{1,\Phi} E = \lambda_{1,\Phi}^{-1/2} \mu^{-1} \operatorname{rot}_{\Gamma_{t,\Phi}} E & E &:= \lambda_{1,\Phi}^{-1/2} \mathrm{A}_{1,\Phi}^* E^* = \lambda_{1,\Phi}^{-1/2} \varepsilon^{-1} \operatorname{rot}_{\Gamma_{n,\Phi}} E^*,
\end{aligned}
$$

we get the formulas

$$
\begin{aligned}
\frac{\partial_{\widetilde{\Psi}} \lambda_{0,\Phi}}{\lambda_{0,\Phi}} &= \left\langle \left(\partial_\Psi \varepsilon + (\operatorname{div} \Psi) \varepsilon - 2 \operatorname{sym}(J_\Psi \varepsilon) \right) H^*, H^* \right\rangle_{\mathrm{L}^2(\Omega_\Phi)}\\
&\quad - \left\langle \left(\partial_\Psi \nu + (\operatorname{div} \Psi) \nu \right) u, u \right\rangle_{\mathrm{L}^2(\Omega_\Phi)},\\
\frac{\partial_{\widetilde{\Psi}} \lambda_{1,\Phi}}{\lambda_{1,\Phi}} &= -\left\langle \left(\partial_\Psi \mu + (\operatorname{div} \Psi) \mu - 2 \operatorname{sym}(J_\Psi \mu) \right) E^*, E^* \right\rangle_{\mathrm{L}^2(\Omega_\Phi)}\\
&\quad - \left\langle \left(\partial_\Psi \varepsilon + (\operatorname{div} \Psi) \varepsilon - 2 \operatorname{sym}(J_\Psi \varepsilon) \right) E, E \right\rangle_{\mathrm{L}^2(\Omega_\Phi)},\\
\frac{\partial_{\widetilde{\Psi}} \lambda_{0,\Phi}}{\lambda_{0,\Phi}} &= -\left\langle \left(\partial_\Psi \nu + (\operatorname{div} \Psi) \nu \right) \nu u^*, u^* \right\rangle_{\mathrm{L}^2(\Omega_\Phi)}\\
&\quad + \left\langle \left(\partial_\Psi \varepsilon + (\operatorname{div} \Psi) \varepsilon - 2 \operatorname{sym}(J_\Psi \varepsilon) \right) H, H \right\rangle_{\mathrm{L}^2(\Omega_\Phi)}.
\end{aligned}
$$

Proof. Again we focus on $\lambda_{1,\Phi}$ and refer for $\lambda_{0,\Phi}$ to Appendix A. By Theorem 2.1, Corollary 2.1, and Remark 2.2 we see

$$
\begin{aligned}
\partial_{\widetilde{\Psi}} \lambda_{1,\Phi} &= \left\langle \left(\partial_{\widetilde{\Psi}} \mu_\Phi^{-1} \right) \operatorname{rot}_{\Gamma_t} \tau_\Phi^1 E,\ \operatorname{rot}_{\Gamma_t} \tau_\Phi^1 E \right\rangle_{\mathrm{L}^2(\Omega)} - \lambda_{1,\Phi} \left\langle \left(\partial_{\widetilde{\Psi}} \varepsilon_\Phi \right) \tau_\Phi^1 E, \tau_\Phi^1 E \right\rangle_{\mathrm{L}^2(\Omega)},\\
&= \left\langle (\det J_\Phi)^{-1} \left(\partial_{\widetilde{\Psi}} \widetilde{\mu^{-1}} - (\widetilde{\operatorname{div} \Psi}) \widetilde{\mu^{-1}} + 2 \operatorname{sym}(\widetilde{\mu^{-1}} \widetilde{J_\Psi}) \right) J_\Phi \operatorname{rot}_{\Gamma_t} \tau_\Phi^1 E, J_\Phi \operatorname{rot}_{\Gamma_t} \tau_\Phi^1 E \right\rangle_{\mathrm{L}^2(\Omega)}\\
&\quad - \lambda_{1,\Phi} \left\langle (\det J_\Phi) \left(\partial_{\widetilde{\Psi}} \widetilde{\varepsilon} + (\widetilde{\operatorname{div} \Psi}) \widetilde{\varepsilon} - 2 \operatorname{sym}(\widetilde{J_\Psi \varepsilon}) \right) J_\Phi^{-\top} \tau_\Phi^1 E, J_\Phi^{-\top} \tau_\Phi^1 E \right\rangle_{\mathrm{L}^2(\Omega)},\\
&= \left\langle (\det J_\Phi) \left(\partial_{\widetilde{\Psi}} \widetilde{\mu^{-1}} - (\widetilde{\operatorname{div} \Psi}) \widetilde{\mu^{-1}} + 2 \operatorname{sym}(\widetilde{\mu^{-1}} \widetilde{J_\Psi}) \right) \widetilde{\operatorname{rot}_{\Gamma_{t,\Phi}} E},\ \widetilde{\operatorname{rot}_{\Gamma_{t,\Phi}} E} \right\rangle_{\mathrm{L}^2(\Omega)}\\
&\quad - \lambda_{1,\Phi} \left\langle (\det J_\Phi) \left(\partial_{\widetilde{\Psi}} \widetilde{\varepsilon} + (\widetilde{\operatorname{div} \Psi}) \widetilde{\varepsilon} - 2 \operatorname{sym}(\widetilde{J_\Psi \varepsilon}) \right) \widetilde{E}, \widetilde{E} \right\rangle_{\mathrm{L}^2(\Omega)},\\
&= \left\langle \left(\partial_\Psi \mu^{-1} - (\operatorname{div} \Psi) \mu^{-1} + 2 \operatorname{sym}(\mu^{-1} J_\Psi) \right) \operatorname{rot}_{\Gamma_{t,\Phi}} E,\ \operatorname{rot}_{\Gamma_{t,\Phi}} E \right\rangle_{\mathrm{L}^2(\Omega_\Phi)}\\
&\quad - \lambda_{1,\Phi} \left\langle \left(\partial_\Psi \varepsilon + (\operatorname{div} \Psi) \varepsilon - 2 \operatorname{sym}(J_\Psi \varepsilon) \right) E, E \right\rangle_{\mathrm{L}^2(\Omega_\Phi)},\\
&= \left\langle \left(-\partial_\Psi \mu - (\operatorname{div} \Psi) \mu + 2 \operatorname{sym}(J_\Psi \mu) \right) \mathrm{A}_{1,\Phi} E, \mathrm{A}_{1,\Phi} E \right\rangle_{\mathrm{L}^2(\Omega_\Phi)}\\
&\quad - \lambda_{1,\Phi} \left\langle \left(\partial_\Psi \varepsilon + (\operatorname{div} \Psi) \varepsilon - 2 \operatorname{sym}(J_\Psi \varepsilon) \right) E, E \right\rangle_{\mathrm{L}^2(\Omega_\Phi)}.
\end{aligned}
$$

□

In the particular case, where ε, μ, and ν are the identity mappings, Theorem 4.2 yields

$$
\begin{aligned}
\frac{\partial_{\widetilde{\Psi}} \lambda_{0,\Phi}}{\lambda_{0,\Phi}} &= \langle (\operatorname{div} \Psi - 2 \operatorname{sym} J_\Psi) H^*, H^* \rangle_{\mathrm{L}^2(\Omega_\Phi)} - \langle (\operatorname{div} \Psi) u, u \rangle_{\mathrm{L}^2(\Omega_\Phi)}\\
&= -\langle (\operatorname{symtr} J_\Psi) H^*, H^* \rangle_{\mathrm{L}^2(\Omega_\Phi)} - \langle (\operatorname{div} \Psi) u, u \rangle_{\mathrm{L}^2(\Omega_\Phi)},\\
\frac{\partial_{\widetilde{\Psi}} \lambda_{1,\Phi}}{\lambda_{1,\Phi}} &= \langle (2 \operatorname{sym} J_\Psi - \operatorname{div} \Psi) E^*, E^* \rangle_{\mathrm{L}^2(\Omega_\Phi)} + \langle (2 \operatorname{sym} J_\Psi - \operatorname{div} \Psi) E, E \rangle_{\mathrm{L}^2(\Omega_\Phi)}\\
&= \langle (\operatorname{symtr} J_\Psi) E^*, E^* \rangle_{\mathrm{L}^2(\Omega_\Phi)} + \langle (\operatorname{symtr} J_\Psi) E, E \rangle_{\mathrm{L}^2(\Omega_\Phi)},\\
\frac{\partial_{\widetilde{\Psi}} \lambda_{0,\Phi}}{\lambda_{0,\Phi}} &= -\langle (\operatorname{div} \Psi) u^*, u^* \rangle_{\mathrm{L}^2(\Omega_\Phi)} + \langle (\operatorname{div} \Psi - 2 \operatorname{sym} J_\Psi) H, H \rangle_{\mathrm{L}^2(\Omega_\Phi)}\\
&= -\langle (\operatorname{div} \Psi) u^*, u^* \rangle_{\mathrm{L}^2(\Omega_\Phi)} - \langle (\operatorname{symtr} J_\Psi) H, H \rangle_{\mathrm{L}^2(\Omega_\Phi)},
\end{aligned}
\tag{31}
$$

with symtr $M := 2\,\mathrm{sym}\,M - (\mathrm{tr}\,M)\cdot \mathrm{id}$, that is,

$$\mathrm{symtr}\,J_\Psi = 2\,\mathrm{sym}\,J_\Psi - (\mathrm{tr}\,J_\Psi)\cdot \mathrm{id} = J_\Psi + J_\Psi^\top - (\mathrm{div}\,\Psi)\cdot \mathrm{id}.$$

Equation (31) are the formulas (10) from the introduction with $H = H^*$ and $B = E^*$.

Author Contributions

Pier Domenico Lamberti: investigation; writing – original draft; methodology; validation; writing – review and editing; formal analysis; conceptualization. **Dirk Pauly:** investigation; writing – original draft; conceptualization; methodology; validation; formal analysis; writing – review and editing. **Michele Zaccaron:** conceptualization; writing – original draft; investigation; formal analysis; validation; methodology; writing – review and editing.

Acknowledgments

The authors are thankful to Prof. Massimo Lanza de Cristoforis for useful discussions and references. The first named author acknowledges support by the project "Perturbation problems and asymptotics for elliptic differential equations: variational and potential theoretic methods" funded by the MUR "Progetti di Ricerca di Rilevante Interesse Nazionale" (PRIN) Bando 2022 grant no. 2022SENJZ3; he is also a member of the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM). The third named author is very thankful for the kind invitation of the second author and the great hospitality of the Fakultät Mathematik, Institut für Analysis of the Technische Universität Dresden during his visiting period when some problems discussed in this paper where addressed. He also acknowledges financial support by "Fondazione Ing. Aldo Gini" for his visit in Dresden. His work was partially realized during his stay at the Czech Technical University in Prague with the support of the EXPRO grant No. 20-17749X of the Czech Science Foundation. He has also received support from the French government under the France 2030 investment plan, as part of the Initiative d'Excellence d'Aix-Marseille Université – A*MIDEX AMX-21-RID-012. Open Access funding enabled and organized by Projekt DEAL.

Funding

This work was supported by the "Perturbation problems and asymptotics for elliptic differential equations: variational and potential theoretic methods" funded by the European Union - Next Generation EU and by MUR Progetti di Ricerca di Rilevante Interesse Nazionale (PRIN) Bando 2022 grant (Grant No. 2022SENJZ3), EXPRO grant No. 20-17749X of the Czech Science Foundation.

Conflicts of Interest

This work does not have any conflicts of interest.

Endnotes

[1] The so-called reduced operators are compact but this does not help too much because their domain depends too heavily on the perturbations.

[2] Note that the Jacobian determinant of a bi-Lipschitz diffeomorphism has a constant sign on the connected components of the domain, see [27], Lemma 6.7, hence it is not restrictive to assume that it is positive almost everywhere.

[3] The related bounded linear operator, where the domain $D(\mathrm{A})$ is endowed with the graph norm, shall be denoted by $\mathrm{A} : D(\mathrm{A}) \to \mathsf{H}_1$.

[4] Note that this assumption is quite strong and, unless one restricts the analysis to suitable families of perturbations Φ, it requires that the eigenvalue under consideration is simple. See Part II of this series of papers for more details concerning multiple eigenvalues.

References

1. D. Henry, *Perturbation of the Boundary in Boundary-Value Problems of Partial Differential Equations*, vol. 318, ed. J. Hale and A. L. Pereira (Cambridge University Press, 2005).

2. J. M. Arrieta, F. Ferraresso, and P. D. Lamberti, "Boundary Homogenization for a Triharmonic Intermediate Problem," *Mathematical Methods in the Applied Sciences* 41, no. 3 (2018): 979–985.

3. J. M. Arrieta, F. Ferraresso, and P. D. Lamberti, "Spectral Analysis of the Biharmonic Operator Subject to Neumann Boundary Conditions on Dumbbell Domains," *Integral Equations and Operator Theory* 89, no. 3 (2017): 377–408.

4. D. Buoso and P. D. Lamberti, "Eigenvalues of Polyharmonic Operators on Variable Domains," *ESAIM. Control, Optimisation and Calculus of Variations* 19, no. 4 (2013): 1225–1235.

5. F. Ferraresso, "On the Spectral Instability for Weak Intermediate Triharmonic Problems," *Mathematicsl Methods in the Applied Sciences* 45, no. 10 (2022): 5864–5891.

6. F. Ferraresso and P. D. Lamberti, “On a Babuška Paradox for Polyharmonic Operators: Spectral Stability and Boundary Homogenization for Intermediate Problems,” *Integral Equations and Operator Theory* 91, no. 6 (2019): 55.

7. D. Buoso, “Shape Differentiability of the Eigenvalues of Elliptic Systems,” in *Integral Methods in Science and Engineering* (Birkhäuser/Springer, 2015), 91–97.

8. D. Buoso and P. D. Lamberti, “Shape Sensitivity Analysis of the Eigenvalues of the Reissner-Mindlin System,” *SIAM Journal on Mathematical Analysis* 47, no. 1 (2015): 407–426.

9. S. Bögli, F. Ferraresso, M. Marletta, and C. Tretter, “Spectral Analysis and Domain Truncation for Maxwell’s Equations,” *Journal de Mathématiques Pures et Appliquées* 170 (2023): 96–135.

10. S. Jimbo, “Hadamard Variation for Electromagnetic Frequencies,” in *Geometric Properties for Parabolic and Elliptic PDE’S*, vol. 2 (Springer, 2013), 179–199.

11. S. G. Johnson, M. Ibanescu, M. A. Skorobogatiy, O. Weisberg, J. D. Joannopoulos, and Y. Fink, “Perturbation Theory for Maxwell’s Equations With Shifting Material Boundaries,” *Physical Review E* 65, no. 6 (2002): 066611.

12. P. D. Lamberti and M. Zaccaron, “Shape Sensitivity Analysis for Electromagnetic Cavities,” *Mathematical Methods in the Applied Sciences* 44, no. 13 (2021): 10477–10500.

13. P. D. Lamberti and M. Zaccaron, “Spectral Stability of the Curlcurl Operator via Uniform Gaffney Inequalities on Perturbed Electromagnetic Cavities,” *Mathematics in Engineering* 5, no. 1 (2023): 1–31.

14. M. Zaccaron, “A Remark on the Hadamard-Hirakawa Formula for an Electromagnetic Cavity Problem,” *Bollettino dell’Unione Matematica Italiana* 17 (2024): 527–535.

15. M. Zaccaron, *Spectral Sensitivity Analysis for Electromagnetic Cavities* PhD Thesis (Università degli Studi di Padova, Dipartimento di Matematica “Tullio Levi-Civita”, 2022).

16. H. Hirakawa, *Denki Rikigaku* (Baifukan, 1973).

17. J. G. Esteve, F. Falceto, and C. García Canal, “Generalization of the Hellmann-Feynman Theorem,” *Physics Letters A* 374, no. 6 (2010): 819–822.

18. P. D. Lamberti and M. de Lanza Cristoforis, “A Real Analyticity Result for Symmetric Functions of the Eigenvalues of a Domain Dependent Dirichlet Problem for the Laplace Operator,” *Journal of Nonlinear and Convex Analysis* 5, no. 1 (2004): 19–42.

19. S. Bauer, D. Pauly, and M. Schomburg, “The Maxwell Compactness Property in Bounded Weak Lipschitz Domains With Mixed Boundary Conditions,” *SIAM Journal on Mathematical Analysis* 48, no. 4 (2016): 2912–2943.

20. D. Pauly, “On the Maxwell Constants in 3D,” *Mathematicsl Methods in the Applied Sciences* 40, no. 2 (2017): 435–447.

21. D. Pauly, “A Global div-Curl-Lemma for Mixed Boundary Conditions in Weak Lipschitz Domains and a Corresponding Generalized A^*_0-A_1-Lemma in Hilbert Spaces,” *Analysis* 39 (2019): 33–58.

22. D. Pauly and M. Schomburg, “Hilbert Complexes With Mixed Boundary Conditions – Part 1: De Rham Complex,” *Mathematical Methods in the Applied Sciences* 45, no. 5 (2022): 2465–2507.

23. P. D. Lamberti and M. de Lanza Cristoforis, “A Real Analyticity Result for Symmetric Functions of the Eigenvalues of a Domain-Dependent Neumann Problem for the Laplace Operator,” *Mediterranean Journal of Mathematics* 4, no. 4 (2007): 435–449.

24. R. A. Adams and J. J. F. Fournier, *Sobolev Spaces*, vol. 140, 2nd ed. (Elsevier/Academic Press, 2003).

25. S. Agmon, *Lectures on Elliptic Boundary Value Problems* (Van Nostrand, 1965).

26. J. Wloka, *Partial Differential Equations* (Teubner, 1982).

27. Y. G. Reshetnyak, “Space Mappings With Bounded Distortion,” in *Translations of Mathematical Monographs* (American Mathematical Society, 1989).

28. S. Bauer, D. Pauly, and M. Schomburg, “Weck’s Selection Theorem: The Maxwell Compactness Property for Bounded Weak Lipschitz Domains With Mixed Boundary Conditions in Arbitrary Dimensions,” in *Maxwell’s Equations: Analysis and Numerics*, vol. 24 (De Gruyter, 2019), 77–104.

29. D. Pauly, “On the Maxwell and Friedrichs/Poincare Constants in ND,” *Mathematische Zeitschrift* 293, no. 3 (2019): 957–987.

30. D. Pauly, “Solution Theory, Variational Formulations, and Functional a Posteriori Error Estimates for General First Order Systems With Applications to Electro-Magneto-Statics and More,” *Numerical Functional Analysis and Optimization* 41, no. 1 (2020): 16–112.

31. D. Pauly and M. Waurick, “The Index of Some Mixed Order Dirac Type Operators and Generalised Dirichlet–Neumann Tensor Fields,” *Mathematische Zeitschrift* 301, no. 2 (2022): 1739–1819.

32. D. Pauly and M. Schomburg, “Hilbert Complexes With Mixed Boundary Conditions – Part 2: Elasticity Complex,” *Mathematical Methods in the Applied Sciences* 45, no. 16 (2022): 8971–9005.

33. D. Pauly and M. Schomburg, "Hilbert Complexes With Mixed Boundary Conditions Part 3: Biharmonic Complexes," *Mathematical Methods in the Applied Sciences* 47, no. 6 (2024): 3847–3892.

34. D. Pauly and W. Zulehner, "The divDiv-Complex and Applications to Biharmonic Equations," *Applicable Analysis* 99, no. 9 (2020): 1579–1630.

35. D. Pauly and W. Zulehner, "The Elasticity Complex: Compact Embeddings and Regular Decompositions," *Applicable Analysis* 102, no. 16 (2023): 4393–4421.

36. N. Weck, "Maxwell's Boundary Value Problems on Riemannian Manifolds With Nonsmooth Boundaries," *Journal of Mathematical Analysis and Applications* 46 (1974): 410–437.

37. R. Picard, "An Elementary Proof for a Compact Imbedding Result in Generalized Electromagnetic Theory," *Mathematische Zeitschrift* 187 (1984): 151–164.

38. C. Weber, "A Local Compactness Theorem for Maxwell's Equations," *Mathematical Methods in the Applied Sciences* 2 (1980): 12–25.

39. P. Fernandes and G. Gilardi, "Magnetostatic and Electrostatic Problems in Inhomogeneous Anisotropic Media With Irregular Boundary and Mixed Boundary Conditions," *Mathematical Models and Methods in Applied Sciences* 7, no. 7 (1997): 957–991.

40. F. Jochmann, "A Compactness Result for Vector Fields With Divergence and Curl in $\mathsf{L}^q(\Omega)$ Involving Mixed Boundary Conditions," *Applicable Analysis* 66 (1997): 189–203.

Appendix A

Computations of Shape Derivatives

Recall from Section 3.1 the transformed matrices

$$\varepsilon_\Phi = \tau^2_\Phi \varepsilon \tau^1_{\Phi^{-1}} = (\det J_\Phi) J_\Phi^{-1} \widetilde{\varepsilon} J_\Phi^{-\top}, \qquad \nu_\Phi = \tau^3_\Phi \nu \tau^0_{\Phi^{-1}} = (\det J_\Phi)\widetilde{\nu},$$
$$\varepsilon_\Phi^{-1} = \tau^1_\Phi \varepsilon^{-1} \tau^2_{\Phi^{-1}} = (\det J_\Phi)^{-1} J_\Phi^\top \widetilde{\varepsilon^{-1}} J_\Phi.$$

Note that generally

$$\partial_v f(x) = f'(x)v$$

and that $\partial_v f(x) = f'(x)v = f v$ holds for bounded linear f.

Lemma A.1. *Let $k \in \mathbb{R}$. It holds*

$$\partial_{\widetilde{\Psi}} J_\Phi = J_{\widetilde{\Psi}} = \widetilde{J_\Psi} J_\Phi, \qquad \partial_{\widetilde{\Psi}} J_\Phi^{-1} = -J_\Phi^{-1} J_{\widetilde{\Psi}} J_\Phi^{-1} = -J_\Phi^{-1} \widetilde{J_\Psi},$$
$$\partial_{\widetilde{\Psi}} J_\Phi^\top = J_{\widetilde{\Psi}}^\top = J_\Phi^\top \widetilde{J_\Psi^\top}, \qquad \partial_{\widetilde{\Psi}} J_\Phi^{-\top} = -J_\Phi^{-\top} J_{\widetilde{\Psi}}^\top J_\Phi^{-\top} = -\widetilde{J_\Psi^\top} J_\Phi^{-\top},$$

and

$$\partial_{\widetilde{\Psi}} (\det J_\Phi) = (\det J_\Phi) \widetilde{\operatorname{div} \Psi}, \qquad \partial_{\widetilde{\Psi}} \big((\det J_\Phi)^k\big) = k(\det J_\Phi)^k \widetilde{\operatorname{div} \Psi}.$$

Moreover, if ε and ν are of class C^1, it holds

$$\partial_{\widetilde{\Psi}} \varepsilon_\Phi = (\det J_\Phi) J_\Phi^{-1} \Big(\partial_{\widetilde{\Psi}} \widetilde{\varepsilon} + (\widetilde{\operatorname{div} \Psi}) \widetilde{\varepsilon} - 2 \operatorname{sym}(\widetilde{J_\Psi} \widetilde{\varepsilon}) \Big) J_\Phi^{-\top},$$
$$\partial_{\widetilde{\Psi}} \varepsilon_\Phi^{-1} = (\det J_\Phi)^{-1} J_\Phi^\top \Big(\partial_{\widetilde{\Psi}} \widetilde{\varepsilon^{-1}} - (\widetilde{\operatorname{div} \Psi}) \widetilde{\varepsilon^{-1}} + 2 \operatorname{sym}(\widetilde{\varepsilon^{-1}} \widetilde{J_\Psi}) \Big) J_\Phi,$$
$$\partial_{\widetilde{\Psi}} \nu_\Phi = (\det J_\Phi) \Big(\partial_{\widetilde{\Psi}} \widetilde{\nu} + (\widetilde{\operatorname{div} \Psi}) \widetilde{\nu} \Big) = \nu_\Phi \left(\frac{\partial_{\widetilde{\Psi}} \widetilde{\nu}}{\widetilde{\nu}} + (\widetilde{\operatorname{div} \Psi}) \right),$$
$$\partial_{\widetilde{\Psi}} \nu_\Phi^k = k(\det J_\Phi)^k \widetilde{\nu}^{k-1} \Big(\partial_{\widetilde{\Psi}} \widetilde{\nu} + (\widetilde{\operatorname{div} \Psi}) \widetilde{\nu} \Big) = k \nu_\Phi^k \left(\frac{\partial_{\widetilde{\Psi}} \widetilde{\nu}}{\widetilde{\nu}} + (\widetilde{\operatorname{div} \Psi}) \right).$$

Remark A.1. In particular, we have for $\varepsilon = \mathrm{id}$

$$\partial_{\widetilde{\Psi}} \mathrm{id}_\Phi = (\det J_\Phi) J_\Phi^{-1} \Big(\widetilde{\operatorname{div} \Psi} - 2 \operatorname{sym} \widetilde{J_\Psi} \Big) J_\Phi^{-\top} = -(\det J_\Phi) J_\Phi^{-1} (\widetilde{\operatorname{symtr} J_\Psi}) J_\Phi^{-\top},$$
$$\partial_{\widetilde{\Psi}} \mathrm{id}_\Phi^{-1} = (\det J_\Phi)^{-1} J_\Phi^\top \Big(-\widetilde{\operatorname{div} \Psi} + 2 \operatorname{sym} \widetilde{J_\Psi} \Big) J_\Phi = (\det J_\Phi)^{-1} J_\Phi^\top (\widetilde{\operatorname{symtr} J_\Psi}) J_\Phi$$

with symtr from (31), that is, $\operatorname{symtr} J_\Psi = 2 \operatorname{sym} J_\Psi - \operatorname{div} \Psi$.

Remark A.2. Note that

$$\partial_{\widetilde{\Psi}}\varepsilon_\Phi^{-1} = -\varepsilon_\Phi^{-1}(\partial_{\widetilde{\Psi}}\varepsilon_\Phi)\varepsilon_\Phi^{-1}, \qquad \partial_{\widetilde{\Psi}}\nu_\Phi^{-1} = -\nu_\Phi^{-2}\partial_{\widetilde{\Psi}}\nu_\Phi,$$
$$\partial_{\widetilde{\Psi}}\varepsilon_\Phi = -\varepsilon_\Phi(\partial_{\widetilde{\Psi}}\varepsilon_\Phi^{-1})\varepsilon_\Phi, \qquad \partial_{\widetilde{\Psi}}\nu_\Phi = -\nu_\Phi^{2}\partial_{\widetilde{\Psi}}\nu_\Phi^{-1}.$$

Similar formulas hold for ε_Φ and ν_Φ replaced by $\widetilde{\varepsilon}$ and $\widetilde{\nu}$, respectively.

Proof of Lemma A.1. By the chain rule we have $(\widetilde{\Psi})' = \widetilde{\Psi'}\Phi'$, that is, $J_{\widetilde{\Psi}} = \widetilde{J_\Psi}J_\Phi$ and $J_{\widetilde{\Psi}}J_\Phi^{-1} = \widetilde{J_\Psi}$. Since $\det(\mathrm{id} + sT) = 1 + s\operatorname{tr} T + O(s^2)$ we get with $\operatorname{tr} J_\Psi = \operatorname{div}\Psi$

$$\begin{aligned}\det J_{\Phi+s\widetilde{\Psi}} &= \det(J_\Phi + sJ_{\widetilde{\Psi}}) = (\det J_\Phi)\det(\mathrm{id} + sJ_{\widetilde{\Psi}}J_\Phi^{-1}) = (\det J_\Phi)\det(\mathrm{id} + s\widetilde{J_\Psi})\\ &= (\det J_\Phi)\Big(1 + s\operatorname{tr}\widetilde{J_\Psi} + O(s^2)\Big) = (\det J_\Phi)\Big(1 + s\,\widetilde{\operatorname{div}\Psi} + O(s^2)\Big).\end{aligned}$$

Moreover, for topological isomorphisms it holds $\partial_H T^{-1} = (T^{-1})'H = -T^{-1}HT^{-1}$ as

$$(T+H)^{-1} = T^{-1}(\mathrm{id} + HT^{-1})^{-1} = T^{-1}\sum_{n\ge 0}(-HT^{-1})^n = T^{-1} - T^{-1}HT^{-1} + O\big(|H|^2\big).$$

Then the first six and the last two derivatives in the lemma are easily computed. Furthermore, using the latter results we get

$$\begin{aligned}\partial_{\widetilde{\Psi}}\varepsilon_\Phi = \partial_{\widetilde{\Psi}}\big((\det J_\Phi)J_\Phi^{-1}\widetilde{\varepsilon}J_\Phi^{-\top}\big) &= \big(\partial_{\widetilde{\Psi}}(\det J_\Phi)\big)J_\Phi^{-1}\widetilde{\varepsilon}J_\Phi^{-\top} + (\det J_\Phi)(\partial_{\widetilde{\Psi}}J_\Phi^{-1})\widetilde{\varepsilon}J_\Phi^{-\top}\\ &\quad + (\det J_\Phi)J_\Phi^{-1}(\partial_{\widetilde{\Psi}}\widetilde{\varepsilon})J_\Phi^{-\top} + (\det J_\Phi)J_\Phi^{-1}\widetilde{\varepsilon}(\partial_{\widetilde{\Psi}}J_\Phi^{-\top})\\ &= \Big((\det J_\Phi)\widetilde{\operatorname{div}\Psi}\Big)J_\Phi^{-1}\widetilde{\varepsilon}J_\Phi^{-\top} - (\det J_\Phi)J_\Phi^{-1}\widetilde{J_\Psi}\widetilde{\varepsilon}J_\Phi^{-\top}\\ &\quad + (\det J_\Phi)J_\Phi^{-1}(\partial_{\widetilde{\Psi}}\widetilde{\varepsilon})J_\Phi^{-\top} - (\det J_\Phi)J_\Phi^{-1}\widetilde{\varepsilon}\widetilde{J_\Psi^\top}J_\Phi^{-\top}\\ &= (\det J_\Phi)J_\Phi^{-1}\left((\widetilde{\operatorname{div}\Psi})\widetilde{\varepsilon} - \underbrace{\Big(\widetilde{J_\Psi}\widetilde{\varepsilon} + \widetilde{\varepsilon}\widetilde{J_\Psi^\top}\Big)}_{=2\operatorname{sym}(\widetilde{J_\Psi}\widetilde{\varepsilon})} + \partial_{\widetilde{\Psi}}\widetilde{\varepsilon}\right)J_\Phi^{-\top},\end{aligned}$$

and, using this, by the chain rule

$$\begin{aligned}\partial_{\widetilde{\Psi}}\varepsilon_\Phi^{-1} &= -\varepsilon_\Phi^{-1}(\partial_{\widetilde{\Psi}}\varepsilon_\Phi)\varepsilon_\Phi^{-1}\\ &= -(\det J_\Phi)^{-1}J_\Phi^\top\widetilde{\varepsilon^{-1}}\underbrace{J_\Phi J_\Phi^{-1}}_{=\mathrm{id}}\Big((\widetilde{\operatorname{div}\Psi})\widetilde{\varepsilon} - 2\operatorname{sym}(\widetilde{J_\Psi}\widetilde{\varepsilon}) + \partial_{\widetilde{\Psi}}\widetilde{\varepsilon}\Big)\underbrace{J_\Phi^{-\top}J_\Phi^\top}_{=\mathrm{id}}\widetilde{\varepsilon^{-1}}J_\Phi\\ &= -(\det J_\Phi)^{-1}J_\Phi^\top\left((\widetilde{\operatorname{div}\Psi})\widetilde{\varepsilon^{-1}} - 2\underbrace{\widetilde{\varepsilon^{-1}}\operatorname{sym}(\widetilde{J_\Psi}\widetilde{\varepsilon})\widetilde{\varepsilon^{-1}}}_{=\operatorname{sym}(\widetilde{\varepsilon^{-1}}\widetilde{J_\Psi})} + \underbrace{\widetilde{\varepsilon^{-1}}(\partial_{\widetilde{\Psi}}\widetilde{\varepsilon})\widetilde{\varepsilon^{-1}}}_{=-\partial_{\widetilde{\Psi}}\widetilde{\varepsilon^{-1}}}\right)J_\Phi,\end{aligned}$$

finishing the proof. □

Variational Formulations

For A_ℓ and $\mathrm{A}_{\ell,\Phi}$, $\ell \in \{0,1\}$, from Section 3.3 we note the following variational formulations: For all

$$\psi_\Phi \in D(\mathrm{A}_{0,\Phi}) = \mathsf{H}^1_{\Gamma_{t,\Phi}}(\Omega_\Phi), \quad \Psi_\Phi \in D(\mathrm{A}_{1,\Phi}) = \mathsf{R}_{\Gamma_{t,\Phi}}(\Omega_\Phi), \quad \Theta_\Phi \in D(\mathrm{A}^*_{0,\Phi}) = \varepsilon^{-1}\mathsf{D}_{\Gamma_{n,\Phi}}(\Omega_\Phi),$$

it holds

$$\begin{aligned}\lambda_0\langle\nu u, \psi_\Phi\rangle_{\mathsf{L}^2(\Omega_\Phi)} &= \lambda_0\langle u, \psi_\Phi\rangle_{\mathsf{L}^2_\nu(\Omega_\Phi)} = \langle \mathrm{A}^*_{0,\Phi}\mathrm{A}_{0,\Phi}u, \psi_\Phi\rangle_{\mathsf{L}^2_\nu(\Omega_\Phi)}\\ &= \langle \mathrm{A}_{0,\Phi}u, \mathrm{A}_{0,\Phi}\psi_\Phi\rangle_{\mathsf{L}^2_\varepsilon(\Omega_\Phi)} = \langle \varepsilon\nabla_{\Gamma_{t,\Phi}}u, \nabla_{\Gamma_{t,\Phi}}\psi_\Phi\rangle_{\mathsf{L}^2(\Omega_\Phi)},\\ \lambda_1\langle\varepsilon E, \Psi_\Phi\rangle_{\mathsf{L}^2(\Omega_\Phi)} &= \lambda_1\langle E, \Psi_\Phi\rangle_{\mathsf{L}^2_\varepsilon(\Omega_\Phi)} = \langle \mathrm{A}^*_{1,\Phi}\mathrm{A}_{1,\Phi}E, \Psi_\Phi\rangle_{\mathsf{L}^2_\varepsilon(\Omega_\Phi)}\\ &= \langle \mathrm{A}_{1,\Phi}E, \mathrm{A}_{1,\Phi}\Psi_\Phi\rangle_{\mathsf{L}^2_\mu(\Omega_\Phi)} = \langle \mu^{-1}\operatorname{rot}_{\Gamma_{t,\Phi}}E, \operatorname{rot}_{\Gamma_{t,\Phi}}\Psi_\Phi\rangle_{\mathsf{L}^2(\Omega_\Phi)},\\ \lambda_0\langle\varepsilon H, \Theta_\Phi\rangle_{\mathsf{L}^2(\Omega_\Phi)} &= \lambda_0\langle H, \Theta_\Phi\rangle_{\mathsf{L}^2_\varepsilon(\Omega_\Phi)} = \langle \mathrm{A}_{0,\Phi}\mathrm{A}^*_{0,\Phi}H, \Theta_\Phi\rangle_{\mathsf{L}^2_\varepsilon(\Omega_\Phi)}\\ &= \langle \mathrm{A}^*_{0,\Phi}H, \mathrm{A}^*_{0,\Phi}\Theta_\Phi\rangle_{\mathsf{L}^2_\nu(\Omega_\Phi)} = \langle \nu^{-1}\operatorname{div}_{\Gamma_{n,\Phi}}\varepsilon H, \operatorname{div}_{\Gamma_{n,\Phi}}\varepsilon\Theta_\Phi\rangle_{\mathsf{L}^2(\Omega_\Phi)}.\end{aligned}$$

For all

$$\psi \in D(\mathrm{A}_0) = \mathsf{H}^1_{\Gamma_t}(\Omega), \quad \Psi \in D(\mathrm{A}_1) = \mathsf{R}_{\Gamma_t}(\Omega), \quad \Theta \in D(\mathrm{A}_0^*) = \varepsilon_\Phi^{-1}\mathsf{D}_{\Gamma_n}(\Omega),$$

it holds

$$\begin{aligned}
\lambda_0\langle \nu_\Phi \tau^0_\Phi u, \psi\rangle_{\mathsf{L}^2(\Omega)} &= \lambda_0\langle \tau^0_\Phi u, \psi\rangle_{\mathsf{L}^2_{\nu_\Phi}(\Omega)} = \langle \mathrm{A}_0^*\mathrm{A}_0 \tau^0_\Phi u, \psi\rangle_{\mathsf{L}^2_{\nu_\Phi}(\Omega)} \\
&= \langle \mathrm{A}_0 \tau^0_\Phi u, \mathrm{A}_0\psi\rangle_{\mathsf{L}^2_{\varepsilon_\Phi}(\Omega)} = \langle \varepsilon_\Phi \nabla_{\Gamma_t}\tau^0_\Phi u, \nabla_{\Gamma_t}\psi\rangle_{\mathsf{L}^2(\Omega)}, \\
\lambda_1\langle \varepsilon_\Phi \tau^1_\Phi E, \Psi\rangle_{\mathsf{L}^2(\Omega)} &= \lambda_1\langle \tau^1_\Phi E, \Psi\rangle_{\mathsf{L}^2_{\varepsilon_\Phi}(\Omega)} = \langle \mathrm{A}_1^*\mathrm{A}_1 \tau^1_\Phi E, \Psi\rangle_{\mathsf{L}^2_{\varepsilon_\Phi}(\Omega)} \\
&= \langle \mathrm{A}_1 \tau^1_\Phi E, \mathrm{A}_1\Psi\rangle_{\mathsf{L}^2_{\mu_\Phi}(\Omega)} = \langle \mu_\Phi^{-1}\operatorname{rot}_{\Gamma_t}\tau^1_\Phi E, \operatorname{rot}_{\Gamma_t}\Psi\rangle_{\mathsf{L}^2(\Omega)}, \\
\lambda_0\langle \varepsilon_\Phi \tau^1_\Phi H, \Theta\rangle_{\mathsf{L}^2(\Omega)} &= \lambda_0\langle \tau^1_\Phi H, \Theta\rangle_{\mathsf{L}^2_{\varepsilon_\Phi}(\Omega)} = \langle \mathrm{A}_0\mathrm{A}_0^* \tau^1_\Phi H, \Theta\rangle_{\mathsf{L}^2_{\varepsilon_\Phi}(\Omega)} \\
&= \langle \mathrm{A}_0^* \tau^1_\Phi H, \mathrm{A}_0^*\Theta\rangle_{\mathsf{L}^2_{\nu_\Phi}(\Omega)} = \langle \nu_\Phi^{-1}\operatorname{div}_{\Gamma_n}\varepsilon_\Phi\tau^1_\Phi H, \operatorname{div}_{\Gamma_n}\varepsilon_\Phi\Theta\rangle_{\mathsf{L}^2(\Omega)}.
\end{aligned}$$

Hence, more explicitly,

$$\begin{aligned}
\lambda_0\left\langle (\det J_\Phi)\tilde{\nu}\tau^0_\Phi u, \psi\right\rangle_{\mathsf{L}^2(\Omega)} &= \left\langle (\det J_\Phi)\tilde{\varepsilon}J_\Phi^{-\top}\nabla_{\Gamma_t}\tau^0_\Phi u, J_\Phi^{-\top}\nabla_{\Gamma_t}\psi\right\rangle_{\mathsf{L}^2(\Omega)}, \\
\lambda_1\left\langle (\det J_\Phi)\tilde{\varepsilon}J_\Phi^{-\top}\tau^1_\Phi E, J_\Phi^{-\top}\Psi\right\rangle_{\mathsf{L}^2(\Omega)} &= \left\langle (\det J_\Phi)^{-1}\tilde{\mu}^{-1}J_\Phi\operatorname{rot}_{\Gamma_t}\tau^1_\Phi E, J_\Phi\operatorname{rot}_{\Gamma_t}\Psi\right\rangle_{\mathsf{L}^2(\Omega)}, \\
\lambda_0\left\langle (\det J_\Phi)\tilde{\varepsilon}J_\Phi^{-\top}\tau^1_\Phi H, J_\Phi^{-\top}\Theta\right\rangle_{\mathsf{L}^2(\Omega)} &= \left\langle (\det J_\Phi)^{-1}\tilde{\nu}^{-1}\operatorname{div}_{\Gamma_n}\varepsilon_\Phi\tau^1_\Phi H, \operatorname{div}_{\Gamma_n}\varepsilon_\Phi\Theta\right\rangle_{\mathsf{L}^2(\Omega)}.
\end{aligned}$$

Note that $\tau^0_\Phi u = \tilde{u}$, $J_\Phi^{-\top}\tau^1_\Phi E = \widetilde{E}$, and $\varepsilon_\Phi = \tau^2_\Phi \varepsilon \tau^1_{\Phi^{-1}} = (\det J_\Phi)J_\Phi^{-1}\tilde{\varepsilon}J_\Phi^{-\top} = (\operatorname{adj} J_\Phi)\tilde{\varepsilon}J_\Phi^{-\top}$. Thus

$$\begin{aligned}
\lambda_0\langle (\det J_\Phi)\tilde{\nu}\tilde{u}, \psi\rangle_{\mathsf{L}^2(\Omega)} &= \left\langle (\det J_\Phi)\tilde{\varepsilon}J_\Phi^{-\top}\nabla_{\Gamma_t}\tilde{u}, J_\Phi^{-\top}\nabla_{\Gamma_t}\psi\right\rangle_{\mathsf{L}^2(\Omega)}, \\
\lambda_1\left\langle (\det J_\Phi)\widetilde{\varepsilon E}, J_\Phi^{-\top}\Psi\right\rangle_{\mathsf{L}^2(\Omega)} &= \left\langle (\det J_\Phi)^{-1}\tilde{\mu}^{-1}J_\Phi\operatorname{rot}_{\Gamma_t}J_\Phi^{\top}\widetilde{E}, J_\Phi\operatorname{rot}_{\Gamma_t}\Psi\right\rangle_{\mathsf{L}^2(\Omega)}, \\
\lambda_0\left\langle (\det J_\Phi)\widetilde{\varepsilon H}, J_\Phi^{-\top}\Theta\right\rangle_{\mathsf{L}^2(\Omega)} &= \left\langle (\det J_\Phi)^{-1}\tilde{\nu}^{-1}\operatorname{div}_{\Gamma_n}(\operatorname{adj} J_\Phi)\widetilde{\varepsilon H}, \operatorname{div}_{\Gamma_n}(\operatorname{adj} J_\Phi)\tilde{\varepsilon}J_\Phi^{-\top}\Theta\right\rangle_{\mathsf{L}^2(\Omega)}.
\end{aligned}$$

Some Additional Proofs

Proof of Lemma 2.1. Consider the densely defined and closed linear operators

$$\begin{aligned}
\mathrm{A}_0 &:= \nabla_\Gamma : \mathsf{H}^1_\Gamma(\Omega) \subset \mathsf{L}^2(\Omega) \to \mathsf{L}^2(\Omega), \\
\mathrm{A}_1 &:= \operatorname{rot}_\Gamma : \mathsf{R}_\Gamma(\Omega) \subset \mathsf{L}^2(\Omega) \to \mathsf{L}^2(\Omega), \\
\mathrm{A}_2 &:= \operatorname{div}_\Gamma : \mathsf{D}_\Gamma(\Omega) \subset \mathsf{L}^2(\Omega) \to \mathsf{L}^2(\Omega)
\end{aligned}$$

together with their densely defined and closed adjoints

$$\begin{aligned}
\mathrm{A}_0^* &= -\operatorname{div} : \mathbf{D}(\Omega) \subset \mathsf{L}^2(\Omega) \to \mathsf{L}^2(\Omega), \\
\mathrm{A}_1^* &= \operatorname{rot} : \mathbf{R}(\Omega) \subset \mathsf{L}^2(\Omega) \to \mathsf{L}^2(\Omega), \\
\mathrm{A}_2^* &= -\nabla : \mathbf{H}^1(\Omega) \subset \mathsf{L}^2(\Omega) \to \mathsf{L}^2(\Omega),
\end{aligned}$$

and recall that generally $\mathrm{A}_\ell^{**} = \overline{\mathrm{A}_\ell} = \mathrm{A}_\ell$. Then, for example, for the rotor

$$\begin{aligned}
\mathsf{R}_\Gamma(\Omega) &= D(\mathrm{A}_1) = D(\mathrm{A}_1^{**}) \\
&= \left\{\Psi \in \mathsf{L}^2(\Omega) \; : \; \exists\, \Psi_{\mathrm{A}_1^{**}} \in \mathsf{L}^2(\Omega) \quad \forall\, \Theta \in D(\mathrm{A}_1^*) \quad \langle \Psi, \mathrm{A}_1^*\Theta\rangle_{\mathsf{L}^2(\Omega)} = \langle \Psi_{\mathrm{A}_1^{**}}, \Theta\rangle_{\mathsf{L}^2(\Omega)}\right\} \\
&= \left\{\Psi \in \mathsf{L}^2(\Omega) \; : \; \exists\, \Psi_{\mathrm{rot}} \in \mathsf{L}^2(\Omega) \quad \forall\, \Theta \in \mathbf{R}(\Omega) \quad \langle \Psi, \operatorname{rot}\Theta\rangle_{\mathsf{L}^2(\Omega)} = \langle \Psi_{\mathrm{rot}}, \Theta\rangle_{\mathsf{L}^2(\Omega)}\right\} \\
&= \left\{\Psi \in \mathbf{R}(\Omega) \; : \; \forall\, \Theta \in \mathbf{R}(\Omega) \quad \langle \Psi, \operatorname{rot}\Theta\rangle_{\mathsf{L}^2(\Omega)} = \langle \operatorname{rot}\Psi, \Theta\rangle_{\mathsf{L}^2(\Omega)}\right\},
\end{aligned}$$

finishing the proof. □

Longer Proof of Theorem 4.1. By (30) and the quotient rule we compute

$$
\begin{aligned}
\big(\partial_{\widetilde{\Psi}}\lambda_{0,\Phi}\big)\langle \nu_\Phi\tau_\Phi^0 u,\tau_\Phi^0 u\rangle_{\mathrm{L}^2(\Omega)}^2 &= \langle \nu_\Phi\tau_\Phi^0 u,\tau_\Phi^0 u\rangle_{\mathrm{L}^2(\Omega)}\partial_{\widetilde{\Psi}}\langle \varepsilon_\Phi\nabla_{\Gamma_t}\tau_\Phi^0 u,\nabla_{\Gamma_t}\tau_\Phi^0 u\rangle_{\mathrm{L}^2(\Omega)}\\
&\quad - \langle \varepsilon_\Phi\nabla_{\Gamma_t}\tau_\Phi^0 u,\nabla_{\Gamma_t}\tau_\Phi^0 u\rangle_{\mathrm{L}^2(\Omega)}\partial_{\widetilde{\Psi}}\langle \nu_\Phi\tau_\Phi^0 u,\tau_\Phi^0 u\rangle_{\mathrm{L}^2(\Omega)}\\
&= \langle \nu_\Phi\tau_\Phi^0 u,\tau_\Phi^0 u\rangle_{\mathrm{L}^2(\Omega)}\Big(\Big\langle\big(\partial_{\widetilde{\Psi}}\varepsilon_\Phi\big)\nabla_{\Gamma_t}\tau_\Phi^0 u,\nabla_{\Gamma_t}\tau_\Phi^0 u\Big\rangle_{\mathrm{L}^2(\Omega)}\\
&\quad + 2\mathfrak{R}\langle \varepsilon_\Phi\nabla_{\Gamma_t}\tau_\Phi^0 u,\nabla_{\Gamma_t}\partial_{\widetilde{\Psi}}\tau_\Phi^0 u\rangle_{\mathrm{L}^2(\Omega)}\Big)\\
&\quad - \langle \varepsilon_\Phi\nabla_{\Gamma_t}\tau_\Phi^0 u,\nabla_{\Gamma_t}\tau_\Phi^0 u\rangle_{\mathrm{L}^2(\Omega)}\Big(\Big\langle\big(\partial_{\widetilde{\Psi}}\nu_\Phi\big)\tau_\Phi^0 u,\tau_\Phi^0 u\Big\rangle_{\mathrm{L}^2(\Omega)}\\
&\quad + 2\mathfrak{R}\langle \nu_\Phi\tau_\Phi^0 u,\partial_{\widetilde{\Psi}}\tau_\Phi^0 u\rangle_{\mathrm{L}^2(\Omega)}\Big),\\
\big(\partial_{\widetilde{\Psi}}\lambda_{0,\Phi}\big)\langle \varepsilon_\Phi\tau_\Phi^1 H,\tau_\Phi^1 H\rangle_{\mathrm{L}^2(\Omega)}^2 &= \langle \varepsilon_\Phi\tau_\Phi^1 H,\tau_\Phi^1 H\rangle_{\mathrm{L}^2(\Omega)}\partial_{\widetilde{\Psi}}\langle \nu_\Phi^{-1}\operatorname{div}_{\Gamma_n}\varepsilon_\Phi\tau_\Phi^1 H,\operatorname{div}_{\Gamma_n}\varepsilon_\Phi\tau_\Phi^1 H\rangle_{\mathrm{L}^2(\Omega)}\\
&\quad - \langle \nu_\Phi^{-1}\operatorname{div}_{\Gamma_n}\varepsilon_\Phi\tau_\Phi^1 H,\operatorname{div}_{\Gamma_n}\varepsilon_\Phi\tau_\Phi^1 H\rangle_{\mathrm{L}^2(\Omega)}\partial_{\widetilde{\Psi}}\langle \varepsilon_\Phi\tau_\Phi^1 H,\tau_\Phi^1 H\rangle_{\mathrm{L}^2(\Omega)}\\
&= \langle \varepsilon_\Phi\tau_\Phi^1 H,\tau_\Phi^1 H\rangle_{\mathrm{L}^2(\Omega)}\Big(\Big\langle\big(\partial_{\widetilde{\Psi}}\nu_\Phi^{-1}\big)\operatorname{div}_{\Gamma_n}\varepsilon_\Phi\tau_\Phi^1 H,\operatorname{div}_{\Gamma_n}\varepsilon_\Phi\tau_\Phi^1 H\Big\rangle_{\mathrm{L}^2(\Omega)}\\
&\quad + 2\mathfrak{R}\Big\langle \nu_\Phi^{-1}\operatorname{div}_{\Gamma_n}\varepsilon_\Phi\tau_\Phi^1 H,\operatorname{div}_{\Gamma_n}\partial_{\widetilde{\Psi}}(\varepsilon_\Phi\tau_\Phi^1 H)\Big\rangle_{\mathrm{L}^2(\Omega)}\Big)\\
&\quad - \langle \nu_\Phi^{-1}\operatorname{div}_{\Gamma_n}\varepsilon_\Phi\tau_\Phi^1 H,\operatorname{div}_{\Gamma_n}\varepsilon_\Phi\tau_\Phi^1 H\rangle_{\mathrm{L}^2(\Omega)}\Big(\Big\langle\big(\partial_{\widetilde{\Psi}}\varepsilon_\Phi^{-1}\big)\varepsilon_\Phi\tau_\Phi^1 H,\varepsilon_\Phi\tau_\Phi^1 H\Big\rangle_{\mathrm{L}^2(\Omega)}\\
&\quad + 2\mathfrak{R}\Big\langle \tau_\Phi^1 H,\partial_{\widetilde{\Psi}}(\varepsilon_\Phi\tau_\Phi^1 H)\Big\rangle_{\mathrm{L}^2(\Omega)}\Big),
\end{aligned}
$$

and thus using

$$
\begin{aligned}
\langle \varepsilon_\Phi\nabla_{\Gamma_t}\tau_\Phi^0 u,\nabla_{\Gamma_t}\tau_\Phi^0 u\rangle_{\mathrm{L}^2(\Omega)} &= \lambda_{0,\Phi}\langle \nu_\Phi\tau_\Phi^0 u,\tau_\Phi^0 u\rangle_{\mathrm{L}^2(\Omega)},\\
\langle \nu_\Phi^{-1}\operatorname{div}_{\Gamma_n}\varepsilon_\Phi\tau_\Phi^1 H,\operatorname{div}_{\Gamma_n}\varepsilon_\Phi\tau_\Phi^1 H\rangle_{\mathrm{L}^2(\Omega)} &= \lambda_{0,\Phi}\langle \varepsilon_\Phi\tau_\Phi^1 H,\tau_\Phi^1 H\rangle_{\mathrm{L}^2(\Omega)}
\end{aligned}
$$

we see

$$
\begin{aligned}
(\partial_{\widetilde{\Psi}}\lambda_{0,\Phi})\langle \nu_\Phi\tau_\Phi^0 u,\tau_\Phi^0 u\rangle_{\mathrm{L}^2(\Omega)} &= \Big\langle\big(\partial_{\widetilde{\Psi}}\varepsilon_\Phi\big)\nabla_{\Gamma_t}\tau_\Phi^0 u,\nabla_{\Gamma_t}\tau_\Phi^0 u\Big\rangle_{\mathrm{L}^2(\Omega)}\\
&\quad + 2\mathfrak{R}\langle \varepsilon_\Phi\nabla_{\Gamma_t}\tau_\Phi^0 u,\nabla_{\Gamma_t}\partial_{\widetilde{\Psi}}\tau_\Phi^0 u\rangle_{\mathrm{L}^2(\Omega)}\\
&\quad - \lambda_{0,\Phi}\Big(\Big\langle\big(\partial_{\widetilde{\Psi}}\nu_\Phi\big)\tau_\Phi^0 u,\tau_\Phi^0 u\Big\rangle_{\mathrm{L}^2(\Omega)} + 2\mathfrak{R}\langle \nu_\Phi\tau_\Phi^0 u,\partial_{\widetilde{\Psi}}\tau_\Phi^0 u\rangle_{\mathrm{L}^2(\Omega)}\Big)\\
&= \Big\langle\big(\partial_{\widetilde{\Psi}}\varepsilon_\Phi\big)\nabla_{\Gamma_t}\tau_\Phi^0 u,\nabla_{\Gamma_t}\tau_\Phi^0 u\Big\rangle_{\mathrm{L}^2(\Omega)} + 2\mathfrak{R}\langle \mathrm{A}_0\tau_\Phi^0 u,\mathrm{A}_0\partial_{\widetilde{\Psi}}\tau_\Phi^0 u\rangle_{\mathrm{L}^2_{\varepsilon_\Phi}(\Omega)}\\
&\quad - \lambda_{0,\Phi}\Big(\Big\langle\big(\partial_{\widetilde{\Psi}}\nu_\Phi\big)\tau_\Phi^0 u,\tau_\Phi^0 u\Big\rangle_{\mathrm{L}^2(\Omega)} + 2\mathfrak{R}\langle \tau_\Phi^0 u,\partial_{\widetilde{\Psi}}\tau_\Phi^0 u\rangle_{\mathrm{L}^2_{\nu_\Phi}(\Omega)}\Big)\\
&= \Big\langle\big(\partial_{\widetilde{\Psi}}\varepsilon_\Phi\big)\nabla_{\Gamma_t}\tau_\Phi^0 u,\nabla_{\Gamma_t}\tau_\Phi^0 u\Big\rangle_{\mathrm{L}^2(\Omega)} - \lambda_{0,\Phi}\Big\langle\big(\partial_{\widetilde{\Psi}}\nu_\Phi\big)\tau_\Phi^0 u,\tau_\Phi^0 u\Big\rangle_{\mathrm{L}^2(\Omega)}\\
&\quad + 2\mathfrak{R}\Big\langle \underbrace{(\mathrm{A}_0^*\mathrm{A}_0 - \lambda_{0,\Phi})\tau_\Phi^0 u}_{=0},\partial_{\widetilde{\Psi}}\tau_\Phi^0 u\Big\rangle_{\mathrm{L}^2_{\nu_\Phi}(\Omega)}\\
&= \langle \nabla_{\Gamma_t}\tau_\Phi^0 u,\nabla_{\Gamma_t}\tau_\Phi^0 u\rangle_{\mathrm{L}^2_{(\partial_{\widetilde{\Psi}}\varepsilon_\Phi)}(\Omega)} - \lambda_{0,\Phi}\langle \tau_\Phi^0 u,\tau_\Phi^0 u\rangle_{\mathrm{L}^2_{(\partial_{\widetilde{\Psi}}\nu_\Phi)}(\Omega)},\\
\big(\partial_{\widetilde{\Psi}}\lambda_{0,\Phi}\big)\langle \varepsilon_\Phi\tau_\Phi^1 H,\tau_\Phi^1 H\rangle_{\mathrm{L}^2(\Omega)} &= \Big\langle\big(\partial_{\widetilde{\Psi}}\nu_\Phi^{-1}\big)\operatorname{div}_{\Gamma_n}\varepsilon_\Phi\tau_\Phi^1 H,\operatorname{div}_{\Gamma_n}\varepsilon_\Phi\tau_\Phi^1 H\Big\rangle_{\mathrm{L}^2(\Omega)}\\
&\quad + 2\mathfrak{R}\Big\langle \nu_\Phi^{-1}\operatorname{div}_{\Gamma_n}\varepsilon_\Phi\tau_\Phi^1 H,\operatorname{div}_{\Gamma_n}\partial_{\widetilde{\Psi}}(\varepsilon_\Phi\tau_\Phi^1 H)\Big\rangle_{\mathrm{L}^2(\Omega)}\\
&\quad - \lambda_{0,\Phi}\Big(2\mathfrak{R}\Big\langle \tau_\Phi^1 H,\partial_{\widetilde{\Psi}}(\varepsilon_\Phi\tau_\Phi^1 H)\Big\rangle_{\mathrm{L}^2(\Omega)}\\
&\quad + \Big\langle\big(\partial_{\widetilde{\Psi}}\varepsilon_\Phi^{-1}\big)\varepsilon_\Phi\tau_\Phi^1 H,\varepsilon_\Phi\tau_\Phi^1 H\Big\rangle_{\mathrm{L}^2(\Omega)}\Big)\\
&= \Big\langle\big(\partial_{\widetilde{\Psi}}\nu_\Phi^{-1}\big)\operatorname{div}_{\Gamma_n}\varepsilon_\Phi\tau_\Phi^1 H,\operatorname{div}_{\Gamma_n}\varepsilon_\Phi\tau_\Phi^1 H\Big\rangle_{\mathrm{L}^2(\Omega)}\\
&\quad + 2\mathfrak{R}\Big\langle \mathrm{A}_0^*\tau_\Phi^1 H,\mathrm{A}_0^*\varepsilon_\Phi^{-1}\partial_{\widetilde{\Psi}}(\varepsilon_\Phi\tau_\Phi^1 H)\Big\rangle_{\mathrm{L}^2_{\nu_\Phi}(\Omega)}\\
&\quad - \lambda_{0,\Phi}\Big(\Big\langle\big(\partial_{\widetilde{\Psi}}\varepsilon_\Phi^{-1}\big)\varepsilon_\Phi\tau_\Phi^1 H,\varepsilon_\Phi\tau_\Phi^1 H\Big\rangle_{\mathrm{L}^2(\Omega)}\\
&\quad + 2\mathfrak{R}\Big\langle \tau_\Phi^1 H,\varepsilon_\Phi^{-1}\partial_{\widetilde{\Psi}}(\varepsilon_\Phi\tau_\Phi^1 H)\Big\rangle_{\mathrm{L}^2_{\varepsilon_\Phi}(\Omega)}\Big)
\end{aligned}
$$

$$
\begin{aligned}
&= \left\langle \left(\partial_{\widetilde{\Psi}} \nu_{\Phi}^{-1}\right) \operatorname{div}_{\Gamma_n} \varepsilon_{\Phi} \tau_{\Phi}^{1} H, \operatorname{div}_{\Gamma_n} \varepsilon_{\Phi} \tau_{\Phi}^{1} H \right\rangle_{\mathsf{L}^2(\Omega)} \\
&\quad - \lambda_{0,\Phi} \left\langle \left(\partial_{\widetilde{\Psi}} \varepsilon_{\Phi}^{-1}\right) \varepsilon_{\Phi} \tau_{\Phi}^{1} H, \varepsilon_{\Phi} \tau_{\Phi}^{1} H \right\rangle_{\mathsf{L}^2(\Omega)} \\
&\quad + 2\Re \left\langle \underbrace{\left(\mathrm{A}_0 \mathrm{A}_0^* - \lambda_{0,\Phi}\right) \tau_{\Phi}^{1} H}_{=0}, \varepsilon_{\Phi}^{-1} \partial_{\widetilde{\Psi}} (\varepsilon_{\Phi} \tau_{\Phi}^{1} H) \right\rangle_{\mathsf{L}^2_{\varepsilon_\Phi}(\Omega)} \\
&= \langle \operatorname{div}_{\Gamma_n} \varepsilon_{\Phi} \tau_{\Phi}^{1} H, \operatorname{div}_{\Gamma_n} \varepsilon_{\Phi} \tau_{\Phi}^{1} H \rangle_{\mathsf{L}^2_{(\partial_{\widetilde{\Psi}} \nu_{\Phi}^{-1})}(\Omega)} + \lambda_{0,\Phi} \langle \tau_{\Phi}^{1} H, \tau_{\Phi}^{1} H \rangle_{\mathsf{L}^2_{(\partial_{\widetilde{\Psi}} \varepsilon_{\Phi})}(\Omega)}.
\end{aligned}
$$

Note that $\partial_{\widetilde{\Psi}} \varepsilon_{\Phi} = -\varepsilon_{\Phi} (\partial_{\widetilde{\Psi}} \varepsilon_{\Phi}^{-1}) \varepsilon_{\Phi}$ by Remark A.2. Therefore, for normalised eigenfields u, E, and H we obtain the assertions. □

Longer Proof of Theorem 4.2. By Theorem 2.1, Corollary 2.1, Remark 2.2 and Lemma A.1 we see

$$
\begin{aligned}
\partial_{\widetilde{\Psi}} \lambda_{0,\Phi} &= \left\langle \left(\partial_{\widetilde{\Psi}} \varepsilon_{\Phi}\right) \nabla_{\Gamma_t} \tau_{\Phi}^{0} u, \nabla_{\Gamma_t} \tau_{\Phi}^{0} u \right\rangle_{\mathsf{L}^2(\Omega)} - \lambda_{0,\Phi} \left\langle \left(\partial_{\widetilde{\Psi}} \nu_{\Phi}\right) \tau_{\Phi}^{0} u, \tau_{\Phi}^{0} u \right\rangle_{\mathsf{L}^2(\Omega)} \\
&= \left\langle (\det J_{\Phi}) \left(\partial_{\widetilde{\Psi}} \widetilde{\varepsilon} + (\widetilde{\operatorname{div} \Psi}) \widetilde{\varepsilon} - 2 \operatorname{sym}(\widetilde{J_{\Psi} \varepsilon}) \right) J_{\Phi}^{-\top} \nabla_{\Gamma_t} \tau_{\Phi}^{0} u, J_{\Phi}^{-\top} \nabla_{\Gamma_t} \tau_{\Phi}^{0} u \right\rangle_{\mathsf{L}^2(\Omega)} \\
&\quad - \lambda_{0,\Phi} \left\langle (\det J_{\Phi}) \left(\partial_{\widetilde{\Psi}} \widetilde{\nu} + (\widetilde{\operatorname{div} \Psi}) \widetilde{\nu} \right) \tau_{\Phi}^{0} u, \tau_{\Phi}^{0} u \right\rangle_{\mathsf{L}^2(\Omega)} \\
&= \left\langle (\det J_{\Phi}) \left(\partial_{\widetilde{\Psi}} \widetilde{\varepsilon} + (\widetilde{\operatorname{div} \Psi}) \widetilde{\varepsilon} - 2 \operatorname{sym}(\widetilde{J_{\Psi} \varepsilon}) \right) \widetilde{\nabla_{\Gamma_{t,\Phi}} u}, \widetilde{\nabla_{\Gamma_{t,\Phi}} u} \right\rangle_{\mathsf{L}^2(\Omega)} \\
&\quad - \lambda_{0,\Phi} \left\langle (\det J_{\Phi}) \left(\partial_{\widetilde{\Psi}} \widetilde{\nu} + (\widetilde{\operatorname{div} \Psi}) \widetilde{\nu} \right) \widetilde{u}, \widetilde{u} \right\rangle_{\mathsf{L}^2(\Omega)} \\
&= \left\langle \left(\partial_{\Psi} \varepsilon + (\operatorname{div} \Psi) \varepsilon - 2 \operatorname{sym}(J_{\Psi} \varepsilon) \right) \nabla_{\Gamma_{t,\Phi}} u, \nabla_{\Gamma_{t,\Phi}} u \right\rangle_{\mathsf{L}^2(\Omega_{\Phi})} \\
&\quad - \lambda_{0,\Phi} \left\langle \left(\partial_{\Psi} \nu + (\operatorname{div} \Psi) \nu \right) u, u \right\rangle_{\mathsf{L}^2(\Omega_{\Phi})}, \\
\partial_{\widetilde{\Psi}} \lambda_{1,\Phi} &= \left\langle \left(\partial_{\widetilde{\Psi}} \mu_{\Phi}^{-1}\right) \operatorname{rot}_{\Gamma_t} \tau_{\Phi}^{1} E, \ \operatorname{rot}_{\Gamma_t} \tau_{\Phi}^{1} E \right\rangle_{\mathsf{L}^2(\Omega)} - \lambda_{1,\Phi} \left\langle \left(\partial_{\widetilde{\Psi}} \varepsilon_{\Phi}\right) \tau_{\Phi}^{1} E, \tau_{\Phi}^{1} E \right\rangle_{\mathsf{L}^2(\Omega)}, \\
&= \left\langle (\det J_{\Phi})^{-1} \left(\partial_{\widetilde{\Psi}} \widetilde{\mu^{-1}} - (\widetilde{\operatorname{div} \Psi}) \widetilde{\mu^{-1}} + 2 \operatorname{sym}(\widetilde{\mu^{-1}} \widetilde{J_{\Psi}}) \right) J_{\Phi} \operatorname{rot}_{\Gamma_t} \tau_{\Phi}^{1} E, J_{\Phi} \operatorname{rot}_{\Gamma_t} \tau_{\Phi}^{1} E \right\rangle_{\mathsf{L}^2(\Omega)} \\
&\quad - \lambda_{1,\Phi} \left\langle (\det J_{\Phi}) \left(\partial_{\widetilde{\Psi}} \widetilde{\varepsilon} + (\widetilde{\operatorname{div} \Psi}) \widetilde{\varepsilon} - 2 \operatorname{sym}(\widetilde{J_{\Psi} \varepsilon}) \right) J_{\Phi}^{-\top} \tau_{\Phi}^{1} E, J_{\Phi}^{-\top} \tau_{\Phi}^{1} E \right\rangle_{\mathsf{L}^2(\Omega)}, \\
&= \left\langle (\det J_{\Phi}) \left(\partial_{\widetilde{\Psi}} \widetilde{\mu^{-1}} - (\widetilde{\operatorname{div} \Psi}) \widetilde{\mu^{-1}} + 2 \operatorname{sym}(\widetilde{\mu^{-1}} \widetilde{J_{\Psi}}) \right) \widetilde{\operatorname{rot}_{\Gamma_{t,\Phi}} E}, \ \widetilde{\operatorname{rot}_{\Gamma_{t,\Phi}} E} \right\rangle_{\mathsf{L}^2(\Omega)} \\
&\quad - \lambda_{1,\Phi} \left\langle (\det J_{\Phi}) \left(\partial_{\widetilde{\Psi}} \widetilde{\varepsilon} + (\widetilde{\operatorname{div} \Psi}) \widetilde{\varepsilon} - 2 \operatorname{sym}(\widetilde{J_{\Psi} \varepsilon}) \right) \widetilde{E}, \widetilde{E} \right\rangle_{\mathsf{L}^2(\Omega)}, \\
&= \left\langle \left(\partial_{\Psi} \mu^{-1} - (\operatorname{div} \Psi) \mu^{-1} + 2 \operatorname{sym}(\mu^{-1} J_{\Psi}) \right) \operatorname{rot}_{\Gamma_{t,\Phi}} E, \ \operatorname{rot}_{\Gamma_{t,\Phi}} E \right\rangle_{\mathsf{L}^2(\Omega_{\Phi})} \\
&\quad - \lambda_{1,\Phi} \left\langle \left(\partial_{\Psi} \varepsilon + (\operatorname{div} \Psi) \varepsilon - 2 \operatorname{sym}(J_{\Psi} \varepsilon) \right) E, E \right\rangle_{\mathsf{L}^2(\Omega_{\Phi})}, \\
\partial_{\widetilde{\Psi}} \lambda_{0,\Phi} &= \left\langle \left(\partial_{\widetilde{\Psi}} \nu_{\Phi}^{-1}\right) \operatorname{div}_{\Gamma_n} \varepsilon_{\Phi} \tau_{\Phi}^{1} H, \operatorname{div}_{\Gamma_n} \varepsilon_{\Phi} \tau_{\Phi}^{1} H \right\rangle_{\mathsf{L}^2(\Omega)} + \lambda_{0,\Phi} \left\langle \left(\partial_{\widetilde{\Psi}} \varepsilon_{\Phi}\right) \tau_{\Phi}^{1} H, \tau_{\Phi}^{1} H \right\rangle_{\mathsf{L}^2(\Omega)} \\
&= \left\langle (\det J_{\Phi})^{-1} \left(\partial_{\widetilde{\Psi}} \widetilde{\nu^{-1}} - (\widetilde{\operatorname{div} \Psi}) \widetilde{\nu^{-1}} \right) \operatorname{div}_{\Gamma_n} \varepsilon_{\Phi} \tau_{\Phi}^{1} H, \operatorname{div}_{\Gamma_n} \varepsilon_{\Phi} \tau_{\Phi}^{1} H \right\rangle_{\mathsf{L}^2(\Omega)} \\
&\quad + \lambda_{0,\Phi} \left\langle (\det J_{\Phi}) \left(\partial_{\widetilde{\Psi}} \widetilde{\varepsilon} + (\widetilde{\operatorname{div} \Psi}) \widetilde{\varepsilon} - 2 \operatorname{sym}(\widetilde{J_{\Psi} \varepsilon}) \right) J_{\Phi}^{-\top} \tau_{\Phi}^{1} H, J_{\Phi}^{-\top} \tau_{\Phi}^{1} H \right\rangle_{\mathsf{L}^2(\Omega)} \\
&= \left\langle (\det J_{\Phi}) \left(\partial_{\widetilde{\Psi}} \widetilde{\nu^{-1}} - (\widetilde{\operatorname{div} \Psi}) \widetilde{\nu^{-1}} \right) \widetilde{\operatorname{div}_{\Gamma_{n,\Phi}} \varepsilon H}, \widetilde{\operatorname{div}_{\Gamma_{n,\Phi}} \varepsilon H} \right\rangle_{\mathsf{L}^2(\Omega)} \\
&\quad + \lambda_{0,\Phi} \left\langle (\det J_{\Phi}) \left(\partial_{\widetilde{\Psi}} \widetilde{\varepsilon} + (\widetilde{\operatorname{div} \Psi}) \widetilde{\varepsilon} - 2 \operatorname{sym}(\widetilde{J_{\Psi} \varepsilon}) \right) \widetilde{H}, \widetilde{H} \right\rangle_{\mathsf{L}^2(\Omega)} \\
&= \left\langle \left(\partial_{\Psi} \nu^{-1} - (\operatorname{div} \Psi) \nu^{-1} \right) \operatorname{div}_{\Gamma_{n,\Phi}} \varepsilon H, \operatorname{div}_{\Gamma_{n,\Phi}} \varepsilon H \right\rangle_{\mathsf{L}^2(\Omega_{\Phi})} \\
&\quad + \lambda_{0,\Phi} \left\langle \left(\partial_{\Psi} \varepsilon + (\operatorname{div} \Psi) \varepsilon - 2 \operatorname{sym}(J_{\Psi} \varepsilon) \right) H, H \right\rangle_{\mathsf{L}^2(\Omega_{\Phi})},
\end{aligned}
$$

finishing the proof. □