Received: 26 July 2022 | Revised: 26 February 2023 | Accepted: 25 April 2023

DOI: 10.1002/mma.9437

RESEARCH ARTICLE

WILEY

Hilbert complexes with mixed boundary conditions part 3: Biharmonic complexes

Dirk Pauly | **Michael Schomburg**

Institut für Analysis, Technische Universität Dresden, Germany

Correspondence
Dirk Pauly, Institut für Analysis, Technische Universität Dresden, Germany.
Email: dirk.pauly@tu-dresden.de

Communicated by: M. Waurick

Funding information
There are no funders to report for this submission.

We show that the biharmonic Hilbert complex with mixed boundary conditions on bounded strong Lipschitz domains is closed and compact. The crucial results are compact embeddings that follow by abstract arguments using functional analysis together with particular regular decompositions. Higher Sobolev order results are also proved.

KEYWORDS
biharmonic complex, compact embeddings, Hilbert complexes, mixed boundary conditions, regular decompositions, regular potentials

MSC CLASSIFICATION
35A23, 35Q61, 58Axx, 58Jxx

1 | INTRODUCTION

In [1], we investigated the de Rham Hilbert complex with mixed boundary conditions on bounded strong Lipschitz domains

$$\cdots \xrightarrow{\cdots} \mathsf{L}^{q-1,2}(\Omega) \xrightarrow{\mathrm{d}^{q-1}} \mathsf{L}^{q,2}(\Omega) \xrightarrow{\mathrm{d}^{q}} \mathsf{L}^{q+1,2}(\Omega) \xrightarrow{\cdots} \cdots,$$

whose 3D version for vector proxies reads

$$\cdots \xrightarrow{\cdots} \mathsf{L}^{2}(\Omega) \xrightarrow{\mathrm{d}^{0} \,\widehat{=}\, \mathrm{grad}} \mathsf{L}^{2}(\Omega) \xrightarrow{\mathrm{d}^{1} \,\widehat{=}\, \mathrm{rot}} \mathsf{L}^{2}(\Omega) \xrightarrow{\mathrm{d}^{2} \,\widehat{=}\, \mathrm{div}} \mathsf{L}^{2}(\Omega) \xrightarrow{\cdots} \cdots.$$

In [2], we extended our studies and results to the elasticity complex

$$\cdots \xrightarrow{\cdots} \mathsf{L}^{2}(\Omega) \xrightarrow{\mathrm{symGrad}} \mathsf{L}^{2}_{\mathbb{S}}(\Omega) \xrightarrow{\mathrm{RotRot}^{\top}_{\mathbb{S}}} \mathsf{L}^{2}_{\mathbb{S}}(\Omega) \xrightarrow{\mathrm{Div}_{\mathbb{S}}} \mathsf{L}^{2}(\Omega) \xrightarrow{\cdots} \cdots.$$

In this contribution, the third part of the series, we shall investigate the two biharmonic Hilbert complexes with mixed boundary conditions on a bounded strong Lipschitz domain $\Omega \subset \mathbb{R}^3$

$$\cdots \xrightarrow{\cdots} \mathsf{L}^{2}(\Omega) \xrightarrow{\mathrm{Gradgrad}} \mathsf{L}^{2}_{\mathbb{S}}(\Omega) \xrightarrow{\mathrm{Rot}_{\mathbb{S}}} \mathsf{L}^{2}_{\mathbb{T}}(\Omega) \xrightarrow{\mathrm{Div}_{\mathbb{T}}} \mathsf{L}^{2}(\Omega) \xrightarrow{\cdots} \cdots,$$

$$\cdots \xrightarrow{\cdots} \mathsf{L}^{2}(\Omega) \xrightarrow{\mathrm{devGrad}} \mathsf{L}^{2}_{\mathbb{T}}(\Omega) \xrightarrow{\mathrm{symRot}_{\mathbb{T}}} \mathsf{L}^{2}_{\mathbb{S}}(\Omega) \xrightarrow{\mathrm{divDiv}_{\mathbb{S}}} \mathsf{L}^{2}(\Omega) \xrightarrow{\cdots} \cdots.$$

Note that these two complexes are formally dual (adjoint) to each other.

As explained in detail in [1, 2], all these Hilbert complexes share the same geometric structure

$$\cdots \xrightarrow{\cdots} \mathsf{H}_0 \xrightarrow{\mathrm{A}_0} \mathsf{H}_1 \xrightarrow{\mathrm{A}_1} \mathsf{H}_2 \xrightarrow{\cdots} \cdots, \qquad R(\mathrm{A}_0) \subset N(\mathrm{A}_1),$$

where A_0 and A_1 are densely defined and closed (unbounded) linear operators between Hilbert spaces H_ℓ. The corresponding domain Hilbert complex is denoted by

$$\cdots \xrightarrow{\cdots} D(\mathrm{A}_0) \xrightarrow{\mathrm{A}_0} D(\mathrm{A}_1) \xrightarrow{\mathrm{A}_1} \mathsf{H}_2 \xrightarrow{\cdots} \cdots.$$

The goal of this article is to show that the previous biharmonic Hilbert complexes are compact, which is proved by using regular decompositions of the domains of definition of the respective operators as a crucial tool. We shall follow in close lines the rationale from [1, 2]. Along the way, we show the existence of regular potentials and decompositions, compact embeddings, Helmholtz decompositions, closed ranges, Friedrichs/Poincaré type estimates, and bases of the corresponding cohomology groups (generalised Dirichlet/Neumann tensors). Due to the similarity of results, we shall only state those which are most important. In the appendix, we will present some of the crucial proofs, which differ from the proofs of the previously investigated complexes.

2 | BIHARMONIC COMPLEXES I

Throughout this paper, let $\Omega \subset \mathbb{R}^3$ be a *bounded strong Lipschitz domain* with boundary Γ, decomposed into two parts Γ_t and $\Gamma_n := \Gamma \setminus \overline{\Gamma_t}$ with some *relatively open and strong Lipschitz boundary part* $\Gamma_t \subset \Gamma$. More precisely, we assume generally that (Ω, Γ_t) is a *bounded strong Lipschitz pair*. We shall consequently use the notations, methods, and results from our corresponding papers for the de Rham complex [1], for the elasticity complex [2, 3], and for the biharmonic complexes [4]. In particular, we recall [1, Section 2, Section 3] including the notion of *extendable domains*. The standard Lebesgue and Sobolev spaces (scalar or tensor valued) are denoted by $\mathsf{L}^2(\Omega)$ and $\mathsf{H}^k(\Omega)$ with $k \in \mathbb{N}_0$.

We recall that weak and strong boundary conditions coincide for the standard Sobolev spaces with mixed boundary conditions, that is,

$$\mathbf{H}^k_{\Gamma_t}(\Omega) = \mathsf{H}^k_{\Gamma_t}(\Omega); \tag{1}$$

and compare [1, Lemma 3.2, Theorem 4.6]. Below, we shall show that "*strong* = *weak*" holds generally also for the biharmonic complex. Note that $\mathsf{H}^k_{\emptyset}(\Omega) = \mathsf{H}^k(\Omega)$ and $\mathsf{H}^0_{\Gamma_t}(\Omega) = \mathsf{L}^2(\Omega)$.

We introduce as usual Grad, Rot, and Div as "row-wise" incarnations of the classical operators grad, rot, and div from the de Rham complex.

2.1 | Operators

Let Gradgrad, Rot, Div, devGrad, symRot, and divDiv be realised as densely defined (unbounded) linear operators

$$\begin{aligned}
{}_{\mathbb{S}}\mathring{\mathrm{Grad}}\mathrm{grad}_{\Gamma_t} &: D({}_{\mathbb{S}}\mathring{\mathrm{Grad}}\mathrm{grad}_{\Gamma_t}) && \subset \mathsf{L}^2(\Omega) \to \mathsf{L}^2_{\mathbb{S}}(\Omega); && u \mapsto \mathrm{Gradgrad}\, u,\\
{}_{\mathbb{T}}\mathring{\mathrm{Rot}}_{\mathbb{S},\Gamma_t} &: D({}_{\mathbb{T}}\mathring{\mathrm{Rot}}_{\mathbb{S},\Gamma_t}) && \subset \mathsf{L}^2_{\mathbb{S}}(\Omega) \to \mathsf{L}^2_{\mathbb{T}}(\Omega); && S \mapsto \mathrm{Rot}\, S,\\
\mathring{\mathrm{Div}}_{\mathbb{T},\Gamma_t} &: D(\mathring{\mathrm{Div}}_{\mathbb{T},\Gamma_t}) && \subset \mathsf{L}^2_{\mathbb{T}}(\Omega) \to \mathsf{L}^2(\Omega); && T \mapsto \mathrm{Div}\, T,\\
{}_{\mathbb{T}}\mathring{\mathrm{Grad}}_{\Gamma_t} &: D({}_{\mathbb{T}}\mathring{\mathrm{Grad}}_{\Gamma_t}) && \subset \mathsf{L}^2(\Omega) \to \mathsf{L}^2_{\mathbb{T}}(\Omega); && v \mapsto \mathrm{devGrad}\, v,\\
{}_{\mathbb{S}}\mathring{\mathrm{Rot}}_{\mathbb{T},\Gamma_t} &: D({}_{\mathbb{S}}\mathring{\mathrm{Rot}}_{\mathbb{T},\Gamma_t}) && \subset \mathsf{L}^2_{\mathbb{T}}(\Omega) \to \mathsf{L}^2_{\mathbb{S}}(\Omega); && T \mapsto \mathrm{symRot}\, T,\\
\mathrm{div}\mathring{\mathrm{Div}}_{\mathbb{S},\Gamma_t} &: D(\mathrm{div}\mathring{\mathrm{Div}}_{\mathbb{S},\Gamma_t}) && \subset \mathsf{L}^2_{\mathbb{S}}(\Omega) \to \mathsf{L}^2(\Omega); && S \mapsto \mathrm{divDiv}\, S,
\end{aligned}$$

where $\operatorname{sym} S := \frac{1}{2}(S + S^\top)$ and $\operatorname{dev} T := T - \frac{1}{3}(\operatorname{tr} T)\mathrm{id}$, with domains of definition

$$\begin{aligned}
D({}_{\mathbb{S}}\mathring{\mathrm{Grad}}\mathrm{grad}_{\Gamma_t}) &:= \mathsf{C}^\infty_{\Gamma_t}(\Omega), & D({}_{\mathbb{T}}\mathring{\mathrm{Rot}}_{\mathbb{S},\Gamma_t}) &:= \mathsf{C}^\infty_{\mathbb{S},\Gamma_t}(\Omega), & D(\mathring{\mathrm{Div}}_{\mathbb{T},\Gamma_t}) &:= \mathsf{C}^\infty_{\mathbb{T},\Gamma_t}(\Omega),\\
D({}_{\mathbb{T}}\mathring{\mathrm{Grad}}_{\Gamma_t}) &:= \mathsf{C}^\infty_{\Gamma_t}(\Omega), & D({}_{\mathbb{S}}\mathring{\mathrm{Rot}}_{\mathbb{T},\Gamma_t}) &:= \mathsf{C}^\infty_{\mathbb{T},\Gamma_t}(\Omega), & D(\mathrm{div}\mathring{\mathrm{Div}}_{\mathbb{S},\Gamma_t}) &:= \mathsf{C}^\infty_{\mathbb{S},\Gamma_t}(\Omega),
\end{aligned}$$

satisfying the complex properties

$$
{}_{\mathbb{T}}\mathring{\mathrm{Rot}}_{\mathbb{S},\Gamma_t}\,{}_{\mathbb{S}}\mathrm{Grad}\mathring{\mathrm{grad}}_{\Gamma_t}\subset 0,\qquad \mathring{\mathrm{Div}}_{\mathbb{T},\Gamma_t}\,{}_{\mathbb{T}}\mathring{\mathrm{Rot}}_{\mathbb{S},\Gamma_t}\subset 0,
$$
$$
{}_{\mathbb{S}}\mathring{\mathrm{Rot}}_{\mathbb{T},\Gamma_t}\,{}_{\mathbb{T}}\mathring{\mathrm{Grad}}_{\Gamma_t}\subset 0,\qquad \mathrm{div}\mathring{\mathrm{Div}}_{\mathbb{S},\Gamma_t}\,{}_{\mathbb{S}}\mathring{\mathrm{Rot}}_{\mathbb{T},\Gamma_t}\subset 0.
$$

For elementary properties of these operators, see, for example, [3]; in particular, we have a collection of formulas presented in Lemma A.1 (Appendix A). Here, we introduce the Lebesgue Hilbert spaces and the test spaces of symmetric and deviatoric tensor fields

$$
\begin{aligned}
\mathsf{L}^2_{\mathbb{S}}(\Omega) &:= \left\{S\in\mathsf{L}^2(\Omega)\,:\,\mathrm{skw}\,S=0\right\}, & \mathsf{C}^{\infty}_{\mathbb{S},\Gamma_t}(\Omega) &:= \mathsf{C}^{\infty}_{\Gamma_t}(\Omega)\cap\mathsf{L}^2_{\mathbb{S}}(\Omega),\\
\mathsf{L}^2_{\mathbb{T}}(\Omega) &:= \left\{S\in\mathsf{L}^2(\Omega)\,:\,\mathrm{tr}\,T=0\right\}, & \mathsf{C}^{\infty}_{\mathbb{T},\Gamma_t}(\Omega) &:= \mathsf{C}^{\infty}_{\Gamma_t}(\Omega)\cap\mathsf{L}^2_{\mathbb{T}}(\Omega),
\end{aligned}
$$

respectively. We get the first and second biharmonic complexes on smooth tensor fields

$$
\cdots\xrightarrow{\cdots}\mathsf{L}^2(\Omega)\xrightarrow{{}_{\mathbb{S}}\mathrm{Grad}\mathring{\mathrm{grad}}_{\Gamma_t}}\mathsf{L}^2_{\mathbb{S}}(\Omega)\xrightarrow{{}_{\mathbb{T}}\mathring{\mathrm{Rot}}_{\mathbb{S},\Gamma_t}}\mathsf{L}^2_{\mathbb{T}}(\Omega)\xrightarrow{\mathring{\mathrm{Div}}_{\mathbb{T},\Gamma_t}}\mathsf{L}^2(\Omega)\xrightarrow{\cdots}\cdots,
$$
$$
\cdots\xrightarrow{\cdots}\mathsf{L}^2(\Omega)\xrightarrow{{}_{\mathbb{T}}\mathring{\mathrm{Grad}}_{\Gamma_t}}\mathsf{L}^2_{\mathbb{T}}(\Omega)\xrightarrow{{}_{\mathbb{S}}\mathring{\mathrm{Rot}}_{\mathbb{T},\Gamma_t}}\mathsf{L}^2_{\mathbb{S}}(\Omega)\xrightarrow{\mathrm{div}\mathring{\mathrm{Div}}_{\mathbb{S},\Gamma_t}}\mathsf{L}^2(\Omega)\xrightarrow{\cdots}\cdots.
$$

For a more algebraically structured introduction of the latter operators suggested by Rainer Picard, see Appendix B. The closures

$$
\begin{aligned}
{}_{\mathbb{S}}\mathrm{Gradgrad}_{\Gamma_t} &:= \overline{{}_{\mathbb{S}}\mathrm{Grad}\mathring{\mathrm{grad}}_{\Gamma_t}}, & {}_{\mathbb{T}}\mathrm{Rot}_{\mathbb{S},\Gamma_t} &:= \overline{{}_{\mathbb{T}}\mathring{\mathrm{Rot}}_{\mathbb{S},\Gamma_t}}, & \mathrm{Div}_{\mathbb{T},\Gamma_t} &:= \overline{\mathring{\mathrm{Div}}_{\mathbb{T},\Gamma_t}},\\
{}_{\mathbb{T}}\mathrm{Grad}_{\Gamma_t} &:= \overline{{}_{\mathbb{T}}\mathring{\mathrm{Grad}}_{\Gamma_t}}, & {}_{\mathbb{S}}\mathrm{Rot}_{\mathbb{T},\Gamma_t} &:= \overline{{}_{\mathbb{S}}\mathring{\mathrm{Rot}}_{\mathbb{T},\Gamma_t}}, & \mathrm{divDiv}_{\mathbb{S},\Gamma_t} &:= \overline{\mathrm{div}\mathring{\mathrm{Div}}_{\mathbb{S},\Gamma_t}},
\end{aligned}
$$

and Hilbert space adjoints are given by the densely defined and closed linear operators

$$
\begin{aligned}
{}_{\mathbb{S}}\mathrm{Gradgrad}_{\Gamma_t} &: D({}_{\mathbb{S}}\mathrm{Gradgrad}_{\Gamma_t}) &&\subset \mathsf{L}^2(\Omega)\to\mathsf{L}^2_{\mathbb{S}}(\Omega); && u\mapsto \mathrm{Gradgrad}\ u,\\
{}_{\mathbb{S}}\mathrm{Gradgrad}^*_{\Gamma_t} = \mathbf{divDiv}_{\mathbb{S},\Gamma_n} &: D(\mathbf{divDiv}_{\mathbb{S},\Gamma_n}) &&\subset \mathsf{L}^2_{\mathbb{S}}(\Omega)\to\mathsf{L}^2(\Omega); && S\mapsto \mathrm{divDiv}\ S,\\
{}_{\mathbb{T}}\mathrm{Rot}_{\mathbb{S},\Gamma_t} &: D({}_{\mathbb{T}}\mathrm{Rot}_{\mathbb{S},\Gamma_t}) &&\subset \mathsf{L}^2_{\mathbb{S}}(\Omega)\to\mathsf{L}^2_{\mathbb{T}}(\Omega); && S\mapsto \mathrm{Rot}\ S,\\
{}_{\mathbb{T}}\mathrm{Rot}^*_{\mathbb{S},\Gamma_t} = {}_{\mathbb{S}}\mathbf{Rot}_{\mathbb{T},\Gamma_n} &: D({}_{\mathbb{S}}\mathbf{Rot}_{\mathbb{T},\Gamma_n}) &&\subset \mathsf{L}^2_{\mathbb{T}}(\Omega)\to\mathsf{L}^2_{\mathbb{S}}(\Omega); && T\mapsto \mathrm{symRot}\ T,\\
\mathrm{Div}_{\mathbb{T},\Gamma_t} &: D(\mathrm{Div}_{\mathbb{T},\Gamma_t}) &&\subset \mathsf{L}^2_{\mathbb{T}}(\Omega)\to\mathsf{L}^2(\Omega); && T\mapsto \mathrm{Div}\ T,\\
\mathrm{Div}^*_{\mathbb{T},\Gamma_t} = -{}_{\mathbb{T}}\mathbf{Grad}_{\Gamma_n} &: D({}_{\mathbb{T}}\mathbf{Grad}_{\Gamma_n}) &&\subset \mathsf{L}^2(\Omega)\to\mathsf{L}^2_{\mathbb{T}}(\Omega); && v\mapsto -\mathrm{devGrad}\ v,\\
{}_{\mathbb{T}}\mathrm{Grad}_{\Gamma_t} &: D({}_{\mathbb{T}}\mathrm{Grad}_{\Gamma_t}) &&\subset \mathsf{L}^2(\Omega)\to\mathsf{L}^2_{\mathbb{T}}(\Omega); && v\mapsto \mathrm{devGrad}\ v,\\
{}_{\mathbb{T}}\mathrm{Grad}^*_{\Gamma_t} = -\mathbf{Div}_{\mathbb{T},\Gamma_n} &: D(\mathbf{Div}_{\mathbb{T},\Gamma_n}) &&\subset \mathsf{L}^2_{\mathbb{T}}(\Omega)\to\mathsf{L}^2(\Omega); && T\mapsto -\mathrm{Div}\ T,\\
{}_{\mathbb{S}}\mathrm{Rot}_{\mathbb{T},\Gamma_t} &: D\left({}_{\mathbb{S}}\mathrm{Rot}_{\mathbb{T},\Gamma_t}\right) &&\subset \mathsf{L}^2_{\mathbb{T}}(\Omega)\to\mathsf{L}^2_{\mathbb{S}}(\Omega); && T\mapsto \mathrm{symRot}\ T,\\
{}_{\mathbb{S}}\mathrm{Rot}\,^*_{\mathbb{T},\Gamma_t} = {}_{\mathbb{T}}\mathbf{Rot}_{\mathbb{S},\Gamma_n} &: D({}_{\mathbb{T}}\mathbf{Rot}_{\mathbb{S},\Gamma_n}) &&\subset \mathsf{L}^2_{\mathbb{S}}(\Omega)\to\mathsf{L}^2_{\mathbb{T}}(\Omega); && S\mapsto \mathrm{Rot}\ S,\\
\mathrm{divDiv}_{\mathbb{S},\Gamma_t} &: D\left(\mathrm{divDiv}_{\mathbb{S},\Gamma_t}\right) &&\subset \mathsf{L}^2_{\mathbb{S}}(\Omega)\to\mathsf{L}^2(\Omega); && S\mapsto \mathrm{divDiv}\ S,\\
\mathrm{divDiv}^*_{\mathbb{S},\Gamma_t} = {}_{\mathbb{S}}\mathbf{Gradgrad}_{\Gamma_n} &: D({}_{\mathbb{S}}\mathbf{Gradgrad}_{\Gamma_n}) &&\subset \mathsf{L}^2(\Omega)\to\mathsf{L}^2_{\mathbb{S}}(\Omega); && u\mapsto \mathrm{Gradgrad}\ u,
\end{aligned}
$$

with domains of definition

$$
\begin{aligned}
D\left({}_{\mathbb{S}}\mathrm{Gradgrad}_{\Gamma_t}\right) &= \mathsf{H}_{\Gamma_t}(\mathrm{Gradgrad},\Omega), & D(\mathbf{divDiv}_{\mathbb{S},\Gamma_n}) &= \mathbf{H}_{\mathbb{S},\Gamma_n}(\mathrm{divDiv},\Omega),\\
D({}_{\mathbb{T}}\mathrm{Rot}_{\mathbb{S},\Gamma_t}) &= \mathsf{H}_{\mathbb{S},\Gamma_t}(\mathrm{Rot},\Omega), & D({}_{\mathbb{S}}\mathbf{Rot}_{\mathbb{T},\Gamma_n}) &= \mathbf{H}_{\mathbb{T},\Gamma_n}(\mathrm{symRot},\Omega),\\
D(\mathrm{Div}_{\mathbb{T},\Gamma_t}) &= \mathsf{H}_{\mathbb{T},\Gamma_t}(\mathrm{Div},\Omega), & D({}_{\mathbb{T}}\mathbf{Grad}_{\Gamma_n}) &= \mathbf{H}_{\Gamma_n}(\mathrm{devGrad},\Omega),\\
D({}_{\mathbb{T}}\mathrm{Grad}_{\Gamma_t}) &= \mathsf{H}_{\Gamma_t}(\mathrm{devGrad},\Omega), & D(\mathbf{Div}_{\mathbb{T},\Gamma_n}) &= \mathbf{H}_{\mathbb{T},\Gamma_n}(\mathrm{Div},\Omega),\\
D\left({}_{\mathbb{S}}\mathrm{Rot}_{\mathbb{T},\Gamma_t}\right) &= \mathsf{H}_{\mathbb{T},\Gamma_t}(\mathrm{symRot},\Omega), & D({}_{\mathbb{T}}\mathbf{Rot}_{\mathbb{S},\Gamma_n}) &= \mathbf{H}_{\mathbb{S},\Gamma_n}(\mathrm{Rot},\Omega),\\
D\left(\mathrm{divDiv}_{\mathbb{S},\Gamma_t}\right) &= \mathsf{H}_{\mathbb{S},\Gamma_t}(\mathrm{divDiv},\Omega), & D({}_{\mathbb{S}}\mathbf{Gradgrad}_{\Gamma_n}) &= \mathbf{H}_{\Gamma_n}(\mathrm{Gradgrad},\Omega).
\end{aligned}
$$

We shall introduce the latter Sobolev spaces in the next section.

2.2 Sobolev spaces

Let

$$\begin{aligned}
\mathsf{H}(\mathrm{Gradgrad},\Omega) &:= \left\{u \in \mathsf{L}^2(\Omega) : \mathrm{Gradgrad}\ u \in \mathsf{L}^2(\Omega)\right\},\\
\mathsf{H}_{\mathbb{S}}(\mathrm{Rot},\Omega) &:= \left\{S \in \mathsf{L}^2_{\mathbb{S}}(\Omega) : \mathrm{Rot}\ S \in \mathsf{L}^2(\Omega)\right\},\\
\mathsf{H}_{\mathbb{T}}(\mathrm{Div},\Omega) &:= \left\{T \in \mathsf{L}^2_{\mathbb{T}}(\Omega) : \mathrm{Div}\ T \in \mathsf{L}^2(\Omega)\right\},\\
\mathsf{H}(\mathrm{devGrad},\Omega) &:= \left\{v \in \mathsf{L}^2(\Omega) : \mathrm{devGrad}\ v \in \mathsf{L}^2(\Omega)\right\},\\
\mathsf{H}_{\mathbb{T}}(\mathrm{symRot},\Omega) &:= \left\{T \in \mathsf{L}^2_{\mathbb{T}}(\Omega) : \mathrm{symRot}\ T \in \mathsf{L}^2(\Omega)\right\},\\
\mathsf{H}_{\mathbb{S}}(\mathrm{divDiv},\Omega) &:= \left\{S \in \mathsf{L}^2_{\mathbb{S}}(\Omega) : \mathrm{divDiv}\ S \in \mathsf{L}^2(\Omega)\right\}.
\end{aligned}$$

Note that $S \in \mathsf{H}_{\mathbb{S}}(\mathrm{Rot},\Omega)$ implies $\mathrm{RotS} \in \mathsf{L}^2_{\mathbb{T}}(\Omega)$ (cf. Lemma A.1 (Appendix A)) and that we have by Nečas' inequality and a Korn type inequality for dev the regularities

$$\mathsf{H}(\mathrm{Gradgrad},\Omega) = \mathsf{H}^2(\Omega), \qquad \mathsf{H}(\mathrm{devGrad},\Omega) = \mathsf{H}^1(\Omega) \tag{2}$$

with equivalent norms; see, for example, [5, Lemma 8.2] and [4, Lemma 3.2]. Moreover, we define boundary conditions in the *strong sense* as closures of respective test fields, that is,

$$\begin{aligned}
\mathsf{H}_{\Gamma_t}(\mathrm{Gradgrad},\Omega) &:= \overline{\mathsf{C}^\infty_{\Gamma_t}(\Omega)}^{\mathsf{H}^2(\Omega)} = \mathsf{H}^2_{\Gamma_t}(\Omega),\\
\mathsf{H}_{\mathbb{S},\Gamma_t}(\mathrm{Rot},\Omega) &:= \overline{\mathsf{C}^\infty_{\mathbb{S},\Gamma_t}(\Omega)}^{\mathsf{H}_{\mathbb{S}}(\mathrm{Rot},\Omega)},\\
\mathsf{H}_{\mathbb{T},\Gamma_t}(\mathrm{Div},\Omega) &:= \overline{\mathsf{C}^\infty_{\mathbb{T},\Gamma_t}(\Omega)}^{\mathsf{H}_{\mathbb{T}}(\mathrm{Div},\Omega)},\\
\mathsf{H}_{\Gamma_t}(\mathrm{devGrad},\Omega) &:= \overline{\mathsf{C}^\infty_{\Gamma_t}(\Omega)}^{\mathsf{H}^1(\Omega)} = \mathsf{H}^1_{\Gamma_t}(\Omega),\\
\mathsf{H}_{\mathbb{T},\Gamma_t}(\mathrm{symRot},\Omega) &:= \overline{\mathsf{C}^\infty_{\mathbb{T},\Gamma_t}(\Omega)}^{\mathsf{H}_{\mathbb{T}}(\mathrm{symRot},\Omega)},\\
\mathsf{H}_{\mathbb{S},\Gamma_t}(\mathrm{divDiv},\Omega) &:= \overline{\mathsf{C}^\infty_{\mathbb{S},\Gamma_t}(\Omega)}^{\mathsf{H}_{\mathbb{S}}(\mathrm{divDiv},\Omega)}.
\end{aligned}$$

For $\Gamma_t = \emptyset$, we may skip the index $\emptyset$, which is justified by density. Spaces with vanishing differential operator coincide with kernels and are denoted by an additional index 0 at the lower right corner, for example,

$$\mathsf{H}_{\mathbb{S},\Gamma_t,0}(\mathrm{Rot},\Omega) = N({}_{\mathbb{T}}\mathrm{Rot}_{\mathbb{S},\Gamma_t}), \qquad \mathsf{H}_{\mathbb{T},\Gamma_t,0}(\mathrm{Div},\Omega) = N(\mathrm{Div}_{\mathbb{T},\Gamma_t}).$$

We need also the Sobolev spaces with boundary conditions defined in the *weak sense*, that is,

$$\begin{aligned}
\mathbf{H}_{\Gamma_t}(\mathrm{Gradgrad},\Omega) &:= \left\{u \in \mathsf{H}^2(\Omega) : \langle \mathrm{Gradgrad}\ u, \Phi\rangle_{\mathsf{L}^2_{\mathbb{S}}(\Omega)} = \langle u, \mathrm{divDiv}\ \Phi\rangle_{\mathsf{L}^2(\Omega)}\ \forall \Phi \in \mathsf{C}^\infty_{\mathbb{S},\Gamma_n}(\Omega)\right\},\\
\mathbf{H}_{\mathbb{S},\Gamma_t}(\mathrm{Rot},\Omega) &:= \left\{S \in \mathsf{H}_{\mathbb{S}}(\mathrm{Rot},\Omega) : \langle \mathrm{Rot}\ S, \Psi\rangle_{\mathsf{L}^2_{\mathbb{T}}(\Omega)} = \langle S, \mathrm{symRot}\ \Psi\rangle_{\mathsf{L}^2_{\mathbb{S}}(\Omega)}\ \forall \Psi \in \mathsf{C}^\infty_{\mathbb{T},\Gamma_n}(\Omega)\right\},\\
\mathbf{H}_{\mathbb{T},\Gamma_t}(\mathrm{Div},\Omega) &:= \left\{T \in \mathsf{H}_{\mathbb{T}}(\mathrm{Div},\Omega) : \langle \mathrm{Div}\ T, \phi\rangle_{\mathsf{L}^2(\Omega)} = -\langle T, \mathrm{devGrad}\ \phi\rangle_{\mathsf{L}^2_{\mathbb{T}}(\Omega)}\ \forall \phi \in \mathsf{C}^\infty_{\Gamma_n}(\Omega)\right\},\\
\mathbf{H}_{\Gamma_t}(\mathrm{devGrad},\Omega) &:= \left\{v \in \mathsf{H}^1(\Omega) : \langle \mathrm{devGrad}\ v, \Psi\rangle_{\mathsf{L}^2_{\mathbb{T}}(\Omega)} = -\langle v, \mathrm{Div}\ \Psi\rangle_{\mathsf{L}^2(\Omega)}\ \forall \Psi \in \mathsf{C}^\infty_{\mathbb{T},\Gamma_n}(\Omega)\right\},\\
\mathbf{H}_{\mathbb{T},\Gamma_t}(\mathrm{symRot},\Omega) &:= \left\{T \in \mathsf{H}_{\mathbb{T}}(\mathrm{symRot},\Omega) : \langle \mathrm{symRot}\ T, \Phi\rangle_{\mathsf{L}^2_{\mathbb{S}}(\Omega)} = \langle T, \mathrm{Rot}\ \Phi\rangle_{\mathsf{L}^2_{\mathbb{T}}(\Omega)}\ \forall \Phi \in \mathsf{C}^\infty_{\mathbb{S},\Gamma_n}(\Omega)\right\},\\
\mathbf{H}_{\mathbb{S},\Gamma_t}(\mathrm{divDiv},\Omega) &:= \left\{S \in \mathsf{H}_{\mathbb{S}}(\mathrm{divDiv},\Omega) : \langle \mathrm{divDiv}\ S, \phi\rangle_{\mathsf{L}^2(\Omega)} = \langle S, \mathrm{Gradgrad}\ \phi\rangle_{\mathsf{L}^2_{\mathbb{S}}(\Omega)}\ \forall \phi \in \mathsf{C}^\infty_{\Gamma_n}(\Omega)\right\}.
\end{aligned}$$

Note that "*strong* ⊂ *weak*" holds, that is, $\mathsf{H}_{\dots}(\cdots,\Omega) \subset \mathbf{H}_{\dots}(\cdots,\Omega)$, for example,

$$\mathsf{H}_{\mathbb{S},\Gamma_t}(\mathrm{Rot},\Omega) \subset \mathbf{H}_{\mathbb{S},\Gamma_t}(\mathrm{Rot},\Omega), \qquad \mathsf{H}_{\mathbb{T},\Gamma_t}(\mathrm{Div},\Omega) \subset \mathbf{H}_{\mathbb{T},\Gamma_t}(\mathrm{Div},\Omega),$$

and that the complex properties hold in both the strong and the weak case, for example,

$$\mathrm{devGrad}\ \mathsf{H}_{\Gamma_t}(\Omega) \subset \mathsf{H}_{\mathbb{T},\Gamma_t,0}(\mathrm{symRot},\Omega), \qquad \mathrm{Rot}\ \mathbf{H}_{\mathbb{S},\Gamma_t}(\mathrm{Rot},\Omega) \subset \mathbf{H}_{\mathbb{T},\Gamma_t,0}(\mathrm{Div},\Omega),$$

which follows immediately by the definitions. In Remark 2.3 below, we comment on the question whether "*strong* = *weak*" holds in general.

2.3 | Higher order Sobolev spaces

For $k \in \mathbb{N}_0$, we define higher order Sobolev spaces by

$$\begin{aligned}
\mathsf{H}^k_{\mathbb{S}}(\Omega) &:= \mathsf{H}^k(\Omega) \cap \mathsf{L}^2_{\mathbb{S}}(\Omega),\\
\mathsf{H}^k_{\mathbb{T}}(\Omega) &:= \mathsf{H}^k(\Omega) \cap \mathsf{L}^2_{\mathbb{T}}(\Omega),\\
\mathsf{H}^k_{\mathbb{S},\Gamma_t}(\Omega) &:= \overline{\mathsf{C}^\infty_{\mathbb{S},\Gamma_t}(\Omega)}^{\mathsf{H}^k(\Omega)} = \mathsf{H}^k_{\Gamma_t}(\Omega) \cap \mathsf{L}^2_{\mathbb{S}}(\Omega),\\
\mathsf{H}^k_{\mathbb{T},\Gamma_t}(\Omega) &:= \overline{\mathsf{C}^\infty_{\mathbb{T},\Gamma_t}(\Omega)}^{\mathsf{H}^k(\Omega)} = \mathsf{H}^k_{\Gamma_t}(\Omega) \cap \mathsf{L}^2_{\mathbb{T}}(\Omega),\\
\mathsf{H}^k(\mathrm{Gradgrad},\Omega) &:= \left\{u \in \mathsf{H}^k(\Omega) : \mathrm{Gradgrad}\ u \in \mathsf{H}^k(\Omega)\right\},\\
\mathsf{H}^k_{\Gamma_t}(\mathrm{Gradgrad},\Omega) &:= \left\{u \in \mathsf{H}^k_{\Gamma_t}(\Omega) \cap \mathsf{H}_{\Gamma_t}(\mathrm{Gradgrad},\Omega) : \mathrm{Gradgrad}\ u \in \mathsf{H}^k_{\Gamma_t}(\Omega)\right\},\\
\mathsf{H}^k_{\mathbb{S}}(\mathrm{Rot},\Omega) &:= \left\{S \in \mathsf{H}^k_{\mathbb{S}}(\Omega) : \mathrm{Rot}\ S \in \mathsf{H}^k(\Omega)\right\},\\
\mathsf{H}^k_{\mathbb{S},\Gamma_t}(\mathrm{Rot},\Omega) &:= \left\{S \in \mathsf{H}^k_{\Gamma_t}(\Omega) \cap \mathsf{H}_{\mathbb{S},\Gamma_t}(\mathrm{Rot},\Omega) : \mathrm{Rot}\ S \in \mathsf{H}^k_{\Gamma_t}(\Omega)\right\},\\
\mathsf{H}^k_{\mathbb{T}}(\mathrm{Div},\Omega) &:= \left\{T \in \mathsf{H}^k_{\mathbb{T}}(\Omega) : \mathrm{Div}\ T \in \mathsf{H}^k(\Omega)\right\},\\
\mathsf{H}^k_{\mathbb{T},\Gamma_t}(\mathrm{Div},\Omega) &:= \left\{T \in \mathsf{H}^k_{\Gamma_t}(\Omega) \cap \mathsf{H}_{\mathbb{T},\Gamma_t}(\mathrm{Div},\Omega) : \mathrm{Div}\ T \in \mathsf{H}^k_{\Gamma_t}(\Omega)\right\},\\
\mathsf{H}^k(\mathrm{devGrad},\Omega) &:= \left\{v \in \mathsf{H}^k(\Omega) : \mathrm{devGrad}\ v \in \mathsf{H}^k(\Omega)\right\},\\
\mathsf{H}^k_{\Gamma_t}(\mathrm{devGrad},\Omega) &:= \left\{v \in \mathsf{H}^k_{\Gamma_t}(\Omega) \cap \mathsf{H}_{\Gamma_t}(\mathrm{devGrad},\Omega) : \mathrm{devGrad}\ v \in \mathsf{H}^k_{\Gamma_t}(\Omega)\right\},\\
\mathsf{H}^k_{\mathbb{T}}(\mathrm{symRot},\Omega) &:= \left\{T \in \mathsf{H}^k_{\mathbb{T}}(\Omega) : \mathrm{symRot}\ T \in \mathsf{H}^k(\Omega)\right\},\\
\mathsf{H}^k_{\mathbb{T},\Gamma_t}(\mathrm{symRot},\Omega) &:= \left\{T \in \mathsf{H}^k_{\Gamma_t}(\Omega) \cap \mathsf{H}_{\mathbb{T},\Gamma_t}(\mathrm{symRot},\Omega) : \mathrm{symRot}\ T \in \mathsf{H}^k_{\Gamma_t}(\Omega)\right\},\\
\mathsf{H}^k_{\mathbb{S}}(\mathrm{divDiv},\Omega) &:= \left\{S \in \mathsf{H}^k_{\mathbb{S}}(\Omega) : \mathrm{divDiv}\ S \in \mathsf{H}^k(\Omega)\right\},\\
\mathsf{H}^k_{\mathbb{S},\Gamma_t}(\mathrm{divDiv},\Omega) &:= \left\{S \in \mathsf{H}^k_{\Gamma_t}(\Omega) \cap \mathsf{H}_{\mathbb{S},\Gamma_t}(\mathrm{divDiv},\Omega) : \mathrm{divDiv}\ S \in \mathsf{H}^k_{\Gamma_t}(\Omega)\right\}.
\end{aligned}$$

For the first reading, we recommend to only regard the case $k = 0$ from Section 2.2.

Note that, for example, for the latter divDiv-Sobolev spaces, we have $\mathsf{H}^k_{\mathbb{S},\emptyset}(\mathrm{divDiv},\Omega) = \mathsf{H}^k_{\mathbb{S}}(\mathrm{divDiv},\Omega)$ and $\mathsf{H}^0_{\mathbb{S},\emptyset}(\mathrm{divDiv},\Omega) = \mathsf{H}_{\mathbb{S}}(\mathrm{divDiv},\Omega)$ as well as $\mathsf{H}^0_{\mathbb{S},\Gamma_t}(\mathrm{divDiv},\Omega) = \mathsf{H}_{\mathbb{S},\Gamma_t}(\mathrm{divDiv},\Omega)$. For $\Gamma_t \neq \emptyset$, it holds

$$\mathsf{H}^k_{\mathbb{S},\Gamma_t}(\mathrm{divDiv},\Omega) = \left\{S \in \mathsf{H}^k_{\mathbb{S},\Gamma_t}(\Omega) : \mathrm{divDiv}\ S \in \mathsf{H}^k_{\Gamma_t}(\Omega)\right\}, \qquad k \geq 2,$$

but for $\Gamma_t \neq \emptyset$ and $k=0$ and $k=1$

$$\begin{aligned}\mathsf{H}^0_{\mathbb{S},\Gamma_t}(\operatorname{divDiv},\Omega) &= \mathsf{H}_{\mathbb{S},\Gamma_t}(\operatorname{divDiv},\Omega)\\ &\subsetneq \left\{ S \in \underbrace{\mathsf{H}^0_{\mathbb{S},\Gamma_t}(\Omega)}_{=\mathsf{L}^2_{\mathbb{S}}(\Omega)} \,:\, \operatorname{divDiv} S \in \underbrace{\mathsf{H}^0_{\Gamma_t}(\Omega)}_{=\mathsf{L}^2(\Omega)} \right\} = \mathsf{H}_{\mathbb{S}}(\operatorname{divDiv},\Omega),\\ \mathsf{H}^1_{\mathbb{S},\Gamma_t}(\operatorname{divDiv},\Omega) &\subsetneq \left\{ S \in \mathsf{H}^1_{\mathbb{S},\Gamma_t}(\Omega) \,:\, \operatorname{divDiv} S \in \mathsf{H}^1_{\Gamma_t}(\Omega) \right\},\end{aligned}$$

respectively. As before, we introduce the kernels

$$\begin{aligned}\mathsf{H}^k_{\mathbb{S},\Gamma_t,0}(\operatorname{divDiv},\Omega) &:= \mathsf{H}^k_{\Gamma_t}(\Omega) \cap \mathsf{H}_{\mathbb{S},\Gamma_t,0}(\operatorname{divDiv},\Omega) = \mathsf{H}^k_{\mathbb{S},\Gamma_t}(\operatorname{divDiv},\Omega) \cap \mathsf{H}_{\mathbb{S},0}(\operatorname{divDiv},\Omega)\\ &= \left\{ S \in \mathsf{H}^k_{\mathbb{S},\Gamma_t}(\operatorname{divDiv},\Omega) \,:\, \operatorname{divDiv} S = 0 \right\}.\end{aligned}$$

The corresponding remarks and definitions extend also to the $\mathsf{H}^k_{\Gamma_t}(\operatorname{Gradgrad},\Omega)$, $\mathsf{H}^k_{\mathbb{S},\Gamma_t}(\operatorname{Rot},\Omega)$, $\mathsf{H}^k_{\mathbb{T},\Gamma_t}(\operatorname{Div},\Omega)$, $\mathsf{H}^k_{\Gamma_t}(\operatorname{devGrad},\Omega)$, and $\mathsf{H}^k_{\mathbb{T},\Gamma_t}(\operatorname{symRot},\Omega)$-spaces. In particular, we have for $\Gamma_t \neq \emptyset$ and $k \geq 1$ and, for example, $\mathsf{H}^k_{\mathbb{S},\Gamma_t}(\operatorname{Rot},\Omega)$, the observations

$$\begin{aligned}\mathsf{H}^k_{\mathbb{S},\Gamma_t}(\operatorname{Rot},\Omega) &= \left\{ S \in \mathsf{H}^k_{\mathbb{S},\Gamma_t}(\Omega) \,:\, \operatorname{Rot} S \in \mathsf{H}^k_{\Gamma_t}(\Omega) \right\},\\ \mathsf{H}^0_{\mathbb{S},\Gamma_t}(\operatorname{Rot},\Omega) &= \mathsf{H}_{\mathbb{S},\Gamma_t}(\operatorname{Rot},\Omega) \subsetneq \left\{ S \in \underbrace{\mathsf{H}^0_{\mathbb{S},\Gamma_t}(\Omega)}_{=\mathsf{L}^2_{\mathbb{S}}(\Omega)} \,:\, \operatorname{Rot} S \in \underbrace{\mathsf{H}^0_{\Gamma_t}(\Omega)}_{=\mathsf{L}^2(\Omega)} \right\} = \mathsf{H}_{\mathbb{S}}(\operatorname{Rot},\Omega),\\ \mathsf{H}^k_{\mathbb{S},\Gamma_t,0}(\operatorname{Rot},\Omega) &= \mathsf{H}^k_{\Gamma_t}(\Omega) \cap \mathsf{H}_{\mathbb{S},\Gamma_t,0}(\operatorname{Rot},\Omega) = \mathsf{H}^k_{\mathbb{S},\Gamma_t}(\operatorname{Rot},\Omega) \cap \mathsf{H}_{\mathbb{S},0}(\operatorname{Rot},\Omega)\\ &= \left\{ S \in \mathsf{H}^k_{\mathbb{S},\Gamma_t}(\operatorname{Rot},\Omega) \,:\, \operatorname{Rot} S = 0 \right\}.\end{aligned}$$

Analogously, we define the Sobolev spaces $\mathbf{H}^k_{\Gamma_t}(\operatorname{Gradgrad},\Omega)$, $\mathbf{H}^k_{\mathbb{S},\Gamma_t}(\operatorname{Rot},\Omega)$, $\mathbf{H}^k_{\mathbb{T},\Gamma_t}(\operatorname{Div},\Omega)$, $\mathbf{H}^k_{\Gamma_t}(\operatorname{devGrad},\Omega)$, $\mathbf{H}^k_{\mathbb{T},\Gamma_t}(\operatorname{symRot},\Omega)$, and $\mathbf{H}^k_{\mathbb{S},\Gamma_t}(\operatorname{divDiv},\Omega)$ using the respective Sobolev spaces with weak boundary conditions $\mathbf{H}^{\cdots}_{\cdots}(\cdots,\Omega)$ in the definitions, for example,

$$\begin{aligned}\mathbf{H}^k_{\mathbb{T},\Gamma_t}(\operatorname{symRot},\Omega) &:= \left\{ T \in \mathbf{H}^k_{\Gamma_t}(\Omega) \cap \mathbf{H}_{\mathbb{T},\Gamma_t}(\operatorname{symRot},\Omega) \,:\, \operatorname{symRot} T \in \mathbf{H}^k_{\Gamma_t}(\Omega) \right\}\\ &= \left\{ T \in \mathsf{H}^k_{\Gamma_t}(\Omega) \cap \mathbf{H}_{\mathbb{T},\Gamma_t}(\operatorname{symRot},\Omega) \,:\, \operatorname{symRot} T \in \mathsf{H}^k_{\Gamma_t}(\Omega) \right\},\end{aligned}$$

where we have used (1). Note that again "*strong* $\subset$ *weak*" holds, that is, $\mathsf{H}^{\cdots}_{\cdots}(\cdots,\Omega) \subset \mathbf{H}^{\cdots}_{\cdots}(\cdots,\Omega)$, for example, $\mathsf{H}^k_{\mathbb{S},\Gamma_t}(\operatorname{Rot},\Omega) \subset \mathbf{H}^k_{\mathbb{S},\Gamma_t}(\operatorname{Rot},\Omega)$, and that the complex properties hold in both the strong and the weak case, for example,

$$\operatorname{Gradgrad} \mathsf{H}^{k+2}_{\Gamma_t}(\Omega) \subset \mathsf{H}^k_{\mathbb{S},\Gamma_t,0}(\operatorname{Rot},\Omega), \qquad \operatorname{symRot} \mathbf{H}^k_{\mathbb{T},\Gamma_t}(\operatorname{symRot},\Omega) \subset \mathbf{H}^k_{\mathbb{S},\Gamma_t,0}(\operatorname{divDiv},\Omega).$$

In the forthcoming sections, we shall also investigate whether indeed "*strong* = *weak*" holds. We start with a simple implication from (1).

Corollary 2.1. *$\mathbf{H}^k_{\mathbb{S},\Gamma_t}(\Omega) = \mathsf{H}^k_{\mathbb{S},\Gamma_t}(\Omega)$ and $\mathbf{H}^k_{\mathbb{T},\Gamma_t}(\Omega) = \mathsf{H}^k_{\mathbb{T},\Gamma_t}(\Omega)$, that is, weak and strong boundary conditions coincide for the standard Sobolev spaces of symmetric and deviatoric tensor fields with mixed boundary conditions, respectively.*

As in (2) *and with Corollary* 2.1, *we get the following*.

Lemma 2.2 (Higher order weak and strong partial boundary conditions coincide)**.**

(i) For $k \geq 0$, it holds

$$\mathsf{H}^k_{\Gamma_t}(\operatorname{devGrad},\Omega) = \mathsf{H}^{k+1}_{\Gamma_t}(\Omega) = \mathbf{H}^{k+1}_{\Gamma_t}(\Omega),$$
$$\mathsf{H}^k_{\Gamma_t}(\operatorname{Gradgrad},\Omega) = \mathsf{H}^{k+2}_{\Gamma_t}(\Omega) = \mathbf{H}^{k+2}_{\Gamma_t}(\Omega).$$

(ii) For $k \geq 1$, it holds

$$\mathbf{H}^k_{\Gamma_t}(\operatorname{devGrad},\Omega) = \left\{ v \in \mathsf{H}^k_{\Gamma_t}(\Omega) \,:\, \operatorname{devGrad}\ v \in \mathsf{H}^k_{\Gamma_t}(\Omega) \right\} = \mathsf{H}^k_{\Gamma_t}(\operatorname{devGrad},\Omega) = \mathsf{H}^{k+1}_{\Gamma_t}(\Omega),$$
$$\mathbf{H}^k_{\mathbb{S},\Gamma_t}(\operatorname{Rot},\Omega) = \left\{ S \in \mathsf{H}^k_{\mathbb{S},\Gamma_t}(\Omega) \,:\, \operatorname{Rot}\ S \in \mathsf{H}^k_{\Gamma_t}(\Omega) \right\} = \mathsf{H}^k_{\mathbb{S},\Gamma_t}(\operatorname{Rot},\Omega),$$
$$\mathbf{H}^k_{\mathbb{T},\Gamma_t}(\operatorname{symRot},\Omega) = \left\{ T \in \mathsf{H}^k_{\mathbb{T},\Gamma_t}(\Omega) \,:\, \operatorname{symRot}\ T \in \mathsf{H}^k_{\Gamma_t}(\Omega) \right\} = \mathsf{H}^k_{\mathbb{T},\Gamma_t}(\operatorname{symRot},\Omega),$$
$$\mathbf{H}^k_{\mathbb{T},\Gamma_t}(\operatorname{Div},\Omega) = \left\{ T \in \mathsf{H}^k_{\mathbb{T},\Gamma_t}(\Omega) \,:\, \operatorname{Div}\ T \in \mathsf{H}^k_{\Gamma_t}(\Omega) \right\} = \mathsf{H}^k_{\mathbb{T},\Gamma_t}(\operatorname{Div},\Omega).$$

(iii) For $k \geq 2$, it holds

$$\mathbf{H}^k_{\Gamma_t}(\operatorname{Gradgrad},\Omega) = \left\{ u \in \mathsf{H}^k_{\Gamma_t}(\Omega) \,:\, \operatorname{Gradgrad}\ u \in \mathsf{H}^k_{\Gamma_t}(\Omega) \right\} = \mathsf{H}^k_{\Gamma_t}(\operatorname{Gradgrad},\Omega),$$
$$\mathbf{H}^k_{\mathbb{S},\Gamma_t}(\operatorname{divDiv},\Omega) = \left\{ S \in \mathsf{H}^k_{\mathbb{S},\Gamma_t}(\Omega) \,:\, \operatorname{divDiv}\ S \in \mathsf{H}^k_{\Gamma_t}(\Omega) \right\} = \mathsf{H}^k_{\mathbb{S},\Gamma_t}(\operatorname{divDiv},\Omega).$$

Remark 2.3 (Weak and strong partial boundary conditions coincide)**.** In [4, 5], we could prove the corresponding results "*strong* = *weak*" for the whole two biharmonic complexes but only with empty or full boundary conditions ($\Gamma_t = \emptyset$ or $\Gamma_t = \Gamma$). Therefore, in these special cases, the adjoints are well defined on the spaces with strong boundary conditions as well.

Lemma 2.2 shows that for higher values of k indeed "*strong* = *weak*" holds. Thus, to show "*strong* = *weak*" in general, we only have to prove that equality holds in the remaining cases $k = 0$ and $k = 1$; that is, we only have to show

$$\begin{aligned}
&\mathbf{H}_{\Gamma_t}(\operatorname{devGrad},\Omega) \subset \mathsf{H}_{\Gamma_t}(\operatorname{devGrad},\Omega), && \mathbf{H}_{\Gamma_t}(\operatorname{Gradgrad},\Omega) \subset \mathsf{H}_{\Gamma_t}(\operatorname{Gradgrad},\Omega),\\
&\mathbf{H}_{\mathbb{T},\Gamma_t}(\operatorname{Div},\Omega) \subset \mathsf{H}_{\mathbb{T},\Gamma_t}(\operatorname{Div},\Omega), && \mathbf{H}^1_{\Gamma_t}(\operatorname{Gradgrad},\Omega) \subset \mathsf{H}^1_{\Gamma_t}(\operatorname{Gradgrad},\Omega),\\
&\mathbf{H}_{\mathbb{S},\Gamma_t}(\operatorname{Rot},\Omega) \subset \mathsf{H}_{\mathbb{S},\Gamma_t}(\operatorname{Rot},\Omega), && \mathbf{H}_{\mathbb{S},\Gamma_t}(\operatorname{divDiv},\Omega) \subset \mathsf{H}_{\mathbb{S},\Gamma_t}(\operatorname{divDiv},\Omega),\\
&\mathbf{H}_{\mathbb{T},\Gamma_t}(\operatorname{symRot},\Omega) \subset \mathsf{H}_{\mathbb{T},\Gamma_t}(\operatorname{symRot},\Omega), && \mathbf{H}^1_{\mathbb{S},\Gamma_t}(\operatorname{divDiv},\Omega) \subset \mathsf{H}^1_{\mathbb{S},\Gamma_t}(\operatorname{divDiv},\Omega).
\end{aligned}$$

The most delicate situation appears due to the second-order nature of $\operatorname{divDiv}_{\mathbb{S}}$. In Corollary 3.11, we shall show using regular decompositions that these results (weak and strong boundary conditions coincide for the biharmonic complexes for all $k \geq 0$) indeed hold true.

2.4 | More Sobolev spaces

For $k \in \mathbb{N}$, we introduce also slightly less regular higher order Sobolev spaces by

$$\mathsf{H}^{k,k-1}_{\Gamma_t}(\operatorname{Gradgrad},\Omega) := \left\{ u \in \mathsf{H}^k_{\Gamma_t}(\Omega) \cap \mathsf{H}_{\Gamma_t}(\operatorname{Gradgrad},\Omega) \,:\, \operatorname{Gradgrad}\ u \in \mathsf{H}^{k-1}_{\Gamma_t}(\Omega) \right\},$$
$$\mathbf{H}^{k,k-1}_{\Gamma_t}(\operatorname{Gradgrad},\Omega) := \left\{ u \in \mathbf{H}^k_{\Gamma_t}(\Omega) \cap \mathbf{H}_{\Gamma_t}(\operatorname{Gradgrad},\Omega) \,:\, \operatorname{Gradgrad}\ u \in \mathbf{H}^{k-1}_{\Gamma_t}(\Omega) \right\},$$
$$\mathsf{H}^{k,k-1}_{\mathbb{S},\Gamma_t}(\operatorname{divDiv},\Omega) := \left\{ S \in \mathsf{H}^k_{\Gamma_t}(\Omega) \cap \mathsf{H}_{\mathbb{S},\Gamma_t}(\operatorname{divDiv},\Omega) \,:\, \operatorname{divDiv}\ S \in \mathsf{H}^{k-1}_{\Gamma_t}(\Omega) \right\},$$
$$\mathbf{H}^{k,k-1}_{\mathbb{S},\Gamma_t}(\operatorname{divDiv},\Omega) := \left\{ S \in \mathbf{H}^k_{\Gamma_t}(\Omega) \cap \mathbf{H}_{\mathbb{S},\Gamma_t}(\operatorname{divDiv},\Omega) \,:\, \operatorname{divDiv}\ S \in \mathbf{H}^{k-1}_{\Gamma_t}(\Omega) \right\},$$

and we extend all conventions of our notations. These spaces can be ignored at the first reading.

We have for the kernels of $\operatorname{divDiv}_{\mathbb{S}}$

$$\mathsf{H}^{k,k-1}_{\mathbb{S},\Gamma_t,0}(\operatorname{divDiv},\Omega)=\mathsf{H}^{k}_{\mathbb{S},\Gamma_t,0}(\operatorname{divDiv},\Omega),\qquad \mathbf{H}^{k,k-1}_{\mathbb{S},\Gamma_t,0}(\operatorname{divDiv},\Omega)=\mathbf{H}^{k}_{\mathbb{S},\Gamma_t,0}(\operatorname{divDiv},\Omega),$$

and by Nečas' inequality (cf. (2)),

$$\mathsf{H}^{k,k-1}_{\Gamma_t}(\operatorname{Gradgrad},\Omega)=\mathsf{H}^{k+1}_{\Gamma_t}(\Omega)=\mathbf{H}^{k+1}_{\Gamma_t}(\Omega)\subset\mathbf{H}^{k,k-1}_{\Gamma_t}(\operatorname{Gradgrad},\Omega).$$

The intersection with $\mathsf{H}_{\Gamma_t}(\operatorname{Gradgrad},\Omega)$, $\mathbf{H}_{\Gamma_t}(\operatorname{Gradgrad},\Omega)$, and $\mathsf{H}_{\mathbb{S},\Gamma_t}(\operatorname{divDiv},\Omega)$, $\mathbf{H}_{\mathbb{S},\Gamma_t}(\operatorname{divDiv},\Omega)$, respectively, is only needed if $k=1$. As before, we observe $\mathsf{H}^{k,k-1}_{\mathbb{S},\Gamma_t}(\operatorname{divDiv},\Omega)\subset\mathbf{H}^{k,k-1}_{\mathbb{S},\Gamma_t}(\operatorname{divDiv},\Omega)$, that is, "*strong* $\subset$ *weak*," and in both cases (weak and strong), the complex properties hold, for example, $\operatorname{Gradgrad}\mathsf{H}^{k,k-1}_{\Gamma_t}(\operatorname{Gradgrad},\Omega)\subset\mathsf{H}^{k-1}_{\mathbb{S},\Gamma_t,0}(\operatorname{Rot},\Omega)$.

Similar to Lemma 2.2, we have the following.

Lemma 2.4. (Higher order weak and strong partial boundary conditions coincide)**.** *For $k\geq 2$,*

$$\mathbf{H}^{k,k-1}_{\Gamma_t}(\operatorname{Gradgrad},\Omega)=\left\{u\in\mathsf{H}^{k}_{\Gamma_t}(\Omega)\,:\,\operatorname{Gradgrad}\,u\in\mathsf{H}^{k-1}_{\Gamma_t}(\Omega)\right\}=\mathsf{H}^{k,k-1}_{\Gamma_t}(\operatorname{Gradgrad},\Omega)=\mathsf{H}^{k+1}_{\Gamma_t}(\Omega),$$
$$\mathbf{H}^{k,k-1}_{\mathbb{S},\Gamma_t}(\operatorname{divDiv},\Omega)=\left\{S\in\mathsf{H}^{k}_{\mathbb{S},\Gamma_t}(\Omega)\,:\,\operatorname{divDiv}\,S\in\mathsf{H}^{k-1}_{\Gamma_t}(\Omega)\right\}=\mathsf{H}^{k,k-1}_{\mathbb{S},\Gamma_t}(\operatorname{divDiv},\Omega).$$

2.5 | Some biharmonic complexes

By definition, we have densely defined and closed (unbounded) linear operators defining six dual pairs

$$\begin{aligned}
({}_{\mathbb{S}}\operatorname{Gradgrad}_{\Gamma_t},{}_{\mathbb{S}}\operatorname{Gradgrad}^*_{\Gamma_t})&=({}_{\mathbb{S}}\operatorname{Gradgrad}_{\Gamma_t},\mathbf{divDiv}_{\mathbb{S},\Gamma_n}),\\
({}_{\mathbb{T}}\operatorname{Rot}_{\mathbb{S},\Gamma_t},{}_{\mathbb{T}}\operatorname{Rot}^*_{\mathbb{S},\Gamma_t})&=({}_{\mathbb{T}}\operatorname{Rot}_{\mathbb{S},\Gamma_t},{}_{\mathbb{S}}\mathbf{Rot}_{\mathbb{T},\Gamma_n}),\\
(\operatorname{Div}_{\mathbb{T},\Gamma_t},\operatorname{Div}^*_{\mathbb{T},\Gamma_t})&=(\operatorname{Div}_{\mathbb{T},\Gamma_t},-{}_{\mathbb{T}}\mathbf{Grad}_{\Gamma_n}),\\
({}_{\mathbb{T}}\operatorname{Grad}_{\Gamma_t},{}_{\mathbb{T}}\operatorname{Grad}^*_{\Gamma_t})&=({}_{\mathbb{T}}\operatorname{Grad}_{\Gamma_t},-\mathbf{Div}_{\mathbb{T},\Gamma_n}),\\
({}_{\mathbb{S}}\operatorname{Rot}_{\mathbb{T},\Gamma_t},{}_{\mathbb{S}}\operatorname{Rot}^{\ *}_{\mathbb{T},\Gamma_t})&=({}_{\mathbb{S}}\operatorname{Rot}_{\mathbb{T},\Gamma_t},{}_{\mathbb{T}}\mathbf{Rot}_{\mathbb{S},\Gamma_n}),\\
(\operatorname{divDiv}_{\mathbb{S},\Gamma_t},\operatorname{divDiv}^*_{\mathbb{S},\Gamma_t})&=(\operatorname{divDiv}_{\mathbb{S},\Gamma_t},{}_{\mathbb{S}}\mathbf{Gradgrad}_{\Gamma_n}).
\end{aligned}$$

Pauly and Schomburg [1, Remark 2.5, Remark 2.6] show the complex properties

$$\begin{aligned}
{}_{\mathbb{T}}\operatorname{Rot}_{\mathbb{S},\Gamma_t\,\mathbb{S}}\operatorname{Gradgrad}_{\Gamma_t}&\subset 0, & \operatorname{Div}_{\mathbb{T},\Gamma_t\,\mathbb{T}}\operatorname{Rot}_{\mathbb{S},\Gamma_t}&\subset 0,\\
{}_{\mathbb{S}}\operatorname{Rot}_{\mathbb{T},\Gamma_t\,\mathbb{T}}\operatorname{Grad}_{\Gamma_t}&\subset 0, & \operatorname{divDiv}_{\mathbb{S},\Gamma_t\,\mathbb{S}}\operatorname{Rot}_{\mathbb{T},\Gamma_t}&\subset 0,\\
\mathbf{divDiv}_{\mathbb{S},\Gamma_n\,\mathbb{S}}\mathbf{Rot}_{\mathbb{T},\Gamma_n}&\subset 0, & -{}_{\mathbb{S}}\mathbf{Rot}_{\mathbb{T},\Gamma_n\,\mathbb{T}}\mathbf{Grad}_{\Gamma_n}&\subset 0,\\
-\mathbf{Div}_{\mathbb{T},\Gamma_n\,\mathbb{T}}\mathbf{Rot}_{\mathbb{S},\Gamma_n}&\subset 0, & {}_{\mathbb{T}}\mathbf{Rot}_{\mathbb{S},\Gamma_n\,\mathbb{S}}\mathbf{Gradgrad}_{\Gamma_n}&\subset 0.
\end{aligned}$$

Hence, we get the two primal and dual biharmonic Hilbert complexes

$$\cdots \underset{\cdots}{\overset{\cdots}{\rightleftarrows}} \mathsf{L}^2(\Omega) \underset{\mathbf{divDiv}_{\mathbb{S},\Gamma_n}}{\overset{{}_{\mathbb{S}}\operatorname{Gradgrad}_{\Gamma_t}}{\rightleftarrows}} \mathsf{L}^2_{\mathbb{S}}(\Omega) \underset{{}_{\mathbb{S}}\mathbf{Rot}_{\mathbb{T},\Gamma_n}}{\overset{{}_{\mathbb{T}}\operatorname{Rot}_{\mathbb{S},\Gamma_t}}{\rightleftarrows}} \mathsf{L}^2_{\mathbb{T}}(\Omega) \underset{-\,{}_{\mathbb{T}}\mathbf{Grad}_{\Gamma_n}}{\overset{\operatorname{Div}_{\mathbb{T},\Gamma_t}}{\rightleftarrows}} \mathsf{L}^2(\Omega) \underset{\cdots}{\overset{\cdots}{\rightleftarrows}} \cdots, \tag{3}$$

$$\cdots \underset{\cdots}{\overset{\cdots}{\rightleftarrows}} \mathsf{L}^2(\Omega) \underset{-\,\mathbf{Div}_{\mathbb{T},\Gamma_n}}{\overset{{}_{\mathbb{T}}\operatorname{Grad}_{\Gamma_t}}{\rightleftarrows}} \mathsf{L}^2_{\mathbb{T}}(\Omega) \underset{{}_{\mathbb{T}}\mathbf{Rot}_{\mathbb{S},\Gamma_n}}{\overset{{}_{\mathbb{S}}\operatorname{Rot}_{\mathbb{T},\Gamma_t}}{\rightleftarrows}} \mathsf{L}^2_{\mathbb{S}}(\Omega) \underset{{}_{\mathbb{S}}\mathbf{Gradgrad}_{\Gamma_n}}{\overset{\operatorname{divDiv}_{\mathbb{S},\Gamma_t}}{\rightleftarrows}} \mathsf{L}^2(\Omega) \underset{\cdots}{\overset{\cdots}{\rightleftarrows}} \cdots. \tag{4}$$

The long primal and dual biharmonic Hilbert complexes (cf. [1, (12)]) read

$$\mathbb{P}^1_{\Gamma_t} \underset{\pi_{\mathbb{P}^1_{\Gamma_t}}}{\overset{\iota_{\mathbb{P}^1_{\Gamma_t}}}{\rightleftarrows}} \mathsf{L}^2(\Omega) \underset{\mathbf{divDiv}_{\mathbb{S},\Gamma_n}}{\overset{{}_{\mathbb{S}}\operatorname{Gradgrad}_{\Gamma_t}}{\rightleftarrows}} \mathsf{L}^2_{\mathbb{S}}(\Omega) \underset{{}_{\mathbb{S}}\mathbf{Rot}_{\mathbb{T},\Gamma_n}}{\overset{{}_{\mathbb{T}}\operatorname{Rot}_{\mathbb{S},\Gamma_t}}{\rightleftarrows}} \mathsf{L}^2_{\mathbb{T}}(\Omega) \underset{-\,{}_{\mathbb{T}}\mathbf{Grad}_{\Gamma_n}}{\overset{\operatorname{Div}_{\mathbb{T},\Gamma_t}}{\rightleftarrows}} \mathsf{L}^2(\Omega) \underset{\iota_{\mathbb{RT}_{\Gamma_n}}}{\overset{\pi_{\mathbb{RT}_{\Gamma_n}}}{\rightleftarrows}} \mathbb{RT}_{\Gamma_n} \tag{5}$$

$$\mathbb{RT}_{\Gamma_t} \underset{\pi_{\mathbb{RT}_{\Gamma_t}}}{\overset{\iota_{\mathbb{RT}_{\Gamma_t}}}{\rightleftarrows}} \mathsf{L}^2(\Omega) \underset{-\,\mathbf{Div}_{\mathbb{T},\Gamma_n}}{\overset{{}_{\mathbb{T}}\mathrm{Grad}_{\Gamma_t}}{\rightleftarrows}} \mathsf{L}^2_{\mathbb{T}}(\Omega) \underset{{}_{\mathbb{T}}\mathbf{Rot}_{\mathbb{S},\Gamma_n}}{\overset{{}_{\mathbb{S}}\mathrm{Rot}_{\mathbb{T},\Gamma_t}}{\rightleftarrows}} \mathsf{L}^2_{\mathbb{S}}(\Omega) \underset{{}_{\mathbb{S}}\mathbf{Gradgrad}_{\Gamma_n}}{\overset{\mathrm{divDiv}_{\mathbb{S},\Gamma_t}}{\rightleftarrows}} \mathsf{L}^2(\Omega) \underset{\iota_{\mathbb{P}^1_{\Gamma_n}}}{\overset{\pi_{\mathbb{P}^1_{\Gamma_n}}}{\rightleftarrows}} \mathbb{P}^1_{\Gamma_n} \tag{6}$$

with the additional complex properties

$$R(\iota_{\mathbb{RT}_{\Gamma_t}}) = N\left({}_{\mathbb{T}}\mathrm{Grad}_{\Gamma_t}\right) = \mathbb{RT}_{\Gamma_t}, \qquad \overline{R(\mathrm{Div}_{\mathbb{T},\Gamma_n})} = \mathbb{RT}_{\Gamma_t}^{\perp_{\mathsf{L}^2(\Omega)}},$$
$$R(\iota_{\mathbb{P}^1_{\Gamma_t}}) = N\left({}_{\mathbb{S}}\mathrm{Gradgrad}_{\Gamma_t}\right) = \mathbb{P}^1_{\Gamma_t}, \quad \overline{R(\mathrm{divDiv}_{\mathbb{S},\Gamma_n})} = (\mathbb{P}^1_{\Gamma_t})^{\perp_{\mathsf{L}^2(\Omega)}},$$

where

$$\mathbb{RT}_\Sigma = \begin{cases} \{0\} & \text{if } \Sigma \neq \emptyset, \\ \mathbb{RT} & \text{if } \Sigma = \emptyset, \end{cases} \qquad \text{with} \qquad \mathbb{RT} := \left\{ \mathbb{R}^{3^3} \ni x \mapsto \mathrm{a}x + q \,:\, \mathrm{a} \in \mathbb{R}^3, \, q \in \mathbb{R}^3 \right\},$$
$$\mathbb{P}^1{}_\Sigma = \begin{cases} \{0\} & \text{if } \Sigma \neq \emptyset, \\ \mathbb{P}^1 & \text{if } \Sigma = \emptyset, \end{cases} \qquad \text{with} \qquad \mathbb{P}^1 := \left\{ \mathbb{R}^{3^3} \ni x \mapsto q \cdot x + \mathrm{a} \,:\, \mathrm{a} \in \mathbb{R}^3, \, q \in \mathbb{R}^3 \right\}$$

denote the global Raviart–Thomas fields and the global polynomials of degree less or equal to 1 in Ω, respectively. We have $\dim \mathbb{RT} = \dim \mathbb{P}^1 = 4$. Note that, for example, by Lemma 2.2 (i), it holds

$$N\left({}_{\mathbb{S}}\mathrm{Gradgrad}_{\Gamma_t}\right) = \left\{ u \in \mathsf{H}^2_{\Gamma_t}(\Omega) \,:\, \mathrm{Gradgrad}\; u = 0 \right\}.$$

More generally, in addition to (5) and (6), we shall discuss for $k \in \mathbb{N}_0$ the higher Sobolev order (long primal and formally dual) biharmonic Hilbert complexes (omitting Ω in the notation)

$$\mathbb{P}^1_{\Gamma_t} \xrightarrow{\iota_{\mathbb{P}^1_{\Gamma_t}}} \mathsf{H}^k_{\Gamma_t} \xrightarrow{{}_{\mathbb{S}}\mathrm{Gradgrad}^k_{\Gamma_t}} \mathsf{H}^k_{\mathbb{S},\Gamma_t} \xrightarrow{{}_{\mathbb{T}}\mathrm{Rot}^k_{\mathbb{S},\Gamma_t}} \mathsf{H}^k_{\mathbb{T},\Gamma_t} \xrightarrow{\mathrm{Div}^k_{\mathbb{T},\Gamma_t}} \mathsf{H}^k_{\Gamma_t} \xrightarrow{\pi_{\mathbb{RT}_{\Gamma_n}}} \mathbb{RT}_{\Gamma_n},$$

$$\mathbb{P}^1_{\Gamma_t} \xleftarrow{\pi_{\mathbb{P}^1_{\Gamma_t}}} \mathsf{H}^k_{\Gamma_n} \xleftarrow{\mathbf{divDiv}^k_{\mathbb{S},\Gamma_n}} \mathsf{H}^k_{\mathbb{S},\Gamma_n} \xleftarrow{{}_{\mathbb{S}}\mathbf{Rot}^k_{\mathbb{T},\Gamma_n}} \mathsf{H}^k_{\mathbb{T},\Gamma_n} \xleftarrow{-\,{}_{\mathbb{T}}\mathbf{Grad}^k_{\Gamma_n}} \mathsf{H}^k_{\Gamma_n} \xleftarrow{\iota_{\mathbb{RT}_{\Gamma_n}}} \mathbb{RT}_{\Gamma_n}$$

and

$$\mathbb{RT}_{\Gamma_t} \xrightarrow{\iota_{\mathbb{RT}_{\Gamma_t}}} \mathsf{H}^k_{\Gamma_t} \xrightarrow{{}_{\mathbb{T}}\mathrm{Grad}^k_{\Gamma_t}} \mathsf{H}^k_{\mathbb{T},\Gamma_t} \xrightarrow{{}_{\mathbb{S}}\mathrm{Rot}^k_{\mathbb{T},\Gamma_t}} \mathsf{H}^k_{\mathbb{S},\Gamma_t} \xrightarrow{\mathrm{divDiv}^k_{\mathbb{S},\Gamma_t}} \mathsf{H}^k_{\Gamma_t} \xrightarrow{\pi_{\mathbb{P}^1_{\Gamma_n}}} \mathbb{P}^1_{\Gamma_n},$$

$$\mathbb{RT}_{\Gamma_t} \xleftarrow{\pi_{\mathbb{RT}_{\Gamma_t}}} \mathsf{H}^k_{\Gamma_n} \xleftarrow{-\,\mathbf{Div}^k_{\mathbb{T},\Gamma_n}} \mathsf{H}^k_{\mathbb{T},\Gamma_n} \xleftarrow{{}_{\mathbb{T}}\mathbf{Rot}^k_{\mathbb{S},\Gamma_n}} \mathsf{H}^k_{\mathbb{S},\Gamma_n} \xleftarrow{{}_{\mathbb{S}}\mathbf{Gradgrad}^k_{\Gamma_n}} \mathsf{H}^k_{\Gamma_n} \xleftarrow{\iota_{\mathbb{P}^1_{\Gamma_n}}} \mathbb{P}^1_{\Gamma_n}$$

with associated domain complexes

$$\mathbb{P}^1_{\Gamma_t} \xrightarrow{\iota_{\mathbb{P}^1_{\Gamma_t}}} \mathsf{H}^k_{\Gamma_t}(\mathrm{Gradgrad}) \xrightarrow{{}_{\mathbb{S}}\mathrm{Gradgrad}^k_{\Gamma_t}} \mathsf{H}^k_{\mathbb{S},\Gamma_t}(\mathrm{Rot}) \xrightarrow{{}_{\mathbb{T}}\mathrm{Rot}^k_{\mathbb{S},\Gamma_t}} \mathsf{H}^k_{\mathbb{T},\Gamma_t}(\mathrm{Div}) \xrightarrow{\mathrm{Div}^k_{\mathbb{T},\Gamma_t}} \mathsf{H}^k_{\Gamma_t} \xrightarrow{\pi_{\mathbb{RT}_{\Gamma_n}}} \mathbb{RT}_{\Gamma_n},$$

$$\mathbb{P}^1_{\Gamma_t} \xleftarrow{\pi_{\mathbb{P}^1_{\Gamma_t}}} \mathsf{H}^k_{\Gamma_n} \xleftarrow{\mathbf{divDiv}^k_{\mathbb{S},\Gamma_n}} \mathbf{H}^k_{\mathbb{S},\Gamma_n}(\mathrm{divDiv}) \xleftarrow{{}_{\mathbb{S}}\mathbf{Rot}^k_{\mathbb{T},\Gamma_n}} \mathbf{H}^k_{\mathbb{T},\Gamma_n}(\mathrm{symRot}) \xleftarrow{-\,{}_{\mathbb{T}}\mathbf{Grad}^k_{\Gamma_n}} \mathbf{H}^k_{\Gamma_n}(\mathrm{devGrad}) \xleftarrow{\iota_{\mathbb{RT}_{\Gamma_n}}} \mathbb{RT}_{\Gamma_n}$$

and

$$\mathbb{RT}_{\Gamma_t} \xrightarrow{\iota_{\mathbb{RT}_{\Gamma_t}}} \mathsf{H}^k_{\Gamma_t}(\mathrm{devGrad}) \xrightarrow{{}_{\mathbb{T}}\mathrm{Grad}^k_{\Gamma_t}} \mathsf{H}^k_{\mathbb{T},\Gamma_t}(\mathrm{symRot}) \xrightarrow{{}_{\mathbb{S}}\mathrm{Rot}^k_{\mathbb{T},\Gamma_t}} \mathsf{H}^k_{\mathbb{S},\Gamma_t}(\mathrm{divDiv}) \xrightarrow{\mathrm{divDiv}^k_{\mathbb{S},\Gamma_t}} \mathsf{H}^k_{\Gamma_t} \xrightarrow{\pi_{\mathbb{P}^1_{\Gamma_n}}} \mathbb{P}^1_{\Gamma_n},$$

$$\mathbb{RT}_{\Gamma_t} \xleftarrow{\pi_{\mathbb{RT}_{\Gamma_t}}} \mathsf{H}^k_{\Gamma_n} \xleftarrow{-\,\mathbf{Div}^k_{\mathbb{T},\Gamma_n}} \mathbf{H}^k_{\mathbb{T},\Gamma_n}(\mathrm{Div}) \xleftarrow{{}_{\mathbb{T}}\mathbf{Rot}^k_{\mathbb{S},\Gamma_n}} \mathbf{H}^k_{\mathbb{S},\Gamma_n}(\mathrm{Rot}) \xleftarrow{{}_{\mathbb{S}}\mathbf{Gradgrad}^k_{\Gamma_n}} \mathbf{H}^k_{\Gamma_n}(\mathrm{Gradgrad}) \xleftarrow{\iota_{\mathbb{P}^1_{\Gamma_n}}} \mathbb{P}^1_{\Gamma_n}.$$

Additionally, for $k \geq 1$ we will also discuss the following variants of the biharmonic complexes

$$\mathbb{P}^1_{\Gamma_t} \xrightarrow{\iota_{\mathbb{P}^1_{\Gamma_t}}} \mathsf{H}^k_{\Gamma_t} \xrightarrow{{}_{\mathbb{S}}\mathrm{Gradgrad}^{k,k-1}_{\Gamma_t}} \mathsf{H}^{k-1}_{\mathbb{S},\Gamma_t} \xrightarrow{{}_{\mathbb{T}}\mathrm{Rot}^{k-1}_{\mathbb{S},\Gamma_t}} \mathsf{H}^{k-1}_{\mathbb{T},\Gamma_t} \xrightarrow{\mathrm{Div}^{k-1}_{\mathbb{T},\Gamma_t}} \mathsf{H}^{k-1}_{\Gamma_t} \xrightarrow{\pi_{\mathbb{RT}_{\Gamma_n}}} \mathbb{RT}_{\Gamma_n},$$

$$\mathbb{P}^1_{\Gamma_t} \xleftarrow{\pi_{\mathbb{P}^1_{\Gamma_t}}} \mathsf{H}^{k-1}_{\Gamma_n} \xleftarrow{\mathbf{divDiv}^{k,k-1}_{\mathbb{S},\Gamma_n}} \mathsf{H}^k_{\mathbb{S},\Gamma_n} \xleftarrow{{}_{\mathbb{S}}\mathbf{Rot}^k_{\mathbb{T},\Gamma_n}} \mathsf{H}^k_{\mathbb{T},\Gamma_n} \xleftarrow{-\,{}_{\mathbb{T}}\mathbf{Grad}^k_{\Gamma_n}} \mathsf{H}^k_{\Gamma_n} \xleftarrow{\iota_{\mathbb{RT}_{\Gamma_n}}} \mathbb{RT}_{\Gamma_n}$$

and

$$\mathbb{RT}_{\Gamma_t} \xrightarrow{\iota_{\mathbb{RT}_{\Gamma_t}}} \mathsf{H}^k_{\Gamma_t} \xrightarrow{{}_{\mathbb{T}}\mathrm{Grad}^k_{\Gamma_t}} \mathsf{H}^k_{\mathbb{T},\Gamma_t} \xrightarrow{{}_{\mathbb{S}}\mathrm{Rot}^k_{\mathbb{T},\Gamma_t}} \mathsf{H}^k_{\mathbb{S},\Gamma_t} \xrightarrow{\mathrm{divDiv}^{k,k-1}_{\mathbb{S},\Gamma_t}} \mathsf{H}^{k-1}_{\Gamma_t} \xrightarrow{\pi_{\mathbb{P}^1_{\Gamma_n}}} \mathbb{P}^1_{\Gamma_n},$$

$$\mathbb{RT}_{\Gamma_t} \xleftarrow{\pi_{\mathbb{RT}_{\Gamma_t}}} \mathsf{H}^{k-1}_{\Gamma_n} \xleftarrow{-\,\mathbf{Div}^{k-1}_{\mathbb{T},\Gamma_n}} \mathsf{H}^{k-1}_{\mathbb{T},\Gamma_n} \xleftarrow{{}_{\mathbb{T}}\mathbf{Rot}^{k-1}_{\mathbb{S},\Gamma_n}} \mathsf{H}^{k-1}_{\mathbb{S},\Gamma_n} \xleftarrow{{}_{\mathbb{S}}\mathbf{Gradgrad}^{k,k-1}_{\Gamma_n}} \mathsf{H}^k_{\Gamma_n} \xleftarrow{\iota_{\mathbb{P}^1_{\Gamma_n}}} \mathbb{P}^1_{\Gamma_n}$$

with associated domain complexes

$$\mathbb{P}^1_{\Gamma_t} \xrightarrow{\iota_{\mathbb{P}^1_{\Gamma_t}}} \mathsf{H}^{k,k-1}_{\Gamma_t}(\mathrm{Gradgrad}) \xrightarrow{{}_{\mathbb{S}}\mathrm{Gradgrad}^{k,k-1}_{\Gamma_t}} \mathsf{H}^{k-1}_{\mathbb{S},\Gamma_t}(\mathrm{Rot}) \xrightarrow{{}_{\mathbb{T}}\mathrm{Rot}^{k-1}_{\mathbb{S},\Gamma_t}} \mathsf{H}^{k-1}_{\mathbb{T},\Gamma_t}(\mathrm{Div}) \xrightarrow{\mathrm{Div}^{k-1}_{\mathbb{T},\Gamma_t}} \mathsf{H}^{k-1}_{\Gamma_t} \xrightarrow{\pi_{\mathbb{RT}_{\Gamma_n}}} \mathbb{RT}_{\Gamma_n},$$

$$\mathbb{P}^1_{\Gamma_t} \xleftarrow{\pi_{\mathbb{P}^1_{\Gamma_t}}} \mathsf{H}^{k-1}_{\Gamma_n} \xleftarrow{\mathbf{divDiv}^{k,k-1}_{\mathbb{S},\Gamma_n}} \boldsymbol{\mathsf{H}}^{k,k-1}_{\mathbb{S},\Gamma_n}(\mathrm{divDiv}) \xleftarrow{{}_{\mathbb{S}}\mathbf{Rot}^k_{\mathbb{T},\Gamma_n}} \boldsymbol{\mathsf{H}}^k_{\mathbb{T},\Gamma_n}(\mathrm{symRot}) \xleftarrow{-\,{}_{\mathbb{T}}\mathbf{Grad}^k_{\Gamma_n}} \boldsymbol{\mathsf{H}}^k_{\Gamma_n}(\mathrm{devGrad}) \xleftarrow{\iota_{\mathbb{RT}_{\Gamma_n}}} \mathbb{RT}_{\Gamma_n}$$

and

$$\mathbb{RT}_{\Gamma_t} \xrightarrow{\iota_{\mathbb{RT}_{\Gamma_t}}} \mathsf{H}^k_{\Gamma_t}(\mathrm{devGrad}) \xrightarrow{{}_{\mathbb{T}}\mathrm{Grad}^k_{\Gamma_t}} \mathsf{H}^k_{\mathbb{T},\Gamma_t}(\mathrm{symRot}) \xrightarrow{{}_{\mathbb{S}}\mathrm{Rot}^k_{\mathbb{T},\Gamma_t}} \mathsf{H}^{k,k-1}_{\mathbb{S},\Gamma_t}(\mathrm{divDiv}) \xrightarrow{\mathrm{divDiv}^{k,k-1}_{\mathbb{S},\Gamma_t}} \mathsf{H}^{k-1}_{\Gamma_t} \xrightarrow{\pi_{\mathbb{P}^1_{\Gamma_n}}} \mathbb{P}^1_{\Gamma_n},$$

$$\mathbb{RT}_{\Gamma_t} \xleftarrow{\pi_{\mathbb{RT}_{\Gamma_t}}} \mathsf{H}^{k-1}_{\Gamma_n} \xleftarrow{-\,\mathbf{Div}^{k-1}_{\mathbb{T},\Gamma_n}} \boldsymbol{\mathsf{H}}^{k-1}_{\mathbb{T},\Gamma_n}(\mathrm{Div}) \xleftarrow{{}_{\mathbb{T}}\mathbf{Rot}^{k-1}_{\mathbb{S},\Gamma_n}} \boldsymbol{\mathsf{H}}^{k-1}_{\mathbb{S},\Gamma_n}(\mathrm{Rot}) \xleftarrow{{}_{\mathbb{S}}\mathbf{Gradgrad}^{k,k-1}_{\Gamma_n}} \boldsymbol{\mathsf{H}}^k_{\Gamma_n}(\mathrm{Gradgrad}) \xleftarrow{\iota_{\mathbb{P}^1_{\Gamma_n}}} \mathbb{P}^1_{\Gamma_n}.$$

Here, we have introduced the densely defined and closed linear operators

$$\begin{aligned}
{}_{\mathbb{S}}\mathrm{Gradgrad}^k_{\Gamma_t} &: D\left({}_{\mathbb{S}}\mathrm{Gradgrad}^k_{\Gamma_t}\right) \subset \mathsf{H}^k_{\Gamma_t}(\Omega) \to \mathsf{H}^k_{\mathbb{S},\Gamma_t}(\Omega); && u \mapsto \mathrm{Gradgrad}\ u,\\
\mathbf{divDiv}^k_{\mathbb{S},\Gamma_n} &: D\left(\mathbf{divDiv}^k_{\mathbb{S},\Gamma_n}\right) \subset \mathsf{H}^k_{\mathbb{S},\Gamma_n}(\Omega) \to \mathsf{H}^k_{\Gamma_n}(\Omega); && S \mapsto \mathrm{divDiv}\ S,\\
{}_{\mathbb{T}}\mathrm{Rot}^k_{\mathbb{S},\Gamma_t}, &: D\left({}_{\mathbb{T}}\mathrm{Rot}^k_{\mathbb{S},\Gamma_t}\right) \subset \mathsf{H}^k_{\mathbb{S},\Gamma_t}(\Omega) \to \mathsf{H}^k_{\mathbb{T},\Gamma_t}(\Omega); && S \mapsto \mathrm{Rot}\ S,\\
{}_{\mathbb{S}}\mathbf{Rot}^k_{\mathbb{T},\Gamma_n} &: D\left({}_{\mathbb{S}}\mathbf{Rot}^k_{\mathbb{T},\Gamma_n}\right) \subset \mathsf{H}^k_{\mathbb{T},\Gamma_n}(\Omega) \to \mathsf{H}^k_{\mathbb{S},\Gamma_n}(\Omega); && T \mapsto \mathrm{symRot}\ T,\\
\mathrm{Div}^k_{\mathbb{T},\Gamma_t} &: D\left(\mathrm{Div}^k_{\mathbb{T},\Gamma_t}\right) \subset \mathsf{H}^k_{\mathbb{T},\Gamma_t}(\Omega) \to \mathsf{H}^k_{\Gamma_t}(\Omega); && T \mapsto \mathrm{Div}\ T,\\
{}_{\mathbb{T}}\mathbf{Grad}^k_{\Gamma_n} &: D\left({}_{\mathbb{T}}\mathbf{Grad}^k_{\Gamma_n}\right) \subset \mathsf{H}^k_{\Gamma_n}(\Omega) \to \mathsf{H}^k_{\mathbb{T},\Gamma_n}(\Omega); && v \mapsto \mathrm{devGrad}\ v,\\
{}_{\mathbb{T}}\mathrm{Grad}^k_{\Gamma_t} &: D\left({}_{\mathbb{T}}\mathrm{Grad}^k_{\Gamma_t}\right) \subset \mathsf{H}^k_{\Gamma_t}(\Omega) \to \mathsf{H}^k_{\mathbb{T},\Gamma_t}(\Omega); && v \mapsto \mathrm{devGrad}\ v,\\
\mathbf{Div}^k_{\mathbb{T},\Gamma_n} &: D\left(\mathbf{Div}^k_{\mathbb{T},\Gamma_n}\right) \subset \mathsf{H}^k_{\mathbb{T},\Gamma_n}(\Omega) \to \mathsf{H}^k_{\Gamma_n}(\Omega); && T \mapsto \mathrm{Div}\ T,\\
{}_{\mathbb{S}}\mathrm{Rot}^k_{\mathbb{T},\Gamma_t} &: D\left({}_{\mathbb{S}}\mathrm{Rot}^k_{\mathbb{T},\Gamma_t}\right) \subset \mathsf{H}^k_{\mathbb{T},\Gamma_t}(\Omega) \to \mathsf{H}^k_{\mathbb{S},\Gamma_t}(\Omega); && T \mapsto \mathrm{symRot}\ T,\\
{}_{\mathbb{T}}\mathbf{Rot}^k_{\mathbb{S},\Gamma_n} &: D\left({}_{\mathbb{T}}\mathbf{Rot}^k_{\mathbb{S},\Gamma_n}\right) \subset \mathsf{H}^k_{\mathbb{S},\Gamma_n}(\Omega) \to \mathsf{H}^k_{\mathbb{T},\Gamma_n}(\Omega); && S \mapsto \mathrm{Rot}\ S,\\
\mathrm{divDiv}^k_{\mathbb{S},\Gamma_t} &: D\left(\mathrm{divDiv}^k_{\mathbb{S},\Gamma_t}\right) \subset \mathsf{H}^k_{\mathbb{S},\Gamma_t}(\Omega \to \mathsf{H}^k_{\Gamma_t}(\Omega); && S \mapsto \mathrm{divDiv}\ S,\\
{}_{\mathbb{S}}\mathbf{Gradgrad}^k_{\Gamma_n} &: D\left({}_{\mathbb{S}}\mathbf{Gradgrad}^k_{\Gamma_n}\right) \subset \mathsf{H}^k_{\Gamma_n}(\Omega) \to \mathsf{H}^k_{\mathbb{S},\Gamma_n}(\Omega); && u \mapsto \mathrm{Gradgrad}\ u,
\end{aligned}$$

with domains of definition

$$\begin{aligned}
D\left({}_{\mathbb{S}}\mathrm{Gradgrad}^k_{\Gamma_t}\right) &= \mathsf{H}^k_{\Gamma_t}(\mathrm{Gradgrad},\Omega), & D\left(\mathbf{divDiv}^k_{\mathbb{S},\Gamma_n}\right) &= \boldsymbol{\mathsf{H}}^k_{\mathbb{S},\Gamma_n}(\mathrm{divDiv},\Omega),\\
D\left({}_{\mathbb{T}}\mathrm{Rot}^k_{\mathbb{S},\Gamma_t}\right) &= \mathsf{H}^k_{\mathbb{S},\Gamma_t}(\mathrm{Rot},\Omega), & D\left({}_{\mathbb{S}}\mathbf{Rot}^k_{\mathbb{T},\Gamma_n}\right) &= \boldsymbol{\mathsf{H}}^k_{\mathbb{T},\Gamma_n}(\mathrm{symRot},\Omega),\\
D\left(\mathrm{Div}^k_{\mathbb{T},\Gamma_t}\right) &= \mathsf{H}^k_{\mathbb{T},\Gamma_t}(\mathrm{Div},\Omega), & D\left({}_{\mathbb{T}}\mathbf{Grad}^k_{\Gamma_n}\right) &= \boldsymbol{\mathsf{H}}^k_{\Gamma_n}(\mathrm{devGrad},\Omega),\\
D\left({}_{\mathbb{T}}\mathrm{Grad}^k_{\Gamma_t}\right) &= \mathsf{H}^k_{\Gamma_t}(\mathrm{devGrad},\Omega), & D\left(\mathbf{Div}^k_{\mathbb{T},\Gamma_n}\right) &= \boldsymbol{\mathsf{H}}^k_{\mathbb{T},\Gamma_n}(\mathrm{Div},\Omega),\\
D\left({}_{\mathbb{S}}\mathrm{Rot}^k_{\mathbb{T},\Gamma_t}\right) &= \mathsf{H}^k_{\mathbb{T},\Gamma_t}(\mathrm{symRot},\Omega), & D\left({}_{\mathbb{T}}\mathbf{Rot}^k_{\mathbb{S},\Gamma_n}\right) &= \boldsymbol{\mathsf{H}}^k_{\mathbb{S},\Gamma_n}(\mathrm{Rot},\Omega),\\
D\left(\mathrm{divDiv}^k_{\mathbb{S},\Gamma_t}\right) &= \mathsf{H}^k_{\mathbb{S},\Gamma_t}(\mathrm{divDiv},\Omega), & D\left({}_{\mathbb{S}}\mathbf{Gradgrad}^k_{\Gamma_n}\right) &= \boldsymbol{\mathsf{H}}^k_{\Gamma_n}(\mathrm{Gradgrad},\Omega).
\end{aligned}$$

Moreover,

$$
\begin{aligned}
{}_{\mathbb{S}}\mathrm{Gradgrad}^{k,k-1}_{\Gamma_t} &: D\left({}_{\mathbb{S}}\mathrm{Gradgrad}^{k,k-1}_{\Gamma_t}\right) \subset \mathsf{H}^{k}_{\Gamma_t}(\Omega) \to \mathsf{H}^{k-1}_{\mathbb{S},\Gamma_t}(\Omega); \quad u \mapsto \mathrm{Gradgrad}\ u,\\
{}_{\mathbb{S}}\mathbf{Gradgrad}^{k,k-1}_{\Gamma_n} &: D\left({}_{\mathbb{S}}\mathbf{Gradgrad}^{k,k-1}_{\Gamma_n}\right) \subset \mathsf{H}^{k}_{\Gamma_n}(\Omega) \to \mathsf{H}^{k-1}_{\mathbb{S},\Gamma_n}(\Omega); \quad u \mapsto \mathrm{Gradgrad}\ u,\\
\mathrm{divDiv}^{k,k-1}_{\mathbb{S},\Gamma_t} &: D\left(\mathrm{divDiv}^{k,k-1}_{\mathbb{S},\Gamma_t}\right) \subset \mathsf{H}^{k}_{\mathbb{S},\Gamma_t}(\Omega) \to \mathsf{H}^{k-1}_{\Gamma_t}(\Omega); \quad S \mapsto \mathrm{divDiv}\ S,\\
\mathbf{divDiv}^{k,k-1}_{\mathbb{S},\Gamma_n} &: D\left(\mathbf{divDiv}^{k,k-1}_{\mathbb{S},\Gamma_n}\right) \subset \mathsf{H}^{k}_{\mathbb{S},\Gamma_n}(\Omega) \to \mathsf{H}^{k-1}_{\Gamma_n}(\Omega); \quad S \mapsto \mathrm{divDiv}\ S,
\end{aligned}
$$

with domains of definition

$$
\begin{aligned}
D\left({}_{\mathbb{S}}\mathrm{Gradgrad}^{k,k-1}_{\Gamma_t}\right) &= \mathsf{H}^{k,k-1}_{\Gamma_t}(\mathrm{Gradgrad},\Omega), \quad D\left(\mathrm{divDiv}^{k,k-1}_{\mathbb{S},\Gamma_t}\right) = \mathsf{H}^{k,k-1}_{\mathbb{S},\Gamma_t}(\mathrm{divDiv},\Omega),\\
D\left({}_{\mathbb{S}}\mathbf{Gradgrad}^{k,k-1}_{\Gamma_n}\right) &= \mathbf{H}^{k,k-1}_{\Gamma_n}(\mathrm{Gradgrad},\Omega), \quad D\left(\mathbf{divDiv}^{k,k-1}_{\mathbb{S},\Gamma_n}\right) = \mathbf{H}^{k,k-1}_{\mathbb{S},\Gamma_n}(\mathrm{divDiv},\Omega).
\end{aligned}
$$

2.6 | Dirichlet/Neumann fields

We also introduce the cohomology spaces of biharmonic Dirichlet/Neumann tensor fields (generalised harmonic tensors)

$$
\begin{aligned}
\mathcal{H}_{\mathbb{S},\Gamma_t,\Gamma_n,\varepsilon}(\Omega) &:= N({}_{\mathbb{T}}\mathrm{Rot}_{\mathbb{S},\Gamma_t}) \cap N\left(\mathrm{divDiv}_{\mathbb{S},\Gamma_n}\varepsilon\right) = \mathsf{H}_{\mathbb{S},\Gamma_t,0}(\mathrm{Rot},\Omega) \cap \varepsilon^{-1}\mathsf{H}_{\mathbb{S},\Gamma_n,0}(\mathrm{divDiv},\Omega),\\
\mathcal{H}_{\mathbb{T},\Gamma_n,\Gamma_t,\mu}(\Omega) &:= N({}_{\mathbb{S}}\mathrm{Rot}_{\mathbb{T},\Gamma_n}) \cap N(\mathrm{Div}_{\mathbb{T},\Gamma_t}\mu) = \mathsf{H}_{\mathbb{T},\Gamma_n,0}(\mathrm{symRot},\Omega) \cap \mu^{-1}\mathsf{H}_{\mathbb{T},\Gamma_t,0}(\mathrm{Div},\Omega).
\end{aligned}
$$

Here, $\varepsilon : \mathsf{L}^2_{\mathbb{S}}(\Omega) \to \mathsf{L}^2_{\mathbb{S}}(\Omega)$ is a symmetric and positive topological isomorphism (symmetric and positive bijective bounded linear operator), which introduces a new inner product

$$
\langle \cdot, \cdot \rangle_{\mathsf{L}^2_{\mathbb{S},\varepsilon}(\Omega)} := \langle \varepsilon \cdot, \cdot \rangle_{\mathsf{L}^2_{\mathbb{S}}(\Omega)},
$$

where $\mathsf{L}^2_{\mathbb{S},\varepsilon}(\Omega) := \mathsf{L}^2_{\mathbb{S}}(\Omega)$ (as linear space) equipped with the inner product $\langle \cdot, \cdot \rangle_{\mathsf{L}^2_{\mathbb{S},\varepsilon}(\Omega)}$. Such *weights* ε and also $\mu : \mathsf{L}^2_{\mathbb{T}}(\Omega) \to \mathsf{L}^2_{\mathbb{T}}(\Omega)$ are called *admissible*. Typical examples are given by symmetric, L_∞-bounded, and uniformly positive definite tensor fields $\varepsilon, \mu : \Omega \to \mathbb{R}^{(3\times3)\times(3\times3)}$ with appropriate algebraic properties.

3 | BIHARMONIC COMPLEXES II

3.1 | Regular potentials and decompositions I

3.1.1 | Extendable domains

The next theorem is a crucial result. Its proof is based on [4, Theorem 3.10], where the stated results for $\Gamma_t = \Gamma$ and $\Gamma_t = \emptyset$ have been shown, and the arguments used in, for example, [1, Lemma 4.4] for partial boundary conditions. See Appendix C for a detailed proof.

Theorem 3.1 (Regular potential operators for extendable domains). *Let (Ω, Γ_t) be an extendable bounded strong Lipschitz pair and let $k \geq 0$. Then there exist bounded linear regular potential operators*

$$
\begin{aligned}
\mathcal{P}^k_{{}_{\mathbb{S}}\mathrm{Gradgrad},\Gamma_t} &: \mathbf{H}^k_{\mathbb{S},\Gamma_t,0}(\mathrm{Rot},\Omega) \to \mathsf{H}^{k+2}_{\Gamma_t}(\Omega) \cap \mathsf{H}^{k+2}(\mathbb{R}^3),\\
\mathcal{P}^k_{{}_{\mathbb{T}}\mathrm{Rot}_{\mathbb{S}},\Gamma_t} &: \mathbf{H}^k_{\mathbb{T},\Gamma_t,0}(\mathrm{Div},\Omega) \to \mathsf{H}^{k+1}_{\mathbb{S},\Gamma_t}(\Omega) \cap \mathsf{H}^{k+1}(\mathbb{R}^3),\\
\mathcal{P}^k_{\mathrm{Div}_{\mathbb{T}},\Gamma_t} &: \mathsf{H}^k_{\Gamma_t}(\Omega) \cap (\mathbb{RT}_{\Gamma_n})^{\perp_{\mathsf{L}^2(\Omega)}} \to \mathsf{H}^{k+1}_{\mathbb{T},\Gamma_t}(\Omega) \cap \mathsf{H}^{k+1}(\mathbb{R}^3),\\
\mathcal{P}^k_{{}_{\mathbb{T}}\mathrm{Grad},\Gamma_t} &: \mathbf{H}^k_{\mathbb{T},\Gamma_t,0}(\mathrm{symRot},\Omega) \to \mathsf{H}^{k+1}_{\Gamma_t}(\Omega) \cap \mathsf{H}^{k+1}(\mathbb{R}^3),\\
\mathcal{P}^k_{{}_{\mathbb{S}}\mathrm{Rot}_{\mathbb{T}},\Gamma_t} &: \mathbf{H}^k_{\mathbb{S},\Gamma_t,0}(\mathrm{divDiv},\Omega) \to \mathsf{H}^{k+1}_{\mathbb{T},\Gamma_t}(\Omega) \cap \mathsf{H}^{k+1}(\mathbb{R}^3),\\
\mathcal{P}^k_{\mathrm{divDiv}_{\mathbb{S}},\Gamma_t} &: \mathsf{H}^k_{\Gamma_t}(\Omega) \cap \left(\mathbb{P}^1_{\Gamma_n}\right)^{\perp_{\mathsf{L}^2(\Omega)}} \to \mathsf{H}^{k+2}_{\mathbb{S},\Gamma_t}(\Omega) \cap \mathsf{H}^{k+2}(\mathbb{R}^3).
\end{aligned}
$$

In particular, $\mathcal{P}_{:::}$ are right inverses for ${}_{\mathbb{S}}\mathrm{Gradgrad}$, ${}_{\mathbb{T}}\mathrm{Rot}_{\mathbb{S}}$, $\mathrm{Div}_{\mathbb{T}}$, ${}_{\mathbb{T}}\mathrm{Grad}$, ${}_{\mathbb{S}}\mathrm{Rot}_{\mathbb{T}}$, *and* $\mathrm{divDiv}_{\mathbb{S}}$, *respectively, that is,*

$$\mathrm{Gradgrad}\ \mathcal{P}^k_{{}_{\mathbb{S}}\mathrm{Gradgrad},\Gamma_t} = \mathrm{id}_{\mathbf{H}^k_{\mathbb{S},\Gamma_t,0}(\mathrm{Rot},\Omega)}, \qquad \mathrm{devGrad}\ \mathcal{P}^k_{{}_{\mathbb{T}}\mathrm{Grad},\Gamma_t} = \mathrm{id}_{\mathbf{H}^k_{\mathbb{T},\Gamma_t,0}(\mathrm{symRot},\Omega)},$$

$$\mathrm{Rot}\ \mathcal{P}^k_{{}_{\mathbb{T}}\mathrm{Rot}_{\mathbb{S}},\Gamma_t} = \mathrm{id}_{\mathbf{H}^k_{\mathbb{T},\Gamma_t,0}(\mathrm{Div},\Omega)}, \qquad \mathrm{symRot}\ \mathcal{P}^k_{{}_{\mathbb{S}}\mathrm{Rot}_{\mathbb{T}},\Gamma_t} = \mathrm{id}_{\mathbf{H}^k_{\mathbb{S},\Gamma_t,0}(\mathrm{divDiv},\Omega)},$$

$$\mathrm{Div}\ \mathcal{P}^k_{\mathrm{Div}_{\mathbb{T}},\Gamma_t} = \mathrm{id}_{\mathsf{H}^k_{\Gamma_t}(\Omega)\cap\left(\mathbb{RT}_{\Gamma_n}\right)^{\perp_{\mathsf{L}^2(\Omega)}}}, \qquad \mathrm{divDiv}\ \mathcal{P}^k_{\mathrm{divDiv}_{\mathbb{S}},\Gamma_t} = \mathrm{id}_{\mathsf{H}^k_{\Gamma_t}(\Omega)\cap\left(\mathbb{P}^1_{\Gamma_n}\right)^{\perp_{\mathsf{L}^2(\Omega)}}}.$$

Without loss of generality, $\mathcal{P}_{:::}$ map to tensor fields with a fixed compact support in $\mathbb{R}^3$.

Remark 3.2. Note that $\mathrm{A}_n\mathcal{P}_{\mathrm{A}_n} = \mathrm{id}_{R(\mathrm{A}_n)}$ is a general property of a (bounded regular) potential operator $\mathcal{P}_{\mathrm{A}_n} : R(\mathrm{A}_n) \to \mathsf{H}^+_n$ with $\mathsf{H}^+_n \subset D(\mathrm{A}_n)$ (cf. [1, Section 2.3]).

As a simple consequence of the complex properties, the general results for regular potentials and decompositions from, for example, [1, Section 2.3] and Theorem 3.1, we obtain a few corollaries.

Corollary 3.3 (Regular potentials for extendable domains). *Let (Ω, Γ_t) be an extendable bounded strong Lipschitz pair and let $k \geq 0$. Then the regular potentials representations*

$$\begin{aligned}
\mathbf{H}^k_{\mathbb{S},\Gamma_t,0}(\mathrm{Rot},\Omega) = \mathsf{H}^k_{\mathbb{S},\Gamma_t,0}(\mathrm{Rot},\Omega) &= \mathrm{Gradgrad}\ \mathsf{H}^k_{\Gamma_t}(\mathrm{Gradgrad},\Omega) = \mathrm{Gradgrad}\ \mathsf{H}^{k+2}_{\Gamma_t}(\Omega)\\
&= \mathrm{Gradgrad}\ \mathsf{H}^{k+1,k}_{\Gamma_t}(\mathrm{Gradgrad},\Omega)\\
&= R\left({}_{\mathbb{S}}\mathrm{Gradgrad}^k_{\Gamma_t}\right) = R\left({}_{\mathbb{S}}\mathrm{Gradgrad}^{k+1,k}_{\Gamma_t}\right),\\
\mathbf{H}^k_{\mathbb{T},\Gamma_t,0}(\mathrm{Div},\Omega) = \mathsf{H}^k_{\mathbb{T},\Gamma_t,0}(\mathrm{Div},\Omega) &= \mathrm{Rot}\ \mathsf{H}^k_{\mathbb{S},\Gamma_t}(\mathrm{Rot},\Omega) = \mathrm{Rot}\ \mathsf{H}^{k+1}_{\mathbb{S},\Gamma_t}(\Omega)\\
&= R\left({}_{\mathbb{T}}\mathrm{Rot}^k_{\mathbb{S},\Gamma_t}\right),\\
\mathsf{H}^k_{\Gamma_t}(\Omega)\cap\left(\mathbb{RT}_{\Gamma_n}\right)^{\perp_{\mathsf{L}^2(\Omega)}} &= \mathrm{Div}\mathsf{H}^k_{\mathbb{T},\Gamma_t}(\mathrm{Div},\Omega) = \mathrm{Div}\ \mathsf{H}^{k+1}_{\mathbb{T},\Gamma_t}(\Omega)\\
&= R\left(\mathrm{Div}^k_{\mathbb{T},\Gamma_t}\right),\\
\mathbf{H}^k_{\mathbb{T},\Gamma_t,0}(\mathrm{symRot},\Omega) = \mathsf{H}^k_{\mathbb{T},\Gamma_t,0}(\mathrm{symRot},\Omega) &= \mathrm{devGrad}\ \mathsf{H}^k_{\Gamma_t}(\mathrm{devGrad}\ \Omega) = \mathrm{devGrad}\ \mathsf{H}^{k+1}_{\Gamma_t}(\Omega)\\
&= R\left({}_{\mathbb{T}}\mathrm{Grad}^k_{\Gamma_t}\right),\\
\mathbf{H}^k_{\mathbb{S},\Gamma_t,0}(\mathrm{divDiv},\Omega) = \mathsf{H}^k_{\mathbb{S},\Gamma_t,0}(\mathrm{divDiv},\Omega) &= \mathrm{symRot}\ \mathsf{H}^k_{\mathbb{T},\Gamma_t}(\mathrm{symRot},\Omega) = \mathrm{symRot}\ \mathsf{H}^{k+1}_{\mathbb{T},\Gamma_t}(\Omega)\\
&= R\left({}_{\mathbb{S}}\mathrm{Rot}^k_{\mathbb{T},\Gamma_t}\right),\\
\mathsf{H}^k_{\Gamma_t}(\Omega)\cap\left(\mathbb{P}^1_{\Gamma_n}\right)^{\perp_{\mathsf{L}^2(\Omega)}} &= \mathrm{divDiv}\,\mathsf{H}^k_{\mathbb{S},\Gamma_t}(\mathrm{divDiv},\Omega) = \mathrm{divDiv}\,\mathsf{H}^{k+2}_{\mathbb{S},\Gamma_t}(\Omega)\\
&= \mathrm{divDiv}\,\mathsf{H}^{k+1,k}_{\mathbb{S},\Gamma_t}(\mathrm{divDiv},\Omega)\\
&= R\left(\mathrm{divDiv}^k_{\mathbb{S},\Gamma_t}\right) = R\left(\mathrm{divDiv}^{k+1,k}_{\mathbb{S},\Gamma_t}\right)
\end{aligned}$$

hold, and the potentials can be chosen such that they depend continuously on the data. In particular, the latter spaces are closed subspaces of $\mathsf{H}^k_{\mathbb{S}}(\Omega)$, $\mathsf{H}^k_{\mathbb{T}}(\Omega)$, *and* $\mathsf{H}^k(\Omega)$, *respectively.*

Corollary 3.4. (Regular decompositions for extendable domains)**.** *Let (Ω, Γ_t) be an extendable bounded strong Lipschitz pair and let $k \geq 0$. Then the bounded regular decompositions*

$$\begin{aligned}
\mathbf{H}^k_{\mathbb{S},\Gamma_t}(\mathrm{Rot},\Omega) &= \mathsf{H}^{k+1}_{\mathbb{S},\Gamma_t}(\Omega) + \mathrm{Gradgrad}\ \mathsf{H}^{k+2}_{\Gamma_t}(\Omega) = R\left(\mathcal{P}^k_{{}_{\mathbb{T}}\mathrm{Rot}_{\mathbb{S}},\Gamma_t}\right) \dotplus \mathsf{H}^k_{\mathbb{S},\Gamma_t,0}(\mathrm{Rot},\Omega)\\
&= R\left(\mathcal{P}^k_{{}_{\mathbb{T}}\mathrm{Rot}_{\mathbb{S}},\Gamma_t}\right) \dotplus \mathrm{Gradgrad}\ \mathsf{H}^{k+2}_{\Gamma_t}(\Omega)\\
&= R\left(\mathcal{P}^k_{{}_{\mathbb{T}}\mathrm{Rot}_{\mathbb{S}},\Gamma_t}\right) \dotplus \mathrm{Gradgrad}\ R\left(\mathcal{P}^k_{{}_{\mathbb{S}}\mathrm{Gradgrad},\Gamma_t}\right),\\
\mathbf{H}^k_{\mathbb{T},\Gamma_t}(\mathrm{Div},\Omega) &= \mathsf{H}^{k+1}_{\mathbb{T},\Gamma_t}(\Omega) + \mathrm{Rot}\ \mathsf{H}^{k+1}_{\mathbb{S},\Gamma_t}(\Omega) = R\left(\mathcal{P}^k_{\mathrm{Div}_{\mathbb{T}},\Gamma_t}\right) \dotplus \mathsf{H}^k_{\mathbb{T},\Gamma_t,0}(\mathrm{Div},\Omega)\\
&= R\left(\mathcal{P}^k_{\mathrm{Div}_{\mathbb{T}},\Gamma_t}\right) \dotplus \mathrm{Rot}\ \mathsf{H}^{k+1}_{\mathbb{S},\Gamma_t}(\Omega)\\
&= R\left(\mathcal{P}^k_{\mathrm{Div}_{\mathbb{T}},\Gamma_t}\right) \dotplus \mathrm{Rot}\ R\left(\mathcal{P}^k_{{}_{\mathbb{T}}\mathrm{Rot}_{\mathbb{S}},\Gamma_t}\right),\\
\mathbf{H}^k_{\mathbb{T},\Gamma_t}(\mathrm{symRot},\Omega) &= \mathsf{H}^{k+1}_{\mathbb{T},\Gamma_t}(\Omega) + \mathrm{devGrad}\ \mathsf{H}^{k+1}_{\Gamma_t}(\Omega) = R\left(\mathcal{P}^k_{{}_{\mathbb{S}}\mathrm{Rot}_{\mathbb{T}},\Gamma_t}\right) \dotplus \mathsf{H}^k_{\mathbb{T},\Gamma_t,0}(\mathrm{symRot},\Omega)\\
&= R\left(\mathcal{P}^k_{{}_{\mathbb{S}}\mathrm{Rot}_{\mathbb{T}},\Gamma_t}\right) \dotplus \mathrm{devGrad}\ \mathsf{H}^{k+1}_{\Gamma_t}(\Omega)\\
&= R\left(\mathcal{P}^k_{{}_{\mathbb{S}}\mathrm{Rot}_{\mathbb{T}},\Gamma_t}\right) \dotplus \mathrm{devGrad}\ R\left(\mathcal{P}^k_{{}_{\mathbb{T}}\mathrm{Grad},\Gamma_t}\right),\\
\mathbf{H}^k_{\mathbb{S},\Gamma_t}(\mathrm{divDiv},\Omega) &= \mathsf{H}^{k+2}_{\mathbb{S},\Gamma_t}(\Omega) + \mathrm{symRot}\ \mathsf{H}^{k+1}_{\mathbb{T},\Gamma_t}(\Omega) = R\left(\mathcal{P}^k_{\mathrm{divDiv}_{\mathbb{S}},\Gamma_t}\right) \dotplus \mathsf{H}^k_{\mathbb{S},\Gamma_t,0}(\mathrm{divDiv},\Omega)\\
&= R\left(\mathcal{P}^k_{\mathrm{divDiv}_{\mathbb{S}},\Gamma_t}\right) \dotplus \mathrm{symRot}\ \mathsf{H}^{k+1}_{\mathbb{T},\Gamma_t}(\Omega)\\
&= R\left(\mathcal{P}^k_{\mathrm{divDiv}_{\mathbb{S}},\Gamma_t}\right) \dotplus \mathrm{symRot}\ R\left(\mathcal{P}^k_{{}_{\mathbb{S}}\mathrm{Rot}_{\mathbb{T}},\Gamma_t}\right)
\end{aligned}$$

hold with bounded linear regular decomposition operators

$$\begin{aligned}
\mathcal{Q}^{k,1}_{{}_{\mathbb{T}}\mathrm{Rot}_{\mathbb{S}},\Gamma_t} &:= \mathcal{P}^k_{{}_{\mathbb{T}}\mathrm{Rot}_{\mathbb{S}},\Gamma_t}\mathrm{Rot} : \mathbf{H}^k_{\mathbb{S},\Gamma_t}(\mathrm{Rot},\Omega) \to \mathsf{H}^{k+1}_{\mathbb{S},\Gamma_t}(\Omega),\\
\mathcal{Q}^{k,0}_{{}_{\mathbb{T}}\mathrm{Rot}_{\mathbb{S}},\Gamma_t} &:= \mathcal{P}^k_{\mathbb{S}\mathrm{Gradgrad},\Gamma_t}\left(1 - \mathcal{Q}^{k,1}_{{}_{\mathbb{T}}\mathrm{Rot}_{\mathbb{S}}\Gamma_t}\right) : \mathbf{H}^k_{\mathbb{S},\Gamma_t}(\mathrm{Rot},\Omega) \to \mathsf{H}^{k+2}_{\Gamma_t}(\Omega),\\
\mathcal{Q}^{k,1}_{\mathrm{Div}_{\mathbb{T}},\Gamma_t} &:= \mathcal{P}^k_{\mathrm{Div}_{\mathbb{T}},\Gamma_t}\mathrm{Div} : \mathbf{H}^k_{\mathbb{T},\Gamma_t}(\mathrm{Div},\Omega) \to \mathsf{H}^{k+1}_{\mathbb{T},\Gamma_t}(\Omega),\\
\mathcal{Q}^{k,0}_{\mathrm{Div}_{\mathbb{T}},\Gamma_t} &:= \mathcal{P}^k_{{}_{\mathbb{T}}\mathrm{Rot}_{\mathbb{S}},\Gamma_t}\left(1 - \mathcal{Q}^{k,1}_{\mathrm{Div}_{\mathbb{T}},\Gamma_t}\right) : \mathbf{H}^k_{\mathbb{T},\Gamma_t}(\mathrm{Div},\Omega) \to \mathsf{H}^{k+1}_{\mathbb{S},\Gamma_t}(\Omega),\\
\mathcal{Q}^{k,1}_{\mathbb{S}\mathrm{Rot}_{\mathbb{T}},\Gamma_t} &:= \mathcal{P}^k_{\mathbb{S}\mathrm{Rot}_{\mathbb{T}},\Gamma_t}\mathrm{symRot} : \mathbf{H}^k_{\mathbb{T},\Gamma_t}(\mathrm{symRot},\Omega) \to \mathsf{H}^{k+1}_{\mathbb{T},\Gamma_t}(\Omega),\\
\mathcal{Q}^{k,0}_{{}_{\mathbb{S}}\mathrm{Rot}_{\mathbb{T}},\Gamma_t} &:= \mathcal{P}^k_{{}_{\mathbb{T}}\mathrm{Grad},\Gamma_t}\left(1 - \mathcal{Q}^{k,1}_{\mathbb{S}\mathrm{Rot}_{\mathbb{T}},\Gamma_t}\right) : \mathbf{H}^k_{\mathbb{T},\Gamma_t}(\mathrm{symRot},\Omega) \to \mathsf{H}^{k+1}_{\Gamma_t}(\Omega),\\
\mathcal{Q}^{k,1}_{\mathrm{divDiv}_{\mathbb{S}},\Gamma_t} &:= \mathcal{P}^k_{\mathrm{divDiv}_{\mathbb{S}},\Gamma_t}\mathrm{divDiv} : \mathbf{H}^k_{\mathbb{S},\Gamma_t}(\mathrm{divDiv},\Omega) \to \mathsf{H}^{k+2}_{\mathbb{S},\Gamma_t}(\Omega),\\
\mathcal{Q}^{k,0}_{\mathrm{divDiv}_{\mathbb{S}},\Gamma_t} &:= \mathcal{P}^k_{{}_{\mathbb{S}}\mathrm{Rot}_{\mathbb{T}},\Gamma_t}\left(1 - \mathcal{Q}^{k,1}_{\mathrm{divDiv}_{\mathbb{S}},\Gamma_t}\right) : \mathbf{H}^k_{\mathbb{S},\Gamma_t}(\mathrm{divDiv},\Omega) \to \mathsf{H}^{k+1}_{\mathbb{T},\Gamma_t}(\Omega),
\end{aligned}$$

satisfying

$$\begin{aligned}
\mathcal{Q}^{k,1}_{{}_{\mathbb{T}}\mathrm{Rot}_{\mathbb{S}},\Gamma_t} + \mathrm{Gradgrad}\ \mathcal{Q}^{k,0}_{{}_{\mathbb{T}}\mathrm{Rot}_{\mathbb{S}},\Gamma_t} &= \mathrm{id}_{\mathbf{H}^k_{\mathbb{S},\Gamma_t}(\mathrm{Rot},\Omega)},\\
\mathcal{Q}^{k,1}_{\mathrm{Div}_{\mathbb{T}},\Gamma_t} + \mathrm{Rot}\ \mathcal{Q}^{k,0}_{\mathrm{Div}_{\mathbb{T}},\Gamma_t} &= \mathrm{id}_{\mathbf{H}^k_{\mathbb{T},\Gamma_t}(\mathrm{Div},\Omega)},\\
\mathcal{Q}^{k,1}_{{}_{\mathbb{S}}\mathrm{Rot}_{\mathbb{T}},\Gamma_t} + \mathrm{devGrad}\ \mathcal{Q}^{k,0}_{{}_{\mathbb{S}}\mathrm{Rot}_{\mathbb{T}},\Gamma_t} &= \mathrm{id}_{\mathbf{H}^k_{\mathbb{T},\Gamma_t}(\mathrm{symRot},\Omega)},\\
\mathcal{Q}^{k,1}_{\mathrm{divDiv}_{\mathbb{S}},\Gamma_t} + \mathrm{symRot}\ \mathcal{Q}^{k,0}_{\mathrm{divDiv}_{\mathbb{S}},\Gamma_t} &= \mathrm{id}_{\mathbf{H}^k_{\mathbb{S},\Gamma_t}(\mathrm{divDiv},\Omega)}.
\end{aligned}$$

Remark 3.5. Note that for (bounded linear) potential operators $\mathcal{P}_{\mathrm{A}_{n-1}}$ and $\mathcal{P}_{\mathrm{A}_n}$, the identity

$$\mathcal{Q}^1_{\mathrm{A}_n} + \mathrm{A}_{n-1}\mathcal{Q}^0_{\mathrm{A}_n} = \mathrm{id}_{D(\mathrm{A}_n)} \quad \text{with} \quad \begin{aligned} \mathcal{Q}^1_{\mathrm{A}_n} &:= \mathcal{P}_{\mathrm{A}_n}\mathrm{A}_n : D(\mathrm{A}_n) \to \mathsf{H}^+_n, \\ \mathcal{Q}^0_{\mathrm{A}_n} &:= \mathcal{P}_{\mathrm{A}_{n-1}}\left(1 - \mathcal{Q}^1_{\mathrm{A}_n}\right) : D(\mathrm{A}_n) \to \mathsf{H}^+_{n-1} \end{aligned}$$

is a general structure of a (bounded) regular decomposition. Moreover,

- **(i)** $R(\mathcal{Q}^1_{\mathrm{A}_n}) = R(\mathcal{P}_{\mathrm{A}_n})$ and $R(\mathcal{Q}^0_{\mathrm{A}_n}) = R(\mathcal{P}_{\mathrm{A}_{n-1}})$.
- **(ii)** $N(\mathrm{A}_n)$ is invariant under $\mathcal{Q}^1_{\mathrm{A}_n}$, as $\mathrm{A}_n = \mathrm{A}_n\mathcal{Q}^1_{\mathrm{A}_n}$ holds by the complex property.
- **(iii)** $\mathcal{Q}^1_{\mathrm{A}_n}$ and $\mathrm{A}_{n-1}\mathcal{Q}^0_{\mathrm{A}_n} = 1 - \mathcal{Q}^1_{\mathrm{A}_n}$ are projections.
- **(iv)** There exists $c > 0$ such that for all $x \in D(\mathrm{A}_n)$

$$\left|\mathcal{Q}^1_{\mathrm{A}_n} x\right|_{\mathsf{H}^+_n} \leq c|\mathrm{A}_n x|_{\mathsf{H}_{n+1}}.$$

- **(iv′)** In particular, $\mathcal{Q}^1_{\mathrm{A}_n}|_{N(\mathrm{A}_n)} = 0$.

Corollary 3.6. (Weak and strong partial boundary conditions coincide for extendable domains). *Let* (Ω, Γ_t) *be an extendable bounded strong Lipschitz pair and let* $k \geq 0$. *Then weak and strong boundary conditions coincide, that is,*

$$\begin{aligned}
\mathbf{H}^k_{\Gamma_t}(\mathrm{Gradgrad}, \Omega) &= \mathsf{H}^k_{\Gamma_t}(\mathrm{Gradgrad}, \Omega) = \mathsf{H}^{k+2}_{\Gamma_t}(\Omega) = \mathbf{H}^{k+2}_{\Gamma_t}(\Omega), \\
\mathbf{H}^k_{\mathbb{S},\Gamma_t}(\mathrm{Rot}, \Omega) &= \mathsf{H}^k_{\mathbb{S},\Gamma_t}(\mathrm{Rot}, \Omega), \\
\mathbf{H}^k_{\mathbb{T},\Gamma_t}(\mathrm{Div}, \Omega) &= \mathsf{H}^k_{\mathbb{T},\Gamma_t}(\mathrm{Div}, \Omega), \\
\mathbf{H}^k_{\Gamma_t}(\mathrm{devGrad}, \Omega) &= \mathsf{H}^k_{\Gamma_t}(\mathrm{devGrad}, \Omega) = \mathsf{H}^{k+1}_{\Gamma_t}(\Omega) = \mathbf{H}^{k+1}_{\Gamma_t}(\Omega), \\
\mathbf{H}^k_{\mathbb{T},\Gamma_t}(\mathrm{symRot}, \Omega) &= \mathsf{H}^k_{\mathbb{T},\Gamma_t}(\mathrm{symRot}, \Omega), \\
\mathbf{H}^k_{\mathbb{S},\Gamma_t}(\mathrm{divDiv}, \Omega) &= \mathsf{H}^k_{\mathbb{S},\Gamma_t}(\mathrm{divDiv}, \Omega).
\end{aligned}$$

Similar versions of Corollary 3.4 and Corollary 3.6 are available for the nonstandard Sobolev spaces of the form $\mathsf{H}^{k,k-1}_{\ldots}(\cdots, \Omega)$ (cf. Section 2.4). Note that

$$\mathbf{H}^{k,k-1}_{\Gamma_t}(\mathrm{Gradgrad}, \Omega) = \mathsf{H}^{k+1}_{\Gamma_t}(\Omega) \tag{7}$$

as $\mathbf{H}^{k,k-1}_{\Gamma_t}(\mathrm{Gradgrad}, \Omega) \subset \mathbf{H}^{k-1}_{\Gamma_t}(\mathrm{Gradgrad}, \Omega) = \mathsf{H}^{k+1}_{\Gamma_t}(\Omega) \subset \mathbf{H}^{k,k-1}_{\Gamma_t}(\mathrm{Gradgrad}, \Omega)$.

Corollary 3.7. (Corollary 3.4 and Corollary 3.6 for nonstandard Sobolev spaces). *Let* (Ω, Γ_t) *be an extendable bounded strong Lipschitz pair and let* $k \geq 1$. *Then the bounded regular decompositions*

$$\begin{aligned}
\mathbf{H}^{k,k-1}_{\mathbb{S},\Gamma_t}(\mathrm{divDiv}, \Omega) &= \mathsf{H}^{k+1}_{\mathbb{S},\Gamma_t}(\Omega) + \mathrm{symRot}\ \mathsf{H}^{k+1}_{\mathbb{T},\Gamma_t}(\Omega) = R\left(\mathcal{P}^{k-1}_{\mathrm{divDiv}_{\mathbb{S}},\Gamma_t}\right) \dotplus \mathsf{H}^k_{\mathbb{S},\Gamma_t,0}(\mathrm{divDiv}, \Omega) \\
&= R\left(\mathcal{P}^{k-1}_{\mathrm{divDiv}_{\mathbb{S}},\Gamma_t}\right) \dotplus \mathrm{symRot}\ \mathsf{H}^{k+1}_{\mathbb{T},\Gamma_t}(\Omega) \\
&= R\left(\mathcal{P}^{k-1}_{\mathrm{divDiv}_{\mathbb{S}},\Gamma_t}\right) \dotplus \mathrm{symRot}\ R\left(\mathcal{P}^k_{{}_{\mathbb{S}}\mathrm{Rot}_{\mathbb{T}},\Gamma_t}\right) = \mathsf{H}^{k,k-1}_{\mathbb{S},\Gamma_t}(\mathrm{divDiv}, \Omega)
\end{aligned}$$

hold with bounded linear regular decomposition operators

$$\begin{aligned}
\mathcal{Q}^{k,k-1,1}_{\mathrm{divDiv}_{\mathbb{S}},\Gamma_t} &:= \mathcal{P}^{k-1}_{\mathrm{divDiv}_{\mathbb{S}},\Gamma_t}\mathrm{divDiv} : \mathbf{H}^{k,k-1}_{\mathbb{S},\Gamma_t}(\mathrm{divDiv}, \Omega) \to \mathsf{H}^{k+1}_{\mathbb{S},\Gamma_t}(\Omega), \\
\mathcal{Q}^{k,k-1,0}_{\mathrm{divDiv}_{\mathbb{S}},\Gamma_t} &:= \mathcal{P}^k_{{}_{\mathbb{S}}\mathrm{Rot}_{\mathbb{T}},\Gamma_t}\left(1 - \mathcal{Q}^{k,k-1,1}_{\mathrm{divDiv}_{\mathbb{S}},\Gamma_t}\right) : \mathbf{H}^{k,k-1}_{\mathbb{S},\Gamma_t}(\mathrm{divDiv}, \Omega) \to \mathsf{H}^{k+1}_{\mathbb{T},\Gamma_t}(\Omega)
\end{aligned}$$

satisfying $Q^{k,k-1,1}_{\operatorname{divDiv}_{\mathbb{S}},\Gamma_t} + \operatorname{symRot} Q^{k,k-1,0}_{\operatorname{divDiv}_{\mathbb{S}},\Gamma_t} = \operatorname{id}_{\mathbf{H}^{k,k-1}_{\mathbb{S},\Gamma_t}(\operatorname{divDiv},\Omega)}$. *In particular, weak and strong boundary conditions coincide also for the nonstandard Sobolev spaces.*

Recall the Hilbert complexes and cohomology groups from Section 2.5 and Section 2.6.

Theorem 3.8. (Closed and exact Hilbert complexes for extendable domains)**.** *Let* (Ω, Γ_t) *be an extendable bounded strong Lipschitz pair and let* $k \geq 0$*. Both biharmonic domain complexes*

$$\mathbb{P}^1_{\Gamma_t} \xrightarrow{\iota_{\mathbb{P}^1_{\Gamma_t}}} \mathsf{H}^{k+2}_{\Gamma_t} \xrightarrow{{}_{\mathbb{S}}\operatorname{Gradgrad}^k_{\Gamma_t}} \mathsf{H}^k_{\mathbb{S},\Gamma_t}(\operatorname{Rot}) \xrightarrow{{}_{\mathbb{T}}\operatorname{Rot}^k_{\mathbb{S},\Gamma_t}} \mathsf{H}^k_{\mathbb{T},\Gamma_t}(\operatorname{Div}) \xrightarrow{\operatorname{Div}^k_{\mathbb{T},\Gamma_t}} \mathsf{H}^k_{\Gamma_t} \xrightarrow{\pi_{\mathbb{RT}_{\Gamma_n}}} \mathbb{RT}_{\Gamma_n},$$

$$\mathbb{P}^1_{\Gamma_t} \xleftarrow{\pi_{\mathbb{P}^1_{\Gamma_t}}} \mathsf{H}^k_{\Gamma_n} \xleftarrow{\operatorname{divDiv}^k_{\mathbb{S},\Gamma_n}} \mathsf{H}^k_{\mathbb{S},\Gamma_n}(\operatorname{divDiv}) \xleftarrow{{}_{\mathbb{S}}\operatorname{Rot}^k_{\mathbb{T},\Gamma_n}} \mathsf{H}^k_{\mathbb{T},\Gamma_n}(\operatorname{symRot}) \xleftarrow{-\,{}_{\mathbb{T}}\operatorname{Grad}^k_{\Gamma_n}} \mathsf{H}^{k+1}_{\Gamma_n} \xleftarrow{\iota_{\mathbb{RT}_{\Gamma_n}}} \mathbb{RT}_{\Gamma_n}$$

and, for $k \geq 1$*,*

$$\mathbb{P}^1_{\Gamma_t} \xrightarrow{\iota_{\mathbb{P}^1_{\Gamma_t}}} \mathsf{H}^{k+1}_{\Gamma_t} \xrightarrow{{}_{\mathbb{S}}\operatorname{Gradgrad}^{k,k-1}_{\Gamma_t}} \mathsf{H}^{k-1}_{\mathbb{S},\Gamma_t}(\operatorname{Rot}) \xrightarrow{{}_{\mathbb{T}}\operatorname{Rot}^{k-1}_{\mathbb{S},\Gamma_t}} \mathsf{H}^{k-1}_{\mathbb{T},\Gamma_t}(\operatorname{Div}) \xrightarrow{\operatorname{Div}^{k-1}_{\mathbb{T},\Gamma_t}} \mathsf{H}^{k-1}_{\Gamma_t} \xrightarrow{\pi_{\mathbb{RT}_{\Gamma_n}}} \mathbb{RT}_{\Gamma_n},$$

$$\mathbb{P}^1_{\Gamma_t} \xleftarrow{\pi_{\mathbb{P}^1_{\Gamma_t}}} \mathsf{H}^{k-1}_{\Gamma_n} \xleftarrow{\operatorname{divDiv}^{k,k-1}_{\mathbb{S},\Gamma_n}} \mathsf{H}^{k,k-1}_{\mathbb{S},\Gamma_n}(\operatorname{divDiv}) \xleftarrow{{}_{\mathbb{S}}\operatorname{Rot}^k_{\mathbb{T},\Gamma_n}} \mathsf{H}^k_{\mathbb{T},\Gamma_n}(\operatorname{symRot}) \xleftarrow{-\,{}_{\mathbb{T}}\operatorname{Grad}^k_{\Gamma_n}} \mathsf{H}^{k+1}_{\Gamma_n} \xleftarrow{\iota_{\mathbb{RT}_{\Gamma_n}}} \mathbb{RT}_{\Gamma_n}$$

are exact and closed Hilbert complexes. In particular, all ranges are closed, all cohomology groups (Dirichlet/Neumann fields) are trivial, and the operators from Theorem 3.1 are associated bounded regular potential operators.

3.1.2 | General strong Lipschitz domains

From now on, we drop the additional condition "extendable domain," thus (Ω, Γ_t) is a bounded strong Lipschitz pair.

Lemma 3.9. (cutting lemma)**.** *Let* $\varphi \in C^\infty(\mathbb{R}^3)$ *and let* $k \geq 0$*.*

(i) *If* $S \in \mathbf{H}^k_{\mathbb{S},\Gamma_t}(\operatorname{Rot},\Omega)$*, then* $\varphi S \in \mathbf{H}^k_{\mathbb{S},\Gamma_t}(\operatorname{Rot},\Omega)$ *and* $\operatorname{Rot}(\varphi S) = \varphi \operatorname{Rot}\ S - S \operatorname{spn} \operatorname{grad}\varphi$.
(ii) *If* $T \in \mathbf{H}^k_{\mathbb{T},\Gamma_t}(\operatorname{Div},\Omega)$*, then* $\varphi T \in \mathbf{H}^k_{\mathbb{T},\Gamma_t}(\operatorname{Div},\Omega)$ *and* $\operatorname{Div}\ (\varphi T) = \varphi \operatorname{Div}\ T + T \operatorname{grad}\varphi$.
(iii) *If* $T \in \mathbf{H}^k_{\mathbb{T},\Gamma_t}(\operatorname{symRot},\Omega)$*, then* $\varphi T \in \mathbf{H}^k_{\mathbb{T},\Gamma_t}(\operatorname{symRot},\Omega)$ *and* $\operatorname{symRot}(\varphi T) = \varphi \operatorname{symRot}\ T - \operatorname{sym}(T \operatorname{spn} \operatorname{grad}\varphi)$.
(iv) *If* $k \geq 1$ *and* $S \in \mathbf{H}^{k,k-1}_{\mathbb{S},\Gamma_t}(\operatorname{divDiv},\Omega)$*, then* $\varphi S \in \mathbf{H}^{k,k-1}_{\mathbb{S},\Gamma_t}(\operatorname{divDiv},\Omega)$ *and*

$$\operatorname{divDiv}(\varphi S) = \varphi \operatorname{divDiv}\ S + 2 \operatorname{grad}\varphi \cdot \operatorname{Div}\ S + \operatorname{Gradgrad}\varphi : S.$$

In particular, this holds for $S \in \mathbf{H}^k_{\mathbb{S},\Gamma_t}(\operatorname{divDiv},\Omega)$*. Note that* $\cdot$ *and* $:$ *denote the point-wise scalar product for vectors fields and tensor (matrix) fields, respectively.*

We proceed by showing crucial regular decompositions for the biharmonic complexes extending the results of Corollary 3.4 and Corollary 3.7 to our general setting. The proof is based on Corollary 3.4 together with a partition of unity.

Lemma 3.10. (Regular decompositions)**.** *Let* $k \geq 0$*. Then the bounded regular decompositions*

$$\begin{aligned}
\mathbf{H}^k_{\mathbb{S},\Gamma_t}(\operatorname{Rot},\Omega) &= \mathsf{H}^{k+1}_{\mathbb{S},\Gamma_t}(\Omega) + \operatorname{Gradgrad}\ \mathsf{H}^{k+2}_{\Gamma_t}(\Omega),\\
\mathbf{H}^k_{\mathbb{T},\Gamma_t}(\operatorname{Div},\Omega) &= \mathsf{H}^{k+1}_{\mathbb{T},\Gamma_t}(\Omega) + \operatorname{Rot}\ \mathsf{H}^{k+1}_{\mathbb{S},\Gamma_t}(\Omega),\\
\mathbf{H}^k_{\mathbb{T},\Gamma_t}(\operatorname{symRot},\Omega) &= \mathsf{H}^{k+1}_{\mathbb{T},\Gamma_t}(\Omega) + \operatorname{devGrad}\ \mathsf{H}^{k+1}_{\Gamma_t}(\Omega),\\
\mathbf{H}^k_{\mathbb{S},\Gamma_t}(\operatorname{divDiv},\Omega) &= \mathsf{H}^{k+2}_{\mathbb{S},\Gamma_t}(\Omega) + \operatorname{symRot}\ \mathsf{H}^{k+1}_{\mathbb{T},\Gamma_t}(\Omega)
\end{aligned}$$

and, for $k \geq 1$, the nonstandard bounded regular decompositions

$$\mathbf{H}^{k}_{\mathbb{S},\Gamma_t}(\operatorname{divDiv},\Omega) \subset \mathbf{H}^{k,k-1}_{\mathbb{S},\Gamma_t}(\operatorname{divDiv},\Omega) = \mathsf{H}^{k+1}_{\mathbb{S},\Gamma_t}(\Omega) + \operatorname{symRot}\,\mathsf{H}^{k+1}_{\mathbb{T},\Gamma_t}(\Omega)$$

hold with bounded linear regular decomposition operators

$$\begin{aligned}
\mathcal{Q}^{k,1}_{{}_{\mathbb{T}}\operatorname{Rot}_{\mathbb{S}},\Gamma_t} &: \mathbf{H}^{k}_{\mathbb{S},\Gamma_t}(\operatorname{Rot},\Omega) \to \mathsf{H}^{k+1}_{\mathbb{S},\Gamma_t}(\Omega), & \mathcal{Q}^{k,0}_{{}_{\mathbb{T}}\operatorname{Rot}_{\mathbb{S}},\Gamma_t} &: \mathbf{H}^{k}_{\mathbb{S},\Gamma_t}(\operatorname{Rot},\Omega) \to \mathsf{H}^{k+2}_{\Gamma_t}(\Omega),\\
\mathcal{Q}^{k,1}_{\operatorname{Div}_{\mathbb{T}},\Gamma_t} &: \mathbf{H}^{k}_{\mathbb{T},\Gamma_t}(\operatorname{Div},\Omega) \to \mathsf{H}^{k+1}_{\mathbb{T},\Gamma_t}(\Omega), & \mathcal{Q}^{k,0}_{\operatorname{Div}_{\mathbb{T}},\Gamma_t} &: \mathbf{H}^{k}_{\mathbb{T},\Gamma_t}(\operatorname{Div},\Omega) \to \mathsf{H}^{k+1}_{\mathbb{S},\Gamma_t}(\Omega),\\
\mathcal{Q}^{k,1}_{{}_{\mathbb{S}}\operatorname{Rot}_{\mathbb{T}},\Gamma_t} &: \mathbf{H}^{k}_{\mathbb{T},\Gamma_t}(\operatorname{symRot},\Omega) \to \mathsf{H}^{k+1}_{\mathbb{T},\Gamma_t}(\Omega), & \mathcal{Q}^{k,0}_{{}_{\mathbb{S}}\operatorname{Rot}_{\mathbb{T}},\Gamma_t} &: \mathbf{H}^{k}_{\mathbb{T},\Gamma_t}(\operatorname{symRot},\Omega) \to \mathsf{H}^{k+1}_{\Gamma_t}(\Omega),\\
\mathcal{Q}^{k,1}_{\operatorname{divDiv}_{\mathbb{S}},\Gamma_t} &: \mathbf{H}^{k}_{\mathbb{S},\Gamma_t}(\operatorname{divDiv},\Omega) \to \mathsf{H}^{k+2}_{\mathbb{S},\Gamma_t}(\Omega), & \mathcal{Q}^{k,0}_{\operatorname{divDiv}_{\mathbb{S}},\Gamma_t} &: \mathbf{H}^{k}_{\mathbb{S},\Gamma_t}(\operatorname{divDiv},\Omega) \to \mathsf{H}^{k+1}_{\mathbb{T},\Gamma_t}(\Omega),\\
\mathcal{Q}^{k,k-1,1}_{\operatorname{divDiv}_{\mathbb{S}},\Gamma_t} &: \mathbf{H}^{k,k-1}_{\mathbb{S},\Gamma_t}(\operatorname{divDiv},\Omega) \to \mathsf{H}^{k+1}_{\mathbb{S},\Gamma_t}(\Omega), & \mathcal{Q}^{k,k-1,0}_{\operatorname{divDiv}_{\mathbb{S}},\Gamma_t} &: \mathbf{H}^{k,k-1}_{\mathbb{S},\Gamma_t}(\operatorname{divDiv},\Omega) \to \mathsf{H}^{k+1}_{\mathbb{T},\Gamma_t}(\Omega)
\end{aligned}$$

satisfying

$$\begin{aligned}
\mathcal{Q}^{k,1}_{{}_{\mathbb{T}}\operatorname{Rot}_{\mathbb{S}},\Gamma_t} + \operatorname{Gradgrad}\,\mathcal{Q}^{k,0}_{{}_{\mathbb{T}}\operatorname{Rot}_{\mathbb{S}},\Gamma_t} &= \mathrm{id}_{\mathbf{H}^{k}_{\mathbb{S},\Gamma_t}(\operatorname{Rot},\Omega)},\\
\mathcal{Q}^{k,1}_{\operatorname{Div}_{\mathbb{T}},\Gamma_t} + \operatorname{Rot}\,\mathcal{Q}^{k,0}_{\operatorname{Div}_{\mathbb{T}},\Gamma_t} &= \mathrm{id}_{\mathbf{H}^{k}_{\mathbb{T},\Gamma_t}(\operatorname{Div},\Omega)},\\
\mathcal{Q}^{k,1}_{{}_{\mathbb{S}}\operatorname{Rot}_{\mathbb{T}},\Gamma_t} + \operatorname{devGrad}\,\mathcal{Q}^{k,0}_{{}_{\mathbb{S}}\operatorname{Rot}_{\mathbb{T}},\Gamma_t} &= \mathrm{id}_{\mathbf{H}^{k}_{\mathbb{T},\Gamma_t}(\operatorname{symRot},\Omega)},\\
\mathcal{Q}^{k,1}_{\operatorname{divDiv}_{\mathbb{S}},\Gamma_t} + \operatorname{symRot}\,\mathcal{Q}^{k,0}_{\operatorname{divDiv}_{\mathbb{S}},\Gamma_t} &= \mathrm{id}_{\mathbf{H}^{k}_{\mathbb{S},\Gamma_t}(\operatorname{divDiv},\Omega)},\\
\mathcal{Q}^{k,k-1,1}_{\operatorname{divDiv}_{\mathbb{S}},\Gamma_t} + \operatorname{symRot}\,\mathcal{Q}^{k,k-1,0}_{\operatorname{divDiv}_{\mathbb{S}},\Gamma_t} &= \mathrm{id}_{\mathbf{H}^{k,k-1}_{\mathbb{S},\Gamma_t}(\operatorname{divDiv},\Omega)}, \qquad k \geq 1.
\end{aligned}$$

It holds $\operatorname{Rot}\,\mathcal{Q}^{k,1}_{{}_{\mathbb{T}}\operatorname{Rot}_{\mathbb{S}},\Gamma_t} = {}_{\mathbb{T}}\mathbf{Rot}^{k}_{\mathbb{S},\Gamma_t}$*,* $\operatorname{Div}\mathcal{Q}^{k,1}_{\operatorname{Div}_{\mathbb{T}},\Gamma_t} = \mathbf{Div}^{k}_{\mathbb{T},\Gamma_t}$*, and* $\operatorname{symRot}\,\mathcal{Q}^{k,1}_{{}_{\mathbb{S}}\operatorname{Rot}_{\mathbb{T}},\Gamma_t} = {}_{\mathbb{S}}\mathbf{Rot}^{k}_{\mathbb{T},\Gamma_t}$ *and thus* $\mathbf{H}^{k}_{\mathbb{S},\Gamma_t,0}(\operatorname{Rot},\Omega)$*,* $\mathbf{H}^{k}_{\mathbb{T},\Gamma_t,0}(\operatorname{Div},\Omega)$*, and* $\mathbf{H}^{k}_{\mathbb{T},\Gamma_t,0}(\operatorname{symRot},\Omega)$ *are invariant under* $\mathcal{Q}^{k,1}_{{}_{\mathbb{T}}\operatorname{Rot}_{\mathbb{S}},\Gamma_t}$*,* $\mathcal{Q}^{k,1}_{\operatorname{Div}_{\mathbb{T}},\Gamma_t}$*, and* $\mathcal{Q}^{k,1}_{{}_{\mathbb{S}}\operatorname{Rot}_{\mathbb{T}},\Gamma_t}$*, respectively. Analogously, we have* $\operatorname{divDiv}\mathcal{Q}^{k,1}_{\operatorname{divDiv}_{\mathbb{S}},\Gamma_t} = \mathbf{divDiv}^{k}_{\mathbb{S},\Gamma_t}$ *and* $\operatorname{divDiv}\mathcal{Q}^{k,k-1,1}_{\operatorname{divDiv}_{\mathbb{S}},\Gamma_t} = \mathbf{divDiv}^{k,k-1}_{\mathbb{S},\Gamma_t}$ *and thus* $\mathbf{H}^{k}_{\mathbb{S},\Gamma_t,0}(\operatorname{divDiv},\Omega)$ *is invariant under* $\mathcal{Q}^{k,1}_{\operatorname{divDiv}_{\mathbb{S}},\Gamma_t}$ *and* $\mathcal{Q}^{k,k-1,1}_{\operatorname{divDiv}_{\mathbb{S}},\Gamma_t}$*, respectively.*

Corollary 3.6 and (7) are generalised to the following important result.

Corollary 3.11. (Weak and strong partial boundary conditions coincide)**.** *Let $k \geq 0$. Weak and strong boundary conditions coincide, that is,*

$$\begin{aligned}
\mathbf{H}^{k}_{\Gamma_t}(\operatorname{Gradgrad},\Omega) &= \mathsf{H}^{k}_{\Gamma_t}(\operatorname{Gradgrad},\Omega) = \mathsf{H}^{k+2}_{\Gamma_t}(\Omega) = \mathbf{H}^{k+2}_{\Gamma_t}(\Omega),\\
\mathbf{H}^{k,k-1}_{\Gamma_t}(\operatorname{Gradgrad},\Omega) &= \mathsf{H}^{k,k-1}_{\Gamma_t}(\operatorname{Gradgrad},\Omega) = \mathsf{H}^{k+1}_{\Gamma_t}(\Omega) = \mathbf{H}^{k+1}_{\Gamma_t}(\Omega), \qquad k \geq 1,\\
\mathbf{H}^{k}_{\mathbb{S},\Gamma_t}(\operatorname{Rot},\Omega) &= \mathsf{H}^{k}_{\mathbb{S},\Gamma_t}(\operatorname{Rot},\Omega),\\
\mathbf{H}^{k}_{\mathbb{T},\Gamma_t}(\operatorname{Div},\Omega) &= \mathsf{H}^{k}_{\mathbb{T},\Gamma_t}(\operatorname{Div},\Omega),\\
\mathbf{H}^{k}_{\Gamma_t}(\operatorname{devGrad},\Omega) &= \mathsf{H}^{k}_{\Gamma_t}(\operatorname{devGrad},\Omega) = \mathsf{H}^{k+1}_{\Gamma_t}(\Omega) = \mathbf{H}^{k+1}_{\Gamma_t}(\Omega),\\
\mathbf{H}^{k}_{\mathbb{T},\Gamma_t}(\operatorname{symRot},\Omega) &= \mathsf{H}^{k}_{\mathbb{T},\Gamma_t}(\operatorname{symRot},\Omega),\\
\mathbf{H}^{k}_{\mathbb{S},\Gamma_t}(\operatorname{divDiv},\Omega) &= \mathsf{H}^{k}_{\mathbb{S},\Gamma_t}(\operatorname{divDiv},\Omega),\\
\mathbf{H}^{k,k-1}_{\mathbb{S},\Gamma_t}(\operatorname{divDiv},\Omega) &= \mathsf{H}^{k,k-1}_{\mathbb{S},\Gamma_t}(\operatorname{divDiv},\Omega), \qquad k \geq 1.
\end{aligned}$$

In particular, we have ${}_{\mathbb{S}}\mathbf{Gradgrad}^{k}_{\Gamma_t} = {}_{\mathbb{S}}\operatorname{Gradgrad}^{k}_{\Gamma_t}$, ${}_{\mathbb{T}}\mathbf{Rot}^{k}_{\mathbb{S},\Gamma_t} = {}_{\mathbb{T}}\operatorname{Rot}^{k}_{\mathbb{S},\Gamma_t}$, $\mathbf{Div}^{k}_{\mathbb{T},\Gamma_t} = \operatorname{Div}^{k}_{\mathbb{T},\Gamma_t}$, ${}_{\mathbb{T}}\mathbf{Grad}^{k}_{\Gamma_t} = {}_{\mathbb{T}}\operatorname{Grad}^{k}_{\Gamma_t}$, ${}_{\mathbb{S}}\mathbf{Rot}^{k}_{\mathbb{T},\Gamma_t} = {}_{\mathbb{S}}\operatorname{Rot}^{k}_{\mathbb{T},\Gamma_t}$, $\mathbf{divDiv}^{k}_{\mathbb{S},\Gamma_t} = \operatorname{divDiv}^{k}_{\mathbb{S},\Gamma_t}$*, as well as, for $k \geq 1$,* ${}_{\mathbb{S}}\mathbf{Gradgrad}^{k,k-1}_{\Gamma_t} = {}_{\mathbb{S}}\operatorname{Gradgrad}^{k,k-1}_{\Gamma_t}$ *and* $\mathbf{divDiv}^{k,k-1}_{\mathbb{S},\Gamma_t} = \operatorname{divDiv}^{k,k-1}_{\mathbb{S},\Gamma_t}$.

For a detailed proof of Lemma 3.10 and Corollary 3.11, see Appendix C.

3.2 | Mini FA-ToolBox

3.2.1 | Zero order mini FA-ToolBox

Recall Section 2.6 and let ε, μ be admissible. In Section 2.1 (for $\varepsilon = \mu = \mathrm{id}$), we have seen that the densely defined and closed linear operators

$$
\begin{aligned}
\mathrm{A}_{-1} = \iota_{\mathbb{P}^1_{\Gamma_t}} &: \mathbb{P}^1_{\Gamma_t} \to \mathsf{L}^2(\Omega); && p \mapsto p,\\
\mathrm{A}_0 = {}_{\mathbb{S}}\mathrm{Gradgrad}_{\Gamma_t} &: \mathsf{H}^2_{\Gamma_t}(\Omega) \subset \mathsf{L}^2(\Omega) \to \mathsf{L}^2_{\mathbb{S},\varepsilon}(\Omega); && u \mapsto \mathrm{Gradgrad}\ u,\\
\mathrm{A}_1 = \mu^{-1}{}_{\mathbb{T}}\mathrm{Rot}_{\mathbb{S},\Gamma_t} &: \mathsf{H}_{\mathbb{S},\Gamma_t}(\mathrm{Rot},\Omega) \subset \mathsf{L}^2_{\mathbb{S},\varepsilon}(\Omega) \to \mathsf{L}^2_{\mathbb{T},\mu}(\Omega); && S \mapsto \mu^{-1}\mathrm{Rot}\ S,\\
\mathrm{A}_2 = \mathrm{Div}_{\mathbb{T},\Gamma_t}\mu &: \mu^{-1}\mathsf{H}_{\mathbb{T},\Gamma_t}(\mathrm{Div},\Omega) \subset \mathsf{L}^2_{\mathbb{T},\mu}(\Omega) \to \mathsf{L}^2(\Omega); && T \mapsto \mathrm{Div}\ \mu T,\\
\mathrm{A}_3 = \iota^*_{\mathbb{RT}_{\Gamma_n}} &: \mathsf{L}^2(\Omega) \to \mathbb{RT}_{\Gamma_n}; && q \mapsto \pi_{\mathbb{RT}_{\Gamma_n}} q,\\
\mathrm{A}^*_{-1} = \iota^*_{\mathbb{P}^1_{\Gamma_t}} &: \mathsf{L}^2(\Omega) \to \mathbb{P}^1_{\Gamma_t}; && p \mapsto \pi_{\mathbb{P}^1_{\Gamma_t}} p,\\
\mathrm{A}^*_0 = \mathrm{divDiv}_{\mathbb{S},\Gamma_n}\varepsilon &: \varepsilon^{-1}\mathsf{H}_{\mathbb{S},\Gamma_n}(\mathrm{divDiv},\Omega) \subset \mathsf{L}^2_{\mathbb{S},\varepsilon}(\Omega) \to \mathsf{L}^2(\Omega); && S \mapsto \mathrm{divDiv}\ \varepsilon S,\\
\mathrm{A}^*_1 = \varepsilon^{-1}{}_{\mathbb{S}}\mathrm{Rot}_{\mathbb{T},\Gamma_n} &: \mathsf{H}_{\mathbb{T},\Gamma_n}(\mathrm{symRot},\Omega) \subset \mathsf{L}^2_{\mathbb{T},\mu}(\Omega) \to \mathsf{L}^2_{\mathbb{S},\varepsilon}(\Omega); && T \mapsto \varepsilon^{-1}\mathrm{symRot}\ T,\\
\mathrm{A}^*_2 = -{}_{\mathbb{T}}\mathrm{Grad}_{\Gamma_n} &: \mathsf{H}^1_{\Gamma_n}(\Omega) \subset \mathsf{L}^2(\Omega) \to \mathsf{L}^2_{\mathbb{T},\mu}(\Omega); && v \mapsto -\mathrm{devGrad}\ v,\\
\mathrm{A}^*_3 = \iota_{\mathbb{RT}_{\Gamma_n}} &: \mathbb{RT}_{\Gamma_n} \to \mathsf{L}^2(\Omega); && q \mapsto q,
\end{aligned}
$$

where we have used Corollary 3.11, build the long primal and dual elasticity Hilbert complex

$$
\mathbb{P}^1_{\Gamma_t}
\underset{\mathrm{A}^*_{-1}\cong\pi_{\mathbb{P}^1_{\Gamma_t}}}{\overset{\mathrm{A}_{-1}=\iota_{\mathbb{P}^1_{\Gamma_t}}}{\rightleftarrows}}
\mathsf{L}^2(\Omega)
\underset{\mathrm{A}^*_0=\mathrm{divDiv}_{\mathbb{S},\Gamma_n}\varepsilon}{\overset{\mathrm{A}_0={}_{\mathbb{S}}\mathrm{Gradgrad}_{\Gamma_t}}{\rightleftarrows}}
\mathsf{L}^2_{\mathbb{S},\varepsilon}(\Omega)
\underset{\mathrm{A}^*_1=\varepsilon^{-1}\,{}_{\mathbb{S}}\mathrm{Rot}_{\mathbb{T},\Gamma_n}}{\overset{\mathrm{A}_1=\mu^{-1}\,{}_{\mathbb{T}}\mathrm{Rot}_{\mathbb{S},\Gamma_t}}{\rightleftarrows}}
\mathsf{L}^2_{\mathbb{T},\mu}(\Omega)
\underset{\mathrm{A}^*_2=-\,{}_{\mathbb{T}}\mathrm{Grad}_{\Gamma_n}}{\overset{\mathrm{A}_2=\mathrm{Div}_{\mathbb{T},\Gamma_t}\,\mu}{\rightleftarrows}}
\mathsf{L}^2(\Omega)
\underset{\mathrm{A}^*_3=\iota_{\mathbb{RT}_{\Gamma_n}}}{\overset{\mathrm{A}_3\cong\pi_{\mathbb{RT}_{\Gamma_n}}}{\rightleftarrows}}
\mathbb{RT}_{\Gamma_n}, \tag{8}
$$

and compare (5). Note that

$$
\begin{aligned}
\iota_{\mathbb{P}^1_{\Gamma_t}}\mathrm{A}^*_{-1} = \iota_{\mathbb{P}^1_{\Gamma_t}}\iota^*_{\mathbb{P}^1_{\Gamma_t}} = \pi_{\mathbb{P}^1_{\Gamma_t}} &: \mathsf{L}^2(\Omega) \to \mathsf{L}^2(\Omega),\\
\iota_{\mathbb{RT}_{\Gamma_n}}\mathrm{A}_3 = \iota_{\mathbb{RT}_{\Gamma_n}}\iota^*_{\mathbb{RT}_{\Gamma_n}} = \pi_{\mathbb{RT}_{\Gamma_n}} &: \mathsf{L}^2(\Omega) \to \mathsf{L}^2(\Omega)
\end{aligned}
$$

are the actual projectors onto $\mathbb{P}^1_{\Gamma_t}$ and $\mathbb{RT}_{\Gamma_n}$, respectively.

Theorem 3.12. (Compact embeddings). *The embeddings*

$$
\begin{aligned}
D(\mathrm{A}_1) \cap D(\mathrm{A}^*_0) &= \mathsf{H}_{\mathbb{S},\Gamma_t}(\mathrm{Rot},\Omega) \cap \varepsilon^{-1}\mathsf{H}_{\mathbb{S},\Gamma_n}(\mathrm{divDiv},\Omega) \hookrightarrow \mathsf{L}^2_{\mathbb{S},\varepsilon}(\Omega),\\
D(\mathrm{A}_2) \cap D(\mathrm{A}^*_1) &= \mu^{-1}\mathsf{H}_{\mathbb{T},\Gamma_t}(\mathrm{Div},\Omega) \cap \mathsf{H}_{\mathbb{T},\Gamma_n}(\mathrm{symRot},\Omega) \hookrightarrow \mathsf{L}^2_{\mathbb{T},\mu}(\Omega)
\end{aligned}
$$

are compact. Moreover, the compactness does not depend on ε or μ.

See Appendix C for a proof.

Remark 3.13. (Compact embeddings). The embeddings

$$
D(\mathrm{A}_0) \cap D(\mathrm{A}^*_{-1}) = D(\mathrm{A}_0) = \mathsf{H}^2_{\Gamma_t}(\Omega) \hookrightarrow \mathsf{L}^2(\Omega), \quad D(\mathrm{A}_3) \cap D(\mathrm{A}^*_2) = D(\mathrm{A}^*_2) = \mathsf{H}^1_{\Gamma_n}(\Omega) \hookrightarrow \mathsf{L}^2(\Omega)
$$

are compact by Rellich's selection theorem.

Theorem 3.14. (Compact biharmonic complex). *The long primal and dual biharmonic Hilbert complex (8) is compact. In particular, the complex is closed.*

Let us recall for the densely defined and closed linear operators

$$\mathrm{A}_n : D(\mathrm{A}_n) \subset \mathsf{H}_n \to \mathsf{H}_{n+1}, \qquad \mathrm{A}_n^* : D(\mathrm{A}_n^*) \subset \mathsf{H}_{n+1} \to \mathsf{H}_n$$

the corresponding reduced operators

$$\begin{aligned}(\mathrm{A}_n)_\perp &:= \mathrm{A}_n\Big|_{N(\mathrm{A}_n)^{\perp_{\mathsf{H}_n}}} : D((\mathrm{A}_n)_\perp) = D(\mathrm{A}_n) \cap N(\mathrm{A}_n)^{\perp_{\mathsf{H}_n}} \subset N(\mathrm{A}_n)^{\perp_{\mathsf{H}_n}} \to R(\mathrm{A}_n),\\ (\mathrm{A}_n^*)_\perp &:= \mathrm{A}_n^*|_{N(\mathrm{A}_n^*)^{\perp_{\mathsf{H}_{n+1}}}} : D((\mathrm{A}_n^*)_\perp) = D(\mathrm{A}_n^*) \cap N(\mathrm{A}_n^*)^{\perp_{\mathsf{H}_{n+1}}} \subset N(\mathrm{A}_n^*)^{\perp_{\mathsf{H}_{n+1}}} \to R(\mathrm{A}_n^*).\end{aligned}$$

Note that $R(\mathrm{A}_n) = R((\mathrm{A}_n)_\perp) = N(\mathrm{A}_n^*)^{\perp_{\mathsf{H}_{n+1}}}$ and $R(\mathrm{A}_n^*) = R\left((\mathrm{A}_n^*)_\perp\right) = N(\mathrm{A}_n)^{\perp_{\mathsf{H}_n}}$. Here, we consider

$$\begin{aligned}(\mathrm{A}_0)_\perp &= \left({}_{\mathbb{S}}\mathrm{Gradgrad}_{\Gamma_t}\right)_\perp, & (\mathrm{A}_1)_\perp &= \left(\mu^{-1}{}_{\mathbb{T}}\mathrm{Rot}_{\mathbb{S},\Gamma_t}\right)_\perp, & (\mathrm{A}_2)_\perp &= (\mathrm{Div}_{\mathbb{T},\Gamma_t}\mu)_\perp,\\ (\mathrm{A}_0^*)_\perp &= \left(\mathrm{divDiv}_{\mathbb{S},\Gamma_n}\varepsilon\right)_\perp, & (\mathrm{A}_1^*)_\perp &= \left(\varepsilon^{-1}{}_{\mathbb{S}}\mathrm{Rot}_{\mathbb{T},\Gamma_n}\right)_\perp, & (\mathrm{A}_2^*)_\perp &= -\left({}_{\mathbb{T}}\mathrm{Grad}_{\Gamma_n}\right)_\perp,\end{aligned}$$

and

$$\begin{aligned}(\mathrm{A}_{-1})_\perp &= \left(\iota_{\mathbb{P}^1_{\Gamma_t}}\right)_\perp = \mathrm{id}_{\mathbb{P}^1_{\Gamma_t}} : \mathbb{P}^1_{\Gamma_t} \to \mathbb{P}^1_{\Gamma_t},\\ (\mathrm{A}_{-1}^*)_\perp &\cong \left(\pi_{\mathbb{P}^1_{\Gamma_t}}\right)_\perp = \pi_{\mathbb{P}^1_{\Gamma_t}}\Big|_{\mathbb{P}^1_{\Gamma_t}} = \mathrm{id}_{\mathbb{P}^1_{\Gamma_t}} : \mathbb{P}^1_{\Gamma_t} \to \mathbb{P}^1_{\Gamma_t},\\ (\mathrm{A}_3)_\perp &\cong \left(\pi_{\mathbb{RT}_{\Gamma_n}}\right)_\perp = \pi_{\mathbb{RT}_{\Gamma_n}}\Big|_{\mathbb{RT}_{\Gamma_n}} = \mathrm{id}_{\mathbb{RT}_{\Gamma_n}} : \mathbb{RT}_{\Gamma_n} \to \mathbb{RT}_{\Gamma_n},\\ (\mathrm{A}_3^*)_\perp &= \left(\iota_{\mathbb{RT}_{\Gamma_n}}\right)_\perp = \mathrm{id}_{\mathbb{RT}_{\Gamma_n}} : \mathbb{RT}_{\Gamma_n} \to \mathbb{RT}_{\Gamma_n}.\end{aligned}$$

Lemma 2.9 of [1] shows:

Theorem 3.15. (Mini FA-ToolBox). *For the zero order biharmonic complex, it holds*

(i) *The ranges* $R\left({}_{\mathbb{S}}\mathrm{Gradgrad}_{\Gamma_t}\right)$, $R\left(\mu^{-1}{}_{\mathbb{T}}\mathrm{Rot}_{\mathbb{S},\Gamma_t}\right)$, *and* $R(\mathrm{Div}_{\mathbb{T},\Gamma_t}\mu)$ *are closed.*
(i') *The ranges* $R\left(\mathrm{divDiv}_{\mathbb{S},\Gamma_n}\varepsilon\right)$, $R\left(\varepsilon^{-1}{}_{\mathbb{S}}\mathrm{Rot}_{\mathbb{T},\Gamma_n}\right)$, *and* $R\left({}_{\mathbb{T}}\mathrm{Grad}_{\Gamma_n}\right)$ *are closed.*
(ii) *The inverse operators* $\left({}_{\mathbb{S}}\mathrm{Gradgrad}_{\Gamma_t}\right)_\perp^{-1}$, $\left(\mu^{-1}{}_{\mathbb{T}}\mathrm{Rot}_{\mathbb{S},\Gamma_t}\right)_\perp^{-1}$, *and* $(\mathrm{Div}_{\mathbb{T},\Gamma_t}\mu)_\perp^{-1}$ *are compact.*
(ii) *The inverse operators* $\left(\mathrm{divDiv}_{\mathbb{S},\Gamma_n}\varepsilon\right)_\perp^{-1}$, $\left(\varepsilon^{-1}{}_{\mathbb{S}}\mathrm{Rot}_{\mathbb{T},\Gamma_n}\right)_\perp^{-1}$, *and* $\left({}_{\mathbb{T}}\mathrm{Grad}_{\Gamma_n}\right)_\perp^{-1}$ *are compact.*
(iii) *The cohomology groups of generalised Dirichlet/Neumann tensor fields* $\mathcal{H}_{\mathbb{S},\Gamma_t,\Gamma_n,\varepsilon}(\Omega)$ *and* $\mathcal{H}_{\mathbb{T},\Gamma_n,\Gamma_t,\mu}(\Omega)$ *Are finite-dimensional. Moreover, the dimensions do not depend on* ε *or* μ.
(iv) *The orthonormal Helmholtz type decompositions*

$$\begin{aligned}\mathsf{L}^2_{\mathbb{S},\varepsilon}(\Omega) &= R\left({}_{\mathbb{S}}\mathrm{Gradgrad}_{\Gamma_t}\right) \oplus_{\mathsf{L}^2_{\mathbb{S},\varepsilon}(\Omega)} N\left(\mathrm{divDiv}_{\mathbb{S},\Gamma_n}\varepsilon\right)\\ &= N\left(\mu^{-1}{}_{\mathbb{T}}\mathrm{Rot}_{\mathbb{S},\Gamma_t}\right) \oplus_{\mathsf{L}^2_{\mathbb{S},\varepsilon}(\Omega)} R\left(\varepsilon^{-1}{}_{\mathbb{S}}\mathrm{Rot}_{\mathbb{T},\Gamma_n}\right)\\ &= R\left({}_{\mathbb{S}}\mathrm{Gradgrad}_{\Gamma_t}\right) \oplus_{\mathsf{L}^2_{\mathbb{S},\varepsilon}(\Omega)} \mathcal{H}_{\mathbb{S},\Gamma_t,\Gamma_n,\varepsilon}(\Omega) \oplus_{\mathsf{L}^2_{\mathbb{S},\varepsilon}(\Omega)} R\left(\varepsilon^{-1}{}_{\mathbb{S}}\mathrm{Rot}_{\mathbb{T},\Gamma_n}\right),\\ \mathsf{L}^2_{\mathbb{T},\mu}(\Omega) &= R\left({}_{\mathbb{T}}\mathrm{Grad}_{\Gamma_n}\right) \oplus_{\mathsf{L}^2_{\mathbb{T},\mu}(\Omega)} N(\mathrm{Div}_{\mathbb{T},\Gamma_t}\mu)\\ &= N\left(\varepsilon^{-1}{}_{\mathbb{S}}\mathrm{Rot}_{\mathbb{T},\Gamma_n}\right) \oplus_{\mathsf{L}^2_{\mathbb{T},\mu}(\Omega)} R\left(\mu^{-1}{}_{\mathbb{T}}\mathrm{Rot}_{\mathbb{S},\Gamma_t}\right)\\ &= R\left({}_{\mathbb{T}}\mathrm{Grad}_{\Gamma_n}\right) \oplus_{\mathsf{L}^2_{\mathbb{T},\mu}(\Omega)} \mathcal{H}_{\mathbb{T},\Gamma_n,\Gamma_t,\mu}(\Omega) \oplus_{\mathsf{L}^2_{\mathbb{T},\mu}(\Omega)} R\left(\mu^{-1}{}_{\mathbb{T}}\mathrm{Rot}_{\mathbb{S},\Gamma_t}\right)\end{aligned}$$

hold.

(v) *There exist (optimal)* $c_0, c_1, c_2 > 0$ *such that the Friedrichs/Poincaré type estimates*

$$\begin{aligned}
&\forall u \in \mathsf{H}^2_{\Gamma_t}(\Omega) \cap \left(\mathbb{P}^1_{\Gamma_t}\right)^{\perp_{\mathsf{L}^2(\Omega)}} && |u|_{\mathsf{L}^2(\Omega)} \leq c_0 \, |\mathrm{Gradgrad}\ u|_{\mathsf{L}^2_{\mathbb{S},\varepsilon}(\Omega)},\\
&\forall S \in \varepsilon^{-1}\mathsf{H}_{\mathbb{S},\Gamma_n}(\mathrm{divDiv},\Omega) \cap R\left({}_{\mathbb{S}}\mathrm{Gradgrad}_{\Gamma_t}\right) && |S|_{\mathsf{L}^2_{\mathbb{S},\varepsilon}(\Omega)} \leq c_0 \, |\mathrm{divDiv}\ \varepsilon S|_{\mathsf{L}^2(\Omega)},\\
&\forall S \in \mathsf{H}_{\mathbb{S},\Gamma_t}(\mathrm{Rot},\Omega) \cap R\left(\varepsilon_{\mathbb{S}}^{-1}\mathrm{Rot}_{\mathbb{T},\Gamma_n}\right) && |S|_{\mathsf{L}^2_{\mathbb{S},\varepsilon}(\Omega)} \leq c_1 \left|\mu^{-1}\mathrm{Rot}\ S\right|_{\mathsf{L}^2_{\mathbb{T},\mu}(\Omega)},\\
&\forall T \in \mathsf{H}_{\mathbb{T},\Gamma_n}(\mathrm{symRot},\Omega) \cap R\left(\mu^{-1}{}_{\mathbb{T}}\mathrm{Rot}_{\mathbb{S},\Gamma_t}\right) && |T|_{\mathsf{L}^2_{\mathbb{T},\mu}(\Omega)} \leq c_1 \left|\varepsilon^{-1}\mathrm{symRot}\ T\right|_{\mathsf{L}^2_{\mathbb{S},\varepsilon}(\Omega)},\\
&\forall T \in \mu^{-1}\mathsf{H}_{\mathbb{T},\Gamma_t}(\mathrm{Div},\Omega) \cap R\left({}_{\mathbb{T}}\mathrm{Grad}_{\Gamma_n}\right) && |T|_{\mathsf{L}^2_{\mathbb{T},\mu}(\Omega)} \leq c_2 \, |\mathrm{Div}\ \mu T|_{\mathsf{L}^2(\Omega)},\\
&\forall v \in \mathsf{H}^1_{\Gamma_n}(\Omega) \cap \left(\mathbb{RT}_{\Gamma_n}\right)^{\perp_{\mathsf{L}^2(\Omega)}} && |v|_{\mathsf{L}^2(\Omega)} \leq c_2 \, |\mathrm{devGrad}\ v|_{\mathsf{L}^2_{\mathbb{T},\mu}(\Omega)}
\end{aligned}$$

hold.

(vi) *For all* $S \in \mathsf{H}_{\mathbb{S},\Gamma_t}(\mathrm{Rot},\Omega) \cap \varepsilon^{-1}\mathsf{H}_{\mathbb{S},\Gamma_n}(\mathrm{divDiv},\Omega) \cap \mathcal{H}_{\mathbb{S},\Gamma_t,\Gamma_n,\varepsilon}(\Omega)^{\perp_{\mathsf{L}^2_{\mathbb{S},\varepsilon}(\Omega)}}$*, it holds*

$$|S|^2_{\mathsf{L}^2_{\mathbb{S},\varepsilon}(\Omega)} \leq c_1^2 \left|\mu^{-1}\mathrm{Rot}\, S\right|^2_{\mathsf{L}^2_{\mathbb{T},\mu}(\Omega)} + c_0^2 \, |\mathrm{divDiv}\ \varepsilon S|^2_{\mathsf{L}^2(\Omega)}.$$

(vi') *For all* $T \in \mathsf{H}_{\mathbb{T},\Gamma_n}(\mathrm{symRot},\Omega) \cap \mu^{-1}\mathsf{H}_{\mathbb{T},\Gamma_t}(\mathrm{Div},\Omega) \cap \mathcal{H}_{\mathbb{T},\Gamma_n,\Gamma_t,\mu}(\Omega)^{\perp_{\mathsf{L}^2_{\mathbb{T},\mu}(\Omega)}}$*, it holds*

$$|T|^2_{\mathsf{L}^2_{\mathbb{T},\mu}(\Omega)} \leq c_1^2 \left|\varepsilon^{-1}\mathrm{symRot}\ T\right|^2_{\mathsf{L}^2_{\mathbb{S},\varepsilon}(\Omega)} + c_2^2 \, |\mathrm{Div}\ \mu T|^2_{\mathsf{L}^2(\Omega)}.$$

(vii) $\mathcal{H}_{\mathbb{S},\Gamma_t,\Gamma_n,\varepsilon}(\Omega) = \{0\}$ *and* $\mathcal{H}_{\mathbb{T},\Gamma_n,\Gamma_t,\mu}(\Omega) = \{0\}$*, if* (Ω, Γ_t) *is extendable.*

3.2.2 | Higher order mini FA-ToolBox

For simplicity, let $\varepsilon = \mu = \mathrm{id}$. From Section 2.5, we recall the densely defined and closed higher Sobolev order operators

$$\begin{aligned}
{}_{\mathbb{S}}\mathrm{Gradgrad}^k_{\Gamma_t} &: \mathsf{H}^{k+2}_{\Gamma_t}(\Omega) \subset \mathsf{H}^k_{\Gamma_t}(\Omega) \to \mathsf{H}^k_{\mathbb{S},\Gamma_t}(\Omega),\\
{}_{\mathbb{S}}\mathrm{Gradgrad}^{k,k-1}_{\Gamma_t} &: \mathsf{H}^{k+1}_{\Gamma_t}(\Omega) \subset \mathsf{H}^k_{\Gamma_t}(\Omega) \to \mathsf{H}^{k-1}_{\mathbb{S},\Gamma_t}(\Omega), \qquad k \geq 1,\\
{}_{\mathbb{T}}\mathrm{Rot}^k_{\mathbb{S},\Gamma_t}, &: \mathsf{H}^k_{\mathbb{S},\Gamma_t}(\mathrm{Rot},\Omega) \subset \mathsf{H}^k_{\mathbb{S},\Gamma_t}(\Omega) \to \mathsf{H}^k_{\mathbb{T},\Gamma_t}(\Omega),\\
\mathrm{Div}^k_{\mathbb{T},\Gamma_t} &: \mathsf{H}^k_{\mathbb{T},\Gamma_t}(\mathrm{Div},\Omega) \subset \mathsf{H}^k_{\mathbb{T},\Gamma_t}(\Omega) \to \mathsf{H}^k_{\Gamma_t}(\Omega),\\
{}_{\mathbb{T}}\mathrm{Grad}^k_{\Gamma_n} &: \mathsf{H}^{k+1}_{\Gamma_n}(\Omega) \subset \mathsf{H}^k_{\Gamma_n}(\Omega) \to \mathsf{H}^k_{\mathbb{T},\Gamma_n}(\Omega),\\
{}_{\mathbb{S}}\mathrm{Rot}^k_{\mathbb{T},\Gamma_n} &: \mathsf{H}^k_{\mathbb{T},\Gamma_n}(\mathrm{symRot},\Omega) \subset \mathsf{H}^k_{\mathbb{T},\Gamma_n}(\Omega) \to \mathsf{H}^k_{\mathbb{S},\Gamma_n}(\Omega),\\
\mathrm{divDiv}^k_{\mathbb{S},\Gamma_n} &: \mathsf{H}^k_{\mathbb{S},\Gamma_n}(\mathrm{divDiv},\Omega) \subset \mathsf{H}^k_{\mathbb{S},\Gamma_n}(\Omega) \to \mathsf{H}^k_{\Gamma_n}(\Omega),\\
\mathrm{divDiv}^{k,k-1}_{\mathbb{S},\Gamma_n} &: \mathsf{H}^{k,k-1}_{\mathbb{S},\Gamma_n}(\mathrm{divDiv},\Omega) \subset \mathsf{H}^k_{\mathbb{S},\Gamma_n}(\Omega) \to \mathsf{H}^{k-1}_{\Gamma_n}(\Omega), \qquad k \geq 1,
\end{aligned} \tag{9}$$

building the long biharmonic Hilbert complexes

$$\mathbb{P}^1_{\Gamma_t} \xrightarrow{\iota_{\mathbb{P}^1_{\Gamma_t}}} \mathsf{H}^k_{\Gamma_t}(\Omega) \xrightarrow{{}_{\mathbb{S}}\mathrm{Gradgrad}^k_{\Gamma_t}} \mathsf{H}^k_{\mathbb{S},\Gamma_t}(\Omega) \xrightarrow{{}_{\mathbb{T}}\mathrm{Rot}^k_{\mathbb{S},\Gamma_t}} \mathsf{H}^k_{\mathbb{T},\Gamma_t}(\Omega) \xrightarrow{\mathrm{Div}^k_{\mathbb{T},\Gamma_t}} \mathsf{H}^k_{\Gamma_t}(\Omega) \xrightarrow{\pi_{\mathbb{RT}_{\Gamma_n}}} \mathbb{RT}_{\Gamma_n}, \quad k \geq 0, \tag{10}$$

$$\mathbb{P}^1_{\Gamma_t} \xleftarrow{\pi_{\mathbb{P}^1_{\Gamma_t}}} \mathsf{H}^k_{\Gamma_n}(\Omega) \xleftarrow{\mathrm{divDiv}^k_{\mathbb{S},\Gamma_n}} \mathsf{H}^k_{\mathbb{S},\Gamma_n}(\Omega) \xleftarrow{{}_{\mathbb{S}}\mathrm{Rot}^k_{\mathbb{T},\Gamma_n}} \mathsf{H}^k_{\mathbb{T},\Gamma_n}(\Omega) \xleftarrow{-{}_{\mathbb{T}}\mathrm{Grad}^k_{\Gamma_n}} \mathsf{H}^k_{\Gamma_n}(\Omega) \xleftarrow{\iota_{\mathbb{RT}_{\Gamma_n}}} \mathbb{RT}_{\Gamma_n}, \quad k \geq 0, \tag{11}$$

$$\mathbb{P}^1_{\Gamma_t} \xrightarrow{\iota_{\mathbb{P}^1_{\Gamma_t}}} \mathsf{H}^k_{\Gamma_t}(\Omega) \xrightarrow{{}_{\mathbb{S}}\mathrm{Gradgrad}^{k,k-1}_{\Gamma_t}} \mathsf{H}^{k-1}_{\mathbb{S},\Gamma_t}(\Omega) \xrightarrow{{}_{\mathbb{T}}\mathrm{Rot}^{k-1}_{\mathbb{S},\Gamma_t}} \mathsf{H}^{k-1}_{\mathbb{T},\Gamma_t}(\Omega) \xrightarrow{\mathrm{Div}^{k-1}_{\mathbb{T},\Gamma_t}} \mathsf{H}^{k-1}_{\Gamma_t}(\Omega) \xrightarrow{\pi_{\mathbb{RT}_{\Gamma_n}}} \mathbb{RT}_{\Gamma_n}, \quad k \geq 1, \tag{12}$$

$$\mathbb{P}^1_{\Gamma_t} \xleftarrow{\pi_{\mathbb{P}^1_{\Gamma_t}}} \mathsf{H}^{k-1}_{\Gamma_n}(\Omega) \xleftarrow{\mathrm{divDiv}^{k,k-1}_{\mathbb{S},\Gamma_n}} \mathsf{H}^k_{\mathbb{S},\Gamma_n}(\Omega) \xleftarrow{{}_{\mathbb{S}}\mathrm{Rot}^k_{\mathbb{T},\Gamma_n}} \mathsf{H}^k_{\mathbb{T},\Gamma_n}(\Omega) \xleftarrow{-{}_{\mathbb{T}}\mathrm{Grad}^k_{\Gamma_n}} \mathsf{H}^k_{\Gamma_n}(\Omega) \xleftarrow{\iota_{\mathbb{RT}_{\Gamma_n}}} \mathbb{RT}_{\Gamma_n}, \quad k \geq 1, \tag{13}$$

We start with regular representations implied by Lemma 3.10 and Corollary 3.11.

Theorem 3.16. (Regular representations and closed ranges)**.** *Let $k \geq 0$. Then the regular potential representations*

$$\begin{aligned}
R\left({}_{\mathbb{S}}\mathrm{Gradgrad}^{k+1,k}_{\Gamma_t}\right) = R\left({}_{\mathbb{S}}\mathrm{Gradgrad}^{k}_{\Gamma_t}\right) &= \mathrm{Gradgrad}\ \mathsf{H}^k_{\Gamma_t}(\mathrm{Gradgrad},\Omega) = \mathrm{Gradgrad}\ \mathsf{H}^{k+2}_{\Gamma_t}(\Omega)\\
&= \mathrm{Gradgrad}\ \mathsf{H}^{k+1,k}_{\Gamma_t}(\mathrm{Gradgrad},\Omega)\\
&= \mathsf{H}^k_{\mathbb{S},\Gamma_t}(\Omega) \cap R\left({}_{\mathbb{S}}\mathrm{Gradgrad}_{\Gamma_t}\right)\\
&= \mathsf{H}^k_{\mathbb{S},\Gamma_t}(\Omega) \cap \mathsf{H}_{\mathbb{S},\Gamma_t,0}(\mathrm{Rot},\Omega) \cap \mathcal{H}_{\mathbb{S},\Gamma_t,\Gamma_n,\varepsilon}(\Omega)^{\perp_{\mathsf{L}^2_{\mathbb{S},\varepsilon}(\Omega)}}\\
&= \mathsf{H}^k_{\mathbb{S},\Gamma_t,0}(\mathrm{Rot},\Omega) \cap \mathcal{H}_{\mathbb{S},\Gamma_t,\Gamma_n,\varepsilon}(\Omega)^{\perp_{\mathsf{L}^2_{\mathbb{S},\varepsilon}(\Omega)}},\\
R\left({}_{\mathbb{T}}\mathrm{Rot}^k_{\mathbb{S},\Gamma_t}\right) &= \mathrm{Rot}\ \mathsf{H}^k_{\mathbb{S},\Gamma_t}(\mathrm{Rot},\Omega) = \mathrm{Rot}\ \mathsf{H}^{k+1}_{\mathbb{S},\Gamma_t}(\Omega)\\
&= \mathsf{H}^k_{\mathbb{T},\Gamma_t}(\Omega) \cap R({}_{\mathbb{T}}\mathrm{Rot}_{\mathbb{S},\Gamma_t})\\
&= \mathsf{H}^k_{\mathbb{T},\Gamma_t}(\Omega) \cap \mathsf{H}_{\mathbb{T},\Gamma_t,0}(\mathrm{Div},\Omega) \cap \mathcal{H}_{\mathbb{T},\Gamma_n,\Gamma_t,\mu}(\Omega)^{\perp_{\mathsf{L}^2_{\mathbb{T}}(\Omega)}}\\
&= \mathsf{H}^k_{\mathbb{T},\Gamma_t,0}(\mathrm{Div},\Omega) \cap \mathcal{H}_{\mathbb{T},\Gamma_n,\Gamma_t,\mu}(\Omega)^{\perp_{\mathsf{L}^2_{\mathbb{T}}(\Omega)}},\\
R\left(\mathrm{Div}^k_{\mathbb{T},\Gamma_t}\right) &= \mathrm{Div}\mathsf{H}^k_{\mathbb{T},\Gamma_t}(\mathrm{Div},\Omega) = \mathrm{Div}\ \mathsf{H}^{k+1}_{\mathbb{T},\Gamma_t}(\Omega)\\
&= \mathsf{H}^k_{\Gamma_t}(\Omega) \cap R(\mathrm{Div}_{\mathbb{T},\Gamma_t}) = \mathsf{H}^k_{\Gamma_t}(\Omega) \cap \left(\mathbb{RT}_{\Gamma_n}\right)^{\perp_{\mathsf{L}^2(\Omega)}},\\
R\left({}_{\mathbb{T}}\mathrm{Grad}^k_{\Gamma_t}\right) &= \mathrm{devGrad}\ \mathsf{H}^k_{\Gamma_t}(\mathrm{devGrad},\Omega) = \mathrm{devGrad}\ \mathsf{H}^{k+1}_{\Gamma_t}(\Omega)\\
&= \mathsf{H}^k_{\mathbb{T},\Gamma_t}(\Omega) \cap R\left({}_{\mathbb{T}}\mathrm{Grad}_{\Gamma_t}\right)\\
&= \mathsf{H}^k_{\mathbb{T},\Gamma_t}(\Omega) \cap \mathsf{H}_{\mathbb{T},\Gamma_t,0}(\mathrm{symRot},\Omega) \cap \mathcal{H}_{\mathbb{T},\Gamma_t,\Gamma_n,\mu}(\Omega)^{\perp_{\mathsf{L}^2_{\mathbb{T},\mu}(\Omega)}}\\
&= \mathsf{H}^k_{\mathbb{T},\Gamma_t,0}(\mathrm{symRot},\Omega) \cap \mathcal{H}_{\mathbb{T},\Gamma_t,\Gamma_n,\mu}(\Omega)^{\perp_{\mathsf{L}^2_{\mathbb{T},\mu}(\Omega)}},\\
R\left({}_{\mathbb{S}}\mathrm{Rot}^k_{\mathbb{T},\Gamma_t}\right) &= \mathrm{symRot}\ \mathsf{H}^k_{\mathbb{T},\Gamma_t}(\mathrm{symRot},\Omega) = \mathrm{symRot}\ \mathsf{H}^{k+1}_{\mathbb{T},\Gamma_t}(\Omega)\\
&= \mathsf{H}^k_{\mathbb{S},\Gamma_t}(\Omega) \cap R\left({}_{\mathbb{S}}\mathrm{Rot}_{\mathbb{T},\Gamma_t}\right)\\
&= \mathsf{H}^k_{\mathbb{S},\Gamma_t}(\Omega) \cap \mathsf{H}_{\mathbb{S},\Gamma_t,0}(\mathrm{divDiv},\Omega) \cap \mathcal{H}_{\mathbb{S},\Gamma_n,\Gamma_t,\varepsilon}(\Omega)^{\perp_{\mathsf{L}^2_{\mathbb{S}}(\Omega)}}\\
&= \mathsf{H}^k_{\mathbb{S},\Gamma_t,0}(\mathrm{divDiv},\Omega) \cap \mathcal{H}_{\mathbb{S},\Gamma_n,\Gamma_t,\varepsilon}(\Omega)^{\perp_{\mathsf{L}^2_{\mathbb{S}}(\Omega)}},\\
R\left(\mathrm{divDiv}^{k+1,k}_{\mathbb{S},\Gamma_t}\right) = R\left(\mathrm{divDiv}^{k}_{\mathbb{S},\Gamma_t}\right) &= \mathrm{divDiv}\ \mathsf{H}^k_{\mathbb{S},\Gamma_t}(\mathrm{divDiv},\Omega) = \mathrm{divDiv}\ \mathsf{H}^{k+2}_{\mathbb{S},\Gamma_t}(\Omega)\\
&= \mathrm{divDiv}\ \mathsf{H}^{k+1,k}_{\mathbb{S},\Gamma_t}(\mathrm{divDiv},\Omega)\\
&= \mathsf{H}^k_{\Gamma_t}(\Omega) \cap R\left(\mathrm{divDiv}_{\mathbb{S},\Gamma_t}\right) = \mathsf{H}^k_{\Gamma_t}(\Omega) \cap \left(\mathbb{P}^1_{\Gamma_n}\right)^{\perp_{\mathsf{L}^2(\Omega)}}
\end{aligned}$$

hold. In particular, the latter spaces are closed subspaces of $\mathsf{H}^k_{\mathbb{S}}(\Omega)$, $\mathsf{H}^k_{\mathbb{T}}(\Omega)$, *and* $\mathsf{H}^k(\Omega)$, *respectively, and all ranges of the higher Sobolev order operators in (9) are closed. Moreover, the long biharmonic Hilbert complexes (10)–(13) are closed.*

A proof is given in Appendix C. Note that in Theorem 3.16 we claim nothing about bounded regular potential operators, leaving the question of bounded potentials to the next sections (cf. Theorem 3.24).

The reduced operators corresponding to (9) are

$$
\begin{aligned}
\left({}_{\mathbb{S}}\mathrm{Gradgrad}^{k}_{\Gamma_t}\right)_{\perp} &: D\left(\left({}_{\mathbb{S}}\mathrm{Gradgrad}^{k}_{\Gamma_t}\right)_{\perp}\right) \subset \left(\mathbb{P}^1_{\Gamma_t}\right)^{\perp_{\mathsf{H}^k_{\Gamma_t}(\Omega)}} \to R\left({}_{\mathbb{S}}\mathrm{Gradgrad}^{k}_{\Gamma_t}\right),\\
\left({}_{\mathbb{S}}\mathrm{Gradgrad}^{k,k-1}_{\Gamma_t}\right)_{\perp} &: D\left(\left({}_{\mathbb{S}}\mathrm{Gradgrad}^{k,k-1}_{\Gamma_t}\right)_{\perp}\right) \subset \left(\mathbb{P}^1_{\Gamma_t}\right)^{\perp_{\mathsf{H}^k_{\Gamma_t}(\Omega)}} \to R\left({}_{\mathbb{S}}\mathrm{Gradgrad}^{k-1}_{\Gamma_t}\right),\ k \geq 1,\\
\left({}_{\mathbb{T}}\mathrm{Rot}^{k}_{\mathbb{S},\Gamma_t}\right)_{\perp} &: D\left(\left({}_{\mathbb{T}}\mathrm{Rot}^{k}_{\mathbb{S},\Gamma_t}\right)_{\perp}\right) \subset N\left({}_{\mathbb{T}}\mathrm{Rot}^{k}_{\mathbb{S},\Gamma_t}\right)^{\perp_{\mathsf{H}^k_{\mathbb{S},\Gamma_t}(\Omega)}} \to R\left({}_{\mathbb{T}}\mathrm{Rot}^{k}_{\mathbb{S},\Gamma_t}\right),\\
\left(\mathrm{Div}^{k}_{\mathbb{T},\Gamma_t}\right)_{\perp} &: D\left(\left(\mathrm{Div}^{k}_{\mathbb{T},\Gamma_t}\right)_{\perp}\right) \subset N\left(\mathrm{Div}^{k}_{\mathbb{T},\Gamma_t}\right)^{\perp_{\mathsf{H}^k_{\mathbb{T},\Gamma_t}(\Omega)}} \to R\left(\mathrm{Div}^{k}_{\mathbb{T},\Gamma_t}\right),\\
\left({}_{\mathbb{T}}\mathrm{Grad}^{k}_{\Gamma_n}\right)_{\perp} &: D\left(\left({}_{\mathbb{T}}\mathrm{Grad}^{k}_{\Gamma_n}\right)_{\perp}\right) \subset \left(\mathbb{RT}_{\Gamma_n}\right)^{\perp_{\mathsf{H}^k_{\Gamma_n}(\Omega)}} \to R\left({}_{\mathbb{T}}\mathrm{Grad}^{k}_{\Gamma_n}\right),\\
\left({}_{\mathbb{S}}\mathrm{Rot}^{k}_{\mathbb{T},\Gamma_n}\right)_{\perp} &: D\left(\left({}_{\mathbb{S}}\mathrm{Rot}^{k}_{\mathbb{T},\Gamma_n}\right)_{\perp}\right) \subset N\left({}_{\mathbb{S}}\mathrm{Rot}^{k}_{\mathbb{T},\Gamma_n}\right)^{\perp_{\mathsf{H}^k_{\mathbb{T},\Gamma_n}(\Omega)}} \to R\left({}_{\mathbb{S}}\mathrm{Rot}^{k}_{\mathbb{T},\Gamma_n}\right),\\
\left(\mathrm{divDiv}^{k}_{\mathbb{S},\Gamma_n}\right)_{\perp} &: D\left(\left(\mathrm{divDiv}^{k}_{\mathbb{S},\Gamma_n}\right)_{\perp}\right) \subset N\left(\mathrm{divDiv}^{k}_{\mathbb{S},\Gamma_n}\right)^{\perp_{\mathsf{H}^k_{\mathbb{S},\Gamma_n}(\Omega)}} \to R\left(\mathrm{divDiv}^{k}_{\mathbb{S},\Gamma_n}\right),\\
\left(\mathrm{divDiv}^{k,k-1}_{\mathbb{S},\Gamma_n}\right)_{\perp} &: D\left(\left(\mathrm{divDiv}^{k,k-1}_{\mathbb{S},\Gamma_n}\right)_{\perp}\right) \subset N\left(\mathrm{divDiv}^{k}_{\mathbb{S},\Gamma_n}\right)^{\perp_{\mathsf{H}^k_{\mathbb{S},\Gamma_n}(\Omega)}} \to R(\mathrm{divDiv}^{k-1}_{\mathbb{S},\Gamma_n}),\ k \geq 1.
\end{aligned}
$$

Lemma 2.1 of [1] and Theorem 3.16 yield the following:

Theorem 3.17. (Closed ranges and bounded inverse operators)**.** *Let* $k \geq 0$. *Then,*

(i) $R\left(\left({}_{\mathbb{S}}\mathrm{Gradgrad}^{k}_{\Gamma_t}\right)_{\perp}\right) = R\left({}_{\mathbb{S}}\mathrm{Gradgrad}^{k}_{\Gamma_t}\right) = R\left({}_{\mathbb{S}}\mathrm{Gradgrad}^{k+1,k}_{\Gamma_t}\right) = R\left(\left({}_{\mathbb{S}}\mathrm{Gradgrad}^{k+1,k}_{\Gamma_t}\right)_{\perp}\right)$ *are closed, and equivalently, the inverse operators*

$$
\begin{aligned}
&\left({}_{\mathbb{S}}\mathrm{Gradgrad}^{k}_{\Gamma_t}\right)_{\perp}^{-1} : R\left({}_{\mathbb{S}}\mathrm{Gradgrad}^{k}_{\Gamma_t}\right) \to D\left(\left({}_{\mathbb{S}}\mathrm{Gradgrad}^{k}_{\Gamma_t}\right)_{\perp}\right)\\
\textit{resp.}\quad &\left({}_{\mathbb{S}}\mathrm{Gradgrad}^{k}_{\Gamma_t}\right)_{\perp}^{-1} : R\left({}_{\mathbb{S}}\mathrm{Gradgrad}^{k}_{\Gamma_t}\right) \to D\left({}_{\mathbb{S}}\mathrm{Gradgrad}^{k}_{\Gamma_t}\right),\\
&\left({}_{\mathbb{S}}\mathrm{Gradgrad}^{k+1,k}_{\Gamma_t}\right)_{\perp}^{-1} : R\left({}_{\mathbb{S}}\mathrm{Gradgrad}^{k}_{\Gamma_t}\right) \to D\left(\left({}_{\mathbb{S}}\mathrm{Gradgrad}^{k+1,k}_{\Gamma_t}\right)_{\perp}\right)\\
\textit{resp.}\quad &\left({}_{\mathbb{S}}\mathrm{Gradgrad}^{k+1,k}_{\Gamma_t}\right)_{\perp}^{-1} : R\left({}_{\mathbb{S}}\mathrm{Gradgrad}^{k}_{\Gamma_t}\right) \to D\left({}_{\mathbb{S}}\mathrm{Gradgrad}^{k+1,k}_{\Gamma_t}\right)
\end{aligned}
$$

are bounded. Equivalently, there is $c > 0$ *such that for all* $u \in D\left(\left({}_{\mathbb{S}}\mathrm{Gradgrad}^{k}_{\Gamma_t}\right)_{\perp}\right)$ *resp.* $u \in D\left(\left({}_{\mathbb{S}}\mathrm{Gradgrad}^{k+1,k}_{\Gamma_t}\right)_{\perp}\right)$

$$
|u|_{\mathsf{H}^k(\Omega)} \leq c\,|\mathrm{Gradgrad}\ u|_{\mathsf{H}^k\mathbb{S}(\Omega)} \quad \textit{resp.} \quad |u|_{\mathsf{H}^{k+1}(\Omega)} \leq c\,|\mathrm{Gradgrad}\ u|_{\mathsf{H}^k_{\mathbb{S}}(\Omega)}.
$$

(ii) $R\left({}_{\mathbb{T}}\mathrm{Rot}^{k}_{\mathbb{S},\Gamma_t}\right) = R\left(\left({}_{\mathbb{T}}\mathrm{Rot}^{k}_{\mathbb{S},\Gamma_t}\right)_{\perp}\right)$ *are closed, and equivalently, the inverse operator*

$$
\begin{aligned}
&\left({}_{\mathbb{T}}\mathrm{Rot}^{k}_{\mathbb{S},\Gamma_t}\right)_{\perp}^{-1} : R\left({}_{\mathbb{T}}\mathrm{Rot}^{k}_{\mathbb{S},\Gamma_t}\right) \to D\left(\left({}_{\mathbb{T}}\mathrm{Rot}^{k}_{\mathbb{S},\Gamma_t}\right)_{\perp}\right)\\
\textit{resp.}\quad &\left({}_{\mathbb{T}}\mathrm{Rot}^{k}_{\mathbb{S},\Gamma_t}\right)_{\perp}^{-1} : R\left({}_{\mathbb{T}}\mathrm{Rot}^{k}_{\mathbb{S},\Gamma_t}\right) \to D\left({}_{\mathbb{T}}\mathrm{Rot}^{k}_{\mathbb{S},\Gamma_t}\right)
\end{aligned}
$$

is bounded. Equivalently, there is $c > 0$ *such that for all* $S \in D\left(\left({}_{\mathbb{T}}\mathrm{Rot}^{k}_{\mathbb{S},\Gamma_t}\right)_{\perp}\right)$

$$
|S|_{\mathsf{H}^k_{\mathbb{S}}(\Omega)} \leq c\,|\mathrm{Rot}\,S|_{\mathsf{H}^k_{\mathbb{T}}(\Omega)}.
$$

(iii) $R\left(\mathrm{Div}^k_{\mathbb{T},\Gamma_t}\right) = R\left(\left(\mathrm{Div}^k_{\mathbb{T},\Gamma_t}\right)_\perp\right)$ *are closed, and equivalently, the inverse operator*

$$\left(\mathrm{Div}^k_{\mathbb{T},\Gamma_t}\right)_\perp^{-1} : R\left(\mathrm{Div}^k_{\mathbb{T},\Gamma_t}\right) \to D\left(\left(\mathrm{Div}^k_{\mathbb{T},\Gamma_t}\right)_\perp\right)$$
$$\textit{resp.} \qquad \left(\mathrm{Div}^k_{\mathbb{T},\Gamma_t}\right)_\perp^{-1} : R\left(\mathrm{Div}^k_{\mathbb{T},\Gamma_t}\right) \to D\left(\mathrm{Div}^k_{\mathbb{T},\Gamma_t}\right)$$

is bounded. Equivalently, there is $c > 0$ *such that for all* $T \in D\left(\left(\mathrm{Div}^k_{\mathbb{T},\Gamma_t}\right)_\perp\right)$

$$|T|_{\mathsf{H}^k_{\mathbb{T}}(\Omega)} \leq c\,|\mathrm{Div}\ T|_{\mathsf{H}^k(\Omega)}.$$

(iv) $R\left({}_{\mathbb{T}}\mathrm{Grad}^k_{\Gamma_t}\right) = R\left(\left({}_{\mathbb{T}}Grad^k_{\Gamma_t}\right)_\perp\right)$ *are closed, and equivalently, the inverse operator*

$$\left({}_{\mathbb{T}}\mathrm{Grad}^k_{\Gamma_t}\right)_\perp^{-1} : R\left({}_{\mathbb{T}}\mathrm{Grad}^k_{\Gamma_t}\right) \to D\left(\left({}_{\mathbb{T}}Grad^k_{\Gamma_t}\right)_\perp\right)$$
$$\textit{resp.} \qquad \left({}_{\mathbb{T}}\mathrm{Grad}^k_{\Gamma_t}\right)_\perp^{-1} : R\left({}_{\mathbb{T}}\mathrm{Grad}^k_{\Gamma_t}\right) \to D\left({}_{\mathbb{T}}\mathrm{Grad}^k_{\Gamma_t}\right)$$

is bounded. Equivalently, there is $c > 0$ *such that for all* $v \in D\left(\left({}_{\mathbb{T}}\mathrm{Grad}^k_{\Gamma_t}\right)_\perp\right)$

$$|v|_{\mathsf{H}^k(\Omega)} \leq c\,|\mathrm{devGrad}\ v|_{\mathsf{H}^k_{\mathbb{T}}(\Omega)}.$$

(v) $R\left({}_{\mathbb{S}}\mathrm{Rot}^k_{\mathbb{T},\Gamma_t}\right) = R\left(\left({}_{\mathbb{S}}\mathrm{Rot}^k_{\mathbb{T},\Gamma_t}\right)_\perp\right)$ *are closed, and equivalently, the inverse operator*

$$\left({}_{\mathbb{S}}\mathrm{Rot}^k_{\mathbb{T},\Gamma_t}\right)_\perp^{-1} : R\left({}_{\mathbb{S}}\mathrm{Rot}^k_{\mathbb{T},\Gamma_t}\right) \to D\left(\left({}_{\mathbb{S}}\mathrm{Rot}^k_{\mathbb{T},\Gamma_t}\right)_\perp\right)$$
$$\textit{resp.} \qquad \left({}_{\mathbb{S}}\mathrm{Rot}^k_{\mathbb{T},\Gamma_t}\right)_\perp^{-1} : R\left({}_{\mathbb{S}}\mathrm{Rot}^k_{\mathbb{T},\Gamma_t}\right) \to D\left({}_{\mathbb{S}}\mathrm{Rot}^k_{\mathbb{T},\Gamma_t}\right)$$

is bounded. Equivalently, there is $c > 0$ *such that for all* $T \in D\left(\left({}_{\mathbb{S}}\mathrm{Rot}^k_{\mathbb{T},\Gamma_t}\right)_\perp\right)$

$$|T|_{\mathsf{H}^k_{\mathbb{T}}(\Omega)} \leq c\,|\mathrm{symRot}\ T|_{\mathsf{H}^k_{\mathbb{S}}(\Omega)}.$$

(vi) $R\left(\left(\mathrm{divDiv}^k_{\mathbb{S},\Gamma_t}\right)_\perp\right) = R\left(\mathrm{divDiv}^k_{\mathbb{S},\Gamma_t}\right) = R\left(\mathrm{divDiv}^{k+1,k}_{\mathbb{S},\Gamma_t}\right) = R\left(\left(\mathrm{divDiv}^{k+1,k}_{\mathbb{S},\Gamma_t}\right)_\perp\right)$ *are closed, and equivalently, the inverse operators*

$$\left(\mathrm{divDiv}^k_{\mathbb{S},\Gamma_t}\right)_\perp^{-1} : R\left(\mathrm{divDiv}^k_{\mathbb{S},\Gamma_t}\right) \to D\left(\left(\mathrm{divDiv}^k_{\mathbb{S},\Gamma_t}\right)_\perp\right)$$
$$\textit{resp.} \qquad \left(\mathrm{divDiv}^k_{\mathbb{S},\Gamma_t}\right)_\perp^{-1} : R\left(\mathrm{divDiv}^k_{\mathbb{S},\Gamma_t}\right) \to D\left(\mathrm{divDiv}^k_{\mathbb{S},\Gamma_t}\right),$$
$$\left(\mathrm{divDiv}^{k+1,k}_{\mathbb{S},\Gamma_t}\right)_\perp^{-1} : R\left(\mathrm{divDiv}^k_{\mathbb{S},\Gamma_t}\right) \to D\left(\left(\mathrm{divDiv}^{k+1,k}_{\mathbb{S},\Gamma_t}\right)_\perp\right)$$
$$\textit{resp.} \qquad \left(\mathrm{divDiv}^{k+1,k}_{\mathbb{S},\Gamma_t}\right)_\perp^{-1} : R\left(\mathrm{divDiv}^k_{\mathbb{S},\Gamma_t}\right) \to D\left(\mathrm{divDiv}^{k+1,k}_{\mathbb{S},\Gamma_t}\right)$$

are bounded. Equivalently, there is $c > 0$ *such that for all* $S \in D\left(\left(\mathrm{divDiv}^k_{\mathbb{S},\Gamma_t}\right)_\perp\right)$ *resp.* $S \in D\left(\left(\mathrm{divDiv}^{k+1,k}_{\mathbb{S},\Gamma_t}\right)_\perp\right)$

$$|S|_{\mathsf{H}^k_{\mathbb{S}}(\Omega)} \leq c\,|\mathrm{divDiv}\ S|_{\mathsf{H}^k(\Omega)} \quad \textit{resp.} \quad |S|_{\mathsf{H}^{k+1}_{\mathbb{S}}(\Omega)} \leq c\,|\mathrm{divDiv}\ S|_{\mathsf{H}^k(\Omega)}.$$

Lemma 3.18. (Schwarz' lemma). *Let* $0 \leq |\alpha| \leq k$.

(i) *If* $S \in \mathsf{H}^k_{\mathbb{S},\Gamma_t}(\mathrm{Rot},\Omega)$ *then* $\partial^\alpha S \in \mathsf{H}_{\mathbb{S},\Gamma_t}(\mathrm{Rot},\Omega)$ *and* $\mathrm{Rot}\,\partial^\alpha S = \partial^\alpha \mathrm{Rot}\, S$.
(ii) *If* $T \in \mathsf{H}^k_{\mathbb{T},\Gamma_t}(\mathrm{Div},\Omega)$ *then* $\partial^\alpha T \in \mathsf{H}_{\mathbb{T},\Gamma_t}(\mathrm{Div},\Omega)$ *and* $\mathrm{Div}\partial^\alpha T = \partial^\alpha \mathrm{Div}\ T$.
(iii) *If* $T \in \mathsf{H}^k_{\mathbb{T},\Gamma_t}(\mathrm{symRot},\Omega)$ *then* $\partial^\alpha T \in \mathsf{H}_{\mathbb{T},\Gamma_t}(\mathrm{symRot},\Omega)$ *and* $\mathrm{symRot}\,\partial^\alpha T = \partial^\alpha \mathrm{symRot}\ T$.
(iv) *If* $S \in \mathsf{H}^k_{\mathbb{S},\Gamma_t}(\mathrm{divDiv},\Omega)$ *resp.* $S \in \mathsf{H}^{k+1,k}_{\mathbb{S},\Gamma_t}(\mathrm{divDiv},\Omega)$ *then* $\partial^\alpha S \in \mathsf{H}_{\mathbb{S},\Gamma_t}(\mathrm{divDiv},\Omega)$ *resp.* $\partial^\alpha S \in \mathsf{H}^{1,0}_{\mathbb{S},\Gamma_t}(\mathrm{divDiv},\Omega)$ *and* $\mathrm{divDiv}\,\partial^\alpha S = \partial^\alpha \mathrm{divDiv}\ S$.

Theorem 3.19. (Compact embedding). *Let* $k \geq 0$. *Then the embeddings*

$$
\begin{gathered}
\mathsf{H}^k_{\mathbb{S},\Gamma_t}(\mathrm{Rot},\Omega) \cap \mathsf{H}^k_{\mathbb{S},\Gamma_n}(\mathrm{divDiv},\Omega) \hookrightarrow \mathsf{H}^k_{\mathbb{S}},\Gamma(\Omega),\\
\mathsf{H}^k_{\mathbb{T},\Gamma_t}(\mathrm{Div},\Omega) \cap \mathsf{H}^k_{\mathbb{T},\Gamma_n}(\mathrm{symRot},\Omega) \hookrightarrow \mathsf{H}^k_{\mathbb{T}},\Gamma(\Omega)
\end{gathered}
$$

are compact.

A proof is given in Appendix C.

Remark 3.20. (Compact embedding). For $k \geq 1$ (cf. [3, Remark 4.12]), there is another and slightly more general proof of the first compact embedding using a variant of [1, Lemma 2.22] (cf. [2, Theorem 3.19, Remark 3.20]); see Appendix C for a proof. It utilises the decomposition $\mathsf{H}^{k,k-1}_{\mathbb{S},\Gamma_n}(\mathrm{divDiv},\Omega) = \mathsf{H}^{k+1}_{\mathbb{S},\Gamma_n}(\Omega) + \mathrm{symRot}\ \mathsf{H}^{k+1}_{\mathbb{T},\Gamma_n}(\Omega)$ from Lemma 3.10 and leads immediately to the next (stronger) result.

Theorem 3.21. (Compact embedding). *Let* $k \geq 1$. *Then the embedding*

$$
\mathsf{H}^k_{\mathbb{S},\Gamma_t}(\mathrm{Rot},\Omega) \cap \mathsf{H}^{k,k-1}_{\mathbb{S},\Gamma_n}(\mathrm{divDiv},\Omega) \hookrightarrow \mathsf{H}^k_{\mathbb{S},\Gamma}(\Omega)
$$

is compact.

Theorem 3.22. (Friedrichs/Poincaré type estimate). *Let* $k \geq 0$. *Then there exists* $c > 0$ *such that for all*

$$
\begin{gathered}
S \in \mathsf{H}^k_{\mathbb{S},\Gamma_t}(\mathrm{Rot},\Omega) \cap \mathsf{H}^k_{\mathbb{S},\Gamma_n}(\mathrm{divDiv},\Omega) \cap \mathcal{H}_{\mathbb{S},\Gamma_t,\Gamma_n,\mathrm{id}}(\Omega)^{\perp_{\mathsf{L}^2_{\mathbb{S}}(\Omega)}},\\
T \in \mathsf{H}^k_{\mathbb{T},\Gamma_t}(\mathrm{symRot},\Omega) \cap \mathsf{H}^k_{\mathbb{T},\Gamma_n}(\mathrm{Div},\Omega) \cap \mathcal{H}_{\mathbb{T},\Gamma_t,\Gamma_n,\mathrm{id}}(\Omega)^{\perp_{\mathsf{L}^2_{\mathbb{T}}(\Omega)}},
\end{gathered}
$$

it holds

$$
\begin{gathered}
|S|_{\mathsf{H}^k_{\mathbb{S}}(\Omega)} \leq c\left(|\mathrm{Rot}\ S|_{\ \mathsf{H}^k_{\mathbb{T}}(\Omega)} + |\mathrm{divDiv}\ \ S|_{\ \mathsf{H}^k(\Omega)}\right),\\
|T|_{\mathsf{H}^k_{\mathbb{T}}(\Omega)} \leq c\left(|\mathrm{symRot}\ \ T|_{\ \mathsf{H}^k_{\mathbb{S}}(\Omega)} + |\mathrm{Div}\ \ T|_{\ \mathsf{H}^k(\Omega)}\right),
\end{gathered}
$$

respectively. The orthogonality condition $\mathcal{H}_{\mathbb{S},\Gamma_t,\Gamma_n,\mathrm{id}}(\Omega)^{\perp_{\mathsf{L}^2_{\mathbb{S}}(\Omega)}}$ *and* $\mathcal{H}_{\mathbb{T},\Gamma_t,\Gamma_n,\mathrm{id}}(\Omega)^{\perp_{\mathsf{L}^2_{\mathbb{T}}(\Omega)}}$ *can be replaced by the weaker conditions* $\mathcal{H}^k_{\mathbb{S},\Gamma_t,\Gamma_n,\mathrm{id}}(\Omega)^{\perp_{\mathsf{L}^2_{\mathbb{S}}(\Omega)}}$ *or* $\mathcal{H}^k_{\mathbb{S},\Gamma_t,\Gamma_n,\mathrm{id}}(\Omega)^{\perp_{\mathsf{H}^k_{\mathbb{S}}(\Omega)}}$ *and* $\mathcal{H}^k_{\mathbb{T},\Gamma_t,\Gamma_n,\mathrm{id}}(\Omega)^{\perp_{\mathsf{L}^2_{\mathbb{T}}(\Omega)}}$ *or* $\mathcal{H}^k_{\mathbb{T},\Gamma_t,\Gamma_n,\mathrm{id}}(\Omega)^{\perp_{\mathsf{H}^k_{\mathbb{T}}(\Omega)}}$, *respectively. In particular,*

$$
\begin{aligned}
&\forall S \in \mathsf{H}^k_{\mathbb{S},\Gamma_t}(\mathrm{Rot},\Omega) \cap R\left({}_{\mathbb{S}}\mathrm{Rot}^k_{\mathbb{T},\Gamma_n}\right) && |S|_{\mathsf{H}^k_{\mathbb{S}}(\Omega)} \leq c\,|\mathrm{Rot}\ S|_{\ \mathsf{H}^k_{\mathbb{T}}(\Omega)},\\
&\forall S \in \mathsf{H}^k_{\mathbb{S},\Gamma_n}(\mathrm{divDiv},\Omega) \cap R\left({}_{\mathbb{S}}\mathrm{Gradgrad}^k_{\Gamma_t}\right) && |S|_{\mathsf{H}^k_{\mathbb{S}}(\Omega)} \leq c\,|\mathrm{divDiv}\ \ S|_{\ \mathsf{H}^k(\Omega)},\\
&\forall T \in \mathsf{H}^k_{\mathbb{T},\Gamma_t}(\mathrm{symRot},\Omega) \cap R\left({}_{\mathbb{T}}\mathrm{Rot}^k_{\mathbb{S},\Gamma_n}\right) && |T|_{\mathsf{H}^k_{\mathbb{T}}(\Omega)} \leq c\,|\mathrm{symRot}\ \ T|_{\ \mathsf{H}^k_{\mathbb{S}}(\Omega)},\\
&\forall T \in \mathsf{H}^k_{\mathbb{T},\Gamma_n}(\mathrm{Div},\Omega) \cap R\left({}_{\mathbb{T}}\mathrm{Grad}^k_{\Gamma_t}\right) && |T|_{\mathsf{H}^k_{\mathbb{T}}(\Omega)} \leq c\,|\mathrm{Div}\ \ T|_{\ \mathsf{H}^k(\Omega)}
\end{aligned}
$$

with

$$R\left({}_{\mathbb{S}}\mathrm{Rot}^{k}_{\mathbb{T},\Gamma_n}\right)=\mathsf{H}^{k}_{\mathbb{S},\Gamma_n,0}(\mathrm{divDiv},\Omega)\cap\mathcal{H}_{\mathbb{S},\Gamma_t,\Gamma_n,\mathrm{id}}(\Omega)^{\perp_{\mathsf{L}^2_{\mathbb{S}}(\Omega)}},$$

$$R\left({}_{\mathbb{S}}\mathrm{Gradgrad}^{k+1,k}_{\Gamma_t}\right)=R\left({}_{\mathbb{S}}\mathrm{Gradgrad}^{k}_{\Gamma_t}\right)=\mathsf{H}^{k}_{\mathbb{S},\Gamma_t,0}(\mathrm{Rot},\Omega)\cap\mathcal{H}_{\mathbb{S},\Gamma_t,\Gamma_n,\mathrm{id}}(\Omega)^{\perp_{\mathsf{L}^2_{\mathbb{S}}(\Omega)}},$$

$$R\left({}_{\mathbb{T}}\mathrm{Rot}^{k}_{\mathbb{S},\Gamma_n}\right)=\mathsf{H}^{k}_{\mathbb{T},\Gamma_n,0}(\mathrm{Div},\Omega)\cap\mathcal{H}_{\mathbb{T},\Gamma_t,\Gamma_n,\mathrm{id}}(\Omega)^{\perp_{\mathsf{L}^2_{\mathbb{T}}(\Omega)}},$$

$$R\left({}_{\mathbb{T}}\mathrm{Grad}^{k}_{\Gamma_t}\right)=\mathsf{H}^{k}_{\mathbb{T},\Gamma_t,0}(\mathrm{symRot},\Omega)\cap\mathcal{H}_{\mathbb{T},\Gamma_t,\Gamma_n,\mathrm{id}}(\Omega)^{\perp_{\mathsf{L}^2_{\mathbb{T}}(\Omega)}}.$$

Analogously, for $k\geq 1$, there exists $c>0$ such that

$$|S|_{\mathsf{H}^k_{\mathbb{S}}(\Omega)}\leq c\left(|\mathrm{Rot}\ S|_{\mathsf{H}^k_{\mathbb{T}}(\Omega)}+|\mathrm{divDiv}\ S|_{\mathsf{H}^{k-1}(\Omega)}\right)$$

for all S in $\mathsf{H}^{k}_{\mathbb{S},\Gamma_t}(\mathrm{Rot},\Omega)\cap\mathsf{H}^{k,k-1}_{\mathbb{S},\Gamma_n}(\mathrm{divDiv},\Omega)\cap\mathcal{H}_{\mathbb{S},\Gamma_t,\Gamma_n,\mathrm{id}}(\Omega)^{\perp_{\mathsf{L}^2_{\mathbb{S}}(\Omega)}}$. Moreover,

$$\forall S\in\mathsf{H}^{k,k-1}_{\mathbb{S},\Gamma_n}(\mathrm{divDiv},\Omega)\cap R\left({}_{\mathbb{S}}\mathrm{Gradgrad}^{k}_{\Gamma_t}\right)\ |S|_{\mathsf{H}^k_{\mathbb{S}}(\Omega)}\leq c\,|\mathrm{divDiv}\ S|_{\mathsf{H}^{k-1}(\Omega)}.$$

The proof follows by a standard contradiction argument.

Remark 3.23. (Friedrichs/Poincaré/Korn type estimate)**.** Let $k\geq 0$. Similar to Theorem 3.22 and by Rellich's selection theorem (cf. the estimates in Theorem 3.17), there exists $c>0$ such that for all $v\in\mathsf{H}^{k+1}_{\Gamma_t}(\Omega)\cap\left(\mathbb{RT}_{\Gamma_t}\right)^{\perp_{\mathsf{L}^2(\Omega)}}$ and for all $u\in\mathsf{H}^{k+2}_{\Gamma_t}(\Omega)\cap\left(\mathbb{P}^1_{\Gamma_t}\right)^{\perp_{\mathsf{L}^2(\Omega)}}$

$$|v|_{\mathsf{H}^k(\Omega)}\leq c\,|\mathrm{devGrad}\ v|_{\mathsf{H}^k_{\mathbb{T}}(\Omega)},\qquad |u|_{\mathsf{H}^k(\Omega)}\leq|u|_{\mathsf{H}^{k+1}(\Omega)}\leq c\,|\mathrm{Gradgrad}\ u|_{\mathsf{H}^k_{\mathbb{S}}(\Omega)}.$$

As in Theorem 3.17, $\left(\mathbb{RT}_{\Gamma_t}\right)^{\perp_{\mathsf{L}^2(\Omega)}}$ and $\left(\mathbb{P}^1_{\Gamma_t}\right)^{\perp_{\mathsf{L}^2(\Omega)}}$ can be replaced by $\left(\mathbb{RT}_{\Gamma_t}\right)^{\perp_{\mathsf{H}^k_{\Gamma_t}(\Omega)}}$ and $\left(\mathbb{P}^1_{\Gamma_t}\right)^{\perp_{\mathsf{H}^k_{\Gamma_t}(\Omega)}}$, respectively.

3.3 | Regular potentials and decompositions II

Let $k\geq 0$. According to Theorem 3.17, the inverses of the reduced operators

$$\left({}_{\mathbb{S}}\mathrm{Gradgrad}^{k}_{\Gamma_t}\right)^{-1}_{\perp}:R\left({}_{\mathbb{S}}\mathrm{Gradgrad}^{k}_{\Gamma_t}\right)\to D\left({}_{\mathbb{S}}\mathrm{Gradgrad}^{k}_{\Gamma_t}\right)=\mathsf{H}^{k+2}_{\Gamma_t}(\Omega),$$

$$\left({}_{\mathbb{S}}\mathrm{Gradgrad}^{k+1,k}_{\Gamma_t}\right)^{-1}_{\perp}:R\left({}_{\mathbb{S}}\mathrm{Gradgrad}^{k}_{\Gamma_t}\right)\to D\left({}_{\mathbb{S}}\mathrm{Gradgrad}^{k+1,k}_{\Gamma_t}\right)=\mathsf{H}^{k+2}_{\Gamma_t}(\Omega),$$

$$\left({}_{\mathbb{T}}\mathrm{Rot}^{k}_{\mathbb{S},\Gamma_t}\right)^{-1}_{\perp}:R\left({}_{\mathbb{T}}\mathrm{Rot}^{k}_{\mathbb{S},\Gamma_t}\right)\to D\left({}_{\mathbb{T}}\mathrm{Rot}^{k}_{\mathbb{S},\Gamma_t}\right)=\mathsf{H}^{k}_{\mathbb{S},\Gamma_t}(\mathrm{Rot},\Omega),$$

$$\left(\mathrm{Div}^{k}_{\mathbb{T},\Gamma_t}\right)^{-1}_{\perp}:R\left(\mathrm{Div}^{k}_{\mathbb{T},\Gamma_t}\right)\to D\left(\mathrm{Div}^{k}_{\mathbb{T},\Gamma_t}\right)=\mathsf{H}^{k}_{\mathbb{T},\Gamma_t}(\mathrm{Div},\Omega),$$

$$\left({}_{\mathbb{T}}\mathrm{Grad}^{k}_{\Gamma_t}\right)^{-1}_{\perp}:R\left({}_{\mathbb{T}}\mathrm{Grad}^{k}_{\Gamma_t}\right)\to D\left({}_{\mathbb{T}}\mathrm{Grad}^{k}_{\Gamma_t}\right)=\mathsf{H}^{k+1}_{\Gamma_t}(\Omega),$$

$$\left({}_{\mathbb{S}}\mathrm{Rot}^{k}_{\mathbb{T},\Gamma_t}\right)^{-1}_{\perp}:R\left({}_{\mathbb{S}}\mathrm{Rot}^{k}_{\mathbb{T},\Gamma_t}\right)\to D\left({}_{\mathbb{S}}\mathrm{Rot}^{k}_{\mathbb{T},\Gamma_t}\right)=\mathsf{H}^{k}_{\mathbb{T},\Gamma_t}(\mathrm{symRot},\Omega),$$

$$\left(\mathrm{divDiv}^{k}_{\mathbb{S},\Gamma_t}\right)^{-1}_{\perp}:R\left(\mathrm{divDiv}^{k}_{\mathbb{S},\Gamma_t}\right)\to D\left(\mathrm{divDiv}^{k}_{\mathbb{S},\Gamma_t}\right)=\mathsf{H}^{k}_{\mathbb{S},\Gamma_t}(\mathrm{divDiv},\Omega),$$

$$\left(\mathrm{divDiv}^{k+1,k}_{\mathbb{S},\Gamma_t}\right)^{-1}_{\perp}:R\left(\mathrm{divDiv}^{k}_{\mathbb{S},\Gamma_t}\right)\to D\left(\mathrm{divDiv}^{k+1,k}_{\mathbb{S},\Gamma_t}\right)=\mathsf{H}^{k+1,k}_{\mathbb{S},\Gamma_t}(\mathrm{divDiv},\Omega)$$

are bounded, and we recall the bounded linear regular decomposition operators

$$
\begin{aligned}
\mathcal{Q}^{k,1}_{{}_{\mathbb{T}}\mathrm{Rot}_{\mathbb{S}},\Gamma_t} &: \mathsf{H}^k_{\mathbb{S},\Gamma_t}(\mathrm{Rot},\Omega) \to \mathsf{H}^{k+1}_{\mathbb{S},\Gamma_t}(\Omega), & \mathcal{Q}^{k,0}_{{}_{\mathbb{T}}\mathrm{Rot}_{\mathbb{S}},\Gamma_t} &: \mathsf{H}^k_{\mathbb{S},\Gamma_t}(\mathrm{Rot},\Omega) \to \mathsf{H}^{k+2}_{\Gamma_t}(\Omega),\\
\mathcal{Q}^{k,1}_{\mathrm{Div}_{\mathbb{T}},\Gamma_t} &: \mathsf{H}^k_{\mathbb{T},\Gamma_t}(\mathrm{Div},\Omega) \to \mathsf{H}^{k+1}_{\mathbb{T},\Gamma_t}(\Omega), & \mathcal{Q}^{k,0}_{\mathrm{Div}_{\mathbb{T}},\Gamma_t} &: \mathsf{H}^k_{\mathbb{T},\Gamma_t}(\mathrm{Div},\Omega) \to \mathsf{H}^{k+1}_{\mathbb{S},\Gamma_t}(\Omega),\\
\mathcal{Q}^{k,1}_{{}_{\mathbb{S}}\mathrm{Rot}_{\mathbb{T}},\Gamma_t} &: \mathsf{H}^k_{\mathbb{T},\Gamma_t}(\mathrm{symRot},\Omega) \to \mathsf{H}^{k+1}_{\mathbb{T},\Gamma_t}(\Omega), & \mathcal{Q}^{k,0}_{{}_{\mathbb{S}}\mathrm{Rot}_{\mathbb{T}},\Gamma_t} &: \mathsf{H}^k_{\mathbb{T},\Gamma_t}(\mathrm{symRot},\Omega) \to \mathsf{H}^{k+1}_{\Gamma_t}(\Omega),\\
\mathcal{Q}^{k,1}_{\mathrm{divDiv}_{\mathbb{S}},\Gamma_t} &: \mathsf{H}^k_{\mathbb{S},\Gamma_t}(\mathrm{divDiv},\Omega) \to \mathsf{H}^{k+2}_{\mathbb{S},\Gamma_t}(\Omega), & \mathcal{Q}^{k,0}_{\mathrm{divDiv}_{\mathbb{S}},\Gamma_t} &: \mathsf{H}^k_{\mathbb{S},\Gamma_t}(\mathrm{divDiv},\Omega) \to \mathsf{H}^{k+1}_{\mathbb{T},\Gamma_t}(\Omega),\\
\mathcal{Q}^{k+1,k,1}_{\mathrm{divDiv}_{\mathbb{S}},\Gamma_t} &: \mathsf{H}^{k+1,k}_{\mathbb{S},\Gamma_t}(\mathrm{divDiv},\Omega) \to \mathsf{H}^{k+2}_{\mathbb{S},\Gamma_t}(\Omega), & \mathcal{Q}^{k+1,k,0}_{\mathrm{divDiv}_{\mathbb{S}},\Gamma_t} &: \mathsf{H}^{k+1,k}_{\mathbb{S},\Gamma_t}(\mathrm{divDiv},\Omega) \to \mathsf{H}^{k+2}_{\mathbb{T},\Gamma_t}(\Omega)
\end{aligned}
$$

from Lemma 3.10. Similar to [1, Theorem 4.18, Theorem 5.2] and [2, Theorem 3.24, Theorem 3.25] (cf. [1, Lemma 2.22, Theorem 2.23]), we obtain the following sequence of results:

Theorem 3.24. (Bounded regular potentials from bounded regular decompositions)**.** *For $k \geq 0$, there exist bounded linear regular potential operators*

$$
\begin{aligned}
\mathcal{P}^k_{{}_{\mathbb{S}}\mathrm{Gradgrad},\Gamma_t} &:= \left({}_{\mathbb{S}}\mathrm{Gradgrad}^k_{\Gamma_t}\right)^{-1}_{\perp} : \mathsf{H}^k_{\mathbb{S},\Gamma_t,0}(\mathrm{Rot},\Omega) \cap \mathcal{H}_{\mathbb{S},\Gamma_t,\Gamma_n,\varepsilon}(\Omega)^{\perp_{\mathsf{L}^2_{\mathbb{S},\varepsilon}(\Omega)}} \to \mathsf{H}^{k+2}_{\Gamma_t}(\Omega),\\
\mathcal{P}^{k+1,k}_{{}_{\mathbb{S}}\mathrm{Gradgrad},\Gamma_t} &:= \left({}_{\mathbb{S}}\mathrm{Gradgrad}^{k+1,k}_{\Gamma_t}\right)^{-1}_{\perp} : \mathsf{H}^k_{\mathbb{S},\Gamma_t,0}(\mathrm{Rot},\Omega) \cap \mathcal{H}_{\mathbb{S},\Gamma_t,\Gamma_n,\varepsilon}(\Omega)^{\perp_{\mathsf{L}^2_{\mathbb{S},\varepsilon}(\Omega)}} \to \mathsf{H}^{k+2}_{\Gamma_t}(\Omega),\\
\mathcal{P}^k_{{}_{\mathbb{T}}\mathrm{Rot}_{\mathbb{S}},\Gamma_t} &:= \mathcal{Q}^{k,1}_{{}_{\mathbb{T}}\mathrm{Rot}_{\mathbb{S}},\Gamma_t}\left({}_{\mathbb{T}}\mathrm{Rot}^k_{\mathbb{S},\Gamma_t}\right)^{-1}_{\perp} : \mathsf{H}^k_{\mathbb{T},\Gamma_t,0}(\mathrm{Div},\Omega) \cap \mathcal{H}_{\mathbb{T},\Gamma_n,\Gamma_t,\mu}(\Omega)^{\perp_{\mathsf{L}^2_{\mathbb{T}}(\Omega)}} \to \mathsf{H}^{k+1}_{\mathbb{S},\Gamma_t}(\Omega),\\
\mathcal{P}^k_{\mathrm{Div}_{\mathbb{T}},\Gamma_t} &:= \mathcal{Q}^{k,1}_{\mathrm{Div}_{\mathbb{T}},\Gamma_t}\left(\mathrm{Div}^k_{\mathbb{T},\Gamma_t}\right)^{-1}_{\perp} : \mathsf{H}^k_{\Gamma_t}(\Omega) \cap \left(\mathbb{RT}_{\Gamma_n}\right)^{\perp_{\mathsf{L}^2(\Omega)}} \to \mathsf{H}^{k+1}_{\mathbb{T},\Gamma_t}(\Omega),\\
\mathcal{P}^k_{{}_{\mathbb{T}}\mathrm{Grad},\Gamma_t} &:= ({}_{\mathbb{T}}\mathrm{Grad}^k_{\Gamma_t})^{-1}_{\perp} : \mathsf{H}^k_{\mathbb{T},\Gamma_t,0}(\mathrm{symRot},\Omega) \cap \mathcal{H}_{\mathbb{T},\Gamma_t,\Gamma_n,\mu}(\Omega)^{\perp_{\mathsf{L}^2_{\mathbb{T},\mu}(\Omega)}} \to \mathsf{H}^{k+1}_{\Gamma_t}(\Omega),\\
\mathcal{P}^k_{{}_{\mathbb{S}}\mathrm{Rot}_{\mathbb{T}},\Gamma_t} &:= \mathcal{Q}^{k,1}_{{}_{\mathbb{S}}\mathrm{Rot}_{\mathbb{T}},\Gamma_t}\left({}_{\mathbb{S}}\mathrm{Rot}^k_{\mathbb{T},\Gamma_t}\right)^{-1}_{\perp} : \mathsf{H}^k_{\mathbb{S},\Gamma_t,0}(\mathrm{divDiv},\Omega) \cap \mathcal{H}_{\mathbb{S},\Gamma_n,\Gamma_t,\varepsilon}(\Omega)^{\perp_{\mathsf{L}^2_{\mathbb{S}}(\Omega)}} \to \mathsf{H}^{k+1}_{\mathbb{T},\Gamma_t}(\Omega),\\
\mathcal{P}^k_{\mathrm{divDiv}_{\mathbb{S}},\Gamma_t} &:= \mathcal{Q}^{k,1}_{\mathrm{divDiv}_{\mathbb{S}},\Gamma_t}\left(\mathrm{divDiv}^k_{\mathbb{S},\Gamma_t}\right)^{-1}_{\perp} : \mathsf{H}^k_{\Gamma_t}(\Omega) \cap \left(\mathbb{P}^1{}_{\Gamma_n}\right)^{\perp_{\mathsf{L}^2(\Omega)}} \to \mathsf{H}^{k+2}_{\mathbb{S},\Gamma_t}(\Omega),\\
\mathcal{P}^{k+1,k}_{\mathrm{divDiv}_{\mathbb{S}},\Gamma_t} &:= \mathcal{Q}^{k+1,k,1}_{\mathrm{divDiv}_{\mathbb{S}},\Gamma_t}\left(\mathrm{divDiv}^{k+1,k}_{\mathbb{S},\Gamma_t}\right)^{-1}_{\perp} : \mathsf{H}^k_{\Gamma_t}(\Omega) \cap \left(\mathbb{P}^1{}_{\Gamma_n}\right)^{\perp_{\mathsf{L}^2(\Omega)}} \to \mathsf{H}^{k+2}_{\mathbb{S},\Gamma_t}(\Omega),
\end{aligned}
$$

such that

$$
\begin{aligned}
\mathrm{Gradgrad}\ \mathcal{P}^{k+1,k}_{{}_{\mathbb{S}}\mathrm{Gradgrad},\Gamma_t} = \mathrm{Gradgrad}\ \mathcal{P}^k_{{}_{\mathbb{S}}\mathrm{Gradgrad},\Gamma_t} &= \mathrm{id}|_{\mathsf{H}^k_{\mathbb{S},\Gamma_t,0}(\mathrm{Rot},\Omega)\cap\mathcal{H}_{\mathbb{S},\Gamma_t,\Gamma_n,\varepsilon}(\Omega)^{\perp_{\mathsf{L}^2_{\mathbb{S},\varepsilon}(\Omega)}}},\\
\mathrm{Rot}\ \mathcal{P}^k_{{}_{\mathbb{T}}\mathrm{Rot}_{\mathbb{S}},\Gamma_t} &= \mathrm{id}|_{\mathsf{H}^k_{\mathbb{T},\Gamma_t,0}(\mathrm{Div},\Omega)\cap\mathcal{H}_{\mathbb{T},\Gamma_n,\Gamma_t,\mu}(\Omega)^{\perp_{\mathsf{L}^2_{\mathbb{T}}(\Omega)}}},\\
\mathrm{Div}\ \mathcal{P}^k_{\mathrm{Div}_{\mathbb{T}},\Gamma_t} &= \mathrm{id}|_{\mathsf{H}^k_{\Gamma_t}(\Omega)\cap(\mathbb{RT}_{\Gamma_n})^{\perp_{\mathsf{L}^2(\Omega)}}},\\
\mathrm{devGrad}\ \mathcal{P}^k_{{}_{\mathbb{T}}\mathrm{Grad},\Gamma_t} &= \mathrm{id}|_{\mathsf{H}^k_{\mathbb{T},\Gamma_t,0}(\mathrm{symRot},\Omega)\cap\mathcal{H}_{\mathbb{T},\Gamma_t,\Gamma_n,\mu}(\Omega)^{\perp_{\mathsf{L}^2_{\mathbb{T},\mu}(\Omega)}}},\\
\mathrm{symRot}\ \mathcal{P}^k_{{}_{\mathbb{S}}\mathrm{Rot}_{\mathbb{T}},\Gamma_t} &= \mathrm{id}|_{\mathsf{H}^k_{\mathbb{S},\Gamma_t,0}(\mathrm{divDiv},\Omega)\cap\mathcal{H}_{\mathbb{S},\Gamma_n,\Gamma_t,\varepsilon}(\Omega)^{\perp_{\mathsf{L}^2_{\mathbb{S}}(\Omega)}}},\\
\mathrm{divDiv}\ \mathcal{P}^{k+1,k}_{\mathrm{divDiv}_{\mathbb{S}},\Gamma_t} = \mathrm{divDiv}\ \mathcal{P}^k_{\mathrm{divDiv}_{\mathbb{S}},\Gamma_t} &= \mathrm{id}|_{\mathsf{H}^k_{\Gamma_t}(\Omega)\cap(\mathbb{P}^1{}_{\Gamma_n})^{\perp_{\mathsf{L}^2(\Omega)}}}.
\end{aligned}
$$

In particular, all potentials in Theorem 3.16 can be chosen such that they depend continuously on the data. $\mathcal{P}^{k+1,k}_{{}_{\mathbb{S}}\mathrm{Gradgrad},\Gamma_t}$, $\mathcal{P}^k_{{}_{\mathbb{S}}\mathrm{Gradgrad},\Gamma_t}$, $\mathcal{P}^k_{{}_{\mathbb{S}}\mathrm{Rot}_{\mathbb{T}},\Gamma_t}$, $\mathcal{P}^k_{\mathrm{Div}_{\mathbb{T}},\Gamma_t}$, $\mathcal{P}^k_{{}_{\mathbb{T}}\mathrm{Grad},\Gamma_t}$, $\mathcal{P}^k_{{}_{\mathbb{T}}\mathrm{Rot}_{\mathbb{S}},\Gamma_t}$, $\mathcal{P}^k_{\mathrm{divDiv}_{\mathbb{S}},\Gamma_t}$, *and* $\mathcal{P}^{k+1,k}_{\mathrm{divDiv}_{\mathbb{S}},\Gamma_t}$ *are right inverses of* Gradgrad, Rot, Div, devGrad, symRot, *and* divDiv, *respectively*.

Theorem 3.25. (Bounded regular decompositions from bounded regular potentials)**.** *For $k \geq 0$, the bounded regular decompositions*

$$
\begin{aligned}
\mathsf{H}^{k}_{\mathbb{S},\Gamma_t}(\mathrm{Rot},\Omega) &= \mathsf{H}^{k+1}_{\mathbb{S},\Gamma_t}(\Omega) + \mathsf{H}^{k}_{\mathbb{S},\Gamma_t,0}(\mathrm{Rot},\Omega) = \mathsf{H}^{k+1}_{\mathbb{S},\Gamma_t}(\Omega) + \mathrm{Gradgrad}\ \mathsf{H}^{k+2}_{\Gamma_t}(\Omega) \\
&= R\left(\widetilde{\mathcal{Q}}^{k,1}_{{}_{\mathbb{T}}\mathrm{Rot}_{\mathbb{S}},\Gamma_t}\right) \dotplus \mathsf{H}^{k}_{\mathbb{S},\Gamma_t,0}(\mathrm{Rot},\Omega) \\
&= R\left(\widetilde{\mathcal{Q}}^{k,1}_{{}_{\mathbb{T}}\mathrm{Rot}_{\mathbb{S}},\Gamma_t}\right) \dotplus R\left(\widetilde{\mathcal{N}}^{k}_{{}_{\mathbb{T}}\mathrm{Rot}_{\mathbb{S}},\Gamma_t}\right), \\
\mathsf{H}^{k}_{\mathbb{T},\Gamma_t}(\mathrm{Div},\Omega) &= \mathsf{H}^{k+1}_{\mathbb{T},\Gamma_t}(\Omega) + \mathsf{H}^{k}_{\mathbb{T},\Gamma_t,0}(\mathrm{Div},\Omega) = \mathsf{H}^{k+1}_{\mathbb{T},\Gamma_t}(\Omega) + \mathrm{Rot}\ \mathsf{H}^{k+1}_{\mathbb{S},\Gamma_t}(\Omega) \\
&= R\left(\widetilde{\mathcal{Q}}^{k,1}_{\mathrm{Div}_{\mathbb{T}},\Gamma_t}\right) \dotplus \mathsf{H}^{k}_{\mathbb{T},\Gamma_t,0}(\mathrm{Div},\Omega) \\
&= R\left(\widetilde{\mathcal{Q}}^{k,1}_{\mathrm{Div}_{\mathbb{T}},\Gamma_t}\right) \dotplus R\left(\widetilde{\mathcal{N}}^{k}_{\mathrm{Div}_{\mathbb{T}},\Gamma_t}\right), \\
\mathsf{H}^{k}_{\mathbb{T},\Gamma_t}(\mathrm{symRot},\Omega) &= \mathsf{H}^{k+1}_{\mathbb{T},\Gamma_t}(\Omega) + \mathsf{H}^{k}_{\mathbb{T},\Gamma_t,0}(\mathrm{symRot},\Omega) = \mathsf{H}^{k+1}_{\mathbb{T},\Gamma_t}(\Omega) + \mathrm{devGrad}\ \mathsf{H}^{k+1}_{\Gamma_t}(\Omega) \\
&= R\left(\widetilde{\mathcal{Q}}^{k,1}_{{}_{\mathbb{S}}\mathrm{Rot}_{\mathbb{T}},\Gamma_t}\right) \dotplus \mathsf{H}^{k}_{\mathbb{T},\Gamma_t,0}(\mathrm{symRot},\Omega) \\
&= R\left(\widetilde{\mathcal{Q}}^{k,1}_{{}_{\mathbb{S}}\mathrm{Rot}_{\mathbb{T}},\Gamma_t}\right) \dotplus R\left(\widetilde{\mathcal{N}}^{k}_{{}_{\mathbb{S}}\mathrm{Rot}_{\mathbb{T}},\Gamma_t}\right), \\
\mathsf{H}^{k}_{\mathbb{S},\Gamma_t}(\mathrm{divDiv},\Omega) &= \mathsf{H}^{k+2}_{\mathbb{S},\Gamma_t}(\Omega) + \mathsf{H}^{k}_{\mathbb{S},\Gamma_t,0}(\mathrm{divDiv},\Omega) = \mathsf{H}^{k+2}_{\mathbb{S},\Gamma_t}(\Omega) + \mathrm{symRot}\ \mathsf{H}^{k+1}_{\mathbb{T},\Gamma_t}(\Omega) \\
&= R\left(\widetilde{\mathcal{Q}}^{k,1}_{\mathrm{divDiv}_{\mathbb{S}},\Gamma_t}\right) \dotplus \mathsf{H}^{k}_{\mathbb{S},\Gamma_t,0}(\mathrm{divDiv},\Omega) \\
&= R\left(\widetilde{\mathcal{Q}}^{k,1}_{\mathrm{divDiv}_{\mathbb{S}},\Gamma_t}\right) \dotplus R\left(\widetilde{\mathcal{N}}^{k}_{\mathrm{divDiv}_{\mathbb{S}},\Gamma_t}\right), \\
\mathsf{H}^{k+1,k}_{\mathbb{S},\Gamma_t}(\mathrm{divDiv},\Omega) &= \mathsf{H}^{k+2}_{\mathbb{S},\Gamma_t}(\Omega) + \mathsf{H}^{k+1}_{\mathbb{S},\Gamma_t,0}(\mathrm{divDiv},\Omega) = \mathsf{H}^{k+2}_{\mathbb{S},\Gamma_t}(\Omega) + \mathrm{symRot}\ \mathsf{H}^{k+2}_{\mathbb{T},\Gamma_t}(\Omega) \\
&= R\left(\widetilde{\mathcal{Q}}^{k+1,k,1}_{\mathrm{divDiv}_{\mathbb{S}},\Gamma_t}\right) \dotplus \mathsf{H}^{k+1}_{\mathbb{S},\Gamma_t,0}(\mathrm{divDiv},\Omega) \\
&= R\left(\widetilde{\mathcal{Q}}^{k+1,k,1}_{\mathrm{divDiv}_{\mathbb{S}},\Gamma_t}\right) \dotplus R\left(\widetilde{\mathcal{N}}^{k+1,k}_{\mathrm{divDiv}_{\mathbb{S}},\Gamma_t}\right)
\end{aligned}
$$

hold with bounded linear regular decomposition operators

$$
\begin{aligned}
\widetilde{\mathcal{Q}}^{k,1}_{{}_{\mathbb{T}}\mathrm{Rot}_{\mathbb{S}},\Gamma_t} &:= \mathcal{P}^{k}_{{}_{\mathbb{T}}\mathrm{Rot}_{\mathbb{S}},\Gamma_t}\, {}_{\mathbb{T}}\mathrm{Rot}^{k}_{\mathbb{S},\Gamma_t}, : \mathsf{H}^{k}_{\mathbb{S},\Gamma_t}(\mathrm{Rot},\Omega) \to \mathsf{H}^{k+1}_{\mathbb{S},\Gamma_t}(\Omega), \\
\widetilde{\mathcal{Q}}^{k,1}_{\mathrm{Div}_{\mathbb{T}},\Gamma_t} &:= \mathcal{P}^{k}_{\mathrm{Div}_{\mathbb{T}},\Gamma_t} \mathrm{Div}^{k}_{\mathbb{T},\Gamma_t} : \mathsf{H}^{k}_{\mathbb{T},\Gamma_t}(\mathrm{Div},\Omega) \to \mathsf{H}^{k+1}_{\mathbb{T},\Gamma_t}(\Omega), \\
\widetilde{\mathcal{Q}}^{k,1}_{{}_{\mathbb{S}}\mathrm{Rot}_{\mathbb{T}},\Gamma_t} &:= \mathcal{P}^{k}_{{}_{\mathbb{S}}\mathrm{Rot}_{\mathbb{T}},\Gamma_t} {}_{\mathbb{S}}\mathrm{Rot}^{k}_{\mathbb{T},\Gamma_t} : \mathsf{H}^{k}_{\mathbb{T},\Gamma_t}(\mathrm{symRot},\Omega) \to \mathsf{H}^{k+1}_{\mathbb{T},\Gamma_t}(\Omega), \\
\widetilde{\mathcal{Q}}^{k,1}_{\mathrm{divDiv}_{\mathbb{S}},\Gamma_t} &:= \mathcal{P}^{k}_{\mathrm{divDiv}_{\mathbb{S}},\Gamma_t} \mathrm{divDiv}^{k}_{\mathbb{S},\Gamma_t} : \mathsf{H}^{k}_{\mathbb{S},\Gamma_t}(\mathrm{divDiv},\Omega) \to \mathsf{H}^{k+2}_{\mathbb{S},\Gamma_t}(\Omega), \\
\widetilde{\mathcal{Q}}^{k+1,k,1}_{\mathrm{divDiv}_{\mathbb{S}},\Gamma_t} &:= \mathcal{P}^{k+1,k}_{\mathrm{divDiv}_{\mathbb{S}},\Gamma_t} \mathrm{divDiv}^{k+1,k}_{\mathbb{S},\Gamma_t} : \mathsf{H}^{k+1,k}_{\mathbb{S},\Gamma_t}(\mathrm{divDiv},\Omega) \to \mathsf{H}^{k+2}_{\mathbb{S},\Gamma_t}(\Omega), \\
\widetilde{\mathcal{N}}^{k}_{{}_{\mathbb{T}}\mathrm{Rot}_{\mathbb{S}},\Gamma_t} &: \mathsf{H}^{k}_{\mathbb{S},\Gamma_t}(\mathrm{Rot},\Omega) \to \mathsf{H}^{k}_{\mathbb{S},\Gamma_t,0}(\mathrm{Rot},\Omega), \\
\widetilde{\mathcal{N}}^{k}_{\mathrm{Div}_{\mathbb{T}},\Gamma_t} &: \mathsf{H}^{k}_{\mathbb{T},\Gamma_t}(\mathrm{Div},\Omega) \to \mathsf{H}^{k}_{\mathbb{T},\Gamma_t,0}(\mathrm{Div},\Omega), \\
\widetilde{\mathcal{N}}^{k}_{{}_{\mathbb{S}}\mathrm{Rot}_{\mathbb{T}},\Gamma_t} &: \mathsf{H}^{k}_{\mathbb{T},\Gamma_t}(\mathrm{symRot},\Omega) \to \mathsf{H}^{k}_{\mathbb{T},\Gamma_t,0}(\mathrm{symRot},\Omega), \\
\widetilde{\mathcal{N}}^{k}_{\mathrm{divDiv}_{\mathbb{S}},\Gamma_t} &: \mathsf{H}^{k}_{\mathbb{S},\Gamma_t}(\mathrm{divDiv},\Omega) \to \mathsf{H}^{k}_{\mathbb{S},\Gamma_t,0}(\mathrm{divDiv},\Omega), \\
\widetilde{\mathcal{N}}^{k+1,k}_{\mathrm{divDiv}_{\mathbb{S}},\Gamma_t} &: \mathsf{H}^{k+1,k}_{\mathbb{S},\Gamma_t}(\mathrm{divDiv},\Omega) \to \mathsf{H}^{k+1}_{\mathbb{S},\Gamma_t,0}(\mathrm{divDiv},\Omega)
\end{aligned}
$$

satisfying

$$
\begin{aligned}
\mathrm{id}_{\mathsf{H}^{k}_{\mathbb{S},\Gamma_t}(\mathrm{Rot},\Omega)} &= \widetilde{\mathcal{Q}}^{k,1}_{{}_{\mathbb{T}}\mathrm{Rot}_{\mathbb{S}},\Gamma_t} + \widetilde{\mathcal{N}}^{k}_{{}_{\mathbb{T}}\mathrm{Rot}_{\mathbb{S}},\Gamma_t},\\
\mathrm{id}_{\mathsf{H}^{k}_{\mathbb{T},\Gamma_t}(\mathrm{Div},\Omega)} &= \widetilde{\mathcal{Q}}^{k,1}_{\mathrm{Div}_{\mathbb{T}},\Gamma_t} + \widetilde{\mathcal{N}}^{k}_{\mathrm{Div}_{\mathbb{T}},\Gamma_t},\\
\mathrm{id}_{\mathsf{H}^{k}_{\mathbb{T},\Gamma_t}(\mathrm{symRot},\Omega)} &= \widetilde{\mathcal{Q}}^{k,1}_{{}_{\mathbb{S}}\mathrm{Rot}_{\mathbb{T}},\Gamma_t} + \widetilde{\mathcal{N}}^{k}_{{}_{\mathbb{S}}\mathrm{Rot}_{\mathbb{T}},\Gamma_t},\\
\mathrm{id}_{\mathsf{H}^{k}_{\mathbb{S},\Gamma_t}(\mathrm{divDiv},\Omega)} &= \widetilde{\mathcal{Q}}^{k,1}_{\mathrm{divDiv}_{\mathbb{S}},\Gamma_t} + \widetilde{\mathcal{N}}^{k}_{\mathrm{divDiv}_{\mathbb{S}},\Gamma_t},\\
\mathrm{id}_{\mathsf{H}^{k+1,k}_{\mathbb{S},\Gamma_t}(\mathrm{divDiv},\Omega)} &= \widetilde{\mathcal{Q}}^{k+1,k,1}_{\mathrm{divDiv}_{\mathbb{S}},\Gamma_t} + \widetilde{\mathcal{N}}^{k+1,k}_{\mathrm{divDiv}_{\mathbb{S}},\Gamma_t}.
\end{aligned}
$$

Corollary 3.26. (Bounded regular kernel decompositions)**.** *For $k \geq 0$, the bounded regular kernel decompositions*

$$
\begin{aligned}
\mathsf{H}^{k}_{\mathbb{S},\Gamma_t,0}(\mathrm{Rot},\Omega) &= \mathsf{H}^{k+1}_{\mathbb{S},\Gamma_t,0}(\mathrm{Rot},\Omega) + \mathrm{Gradgrad}\ \mathsf{H}^{k+2}_{\Gamma_t}(\Omega),\\
\mathsf{H}^{k}_{\mathbb{T},\Gamma_t,0}(\mathrm{Div},\Omega) &= \mathsf{H}^{k+1}_{\mathbb{T},\Gamma_t,0}(\mathrm{Div},\Omega) + \mathrm{Rot}\ \mathsf{H}^{k+1}_{\mathbb{S},\Gamma_t}(\Omega),\\
\mathsf{H}^{k}_{\mathbb{T},\Gamma_t,0}(\mathrm{symRot}\Omega) &= \mathsf{H}^{k+1}_{\mathbb{T},\Gamma_t,0}(\mathrm{symRot},\Omega) + \mathrm{devGrad}\ \mathsf{H}^{k+1}_{\Gamma_t}(\Omega),\\
\mathsf{H}^{k}_{\mathbb{S},\Gamma_t,0}(\mathrm{divDiv},\Omega) &= \mathsf{H}^{k+2}_{\mathbb{S},\Gamma_t,0}(\mathrm{divDiv},\Omega) + \mathrm{symRot}\ \mathsf{H}^{k+1}_{\mathbb{T},\Gamma_t}(\Omega)
\end{aligned}
$$

hold.

As in [2, Remark 3.27, Theorem 3.28] and [1, Theorem 4.18, Remark 4.19, Theorem 5.2, Remark 5.3] (cf. [1, Sections 2.3 and 2.4]), there is a collection of results about the bounded regular decomposition operators; see Remark D.1 and Remark D.2 of Appendix D.

Corollary 3.26 shows the following:

Corollary 3.27. (Bounded regular higher order kernel decompositions)**.** *For $k, \ell \geq 0$, the bounded regular kernel decompositions*

$$
\begin{aligned}
N\left({}_{\mathbb{T}}\mathrm{Rot}^{k}_{\mathbb{S},\Gamma_t}\right) &= \mathsf{H}^{k}_{\mathbb{S},\Gamma_t,0}(\mathrm{Rot},\Omega) = \mathsf{H}^{\ell}_{\mathbb{S},\Gamma_t,0}(\mathrm{Rot},\Omega) + \mathrm{Gradgrad}\ \mathsf{H}^{k+2}_{\Gamma_t}(\Omega),\\
N\left(\mathrm{Div}^{k}_{\mathbb{T},\Gamma_t}\right) &= \mathsf{H}^{k}_{\mathbb{T},\Gamma_t,0}(\mathrm{Div},\Omega) = \mathsf{H}^{\ell}_{\mathbb{T},\Gamma_t,0}(\mathrm{Div},\Omega) + \mathrm{Rot}\ \mathsf{H}^{k+1}_{\mathbb{S},\Gamma_t}(\Omega),\\
N\left({}_{\mathbb{S}}\mathrm{Rot}^{k}_{\mathbb{T},\Gamma_t}\right) &= \mathsf{H}^{k}_{\mathbb{T},\Gamma_t,0}(\mathrm{symRot},\Omega) = \mathsf{H}^{\ell}_{\mathbb{T},\Gamma_t,0}(\mathrm{symRot},\Omega) + \mathrm{devGrad}\ \mathsf{H}^{k+1}_{\Gamma_t}(\Omega),\\
N\left(\mathrm{divDiv}^{k}_{\mathbb{S},\Gamma_t}\right) &= \mathsf{H}^{k}_{\mathbb{S},\Gamma_t,0}(\mathrm{divDiv},\Omega) = \mathsf{H}^{\ell}_{\mathbb{S},\Gamma_t,0}(\mathrm{divDiv},\Omega) + \mathrm{symRot}\ \mathsf{H}^{k+1}_{\mathbb{T},\Gamma_t}(\Omega)
\end{aligned}
$$

hold. In particular, for $k = 0$ and all $\ell \geq 0$

$$
\begin{aligned}
N\left({}_{\mathbb{T}}\mathrm{Rot}_{\mathbb{S},\Gamma_t}\right) &= \mathsf{H}_{\mathbb{S},\Gamma_t,0}(\mathrm{Rot},\Omega) = \mathsf{H}^{\ell}_{\mathbb{S},\Gamma_t,0}(\mathrm{Rot},\Omega) + \mathrm{Gradgrad}\ \mathsf{H}^{2}_{\Gamma_t}(\Omega),\\
N(\mathrm{Div}_{\mathbb{T},\Gamma_t}) &= \mathsf{H}_{\mathbb{T},\Gamma_t,0}(\mathrm{Div},\Omega) = \mathsf{H}^{\ell}_{\mathbb{T},\Gamma_t,0}(\mathrm{Div},\Omega) + \mathrm{Rot}\ \mathsf{H}^{1}_{\mathbb{S},\Gamma_t}(\Omega),\\
N\left({}_{\mathbb{S}}\mathrm{Rot}_{\mathbb{T},\Gamma_t}\right) &= \mathsf{H}_{\mathbb{T},\Gamma_t,0}(\mathrm{symRot},\Omega) = \mathsf{H}^{\ell}_{\mathbb{T},\Gamma_t,0}(\mathrm{symRot},\Omega) + \mathrm{devGrad}\ \mathsf{H}^{1}_{\Gamma_t}(\Omega),\\
N\left(\mathrm{divDiv}_{\mathbb{S},\Gamma_t}\right) &= \mathsf{H}_{\mathbb{S},\Gamma_t,0}(\mathrm{divDiv},\Omega) = \mathsf{H}^{\ell}_{\mathbb{S},\Gamma_t,0}(\mathrm{divDiv},\Omega) + \mathrm{symRot}\ \mathsf{H}^{1}_{\mathbb{T},\Gamma_t}(\Omega).
\end{aligned}
$$

3.4 Dirichlet/Neumann fields

From Theorem 3.15 (iv), we recall the slightly modified orthonormal Helmholtz type decompositions

$$
\begin{aligned}
\mathsf{L}^2_{\mathbb{S},\varepsilon}(\Omega) &= R\left({}_{\mathbb{S}}\mathrm{Gradgrad}_{\Gamma_t}\right) \oplus_{\mathsf{L}^2_{\mathbb{S},\varepsilon}(\Omega)} N\left(\mathrm{divDiv}_{\mathbb{S},\Gamma_n}\varepsilon\right)\\
&= N\left({}_{\mathbb{T}}\mathrm{Rot}_{\mathbb{S},\Gamma_t}\right) \oplus_{\mathsf{L}^2_{\mathbb{S},\varepsilon}(\Omega)} R\left(\varepsilon^{-1}{}_{\mathbb{S}}\mathrm{Rot}_{\mathbb{T},\Gamma_n}\right)\\
&= R\left({}_{\mathbb{S}}\mathrm{Gradgrad}_{\Gamma_t}\right) \oplus_{\mathsf{L}^2_{\mathbb{S},\varepsilon}(\Omega)} \mathcal{H}_{\mathbb{S},\Gamma_t,\Gamma_n,\varepsilon}(\Omega) \oplus_{\mathsf{L}^2_{\mathbb{S},\varepsilon}(\Omega)} R\left(\varepsilon^{-1}{}_{\mathbb{S}}\mathrm{Rot}_{\mathbb{T},\Gamma_n}\right),\\
N\left({}_{\mathbb{T}}\mathrm{Rot}_{\mathbb{S},\Gamma_t}\right) &= R\left({}_{\mathbb{S}}\mathrm{Gradgrad}_{\Gamma_t}\right) \oplus_{\mathsf{L}^2_{\mathbb{S},\varepsilon}(\Omega)} \mathcal{H}_{\mathbb{S},\Gamma_t,\Gamma_n,\varepsilon}(\Omega),\\
N\left(\mathrm{divDiv}_{\mathbb{S},\Gamma_n}\varepsilon\right) &= \mathcal{H}_{\mathbb{S},\Gamma_t,\Gamma_n,\varepsilon}(\Omega) \oplus_{\mathsf{L}^2_{\mathbb{S},\varepsilon}(\Omega)} R\left(\varepsilon^{-1}{}_{\mathbb{S}}\mathrm{Rot}_{\mathbb{T},\Gamma_n}\right),\\
\mathsf{L}^2_{\mathbb{T},\mu}(\Omega) &= R\left({}_{\mathbb{T}}\mathrm{Grad}_{\Gamma_t}\right) \oplus_{\mathsf{L}^2_{\mathbb{T},\mu}(\Omega)} N\left(\mathrm{Div}_{\mathbb{T},\Gamma_n}\mu\right)\\
&= N\left({}_{\mathbb{S}}\mathrm{Rot}_{\mathbb{T},\Gamma_t}\right) \oplus_{\mathsf{L}^2_{\mathbb{T},\mu}(\Omega)} R\left(\mu^{-1}{}_{\mathbb{T}}\mathrm{Rot}_{\mathbb{S},\Gamma_n}\right)\\
&= R\left({}_{\mathbb{T}}\mathrm{Grad}_{\Gamma_t}\right) \oplus_{\mathsf{L}^2_{\mathbb{T},\mu}(\Omega)} \mathcal{H}_{\mathbb{T},\Gamma_t,\Gamma_n,\mu}(\Omega) \oplus_{\mathsf{L}^2_{\mathbb{T},\mu}(\Omega)} R\left(\mu^{-1}{}_{\mathbb{T}}\mathrm{Rot}_{\mathbb{S},\Gamma_n}\right),\\
N\left({}_{\mathbb{S}}\mathrm{Rot}_{\mathbb{T},\Gamma_t}\right) &= R\left({}_{\mathbb{T}}\mathrm{Grad}_{\Gamma_t}\right) \oplus_{\mathsf{L}^2_{\mathbb{T},\mu}(\Omega)} \mathcal{H}_{\mathbb{T},\Gamma_t,\Gamma_n,\mu}(\Omega),\\
N\left(\mathrm{Div}_{\mathbb{T},\Gamma_n}\mu\right) &= \mathcal{H}_{\mathbb{T},\Gamma_t,\Gamma_n,\mu}(\Omega) \oplus_{\mathsf{L}^2_{\mathbb{T},\mu}(\Omega)} R\left(\mu^{-1}{}_{\mathbb{T}}\mathrm{Rot}_{\mathbb{S},\Gamma_n}\right).
\end{aligned}
\tag{14}
$$

Let us denote the $\mathsf{L}^2_{\mathbb{S},\varepsilon}(\Omega)$- and $\mathsf{L}^2_{\mathbb{T},\mu}(\Omega)$-orthonormal projectors onto $N\left(\mathrm{divDiv}_{\mathbb{S},\Gamma_n}\varepsilon\right)$, $N({}_{\mathbb{T}}\mathrm{Rot}_{\mathbb{S},\Gamma_t})$ and $N\left(\mathrm{Div}_{\mathbb{T},\Gamma_n}\mu\right)$, $N\left({}_{\mathbb{S}}\mathrm{Rot}_{\mathbb{T},\Gamma_t}\right)$ by

$$
\begin{aligned}
\pi_{N(\mathrm{divDiv}_{\mathbb{S},\Gamma_n}\varepsilon)} &: \mathsf{L}^2_{\mathbb{S},\varepsilon}(\Omega) \to N\left(\mathrm{divDiv}_{\mathbb{S},\Gamma_n}\varepsilon\right), & \pi_{N(\mathrm{Div}_{\mathbb{T},\Gamma_n}\mu)} &: \mathsf{L}^2_{\mathbb{T},\mu}(\Omega) \to N\left(\mathrm{Div}_{\mathbb{T},\Gamma_n}\mu\right),\\
\pi_{N({}_{\mathbb{T}}\mathrm{Rot}_{\mathbb{S},\Gamma_t})} &: \mathsf{L}^2_{\mathbb{S},\varepsilon}(\Omega) \to N\left({}_{\mathbb{T}}\mathrm{Rot}_{\mathbb{S},\Gamma_t}\right), & \pi_{N({}_{\mathbb{S}}\mathrm{Rot}_{\mathbb{T},\Gamma_t})} &: \mathsf{L}^2_{\mathbb{T},\mu}(\Omega) \to N\left({}_{\mathbb{S}}\mathrm{Rot}_{\mathbb{T},\Gamma_t}\right),
\end{aligned}
$$

respectively. Then

$$
\begin{aligned}
\pi_{N(\mathrm{divDiv}_{\mathbb{S},\Gamma_n}\varepsilon)}|_{N({}_{\mathbb{T}}\mathrm{Rot}_{\mathbb{S},\Gamma_t})} &: N\left({}_{\mathbb{T}}\mathrm{Rot}_{\mathbb{S},\Gamma_t}\right) \to \mathcal{H}_{\mathbb{S},\Gamma_t,\Gamma_n,\varepsilon}(\Omega),\\
\pi_{N({}_{\mathbb{T}}\mathrm{Rot}_{\mathbb{S},\Gamma_t})}|_{N(\mathrm{divDiv}_{\mathbb{S},\Gamma_n}\varepsilon)} &: N\left(\mathrm{divDiv}_{\mathbb{S},\Gamma_n}\varepsilon\right) \to \mathcal{H}_{\mathbb{S},\Gamma_t,\Gamma_n,\varepsilon}(\Omega),\\
\pi_{N(\mathrm{Div}_{\mathbb{T},\Gamma_n}\mu)}|_{N({}_{\mathbb{S}}\mathrm{Rot}_{\mathbb{T},\Gamma_t})} &: N\left({}_{\mathbb{S}}\mathrm{Rot}_{\mathbb{T},\Gamma_t}\right) \to \mathcal{H}_{\mathbb{T},\Gamma_t,\Gamma_n,\mu}(\Omega),\\
\pi_{N({}_{\mathbb{S}}\mathrm{Rot}_{\mathbb{T},\Gamma_t})}|_{N(\mathrm{Div}_{\mathbb{T},\Gamma_n}\mu)} &: N\left(\mathrm{Div}_{\mathbb{T},\Gamma_n}\mu\right) \to \mathcal{H}_{\mathbb{T},\Gamma_t,\Gamma_n,\mu}(\Omega)
\end{aligned}
$$

are onto. Moreover,

$$
\begin{aligned}
\pi_{N(\mathrm{divDiv}_{\mathbb{S},\Gamma_n}\varepsilon)}|_{R({}_{\mathbb{S}}\mathrm{Gradgrad}_{\Gamma_t})} &= 0, & \pi_{N(\mathrm{Div}_{\mathbb{T},\Gamma_n}\mu)}|_{R({}_{\mathbb{T}}\mathrm{Grad}_{\Gamma_t})} &= 0,\\
\pi_{N({}_{\mathbb{T}}\mathrm{Rot}_{\mathbb{S},\Gamma_t})}|_{R(\varepsilon^{-1}{}_{\mathbb{S}}\mathrm{Rot}_{\mathbb{T},\Gamma_n})} &= 0, & \pi_{N({}_{\mathbb{S}}\mathrm{Rot}_{\mathbb{T},\Gamma_t})}|_{R(\mu^{-1}{}_{\mathbb{T}}\mathrm{Rot}_{\mathbb{S},\Gamma_n})} &= 0,\\
\pi_{N(\mathrm{divDiv}_{\mathbb{S},\Gamma_n}\varepsilon)}|_{\mathcal{H}_{\mathbb{S},\Gamma_t,\Gamma_n,\varepsilon}(\Omega)} &= \mathrm{id}_{\mathcal{H}_{\mathbb{S},\Gamma_t,\Gamma_n,\varepsilon}(\Omega)}, & \pi_{N(\mathrm{Div}_{\mathbb{T},\Gamma_n}\mu)}|_{\mathcal{H}_{\mathbb{T},\Gamma_t,\Gamma_n,\mu}(\Omega)} &= \mathrm{id}_{\mathcal{H}_{\mathbb{T},\Gamma_t,\Gamma_n,\mu}(\Omega)},\\
\pi_{N({}_{\mathbb{T}}\mathrm{Rot}_{\mathbb{S},\Gamma_t})}|_{\mathcal{H}_{\mathbb{S},\Gamma_t,\Gamma_n,\varepsilon}(\Omega)} &= \mathrm{id}_{\mathcal{H}_{\mathbb{S},\Gamma_t,\Gamma_n,\varepsilon}(\Omega)}, & \pi_{N({}_{\mathbb{S}}\mathrm{Rot}_{\mathbb{T},\Gamma_t})}|_{\mathcal{H}_{\mathbb{T},\Gamma_t,\Gamma_n,\mu}(\Omega)} &= \mathrm{id}_{\mathcal{H}_{\mathbb{T},\Gamma_t,\Gamma_n,\mu}(\Omega)}.
\end{aligned}
$$

Therefore, by Corollary 3.27 and for all $\ell \geq 0$,

$$
\begin{aligned}
\mathcal{H}_{\mathbb{S},\Gamma_t,\Gamma_n,\varepsilon}(\Omega) &= \pi_{N(\mathrm{divDiv}_{\mathbb{S},\Gamma_n}\varepsilon)} N\left({}_{\mathbb{T}}\mathrm{Rot}_{\mathbb{S},\Gamma_t}\right) = \pi_{N(\mathrm{divDiv}_{\mathbb{S},\Gamma_n}\varepsilon)} \mathsf{H}^{\ell}_{\mathbb{S},\Gamma_t,0}(\mathrm{Rot},\Omega),\\
\mathcal{H}_{\mathbb{S},\Gamma_t,\Gamma_n,\varepsilon}(\Omega) &= \pi_{N({}_{\mathbb{T}}\mathrm{Rot}_{\mathbb{S},\Gamma_t})} N\left(\mathrm{divDiv}_{\mathbb{S},\Gamma_n}\varepsilon\right) = \pi_{N({}_{\mathbb{T}}\mathrm{Rot}_{\mathbb{S},\Gamma_t})} \varepsilon^{-1}\mathsf{H}^{\ell}_{\mathbb{S},\Gamma_n,0}(\mathrm{divDiv},\Omega),\\
\mathcal{H}_{\mathbb{T},\Gamma_t,\Gamma_n,\mu}(\Omega) &= \pi_{N(\mathrm{Div}_{\mathbb{T},\Gamma_n}\mu)} N\left({}_{\mathbb{S}}\mathrm{Rot}_{\mathbb{T},\Gamma_t}\right) = \pi_{N(\mathrm{Div}_{\mathbb{T},\Gamma_n}\mu)} \mathsf{H}^{\ell}_{\mathbb{T},\Gamma_t,0}(\mathrm{symRot},\Omega),\\
\mathcal{H}_{\mathbb{T},\Gamma_t,\Gamma_n,\mu}(\Omega) &= \pi_{N({}_{\mathbb{S}}\mathrm{Rot}_{\mathbb{T},\Gamma_t})} N\left(\mathrm{Div}_{\mathbb{T},\Gamma_n}\mu\right) = \pi_{N({}_{\mathbb{S}}\mathrm{Rot}_{\mathbb{T},\Gamma_t})} \mu^{-1}\mathsf{H}^{\ell}_{\mathbb{T},\Gamma_n,0}(\mathrm{Div},\Omega),
\end{aligned}
$$

where we have used

$$
N\left(\mathrm{divDiv}_{\mathbb{S},\Gamma_n}\varepsilon\right) = \varepsilon^{-1}\mathsf{H}_{\mathbb{S},\Gamma_n,0}(\mathrm{divDiv},\Omega), \qquad N\left(\mathrm{Div}_{\mathbb{T},\Gamma_n}\mu\right) = \mu^{-1}\mathsf{H}_{\mathbb{T},\Gamma_n,0}(\mathrm{Div},\Omega).
$$

Hence with

$$\mathsf{H}^{\infty}_{\mathbb{S},\Gamma_t,0}(\mathrm{Rot},\Omega) := \bigcap_{k\geq 0}\mathsf{H}^{k}_{\mathbb{S},\Gamma_t,0}(\mathrm{Rot},\Omega), \qquad \mathsf{H}^{\infty}_{\mathbb{S},\Gamma_n,0}(\mathrm{divDiv},\Omega) := \bigcap_{k\geq 0}\mathsf{H}^{k}_{\mathbb{S},\Gamma_n,0}(\mathrm{divDiv},\Omega),$$
$$\mathsf{H}^{\infty}_{\mathbb{T},\Gamma_t,0}(\mathrm{symRot},\Omega) := \bigcap_{k\geq 0}\mathsf{H}^{k}_{\mathbb{T},\Gamma_t,0}(\mathrm{symRot},\Omega), \qquad \mathsf{H}^{\infty}_{\mathbb{T},\Gamma_n,0}(\mathrm{Div},\Omega) := \bigcap_{k\geq 0}\mathsf{H}^{k}_{\mathbb{T},\Gamma_n,0}(\mathrm{Div},\Omega),$$

and with the finite numbers

$$d_{\Omega,\mathbb{S},\Gamma_t} := \dim \mathcal{H}_{\mathbb{S},\Gamma_t,\Gamma_n,\varepsilon}(\Omega), \qquad d_{\Omega,\mathbb{T},\Gamma_t} := \dim \mathcal{H}_{\mathbb{T},\Gamma_t,\Gamma_n,\mu}(\Omega),$$

we get the following result:

Theorem 3.28. (Smooth pre-bases of Dirichlet/Neumann fields)**.** *It holds*

$$\pi_{N(\mathrm{divDiv}_{\mathbb{S},\Gamma_n}\varepsilon)}\mathsf{H}^{\infty}_{\mathbb{S},\Gamma_t,0}(\mathrm{Rot},\Omega) = \mathcal{H}_{\mathbb{S},\Gamma_t,\Gamma_n,\varepsilon}(\Omega) = \pi_{N({}_{\mathbb{T}}\mathrm{Rot}_{\mathbb{S},\Gamma_t})}\varepsilon^{-1}\mathsf{H}^{\infty}_{\mathbb{S},\Gamma_n,0}(\mathrm{divDiv},\Omega),$$
$$\pi_{N(\mathrm{Div}_{\mathbb{T},\Gamma_n}\mu)}\mathsf{H}^{\infty}_{\mathbb{T},\Gamma_t,0}(\mathrm{symRot},\Omega) = \mathcal{H}_{\mathbb{T},\Gamma_t,\Gamma_n,\mu}(\Omega) = \pi_{N({}_{\mathbb{S}}\mathrm{Rot}_{\mathbb{T},\Gamma_t})}\mu^{-1}\mathsf{H}^{\infty}_{\mathbb{T},\Gamma_n,0}(\mathrm{Div},\Omega).$$

Moreover, there exist smooth ${}_{\mathbb{T}}\mathrm{Rot}_{\mathbb{S},\Gamma_t}$ *and* $\mathrm{divDiv}_{\mathbb{S},\Gamma_n}$ *pre-bases of* $\mathcal{H}_{\mathbb{S},\Gamma_t,\Gamma_n,\varepsilon}(\Omega)$ *and smooth* ${}_{\mathbb{S}}\mathrm{Rot}_{\mathbb{T},\Gamma_t}$ *and* $\mathrm{Div}_{\mathbb{T},\Gamma_n}$ *pre-bases of* $\mathcal{H}_{\mathbb{T},\Gamma_t,\Gamma_n,\mu}(\Omega)$*; that is, there are linear independent smooth fields*

$$\mathcal{B}^{{}_{\mathbb{T}}\mathrm{Rot}_{\mathbb{S},\Gamma_t}}(\Omega) := \left\{B_{\ell}^{{}_{\mathbb{T}}\mathrm{Rot}_{\mathbb{S},\Gamma_t}}\right\}_{\ell=1}^{d_{\Omega,\mathbb{S},\Gamma_t}} \subset \mathsf{H}^{\infty}_{\mathbb{S},\Gamma_t,0}(\mathrm{Rot},\Omega),$$
$$\mathcal{B}^{\mathrm{divDiv}_{\mathbb{S},\Gamma_n}}(\Omega) := \left\{B_{\ell}^{\mathrm{divDiv}_{\mathbb{S},\Gamma_n}}\right\}_{\ell=1}^{d_{\Omega,\mathbb{S},\Gamma_t}} \subset \mathsf{H}^{\infty}_{\mathbb{S},\Gamma_n,0}(\mathrm{divDiv},\Omega),$$
$$\mathcal{B}^{{}_{\mathbb{S}}\mathrm{Rot}_{\mathbb{T},\Gamma_t}}(\Omega) := \left\{B_{\ell}^{{}_{\mathbb{S}}\mathrm{Rot}_{\mathbb{T},\Gamma_t}}\right\}_{\ell=1}^{d_{\Omega,\mathbb{T},\Gamma_t}} \subset \mathsf{H}^{\infty}_{\mathbb{T},\Gamma_t,0}(\mathrm{symRot},\Omega),$$
$$\mathcal{B}^{\mathrm{Div}_{\mathbb{T},\Gamma_n}}(\Omega) := \left\{B_{\ell}^{\mathrm{Div}_{\mathbb{T},\Gamma_n}}\right\}_{\ell=1}^{d_{\Omega,\mathbb{T},\Gamma_t}} \subset \mathsf{H}^{\infty}_{\mathbb{T},\Gamma_n,0}(\mathrm{Div},\Omega),$$

such that $\pi_{N(\mathrm{divDiv}_{\mathbb{S},\Gamma_n}\varepsilon)}\mathcal{B}^{{}_{\mathbb{T}}\mathrm{Rot}_{\mathbb{S},\Gamma_t}}(\Omega)$ *and* $\pi_{N({}_{\mathbb{T}}\mathrm{Rot}_{\mathbb{S},\Gamma_t})}\varepsilon^{-1}\mathcal{B}^{\mathrm{divDiv}_{\mathbb{S},\Gamma_n}}(\Omega)$ *are both bases of* $\mathcal{H}_{\mathbb{S},\Gamma_t,\Gamma_n,\varepsilon}(\Omega)$*, and* $\pi_{N(\mathrm{Div}_{\mathbb{T},\Gamma_n}\mu)}\mathcal{B}^{{}_{\mathbb{S}}\mathrm{Rot}_{\mathbb{T},\Gamma_t}}(\Omega)$ *and* $\pi_{N({}_{\mathbb{S}}\mathrm{Rot}_{\mathbb{T},\Gamma_t})}\mu^{-1}\mathcal{B}^{\mathrm{Div}_{\mathbb{T},\Gamma_n}}(\Omega)$ *are both bases of* $\mathcal{H}_{\mathbb{T},\Gamma_t,\Gamma_n,\mu}(\Omega)$*. In particular,*

$$\mathrm{Lin}\,\pi_{N(\mathrm{divDiv}_{\mathbb{S},\Gamma_n}\varepsilon)}\mathcal{B}^{{}_{\mathbb{T}}\mathrm{Rot}_{\mathbb{S},\Gamma_t}}(\Omega) = \mathcal{H}_{\mathbb{S},\Gamma_t,\Gamma_n,\varepsilon}(\Omega) = \mathrm{Lin}\,\pi_{N({}_{\mathbb{T}}\mathrm{Rot}_{\mathbb{S},\Gamma_t})}\varepsilon^{-1}\mathcal{B}^{\mathrm{divDiv}_{\mathbb{S},\Gamma_n}}(\Omega),$$
$$\mathrm{Lin}\,\pi_{N(\mathrm{Div}_{\mathbb{T},\Gamma_n}\mu)}\mathcal{B}^{{}_{\mathbb{S}}\mathrm{Rot}_{\mathbb{T},\Gamma_t}}(\Omega) = \mathcal{H}_{\mathbb{T},\Gamma_t,\Gamma_n,\mu}(\Omega) = \mathrm{Lin}\,\pi_{N({}_{\mathbb{S}}\mathrm{Rot}_{\mathbb{T},\Gamma_t})}\mu^{-1}\mathcal{B}^{\mathrm{Div}_{\mathbb{T},\Gamma_n}}(\Omega).$$

Note that, for example, $\left(1 - \pi_{N(\mathrm{divDiv}_{\mathbb{S},\Gamma_n}\varepsilon)}\right)$ is the $\mathsf{L}^2_{\mathbb{S},\varepsilon}(\Omega)$-orthonormal projector onto $R\left({}_{\mathbb{S}}\mathrm{Gradgrad}_{\Gamma_t}\right)$. By (14), Theorem 3.16, and Theorem 3.28, we compute

$$\begin{aligned}
\mathsf{H}_{\mathbb{S},\Gamma_t,0}(\mathrm{Rot},\Omega) &= R\left({}_{\mathbb{S}}\mathrm{Gradgrad}_{\Gamma_t}\right) \oplus_{\mathsf{L}^2_{\mathbb{S},\varepsilon}(\Omega)} \mathcal{H}_{\mathbb{S},\Gamma_t,\Gamma_n,\varepsilon}(\Omega)\\
&= R\left({}_{\mathbb{S}}\mathrm{Gradgrad}_{\Gamma_t}\right) \oplus_{\mathsf{L}^2_{\mathbb{S},\varepsilon}(\Omega)} \mathrm{Lin}\,\pi_{N(\mathrm{divDiv}_{\mathbb{S},\Gamma_n}\varepsilon)}\mathcal{B}^{{}_{\mathbb{T}}\mathrm{Rot}_{\mathbb{S},\Gamma_t}}(\Omega)\\
&= R\left({}_{\mathbb{S}}\mathrm{Gradgrad}_{\Gamma_t}\right) + (\pi_{N(\mathrm{divDiv}_{\mathbb{S},\Gamma_n}\varepsilon)} - 1)\mathrm{Lin}\,\mathcal{B}^{{}_{\mathbb{T}}\mathrm{Rot}_{\mathbb{S},\Gamma_t}}(\Omega) + \mathrm{Lin}\,\mathcal{B}^{{}_{\mathbb{T}}\mathrm{Rot}_{\mathbb{S},\Gamma_t}}(\Omega)\\
&= R\left({}_{\mathbb{S}}\mathrm{Gradgrad}_{\Gamma_t}\right) + \mathrm{Lin}\,\mathcal{B}^{{}_{\mathbb{T}}\mathrm{Rot}_{\mathbb{S},\Gamma_t}}(\Omega),\\
\mathsf{H}^k_{\mathbb{S},\Gamma_t,0}(\mathrm{Rot},\Omega) &= R\left({}_{\mathbb{S}}\mathrm{Gradgrad}_{\Gamma_t}\right) \cap \mathsf{H}^k_{\mathbb{S},\Gamma_t,0}(\mathrm{Rot},\Omega) + \mathrm{Lin}\,\mathcal{B}^{{}_{\mathbb{T}}\mathrm{Rot}_{\mathbb{S},\Gamma_t}}(\Omega)\\
&= R\left({}_{\mathbb{S}}\mathrm{Gradgrad}^k_{\Gamma_t}\right) + \mathrm{Lin}\,\mathcal{B}^{{}_{\mathbb{T}}\mathrm{Rot}_{\mathbb{S},\Gamma_t}}(\Omega).
\end{aligned} \tag{15}$$

Similarly, we obtain decompositions of $\mathsf{H}^k_{\mathbb{S},\Gamma_n,0}(\mathrm{divDiv},\Omega)$, $\mathsf{H}_{\mathbb{T},\Gamma_t,0}(\mathrm{symRot},\Omega)$, and $\mathsf{H}^k_{\mathbb{T},\Gamma_n,0}(\mathrm{Div},\Omega)$ using $\mathcal{B}^{\mathrm{divDiv}_{\mathbb{S},\Gamma_n}}(\Omega)$, $\mathcal{B}^{{}_{\mathbb{S}}\mathrm{Rot}_{\mathbb{T},\Gamma_t}}(\Omega)$, and $\mathcal{B}^{\mathrm{Div}_{\mathbb{T},\Gamma_n}}(\Omega)$, respectively. We conclude:

Theorem 3.29. (Bounded regular direct decompositions). *Let $k \geq 0$. Then the bounded regular direct decompositions*

$$\begin{aligned}
\mathsf{H}^k_{\mathbb{S},\Gamma_t}(\mathrm{Rot},\Omega) &= R\left(\widetilde{Q}^{k,1}_{{}_{\mathbb{T}}\mathrm{Rot}_{\mathbb{S}},\Gamma_t}\right) \dotplus \mathsf{H}^k_{\mathbb{S},\Gamma_t,0}(\mathrm{Rot},\Omega),\\
\mathsf{H}^k_{\mathbb{S},\Gamma_t,0}(\mathrm{Rot},\Omega) &= \mathrm{Gradgrad}\ \mathsf{H}^{k+2}_{\Gamma_t}(\Omega) \dotplus \mathrm{Lin}\,\mathcal{B}^{{}_{\mathbb{T}}\mathrm{Rot}_{\mathbb{S},\Gamma_t}}(\Omega),\\
\mathsf{H}^k_{\mathbb{T},\Gamma_n}(\mathrm{Div},\Omega) &= R\left(\widetilde{Q}^{k,1}_{\mathrm{Div}_{\mathbb{T}},\Gamma_n}\right) \dotplus \mathsf{H}^k_{\mathbb{T},\Gamma_n,0}(\mathrm{Div},\Omega),\\
\mathsf{H}^k_{\mathbb{T},\Gamma_n,0}(\mathrm{Div},\Omega) &= \mathrm{Rot}\ \mathsf{H}^{k+1}_{\mathbb{S},\Gamma_n}(\Omega) \dotplus \mathrm{Lin}\,\mathcal{B}^{\mathrm{Div}_{\mathbb{T},\Gamma_n}}(\Omega),\\
\mathsf{H}^k_{\mathbb{T},\Gamma_t}(\mathrm{symRot},\Omega) &= R\left(\widetilde{Q}^{k,1}_{{}_{\mathbb{S}}\mathrm{Rot}_{\mathbb{T}},\Gamma_t}\right) \dotplus \mathsf{H}^k_{\mathbb{T},\Gamma_t,0}(\mathrm{symRot},\Omega),\\
\mathsf{H}^k_{\mathbb{T},\Gamma_t,0}(\mathrm{symRot},\Omega) &= \mathrm{devGrad}\ \mathsf{H}^{k+1}_{\Gamma_t}(\Omega) \dotplus \mathrm{Lin}\,\mathcal{B}^{{}_{\mathbb{S}}\mathrm{Rot}_{\mathbb{T},\Gamma_t}}(\Omega),\\
\mathsf{H}^k_{\mathbb{S},\Gamma_n}(\mathrm{divDiv},\Omega) &= R\left(\widetilde{Q}^{k,1}_{\mathrm{divDiv}_{\mathbb{S}},\Gamma_n}\right) \dotplus \mathsf{H}^k_{\mathbb{S},\Gamma_n,0}(\mathrm{divDiv},\Omega),\\
\mathsf{H}^{k+1,k}_{\mathbb{S},\Gamma_n}(\mathrm{divDiv},\Omega) &= R\left(\widetilde{Q}^{k+1,k,1}_{\mathrm{divDiv}_{\mathbb{S}},\Gamma_n}\right) \dotplus \mathsf{H}^{k+1}_{\mathbb{S},\Gamma_n,0}(\mathrm{divDiv},\Omega),\\
\mathsf{H}^k_{\mathbb{S},\Gamma_n,0}(\mathrm{divDiv},\Omega) &= \mathrm{symRot}\ \mathsf{H}^{k+1}_{\mathbb{T},\Gamma_n}(\Omega) \dotplus \mathrm{Lin}\,\mathcal{B}^{\mathrm{divDiv}_{\mathbb{S},\Gamma_n}}(\Omega)
\end{aligned}$$

hold. Note that $R\left(\widetilde{Q}^{k,1}_{{}_{\mathbb{T}}\mathrm{Rot}_{\mathbb{S}},\Gamma_t}\right) \subset \mathsf{H}^{k+1}_{\mathbb{S},\Gamma_t}(\Omega)$, $R\left(\widetilde{Q}^{k,1}_{\mathrm{Div}_{\mathbb{T}},\Gamma_n}\right) \subset \mathsf{H}^{k+1}_{\mathbb{T},\Gamma_n}(\Omega)$, $R\left(\widetilde{Q}^{k,1}_{{}_{\mathbb{S}}\mathrm{Rot}_{\mathbb{T}},\Gamma_t}\right) \subset \mathsf{H}^{k+1}_{\mathbb{T},\Gamma_t}(\Omega)$, *and* $R\left(\widetilde{Q}^{k,1}_{\mathrm{divDiv}_{\mathbb{S}},\Gamma_n}\right)$, $R\left(\widetilde{Q}^{k+1,k,1}_{\mathrm{divDiv}_{\mathbb{S}},\Gamma_n}\right) \subset \mathsf{H}^{k+2}_{\mathbb{S},\Gamma_n}(\Omega)$.

See Appendix C for a proof.

Remark 3.30. (Bounded regular direct decompositions). In particular, for $k = 0$,

$$\begin{aligned}
\mathsf{H}_{\mathbb{S},\Gamma_t}(\mathrm{Rot},\Omega) &= R\left(\widetilde{Q}^{0,1}_{{}_{\mathbb{T}}\mathrm{Rot}_{\mathbb{S}},\Gamma_t}\right) \dotplus \mathsf{H}_{\mathbb{S},\Gamma_t,0}(\mathrm{Rot},\Omega),\\
\mathsf{H}_{\mathbb{S},\Gamma_t,0}(\mathrm{Rot},\Omega) &= \mathrm{Gradgrad}\ \mathsf{H}^2_{\Gamma_t}(\Omega) \dotplus \mathrm{Lin}\,\mathcal{B}^{{}_{\mathbb{T}}\mathrm{Rot}_{\mathbb{S},\Gamma_t}}(\Omega)\\
&= \mathrm{Gradgrad}\ \mathsf{H}^2_{\Gamma_t}(\Omega) \oplus_{\mathsf{L}^2_{\mathbb{S},\varepsilon}(\Omega)} \mathcal{H}_{\mathbb{S},\Gamma_t,\Gamma_n,\varepsilon}(\Omega),\\
\mathsf{H}_{\mathbb{T},\Gamma_n}(\mathrm{Div},\Omega) &= R\left(\widetilde{Q}^{0,1}_{\mathrm{Div}_{\mathbb{T}},\Gamma_n}\right) \dotplus \mathsf{H}_{\mathbb{T},\Gamma_n,0}(\mathrm{Div},\Omega),\\
\mu^{-1}\mathsf{H}^0_{\mathbb{T},\Gamma_n,0}(\mathrm{Div},\Omega) &= \mu^{-1}\mathrm{Rot}\ \mathsf{H}^1_{\mathbb{S},\Gamma_n}(\Omega) \dotplus \mu^{-1}\mathrm{Lin}\,\mathcal{B}^{\mathrm{Div}_{\mathbb{T},\Gamma_n}}(\Omega)\\
&= \mu^{-1}\mathrm{Rot}\ \mathsf{H}^1_{\mathbb{S},\Gamma_n}(\Omega) \oplus_{\mathsf{L}^2_{\mathbb{T},\mu}(\Omega)} \mathcal{H}_{\mathbb{T},\Gamma_t,\Gamma_n,\mu}(\Omega),\\
\mathsf{H}_{\mathbb{T},\Gamma_t}(\mathrm{symRot},\Omega) &= R\left(\widetilde{Q}^{0,1}_{{}_{\mathbb{S}}\mathrm{Rot}_{\mathbb{T}},\Gamma_t}\right) \dotplus \mathsf{H}_{\mathbb{T},\Gamma_t,0}(\mathrm{symRot},\Omega),\\
\mathsf{H}_{\mathbb{T},\Gamma_t,0}(\mathrm{symRot},\Omega) &= \mathrm{devGrad}\ \mathsf{H}^1_{\Gamma_t}(\Omega) \dotplus \mathrm{Lin}\,\mathcal{B}^{{}_{\mathbb{S}}\mathrm{Rot}_{\mathbb{T},\Gamma_t}}(\Omega)\\
&= \mathrm{devGrad}\ \mathsf{H}^1_{\Gamma_t}(\Omega) \oplus_{\mathsf{L}^2_{\mathbb{T},\mu}(\Omega)} \mathcal{H}_{\mathbb{T},\Gamma_t,\Gamma_n,\mu}(\Omega),\\
\mathsf{H}_{\mathbb{S},\Gamma_n}(\mathrm{divDiv},\Omega) &= R\left(\widetilde{Q}^{0,1}_{\mathrm{divDiv}_{\mathbb{S}},\Gamma_n}\right) \dotplus \mathsf{H}_{\mathbb{S},\Gamma_n,0}(\mathrm{divDiv},\Omega),\\
\varepsilon^{-1}\mathsf{H}_{\mathbb{S},\Gamma_n,0}(\mathrm{divDiv},\Omega) &= \varepsilon^{-1}\mathrm{symRot}\ \mathsf{H}^1_{\mathbb{T},\Gamma_n}(\Omega) \dotplus \varepsilon^{-1}\mathrm{Lin}\,\mathcal{B}^{\mathrm{divDiv}_{\mathbb{S},\Gamma_n}}(\Omega)\\
&= \varepsilon^{-1}\mathrm{symRot}\ \mathsf{H}^1_{\mathbb{T},\Gamma_n}(\Omega) \oplus_{\mathsf{L}^2_{\mathbb{S},\varepsilon}(\Omega)} \mathcal{H}_{\mathbb{S},\Gamma_t,\Gamma_n,\varepsilon}(\Omega),
\end{aligned}$$

and

$$\begin{aligned}
\mathsf{L}^2_{\mathbb{S},\varepsilon}(\Omega) &= \mathsf{H}_{\mathbb{S},\Gamma_t,0}(\mathrm{Rot},\Omega) \oplus_{\mathsf{L}^2_{\mathbb{S},\varepsilon}(\Omega)} \varepsilon^{-1}\mathrm{symRot}\ \mathsf{H}^1_{\mathbb{T},\Gamma_n}(\Omega)\\
&= \mathrm{Gradgrad}\ \mathsf{H}^2_{\Gamma_t}(\Omega) \oplus_{\mathsf{L}^2_{\mathbb{S},\varepsilon}(\Omega)} \varepsilon^{-1}\mathsf{H}_{\mathbb{S},\Gamma_n,0}(\mathrm{divDiv},\Omega),\\
\mathsf{L}^2_{\mathbb{T},\mu}(\Omega) &= \mathsf{H}_{\mathbb{T},\Gamma_t,0}(\mathrm{symRot},\Omega) \oplus_{\mathsf{L}^2_{\mathbb{T},\mu}(\Omega)} \mu^{-1}\mathrm{Rot}\ \mathsf{H}^1_{\mathbb{S},\Gamma_n}(\Omega)\\
&= \mathrm{devGrad}\ \mathsf{H}^1_{\Gamma_t}(\Omega) \oplus_{\mathsf{L}^2_{\mathbb{T},\mu}(\Omega)} \mu^{-1}\mathsf{H}_{\mathbb{T},\Gamma_n,0}(\mathrm{Div},\Omega).
\end{aligned}$$

By the latter theorem, we have bounded linear regular (direct) decompositions

$$\begin{aligned}
\mathsf{H}^{k}_{\mathbb{S},\Gamma_t}(\operatorname{Rot},\Omega) &= R\left(\widetilde{\mathcal{Q}}^{k,1}_{{}_{\mathbb{T}}\operatorname{Rot}_{\mathbb{S}},\Gamma_t}\right) \dotplus \operatorname{Lin}\mathcal{B}^{{}_{\mathbb{T}}\operatorname{Rot}_{\mathbb{S},\Gamma_t}}(\Omega) \dotplus \operatorname{Gradgrad}\mathsf{H}^{k+2}_{\Gamma_t}(\Omega)\\
&= \mathsf{H}^{k+1}_{\mathbb{S},\Gamma_t}(\Omega) + \operatorname{Gradgrad}\mathsf{H}^{k+2}_{\Gamma_t}(\Omega),\\
\mathsf{H}^{k}_{\mathbb{T},\Gamma_n}(\operatorname{Div},\Omega) &= R\left(\widetilde{\mathcal{Q}}^{k,1}_{\operatorname{Div}_{\mathbb{T}},\Gamma_n}\right) \dotplus \operatorname{Lin}\mathcal{B}^{\operatorname{Div}_{\mathbb{T},\Gamma_n}}(\Omega) \dotplus \operatorname{Rot}\mathsf{H}^{k+1}_{\mathbb{S},\Gamma_n}(\Omega)\\
&= \mathsf{H}^{k+1}_{\mathbb{T},\Gamma_n}(\Omega) + \operatorname{Rot}\mathsf{H}^{k+1}_{\mathbb{S},\Gamma_n}(\Omega),\\
\mathsf{H}^{k}_{\mathbb{T},\Gamma_t}(\operatorname{symRot},\Omega) &= R\left(\widetilde{\mathcal{Q}}^{k,1}_{{}_{\mathbb{S}}\operatorname{Rot}_{\mathbb{T}},\Gamma_t}\right) \dotplus \operatorname{Lin}\mathcal{B}^{{}_{\mathbb{S}}\operatorname{Rot}_{\mathbb{T},\Gamma_t}}(\Omega) \dotplus \operatorname{devGrad}\mathsf{H}^{k+1}_{\Gamma_t}(\Omega)\\
&= \mathsf{H}^{k+1}_{\mathbb{T},\Gamma_t}(\Omega) + \operatorname{devGrad}\mathsf{H}^{k+1}_{\Gamma_t}(\Omega),\\
\mathsf{H}^{k}_{\mathbb{S},\Gamma_n}(\operatorname{divDiv},\Omega) &= R\left(\widetilde{\mathcal{Q}}^{k,1}_{\operatorname{divDiv}_{\mathbb{S}},\Gamma_n}\right) \dotplus \operatorname{Lin}\mathcal{B}^{\operatorname{divDiv}_{\mathbb{S},\Gamma_n}}(\Omega) \dotplus \operatorname{symRot}\mathsf{H}^{k+1}_{\mathbb{T},\Gamma_n}(\Omega)\\
&= \mathsf{H}^{k+2}_{\mathbb{S},\Gamma_n}(\Omega) + \operatorname{symRot}\mathsf{H}^{k+1}_{\mathbb{T},\Gamma_n}(\Omega),\\
\mathsf{H}^{k+1,k}_{\mathbb{S},\Gamma_n}(\operatorname{divDiv},\Omega) &= R\left(\widetilde{\mathcal{Q}}^{k+1,k,1}_{\operatorname{divDiv}_{\mathbb{S}},\Gamma_n}\right) \dotplus \operatorname{Lin}\mathcal{B}^{\operatorname{divDiv}_{\mathbb{S},\Gamma_n}}(\Omega) \dotplus \operatorname{symRot}\mathsf{H}^{k+2}_{\mathbb{T},\Gamma_n}(\Omega)\\
&= \mathsf{H}^{k+2}_{\mathbb{S},\Gamma_n}(\Omega) + \operatorname{symRot}\mathsf{H}^{k+2}_{\mathbb{T},\Gamma_n}(\Omega).
\end{aligned}\tag{16}$$

See Remark D.3 for more details on these decompositions and the corresponding bounded linear regular direct decomposition operators. Noting

$$\begin{aligned}
&R\left(\varepsilon^{-1}{}_{\mathbb{S}}\operatorname{Rot}_{\mathbb{T},\Gamma_n}\right)\perp_{\mathsf{L}^2_{\mathbb{S},\varepsilon}(\Omega)}\mathcal{B}^{\mathbb{T}\operatorname{Rot}_{\mathbb{S},\Gamma_t}}(\Omega), &&R\left({}_{\mathbb{S}}\operatorname{Gradgrad}_{\Gamma_t}\right)\perp_{\mathsf{L}^2_{\mathbb{S}}(\Omega)}\mathcal{B}^{\operatorname{divDiv}_{\mathbb{S},\Gamma_n}}(\Omega),\\
&R\left(\mu^{-1}{}_{\mathbb{T}}\operatorname{Rot}_{\mathbb{S},\Gamma_n}\right)\perp_{\mathsf{L}^2_{\mathbb{T},\mu}(\Omega)}\mathcal{B}^{{}_{\mathbb{S}}\operatorname{Rot}_{\mathbb{T},\Gamma_t}}(\Omega), &&R\left({}_{\mathbb{T}}\operatorname{Grad}_{\Gamma_t}\right)\perp_{\mathsf{L}^2_{\mathbb{T}}(\Omega)}\mathcal{B}^{\operatorname{Div}_{\mathbb{T},\Gamma_n}}(\Omega),
\end{aligned}\tag{17}$$

we see the following:

Theorem 3.31. (Alternative Dirichlet/Neumann projections)**.** *It holds*

$$\begin{aligned}
\mathcal{H}_{\mathbb{S},\Gamma_t,\Gamma_n,\varepsilon}(\Omega)\cap\mathcal{B}^{{}_{\mathbb{T}}\operatorname{Rot}_{\mathbb{S},\Gamma_t}}(\Omega)^{\perp_{\mathsf{L}^2_{\mathbb{S},\varepsilon}(\Omega)}} &= \{0\},\\
N\left(\operatorname{divDiv}_{\mathbb{S},\Gamma_n}\varepsilon\right)\cap\mathcal{B}^{{}_{\mathbb{T}}\operatorname{Rot}_{\mathbb{S},\Gamma_t}}(\Omega)^{\perp_{\mathsf{L}^2_{\mathbb{S},\varepsilon}(\Omega)}} &= R\left(\varepsilon^{-1}{}_{\mathbb{S}}\operatorname{Rot}_{\mathbb{T},\Gamma_n}\right),\\
\mathcal{H}_{\mathbb{S},\Gamma_t,\Gamma_n,\varepsilon}(\Omega)\cap\mathcal{B}^{\operatorname{divDiv}_{\mathbb{S},\Gamma_n}}(\Omega)^{\perp_{\mathsf{L}^2_{\mathbb{S}}(\Omega)}} &= \{0\},\\
N\left({}_{\mathbb{T}}\operatorname{Rot}_{\mathbb{S},\Gamma_t}\right)\cap\mathcal{B}^{\operatorname{divDiv}_{\mathbb{S},\Gamma_n}}(\Omega)^{\perp_{\mathsf{L}^2_{\mathbb{S}}(\Omega)}} &= R\left({}_{\mathbb{S}}\operatorname{Gradgrad}_{\Gamma_t}\right),\\
\mathcal{H}_{\mathbb{T},\Gamma_t,\Gamma_n,\mu}(\Omega)\cap\mathcal{B}^{{}_{\mathbb{S}}\operatorname{Rot}_{\mathbb{T},\Gamma_t}}(\Omega)^{\perp_{\mathsf{L}^2_{\mathbb{T},\mu}(\Omega)}} &= \{0\},\\
N(\operatorname{Div}_{\mathbb{T},\Gamma_n}\varepsilon)\cap\mathcal{B}^{{}_{\mathbb{S}}\operatorname{Rot}_{\mathbb{T},\Gamma_t}}(\Omega)^{\perp_{\mathsf{L}^2_{\mathbb{T},\mu}(\Omega)}} &= R\left(\mu^{-1}{}_{\mathbb{T}}\operatorname{Rot}_{\mathbb{S},\Gamma_n}\right),\\
\mathcal{H}_{\mathbb{T},\Gamma_t,\Gamma_n,\mu}(\Omega)\cap\mathcal{B}^{\operatorname{Div}_{\mathbb{T},\Gamma_n}}(\Omega)^{\perp_{\mathsf{L}^2_{\mathbb{T}}(\Omega)}} &= \{0\},\\
N\left({}_{\mathbb{S}}\operatorname{Rot}_{\mathbb{T},\Gamma_t}\right)\cap\mathcal{B}^{\operatorname{Div}_{\mathbb{T},\Gamma_n}}(\Omega)^{\perp_{\mathsf{L}^2_{\mathbb{T}}(\Omega)}} &= R\left({}_{\mathbb{T}}\operatorname{Grad}_{\Gamma_t}\right).
\end{aligned}$$

Moreover, for all $k \geq 0$,

$$\begin{aligned}
N\left(\operatorname{divDiv}^k_{\mathbb{S},\Gamma_n}\varepsilon\right)\cap\mathcal{B}^{{}_{\mathbb{T}}\operatorname{Rot}_{\mathbb{S},\Gamma_t}}(\Omega)^{\perp_{\mathsf{L}^2_{\mathbb{S},\varepsilon}(\Omega)}} &= R\left(\varepsilon^{-1}{}_{\mathbb{S}}\operatorname{Rot}^k_{\mathbb{T},\Gamma_n}\right) = \varepsilon^{-1}\operatorname{symRot}\mathsf{H}^{k+1}_{\mathbb{T},\Gamma_n}(\Omega),\\
N\left({}_{\mathbb{T}}\operatorname{Rot}^k_{\mathbb{S},\Gamma_t}\right)\cap\mathcal{B}^{\operatorname{divDiv}_{\mathbb{S},\Gamma_n}}(\Omega)^{\perp_{\mathsf{L}^2_{\mathbb{S}}(\Omega)}} &= R\left({}_{\mathbb{S}}\operatorname{Gradgrad}^k_{\Gamma_t}\right) = \operatorname{Gradgrad}\mathsf{H}^{k+2}_{\Gamma_t}(\Omega),\\
N\left(\operatorname{Div}^k_{\mathbb{T},\Gamma_n}\varepsilon\right)\cap\mathcal{B}^{\mathbb{S}\operatorname{Rot}_{\mathbb{T},\Gamma_t}}(\Omega)^{\perp_{\mathsf{L}^2_{\mathbb{T},\mu}(\Omega)}} &= R\left(\mu^{-1}{}_{\mathbb{T}}\operatorname{Rot}^k_{\mathbb{S},\Gamma_n}\right) = \mu^{-1}\operatorname{Rot}\mathsf{H}^{k+1}_{\mathbb{S},\Gamma_n}(\Omega),\\
N\left({}_{\mathbb{S}}\operatorname{Rot}^k_{\mathbb{T},\Gamma_t}\right)\cap\mathcal{B}^{\operatorname{Div}_{\mathbb{T},\Gamma_n}}(\Omega)^{\perp_{\mathsf{L}^2_{\mathbb{T}}(\Omega)}} &= R\left({}_{\mathbb{T}}\operatorname{Grad}^k_{\Gamma_t}\right) = \operatorname{devGrad}\mathsf{H}^{k+1}_{\Gamma_t}(\Omega).
\end{aligned}$$

See Appendix C for a proof. Theorem 3.29 implies the following:

Theorem 3.32. (Cohomology groups)**.** *It holds*

$$\frac{N\left({}_{\mathbb{T}}\mathrm{Rot}^k_{\mathbb{S},\Gamma_t}\right)}{R\left({}_{\mathbb{S}}\mathrm{Gradgrad}^k_{\Gamma_t}\right)} \cong \mathrm{Lin}\,\mathcal{B}^{{}_{\mathbb{T}}\mathrm{Rot}_{\mathbb{S},\Gamma_t}}(\Omega) \cong \mathcal{H}_{\mathbb{S},\Gamma_t,\Gamma_n,\varepsilon}(\Omega) \cong \mathrm{Lin}\,\mathcal{B}^{\mathrm{divDiv}_{\mathbb{S},\Gamma_n}}(\Omega) \cong \frac{N\left(\mathrm{divDiv}^k_{\mathbb{S},\Gamma_n}\right)}{R\left({}_{\mathbb{S}}\mathrm{Rot}^k_{\mathbb{T},\Gamma_n}\right)},$$

$$\frac{N\left({}_{\mathbb{S}}\mathrm{Rot}^k_{\mathbb{T},\Gamma_t}\right)}{R\left({}_{\mathbb{T}}\mathrm{Grad}^k_{\Gamma_t}\right)} \cong \mathrm{Lin}\,\mathcal{B}^{{}_{\mathbb{S}}\mathrm{Rot}_{\mathbb{T},\Gamma_t}}(\Omega) \cong \mathcal{H}_{\mathbb{T},\Gamma_t,\Gamma_n,\mu}(\Omega) \cong \mathrm{Lin}\,\mathcal{B}^{\mathrm{Div}_{\mathbb{T},\Gamma_n}}(\Omega) \cong \frac{N\left(\mathrm{Div}^k_{\mathbb{T},\Gamma_n}\right)}{R\left({}_{\mathbb{T}}\mathrm{Rot}^k_{\mathbb{S},\Gamma_n}\right)}.$$

In particular, the dimensions of the cohomology groups (Dirichlet/Neumann fields) are independent of k and ε, μ and it holds

$$d_{\Omega,\mathbb{S},\Gamma_t} = \dim\left(N\left({}_{\mathbb{T}}\mathrm{Rot}^k_{\mathbb{S},\Gamma_t}\right)/R\left({}_{\mathbb{S}}\mathrm{Gradgrad}^k_{\Gamma_t}\right)\right) = \dim\left(N\left(\mathrm{divDiv}^k_{\mathbb{S},\Gamma_n}\right)/R\left({}_{\mathbb{S}}\mathrm{Rot}^k_{\mathbb{T},\Gamma_n}\right)\right),$$

$$d_{\Omega,\mathbb{T},\Gamma_t} = \dim\left(N\left({}_{\mathbb{S}}\mathrm{Rot}^k_{\mathbb{T},\Gamma_t}\right)/R\left({}_{\mathbb{T}}\mathrm{Grad}^k_{\Gamma_t}\right)\right) = \dim\left(N\left(\mathrm{Div}^k_{\mathbb{T},\Gamma_n}\right)/R\left({}_{\mathbb{T}}\mathrm{Rot}^k_{\mathbb{S},\Gamma_n}\right)\right).$$

ACKNOWLEDGEMENTS

We thank the anonymous referee for careful reading and helpful suggestions to improve the quality of the paper. Open Access funding enabled and organized by Projekt DEAL.

CONFLICT OF INTEREST STATEMENT

This work does not have any conflicts of interest.

ORCID

Dirk Pauly https://orcid.org/0000-0003-4155-7297

REFERENCES

1. D. Pauly and M. Schomburg, *Hilbert complexes with mixed boundary conditions—Part 1: de Rham complex*, Math. Methods Appl. Sci. **45** (2022), no. 5, 2465–2507.
2. D. Pauly and M. Schomburg, *Hilbert complexes with mixed boundary conditions—Part 2: Elasticity complex*, Math. Methods Appl. Sci. **45** (2022), no. 16, 8971–9005.
3. D. Pauly and W. Zulehner, *The elasticity complex: Compact embeddings and regular decompositions*, Appl. Anal. **2022** (2022), 1–29.
4. D. Pauly and W. Zulehner, *The divDiv-complex and applications to biharmonic equations*, Appl. Anal. **99** (2020), no. 9, 1579–1630.
5. D. Pauly and M. Waurick, *The index of some mixed order Dirac-type operators and generalised Dirichlet-Neumann tensor fields*, Math. Z. **301** (2022), no. 2, 1739–1819.
6. S. Bauer, D. Pauly, and M. Schomburg, *The Maxwell compactness property in bounded weak Lipschitz domains with mixed boundary conditions*, SIAM J. Math. Anal. **48** (2016), no. 4, 2912–2943.
7. S. Bauer, D. Pauly, and M. Schomburg, *Weck's selection theorem: The Maxwell compactness property for bounded weak Lipschitz domains with mixed boundary conditions in arbitrary dimensions*, Maxwell's Equations: Analysis and Numerics, Radon Ser. Comput. Appl. Math., De Gruyter **24** (2019), 77–104.

How to cite this article: D. Pauly and M. Schomburg, *Hilbert complexes with mixed boundary conditions part 3: Biharmonic complexes*, Math. Meth. Appl. Sci. (2023), 1–46, DOI 10.1002/mma.9437.

APPENDIX A: ELEMENTARY FORMULAS

From [3, 4] and [5], we have the following collection of formulas related to the elasticity and the biharmonic complex.

Lemma A.1 ([5, Lemma 12.10])**.** *Let u, v, w, and S belong to* $\mathsf{C}^{\infty}(\mathbb{R}^{3\,3})$.

- $(\operatorname{spn} v)w = v \times w = -(\operatorname{spn} w)v$ *and* $(\operatorname{spn} v)(\operatorname{spn}^{-1} S) = -Sv$, *if* $\operatorname{sym} S = 0$
- $\operatorname{sym}\operatorname{spn} v = 0$ *and* $\operatorname{dev}(u\,\mathrm{id}) = 0$
- $\operatorname{tr}\operatorname{Grad} v = \operatorname{div} v$ *and* $2\operatorname{skw}\operatorname{Grad} v = \operatorname{spn}\operatorname{rot} v$
- $\operatorname{Div}(u\,\mathrm{id}) = \operatorname{grad} u$ *and* $\operatorname{Rot}(u\,\mathrm{id}) = -\operatorname{spn}\operatorname{grad} u$, *in particular,* $\operatorname{rot}\operatorname{Div}\,(u\,\mathrm{id}) = 0$ *and* $\operatorname{rot}\operatorname{spn}^{-1}\operatorname{Rot}\,(u\,\mathrm{id}) = 0$ *and* $\operatorname{sym}\operatorname{Rot}(u\,\mathrm{id}) = 0$
- $\operatorname{Div}\ \operatorname{spn} v = -\operatorname{rot} v$ *and* $\operatorname{Div}\ \operatorname{skw} S = -\operatorname{rot}\operatorname{spn}^{-1}\operatorname{skw} S$, *in particular* $\operatorname{div}\operatorname{Div}\operatorname{skw} S = 0$
- $\operatorname{Rot}\ \operatorname{spn} v = (\operatorname{div} v)\mathrm{id}(\operatorname{Grad} v)^\top$ *and* $\operatorname{Rot}\operatorname{skw} S = (\operatorname{div}\operatorname{spn}^{-1}\operatorname{skw} S)\,\mathrm{id} - (\operatorname{Grad}\operatorname{spn}^{-1}\operatorname{skw} S)^\top$
- $\operatorname{dev}\operatorname{Rot}\operatorname{spn} v = -(\operatorname{dev}\operatorname{Grad}\ v)^\top$
- $-2\operatorname{Rot}\ \operatorname{sym}\operatorname{Grad} v = 2\operatorname{Rot}\operatorname{skw}\operatorname{Grad} v = -(\operatorname{Grad}\operatorname{rot} v)^\top$
- $2\operatorname{spn}^{-1}\operatorname{skw}\operatorname{Rot} S = \operatorname{Div}\ S^\top - \operatorname{grad}\operatorname{tr} S = \operatorname{Div}\ (S - (\operatorname{tr} S)\mathrm{id})^\top$, *in particular* $\operatorname{rot}\operatorname{Div}\ S^\top = 2\operatorname{rot}\operatorname{spn}^{-1}\operatorname{skw}\operatorname{Rot} S$ *and* $2\operatorname{skw}\operatorname{Rot} S = \operatorname{spn}\operatorname{Div}\ S^\top$, *if* $\operatorname{tr} S = 0$
- $\operatorname{tr}\operatorname{Rot} S = 2\operatorname{div}\operatorname{spn}^{-1}\operatorname{skw} S$, *in particular,* $\operatorname{tr}\operatorname{Rot} S = 0$, *if* $\operatorname{skw} S = 0$, *and* $\operatorname{tr}\operatorname{Rot}\operatorname{sym} S = 0$ *and* $\operatorname{tr}\operatorname{Rot}\operatorname{skw} S = \operatorname{tr}\operatorname{Rot} S$
- $2(\operatorname{Grad}\operatorname{spn}^{-1}\operatorname{skw} S)^\top = (\operatorname{tr}\operatorname{Rot}\operatorname{skw} S)\mathrm{id} - 2\operatorname{Rot}\operatorname{skw} S$
- $3\operatorname{Div}(\operatorname{dev}\operatorname{Grad} v)^\top = 2\operatorname{grad}\operatorname{div} v$
- $2\operatorname{Rot}\ \operatorname{sym}\operatorname{Grad} v = -2\operatorname{Rot}\ \operatorname{skw}\operatorname{Grad} v = -\operatorname{Rot}\ \operatorname{spn}\operatorname{rot} v = (\operatorname{Grad}\operatorname{rot} v)^\top$
- $2\operatorname{Div}\ \operatorname{sym}\operatorname{Rot} S = -2\operatorname{Div}\ \operatorname{skw}\operatorname{Rot} S = \operatorname{rot}\operatorname{Div}\ S^\top$
- $\operatorname{Rot}(\operatorname{Rot}\ \operatorname{sym} S)^\top = \operatorname{sym}\operatorname{Rot}(\operatorname{Rot} S)^\top$
- $\operatorname{Rot}(\operatorname{Rot}\operatorname{skw} S)^\top = \operatorname{skw}\operatorname{Rot}(\operatorname{Rot} S)^\top$

All formulas extend also to distributions.

APPENDIX B: BIHARMONIC COMPLEX OPERATORS REVISITED

Let $\top$ denote the formal operator of matrix transposition, that is,

$$\top S := S^\top,$$

and define

$$spn : \mathbb{R}^3 \to \mathbb{R}^{3\times 3}_{\mathrm{skw}}; \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \mapsto \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}.$$

We recall the operators forming the de Rham complex (classical vector analysis) grad, rot, and div acting on functions and vector fields, respectively, as formal matrix operators

$$\operatorname{grad} := \begin{bmatrix} \partial_1 \\ \partial_2 \\ \partial_3 \end{bmatrix}, rot := \operatorname{spn}\operatorname{grad} = \begin{bmatrix} 0 & -\partial_3 & \partial_2 \\ \partial_3 & 0 & -\partial_1 \\ -\partial_2 & \partial_1 & 0 \end{bmatrix}, \operatorname{div} := \top\operatorname{grad} = \begin{bmatrix} \partial_1 & \partial_2 & \partial_3 \end{bmatrix}.$$

Moreover, we introduce their relatives from the vector de Rham complex acting on vector and tensor fields, respectively, as formal matrix operators

$$\operatorname{Grad} := \top\operatorname{grad}\top, \operatorname{Rot} := \top\operatorname{rot}\top, \operatorname{Div} := \top\operatorname{div}\top.$$

In words, Grad, Rot, and Div act row-wise as the operators grad, rot, and div from the classical de Rham complex. Note that Grad v is just the Jacobian for a vector field v.

Let

$$\iota_{\mathbb{S}} : \mathbb{R}^{3\times 3}_{sym} \to \mathbb{R}^{3\times 3}, \qquad \iota_{\mathbb{T}} : \mathbb{R}^{3^{3\times 3}}_{dev} \to \mathbb{R}^{3\times 3}$$

denote the canonical embedding of symmetric and deviatoric (trace free) (3×3)-matrices into the arbitrary (3×3)-matrices, respectively. Then the adjoints

$$\iota^*_{\mathbb{S}} : \mathbb{R}^{3\times 3} \to \mathbb{R}^{3\times 3}_{sym}, \qquad \iota^*_{\mathbb{T}} : \mathbb{R}^{3\times 3} \to \mathbb{R}^{3\times 3}_{dev}$$

are almost the projectors onto symmetric and deviatoric (3×3)-matrices, respectively; that is, the actual projectors are given by

$$\operatorname{sym} := \iota_{\mathbb{S}}\iota_{\mathbb{S}}^{*} : \mathbb{R}^{3\times3} \to \mathbb{R}^{3\times3};\ S \mapsto \frac{1}{2}(S + S^{\top}),\ \operatorname{dev} := \iota_{\mathbb{T}}\iota_{\mathbb{T}}^{*} : \mathbb{R}^{3\times3} \to \mathbb{R}^{3\times3};\ T \mapsto T - \frac{1}{3}(\operatorname{tr} T)\operatorname{id}.$$

We extend all the latter formal operators to $\mathsf{L}^2(\Omega)$-tensor fields.

In the light of this, in the biharmonic complexes, we are dealing with the operators

$$\begin{aligned} {}_{\mathbb{S}}\operatorname{Gradgrad} &:= \iota_{\mathbb{S}}^{*}\operatorname{Gradgrad}, & {}_{\mathbb{T}}\operatorname{Rot}_{\mathbb{S}} &:= \iota_{\mathbb{T}}^{*}\operatorname{Rot}\iota_{\mathbb{S}}, & \operatorname{Div}_{\mathbb{T}} &:= \operatorname{Div}\ \iota_{\mathbb{T}}, \\ \operatorname{divDiv}_{\mathbb{S}} &:= \operatorname{divDiv}\ \iota_{\mathbb{S}}, & {}_{\mathbb{S}}\operatorname{Rot}_{\mathbb{T}} &:= \iota_{\mathbb{S}}^{*}\operatorname{Rot}\iota_{\mathbb{T}}, & {}_{\mathbb{T}}\operatorname{Grad} &:= \iota_{\mathbb{T}}^{*}\operatorname{Grad}. \end{aligned}$$

Note that

$$\begin{aligned} \iota_{\mathbb{S}}\,{}_{\mathbb{S}}\operatorname{Gradgrad} &= \operatorname{sym}\operatorname{Gradgrad} = \operatorname{Gradgrad} = \top\operatorname{grad}\top\operatorname{grad}, \\ \iota_{\mathbb{T}}\,{}_{\mathbb{T}}\operatorname{Rot}_{\mathbb{S}} &= \operatorname{devRot}_{\mathbb{S}} = \operatorname{Rot}\iota_{\mathbb{S}} = \top\operatorname{rot}\top\iota_{\mathbb{S}} =: \operatorname{Rot}_{\mathbb{S}}, \\ \operatorname{Div}_{\mathbb{T}} &= \top\operatorname{div}\top\iota_{\mathbb{T}}, \\ \iota_{\mathbb{T}}\,{}_{\mathbb{T}}\operatorname{Grad} &= \operatorname{devGrad} = \operatorname{dev}\top\operatorname{grad}\top, \\ \mathbb{S}\operatorname{Rot}_{\mathbb{T}} &= \operatorname{symRot}_{\mathbb{T}} = \operatorname{sym}\top\operatorname{rot}\top\iota_{\mathbb{T}}, \\ \operatorname{divDiv}_{\mathbb{S}} &= \operatorname{div}\top\operatorname{div}\top\iota_{\mathbb{S}}; \end{aligned}$$

in particular, on symmetric tensor fields, we have $\mathbb{T}\operatorname{Rot}_{\mathbb{S}} = \operatorname{devRot} = \operatorname{Rot}$ (cf. [3, Lemma A.1]). Using these formal operators, we introduce their maximal $\mathsf{L}_2(\Omega)$-realisations, that is,

$$\begin{aligned} {}_{\mathbb{S}}\operatorname{Gradgrad} &: D({}_{\mathbb{S}}\operatorname{Gradgrad}) \subset \mathsf{L}^2(\Omega) \to \mathsf{L}^2_{\mathbb{S}}(\Omega), & u &\mapsto \operatorname{Gradgrad}\ u, \\ {}_{\mathbb{T}}\operatorname{Rot}_{\mathbb{S}} &: D({}_{\mathbb{T}}\operatorname{Rot}_{\mathbb{S}}) \subset \mathsf{L}^2_{\mathbb{S}}(\Omega) \to \mathsf{L}^2_{\mathbb{T}}(\Omega), & S &\mapsto \operatorname{Rot}\ S, \\ \operatorname{Div}_{\mathbb{T}} &: D(\operatorname{Div}_{\mathbb{T}}) \subset \mathsf{L}^2_{\mathbb{T}}(\Omega) \to \mathsf{L}^2(\Omega), & T &\mapsto \operatorname{Div}\ T, \\ {}_{\mathbb{T}}\operatorname{Grad} &: D({}_{\mathbb{T}}\operatorname{Grad}) \subset \mathsf{L}^2(\Omega) \to \mathsf{L}^2_{\mathbb{T}}(\Omega), & v &\mapsto \operatorname{devGrad}\ v, \\ {}_{\mathbb{S}}\operatorname{Rot}_{\mathbb{T}} &: D({}_{\mathbb{S}}\operatorname{Rot}_{\mathbb{T}}) \subset \mathsf{L}^2_{\mathbb{T}}(\Omega) \to \mathsf{L}^2_{\mathbb{S}}(\Omega), & T &\mapsto \operatorname{symRot}\ T, \\ \operatorname{divDiv}_{\mathbb{S}} &: D(\operatorname{divDiv}_{\mathbb{S}}) \subset \mathsf{L}^2_{\mathbb{S}}(\Omega) \to \mathsf{L}^2_{\mathbb{S}}(\Omega), & S &\mapsto \operatorname{divDiv}\ S, \end{aligned}$$

which are densely defined and closed (unbounded) linear operators and form the two (formally primal and dual) biharmonic complexes

$$\cdots \xrightarrow{\ \cdots\ } \mathsf{L}^2(\Omega) \xrightarrow{{}_{\mathbb{S}}\operatorname{Gradgrad}} \mathsf{L}^2_{\mathbb{S}}(\Omega) \xrightarrow{{}_{\mathbb{T}}\operatorname{Rot}_{\mathbb{S}}} \mathsf{L}^2_{\mathbb{T}}(\Omega) \xrightarrow{\operatorname{Div}_{\mathbb{T}}} \mathsf{L}^2(\Omega) \xrightarrow{\ \cdots\ } \cdots,$$

$$\cdots \xleftarrow{\ \cdots\ } \mathsf{L}^2(\Omega) \xleftarrow{\operatorname{divDiv}_{\mathbb{S}}} \mathsf{L}^2_{\mathbb{S}}(\Omega) \xleftarrow{{}_{\mathbb{S}}\operatorname{Rot}_{\mathbb{T}}} \mathsf{L}^2_{\mathbb{T}}(\Omega) \xleftarrow{{}_{\mathbb{T}}\operatorname{Grad}} \mathsf{L}^2(\Omega) \xleftarrow{\ \cdots\ } \cdots,$$

and compare [4] for the complex properties.

Finally, the operators

$${}_{\mathbb{S}}\operatorname{Gradgrad}_{\Gamma_t},\ {}_{\mathbb{T}}\operatorname{Rot}_{\mathbb{S}},\Gamma_t,\ \operatorname{Div}_{\mathbb{T},\Gamma_t},\ {}_{\mathbb{T}}\operatorname{Grad}_{\Gamma_t},\ {}_{\mathbb{S}}\operatorname{Rot}_{\mathbb{T},\Gamma_t},\ \operatorname{divDiv}_{\mathbb{S},\Gamma_t}$$

from Section 2.1 are the restrictions of

$${}_{\mathbb{S}}\operatorname{Gradgrad},\ {}_{\mathbb{T}}\operatorname{Rot}_{\mathbb{S}},\ \operatorname{Div}_{\mathbb{T}},\ {}_{\mathbb{T}}\operatorname{Grad},\ {}_{\mathbb{S}}\operatorname{Rot}_{\mathbb{T}},\ \operatorname{divDiv}_{\mathbb{S}}$$

to their domains of definition

$$D\left({}_{\mathbb{S}}\operatorname{Gradgrad}_{\Gamma_t}\right),\ D\left({}_{\mathbb{T}}\operatorname{Rot}_{\mathbb{S},\Gamma_t}\right),\ D(\operatorname{Div}_{\mathbb{T},\Gamma_t}),\ D\left({}_{\mathbb{T}}\operatorname{Grad}_{\Gamma_t}\right),\ D\left({}_{\mathbb{S}}\operatorname{Rot}_{\mathbb{T},\Gamma_t}\right),\ D\left(\operatorname{divDiv}_{\mathbb{S},\Gamma_t}\right),$$

which are the closures of $\mathsf{C}^{\infty}_{\Gamma_t}(\Omega)$, $\mathsf{C}^{\infty}_{\mathbb{S},\Gamma_t}(\Omega)$, and $\mathsf{C}^{\infty}_{\mathbb{T},\Gamma_t}(\Omega)$ in the corresponding graph norms, respectively.

APPENDIX C: SOME PROOFS

Proof of Theorem 3.1. In [4, Theorem 3.10], we have shown the stated results for $\Gamma_t = \Gamma$ and $\Gamma_t = \emptyset$, which is also a crucial ingredient of this proof. Note that in these two special cases always "strong = weak" holds as $A_n^{**} = \overline{A_n} = A_n$ and that this argument fails in the remaining cases of mixed boundary conditions. Therefore, let $\emptyset \subsetneq \Gamma_t \subsetneq \Gamma$. Moreover, recall the notion of an extendable domain from [1, Section 3]. In particular, $\hat{\Omega}$ and the extended domain $\tilde{\Omega}$ are topologically trivial.

- Let $S \in \mathbf{H}^k_{\mathbb{S},\Gamma_t,0}(\mathrm{Rot},\Omega)$. By definition, S can be extended through Γ_t by zero to the larger domain $\tilde{\Omega}$ yielding
$$\tilde{S} \in \mathbf{H}^k_{\mathbb{S},\emptyset,0}(\mathrm{Rot},\tilde{\Omega}) = \mathbf{H}^k_{\mathbb{S},0}(\mathrm{Rot},\tilde{\Omega}) = \mathsf{H}^k_{\mathbb{S},0}(\mathrm{Rot},\tilde{\Omega}).$$
By [4, Theorem 3.10, Remark 3.11] and Stein's or Calderon's extension theorem—see also [1, Lemma 4.3, Lemma 4.4] for the fact that the respective potentials are already defined on the whole of $\mathbb{R}^3$—there exists $\tilde{u} \in \mathsf{H}^{k+2}(\mathbb{R}^3)$ such that Gradgrad $\tilde{u} = \tilde{S}$ in $\tilde{\Omega}$. Since $\tilde{S} = 0$ in $\hat{\Omega}$, $\tilde{u}$ must be a polynomial $p \in \mathbb{P}^1$ in $\hat{\Omega}$. Far outside of $\tilde{\Omega}$, we modify p by a cut-off function such that the resulting function $\tilde{p}$ is compactly supported and $\tilde{p}|_{\hat{\Omega}} = p$. Note that $\tilde{p}$ depends continuously on S by Poincaré's estimate. Then $u := \tilde{u} - \tilde{p} \in \mathsf{H}^{k+2}(\mathbb{R}^3)$ with $u|_{\hat{\Omega}} = 0$. Hence, u belongs to $\mathsf{H}^{k+2}_{\Gamma_t}(\Omega)$ and depends continuously on S. Moreover, u satisfies Gradgrad $u =$ Gradgrad $\tilde{u} = \tilde{S}$ in $\tilde{\Omega}$, in particular Gradgrad $u = S$ in Ω. We put $\mathcal{P}^k_{_{\mathbb{S}}\mathrm{Gradgrad},\Gamma_t} S := u \in \mathsf{H}^{k+2}_{\Gamma_t}(\Omega)$.
- Let $T \in \mathbf{H}^k_{\mathbb{T},\Gamma_t,0}(\mathrm{Div},\Omega)$. By definition, T can be extended through Γ_t by zero to $\tilde{\Omega}$ giving
$$\tilde{T} \in \mathbf{H}^k_{\mathbb{T},\emptyset,0}(\mathrm{Div},\tilde{\Omega}) = \mathbf{H}^k_{\mathbb{T},0}(\mathrm{Div},\tilde{\Omega}) = \mathsf{H}^k_{\mathbb{T},0}(\mathrm{Div},\tilde{\Omega}).$$
By [4, Theorem 3.10], there exists $\tilde{S} \in \mathsf{H}^{k+1}_{\mathbb{S}}(\mathbb{R}^3)$ such that Rot$\tilde{S} = \tilde{T}$ in $\tilde{\Omega}$. Since $\tilde{T} = 0$ in $\hat{\Omega}$, that is, $\tilde{S}|_{\hat{\Omega}} \in \mathsf{H}^{k+1}_{\mathbb{S},0}(\mathrm{Rot},\hat{\Omega})$, we get again by [4, Theorem 3.10] (or the first part of this proof) $\tilde{u} \in \mathsf{H}^{k+3}(\mathbb{R}^3)$ such that Gradgrad $\tilde{u} = \tilde{S}$ in $\hat{\Omega}$. Then $S := \tilde{S} -$ Gradgrad $\tilde{u}$ belongs to $\mathsf{H}^{k+1}_{\mathbb{S}}(\mathbb{R}^3)$ and satisfies $S|_{\hat{\Omega}} = 0$. Thus, $S \in \mathsf{H}^{k+1}_{\mathbb{S},\Gamma_t}(\Omega)$ and depends continuously on T. Furthermore, Rot $S =$ Rot $\tilde{S} = \tilde{T}$ in $\tilde{\Omega}$, in particular Rot $S = T$ in Ω. We set $\mathcal{P}^k_{_{\mathbb{T}}\mathrm{Rot}_{\mathbb{S}},\Gamma_t} T := S \in \mathsf{H}^{k+1}_{\mathbb{S},\Gamma_t}(\Omega)$.
- Let $v \in \mathsf{H}^k_{\Gamma_t}(\Omega)$. By definition, v can be extended through Γ_t by zero to $\tilde{\Omega}$ defining $\tilde{v} \in \mathsf{H}^k(\tilde{\Omega})$. Theorem 3.10 of [4] yields $\tilde{T} \in \mathsf{H}^{k+1}_{\mathbb{T}}(\mathbb{R}^3)$ such that Div $\tilde{T} = \tilde{v}$ in $\tilde{\Omega}$. As $\tilde{v} = 0$ in $\hat{\Omega}$, that is, $\tilde{T}|_{\hat{\Omega}} \in \mathsf{H}^{k+1}_{\mathbb{T},0}(\mathrm{Div},\hat{\Omega})$, we get again by [4, Theorem 3.10] (or the second part of this proof) $\tilde{S} \in \mathsf{H}^{k+2}_{\mathbb{S}}(\mathbb{R}^3)$ such that Rot $\tilde{S} = \tilde{T}$ holds in $\hat{\Omega}$. Then $T := \tilde{T} -$Rot $\tilde{S}$ belongs to $\mathsf{H}^{k+1}_{\mathbb{T}}(\mathbb{R}^3)$ with $T|_{\hat{\Omega}} = 0$. Hence, T belongs to $\mathsf{H}^{k+1}_{\mathbb{T},\Gamma_t}(\Omega)$ and depends continuously on v. Furthermore, Div $T =$ Div $\tilde{T} = \tilde{v}$ in $\tilde{\Omega}$, in particular Div $T = v$ in Ω. Finally, we define $\mathcal{P}^k_{\mathrm{Div}_{\mathbb{T}},\Gamma_t} v := T \in \mathsf{H}^{k+1}_{\mathbb{T},\Gamma_t}(\Omega)$.
- Let $T \in \mathbf{H}^k_{\mathbb{T},\Gamma_t,0}(\mathrm{symRot},\Omega)$. By definition, T can be extended through Γ_t by zero to $\tilde{\Omega}$ yielding
$$\tilde{T} \in \mathbf{H}^k_{\mathbb{T},\emptyset,0}(\mathrm{symRot},\tilde{\Omega}) = \mathbf{H}^k_{\mathbb{T},0}(\mathrm{symRot},\tilde{\Omega}) = \mathsf{H}^k_{\mathbb{T},0}(\mathrm{symRot},\tilde{\Omega}).$$
By [4, Theorem 3.10], there exists $\tilde{v} \in \mathsf{H}^{k+1}(\mathbb{R}^3)$ such that devGrad $\tilde{v} = \tilde{T}$ in $\tilde{\Omega}$. Since $\tilde{T} = 0$ in $\hat{\Omega}$, $\tilde{v}$ must be a Raviart–Thomas field $r \in \mathbb{RT}$ in $\hat{\Omega}$. Far outside of $\tilde{\Omega}$, we modify r by a cut-off function such that the resulting vector field $\tilde{r}$ is compactly supported and $\tilde{r}|_{\hat{\Omega}} = r$. Then $v := \tilde{v} - \tilde{r} \in \mathsf{H}^{k+1}(\mathbb{R}^3)$ with $v|_{\hat{\Omega}} = 0$. Hence, v belongs to $\mathsf{H}^{k+1}_{\Gamma_t}(\Omega)$ and depends continuously on T. Moreover, v satisfies devGrad $v =$ devGrad $\tilde{v} = \tilde{T}$ in $\tilde{\Omega}$, in particular devGrad $v = T$ in Ω. We put $\mathcal{P}^k_{_{\mathbb{T}}Grad,\Gamma_t} T := v \in \mathsf{H}^{k+1}_{\Gamma_t}(\Omega)$.
- Let $S \in \mathbf{H}^k_{\mathbb{S},\Gamma_t,0}(\mathrm{divDiv},\Omega)$. By definition, S can be extended through Γ_t by zero to $\tilde{\Omega}$ giving
$$\tilde{S} \in \mathbf{H}^k_{\mathbb{S},\emptyset,0}(\mathrm{divDiv},\tilde{\Omega}) = \mathbf{H}^k_{\mathbb{S},0}(\mathrm{divDiv},\tilde{\Omega}) = \mathsf{H}^k_{\mathbb{S},0}(\mathrm{divDiv},\tilde{\Omega}).$$
By [4, Theorem 3.10], there exists $\tilde{T} \in \mathsf{H}^{k+1}_{\mathbb{T}}(\mathbb{R}^3)$ such that symRot $\tilde{T} = \tilde{S}$ in $\tilde{\Omega}$. Since $\tilde{S} = 0$ in $\hat{\Omega}$, that is, $\tilde{T}|_{\hat{\Omega}} \in \mathsf{H}^{k+1}_{\mathbb{T},0}(\mathrm{symRot},\hat{\Omega})$, we get again by [4, Theorem 3.10] (or the fourth part of this proof) $\tilde{v} \in \mathsf{H}^{k+2}(\mathbb{R}^3)$ such that devGrad $\tilde{v} = \tilde{T}$ in $\hat{\Omega}$. Then $T := \tilde{T} -$ devGrad $\tilde{v}$ belongs to $\mathsf{H}^{k+1}_{\mathbb{T}}(\mathbb{R}^3)$ and satisfies $T|_{\hat{\Omega}} = 0$. Thus, $T \in \mathsf{H}^{k+1}_{\mathbb{T},\Gamma_t}(\Omega)$ and

depends continuously on S. Furthermore, $\operatorname{symRot} T = \operatorname{symRot} \tilde{T} = \tilde{S}$ in $\tilde{\Omega}$, in particular $\operatorname{symRot} T = S$ in Ω. We set $\mathcal{P}^k_{{}_{\mathbb{S}}\mathrm{Rot}_{\mathbb{T}},\Gamma_t} S := T \in \mathsf{H}^{k+1}_{\mathbb{T},\Gamma_t}(\Omega)$.

- Let $u \in \mathsf{H}^k_{\Gamma_t}(\Omega)$. By definition, u can be extended through Γ_t by zero to $\tilde{\Omega}$ defining $\tilde{u} \in \mathsf{H}^k(\tilde{\Omega})$. Theorem 3.10 of [4] yields $\tilde{S} \in \mathsf{H}^{k+2}_{\mathbb{S}}(\mathbb{R}^3)$ such that $\operatorname{divDiv} \tilde{S} = \tilde{u}$ in $\tilde{\Omega}$. As $\tilde{u} = 0$ in $\hat{\Omega}$, that is, $\tilde{S}|_{\hat{\Omega}} \in \mathsf{H}^{k+2}_{\mathbb{S},0}(\operatorname{divDiv}, \hat{\Omega})$, we get again by [4, Theorem 3.10] (or the fifth part of this proof) $\tilde{T} \in \mathsf{H}^{k+3}_{\mathbb{T}}(\mathbb{R}^3)$ such that $\operatorname{symRot} \tilde{T} = \tilde{S}$ holds in $\hat{\Omega}$. Then $S := \tilde{S} - \operatorname{symRot} \tilde{T}$ belongs to $\mathsf{H}^{k+2}_{\mathbb{S}}(\mathbb{R}^3)$ with $S|_{\hat{\Omega}} = 0$. Hence, S belongs to $\mathsf{H}^{k+2}_{\mathbb{S},\Gamma_t}(\Omega)$ and depends continuously on u. Furthermore, $\operatorname{divDiv} S = \operatorname{divDiv} \tilde{S} = \tilde{u}$ in $\tilde{\Omega}$, in particular $\operatorname{divDiv} S = u$ in Ω. Finally, we define $\mathcal{P}^k_{\mathrm{divDiv}_{\mathbb{S}},\Gamma_t} u := S \in \mathsf{H}^{k+2}_{\mathbb{S},\Gamma_t}(\Omega)$.

The assertion about the compact supports is trivial. □

Proof of Lemma 3.10 and Corollary 3.11. According to [6, Section 4.2] (cf. [7, Section 4.2], [1, Lemma 3.1], [2], or [3]), let (U_ℓ, φ_ℓ) be a partition of unity for Ω, such that

$$\Omega = \bigcup_{\ell=-L}^{L} \Omega_\ell, \qquad \Omega_\ell := \Omega \cap U_\ell, \qquad \varphi_\ell \in \mathsf{C}^\infty_{\partial U_\ell}(U_\ell),$$

and such that $(\Omega_\ell, \widehat{\Gamma}_{t,\ell})$ are extendable bounded strong Lipschitz pairs. Recall

$$\Sigma_\ell := \partial\Omega_\ell \setminus \Gamma, \qquad \Gamma_{t,\ell} := \Gamma_t \cap U_\ell, \qquad \widehat{\Gamma}_{t,\ell} := int(\Gamma_{t,\ell} \cup \overline{\Sigma}_\ell).$$

- Let $k \geq 0$ and let $S \in \mathbf{H}^k_{\mathbb{S},\Gamma_t}(\mathrm{Rot}, \Omega)$. Then by definition, $S|_{\Omega_\ell} \in \mathbf{H}^k_{\mathbb{S},\Gamma_{t,\ell}}(\mathrm{Rot}, \Omega_\ell)$ and we decompose by Corollary 3.4

$$S|_{\Omega_\ell} = S_{\ell,1} + \operatorname{Gradgrad} u_{\ell,0}$$

with $S_{\ell,1} := \mathcal{Q}^{k,1}_{{}_{\mathbb{T}}\mathrm{Rot}_{\mathbb{S}},\Gamma_{t,\ell}} S|_{\Omega_\ell} \in \mathsf{H}^{k+1}_{\mathbb{S},\Gamma_{t,\ell}}(\Omega_\ell)$ and $u_{\ell,0} := \mathcal{Q}^{k,0}_{{}_{\mathbb{T}}\mathrm{Rot}_{\mathbb{S}},\Gamma_{t,\ell}} S|_{\Omega_\ell} \in \mathsf{H}^{k+2}_{\Gamma_{t,\ell}}(\Omega_\ell)$. Lemma 3.9 yields

$$\begin{aligned} \varphi_\ell S|_{\Omega_\ell} &= \varphi_\ell S_{\ell,1} + \varphi_\ell \operatorname{Gradgrad} u_{\ell,0} \\ &= \overbrace{\varphi_\ell S_{\ell,1} - 2\operatorname{sym}\big((\operatorname{grad}\varphi_\ell)(\operatorname{grad} u_{\ell,0})^\top\big) - u_{\ell,0} \operatorname{Gradgrad} \varphi_\ell}^{=:S_\ell} \\ &\quad + \operatorname{Gradgrad} \underbrace{(\varphi_\ell u_{\ell,0})}_{=:u_\ell} \end{aligned}$$

with $S_\ell \in \mathsf{H}^{k+1}_{\mathbb{S},\widehat{\Gamma}_{t,\ell}}(\Omega_\ell)$ and $u_\ell \in \mathsf{H}^{k+2}_{\widehat{\Gamma}_{t,\ell}}(\Omega_\ell)$. Extending S_ℓ and u_ℓ by zero to Ω gives tensor fields $\tilde{S}_\ell \in \mathsf{H}^{k+1}_{\mathbb{S},\Gamma_t}(\Omega)$ and $\tilde{u}_\ell \in \mathsf{H}^{k+2}_{\Gamma_t}(\Omega)$ as well as

$$\begin{aligned} S = \sum_{\ell=-L}^{L} \varphi_\ell S|_{\Omega_\ell} &= \sum_{\ell=-L}^{L} \tilde{S}_\ell + \operatorname{Gradgrad} \sum_{\ell=-L}^{L} \tilde{u}_\ell \\ &\in \mathsf{H}^{k+1}_{\mathbb{S},\Gamma_t}(\Omega) + \operatorname{Gradgrad} \mathsf{H}^{k+2}_{\Gamma_t}(\Omega) \subset \mathsf{H}^k_{\mathbb{S},\Gamma_t}(\mathrm{Rot}, \Omega). \end{aligned}$$

As all operations have been linear and continuous, we set

$$\mathcal{Q}^{k,1}_{{}_{\mathbb{T}}\mathrm{Rot}_{\mathbb{S}},\Gamma_t} S := \sum_{\ell=-L}^{L} \tilde{S}_\ell \in \mathsf{H}^{k+1}_{\mathbb{S},\Gamma_t}(\Omega), \qquad \mathcal{Q}^{k,0}_{{}_{\mathbb{T}}\mathrm{Rot}_{\mathbb{S}},\Gamma_t} S := \sum_{\ell=-L}^{L} \tilde{u}_\ell \in \mathsf{H}^{k+2}_{\Gamma_t}(\Omega).$$

- Let $k \geq 0$ and let $T \in \mathbf{H}^k_{\mathbb{T},\Gamma_t}(\mathrm{Div}, \Omega)$. Then by definition, $T|_{\Omega_\ell} \in \mathbf{H}^k_{\mathbb{T},\Gamma_{t,\ell}}(\mathrm{Div}, \Omega_\ell)$ and we decompose by Corollary 3.4

$$T|_{\Omega_\ell} = T_{\ell,1} + \operatorname{Rot} S_{\ell,0}$$

with $T_{\ell,1} := Q^{k,1}_{\mathrm{Div}_{\mathbb{T}},\Gamma_{t,\ell}} T|_{\Omega_\ell} \in \mathsf{H}^{k+1}_{\mathbb{T},\Gamma_{t,\ell}}(\Omega_\ell)$ and $S_{\ell,0} := Q^{k,0}_{\mathrm{Div}_{\mathbb{T}},\Gamma_{t,\ell}} T|_{\Omega_\ell} \in \mathsf{H}^{k+1}_{\mathbb{S},\Gamma_{t,\ell}}(\Omega_\ell)$. Lemma 3.9 yields

$$\varphi_\ell T|_{\Omega_\ell} = \varphi_\ell T_{\ell,1} + \varphi_\ell \mathrm{Rot}\ S_{\ell,0} = \underbrace{\varphi_\ell T_{\ell,1} + S_{\ell,0}\mathrm{spn}\,\mathrm{grad}\,\varphi_\ell}_{=:T_\ell} + \mathrm{Rot}(\underbrace{\varphi_\ell S_{\ell,0}}_{=:S_\ell})$$

with $T_\ell \in \mathsf{H}^{k+1}_{\mathbb{T},\hat{\Gamma}_{t,\ell}}(\Omega_\ell)$ and $S_\ell \in \mathsf{H}^{k+1}_{\mathbb{S},\hat{\Gamma}_{t,\ell}}(\Omega_\ell)$. Extending T_ℓ and S_ℓ by zero to Ω gives tensor fields $\tilde{T}_\ell \in \mathsf{H}^{k+1}_{\mathbb{T},\Gamma_t}(\Omega)$ and $\tilde{S}_\ell \in \mathsf{H}^{k+1}_{\mathbb{S},\Gamma_t}(\Omega)$ as well as

$$\begin{aligned} T = \sum_{\ell=-L}^{L} \varphi_\ell T|_{\Omega_\ell} &= \sum_{\ell=-L}^{L} \tilde{T}_\ell + \mathrm{Rot} \sum_{\ell=-L}^{L} \tilde{S}_\ell \\ &\in \mathsf{H}^{k+1}_{\mathbb{T},\Gamma_t}(\Omega) + \mathrm{Rot}\ \mathsf{H}^{k+1}_{\mathbb{S},\Gamma_t}(\Omega) \subset \mathsf{H}^{k}_{\mathbb{T},\Gamma_t}(\mathrm{Div},\Omega). \end{aligned}$$

As all operations have been linear and continuous, we set

$$Q^{k,1}_{\mathrm{Div}_{\mathbb{T}},\Gamma_t} T := \sum_{\ell=-L}^{L} \tilde{T}_\ell \in \mathsf{H}^{k+1}_{\mathbb{T},\Gamma_t}(\Omega), \qquad Q^{k,0}_{\mathrm{Div}_{\mathbb{T}},\Gamma_t} T := \sum_{\ell=-L}^{L} \tilde{S}_\ell \in \mathsf{H}^{k+1}_{\mathbb{S},\Gamma_t}(\Omega).$$

- Let $k \geq 0$ and let $T \in \mathbf{H}^{k}_{\mathbb{T},\Gamma_t}(\mathrm{symRot},\Omega)$. Then by definition, $T|_{\Omega_\ell} \in \mathbf{H}^{k}_{\mathbb{T},\Gamma_{t,\ell}}(\mathrm{symRot},\Omega_\ell)$ and we decompose by Corollary 3.4

$$T|_{\Omega_\ell} = T_{\ell,1} + \mathrm{devGrad}\ v_{\ell,0}$$

with $T_{\ell,1} := Q^{k,1}_{\mathbb{S}\mathrm{Rot}_{\mathbb{T}},\Gamma_{t,\ell}} T|_{\Omega_\ell} \in \mathsf{H}^{k+1}_{\mathbb{T},\Gamma_{t,\ell}}(\Omega_\ell)$ and $v_{\ell,0} := Q^{k,0}_{\mathbb{S}\mathrm{Rot}_{\mathbb{T}},\Gamma_{t,\ell}} T|_{\Omega_\ell} \in \mathsf{H}^{k+1}_{\Gamma_{t,\ell}}(\Omega_\ell)$. Lemma 3.9 yields

$$\begin{aligned} \varphi_\ell T|_{\Omega_\ell} &= \varphi_\ell T_{\ell,1} + \varphi_\ell \mathrm{devGrad}\ v_{\ell,0} \\ &= \underbrace{\varphi_\ell T_{\ell,1} + dev\left(v_{\ell,0}(\mathrm{grad}\varphi_\ell)^\top\right)}_{=:T_\ell} + \mathrm{devGrad}(\underbrace{\varphi_\ell v_{\ell,0}}_{=:v_\ell}) \end{aligned}$$

with $T_\ell \in \mathsf{H}^{k+1}_{\mathbb{T},\hat{\Gamma}_{t,\ell}}(\Omega_\ell)$ and $v_\ell \in \mathsf{H}^{k+1}_{\hat{\Gamma}_{t,\ell}}(\Omega_\ell)$. Extending T_ℓ and v_ℓ by zero to Ω gives tensor fields $\tilde{T}_\ell \in \mathsf{H}^{k+1}_{\mathbb{T},\Gamma_t}(\Omega)$ and $\tilde{v}_\ell \in \mathsf{H}^{k+1}_{\Gamma_t}(\Omega)$ as well as

$$\begin{aligned} T = \sum_{\ell=-L}^{L} \varphi_\ell T|_{\Omega_\ell} &= \sum_{\ell=-L}^{L} \tilde{T}_\ell + \mathrm{devGrad} \sum_{\ell=-L}^{L} \tilde{v}_\ell \\ &\in \mathsf{H}^{k+1}_{\mathbb{T},\Gamma_t}(\Omega) + \mathrm{devGrad}\ \mathsf{H}^{k+1}_{\Gamma_t}(\Omega) \subset \mathsf{H}^{k}_{\mathbb{T},\Gamma_t}(\mathrm{symRot},\Omega). \end{aligned}$$

As all operations have been linear and continuous, we set

$$Q^{k,1}_{\mathbb{S}\mathrm{Rot}_{\mathbb{T}},\Gamma_t} T := \sum_{\ell=-L}^{L} \tilde{T}_\ell \in \mathsf{H}^{k+1}_{\mathbb{T},\Gamma_t}(\Omega), \qquad Q^{k,0}_{\mathbb{S}\mathrm{Rot}_{\mathbb{T}},\Gamma_t} T := \sum_{\ell=-L}^{L} \tilde{v}_\ell \in \mathsf{H}^{k+1}_{\Gamma_t}(\Omega).$$

- Let $k \geq 1$ and let $S \in \mathbf{H}^{k,k-1}_{\mathbb{S},\Gamma_t}(\mathrm{divDiv},\Omega)$. Then by definition, $S|_{\Omega_\ell} \in \mathbf{H}^{k,k-1}_{\mathbb{S},\Gamma_{t,\ell}}(\mathrm{divDiv},\Omega_\ell)$ and we decompose by Corollary 3.7

$$S|_{\Omega_\ell} = S_{\ell,1} + \mathrm{symRot}\ T_{\ell,0}$$

with $S_{\ell,1} := Q^{k,k-1,1}_{\mathrm{divDiv}_{\mathbb{S}},\Gamma_{t,\ell}} S|_{\Omega_\ell} \in \mathsf{H}^{k+1}_{\mathbb{S},\Gamma_{t,\ell}}(\Omega_\ell)$ and $T_{\ell,0} := Q^{k,k-1,0}_{\mathrm{divDiv}_{\mathbb{S}},\Gamma_{t,\ell}} S|_{\Omega_\ell} \in \mathsf{H}^{k+1}_{\mathbb{T},\Gamma_{t,\ell}}(\Omega_\ell)$. Thus,

$$\begin{aligned} \varphi_\ell S|_{\Omega_\ell} &= \varphi_\ell S_{\ell,1} + \varphi_\ell \mathrm{symRot}\ T_{\ell,0} \\ &= \underbrace{\varphi_\ell S_{\ell,1} + \mathrm{sym}(T_{\ell,0}\mathrm{spn}\,\mathrm{grad}\,\varphi_\ell)}_{=:S_\ell} + \mathrm{symRot}(\underbrace{\varphi_\ell T_{\ell,0}}_{=:T_\ell}) \end{aligned} \tag{C1}$$

with $S_\ell \in \mathsf{H}^{k+1}_{\mathbb{S},\hat{\Gamma}_{t,\ell}}(\Omega_\ell)$ and $T_\ell \in \mathsf{H}^{k+1}_{\mathbb{T},\hat{\Gamma}_{t,\ell}}(\Omega_\ell)$. Extending S_ℓ and T_ℓ by zero to Ω gives fields $\tilde{S}_\ell \in \mathsf{H}^{k+1}_{\mathbb{S},\Gamma_t}(\Omega)$ and $\tilde{T}_\ell \in \mathsf{H}^{k+1}_{\mathbb{T},\Gamma_t}(\Omega)$ as well as

$$S = \sum_{\ell=-L}^{L} \varphi_\ell S|_{\Omega_\ell} = \sum_{\ell=-L}^{L} \tilde{S}_\ell + \operatorname{symRot} \sum_{\ell=-L}^{L} \tilde{T}_\ell$$
$$\in \mathsf{H}^{k+1}_{\mathbb{S},\Gamma_t}(\Omega) + \operatorname{symRot} \mathsf{H}^{k+1}_{\mathbb{T},\Gamma_t}(\Omega) \subset \mathsf{H}^{k,k-1}_{\mathbb{S},\Gamma_t}(\operatorname{divDiv}, \Omega).$$

As all operations have been linear and continuous, we set

$$\mathcal{Q}^{k,k-1,1}_{\operatorname{divDiv}_{\mathbb{S}},\Gamma_t} S := \sum_{\ell=-L}^{L} \tilde{S}_\ell \in \mathsf{H}^{k+1}_{\mathbb{S},\Gamma_t}(\Omega), \qquad \mathcal{Q}^{k,k-1,0}_{\operatorname{divDiv}_{\mathbb{S}},\Gamma_t} S := \sum_{\ell=-L}^{L} \tilde{T}_\ell \in \mathsf{H}^{k+1}_{\mathbb{T},\Gamma_t}(\Omega).$$

- Let $k \geq 0$ and let $S \in \mathbf{H}^{k}_{\mathbb{S},\Gamma_t}(\operatorname{divDiv}, \Omega)$. Then by definition, $S|_{\Omega_\ell} \in \mathbf{H}^{k}_{\mathbb{S},\Gamma_{t,\ell}}(\operatorname{divDiv}, \Omega_\ell)$ and we decompose by Corollary 3.4

$$S|_{\Omega_\ell} = S_{\ell,1} + \operatorname{symRot} T_{\ell,0}$$

with $S_{\ell,1} := \mathcal{Q}^{k,k-1,1}_{\operatorname{divDiv}_{\mathbb{S}},\Gamma_{t,\ell}} S|_{\Omega_\ell} \in \mathsf{H}^{k+2}_{\mathbb{S},\Gamma_{t,\ell}}(\Omega_\ell)$ and $T_{\ell,0} := \mathcal{Q}^{k,k-1,0}_{\operatorname{divDiv}_{\mathbb{S}},\Gamma_{t,\ell}} S|_{\Omega_\ell} \in \mathsf{H}^{k+1}_{\mathbb{T},\Gamma_{t,\ell}}(\Omega_\ell)$. Now we follow the arguments from (C1). Note that still only $S_\ell \in \mathsf{H}^{k+1}_{\mathbb{S},\hat{\Gamma}_{t,\ell}}(\Omega_\ell)$ holds, that is, we have lost one order of regularity for S_ℓ. Nevertheless, we get

$$S \in \mathsf{H}^{k+1}_{\mathbb{S},\Gamma_t}(\Omega) + \operatorname{symRot} \mathsf{H}^{k+1}_{\mathbb{T},\Gamma_t}(\Omega),$$

and all operations have been linear and continuous. But this implies by the previous step

$$S \in \mathsf{H}^{k+1,k}_{\mathbb{S},\Gamma_t}(\operatorname{divDiv}, \Omega) + \operatorname{symRot} \mathsf{H}^{k+1}_{\mathbb{T},\Gamma_t}(\Omega).$$

Again, by the previous step, we obtain

$$S \in \mathsf{H}^{k+2}_{\mathbb{S},\Gamma_t}(\Omega) + \operatorname{symRot} \mathsf{H}^{k+2}_{\mathbb{T},\Gamma_t}(\Omega) + \operatorname{symRot} \mathsf{H}^{k+1}_{\mathbb{T},\Gamma_t}(\Omega)$$
$$= \mathsf{H}^{k+2}_{\mathbb{S},\Gamma_t}(\Omega) + \operatorname{symRot} \mathsf{H}^{k+1}_{\mathbb{T},\Gamma_t}(\Omega) \subset \mathsf{H}^{k}_{\mathbb{S},\Gamma_t}(\operatorname{divDiv}, \Omega),$$

and all operations have been linear and continuous.

It remains to prove the assertions on the operators devGrad and Gradgrad.

- Let $v \in \mathbf{H}^{k}_{\Gamma_t}(\operatorname{devGrad}, \Omega)$. Then by Corollary 3.6,

$$\varphi_\ell v \in \mathbf{H}^{k}_{\hat{\Gamma}_{t,\ell}}(\operatorname{devGrad}, \Omega_\ell) = \mathsf{H}^{k}_{\hat{\Gamma}_{t,\ell}}(\operatorname{devGrad}, \Omega_\ell) = \mathsf{H}^{k+1}_{\hat{\Gamma}_{t,\ell}}(\Omega_\ell).$$

Extending $\varphi_\ell v$ by zero to Ω yields $v_\ell \in \mathsf{H}^{k+1}_{\Gamma_t}(\Omega)$ and $v = \sum_\ell \varphi_\ell v = \sum_\ell v_\ell \in \mathsf{H}^{k+1}_{\Gamma_t}(\Omega)$.

- Let $u \in \mathbf{H}^{k}_{\Gamma_t}(\operatorname{Gradgrad}, \Omega)$. Then by Corollary 3.6,

$$\varphi_\ell u \in \mathbf{H}^{k}_{\hat{\Gamma}_{t,\ell}}(\operatorname{Gradgrad}, \Omega_\ell) = \mathsf{H}^{k}_{\hat{\Gamma}_{t,\ell}}(\operatorname{Gradgrad}, \Omega_\ell) = \mathsf{H}^{k+2}_{\hat{\Gamma}_{t,\ell}}(\Omega_\ell).$$

Extending $\varphi_\ell u$ by zero to Ω yields $u_\ell \in \mathsf{H}^{k+2}_{\Gamma_t}(\Omega)$ and $u = \sum_\ell \varphi_\ell u = \sum_\ell u_\ell \in \mathsf{H}^{k+2}_{\Gamma_t}(\Omega)$.

- Let $u \in \mathbf{H}^{k,k-1}_{\Gamma_t}(\operatorname{Gradgrad}, \Omega)$. Then $\varphi_\ell u \in \mathbf{H}^{k,k-1}_{\hat{\Gamma}_{t,\ell}}(\operatorname{Gradgrad}, \Omega_\ell) = \mathsf{H}^{k+1}_{\hat{\Gamma}_{t,\ell}}(\Omega_\ell)$ by (7). Extending $\varphi_\ell u$ by zero to Ω yields $u_\ell \in \mathsf{H}^{k+1}_{\Gamma_t}(\Omega)$ and $u = \sum_\ell \varphi_\ell u = \sum_\ell u_\ell \in \mathsf{H}^{k+1}_{\Gamma_t}(\Omega)$.

The proof is finished.

Proof of Theorem 3.12. Note that these types of compact embeddings are independent of ε and μ (cf. [5, Lemma 5.1]). So, let $\varepsilon = \mu = \mathrm{id}$. Lemma 3.10 (for $k = 0$) yields, for example, the bounded regular decomposition

$$D(A_1) = \mathsf{H}_{\mathbb{S},\Gamma_t}(\mathrm{Rot}, \Omega) = \mathsf{H}^1_{\mathbb{S},\Gamma_t}(\Omega) + \mathrm{Gradgrad}\ \mathsf{H}^2_{\Gamma_t}(\Omega)$$

with $\mathsf{H}^+_1 = \mathsf{H}^1_{\mathbb{S},\Gamma_t}(\Omega)$ and $\mathsf{H}^+_0 = \mathsf{H}^2_{\Gamma_t}(\Omega)$ and $\mathsf{H}_1 = \mathsf{L}^2_{\mathbb{S}}(\Omega)$, $\mathsf{H}_0 = \mathsf{L}^2(\Omega)$. Rellich's selection theorem and [3, Corollary 2.12] (cf. [1, Lemma 2.22]) yield that $D(A_1) \cap D(A^*_0) \hookrightarrow \mathsf{H}_1$ is compact. Analogously, we show the compactness of $D(A_2) \cap D(A^*_1) \hookrightarrow \mathsf{H}_2$ using, for example, the bounded regular decomposition $D(A_2) = \mathsf{H}_{\mathbb{T},\Gamma_t}(\mathrm{Div}, \Omega) = \mathsf{H}^1_{\mathbb{T},\Gamma_t}(\Omega) + \mathrm{Rot}\ \mathsf{H}^1_{\mathbb{S},\Gamma_t}(\Omega)$. □

Proof of Theorem 3.16. We only show the representations for $R\left({}_{\mathbb{T}}\mathrm{Rot}^k_{\mathbb{S},\Gamma_t}\right)$ and $R\left(\mathrm{divDiv}^k_{\mathbb{S},\Gamma_t}\right)$. The others follow analogously.

- By Lemma 3.10 and Corollary 3.11, we have

$$R\left({}_{\mathbb{T}}\mathrm{Rot}^k_{\mathbb{S},\Gamma_t}\right) = \mathrm{Rot}\ \mathsf{H}^k_{\mathbb{S},\Gamma_t}(\mathrm{Rot}, \Omega) = \mathrm{Rot}\ \mathsf{H}^{k+1}_{\mathbb{S},\Gamma_t}(\Omega). \tag{C2}$$

  Moreover,

$$\begin{aligned} R\left({}_{\mathbb{T}}\mathrm{Rot}^k_{\mathbb{S},\Gamma_t}\right) &\subset \mathsf{H}^k_{\mathbb{T},\Gamma_t,0}(\mathrm{Div}, \Omega) \cap \mathcal{H}_{\mathbb{T},\Gamma_n,\Gamma_t,\mu}(\Omega)^{\perp_{\mathsf{L}^2_{\mathbb{T}}(\Omega)}} \\ &= \mathsf{H}^k_{\mathbb{T},\Gamma_t}(\Omega) \cap \mathsf{H}_{\mathbb{T},\Gamma_t,0}(\mathrm{Div}, \Omega) \cap \mathcal{H}_{\mathbb{T},\Gamma_n,\Gamma_t,\mu}(\Omega)^{\perp_{\mathsf{L}^2_{\mathbb{T}}(\Omega)}} = \mathsf{H}^k_{\mathbb{T},\Gamma_t}(\Omega) \cap R\left({}_{\mathbb{T}}\mathrm{Rot}_{\mathbb{S},\Gamma_t}\right), \end{aligned}$$

  since by Theorem 3.15 (iv)

$$R\left({}_{\mathbb{T}}\mathrm{Rot}_{\mathbb{S},\Gamma_t}\right) = \mathsf{H}_{\mathbb{T},\Gamma_t,0}(\mathrm{Div}, \Omega) \cap \mathcal{H}_{\mathbb{T},\Gamma_n,\Gamma_t,\mu}(\Omega)^{\perp_{\mathsf{L}^2_{\mathbb{T}}(\Omega)}}. \tag{C3}$$

  Thus, it remains to show

$$\mathsf{H}^k_{\mathbb{T},\Gamma_t,0}(\mathrm{Div}, \Omega) \cap \mathcal{H}_{\mathbb{T},\Gamma_n,\Gamma_t,\mu}(\Omega)^{\perp_{\mathsf{L}^2_{\mathbb{T}}(\Omega)}} \subset \mathrm{Rot}\ \mathsf{H}^k_{\mathbb{S},\Gamma_t}(\mathrm{Rot}, \Omega), \qquad k \geq 1.$$

  For this, let $k \geq 1$ and $T \in \mathsf{H}^k_{\mathbb{T},\Gamma_t,0}(\mathrm{Div}, \Omega) \cap \mathcal{H}_{\mathbb{T},\Gamma_n,\Gamma_t,\mu}(\Omega)^{\perp_{\mathsf{L}^2_{\mathbb{T}}(\Omega)}}$. By (C3) and (C2), we have

$$T \in R\left({}_{\mathbb{T}}\mathrm{Rot}_{\mathbb{S},\Gamma_t}\right) = \mathrm{Rot}\ \mathsf{H}^1_{\mathbb{S},\Gamma_t}(\Omega),$$

  and hence there is $S_1 \in \mathsf{H}^1_{\mathbb{S},\Gamma_t}(\Omega)$ such that $\mathrm{Rot}\ S_1 = T$. We see $S_1 \in \mathsf{H}^1_{\mathbb{S},\Gamma_t}(\mathrm{Rot}, \Omega)$. Hence, we are done for $k = 1$. For $k \geq 2$, we have $T \in \mathrm{Rot}\ \mathsf{H}^1_{\mathbb{S},\Gamma_t}(\mathrm{Rot}, \Omega) = \mathrm{Rot}\ \mathsf{H}^2_{\mathbb{S},\Gamma_t}(\Omega)$ by (C2). Thus there is $S_2 \in \mathsf{H}^2_{\mathbb{S},\Gamma_t}(\Omega)$ such that $\mathrm{Rot} S_2 = T$. Then $S_2 \in \mathsf{H}^2_{\mathbb{S},\Gamma_t}(\mathrm{Rot}, \Omega)$, and we are done for $k = 2$. After finitely many steps, we observe that T belongs to $\mathrm{Rot}\ \mathsf{H}^k_{\mathbb{S},\Gamma_t}(\mathrm{Rot}, \Omega)$.
- By Lemma 3.10 and Corollary 3.11, we have

$$\begin{aligned} \mathrm{divDiv}\mathsf{H}^{k+2}_{\mathbb{S},\Gamma_t}(\Omega) &\subset \mathrm{divDiv}\ \mathsf{H}^{k+1,k}_{\mathbb{S},\Gamma_t}(\mathrm{divDiv}, \Omega) = R\left(\mathrm{divDiv}^{k+1,k}_{\mathbb{S},\Gamma_t}\right) \\ &\subset \mathrm{divDiv}\ \mathsf{H}^k_{\mathbb{S},\Gamma_t}(\mathrm{divDiv}, \Omega) = R\left(\mathrm{divDiv}^k_{\mathbb{S},\Gamma_t}\right) = \mathrm{divDiv}\ \mathsf{H}^{k+2}_{\mathbb{S},\Gamma_t}(\Omega). \end{aligned}$$

  In particular,

$$R\left(\mathrm{divDiv}^k_{\mathbb{S},\Gamma_t}\right) = \mathrm{divDiv}\ \mathsf{H}^k_{\mathbb{S},\Gamma_t}(\mathrm{divDiv}, \Omega) = \mathrm{divDiv}\ \mathsf{H}^{k+2}_{\mathbb{S},\Gamma_t}(\Omega). \tag{C4}$$

  Moreover,

$$R\left(\mathrm{divDiv}^k_{\mathbb{S},\Gamma_t}\right) \subset \mathsf{H}^k_{\Gamma_t}(\Omega) \cap \left(\mathbb{P}^1{}_{\Gamma_n}\right)^{\perp_{\mathsf{L}^2(\Omega)}} = \mathsf{H}^k_{\Gamma_t}(\Omega) \cap R\left(\mathrm{divDiv}_{\mathbb{S},\Gamma_t}\right),$$

since

$$R\left(\operatorname{divDiv}_{\mathbb{S},\Gamma_t}\right) = \mathsf{L}^2(\Omega) \cap \left(\mathbb{P}^1{}_{\Gamma_n}\right)^{\perp_{\mathsf{L}^2(\Omega)}}. \tag{C5}$$

Thus, it remains to show

$$\mathsf{H}^k_{\Gamma_t}(\Omega) \cap \left(\mathbb{P}^1_{\Gamma_n}\right)^{\perp_{\mathsf{L}^2(\Omega)}} \subset \operatorname{divDiv}\ \mathsf{H}^k_{\mathbb{S},\Gamma_t}(\operatorname{divDiv},\Omega), \qquad k \geq 1.$$

For this, let $k \geq 1$ and $u \in \mathsf{H}^k_{\Gamma_t}(\Omega) \cap \left(\mathbb{P}^1{}_{\Gamma_n}\right)^{\perp_{\mathsf{L}^2(\Omega)}}$. By (C5) and (C4), we have

$$u \in R\left(\operatorname{divDiv}_{\mathbb{S},\Gamma_t}\right) = \operatorname{divDiv}\ \mathsf{H}^2_{\mathbb{S},\Gamma_t}(\Omega),$$

and hence there is $S_1 \in \mathsf{H}^2_{\mathbb{S},\Gamma_t}(\Omega)$ such that $\operatorname{divDiv}\ S_1 = u$. We see $S_1 \in \mathsf{H}^2_{\mathbb{S},\Gamma_t}(\operatorname{divDiv},\Omega)$ resp. $S_1 \in \mathsf{H}^1_{\mathbb{S},\Gamma_t}(\operatorname{divDiv},\Omega)$ if $k = 1$. Hence, we are done for $k = 1$ and $k = 2$. For $k \geq 2$, we have $u \in \operatorname{divDiv}\ \mathsf{H}^2_{\mathbb{S},\Gamma_t}(\operatorname{divDiv},\Omega) = \operatorname{divDiv}\ \mathsf{H}^4_{\mathbb{S},\Gamma_t}(\Omega)$ by (C4). Thus, there is $S_2 \in \mathsf{H}^4_{\mathbb{S},\Gamma_t}(\Omega)$ such that $\operatorname{divDiv} S_2 = u$. Then $S_2 \in \mathsf{H}^4_{\mathbb{S},\Gamma_t}(\operatorname{divDiv},\Omega)$ resp. $S_2 \in \mathsf{H}^3_{\mathbb{S},\Gamma_t}(\operatorname{divDiv},\Omega)$ if $k = 3$, and we are done for $k = 3$ and $k = 4$. After finitely many steps, we observe that u belongs to $\operatorname{divDiv}\ \mathsf{H}^k_{\mathbb{S},\Gamma_t}(\operatorname{divDiv},\Omega)$, finishing the proof. □

Proof of Theorem 3.19. We follow in close lines the proof of [3, Theorem 4.11] (cf. [1, Theorem 4.16] and [2, Theorem 3.19]), using induction. The case $k = 0$ is given by Theorem 3.12. Let $k \geq 1$ and let (S_ℓ) be a bounded sequence in $\mathsf{H}^k_{\mathbb{S},\Gamma_t}(\operatorname{Rot},\Omega) \cap \mathsf{H}^k_{\mathbb{S},\Gamma_n}(\operatorname{divDiv},\Omega)$. Note that

$$\mathsf{H}^k_{\mathbb{S},\Gamma_t}(\operatorname{Rot},\Omega) \cap \mathsf{H}^k_{\mathbb{S},\Gamma_n}(\operatorname{divDiv},\Omega) \subset \mathsf{H}^k_{\mathbb{S},\Gamma_t}(\Omega) \cap \mathsf{H}^k_{\mathbb{S},\Gamma_n}(\Omega) = \mathsf{H}^k_{\mathbb{S},\Gamma}(\Omega).$$

By assumption and w.l.o.g., we have that (S_ℓ) is a Cauchy sequence in $\mathsf{H}^{k-1}_{\mathbb{S},\Gamma}(\Omega)$. Moreover, for all $|\alpha| = k$, we have $\partial^\alpha S_\ell \in \mathsf{H}_{\mathbb{S},\Gamma_t}(\operatorname{Rot},\Omega) \cap \mathsf{H}_{\mathbb{S},\Gamma_n}(\operatorname{divDiv},\Omega)$ with $\operatorname{Rot}\ \partial^\alpha S_\ell = \partial^\alpha \operatorname{Rot}\ S_\ell$ and $\operatorname{divDiv}\ \partial^\alpha S_\ell = \partial^\alpha \operatorname{divDiv}\ S_\ell$ by Lemma 3.18. Hence, $(\partial^\alpha S_\ell)$ is a bounded sequence in the zero order space $\mathsf{H}_{\mathbb{S},\Gamma_t}(\operatorname{Rot},\Omega) \cap \mathsf{H}_{\mathbb{S},\Gamma_n}(\operatorname{divDiv},\Omega)$. Thus, w.l.o.g. $(\partial^\alpha S_\ell)$ is a Cauchy sequence in $\mathsf{L}^2_{\mathbb{S}}(\Omega)$ by Theorem 3.12. Finally, (S_ℓ) is a Cauchy sequence in $\mathsf{H}^k\mathbb{S},\Gamma(\Omega)$. Analogously, we show the assertion for the second compact embedding. □

Proof of Remark 3.20. Let (S_ℓ) be a bounded sequence in $\mathsf{H}^k_{\mathbb{S},\Gamma_t}(\operatorname{Rot},\Omega) \cap \mathsf{H}^k_{\mathbb{S},\Gamma_n}(\operatorname{divDiv},\Omega)$. In particular, (S_ℓ) is bounded in $\mathsf{H}^k_{\mathbb{S},\Gamma_t}(\operatorname{Rot},\Omega) \cap \mathsf{H}^{k,k-1}_{\mathbb{S},\Gamma_n}(\operatorname{divDiv},\Omega)$. According to Lemma 3.10, that is,

$$\mathsf{H}^{k,k-1}_{\mathbb{S},\Gamma_t}(\operatorname{divDiv},\Omega) = \mathsf{H}^{k+1}_{\mathbb{S},\Gamma_t}(\Omega) + \operatorname{symRot}\ \mathsf{H}^{k+1}_{\mathbb{T},\Gamma_t}(\Omega),$$

we decompose $S_\ell = \tilde{S}_\ell + \operatorname{symRot}\ T_\ell$ with $\tilde{S}_\ell \in \mathsf{H}^{k+1}_{\mathbb{S},\Gamma_t}(\Omega)$ and $T_\ell \in \mathsf{H}^{k+1}_{\mathbb{T},\Gamma_t}(\Omega)$. By the boundedness of the regular decomposition operators, $(\tilde{S}_\ell)$ and (T_ℓ) are bounded in $\mathsf{H}^{k+1}_{\mathbb{S},\Gamma_t}(\Omega)$ and $\mathsf{H}^{k+1}_{\mathbb{T},\Gamma_t}(\Omega)$, respectively. W.l.o.g. $(\tilde{S}_\ell)$ and (T_ℓ) converge in $\mathsf{H}^k_{\mathbb{S},\Gamma_t}(\Omega)$ and $\mathsf{H}^k_{\mathbb{T},\Gamma_t}(\Omega)$, respectively. For all $0 \leq |\alpha| \leq k$, Lemma 3.18 yields $(\partial^\alpha S_\ell) \subset \mathsf{H}_{\mathbb{S},\Gamma_t}(\operatorname{Rot},\Omega)$ and $\operatorname{Rot}\ \partial^\alpha S_\ell = \partial^\alpha \operatorname{Rot}\ S_\ell$. With the notations $S_{\ell,l} := S_\ell - S_l$, $\tilde{S}_{\ell,l} := \tilde{S}_\ell - \tilde{S}_l$, and $T_{\ell,l} := T_\ell - T_l$, we get

$$\begin{aligned}
\left|S_{\ell,l}\right|^2_{\mathsf{H}^k_{\mathbb{S}}(\Omega)} &= \langle S_{\ell,l}, \tilde{S}_{\ell,l}\rangle_{\mathsf{H}^k_{\mathbb{S}}(\Omega)} + \langle S_{\ell,l}, \operatorname{symRot}\ T_{\ell,l}\rangle_{\mathsf{H}^k_{\mathbb{S}}(\Omega)} \\
&= \langle S_{\ell,l}, \tilde{S}_{\ell,l}\rangle_{\mathsf{H}^k_{\mathbb{S}}(\Omega)} + \langle \operatorname{Rot}\ S_{\ell,l}, T_{\ell,l}\rangle_{\mathsf{H}^k_{\mathbb{T}}(\Omega)} \leq c\left(\left|\tilde{S}_{\ell,l}\right|_{\mathsf{H}^k_{\mathbb{S}}(\Omega)} + \left|T_{\ell,l}\right|_{\mathsf{H}^k_{\mathbb{T}}(\Omega)}\right) \to 0,
\end{aligned}$$

completing the proof. □

Proof of Theorem 3.29. Theorem 3.25 and (15) show

$$\mathsf{H}^k_{\mathbb{S},\Gamma_t}(\mathrm{Rot},\Omega) = R\left(\widetilde{\mathcal{Q}}^{k,1}_{{}_\mathbb{T}\mathrm{Rot}_{\mathbb{S},\Gamma_t}}\right) \dotplus \mathsf{H}^k_{\mathbb{S},\Gamma_t,0}(\mathrm{Rot},\Omega),$$
$$\mathsf{H}^k_{\mathbb{S},\Gamma_t,0}(\mathrm{Rot},\Omega) = \mathrm{Gradgrad}\ \mathsf{H}^{k+2}_{\Gamma_t}(\Omega) + \mathrm{Lin}\,\mathcal{B}^{{}_\mathbb{T}\mathrm{Rot}_{\mathbb{S},\Gamma_t}}(\Omega).$$

To prove the directness of the second sum, let

$$\sum_{\ell=1}^{d_{\Omega,\mathbb{S},\Gamma_t}} \lambda_\ell B_\ell^{{}_\mathbb{T}\mathrm{Rot}_{\mathbb{S},\Gamma_t}} \in \mathrm{Gradgrad}\ \mathsf{H}^{k+2}_{\Gamma_t}(\Omega) \cap \mathrm{Lin}\,\mathcal{B}^{{}_\mathbb{T}\mathrm{Rot}_{\mathbb{S},\Gamma_t}}(\Omega).$$

Then $0 = \sum_\ell \lambda_\ell \pi_{N(\mathrm{divDiv}_{\mathbb{S},\Gamma_n}\varepsilon)} B_\ell^{{}_\mathbb{T}\mathrm{Rot}_{\mathbb{S},\Gamma_t}} \in \mathrm{Lin}\pi_{N(\mathrm{divDiv}_{\mathbb{S},\Gamma_n}\varepsilon)}\mathcal{B}^{{}_\mathbb{T}\mathrm{Rot}_{\mathbb{S},\Gamma_t}}$ and therefore $\lambda_\ell = 0$ for all ℓ as $\pi_{N(\mathrm{divDiv}_{\mathbb{S},\Gamma_n}\varepsilon)}\mathcal{B}^{{}_\mathbb{T}\mathrm{Rot}_{\mathbb{S},\Gamma_t}}$ is a basis of $\mathcal{H}^{\mathbb{S},\Gamma_t,\Gamma_n,\varepsilon}(\Omega)$ by Theorem 3.28. Concerning the boundedness of the decompositions, let

$$\mathsf{H}^k_{\mathbb{S},\Gamma_t,0}(\mathrm{Rot},\Omega) \ni S = \mathrm{Gradgrad}\ u + B, \qquad u \in \mathsf{H}^{k+2}_{\Gamma_t}(\Omega),\ B \in \mathrm{Lin}\,\mathcal{B}^{{}_\mathbb{T}\mathrm{Rot}_{\mathbb{S},\Gamma_t}}(\Omega).$$

By Theorem 3.24 Gradgrad $u \in R({}_\mathbb{S}\mathrm{Gradgrad}^k_{\Gamma_t})$ and $\tilde{u} := \mathcal{P}^k_{{}_\mathbb{S}\mathrm{Gradgrad},\Gamma_t}\mathrm{Gradgrad}\ u \in \mathsf{H}^{k+2}_{\Gamma_t}(\Omega)$ solves Gradgrad $\tilde{u} =$ Gradgrad u with $|\tilde{u}|_{\mathsf{H}^{k+2}(\Omega)} \le c\,|\mathrm{Gradgrad}\ u|_{\mathsf{H}^k_\mathbb{S}(\Omega)}$. Therefore,

$$|\tilde{u}|_{\mathsf{H}^{k+2}(\Omega)} + |B|_{\mathsf{H}^k_\mathbb{S}(\Omega)} \le c\left(|\mathrm{Gradgrad} u|_{\mathsf{H}^k_\mathbb{S}(\Omega)} + |B|_{\mathsf{H}^k_\mathbb{S}(\Omega)}\right) \le c\left(|S|_{\mathsf{H}^k_\mathbb{S}(\Omega)} + |B|_{\mathsf{H}^k_\mathbb{S}(\Omega)}\right).$$

Note that the mapping

$$\begin{array}{rccc} I_{\pi_{N(\mathrm{divDiv}_{\mathbb{S},\Gamma_n}\varepsilon)}}: & \mathrm{Lin}\,\mathcal{B}^{{}_\mathbb{T}\mathrm{Rot}_{\mathbb{S},\Gamma_t}}(\Omega) & \to & \mathrm{Lin}\pi_{N(\mathrm{divDiv}_{\mathbb{S},\Gamma_n}\varepsilon)}\mathcal{B}^{{}_\mathbb{T}\mathrm{Rot}_{\mathbb{S},\Gamma_t}}(\Omega) = \mathcal{H}_{\mathbb{S},\Gamma_t,\Gamma_n,\varepsilon}(\Omega) \\ & B_\ell^{{}_\mathbb{T}\mathrm{Rot}_{\mathbb{S},\Gamma_t}} & \mapsto & \pi_{N(\mathrm{divDiv}_{\mathbb{S},\Gamma_n}\varepsilon)}\mathcal{B}^{{}_\mathbb{T}\mathrm{Rot}_{\mathbb{S},\Gamma_t}}{}_\ell \end{array}$$

is a topological isomorphism (between finite dimensional spaces and with arbitrary norms). Thus,

$$|B|_{\mathsf{H}^k_\mathbb{S}(\Omega)} \le c|B|_{\mathsf{L}^2_\mathbb{S}(\Omega)} \le c\left|\pi_{N(\mathrm{divDiv}_{\mathbb{S},\Gamma_n}\varepsilon)}B\right|_{\mathsf{L}^2_\mathbb{S}(\Omega)} = c\left|\pi_{N(\mathrm{divDiv}_{\mathbb{S},\Gamma_n}\varepsilon)}S\right|_{\mathsf{L}^2_\mathbb{S}(\Omega)} \le c|S|_{\mathsf{L}^2_\mathbb{S}(\Omega)} \le c|S|_{\mathsf{H}^k_\mathbb{S}(\Omega)}.$$

Finally, we see $S = \mathrm{Gradgrad}\ \tilde{u} + B \in \mathrm{Gradgrad}\ \mathsf{H}^{k+2}_{\Gamma_t}(\Omega) \dotplus \mathrm{Lin}\,\mathcal{B}^{{}_\mathbb{T}\mathrm{Rot}_{\mathbb{S},\Gamma_t}}(\Omega)$ and

$$|\tilde{u}|_{\mathsf{H}^{k+2}(\Omega)} + |B|_{\mathsf{H}^k_\mathbb{S}(\Omega)} \le c|S|_{\mathsf{H}^k_\mathbb{S}(\Omega)}.$$

The other assertions for $\mathsf{H}^k_{\mathbb{T},\Gamma_n}(\mathrm{Div},\Omega)$, $\mathsf{H}^k_{\mathbb{T},\Gamma_t}(\mathrm{symRot},\Omega)$, $\mathsf{H}^k_{\mathbb{S},\Gamma_n}(\mathrm{divDiv},\Omega)$, and $\mathsf{H}^{k+1,k}_{\mathbb{S},\Gamma_n}(\mathrm{divDiv},\Omega)$ follow analogously. □

Proof of Theorem 3.31. For $k = 0$ and $S \in \mathcal{H}_{\mathbb{S},\Gamma_t,\Gamma_n,\varepsilon}(\Omega) \cap \mathcal{B}^{{}_\mathbb{T}\mathrm{Rot}_{\mathbb{S},\Gamma_t}}(\Omega)^{\perp_{\mathsf{L}^2_{\mathbb{S},\varepsilon}(\Omega)}}$ we have

$$\begin{aligned} 0 = \left\langle S, B_\ell^{{}_\mathbb{T}\mathrm{Rot}_{\mathbb{S},\Gamma_t}}\right\rangle_{\mathsf{L}^2_{\mathbb{S},\varepsilon}(\Omega)} &= \left\langle \pi_{N(\mathrm{divDiv}_{\mathbb{S},\Gamma_n}\varepsilon)}S, B_\ell^{{}_\mathbb{T}\mathrm{Rot}_{\mathbb{S},\Gamma_t}}\right\rangle_{\mathsf{L}^2_{\mathbb{S},\varepsilon}(\Omega)} \\ &= \left\langle S, \pi_{N(\mathrm{divDiv}_{\mathbb{S},\Gamma_n}\varepsilon)}B_\ell^{{}_\mathbb{T}\mathrm{Rot}_{\mathbb{S},\Gamma_t}}\right\rangle_{\mathsf{L}^2_{\mathbb{S},\varepsilon}(\Omega)} \end{aligned}$$

and hence $S = 0$ by Theorem 3.28. Analogously, we see for $S \in \mathcal{H}_{\mathbb{S},\Gamma_t,\Gamma_n,\varepsilon}(\Omega) \cap \mathcal{B}^{\mathrm{divDiv}_{\mathbb{S},\Gamma_n}}(\Omega)^{\perp_{\mathsf{L}^2_\mathbb{S}(\Omega)}}$

$$\begin{aligned} 0 = \left\langle S, B_\ell^{\mathrm{divDiv}_{\mathbb{S},\Gamma_n}}\right\rangle_{\mathsf{L}^2_\mathbb{S}(\Omega)} &= \left\langle \pi_{N({}_\mathbb{T}\mathrm{Rot}_{\mathbb{S},\Gamma_t})}S, \varepsilon^{-1}B_\ell^{\mathrm{divDiv}_{\mathbb{S},\Gamma_n}}\right\rangle_{\mathsf{L}^2_{\mathbb{S},\varepsilon}(\Omega)} \\ &= \left\langle S, \pi_{N({}_\mathbb{T}\mathrm{Rot}_{\mathbb{S},\Gamma_t})}\varepsilon^{-1}B_\ell^{\mathrm{divDiv}_{\mathbb{S},\Gamma_n}}\right\rangle_{\mathsf{L}^2_{\mathbb{S},\varepsilon}(\Omega)} \end{aligned}$$

and thus $S = 0$ again by Theorem 3.28. According to (14), we can decompose

$$N\left(\operatorname{divDiv}_{\mathbb{S},\Gamma_n}\varepsilon\right) = R(\varepsilon^{-1}\mathbb{S}\operatorname{Rot}_{\mathbb{T},\Gamma_n}) \oplus_{\mathsf{L}^2_{\mathbb{S},\varepsilon}(\Omega)} \mathcal{H}_{\mathbb{S},\Gamma_t,\Gamma_n,\varepsilon}(\Omega),$$
$$N\left({}_{\mathbb{T}}\operatorname{Rot}_{\mathbb{S},\Gamma_t}\right) = R\left({}_{\mathbb{S}}\operatorname{Gradgrad}_{\Gamma_t}\right) \oplus_{\mathsf{L}^2_{\mathbb{S},\varepsilon}(\Omega)} \mathcal{H}_{\mathbb{S},\Gamma_t,\Gamma_n,\varepsilon}(\Omega),$$

which shows by (17) the other two assertions. Let $k \geq 0$. The case $k = 0$ and Theorem 3.16 show

$$\begin{aligned}
N\left(\operatorname{divDiv}^k_{\mathbb{S},\Gamma_n}\varepsilon\right) \cap \mathcal{B}^{{}_{\mathbb{T}}\operatorname{Rot}_{\mathbb{S},\Gamma_t}}(\Omega)^{\perp_{\mathsf{L}^2_{\mathbb{S},\varepsilon}(\Omega)}} &= \varepsilon^{-1}\mathsf{H}^k_{\mathbb{S},\Gamma_n}(\Omega) \cap N\left(\operatorname{divDiv}_{\mathbb{S},\Gamma_n}\varepsilon\right) \cap \mathcal{B}^{{}_{\mathbb{T}}\operatorname{Rot}_{\mathbb{S},\Gamma_t}}(\Omega)^{\perp_{\mathsf{L}^2_{\mathbb{S},\varepsilon}(\Omega)}} \\
&= \varepsilon^{-1}\mathsf{H}^k_{\mathbb{S},\Gamma_n}(\Omega) \cap R\left(\varepsilon^{-1}{}_{\mathbb{S}}\operatorname{Rot}_{\mathbb{T},\Gamma_n}\right) \\
&= R\left(\varepsilon^{-1}{}_{\mathbb{S}}\operatorname{Rot}^k_{\mathbb{T},\Gamma_n}\right) = \varepsilon^{-1}\operatorname{symRot}\,\mathsf{H}^{k+1}_{\mathbb{T},\Gamma_n}(\Omega), \\
N\left({}_{\mathbb{T}}\operatorname{Rot}^k_{\mathbb{S},\Gamma_t}\right) \cap \mathcal{B}^{\operatorname{divDiv}_{\mathbb{S},\Gamma_n}}(\Omega)^{\perp_{\mathsf{L}^2_{\mathbb{S}}(\Omega)}} &= \mathsf{H}^k_{\mathbb{S},\Gamma_t}(\Omega) \cap N\left({}_{\mathbb{T}}\operatorname{Rot}_{\mathbb{S},\Gamma_t}\right) \cap \mathcal{B}^{\operatorname{divDiv}_{\mathbb{S},\Gamma_n}}(\Omega)^{\perp_{\mathsf{L}^2_{\mathbb{S}}(\Omega)}} \\
&= \mathsf{H}^k_{\mathbb{S},\Gamma_t}(\Omega) \cap R\left({}_{\mathbb{S}}\operatorname{Gradgrad}_{\Gamma_t}\right) \\
&= R\left({}_{\mathbb{S}}\operatorname{Gradgrad}^k_{\Gamma_t}\right) = \operatorname{Gradgrad}\,\mathsf{H}^{k+2}_{\Gamma_t}(\Omega).
\end{aligned}$$

Analogously, we prove the assertions for the remaining $\mathsf{L}^2_{\mathbb{T},\mu}(\Omega)$-related spaces. □

APPENDIX D: SOME TECHNICAL REMARKS

Remark D.1 (Bounded regular decompositions from bounded regular potentials). It holds

$$\begin{aligned}
\operatorname{Rot}\,\widetilde{\mathcal{Q}}^{k,1}_{{}_{\mathbb{T}}\operatorname{Rot}_{\mathbb{S}},\Gamma_t} &= \operatorname{Rot}\,\mathcal{Q}^{k,1}_{{}_{\mathbb{T}}\operatorname{Rot}_{\mathbb{S}},\Gamma_t} = {}_{\mathbb{T}}\operatorname{Rot}^k_{\mathbb{S},\Gamma_t}, \\
\operatorname{Div}\,\widetilde{\mathcal{Q}}^{k,1}_{\operatorname{Div}_{\mathbb{T}},\Gamma_t} &= \operatorname{Div}\,\mathcal{Q}^{k,1}_{\operatorname{Div}_{\mathbb{T}},\Gamma_t} = \operatorname{Div}^k_{\mathbb{T},\Gamma_t}, \\
\operatorname{symRot}\,\widetilde{\mathcal{Q}}^{k,1}_{{}_{\mathbb{S}}\operatorname{Rot}_{\mathbb{T}},\Gamma_t} &= \operatorname{symRot}\,\mathcal{Q}^{k,1}_{{}_{\mathbb{S}}\operatorname{Rot}_{\mathbb{T}},\Gamma_t} = {}_{\mathbb{S}}\operatorname{Rot}^k_{\mathbb{T},\Gamma_t}, \\
\operatorname{divDiv}\,\widetilde{\mathcal{Q}}^{k,1}_{\operatorname{divDiv}_{\mathbb{S}},\Gamma_t} &= \operatorname{divDiv}\,\mathcal{Q}^{k,1}_{\operatorname{divDiv}_{\mathbb{S}},\Gamma_t} = \operatorname{divDiv}^k_{\mathbb{S},\Gamma_t}, \\
\operatorname{divDiv}\,\widetilde{\mathcal{Q}}^{k+1,k,1}_{\operatorname{divDiv}_{\mathbb{S}},\Gamma_t} &= \operatorname{divDiv}\,\mathcal{Q}^{k+1,k,1}_{\operatorname{divDiv}_{\mathbb{S}},\Gamma_t} = \operatorname{divDiv}^{k+1,k}_{\mathbb{S},\Gamma_t}.
\end{aligned}$$

Therefore, the kernels $\mathsf{H}^k_{\mathbb{S},\Gamma_t,0}(\operatorname{Rot},\Omega)$, $\mathsf{H}^k_{\mathbb{T},\Gamma_t,0}(\operatorname{Div},\Omega)$, $\mathsf{H}^k_{\mathbb{T},\Gamma_t,0}(\operatorname{symRot},\Omega)$, and $\mathsf{H}^k_{\mathbb{S},\Gamma_t,0}(\operatorname{divDiv},\Omega)$, $\mathsf{H}^{k+1}_{\mathbb{S},\Gamma_t,0}(\operatorname{divDiv},\Omega)$ are invariant under $\mathcal{Q}^{k,1}_{{}_{\mathbb{T}}\operatorname{Rot}_{\mathbb{S}},\Gamma_t}$, $\widetilde{\mathcal{Q}}^{k,1}_{{}_{\mathbb{T}}\operatorname{Rot}_{\mathbb{S}},\Gamma_t}$, $\mathcal{Q}^{k,1}_{\operatorname{Div}_{\mathbb{T}},\Gamma_t}$, $\widetilde{\mathcal{Q}}^{k,1}_{\operatorname{Div}_{\mathbb{T}},\Gamma_t}$, $\mathcal{Q}^{k,1}_{{}_{\mathbb{S}}\operatorname{Rot}_{\mathbb{T}},\Gamma_t}$, $\widetilde{\mathcal{Q}}^{k,1}_{{}_{\mathbb{S}}\operatorname{Rot}_{\mathbb{T}},\Gamma_t}$, $\mathcal{Q}^{k,1}_{\operatorname{divDiv}_{\mathbb{S}},\Gamma_t}$, $\widetilde{\mathcal{Q}}^{k,1}_{\operatorname{divDiv}_{\mathbb{S}},\Gamma_t}$, and $\mathcal{Q}^{k+1,k,1}_{\operatorname{divDiv}_{\mathbb{S}},\Gamma_t}$, $\widetilde{\mathcal{Q}}^{k+1,k,1}_{\operatorname{divDiv}_{\mathbb{S}},\Gamma_t}$, respectively. Moreover,

$$\begin{aligned}
R\left(\widetilde{\mathcal{Q}}^{k,1}_{{}_{\mathbb{T}}\operatorname{Rot}_{\mathbb{S}},\Gamma_t}\right) &= R\left(\mathcal{P}^k_{{}_{\mathbb{T}}\operatorname{Rot}_{\mathbb{S}},\Gamma_t}\right), & \widetilde{\mathcal{Q}}^{k,1}_{{}_{\mathbb{T}}\operatorname{Rot}_{\mathbb{S}},\Gamma_t} &= \mathcal{Q}^{k,1}_{{}_{\mathbb{T}}\operatorname{Rot}_{\mathbb{S}},\Gamma_t}\left({}_{\mathbb{T}}\operatorname{Rot}^k_{\mathbb{S},\Gamma_t}\right)^{-1}_{\perp}\,{}_{\mathbb{T}}\operatorname{Rot}^k_{\mathbb{S},\Gamma_t}, \\
R\left(\widetilde{\mathcal{Q}}^{k,1}_{\operatorname{Div}_{\mathbb{T}},\Gamma_t}\right) &= R\left(\mathcal{P}^k_{\operatorname{Div}_{\mathbb{T}},\Gamma_t}\right), & \widetilde{\mathcal{Q}}^{k,1}_{\operatorname{Div}_{\mathbb{T}},\Gamma_t} &= \mathcal{Q}^{k,1}_{\operatorname{Div}_{\mathbb{T}},\Gamma_t}\left(\operatorname{Div}^k_{\mathbb{T},\Gamma_t}\right)^{-1}_{\perp}\operatorname{Div}^k_{\mathbb{T},\Gamma_t}, \\
R\left(\widetilde{\mathcal{Q}}^{k,1}_{{}_{\mathbb{S}}\operatorname{Rot}_{\mathbb{T}},\Gamma_t}\right) &= R(\mathcal{P}^k_{{}_{\mathbb{S}}\operatorname{Rot}_{\mathbb{T}},\Gamma_t}), & \widetilde{\mathcal{Q}}^{k,1}_{{}_{\mathbb{S}}\operatorname{Rot}_{\mathbb{T}},\Gamma_t} &= \mathcal{Q}^{k,1}_{{}_{\mathbb{S}}\operatorname{Rot}_{\mathbb{T}},\Gamma_t}\left({}_{\mathbb{S}}\operatorname{Rot}_{\mathbb{T},\Gamma_t}{}^k\right)^{-1}_{\perp}\,{}_{\mathbb{S}}\operatorname{Rot}_{\mathbb{T},\Gamma_t}{}^k, \\
R\left(\widetilde{\mathcal{Q}}^{k,1}_{\operatorname{divDiv}_{\mathbb{S}},\Gamma_t}\right) &= R\left(\mathcal{P}^k_{\operatorname{divDiv}_{\mathbb{S}},\Gamma_t}\right), & \widetilde{\mathcal{Q}}^{k,1}_{\operatorname{divDiv}_{\mathbb{S}},\Gamma_t} &= \mathcal{Q}^{k,1}_{\operatorname{divDiv}_{\mathbb{S}},\Gamma_t}\left(\operatorname{divDiv}^k_{\mathbb{S},\Gamma_t}\right)^{-1}_{\perp}\operatorname{divDiv}^k_{\mathbb{S},\Gamma_t}, \\
R\left(\widetilde{\mathcal{Q}}^{k+1,k,1}_{\operatorname{divDiv}_{\mathbb{S}},\Gamma_t}\right) &= R\left(\mathcal{P}^{k+1,k}_{\operatorname{divDiv}_{\mathbb{S}},\Gamma_t}\right), & \widetilde{\mathcal{Q}}^{k+1,k,1}_{\operatorname{divDiv}_{\mathbb{S}},\Gamma_t} &= \mathcal{Q}^{k+1,k,1}_{\operatorname{divDiv}_{\mathbb{S}},\Gamma_t}\left(\operatorname{divDiv}^{k+1,k}_{\mathbb{S},\Gamma_t}\right)^{-1}_{\perp}\operatorname{divDiv}^{k+1,k}_{\mathbb{S},\Gamma_t}.
\end{aligned}$$

Hence, $\widetilde{\mathcal{Q}}^{k,1}_{{}_{\mathbb{T}}\operatorname{Rot}_{\mathbb{S}},\Gamma_t}$, $\widetilde{\mathcal{Q}}^{k,1}_{\operatorname{Div}_{\mathbb{T}},\Gamma_t}$, $\widetilde{\mathcal{Q}}^{k,1}_{{}_{\mathbb{S}}\operatorname{Rot}_{\mathbb{T}},\Gamma_t}$, $\widetilde{\mathcal{Q}}^{k,1}_{\operatorname{divDiv}_{\mathbb{S}},\Gamma_t}$, and $\widetilde{\mathcal{Q}}^{k+1,k,1}_{\operatorname{divDiv}_{\mathbb{S}},\Gamma_t}$ coincide with $\mathcal{Q}^{k,1}_{{}_{\mathbb{T}}\operatorname{Rot}_{\mathbb{S}},\Gamma_t}$, $\mathcal{Q}^{k,1}_{\operatorname{Div}_{\mathbb{T}},\Gamma_t}$, $\mathcal{Q}^{k,1}_{{}_{\mathbb{S}}\operatorname{Rot}_{\mathbb{T}},\Gamma_t}$, $\mathcal{Q}^{k,1}_{\operatorname{divDiv}_{\mathbb{S}},\Gamma_t}$, and $\mathcal{Q}^{k+1,k,1}_{\operatorname{divDiv}_{\mathbb{S}},\Gamma_t}$ on the reduced domains of definition

$$D\left(\left({}_{\mathbb{T}}\operatorname{Rot}^k_{\mathbb{S},\Gamma_t}\right)_{\perp}\right),\ D\left(\left(\operatorname{Div}^k_{\mathbb{T},\Gamma_t}\right)_{\perp}\right),\ D\left(\left({}_{\mathbb{S}}\operatorname{Rot}^k_{\mathbb{T},\Gamma_t}\right)_{\perp}\right),\ D\left(\left(\operatorname{divDiv}^k_{\mathbb{S},\Gamma_t}\right)_{\perp}\right),\ D\left(\left(\operatorname{divDiv}^{k+1,k}_{\mathbb{S},\Gamma_t}\right)_{\perp}\right),$$

respectively. Thus, $\widetilde{\mathcal{Q}}^{k,1}_{{}_{\mathbb{T}}\mathrm{Rot}_{\mathbb{S}},\Gamma_t}$, $\widetilde{\mathcal{Q}}^{k,1}_{\mathrm{Div}_{\mathbb{T}},\Gamma_t}$, $\widetilde{\mathcal{Q}}^{k,1}_{{}_{\mathbb{S}}\mathrm{Rot}_{\mathbb{T}},\Gamma_t}$, $\widetilde{\mathcal{Q}}^{k,1}_{\mathrm{divDiv}_{\mathbb{S}},\Gamma_t}$, and $\widetilde{\mathcal{Q}}^{k+1,k,1}_{\mathrm{divDiv}_{\mathbb{S}},\Gamma_t}$ may differ from $\mathcal{Q}^{k,1}_{{}_{\mathbb{T}}\mathrm{Rot}_{\mathbb{S}},\Gamma_t}$, $\mathcal{Q}^{k,1}_{\mathrm{Div}_{\mathbb{T}},\Gamma_t}$, $\mathcal{Q}^{k,1}_{{}_{\mathbb{S}}\mathrm{Rot}_{\mathbb{T}},\Gamma_t}$, $\mathcal{Q}^{k,1}_{\mathrm{divDiv}_{\mathbb{S}},\Gamma_t}$, and $\mathcal{Q}^{k+1,k,1}_{\mathrm{divDiv}_{\mathbb{S}},\Gamma_t}$ only on the kernels

$$N\left({}_{\mathbb{T}}\mathrm{Rot}^k_{\mathbb{S},\Gamma_t}\right) = \mathsf{H}^k_{\mathbb{S},\Gamma_t,0}(\mathrm{Rot},\Omega),\ N\left(\mathrm{Div}^k_{\mathbb{T},\Gamma_t}\right) = \mathsf{H}^k_{\mathbb{T},\Gamma_t,0}(\mathrm{Div},\Omega),\ N\left({}_{\mathbb{S}}\mathrm{Rot}^k_{\mathbb{T},\Gamma_t}\right) = \mathsf{H}^k_{\mathbb{T},\Gamma_t,0}(\mathrm{symRot},\Omega),$$

and $N\left(\mathrm{divDiv}^k_{\mathbb{S},\Gamma_t}\right) = \mathsf{H}^k_{\mathbb{S},\Gamma_t,0}(\mathrm{divDiv},\Omega)$, $N\left(\mathrm{divDiv}^{k+1,k}_{\mathbb{S},\Gamma_t}\right) = \mathsf{H}^{k+1}_{\mathbb{S},\Gamma_t,0}(\mathrm{divDiv},\Omega)$, respectively.

Remark D.2 (Projections). Recall Theorem 3.25, for example, for $\mathrm{divDiv}^k_{\mathbb{S},\Gamma_t}$

$$\mathsf{H}^k_{\mathbb{S},\Gamma_t}(\mathrm{divDiv},\Omega) = R\left(\widetilde{\mathcal{Q}}^{k,1}_{\mathrm{divDiv}_{\mathbb{S}},\Gamma_t}\right) \dotplus R\left(\widetilde{\mathcal{N}}^k_{\mathrm{divDiv}_{\mathbb{S}},\Gamma_t}\right).$$

(i) $\widetilde{\mathcal{Q}}^{k,1}_{\mathrm{divDiv}_{\mathbb{S}},\Gamma_t}$ and $\widetilde{\mathcal{N}}^k_{\mathrm{divDiv}_{\mathbb{S}},\Gamma_t} = 1 - \widetilde{\mathcal{Q}}^{k,1}_{\mathrm{divDiv}_{\mathbb{S}},\Gamma_t}$ are projections.

(i') $\widetilde{\mathcal{Q}}^{k,1}_{\mathrm{divDiv}_{\mathbb{S}},\Gamma_t}\widetilde{\mathcal{N}}^k_{\mathrm{divDiv}_{\mathbb{S}},\Gamma_t} = \widetilde{\mathcal{N}}^k_{\mathrm{divDiv}_{\mathbb{S}},\Gamma_t}\widetilde{\mathcal{Q}}^{k,1}_{\mathrm{divDiv}_{\mathbb{S}},\Gamma_t} = 0$.

(ii) For $I_\pm := \widetilde{\mathcal{Q}}^{k,1}_{\mathrm{divDiv}_{\mathbb{S}},\Gamma_t} \pm \widetilde{\mathcal{N}}^k_{\mathrm{divDiv}_{\mathbb{S}},\Gamma_t}$, it holds $I_+ = I_-^2 = \mathrm{id}_{\mathsf{H}^k_{\mathbb{S},\Gamma_t}(\mathrm{divDiv},\Omega)}$. Therefore, I_+, I_-^2, as well as $I_- = 2\widetilde{\mathcal{Q}}^{k,1}_{\mathrm{divDiv}_{\mathbb{S}},\Gamma_t} - \mathrm{id}_{\mathsf{H}^k_{\mathbb{S},\Gamma_t}(\mathrm{divDiv},\Omega)}$ are topological isomorphisms on $\mathsf{H}^k_{\mathbb{S},\Gamma_t}(\mathrm{divDiv},\Omega)$.

(iii) There exists $c > 0$ such that for all $S \in \mathsf{H}^k_{\mathbb{S},\Gamma_t}(\mathrm{divDiv},\Omega)$

$$\begin{aligned} c\left|\widetilde{\mathcal{Q}}^{k,1}_{\mathrm{divDiv}_{\mathbb{S}},\Gamma_t}S\right|_{\mathsf{H}^{k+2}_{\mathbb{S}}(\Omega)} &\le |\mathrm{divDiv}\ S|_{\mathsf{H}^k(\Omega)} \le |S|_{\mathsf{H}^k_{\mathbb{S}}(\mathrm{divDiv},\Omega)}, \\ \left|\widetilde{\mathcal{N}}^k_{\mathrm{divDiv}_{\mathbb{S}},\Gamma_t}S\right|_{\mathsf{H}^k_{\mathbb{S}}(\Omega)} &\le |S|_{\mathsf{H}^k_{\mathbb{S}}(\Omega)} + \left|\widetilde{\mathcal{Q}}^{k,1}_{\mathrm{divDiv}_{\mathbb{S}},\Gamma_t}S\right|_{\mathsf{H}^k_{\mathbb{S}}(\Omega)}. \end{aligned}$$

(iii') For $S \in \mathsf{H}^k_{\mathbb{S},\Gamma_t,0}(\mathrm{divDiv},\Omega)$, we have $\widetilde{\mathcal{Q}}^{k,1}_{\mathrm{divDiv}_{\mathbb{S}},\Gamma_t}S = 0$ and $\widetilde{\mathcal{N}}^k_{\mathrm{divDiv}_{\mathbb{S}},\Gamma_t}S = S$. In particular, $\widetilde{\mathcal{N}}^k_{\mathrm{divDiv}_{\mathbb{S}},\Gamma_t}$ is onto.

Similar results to (i)–(iii′) hold also for ${}_{\mathbb{T}}\mathrm{Rot}^k_{\mathbb{S},\Gamma_t}$, $\mathrm{Div}^k_{\mathbb{T},\Gamma_t}$, ${}_{\mathbb{S}}\mathrm{Rot}^k_{\mathbb{T},\Gamma_t}$, and $\mathrm{divDiv}^{k+1,k}_{\mathbb{S},\Gamma_t}$. In particular, $\widetilde{\mathcal{Q}}^{k,1}_{{}_{\mathbb{T}}\mathrm{Rot}_{\mathbb{S}},\Gamma_t}$, $\widetilde{\mathcal{Q}}^{k,1}_{\mathrm{Div}_{\mathbb{T}},\Gamma_t}$, $\widetilde{\mathcal{Q}}^{k,1}_{{}_{\mathbb{S}}\mathrm{Rot}_{\mathbb{T}},\Gamma_t}$, $\widetilde{\mathcal{Q}}^{k+1,k,1}_{\mathrm{divDiv}_{\mathbb{S}},\Gamma_t}$, and $\widetilde{\mathcal{N}}^{k,1}_{{}_{\mathbb{T}}\mathrm{Rot}_{\mathbb{S}},\Gamma_t}$, $\widetilde{\mathcal{N}}^{k,1}_{\mathrm{Div}_{\mathbb{T}},\Gamma_t}$, $\widetilde{\mathcal{N}}^{k,1}_{{}_{\mathbb{S}}\mathrm{Rot}_{\mathbb{T}},\Gamma_t}$, and $\widetilde{\mathcal{N}}^{k+1,k,1}_{\mathrm{divDiv}_{\mathbb{S}},\Gamma_t}$ are projections and there exists $c > 0$ such that for all $S \in \mathsf{H}^k_{\mathbb{S},\Gamma_t}(\mathrm{Rot},\Omega)$, $T \in \mathsf{H}^k_{\mathbb{T},\Gamma_t}(\mathrm{Div},\Omega)$, $\hat{T} \in \mathsf{H}^k_{\mathbb{T},\Gamma_t}(\mathrm{symRot},\Omega)$, and $\hat{S} \in \mathsf{H}^{k+1,k}_{\mathbb{S},\Gamma_t}(\mathrm{divDiv},\Omega)$

$$\begin{aligned} \left|\widetilde{\mathcal{Q}}^{k,1}_{{}_{\mathbb{T}}\mathrm{Rot}_{\mathbb{S}},\Gamma_t}S\right|_{\mathsf{H}^{k+1}_{\mathbb{S}}(\Omega)} &\le c\,|\mathrm{Rot}\ S|_{\mathsf{H}^k_{\mathbb{T}}(\Omega)}, &\quad \left|\widetilde{\mathcal{Q}}^{k,1}_{{}_{\mathbb{S}}\mathrm{Rot}_{\mathbb{T}},\Gamma_t}\hat{T}\right|_{\mathsf{H}^{k+1}_{\mathbb{T}}(\Omega)} &\le c\left|\mathrm{symRot}\ \hat{T}\right|_{\mathsf{H}^k_{\mathbb{S}}(\Omega)}, \\ \left|\widetilde{\mathcal{Q}}^{k,1}_{\mathrm{Div}_{\mathbb{T}},\Gamma_t}T\right|_{\mathsf{H}^{k+1}_{\mathbb{T}}(\Omega)} &\le c\,|\mathrm{Div}\ T|_{\mathsf{H}^k(\Omega)}, &\quad \left|\widetilde{\mathcal{Q}}^{k+1,k,1}_{\mathrm{divDiv}_{\mathbb{S}},\Gamma_t}\hat{S}\right|_{\mathsf{H}^{k+2}_{\mathbb{S}}(\Omega)} &\le c\left|\mathrm{divDiv}\ \hat{S}\right|_{\mathsf{H}^k(\Omega)}. \end{aligned}$$

Remark D.3 (Bounded regular direct decompositions). By Theorem 3.29, we have, for example,

$$\begin{aligned} \mathsf{H}^k_{\mathbb{S},\Gamma_t}(\mathrm{Rot},\Omega) &= R\left(\widetilde{\mathcal{Q}}^{k,1}_{{}_{\mathbb{T}}\mathrm{Rot}_{\mathbb{S}},\Gamma_t}\right) \dotplus \mathrm{Lin}\,\mathcal{B}^{{}_{\mathbb{T}}\mathrm{Rot}_{\mathbb{S},\Gamma_t}}(\Omega) \dotplus \mathrm{Gradgrad}\ \mathsf{H}^{k+2}_{\Gamma_t}(\Omega) \\ &= \mathsf{H}^{k+1}_{\mathbb{S},\Gamma_t}(\Omega) + \mathrm{Gradgrad}\ \mathsf{H}^{k+2}_{\Gamma_t}(\Omega) \end{aligned}$$

with bounded linear regular direct decomposition operators

$$\begin{aligned} \widehat{\mathcal{Q}}^{k,1}_{{}_{\mathbb{T}}\mathrm{Rot}_{\mathbb{S}},\Gamma_t} &: \mathsf{H}^k_{\mathbb{S},\Gamma_t}(\mathrm{Rot},\Omega) \to R\left(\widetilde{\mathcal{Q}}^{k,1}_{{}_{\mathbb{T}}\mathrm{Rot}_{\mathbb{S}},\Gamma_t}\right) \subset \mathsf{H}^{k+1}_{\mathbb{S},\Gamma_t}(\Omega), \\ \widehat{\mathcal{Q}}^{k,\infty}_{{}_{\mathbb{T}}\mathrm{Rot}_{\mathbb{S}},\Gamma_t} &: \mathsf{H}^k_{\mathbb{S},\Gamma_t}(\mathrm{Rot},\Omega) \to \mathrm{Lin}\,\mathcal{B}^{{}_{\mathbb{T}}\mathrm{Rot}_{\mathbb{S},\Gamma_t}}(\Omega) \subset \mathsf{H}^\infty_{\mathbb{S},\Gamma_t,0}(\mathrm{Rot},\Omega) \subset \mathsf{H}^{k+1}_{\mathbb{S},\Gamma_t}(\Omega), \\ \widehat{\mathcal{Q}}^{k,0}_{{}_{\mathbb{T}}\mathrm{Rot}_{\mathbb{S}},\Gamma_t} &: \mathsf{H}^k_{\mathbb{S},\Gamma_t}(\mathrm{Rot},\Omega) \to \mathsf{H}^{k+2}_{\Gamma_t}(\Omega) \end{aligned}$$

satisfying $\widehat{\mathcal{Q}}^{k,1}_{{}_{\mathbb{T}}\mathrm{Rot}_{\mathbb{S}},\Gamma_t} + \widehat{\mathcal{Q}}^{k,\infty}_{{}_{\mathbb{T}}\mathrm{Rot}_{\mathbb{S}},\Gamma_t} + \mathrm{Gradgrad}\ \widehat{\mathcal{Q}}^{k,0}_{{}_{\mathbb{T}}\mathrm{Rot}_{\mathbb{S}},\Gamma_t} = \mathrm{id}_{\mathsf{H}^k_{\mathbb{S},\Gamma_t}(\mathrm{Rot},\Omega)}$.

A closer inspection of the proof allows for a more precise description of these bounded decomposition operators. For this, let $S \in \mathsf{H}^{k}_{\mathbb{S},\Gamma_t}(\mathrm{Rot},\Omega)$. According to Theorem 3.25 and Remark D.2, we decompose

$$S = S_R + S_N := \widetilde{\mathcal{Q}}^{k,1}_{{}_{\mathbb{T}}\mathrm{Rot}_{\mathbb{S}},\Gamma_t} S + \widetilde{\mathcal{N}}^{k}_{{}_{\mathbb{T}}\mathrm{Rot}_{\mathbb{S}},\Gamma_t} S \in R\left(\widetilde{\mathcal{Q}}^{k,1}_{{}_{\mathbb{T}}\mathrm{Rot}_{\mathbb{S}},\Gamma_t}\right) \dotplus R\left(\widetilde{\mathcal{N}}^{k}_{{}_{\mathbb{T}}\mathrm{Rot}_{\mathbb{S}},\Gamma_t}\right)$$

with $R\left(\widetilde{\mathcal{N}}^{k}_{{}_{\mathbb{T}}\mathrm{Rot}_{\mathbb{S}},\Gamma_t}\right) = \mathsf{H}^{k}_{\mathbb{S},\Gamma_t,0}(\mathrm{Rot},\Omega) = N\left({}_{\mathbb{T}}\mathrm{Rot}^{k}_{\mathbb{S},\Gamma_t}\right)$. By Theorem 3.29 we further decompose

$$\mathsf{H}^{k}_{\mathbb{S},\Gamma_t,0}(\mathrm{Rot},\Omega) \ni S_N = \mathrm{Gradgrad}\ \tilde{u} + Bin\mathrm{Gradgrad}\ \mathsf{H}^{k+2}_{\Gamma_t}(\Omega) \dotplus \mathrm{Lin}\,\mathcal{B}^{{}_{\mathbb{T}}\mathrm{Rot}_{\mathbb{S},\Gamma_t}}(\Omega).$$

Then $\pi_{N(\mathrm{divDiv}_{\mathbb{S},\Gamma_n}\varepsilon)} S_N = \pi_{N(\mathrm{divDiv}_{\mathbb{S},\Gamma_n}\varepsilon)} B \in \mathcal{H}_{\mathbb{S},\Gamma_t,\Gamma_n,\varepsilon}(\Omega)$ and thus

$$B = I^{-1}_{\pi_{N(\mathrm{divDiv}_{\mathbb{S},\Gamma_n}\varepsilon)}} \pi_{N(\mathrm{divDiv}_{\mathbb{S},\Gamma_n}\varepsilon)} S_N \in \mathrm{Lin}\,\mathcal{B}^{{}_{\mathbb{T}}\mathrm{Rot}_{\mathbb{S},\Gamma_t}}(\Omega).$$

Therefore,

$$\begin{aligned}\tilde{u} = \mathcal{P}^{k}_{{}_{\mathbb{S}}\mathrm{Gradgrad},\Gamma_t} \mathrm{Gradgrad}\tilde{u} &= \mathcal{P}^{k}_{{}_{\mathbb{S}}\mathrm{Gradgrad},\Gamma_t}(S_N - B)\\ &= \mathcal{P}^{k}_{{}_{\mathbb{S}}\mathrm{Gradgrad},\Gamma_t}\left(1 - I^{-1}_{\pi_{N(\mathrm{divDiv}_{\mathbb{S},\Gamma_n}\varepsilon)}} \pi_{N(\mathrm{divDiv}_{\mathbb{S},\Gamma_n}\varepsilon)}\right) S_N.\end{aligned}$$

Finally, we see

$$\begin{aligned}\widehat{\mathcal{Q}}^{k,1}_{{}_{\mathbb{T}}\mathrm{Rot}_{\mathbb{S}},\Gamma_t} &= \widetilde{\mathcal{Q}}^{k,1}_{{}_{\mathbb{T}}\mathrm{Rot}_{\mathbb{S}},\Gamma_t} = \mathcal{P}^{k}_{{}_{\mathbb{T}}\mathrm{Rot}_{\mathbb{S}},\Gamma_t} {}_{\mathbb{T}}\mathrm{Rot}^{k}_{\mathbb{S},\Gamma_t}, = \mathcal{Q}^{k,1}_{{}_{\mathbb{T}}\mathrm{Rot}_{\mathbb{S}},\Gamma_t}\left({}_{\mathbb{T}}\mathrm{Rot}^{k}_{\mathbb{S},\Gamma_t}\right)^{-1}_{\perp} {}_{\mathbb{T}}\mathrm{Rot}^{k}_{\mathbb{S},\Gamma_t},\\ \widehat{\mathcal{Q}}^{k,\infty}_{{}_{\mathbb{T}}\mathrm{Rot}_{\mathbb{S}},\Gamma_t} &= I^{-1}_{\pi_{N(\mathrm{divDiv}_{\mathbb{S},\Gamma_n}\varepsilon)}} \pi_{N(\mathrm{divDiv}_{\mathbb{S},\Gamma_n}\varepsilon)} \widetilde{\mathcal{N}}^{k}_{{}_{\mathbb{T}}\mathrm{Rot}_{\mathbb{S}},\Gamma_t},\\ \widehat{\mathcal{Q}}^{k,0}_{{}_{\mathbb{T}}\mathrm{Rot}_{\mathbb{S}},\Gamma_t} &= \mathcal{P}^{k}_{{}_{\mathbb{S}}\mathrm{Gradgrad},\Gamma_t}\left(1 - I^{-1}_{\pi_{N(\mathrm{divDiv}_{\mathbb{S},\Gamma_n}\varepsilon)}} \pi_{N(\mathrm{divDiv}_{\mathbb{S},\Gamma_n}\varepsilon)}\right) \widetilde{\mathcal{N}}^{k}_{{}_{\mathbb{T}}\mathrm{Rot}_{\mathbb{S}},\Gamma_t}\end{aligned}$$

with $\widetilde{\mathcal{N}}^{k}_{{}_{\mathbb{T}}\mathrm{Rot}_{\mathbb{S}},\Gamma_t} = 1 - \widetilde{\mathcal{Q}}^{k,1}_{{}_{\mathbb{T}}\mathrm{Rot}_{\mathbb{S}},\Gamma_t}$. Analogously, we have for the other spaces

$$\begin{aligned}\mathsf{H}^{k}_{\mathbb{T},\Gamma_n}(\mathrm{Div},\Omega) &= R\left(\widetilde{\mathcal{Q}}^{k,1}_{\mathrm{Div}_{\mathbb{T},\Gamma_n}}\right) \dotplus \mathrm{Lin}\,\mathcal{B}^{\mathrm{Div}_{\mathbb{T},\Gamma_n}}(\Omega) \dotplus \mathrm{Rot}\ \mathsf{H}^{k+1}_{\mathbb{S},\Gamma_n}(\Omega)\\ &= \mathsf{H}^{k+1}_{\mathbb{T},\Gamma_n}(\Omega) + \mathrm{Rot}\ \mathsf{H}^{k+1}_{\mathbb{S},\Gamma_n}(\Omega),\\ \mathsf{H}^{k}_{\mathbb{T},\Gamma_t}(\mathrm{symRot},\Omega) &= R\left(\widetilde{\mathcal{Q}}^{k,1}_{{}_{\mathbb{S}}\mathrm{Rot}_{\mathbb{T}},\Gamma_t}\right) \dotplus \mathrm{Lin}\,\mathcal{B}^{{}_{\mathbb{S}}\mathrm{Rot}_{\mathbb{T},\Gamma_t}}(\Omega) \dotplus \mathrm{devGrad}\ \mathsf{H}^{k+1}_{\Gamma_t}(\Omega)\\ &= \mathsf{H}^{k+1}_{\mathbb{T},\Gamma_t}(\Omega) + \mathrm{devGrad}\ \mathsf{H}^{k+1}_{\Gamma_t}(\Omega),\\ \mathsf{H}^{k}_{\mathbb{S},\Gamma_n}(\mathrm{divDiv},\Omega) &= R\left(\widetilde{\mathcal{Q}}^{k,1}_{\mathrm{divDiv}_{\mathbb{S}},\Gamma_n}\right) \dotplus \mathrm{Lin}\,\mathcal{B}^{\mathrm{divDiv}_{\mathbb{S},\Gamma_n}}(\Omega) \dotplus \mathrm{symRot}\ \mathsf{H}^{k+1}_{\mathbb{T},\Gamma_n}(\Omega)\\ &= \mathsf{H}^{k+2}_{\mathbb{S},\Gamma_n}(\Omega) + \mathrm{symRot}\ \mathsf{H}^{k+1}_{\mathbb{T},\Gamma_n}(\Omega),\\ \mathsf{H}^{k+1,k}_{\mathbb{S},\Gamma_n}(\mathrm{divDiv},\Omega) &= R\left(\widetilde{\mathcal{Q}}^{k+1,k,1}_{\mathrm{divDiv}_{\mathbb{S}},\Gamma_n}\right) \dotplus \mathrm{Lin}\,\mathcal{B}^{\mathrm{divDiv}_{\mathbb{S},\Gamma_n}}(\Omega) \dotplus \mathrm{symRot}\ \mathsf{H}^{k+2}_{\mathbb{T},\Gamma_n}(\Omega)\\ &= \mathsf{H}^{k+2}_{\mathbb{S},\Gamma_n}(\Omega) + \mathrm{symRot}\ \mathsf{H}^{k+2}_{\mathbb{T},\Gamma_n}(\Omega)\end{aligned}$$

with bounded linear regular direct decomposition operators

$$
\begin{aligned}
\widehat{\mathcal{Q}}^{k,1}_{\mathrm{Div}_{\mathbb{T}},\Gamma_n} &: \mathsf{H}^k_{\mathbb{T},\Gamma_n}(\mathrm{Div},\Omega) \to R\left(\widetilde{\mathcal{Q}}^{k,1}_{\mathrm{Div}_{\mathbb{T}},\Gamma_n}\right) \subset \mathsf{H}^{k+1}_{\mathbb{T},\Gamma_n}(\Omega),\\
\widehat{\mathcal{Q}}^{k,\infty}_{\mathrm{Div}_{\mathbb{T}},\Gamma_n} &: \mathsf{H}^k_{\mathbb{T},\Gamma_n}(\mathrm{Div},\Omega) \to \mathrm{Lin}\,\mathcal{B}^{\mathrm{Div}_{\mathbb{T},\Gamma_n}}(\Omega) \subset \mathsf{H}^{\infty}_{\mathbb{T},\Gamma_n,0}(\mathrm{Div},\Omega) \subset \mathsf{H}^{k+1}_{\mathbb{T},\Gamma_n}(\Omega),\\
\widehat{\mathcal{Q}}^{k,0}_{\mathrm{Div}_{\mathbb{T}},\Gamma_n} &: \mathsf{H}^k_{\mathbb{T},\Gamma_n}(\mathrm{Div},\Omega) \to \mathsf{H}^{k+1}_{\mathbb{S},\Gamma_n}(\Omega),\\
\widehat{\mathcal{Q}}^{k,1}_{{}_{\mathbb{S}}\mathrm{Rot}_{\mathbb{T}},\Gamma_t} &: \mathsf{H}^k_{\mathbb{T},\Gamma_t}(\mathrm{symRot},\Omega) \to R\left(\widetilde{\mathcal{Q}}^{k,1}_{{}_{\mathbb{S}}\mathrm{Rot}_{\mathbb{T}},\Gamma_t}\right) \subset \mathsf{H}^{k+1}_{\mathbb{T},\Gamma_t}(\Omega),\\
\widehat{\mathcal{Q}}^{k,\infty}_{{}_{\mathbb{S}}\mathrm{Rot}_{\mathbb{T}},\Gamma_t} &: \mathsf{H}^k_{\mathbb{T},\Gamma_t}(\mathrm{symRot},\Omega) \to \mathrm{Lin}\,\mathcal{B}^{{}_{\mathbb{S}}\mathrm{Rot}_{\mathbb{T}},\Gamma_t}(\Omega) \subset \mathsf{H}^{\infty}_{\mathbb{T},\Gamma_t,0}(\mathrm{symRot},\Omega) \subset \mathsf{H}^{k+1}_{\mathbb{T},\Gamma_t}(\Omega),\\
\widehat{\mathcal{Q}}^{k,0}_{{}_{\mathbb{S}}\mathrm{Rot}_{\mathbb{T}},\Gamma_t} &: \mathsf{H}^k_{\mathbb{T},\Gamma_t}(\mathrm{symRot},\Omega) \to \mathsf{H}^{k+1}_{\Gamma_t}(\Omega),\\
\widehat{\mathcal{Q}}^{k,1}_{\mathrm{divDiv}_{\mathbb{S}},\Gamma_n} &: \mathsf{H}^k_{\mathbb{S},\Gamma_n}(\mathrm{divDiv},\Omega) \to R\left(\widetilde{\mathcal{Q}}^{k,1}_{\mathrm{divDiv}_{\mathbb{S}},\Gamma_n}\right) \subset \mathsf{H}^{k+2}_{\mathbb{S},\Gamma_n}(\Omega),\\
\widehat{\mathcal{Q}}^{k,\infty}_{\mathrm{divDiv}_{\mathbb{S}},\Gamma_n} &: \mathsf{H}^k_{\mathbb{S},\Gamma_n}(\mathrm{divDiv},\Omega) \to \mathrm{Lin}\,\mathcal{B}^{\mathrm{divDiv}_{\mathbb{S},\Gamma_n}}(\Omega) \subset \mathsf{H}^{\infty}_{\mathbb{S},\Gamma_n,0}(\mathrm{divDiv},\Omega) \subset \mathsf{H}^{k+2}_{\mathbb{S},\Gamma_n}(\Omega),\\
\widehat{\mathcal{Q}}^{k,0}_{\mathrm{divDiv}_{\mathbb{S}},\Gamma_n} &: \mathsf{H}^k_{\mathbb{S},\Gamma_n}(\mathrm{divDiv},\Omega) \to \mathsf{H}^{k+1}_{\mathbb{T},\Gamma_n}(\Omega),\\
\widehat{\mathcal{Q}}^{k+1,k,1}_{\mathrm{divDiv}_{\mathbb{S}},\Gamma_n} &: \mathsf{H}^{k+1,k}_{\mathbb{S},\Gamma_n}(\mathrm{divDiv},\Omega) \to R\left(\widetilde{\mathcal{Q}}^{k+1,k,1}_{\mathrm{divDiv}_{\mathbb{S}},\Gamma_n}\right) \subset \mathsf{H}^{k+2}_{\mathbb{S},\Gamma_n}(\Omega),\\
\widehat{\mathcal{Q}}^{k+1,k,\infty}_{\mathrm{divDiv}_{\mathbb{S}},\Gamma_n} &: \mathsf{H}^{k+1,k}_{\mathbb{S},\Gamma_n}(\mathrm{divDiv},\Omega) \to \mathrm{Lin}\,\mathcal{B}^{\mathrm{divDiv}_{\mathbb{S},\Gamma_n}}(\Omega) \subset \mathsf{H}^{\infty}_{\mathbb{S},\Gamma_n,0}(\mathrm{divDiv},\Omega) \subset \mathsf{H}^{k+2}_{\mathbb{S},\Gamma_n}(\Omega),\\
\widehat{\mathcal{Q}}^{k+1,k,0}_{\mathrm{divDiv}_{\mathbb{S}},\Gamma_n} &: \mathsf{H}^{k+1,k}_{\mathbb{S},\Gamma_n}(\mathrm{divDiv},\Omega) \to \mathsf{H}^{k+2}_{\mathbb{T},\Gamma_n}(\Omega)
\end{aligned}
$$

satisfying

$$
\begin{aligned}
\widehat{\mathcal{Q}}^{k,1}_{\mathrm{Div}_{\mathbb{T}},\Gamma_n} + \widehat{\mathcal{Q}}^{k,\infty}_{\mathrm{Div}_{\mathbb{T}},\Gamma_n} + \mathrm{Rot}\ \widehat{\mathcal{Q}}^{k,0}_{\mathrm{Div}_{\mathbb{T}},\Gamma_n} &= \mathrm{id}_{\mathsf{H}^k_{\mathbb{T},\Gamma_n}(\mathrm{Div},\Omega)},\\
\widehat{\mathcal{Q}}^{k,1}_{{}_{\mathbb{S}}\mathrm{Rot}_{\mathbb{T}},\Gamma_t} + \widehat{\mathcal{Q}}^{k,\infty}_{{}_{\mathbb{S}}\mathrm{Rot}_{\mathbb{T}},\Gamma_t} + \mathrm{devGrad}\ \widehat{\mathcal{Q}}^{k,0}_{{}_{\mathbb{S}}\mathrm{Rot}_{\mathbb{T}},\Gamma_t} &= \mathrm{id}_{\mathsf{H}^k_{\mathbb{T},\Gamma_t}(\mathrm{symRot},\Omega)},\\
\widehat{\mathcal{Q}}^{k,1}_{\mathrm{divDiv}_{\mathbb{S}},\Gamma_n} + \widehat{\mathcal{Q}}^{k,\infty}_{\mathrm{divDiv}_{\mathbb{S}},\Gamma_n} + \mathrm{symRot}\ \widehat{\mathcal{Q}}^{k,0}_{\mathrm{divDiv}_{\mathbb{S}},\Gamma_n} &= \mathrm{id}_{\mathsf{H}^k_{\mathbb{S},\Gamma_n}(\mathrm{divDiv},\Omega)},\\
\widehat{\mathcal{Q}}^{k+1,k,1}_{\mathrm{divDiv}_{\mathbb{S}},\Gamma_n} + \widehat{\mathcal{Q}}^{k+1,k,\infty}_{\mathrm{divDiv}_{\mathbb{S}},\Gamma_n} + \mathrm{symRot}\ \widehat{\mathcal{Q}}^{k+1,k,0}_{\mathrm{divDiv}_{\mathbb{S}},\Gamma_n} &= \mathrm{id}_{\mathsf{H}^{k+1,k}_{\mathbb{S},\Gamma_n}(\mathrm{divDiv},\Omega)}
\end{aligned}
$$

and

$$
\begin{aligned}
\widehat{\mathcal{Q}}^{k,1}_{\mathrm{Div}_{\mathbb{T}},\Gamma_n} &= \widetilde{\mathcal{Q}}^{k,1}_{\mathrm{Div}_{\mathbb{T}},\Gamma_n} = \mathcal{P}^k_{\mathrm{Div}_{\mathbb{T}},\Gamma_n}\mathrm{Div}^k_{\mathbb{T},\Gamma_n} = \mathcal{Q}^{k,1}_{\mathrm{Div}_{\mathbb{T}},\Gamma_n}\left(\mathrm{Div}^k_{\mathbb{T},\Gamma_n}\right)^{-1}_{\perp}\mathrm{Div}^k_{\mathbb{T},\Gamma_n},\\
\widehat{\mathcal{Q}}^{k,\infty}_{\mathrm{Div}_{\mathbb{T}},\Gamma_n} &= I^{-1}_{\pi_{N\left({}_{\mathbb{S}}\mathrm{Rot}_{\mathbb{T},\Gamma_t}\right)}}\pi_{N\left({}_{\mathbb{S}}\mathrm{Rot}_{\mathbb{T}},\Gamma_t\right)}\widetilde{\mathcal{N}}^k_{\mathrm{Div}_{\mathbb{T}},\Gamma_n},\\
\widehat{\mathcal{Q}}^{k,0}_{\mathrm{Div}_{\mathbb{T}},\Gamma_n} &= \widetilde{\mathcal{Q}}^{k,1}_{{}_{\mathbb{S}}\mathrm{Rot}_{\mathbb{T}},\Gamma_t} = \mathcal{P}^k_{{}_{\mathbb{S}}\mathrm{Rot}_{\mathbb{T}},\Gamma_t}{}_{\mathbb{S}}\mathrm{Rot}^k_{\mathbb{T},\Gamma_t} = \mathcal{Q}^{k,1}_{{}_{\mathbb{S}}\mathrm{Rot}_{\mathbb{T}},\Gamma_t}\left({}_{\mathbb{S}}\mathrm{Rot}^k_{\mathbb{T},\Gamma_t}\right)^{-1}_{\perp}{}_{\mathbb{S}}\mathrm{Rot}^k_{\mathbb{T},\Gamma_t},\\
\widehat{\mathcal{Q}}^{k,\infty}_{{}_{\mathbb{S}}\mathrm{Rot}_{\mathbb{T}},\Gamma_t} &= I^{-1}_{\pi_{N\left(\mathrm{Div}_{\mathbb{T},\Gamma_n}\mu\right)}}\pi_{N\left(\mathrm{Div}_{\mathbb{T},\Gamma_n}\mu\right)}\widetilde{\mathcal{N}}^k_{{}_{\mathbb{S}}\mathrm{Rot}_{\mathbb{T}},\Gamma_t},\\
\widehat{\mathcal{Q}}^{k,0}_{\mathbb{S}\mathrm{Rot}_{\mathbb{T}},\Gamma_t} &= \mathcal{P}^k_{{}_{\mathbb{T}}\mathrm{Grad},\Gamma_t}\left(1 - I^{-1}_{\pi_{N\left(\mathrm{Div}_{\mathbb{T},\Gamma_n}\mu\right)}}\pi_{N\left(\mathrm{Div}_{\mathbb{T},\Gamma_n}\mu\right)}\right)\widetilde{\mathcal{N}}^k_{{}_{\mathbb{S}}\mathrm{Rot}_{\mathbb{T}},\Gamma_t},\\
\widehat{\mathcal{Q}}^{k,1}_{\mathrm{divDiv}_{\mathbb{S}},\Gamma_n} &= \widetilde{\mathcal{Q}}^{k,1}_{\mathrm{divDiv}_{\mathbb{S}},\Gamma_n} = \mathcal{P}^k_{\mathrm{divDiv}_{\mathbb{S}},\Gamma_n}\mathrm{divDiv}^k_{\mathbb{S},\Gamma_n} = \mathcal{Q}^{k,1}_{\mathrm{divDiv}_{\mathbb{S}},\Gamma_n}\left(\mathrm{divDiv}^k_{\mathbb{S},\Gamma_n}\right)^{-1}_{\perp}\mathrm{divDiv}^k_{\mathbb{S},\Gamma_n},\\
\widehat{\mathcal{Q}}^{k,\infty}_{\mathrm{divDiv}_{\mathbb{S}},\Gamma_n} &= I^{-1}_{\pi_{N\left({}_{\mathbb{T}}\mathrm{Rot}_{\mathbb{S},\Gamma_t}\right)}}\pi_{N\left({}_{\mathbb{T}}\mathrm{Rot}_{\mathbb{S},\Gamma_t}\right)}\widetilde{\mathcal{N}}^k_{\mathrm{divDiv}_{\mathbb{S}},\Gamma_n},\\
\widehat{\mathcal{Q}}^{k,0}_{\mathrm{divDiv}_{\mathbb{S}},\Gamma_n} &= \mathcal{P}^k_{{}_{\mathbb{S}}\mathrm{Rot}_{\mathbb{T}},\Gamma_n}\left(1 - I^{-1}_{\pi_{N\left({}_{\mathbb{T}}\mathrm{Rot}_{\mathbb{S},\Gamma_t}\right)}}\pi_{N\left({}_{\mathbb{T}}\mathrm{Rot}_{\mathbb{S},\Gamma_t}\right)}\right)\widetilde{\mathcal{N}}^k_{\mathrm{divDiv}_{\mathbb{S}},\Gamma_n},\\
\widehat{\mathcal{Q}}^{k+1,k,1}_{\mathrm{divDiv}_{\mathbb{S}},\Gamma_n} &= \widetilde{\mathcal{Q}}^{k+1,k,1}_{\mathrm{divDiv}_{\mathbb{S}},\Gamma_n} = \mathcal{P}^{k+1,k}_{\mathrm{divDiv}_{\mathbb{S}},\Gamma_n}\mathrm{divDiv}^{k+1,k}_{\mathbb{S},\Gamma_n} = \mathcal{Q}^{k+1,k,1}_{\mathrm{divDiv}_{\mathbb{S}},\Gamma_n}\left(\mathrm{divDiv}^{k+1,k}_{\mathbb{S},\Gamma_n}\right)^{-1}_{\perp}\mathrm{divDiv}^{k+1,k}_{\mathbb{S},\Gamma_n},\\
\widehat{\mathcal{Q}}^{k+1,k,\infty}_{\mathrm{divDiv}_{\mathbb{S}},\Gamma_n} &= I^{-1}_{\pi_{N\left({}_{\mathbb{T}}\mathrm{Rot}_{\mathbb{S},\Gamma_t}\right)}}\pi_{N\left({}_{\mathbb{T}}\mathrm{Rot}_{\mathbb{S},\Gamma_t}\right)}\widetilde{\mathcal{N}}^{k+1,k}_{\mathrm{divDiv}_{\mathbb{S}},\Gamma_n},\\
\widehat{\mathcal{Q}}^{k+1,k,0}_{\mathrm{divDiv}_{\mathbb{S}},\Gamma_n} &= \mathcal{P}^{k+1}_{{}_{\mathbb{S}}\mathrm{Rot}_{\mathbb{T}},\Gamma_n}\left(1 - I^{-1}_{\pi_{N\left({}_{\mathbb{T}}\mathrm{Rot}_{\mathbb{S},\Gamma_t}\right)}}\pi_{N\left({}_{\mathbb{T}}\mathrm{Rot}_{\mathbb{S},\Gamma_t}\right)}\right)\widetilde{\mathcal{N}}^{k+1,k}_{\mathrm{divDiv}_{\mathbb{S}},\Gamma_n}
\end{aligned}
$$

with

$$\widetilde{\mathcal{N}}^{k}_{\mathrm{Div}_{\mathbb{T}},\Gamma_n} = 1 - \widetilde{\mathcal{Q}}^{k,1}_{\mathrm{Div}_{\mathbb{T}},\Gamma_n}, \quad \widetilde{\mathcal{N}}^{k}_{\mathrm{divDiv}_{\mathbb{S}},\Gamma_n} = 1 - \widetilde{\mathcal{Q}}^{k,1}_{\mathrm{divDiv}_{\mathbb{S}},\Gamma_n},$$
$$\widetilde{\mathcal{N}}^{k}_{{}_{\mathbb{S}}\mathrm{Rot}_{\mathbb{T}},\Gamma_t} = 1 - \widetilde{\mathcal{Q}}^{k,1}_{{}_{\mathbb{S}}\mathrm{Rot}_{\mathbb{T}},\Gamma_t}, \quad \widetilde{\mathcal{N}}^{k+1,k}_{\mathrm{divDiv}_{\mathbb{S}},\Gamma_n} = 1 - \widetilde{\mathcal{Q}}^{k+1,k,1}_{\mathrm{divDiv}_{\mathbb{S}},\Gamma_n},$$

and

$$\begin{aligned}
I_{\pi_{N(\mathrm{divDiv}_{\mathbb{S},\Gamma_n}\varepsilon)}} : \quad & \mathrm{Lin}\,\mathcal{B}^{{}^{\mathbb{T}}\mathrm{Rot}_{\mathbb{S},\Gamma_t}}(\Omega) && \to && \mathrm{Lin}\,\pi_{N(\mathrm{divDiv}_{\mathbb{S},\Gamma_n}\varepsilon)}\mathcal{B}^{{}^{\mathbb{T}}\mathrm{Rot}_{\mathbb{S},\Gamma_t}}(\Omega) = \mathcal{H}_{\mathbb{S},\Gamma_t,\Gamma_n,\varepsilon}(\Omega) \\
& B_{\ell}^{{}^{\mathbb{T}}\mathrm{Rot}_{\mathbb{S},\Gamma_t}} && \mapsto && \pi_{N(\mathrm{divDiv}_{\mathbb{S},\Gamma_n}\varepsilon)}B_{\ell}^{{}^{\mathbb{T}}\mathrm{Rot}_{\mathbb{S},\Gamma_t}}, \\
I_{\pi_{N({}_{\mathbb{T}}\mathrm{Rot}_{\mathbb{S},\Gamma_t})}} : \quad & \mathrm{Lin}\,\mathcal{B}^{\mathrm{divDiv}_{\mathbb{S},\Gamma_n}}(\Omega) && \to && \mathrm{Lin}\,\pi_{N({}_{\mathbb{T}}\mathrm{Rot}_{\mathbb{S},\Gamma_t})}\varepsilon^{-1}\mathcal{B}^{\mathrm{divDiv}_{\mathbb{S},\Gamma_n}}(\Omega) = \mathcal{H}_{\mathbb{S},\Gamma_t,\Gamma_n,\varepsilon}(\Omega) \\
& B_{\ell}^{\mathrm{divDiv}_{\mathbb{S},\Gamma_n}} && \mapsto && \pi_{N({}_{\mathbb{T}}\mathrm{Rot}_{\mathbb{S},\Gamma_t})}\varepsilon^{-1}B_{\ell}^{\mathrm{divDiv}_{\mathbb{S},\Gamma_n}}, \\
I_{\pi_{N(\mathrm{Div}_{\mathbb{T},\Gamma_n}\mu)}} : \quad & \mathrm{Lin}\,\mathcal{B}^{{}^{\mathbb{S}}\mathrm{Rot}_{\mathbb{T}},\Gamma_t}(\Omega) && \to && \mathrm{Lin}\,\pi_{N(\mathrm{Div}_{\mathbb{T},\Gamma_n}\mu)}\mathcal{B}^{{}^{\mathbb{S}}\mathrm{Rot}_{\mathbb{T}},\Gamma_t}(\Omega) = \mathcal{H}_{\mathbb{T},\Gamma_{\sqcup},\Gamma_{\setminus},\mu}(\Omega) \\
& B_{\ell}^{{}^{\mathbb{S}}\mathrm{Rot}_{\mathbb{T}},\Gamma_t} && \mapsto && \pi_{N(\mathrm{Div}_{\mathbb{T},\Gamma_n}\mu)}B_{\ell}^{{}^{\mathbb{S}}\mathrm{Rot}_{\mathbb{T}},\Gamma_t}, \\
I_{\pi_{N({}^{\mathbb{S}}\mathrm{Rot}_{\mathbb{T},\Gamma_t})}} : \quad & \mathrm{Lin}\,\mathcal{B}^{\mathrm{Div}_{\mathbb{T},\Gamma_n}}(\Omega) && \to && \mathrm{Lin}\,\pi_{N({}_{\mathbb{S}}\mathrm{Rot}_{\mathbb{T}},\Gamma_t)}\mu^{-1}\mathcal{B}^{\mathrm{Div}_{\mathbb{T},\Gamma_n}}(\Omega) = \mathcal{H}_{\mathbb{T},\Gamma_t,\Gamma_n,\mu}(\Omega) \\
& B_{\ell}^{\mathrm{Div}_{\mathbb{T},\Gamma_n}} && \mapsto && \pi_{N({}_{\mathbb{S}}\mathrm{Rot}_{\mathbb{T}},\Gamma_t)}\mu^{-1}B_{\ell}^{\mathrm{Div}_{\mathbb{T},\Gamma_n}}.
\end{aligned}$$