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Hilbert complexes with mixed boundary conditions part 1: de Rham complex

Dirk Pauly | **Michael Schomburg**

Fakultät für Mathematik, Universität Duisburg-Essen, Germany

Correspondence
Dirk Pauly, Fakultät für Mathematik, Universität Duisburg-Essen, Germany.
Email: dirk.pauly@uni-due.de

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We show that the de Rham Hilbert complex with mixed boundary conditions on bounded strong Lipschitz domains is closed and compact. The crucial results are compact embeddings which follow by abstract arguments using functional analysis together with particular regular decompositions. Higher Sobolev order results are proved as well.

KEYWORDS

compact embeddings, de Rham complex, Hilbert complexes, mixed boundary conditions, regular decompositions, regular potentials

MSC CLASSIFICATION

35A23; 35Q61; 58A12; 47B02

1 | INTRODUCTION

In this paper, we prove regular decompositions and resulting compact embeddings for the *de Rham complex* (of vector fields)

$$\cdots \xrightarrow{\cdots} \mathsf{L}^2(\Omega) \xrightarrow{\text{grad}} \mathsf{L}^2(\Omega) \xrightarrow{\text{rot}} \mathsf{L}^2(\Omega) \xrightarrow{\div} \mathsf{L}^2(\Omega) \xrightarrow{\cdots} \cdots,$$

and, more generally, for the *de Rham complex* (of differential forms)

$$\cdots \xrightarrow{\cdots} \mathsf{L}^{q-1,2}(\Omega) \xrightarrow{\mathrm{d}^{q-1}} \mathsf{L}^{q,2}(\Omega) \xrightarrow{\mathrm{d}^{q}} \mathsf{L}^{q+1,2}(\Omega) \xrightarrow{\cdots} \cdots.$$

In forthcoming papers, we shall extend our results to other more complicated complexes as well, such as the elasticity complex

$$\cdots \xrightarrow{\cdots} \mathsf{L}^2(\Omega) \xrightarrow{\text{symGrad}} \mathsf{L}^2_{\mathbb{S}}(\Omega) \xrightarrow{\text{RotRot}^{\top}_{\mathbb{S}}} \mathsf{L}^2_{\mathbb{S}}(\Omega) \xrightarrow{\text{Div}_{\mathbb{S}}} \mathsf{L}^2(\Omega) \xrightarrow{\cdots} \cdots,$$

or the primal and dual biharmonic complexes

$$\cdots \xrightarrow{\cdots} \mathsf{L}^2(\Omega) \xrightarrow{\text{Gradgrad}} \mathsf{L}^2_{\mathbb{S}}(\Omega) \xrightarrow{\text{Rot}_{\mathbb{S}}} \mathsf{L}^2_{\mathbb{T}}(\Omega) \xrightarrow{\text{Div}_{\mathbb{T}}} \mathsf{L}^2(\Omega) \xrightarrow{\cdots} \cdots,$$

$$\cdots \xrightarrow{\cdots} \mathsf{L}^2(\Omega) \xrightarrow{\text{devGrad}} \mathsf{L}^2_{\mathbb{T}}(\Omega) \xrightarrow{\text{symRot}_{\mathbb{T}}} \mathsf{L}^2_{\mathbb{S}}(\Omega) \xrightarrow{\text{divDiv}_{\mathbb{S}}} \mathsf{L}^2(\Omega) \xrightarrow{\cdots} \cdots,$$

which is possible due to the general structure and our unified approach and methods. All complexes are considered with mixed boundary conditions on a bounded strong Lipschitz domain $\Omega \subset \mathbb{R}^d$. Some of our results hold also for higher

Sobolev orders. Note that the first three complexes are formally symmetric and that the last two complexes are formally adjoint or dual to each other.

These *Hilbert complexes* share the same geometric sequence (complex) structure

$$\cdots \xrightarrow{\cdots} \mathsf{H}_0 \xrightarrow{\mathrm{A}_0} \mathsf{H}_1 \xrightarrow{\mathrm{A}_1} \mathsf{H}_2 \xrightarrow{\cdots} \cdots, \qquad R(\mathrm{A}_0) \subset N(\mathrm{A}_1),$$

where A_0 and A_1 are densely defined and closed (unbounded) linear operators between Hilbert spaces H_ℓ. The corresponding *domain Hilbert complex* is denoted by

$$\cdots \xrightarrow{\cdots} D(\mathrm{A}_0) \xrightarrow{\mathrm{A}_0} D(\mathrm{A}_1) \xrightarrow{\mathrm{A}_1} \mathsf{H}_2 \xrightarrow{\cdots} \cdots .$$

In fact, we show that the assumptions of Lemma 2.22 hold, which provides an elegant, abstract, and short way to prove the crucial compact embeddings

$$D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) \hookrightarrow \mathsf{H}_1 \tag{1}$$

for the de Rham Hilbert complexes, cf. Theorems 4.8, Theorem 4.16, and Theorem 5.4, Theorem 5.7. In principle, our general technique—compact embeddings by regular decompositions and Rellich's selection theorem—works for all Hilbert complexes known in the literature; see, for example, Arnold and Hu[1] for a comprehensive list of such Hilbert complexes.

Roughly speaking, a regular decomposition has the form

$$D(\mathrm{A}_1) = \mathsf{H}_1^+ + \mathrm{A}_0\mathsf{H}_0^+$$

with regular subspaces $\mathsf{H}_0^+ \subset D(\mathrm{A}_0)$ and $\mathsf{H}_1^+ \subset D(\mathrm{A}_1)$ such that the embeddings $\mathsf{H}_0^+ \hookrightarrow \mathsf{H}_0$ and $\mathsf{H}_1^+ \hookrightarrow \mathsf{H}_1$ are compact. The compactness is typically and simply given by Rellich's selection theorem, which justifies the notion 'regular,' by applying A_1 any regular decomposition implies regular potentials

$$R(\mathrm{A}_1) = \mathrm{A}_1\mathsf{H}_1^+$$

by the complex property. The respective regular potential and decomposition operators

$$\mathcal{P}_{\mathrm{A}_1} : R(\mathrm{A}_1) \to \mathsf{H}_1^+, \qquad \mathcal{Q}_{\mathrm{A}_1}^1 : D(\mathrm{A}_1) \to \mathsf{H}_1^+, \qquad \mathcal{Q}_{\mathrm{A}_1}^0 : D(\mathrm{A}_1) \to \mathsf{H}_0^+$$

are bounded and satisfy $\mathrm{A}_1\mathcal{P}_{\mathrm{A}_1} = \mathrm{id}_{R(\mathrm{A}_1)}$ as well as $\mathrm{id}_{D(\mathrm{A}_1)} = \mathcal{Q}_{\mathrm{A}_1}^1 + \mathrm{A}_0\mathcal{Q}_{\mathrm{A}_1}^0$.

Note that (1) implies several important results related to the particular Hilbert complex by the so-called FA-ToolBox, cf. previous studies[2–5] and other works.[6–8] Upon others, one gets Friedrichs/Poincaré type estimates, closed ranges, compact resolvents, Helmholtz-type decompositions, comprehensive solution theories, div-curl lemmas, discrete point spectra, eigenvector expansions, a posteriori error estimates, and index theorems for related Dirac type operators. See Theorem 4.9 and Theorem 5.5 for a selection of such results.

For an historical overview on the compact embeddings (1) corresponding to the de Rham complex and Maxwell's equations, also called Weck's or Weber-Weck-Picard's selection theorem, see, for example, the introductions in Bauer et al. and Neff et al,[9,10] the original papers,[11–16] and the recent state of the art results for mixed boundary conditions and bounded weak Lipschitz domains in other works.[9,17,18] Compact embeddings (1) corresponding to the biharmonic and the elasticity complex are given in Pauly and Zulehner[8] and their other works,[6,7] respectively. Note that in the recent paper,[1] similar results have been shown for the special case of no or full boundary conditions using an alternative and more algebraic approach, the so-called Bernstein–Gelfand–Gelfand (BGG) resolution.

2 | FAT: FA-TOOLBOX

We collect and present some old and new results from the so-called functional analysis toolbox (FA-ToolBox).

2.1 | FAT I: Linear operators, adjoints, and fundamental lemmas

We shall work with bounded and unbounded linear operators. For this, let H_0 and H_1 be Hilbert spaces. For a *bounded* linear operator A, we use the notation

$$\mathrm{A} : D(\mathrm{A}) \to \mathsf{H}_1 \tag{2}$$

where $D(\mathrm{A}) \subset \mathsf{H}_0$ is the domain of definition of A. Its *unbounded* version will be denoted by

$$\mathrm{A} : D(\mathrm{A}) \subset \mathsf{H}_0 \to \mathsf{H}_1. \tag{3}$$

Kernel and range of A shall be denoted by $N(\mathrm{A})$ and $R(\mathrm{A})$, respectively. Note that—equipped with the standard graph inner product—$D(\mathrm{A})$ becomes a Hilbert space as long as A is closed. The difference of the latter two versions of A comes from using the norm of $D(\mathrm{A})$ or simply the norm of H_0, respectively. Generally, inner product, norm, orthogonality, and orthogonal sum in a Hilbert space H shall be denoted by $\langle\cdot,\cdot\rangle_{\mathsf{H}}$, $|\cdot|_{\mathsf{H}}$, $\perp_{\mathsf{H}}$, and $\oplus_{\mathsf{H}}$, respectively. By $\dotplus$, we indicate a direct sum. The dual space of a Banach or Hilbert space H will be written as H'.

There are at least three different adjoints. The bounded linear operator (2) has the *Banach space adjoint* $\mathrm{A}' : \mathsf{H}_1' \to D(\mathrm{A})'$, which—as usual—may be identified with its modification

$$\mathrm{A}'\mathcal{R}_{\mathsf{H}_1} : \mathsf{H}_1 \to D(\mathrm{A})',$$

where $\mathcal{R}_{\mathsf{H}_1} : \mathsf{H}_1 \to \mathsf{H}_1'$ denotes the Riesz isomorphism of H_1. Another option is the *Hilbert space adjoint* defined by

$$\mathrm{A}^* := \mathcal{R}_{D(\mathrm{A})}^{-1}\mathrm{A}'\mathcal{R}_{\mathsf{H}_1} : \mathsf{H}_1 \to D(\mathrm{A}).$$

On the other hand, the unbounded linear operator (3) has the *Hilbert space adjoint*

$$\mathrm{A}^* : D(\mathrm{A}^*) \subset \mathsf{H}_1 \to \mathsf{H}_0,$$

provided that A is densely defined. A^* is always closed and characterised by

$$\forall x \in D(\mathrm{A}) \qquad \forall y \in D(\mathrm{A}^*) \qquad \langle \mathrm{A}x, y\rangle_{\mathsf{H}_1} = \langle x, \mathrm{A}^* y\rangle_{\mathsf{H}_0}.$$

Note that the different adjoints are strongly related through the respective Riesz isomorphisms. If the unbounded operator A is densely defined and closed, so is A^*. In this case, $\mathrm{A}^{**} = \overline{\mathrm{A}} = \mathrm{A}$, and we call $(\mathrm{A}, \mathrm{A}^*)$ a dual pair.

Let us recall a small part of the co-called FA-ToolBox from, for example, Pauly;[3, Lemma 4.1, Lemma 4.3] see also previous studies,[2,4,5,7,8] for more details. For this, let A from (3) be *densely defined* and *closed*. Moreover, let

$$\begin{aligned}
\mathrm{A}_\perp &:= \mathcal{A} := \mathrm{A}|_{N(\mathrm{A})^{\perp_{\mathsf{H}_0}}} : D(\mathrm{A}_\perp) \subset N(\mathrm{A})^{\perp_{\mathsf{H}_0}} \to N(\mathrm{A}^*)^{\perp_{\mathsf{H}_1}}, & D(\mathrm{A}_\perp) &:= D(\mathrm{A}) \cap N(\mathrm{A})^{\perp_{\mathsf{H}_0}},\\
\mathrm{A}_\perp^* &:= \mathcal{A}^* := \mathrm{A}^*|_{N(\mathrm{A}^*)^{\perp_{\mathsf{H}_1}}} : D(\mathrm{A}_\perp^*) \subset N(\mathrm{A}^*)^{\perp_{\mathsf{H}_1}} \to N(\mathrm{A})^{\perp_{\mathsf{H}_0}}, & D(\mathrm{A}_\perp^*) &:= D(\mathrm{A}^*) \cap N(\mathrm{A}^*)^{\perp_{\mathsf{H}_1}}
\end{aligned}$$

denote the reduced operators, which are densely defined, closed, and injective. Note that by the projection theorem, we have the orthogonal Helmholtz-type decompositions

$$\begin{aligned}
\mathsf{H}_0 &= N(\mathrm{A}) \oplus_{\mathsf{H}_0} N(\mathrm{A})^{\perp_{\mathsf{H}_0}}, \qquad N(\mathrm{A})^{\perp_{\mathsf{H}_0}} = \overline{R(\mathrm{A}^*)}, \qquad N(\mathrm{A}) = R(\mathrm{A}^*)^{\perp_{\mathsf{H}_0}},\\
D(\mathrm{A}) &= N(\mathrm{A}) \oplus_{\mathsf{H}_0} D(\mathrm{A}_\perp),\\
\mathsf{H}_1 &= N(\mathrm{A}^*) \oplus_{\mathsf{H}_1} N(\mathrm{A}^*)^{\perp_{\mathsf{H}_1}}, \qquad N(\mathrm{A}^*)^{\perp_{\mathsf{H}_1}} = \overline{R(\mathrm{A})}, \qquad N(\mathrm{A}^*) = R(\mathrm{A})^{\perp_{\mathsf{H}_1}},\\
D(\mathrm{A}^*) &= N(\mathrm{A}^*) \oplus_{\mathsf{H}_1} D(\mathrm{A}_\perp^*),
\end{aligned} \tag{4}$$

and thus, $R(\mathrm{A}_\perp) = R(\mathrm{A})$ and $R(\mathrm{A}_\perp^*) = R(\mathrm{A}^*)$.

Lemma 2.1 (fundamental lemma 1)**.** *The following assertions are equivalent:*

(i) $\exists\ c_{\mathrm{A}} > 0 \qquad \forall x \in D(\mathrm{A}_\perp) \qquad |x|_{\mathsf{H}_0} \leq c_{\mathrm{A}} |\mathrm{A}x|_{\mathsf{H}_1}$

(i') $\exists c_{\mathrm{A}^*} > 0 \quad \forall y \in D(\mathrm{A}_\perp^*) \qquad |y|_{\mathsf{H}_1} \le c_{\mathrm{A}^*} |\mathrm{A}^* x|_{\mathsf{H}_0}$
(ii) $R(\mathrm{A}) = R(\mathrm{A}_\perp)$ *is closed.*
(ii') $R(\mathrm{A}^*) = R(\mathrm{A}_\perp^*)$ *is closed.*
(iii) $\mathrm{A}_\perp^{-1} : R(\mathrm{A}) \to D(\mathrm{A}_\perp)$ *is continuous.*
(iii') $(\mathrm{A}_\perp^*)^{-1} : R(\mathrm{A}^*) \to D(\mathrm{A}_\perp^*)$ *is continuous.*

Moreover, for the 'best' constants, it holds $\left|\mathrm{A}_\perp^{-1}\right|_{R(\mathrm{A}),\mathsf{H}_0} = c_{\mathrm{A}} = c_{\mathrm{A}^*} = \left|(\mathrm{A}_\perp^*)^{-1}\right|_{R(\mathrm{A}^*),\mathsf{H}_1}$.

Lemma 2.2 (fundamental lemma 2). *Let* $D(\mathrm{A}_\perp) \hookrightarrow \mathsf{H}_0$ *be compact. Then each of (i)–(iii') in Lemma 2.1 holds.*

Lemma 2.3 (fundamental lemma 3). *The following assertions are equivalent:*

(i) $D(\mathrm{A}_\perp) \hookrightarrow \mathsf{H}_0$ *is compact.*
(i') $D(\mathrm{A}_\perp^*) \hookrightarrow \mathsf{H}_1$ *is compact.*
(ii) $\mathrm{A}_\perp^{-1} : R(\mathrm{A}) \to \mathsf{H}_0$ *is compact.*
(ii') $(\mathrm{A}_\perp^*)^{-1} : R(\mathrm{A}^*) \to \mathsf{H}_1$ *is compact.*

Remark 2.4. $D(\mathrm{A}) \hookrightarrow \mathsf{H}_0$ compact implies $D(\mathrm{A}_\perp) \hookrightarrow \mathsf{H}_0$ compact, and $D(\mathrm{A}^*) \hookrightarrow \mathsf{H}_1$ compact implies $D(\mathrm{A}_\perp^*) \hookrightarrow \mathsf{H}_1$ compact.

2.2 | FAT II: Hilbert complexes and Mini FA-ToolBox

We continue to make use of parts of the FA-ToolBox from, e.g.,[2–5] and,[6–8] together with an extension suited for so called (bounded linear) regular potential operators and regular decompositions introduced in Pauly and Zulehner.[8] Lemma 2.22 provides an elegant, abstract, and short way to prove compact embedding results for an arbitrary Hilbert complex.

For this, let $\mathsf{H}_0, \mathsf{H}_1, \mathsf{H}_2$ be Hilbert spaces and let

$$\cdots \underset{\cdots}{\overset{\cdots}{\rightleftarrows}} \mathsf{H}_0 \underset{\mathrm{A}_0^*}{\overset{\mathrm{A}_0}{\rightleftarrows}} \mathsf{H}_1 \underset{\mathrm{A}_1^*}{\overset{\mathrm{A}_1}{\rightleftarrows}} \mathsf{H}_2 \underset{\cdots}{\overset{\cdots}{\rightleftarrows}} \cdots \tag{5}$$

be a *primal and dual Hilbert complex*, that is,

$$\mathrm{A}_0 : D(\mathrm{A}_0) \subset \mathsf{H}_0 \to \mathsf{H}_1, \qquad \mathrm{A}_1 : D(\mathrm{A}_1) \subset \mathsf{H}_1 \to \mathsf{H}_2$$

are *densely defined* and *closed* (unbounded) linear operators satisfying the *complex property*

$$\mathrm{A}_1\mathrm{A}_0 \subset 0, \tag{6}$$

together with (densely defined and closed Hilbert space) adjoints

$$\mathrm{A}_0^* : D(\mathrm{A}_0^*) \subset \mathsf{H}_1 \to \mathsf{H}_0, \qquad \mathrm{A}_1^* : D(\mathrm{A}_1^*) \subset \mathsf{H}_2 \to \mathsf{H}_1.$$

Remark 2.5. Note that the complex property (6) is equivalent to $R(\mathrm{A}_0) \subset N(\mathrm{A}_1)$, which is equivalent to the dual complex property $R(\mathrm{A}_1^*) \subset N(\mathrm{A}_0^*)$ as

$$R(\mathrm{A}_1^*) \subset \overline{R(\mathrm{A}_1^*)} = N(\mathrm{A}_1)^{\perp_{\mathsf{H}_1}} \subset R(\mathrm{A}_0)^{\perp_{\mathsf{H}_1}} = N(\mathrm{A}_0^*)$$

and vice versa.

Remark 2.6. Let $\mathrm{A}_0, \mathrm{A}_1$ be given by the closures of densely defined (unbounded) linear operators

$$\mathring{\mathrm{A}}_0 : D(\mathring{\mathrm{A}}_0) \subset \mathsf{H}_0 \to \mathsf{H}_1, \qquad \mathring{\mathrm{A}}_1 : D(\mathring{\mathrm{A}}_1) \subset \mathsf{H}_1 \to \mathsf{H}_2$$

satisfying the complex property $\mathring{\mathrm{A}}_1\mathring{\mathrm{A}}_0 \subset 0$. Then $\mathrm{A}_0 = \overline{\mathring{\mathrm{A}}_0}$ and $\mathrm{A}_1 = \overline{\mathring{\mathrm{A}}_1}$ are densely defined and closed (unbounded) linear operators satisfying the complex property $\mathrm{A}_1\mathrm{A}_0 \subset 0$, since $N(\mathrm{A}_1)$ is closed and thus $R(\mathring{\mathrm{A}}_0) \subset N(\mathring{\mathrm{A}}_1) \subset N(\mathrm{A}_1)$ implies $R(\mathrm{A}_0) \subset N(\mathrm{A}_1)$.

As in (4) and defining the cohomology group

$$N_{0,1} := N(\mathrm{A}_1) \cap N(\mathrm{A}_0^*)$$

we get the following orthogonal Helmholtz-type decompositions.

Lemma 2.7 (Helmholtz decomposition lemma)**.** *The refined orthogonal Helmholtz-type decompositions*

$$\begin{aligned}
\mathsf{H}_1 &= \overline{R(\mathrm{A}_0)} \oplus_{\mathsf{H}_1} N(\mathrm{A}_0^*), & \mathsf{H}_1 &= N(\mathrm{A}_1) \oplus_{\mathsf{H}_1} \overline{R(\mathrm{A}_1^*)},\\
N(\mathrm{A}_1) &= \overline{R(\mathrm{A}_0)} \oplus_{\mathsf{H}_1} N_{0,1}, & N(\mathrm{A}_0^*) &= N_{0,1} \oplus_{\mathsf{H}_1} \overline{R(\mathrm{A}_1^*)},\\
D(\mathrm{A}_1) &= \overline{R(\mathrm{A}_0)} \oplus_{\mathsf{H}_1} \big(D(\mathrm{A}_1) \cap N(\mathrm{A}_0^*)\big), & D(\mathrm{A}_0^*) &= \big(N(\mathrm{A}_1) \cap D(\mathrm{A}_0^*)\big) \oplus_{\mathsf{H}_1} \overline{R(\mathrm{A}_1^*)},\\
D(\mathrm{A}_0^*) &= D\big((\mathrm{A}_0^*)_\perp\big) \oplus_{\mathsf{H}_1} N(\mathrm{A}_0^*), & D(\mathrm{A}_1) &= N(\mathrm{A}_1) \oplus_{\mathsf{H}_1} D\left((\mathrm{A}_1)_\perp\right),
\end{aligned} \tag{7}$$

as well as $R\big((\mathrm{A}_0^*)_\perp\big) = R(\mathrm{A}_0^*)$ *and* $R\left((\mathrm{A}_1)_\perp\right) = R(\mathrm{A}_1)$ *hold. Moreover,*

$$\begin{aligned}
\mathsf{H}_1 &= \overline{R(\mathrm{A}_0)} \oplus_{\mathsf{H}_1} N_{0,1} \oplus_{\mathsf{H}_1} \overline{R(\mathrm{A}_1^*)},\\
D(\mathrm{A}_0^*) &= D\big((\mathrm{A}_0^*)_\perp\big) \oplus_{\mathsf{H}_1} N_{0,1} \oplus_{\mathsf{H}_1} \overline{R(\mathrm{A}_1^*)},\\
D(\mathrm{A}_1) &= \overline{R(\mathrm{A}_0)} \oplus_{\mathsf{H}_1} N_{0,1} \oplus_{\mathsf{H}_1} D\left((\mathrm{A}_1)_\perp\right),\\
D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) &= D\big((\mathrm{A}_0^*)_\perp\big) \oplus_{\mathsf{H}_1} N_{0,1} \oplus_{\mathsf{H}_1} D\left((\mathrm{A}_1)_\perp\right).
\end{aligned} \tag{8}$$

As

$$\begin{aligned}
D\left((\mathrm{A}_1)_\perp\right) &= D(\mathrm{A}_1) \cap \overline{R(\mathrm{A}_1^*)} \subset D(\mathrm{A}_1) \cap N(\mathrm{A}_0^*) \subset D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*),\\
D\big((\mathrm{A}_0^*)_\perp\big) &= \overline{R(\mathrm{A}_0)} \cap D(\mathrm{A}_0^*) \subset N(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) \subset D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*)
\end{aligned}$$

with continuous embeddings, we get the following result.

Lemma 2.8 (compactness lemma)**.** *The following assertions are equivalent:*

- *(i)* $D\left((\mathrm{A}_0)_\perp\right) \hookrightarrow \mathsf{H}_0$, $D\left((\mathrm{A}_1)_\perp\right) \hookrightarrow \mathsf{H}_1$, *and* $N_{0,1} \hookrightarrow \mathsf{H}_1$ *are compact.*
- *(ii)* $D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) \hookrightarrow \mathsf{H}_1$ *is compact.*

In this case, the cohomology group $N_{0,1}$ *has finite dimension.*

Summarising the latter results, we get the following theorem.

Theorem 2.9 (mini FAT)**.** *Let* $D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) \hookrightarrow \mathsf{H}_1$ *be compact. Then:*

- *(i) The ranges* $R(A_0)$, $R(\mathrm{A}_0^*)$ *and* $R(A_1)$, $R(\mathrm{A}_1^*)$ *are closed.*
- *(ii) The inverse operators* $(\mathrm{A}_0)_\perp^{-1}$, $(\mathrm{A}_0^*)_\perp^{-1}$ *and* $(\mathrm{A}_1)_\perp^{-1}$, $(\mathrm{A}_1^*)_\perp^{-1}$ *are compact.*
- *(iii) The cohomology group* $N_{0,1} = N(\mathrm{A}_1) \cap N(\mathrm{A}_0^*)$ *has finite dimension.*
- *(iv) The orthogonal Helmholtz-type decomposition* $\mathsf{H}_1 = R(\mathrm{A}_0) \oplus_{\mathsf{H}_1} N_{0,1} \oplus_{\mathsf{H}_1} R(\mathrm{A}_1^*)$ *holds.*
- *(v) There exist* $c_{\mathrm{A}_0}, c_{\mathrm{A}_1} > 0$ *such that*

$$\begin{aligned}
&\forall x \in D\left((\mathrm{A}_0)_\perp\right) = D(\mathrm{A}_0) \cap N(\mathrm{A}_0)^{\perp_{\mathsf{H}_0}} = D(\mathrm{A}_0) \cap R(\mathrm{A}_0^*) && |x|_{\mathsf{H}_0} \leq c_{\mathrm{A}_0} |\mathrm{A}_0 x|_{\mathsf{H}_1},\\
&\forall y \in D\big((\mathrm{A}_0^*)_\perp\big) = D(\mathrm{A}_0^*) \cap N(\mathrm{A}_0^*)^{\perp_{\mathsf{H}_1}} = D(\mathrm{A}_0^*) \cap R(\mathrm{A}_0) && |y|_{\mathsf{H}_1} \leq c_{\mathrm{A}_0} |\mathrm{A}_0^* y|_{\mathsf{H}_0},\\
&\forall y \in D\left((\mathrm{A}_1)_\perp\right) = D(\mathrm{A}_1) \cap N(\mathrm{A}_1)^{\perp_{\mathsf{H}_1}} = D(\mathrm{A}_1) \cap R(\mathrm{A}_1^*) && |y|_{\mathsf{H}_1} \leq c_{\mathrm{A}_1} |\mathrm{A}_1 y|_{\mathsf{H}_2},\\
&\forall z \in D\big((\mathrm{A}_1^*)_\perp\big) = D(\mathrm{A}_1^*) \cap N(\mathrm{A}_1^*)^{\perp_{\mathsf{H}_2}} = D(\mathrm{A}_1^*) \cap R(\mathrm{A}_1) && |z|_{\mathsf{H}_2} \leq c_{\mathrm{A}_1} |\mathrm{A}_1^* z|_{\mathsf{H}_1}.
\end{aligned}$$

- *(v') With* c_{A_0} *and* c_{A_1} *from (v), it holds*

$$\forall y \in D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) \cap N_{0,1}^{\perp_{\mathsf{H}_1}} \qquad |y|_{\mathsf{H}_1}^2 \leq c_{\mathrm{A}_0}^2 |\mathrm{A}_0^* y|_{\mathsf{H}_0}^2 + c_{\mathrm{A}_1}^2 |\mathrm{A}_1 y|_{\mathsf{H}_2}^2.$$

Definition 2.10. The Hilbert complex (5) is called

- closed, if $R(\mathrm{A}_0)$ and $R(\mathrm{A}_1)$ are closed,
- compact, if the embedding $D(\mathrm{A}_1)\cap D(\mathrm{A}_0^*)\hookrightarrow \mathsf{H}_1$ is compact.

Remark 2.11. A compact Hilbert complex is already closed.

2.3 | FAT III: Bounded regular decompositions and potentials

Bounded regular decompositions and bounded regular potentials are very powerful tools. In particular, compact embeddings can easily be proved, cf. Lemma 2.22, which then—in combination with the FA-ToolBox—immediately lead to a comprehensive list of important results for the underlying Hilbert complex, cf. Theorem 2.9 and Pauly.[5]

Throughout this subsection, let A_0 and A_1 be *densely defined* and *closed* linear operators satisfying the *complex property*, that is, $R(\mathrm{A}_0)\subset N(\mathrm{A}_1)$. Moreover, we fix some *regular subspaces* H_0^+, H_1^+ and H_2^+, such that either

$$\begin{aligned} \mathsf{H}_0^+ \hookrightarrow D(\mathrm{A}_0)\hookrightarrow \mathsf{H}_0 \text{ and } \mathsf{H}_1^+ \hookrightarrow D(\mathrm{A}_1)\hookrightarrow \mathsf{H}_1,\\ \text{or } \mathsf{H}_1^+ \hookrightarrow D(\mathrm{A}_0^*)\hookrightarrow \mathsf{H}_1 \text{ and } \mathsf{H}_2^+ \hookrightarrow D(\mathrm{A}_1^*)\hookrightarrow \mathsf{H}_2 \end{aligned} \tag{9}$$

hold with continuous embeddings. In the following, we consider *regular decompositions* of $D(\mathrm{A}_1)$ and $D(\mathrm{A}_0^*)$ of the following type

$$D(\mathrm{A}_1)=\mathsf{H}_1^+ + \mathrm{A}_0\mathsf{H}_0^+,\qquad D(\mathrm{A}_0^*)=\mathsf{H}_1^+ + \mathrm{A}_1^*\mathsf{H}_2^+. \tag{10}$$

For the rest of this subsection, we concentrate on the first regular decomposition in (10). Analogous results hold true for the second regular decomposition in (10), and we leave the corresponding reformulations to the interested reader.

Definition 2.12 (bounded regular decompositions)**.** In (10), we call the regular decomposition $D(\mathrm{A}_1)=\mathsf{H}_1^+ + \mathrm{A}_0\mathsf{H}_0^+$ bounded, if there exist bounded linear operators

$$\mathcal{Q}_{\mathrm{A}_1,1}\,:\,D(\mathrm{A}_1)\to \mathsf{H}_1^+,\qquad \mathcal{Q}_{\mathrm{A}_1,0}\,:\,D(\mathrm{A}_1)\to \mathsf{H}_0^+,$$

such that

$$\mathcal{Q}_{\mathrm{A}_1,1}+\mathrm{A}_0\mathcal{Q}_{\mathrm{A}_1,0}=\mathrm{id}_{D(\mathrm{A}_1)}.$$

$\mathcal{Q}_{\mathrm{A}_1,1}$ and $\mathcal{Q}_{\mathrm{A}_1,0}$ are then called bounded linear regular decomposition operators.

More precisely, for each $x\in D(\mathrm{A}_1)$, there exist two potentials

$$x_1\,:=\mathcal{Q}_{\mathrm{A}_1,1}x\in \mathsf{H}_1^+,\qquad z\,:=\mathcal{Q}_{\mathrm{A}_1,0}x\in \mathsf{H}_0^+,$$

such that $x=x_1+\mathrm{A}_0z$ and $|x_1|_{\mathsf{H}_1^+}+|z|_{\mathsf{H}_0^+}\leq c|x|_{D(\mathrm{A}_1)}$ with some $c>0$ independent of x,x_1,z.

Definition 2.13 (weak bounded regular decompositions)**.** $D(\mathrm{A}_1)=\mathsf{H}_1^+ + N(\mathrm{A}_1)$ is called a weak bounded regular decomposition, if there exist bounded linear operators

$$\mathcal{Q}_{\mathrm{A}_1,1}\,:\,D(\mathrm{A}_1)\to \mathsf{H}_1^+,\qquad \mathcal{N}_{\mathrm{A}_1}\,:\,D(\mathrm{A}_1)\to N(\mathrm{A}_1)$$

such that $\mathcal{Q}_{\mathrm{A}_1,1}+\mathcal{N}_{\mathrm{A}_1}=\mathrm{id}_{D(\mathrm{A}_1)}$. $\mathcal{Q}_{\mathrm{A}_1,1}$ and $\mathcal{N}_{\mathrm{A}_1}$ are again called bounded linear regular decomposition operators.

More precisely, for each $x\in D(\mathrm{A}_1)$, there exist

$$x_1\,:=\mathcal{Q}_{\mathrm{A}_1,1}x\in \mathsf{H}_1^+,\qquad x_0\,:=\mathcal{N}_{\mathrm{A}_1}x\in N(\mathrm{A}_1),$$

such that $x=x_1+x_0$ and $|x_1|_{\mathsf{H}_1^+}+|x_0|_{\mathsf{H}_1}\leq c|x|_{D(\mathrm{A}_1)}$ with some $c>0$ independent of x,x_1,x_0.

Remark 2.14. (bounded regular decompositions). For bounded regular decompositions, it holds:

(i) For $\mathcal{Q}_{\mathrm{A}_1,1}$ from Definition 2.12 or Definition 2.13, we have $\mathrm{A}_1\mathcal{Q}_{\mathrm{A}_1,1} = \mathrm{A}_1$ by the complex property. Hence, $N(\mathrm{A}_1)$ is invariant under $\mathcal{Q}_{\mathrm{A}_1,1}$, that is, $\mathcal{Q}_{\mathrm{A}_1,1}N(\mathrm{A}_1) \subset N(\mathrm{A}_1)$.
(ii) A bounded regular decomposition from Definition 2.12 implies a weak bounded regular decomposition from Definition 2.13 by setting $\mathcal{N}_{\mathrm{A}_1} := \mathrm{A}_0\mathcal{Q}_{\mathrm{A}_1,0}$ since $\mathrm{A}_0\mathsf{H}_0^+ \subset N(\mathrm{A}_1)$ holds by the complex property.

Definition 2.15 (bounded regular potentials)**.** We call $R(\mathrm{A}_1) = \mathrm{A}_1\mathsf{H}_1^+$ a bounded regular potential representation, if there exists a bounded linear operator

$$\mathcal{P}_{\mathrm{A}_1} : R(\mathrm{A}_1) \to \mathsf{H}_1^+ \qquad \text{with} \qquad \mathrm{A}_1\mathcal{P}_{\mathrm{A}_1} = \mathrm{id}_{R(\mathrm{A}_1)}.$$

We say that $\mathcal{P}_{\mathrm{A}_1}$ is a bounded linear regular potential operator of A_1. In particular, $\mathcal{P}_{\mathrm{A}_1}$ is a bounded linear right inverse of A_1.

Analogously, we extend the latter definition to the operators A_0, A_0^* and A_1^*.

Remark 2.16. (bounded regular potentials). We state two simple facts about potential operators:

(i) Let a linear operator

$$\mathcal{P}_{\mathrm{A}_0} : N(\mathrm{A}_1) \cap N_{0,1}^{\perp_{\mathsf{H}_1}} \to D(\mathrm{A}_0) \qquad \text{with} \qquad \mathrm{A}_0\mathcal{P}_{\mathrm{A}_0} = \mathrm{id}_{N(\mathrm{A}_1)\cap N_{0,1}^{\perp_{\mathsf{H}_1}}}$$

be given. Then $R(\mathrm{A}_0)$ is closed as $\overline{R(\mathrm{A}_0)} = N(\mathrm{A}_1) \cap N_{0,1}^{\perp_{\mathsf{H}_1}} = R(\mathrm{A}_0\mathcal{P}_{\mathrm{A}_0}) \subset R(\mathrm{A}_0)$.
(ii) Let a bounded linear operator

$$\mathcal{P}_{\mathrm{A}_0} : N(\mathrm{A}_1) \cap N_{0,1}^{\perp_{\mathsf{H}_1}} \to \mathsf{H}_0^+ \qquad \text{with} \qquad \mathrm{A}_0\mathcal{P}_{\mathrm{A}_0} = \mathrm{id}_{N(\mathrm{A}_1)\cap N_{0,1}^{\perp_{\mathsf{H}_1}}}$$

be given. Then (as above) $R(\mathrm{A}_0) = N(\mathrm{A}_1) \cap N_{0,1}^{\perp_{\mathsf{H}_1}} = \mathrm{A}_0\mathsf{H}_0^+$ is closed and

$$\mathcal{P}_{\mathrm{A}_0} : R(\mathrm{A}_0) \to \mathsf{H}_0^+ \qquad \text{with} \qquad \mathrm{A}_0\mathcal{P}_{\mathrm{A}_0} = \mathrm{id}_{R(\mathrm{A}_0)}$$

is a bounded linear regular potential operator of A_0.

Lemma 2.17 (bounded regular potentials by weak bounded regular decompositions)**.** *Let $R(A_1)$ be closed, and let $D(\mathrm{A}_1) = \mathsf{H}_1^+ + N(\mathrm{A}_1)$ be a weak bounded regular decomposition. Then the bounded regular potential representation $R(\mathrm{A}_1) = \mathrm{A}_1\mathsf{H}_1^+$ holds and*

$$\mathcal{P}_{\mathrm{A}_1} := \mathcal{Q}_{\mathrm{A}_1,1}(\mathrm{A}_1)_\perp^{-1} : R(\mathrm{A}_1) \to \mathsf{H}_1^+ \qquad \textit{with} \qquad \mathrm{A}_1\mathcal{P}_{\mathrm{A}_1} = \mathrm{id}_{R(\mathrm{A}_1)}$$

is a respective bounded linear regular potential operator of A_1.

Proof. As $R(\mathrm{A}_1)$ is closed, Lemma 2.1 shows that $(\mathrm{A}_1)_\perp^{-1} : R(\mathrm{A}_1) \to D(\mathrm{A}_1)$ is bounded. Hence, so is $\mathcal{P}_{\mathrm{A}_1}$. Moreover, $\mathrm{A}_1\mathcal{P}_{\mathrm{A}_1} = \mathrm{A}_1\mathcal{Q}_{\mathrm{A}_1,1}(\mathrm{A}_1)_\perp^{-1} = \mathrm{A}_1(\mathrm{A}_1)_\perp^{-1} = \mathrm{id}_{R(\mathrm{A}_1)}$ by Remark 2.14. □

Lemma 2.18 (weak bounded regular decompositions by bounded regular potentials)**.** *Let a bounded regular potential representation $R(\mathrm{A}_1) = \mathrm{A}_1\mathsf{H}_1^+$ be given with bounded linear regular potential operator $\mathcal{P}_{\mathrm{A}_1} : R(\mathrm{A}_1) \to \mathsf{H}_1^+$ satisfying $\mathrm{A}_1\mathcal{P}_{\mathrm{A}_1} = \mathrm{id}_{R(\mathrm{A}_1)}$. Then*

$$\mathcal{Q}_{\mathrm{A}_1,1} := \mathcal{P}_{\mathrm{A}_1}\mathrm{A}_1 : D(\mathrm{A}_1) \to \mathsf{H}_1^+, \qquad \mathcal{N}_{\mathrm{A}_1} := \mathrm{id}_{D(\mathrm{A}_1)} - \mathcal{Q}_{\mathrm{A}_1,1} : D(\mathrm{A}_1) \to N(\mathrm{A}_1)$$

are bounded linear regular decomposition operators with

$$\mathcal{Q}_{\mathrm{A}_1,1} + \mathcal{N}_{\mathrm{A}_1} = \mathrm{id}_{D(\mathrm{A}_1)}$$

and implying the weak bounded regular decompositions

$$D(\mathrm{A}_1) = \mathsf{H}_1^+ + N(\mathrm{A}_1) = R(\mathcal{Q}_{\mathrm{A}_1,1}) \dot{+} N(\mathrm{A}_1) = R(\mathcal{Q}_{\mathrm{A}_1,1}) \dot{+} R(\mathcal{N}_{\mathrm{A}_1}).$$

It holds $\mathrm{A}_1\mathcal{Q}_{\mathrm{A}_1,1} = \mathrm{A}_1$*, that is, N(A₁) is invariant under* $\mathcal{Q}_{\mathrm{A}_1,1}$*. Note that* $R(\mathcal{Q}_{\mathrm{A}_1,1}) = R(\mathcal{P}_{\mathrm{A}_1})$.

Proof. $\mathcal{Q}_{\mathrm{A}_1,1}$ and $\mathcal{N}_{\mathrm{A}_1}$ are bounded. Let $x \in D(\mathrm{A}_1)$. Then $\mathrm{A}_1 x \in R(\mathrm{A}_1)$ and $\mathcal{P}_{\mathrm{A}_1}\mathrm{A}_1 x \in \mathsf{H}_1^+$ with $\tilde{x} := x - \mathcal{P}_{\mathrm{A}_1}\mathrm{A}_1 x \in N(\mathrm{A}_1)$. For the directness, let $x = \mathcal{Q}_{\mathrm{A}_1,1}\varphi = \mathcal{P}_{\mathrm{A}_1}\mathrm{A}_1\varphi \in N(\mathrm{A}_1)$ with $\varphi \in D(\mathrm{A}_1)$. Then $0 = \mathrm{A}_1 x = \mathrm{A}_1\varphi$, and hence, $x = 0$. □

Remark 2.19. Note that $\mathcal{Q}_{\mathrm{A}_1,1}^2 = \mathcal{Q}_{\mathrm{A}_1,1}$ and $\mathcal{Q}_{\mathrm{A}_1,1}\mathcal{N}_{\mathrm{A}_1} = \mathcal{N}_{\mathrm{A}_1}\mathcal{Q}_{\mathrm{A}_1,1} = 0$ hold for the special bounded linear regular decomposition operator $\mathcal{Q}_{\mathrm{A}_1,1} = \mathcal{P}_{\mathrm{A}_1}\mathrm{A}_1$ from the latter lemma. Hence,

(i) $\mathcal{Q}_{\mathrm{A}_1,1}$ and $\mathcal{N}_{\mathrm{A}_1}$ are projections.
(ii) For $I_\pm := \mathcal{Q}_{\mathrm{A}_1,1} \pm \mathcal{N}_{\mathrm{A}_1}$, we observe $I_+ = I_-^2 = \mathrm{id}_{D(\mathrm{A}_1)}$. Thus, the operators I_+, I_-^2, as well as $I_- = 2\mathcal{Q}_{\mathrm{A}_1,1} - \mathrm{id}_{D(\mathrm{A}_1)}$ are topological isomorphisms on $D(\mathrm{A}_1)$.
(iii) There exists $c > 0$ such that for $x \in D(\mathrm{A}_1)$, it holds

$$c|\mathcal{Q}_{\mathrm{A}_1,1}x|_{\mathsf{H}_1^+} \le |\mathrm{A}_1 x|_{\mathsf{H}_2} \le |x|_{D(\mathrm{A}_1)}, \quad |\mathcal{N}_{\mathrm{A}_1}x|_{\mathsf{H}_1} \le |x|_{\mathsf{H}_1} + |\mathcal{Q}_{\mathrm{A}_1,1}x|_{\mathsf{H}_1}.$$

(iii') For $x \in N(\mathrm{A}_1)$, we have $\mathcal{Q}_{\mathrm{A}_1,1}x = 0$ and $\mathcal{N}_{\mathrm{A}_1}x = x$, that is, $\mathcal{Q}_{\mathrm{A}_1,1}|_{N(\mathrm{A}_1)} = 0$ as well as $\mathcal{N}_{\mathrm{A}_1}|_{N(\mathrm{A}_1)} = \mathrm{id}_{N(\mathrm{A}_1)}$. In particular, $\mathcal{N}_{\mathrm{A}_1}$ is onto.

Corollary 2.20 (bounded regular decompositions by bounded regular potentials)**.** *Let the complex be exact, that is,* $N(\mathrm{A}_1) = R(\mathrm{A}_0)$*, and let* $R(\mathrm{A}_1) = \mathrm{A}_1\mathsf{H}_1^+$ *as well as* $R(\mathrm{A}_0) = \mathrm{A}_0\mathsf{H}_0^+$ *be bounded regular potential representations with bounded linear regular potential operators* $\mathcal{P}_{\mathrm{A}_1} : R(\mathrm{A}_1) \to \mathsf{H}_1^+$ *and* $\mathcal{P}_{\mathrm{A}_0} : R(\mathrm{A}_0) \to \mathsf{H}_0^+$ *satisfying* $\mathrm{A}_1\mathcal{P}_{\mathrm{A}_1} = \mathrm{id}_{R(\mathrm{A}_1)}$ *and* $\mathrm{A}_0\mathcal{P}_{\mathrm{A}_0} = \mathrm{id}_{R(\mathrm{A}_0)}$*, respectively. Then*

$$\mathcal{Q}_{\mathrm{A}_1,1} : D(\mathrm{A}_1) \to \mathsf{H}_1^+, \qquad \mathcal{Q}_{\mathrm{A}_1,0} := \mathcal{P}_{\mathrm{A}_0}\mathcal{N}_{\mathrm{A}_1} : D(\mathrm{A}_1) \to \mathsf{H}_0^+$$

with $\mathcal{Q}_{\mathrm{A}_1,1} = \mathcal{P}_{\mathrm{A}_1}\mathrm{A}_1$ *and* $\mathcal{N}_{\mathrm{A}_1} = \mathrm{id}_{D(\mathrm{A}_1)} - \mathcal{Q}_{\mathrm{A}_1,1}$ *from Lemma 2.18 are bounded linear regular decomposition operators with*

$$\mathcal{Q}_{\mathrm{A}_1,1} + \mathrm{A}_0\mathcal{Q}_{\mathrm{A}_1,0} = \mathrm{id}_{D(\mathrm{A}_1)}$$

and implying bounded regular decompositions

$$D(\mathrm{A}_1) = \mathsf{H}_1^+ + \mathrm{A}_0\mathsf{H}_0^+ = R(\mathcal{Q}_{\mathrm{A}_1,1}) \dot{+} \mathrm{A}_0\mathsf{H}_0^+ = R(\mathcal{Q}_{\mathrm{A}_1,1}) \dot{+} \mathrm{A}_0 R(\mathcal{Q}_{\mathrm{A}_1,0}).$$

It holds $\mathrm{A}_1\mathcal{Q}_{\mathrm{A}_1,1} = \mathrm{A}_1$*; that is, N(A₁) is invariant under* $\mathcal{Q}_{\mathrm{A}_1,1}$*. Note that* $R(\mathcal{Q}_{\mathrm{A}_1,1}) = R(\mathcal{P}_{\mathrm{A}_1})$ *and* $R(\mathcal{Q}_{\mathrm{A}_1,0}) = R(\mathcal{P}_{\mathrm{A}_0})$.

Proof. $\mathcal{Q}_{\mathrm{A}_1,1}$ and $\mathcal{Q}_{\mathrm{A}_1,0}$ are bounded. Let $x \in D(\mathrm{A}_1)$. Then $\mathrm{A}_1 x \in R(\mathrm{A}_1)$ and $\mathcal{P}_{\mathrm{A}_1}\mathrm{A}_1 x \in \mathsf{H}_1^+$ with $\tilde{x} := x - \mathcal{P}_{\mathrm{A}_1}\mathrm{A}_1 x \in N(\mathrm{A}_1) = R(\mathrm{A}_0)$. Thus, $z := \mathcal{P}_{\mathrm{A}_0}\tilde{x} \in \mathsf{H}_0^+$ and $\mathrm{A}_0 z = \tilde{x}$, that is,

$$x = \mathcal{P}_{\mathrm{A}_1}\mathrm{A}_1 x + \tilde{x} = \mathcal{P}_{\mathrm{A}_1}\mathrm{A}_1 x + \mathrm{A}_0\mathcal{P}_{\mathrm{A}_0}\tilde{x} = \mathcal{P}_{\mathrm{A}_1}\mathrm{A}_1 x + \mathrm{A}_0\mathcal{P}_{\mathrm{A}_0}(1 - \mathcal{P}_{\mathrm{A}_1}\mathrm{A}_1)x.$$

Directness is clear by Lemma 2.18 as $\mathrm{A}_0\mathsf{H}_0^+ \subset N(\mathrm{A}_1)$ holds by the complex property. □

Remark 2.21. There exists $c > 0$ such that for $x \in D(\mathrm{A}_1)$, it holds

$$c|\mathcal{Q}_{\mathrm{A}_1,1}x|_{\mathsf{H}_1^+} \le |\mathrm{A}_1 x|_{\mathsf{H}_2} \le |x|_{D(\mathrm{A}_1)}, \quad c|\mathcal{Q}_{\mathrm{A}_1,0}x|_{\mathsf{H}_0^+} \le |\mathcal{N}_{\mathrm{A}_1}x|_{\mathsf{H}_1} \le |x|_{\mathsf{H}_1} + |\mathcal{Q}_{\mathrm{A}_1,1}x|_{\mathsf{H}_1}.$$

Note that $\mathcal{Q}_{\mathrm{A}_1,1}|_{N(\mathrm{A}_1)} = 0$.

2.4 | FAT IV: Compactness results and mini FA-ToolBox

From Pauly and Zulehner,[8, Theorem 2.8, Corollary 2.9] we cite the following compactness result.

Lemma 2.22 (compact embedding by regular decompositions). *Let A_0 and A_1 be densely defined and closed linear operators satisfying the complex property, that is, $R(A_0) \subset N(A_1)$. Moreover, let*

(i) either the bounded regular decomposition $D(\mathrm{A}_1) = \mathsf{H}_1^+ + \mathrm{A}_0\mathsf{H}_0^+$ hold with compact embeddings $\mathsf{H}_0^+ \hookrightarrow \mathsf{H}_0$ and $\mathsf{H}_1^+ \hookrightarrow \mathsf{H}_1$,
(ii) or the bounded regular decomposition $D(\mathrm{A}_0^*) = \mathsf{H}_1^+ + \mathrm{A}_1^*\mathsf{H}_2^+$ hold with compact embeddings $\mathsf{H}_1^+ \hookrightarrow \mathsf{H}_1$ and $\mathsf{H}_2^+ \hookrightarrow \mathsf{H}_2$.

Then the embedding $D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) \hookrightarrow \mathsf{H}_1$ is compact.

For convenience, we repeat the proof of Pauly and Zulehner.[8, Theorem 2.8]

Proof. Let $(x_n) \subset D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*)$ be a bounded sequence; that is, there exists $c > 0$ such that for all n we have $|x_n|_{\mathsf{H}_1} + |\mathrm{A}_1 x_n|_{\mathsf{H}_2} + |\mathrm{A}_0^* x_n|_{\mathsf{H}_0} \leq c$. By assumption, we decompose $x_n = p_{1,n} + \mathrm{A}_0 p_{0,n}$ with $p_{1,n} \in \mathsf{H}_1^+$ and $p_{0,n} \in \mathsf{H}_0^+$ satisfying $|p_{1,n}|_{\mathsf{H}_1^+} + |p_{0,n}|_{\mathsf{H}_0^+} \leq c|x_n|_{D(\mathrm{A}_1)} \leq c$. Hence, $(p_{\ell,n}) \subset \mathsf{H}_\ell^+$ is bounded in H_ℓ^+, $\ell = 0, 1$, and thus, we can extract convergent subsequences, again denoted by $(p_{\ell,n})$, such that $(p_{\ell,n})$ are convergent in H_ℓ, $\ell = 0, 1$. Then with $x_{n,m} := x_n - x_m$ and $p_{\ell,n,m} := p_{\ell,n} - p_{\ell,m}$, we get

$$|x_{n,m}|_{\mathsf{H}_1}^2 = \langle x_{n,m}, p_{1,n,m}\rangle_{\mathsf{H}_1} + \langle \mathrm{A}_0^* x_{n,m}, p_{0,n,m}\rangle_{\mathsf{H}_0} \leq c\left(|p_{1,n,m}|_{\mathsf{H}_1} + |p_{0,n,m}|_{\mathsf{H}_0}\right),$$

which shows that (x_n) is a Cauchy sequence in H_1. Hence, we have shown (i), and (ii) follows analogously. □

Theorem 2.23 (mini FAT by regular decompositions). *Let the assumptions of Lemma 2.22 (i) hold with the bounded linear regular decomposition operators $\mathcal{Q}_{\mathrm{A}_1,1} : D(\mathrm{A}_1) \to \mathsf{H}_1^+$ as well as $\mathcal{Q}_{\mathrm{A}_1,0} : D(\mathrm{A}_1) \to \mathsf{H}_0^+$. Then,*

(i) The embedding $D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) \hookrightarrow \mathsf{H}_1$ is compact.
(ii) The assertions of Theorem 2.9 (mini FAT) hold.
(iii) The bounded regular potential representation $R(\mathrm{A}_1) = \mathrm{A}_1\mathsf{H}_1^+$ holds with bounded linear regular potential operator $\mathcal{P}_{\mathrm{A}_1} = \mathcal{Q}_{\mathrm{A}_1,1}(\mathrm{A}_1)_\perp^{-1} : R(\mathrm{A}_1) \to \mathsf{H}_1^+$ satisfying $\mathrm{A}_1\mathcal{P}_{\mathrm{A}_1} = \mathrm{id}_{R(\mathrm{A}_1)}$.
(iv) $\widetilde{\mathcal{Q}}_{\mathrm{A}_1,1} = \mathcal{P}_{\mathrm{A}_1}\mathrm{A}_1 : D(\mathrm{A}_1) \to \mathsf{H}_1^+$ and $\widetilde{\mathcal{N}}_{\mathrm{A}_1} = \mathrm{id}_{D(\mathrm{A}_1)} - \widetilde{\mathcal{Q}}_{\mathrm{A}_1,1} : D(\mathrm{A}_1) \to N(\mathrm{A}_1)$ are bounded linear regular decomposition operators with $\widetilde{\mathcal{Q}}_{\mathrm{A}_1,1} + \widetilde{\mathcal{N}}_{\mathrm{A}_1} = \mathrm{id}_{D(\mathrm{A}_1)}$ and the bounded regular decompositions

$$D(\mathrm{A}_1) = \mathsf{H}_1^+ + \mathrm{A}_0\mathsf{H}_0^+ = \mathsf{H}_1^+ + N(\mathrm{A}_1) = R(\widetilde{\mathcal{Q}}_{\mathrm{A}_1,1}) \dotplus N(\mathrm{A}_1) = R(\widetilde{\mathcal{Q}}_{\mathrm{A}_1,1}) \dotplus R(\widetilde{\mathcal{N}}_{\mathrm{A}_1})$$

hold. Moreover, $R(\widetilde{\mathcal{Q}}_{\mathrm{A}_1,1}) = R(\mathcal{P}_{\mathrm{A}_1})$.
(iv') $\mathrm{A}_1\widetilde{\mathcal{Q}}_{\mathrm{A}_1,1} = \mathrm{A}_1\mathcal{Q}_{\mathrm{A}_1,1} = \mathrm{A}_1$; that is, $N(A_1)$ is invariant under $\mathcal{Q}_{\mathrm{A}_1,1}$ and $\widetilde{\mathcal{Q}}_{\mathrm{A}_1,1}$. It holds $\widetilde{\mathcal{Q}}_{\mathrm{A}_1,1} = \mathcal{Q}_{\mathrm{A}_1,1}(\mathrm{A}_1)_\perp^{-1}\mathrm{A}_1$. Hence, $\widetilde{\mathcal{Q}}_{\mathrm{A}_1,1}|_{D((\mathrm{A}_1)_\perp)} = \mathcal{Q}_{\mathrm{A}_1,1}|_{D((\mathrm{A}_1)_\perp)}$, and thus, $\widetilde{\mathcal{Q}}_{\mathrm{A}_1,1}$ may differ from $\mathcal{Q}_{\mathrm{A}_1,1}$ only on $N(A_1)$.

Proof. (i) and (ii) are trivial. (iii) follows by Lemma 2.17, and Lemma 2.18 shows (iv). It holds

$$\begin{aligned}\widetilde{\mathcal{Q}}_{\mathrm{A}_1,1}|_{D((\mathrm{A}_1)_\perp)} = \mathcal{Q}_{\mathrm{A}_1,1}(\mathrm{A}_1)_\perp^{-1}\mathrm{A}_1|_{D((\mathrm{A}_1)_\perp)} &= \mathcal{Q}_{\mathrm{A}_1,1}(\mathrm{A}_1)_\perp^{-1}(\mathrm{A}_1)_\perp \\ &= \mathcal{Q}_{\mathrm{A}_1,1}\mathrm{id}_{D((\mathrm{A}_1)_\perp)} = \mathcal{Q}_{\mathrm{A}_1,1}|_{D((\mathrm{A}_1)_\perp)},\end{aligned}$$

which shows the last assertion of (iv'). □

Corollary 2.24 (mini FAT by regular decompositions). *Let the assumptions of Lemma 2.22 (ii) hold with the bounded linear regular decomposition operators $\mathcal{Q}_{\mathrm{A}_0^*,1} : D(\mathrm{A}_1) \to \mathsf{H}_1^+$ as well as $\mathcal{Q}_{\mathrm{A}_0^*,2} : D(\mathrm{A}_1) \to \mathsf{H}_2^+$. Then (i) and (ii) of Theorem 2.23 hold. Moreover,*

(iii) The bounded regular potential representation $R(\mathrm{A}_0^*) = \mathrm{A}_0^*\mathsf{H}_1^+$ holds with bounded linear regular potential operator $\mathcal{P}_{\mathrm{A}_0^*} = \mathcal{Q}_{\mathrm{A}_0^*,1}(\mathrm{A}_0^*)_\perp^{-1} : R(\mathrm{A}_0^*) \to \mathsf{H}_1^+$ satisfying $\mathrm{A}_0^*\mathcal{P}_{\mathrm{A}_0^*} = \mathrm{id}_{R(\mathrm{A}_0^*)}$.

(iv) $\widetilde{\mathcal{Q}}_{\mathrm{A}_0^*,1} = \mathcal{P}_{\mathrm{A}_0^*}\mathrm{A}_0^* : D(\mathrm{A}_0^*) \to \mathsf{H}_1^+$ *and* $\widetilde{\mathcal{N}}_{\mathrm{A}_0^*} = \mathrm{id}_{D(\mathrm{A}_0^*)} - \widetilde{\mathcal{Q}}_{\mathrm{A}_0^*,1} : D(\mathrm{A}_0^*) \to N(\mathrm{A}_0^*)$ *are bounded linear regular decomposition operators with* $\widetilde{\mathcal{Q}}_{\mathrm{A}_0^*,1} + \widetilde{\mathcal{N}}_{\mathrm{A}_0^*} = \mathrm{id}_{D(\mathrm{A}_0^*)}$ *and the bounded regular decompositions*

$$D(\mathrm{A}_0^*) = \mathsf{H}_1^+ + \mathrm{A}_1^*\mathsf{H}_2^+ = \mathsf{H}_1^+ + N(\mathrm{A}_0^*) = R(\widetilde{\mathcal{Q}}_{\mathrm{A}_0^*,1}) \dotplus N(\mathrm{A}_0^*) = R(\widetilde{\mathcal{Q}}_{\mathrm{A}_0^*,1}) \dotplus R(\widetilde{\mathcal{N}}_{\mathrm{A}_0^*})$$

hold. Moreover, $R(\widetilde{\mathcal{Q}}_{\mathrm{A}_0^*,1}) = R(\mathcal{P}_{\mathrm{A}_0^*})$.

(iv') $\mathrm{A}_0^*\widetilde{\mathcal{Q}}_{\mathrm{A}_0^*,1} = \mathrm{A}_0^*\mathcal{Q}_{\mathrm{A}_0^*,1} = \mathrm{A}_0^*$*; that is,* $N(\mathrm{A}_0^*)$ *is invariant under* $\mathcal{Q}_{\mathrm{A}_0^*,1}$ *and* $\widetilde{\mathcal{Q}}_{\mathrm{A}_0^*,1}$*. It holds* $\widetilde{\mathcal{Q}}_{\mathrm{A}_0^*,1} = \mathcal{Q}_{\mathrm{A}_0^*,1}(\mathrm{A}_0^*)_\perp^{-1}\mathrm{A}_0^*$*. Hence,* $\widetilde{\mathcal{Q}}_{\mathrm{A}_0^*,1}|_{D((\mathrm{A}_0^*)_\perp)} = \mathcal{Q}_{\mathrm{A}_0^*,1}|_{D((\mathrm{A}_0^*)_\perp)}$*, and thus,* $\widetilde{\mathcal{Q}}_{\mathrm{A}_0^*,1}$ *may differ from* $\mathcal{Q}_{\mathrm{A}_0^*,1}$ *only on* $N(\mathrm{A}_0^*)$.

2.5 | FAT V: Long Hilbert complexes

As a typical situation in 3D (extending literally to any dimension), we have a *long primal and dual Hilbert complex*

$$\mathsf{H}_{-1} \underset{\mathrm{A}_{-1}^*}{\overset{\mathrm{A}_{-1}}{\rightleftarrows}} \mathsf{H}_0 \underset{\mathrm{A}_0^*}{\overset{\mathrm{A}_0}{\rightleftarrows}} \mathsf{H}_1 \underset{\mathrm{A}_1^*}{\overset{\mathrm{A}_1}{\rightleftarrows}} \mathsf{H}_2 \underset{\mathrm{A}_2^*}{\overset{\mathrm{A}_2}{\rightleftarrows}} \mathsf{H}_3 \underset{\mathrm{A}_3^*}{\overset{\mathrm{A}_3}{\rightleftarrows}} \mathsf{H}_4.$$

Here, $\mathrm{A}_0, \mathrm{A}_1, \mathrm{A}_2$ are densely defined and closed (unbounded) linear operators between three Hilbert spaces $\mathsf{H}_0, \mathsf{H}_1, \mathsf{H}_2$ satisfying the complex properties

$$R(\mathrm{A}_0) \subset N(\mathrm{A}_1), \qquad R(\mathrm{A}_1) \subset N(\mathrm{A}_2).$$

$\mathrm{A}_0^*, \mathrm{A}_1^*, \mathrm{A}_2^*$ are the corresponding (Hilbert space) adjoints. Moreover, $\mathrm{A}_{-1}, \mathrm{A}_4$ and H_{-1}, H_4 are particular operators and kernels, respectively, that is,

$$\mathsf{H}_{-1} := N(\mathrm{A}_0) = R(\mathrm{A}_0^*)^{\perp_{\mathsf{H}_0}}, \qquad \mathsf{H}_4 := N(\mathrm{A}_2^*) = R(\mathrm{A}_2)^{\perp_{\mathsf{H}_3}}$$

with corresponding bounded embeddings

$$\mathrm{A}_{-1} := \iota_{N(\mathrm{A}_0)} : N(\mathrm{A}_0) \to \mathsf{H}_0, \qquad \mathrm{A}_3^* := \iota_{N(\mathrm{A}_2^*)} : N(\mathrm{A}_2^*) \to \mathsf{H}_3.$$

Remark 2.25. It holds $\mathrm{A}_{-1}^* = \iota_{N(\mathrm{A}_0)}^* = \pi_{N(\mathrm{A}_0)} : \mathsf{H}_0 \to N(\mathrm{A}_0)$, the 'orthonormal projection' onto the kernel of A_0. To see this, we note $\mathrm{A}_{-1}^* : \mathsf{H}_0 \to N(\mathrm{A}_0)$ and for $x \in \mathsf{H}_0$ and $\varphi \in N(\mathrm{A}_0)$

$$\langle \mathrm{A}_{-1}\varphi, x\rangle_{\mathsf{H}_0} = \langle \varphi, x\rangle_{\mathsf{H}_0} = \langle \pi_{N(\mathrm{A}_0)}\varphi, x\rangle_{\mathsf{H}_0} = \langle \varphi, \pi_{N(\mathrm{A}_0)}x\rangle_{\mathsf{H}_0} = \langle \varphi, \pi_{N(\mathrm{A}_0)}x\rangle_{N(\mathrm{A}_0)}.$$

Actually, the correct orthonormal projection onto $N(\mathrm{A}_0)$ is then given by the self-adjoint bounded linear operator $\mathrm{A}_{-1}\mathrm{A}_{-1}^* = \iota_{N(\mathrm{A}_0)}\iota_{N(\mathrm{A}_0)}^* = \pi_{N(\mathrm{A}_0)} : \mathsf{H}_0 \to \mathsf{H}_0$ with $R(\pi_{N(\mathrm{A}_0)}) = N(\mathrm{A}_0)$. Analogously, $\mathrm{A}_3 = \iota_{N(\mathrm{A}_2^*)}^* = \pi_{N(\mathrm{A}_2^*)} : \mathsf{H}_3 \to N(\mathrm{A}_2^*)$ and $\mathrm{A}_3^*\mathrm{A}_3 = \iota_{N(\mathrm{A}_2^*)}\iota_{N(\mathrm{A}_2^*)}^* = \pi_{N(\mathrm{A}_2^*)} : \mathsf{H}_3 \to \mathsf{H}_3$, respectively, with $R(\pi_{N(\mathrm{A}_2^*)}) = N(\mathrm{A}_2^*)$.

The latter arguments show that the long primal and dual Hilbert complex (11) reads

$$N(\mathrm{A}_0) \underset{\mathrm{A}_{-1}^*=\pi_{N(\mathrm{A}_0)}}{\overset{\mathrm{A}_{-1}=\iota_{N(\mathrm{A}_0)}}{\rightleftarrows}} \mathsf{H}_0 \underset{\mathrm{A}_0^*}{\overset{\mathrm{A}_0}{\rightleftarrows}} \mathsf{H}_1 \underset{\mathrm{A}_1^*}{\overset{\mathrm{A}_1}{\rightleftarrows}} \mathsf{H}_2 \underset{\mathrm{A}_2^*}{\overset{\mathrm{A}_2}{\rightleftarrows}} \mathsf{H}_3 \underset{\mathrm{A}_3^*=\iota_{N(\mathrm{A}_2^*)}}{\overset{\mathrm{A}_3=\pi_{N(\mathrm{A}_2^*)}}{\rightleftarrows}} N(\mathrm{A}_2^*)$$

with the complex properties

$$\begin{aligned} &R(\mathrm{A}_{-1}) = N(\mathrm{A}_0), && R(\mathrm{A}_0) \subset N(\mathrm{A}_1), && R(\mathrm{A}_1) \subset N(\mathrm{A}_2), && \overline{R(\mathrm{A}_2)} = N(\mathrm{A}_3),\\ &\overline{R(\mathrm{A}_0^*)} = N(\mathrm{A}_{-1}^*), && R(\mathrm{A}_1^*) \subset N(\mathrm{A}_0^*), && R(\mathrm{A}_2^*) \subset N(\mathrm{A}_1^*), && R(\mathrm{A}_3^*) = N(\mathrm{A}_2^*). \end{aligned}$$

Definition 2.26. The long Hilbert complex (12) is called

- closed, if $R(\mathrm{A}_0)$, $R(\mathrm{A}_1)$, and $R(\mathrm{A}_2)$ are closed,

- compact, if the embeddings $D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) \hookrightarrow \mathsf{H}_1$ and $D(\mathrm{A}_2) \cap D(\mathrm{A}_1^*) \hookrightarrow \mathsf{H}_1$ as well as

$$D(\mathrm{A}_0) \cap D(\mathrm{A}_{-1}^*) = D(\mathrm{A}_0) \hookrightarrow \mathsf{H}_0, \qquad D(\mathrm{A}_3) \cap D(\mathrm{A}_2^*) = D(\mathrm{A}_2^*) \hookrightarrow \mathsf{H}_3$$

are compact.

Remark 2.27. A compact long Hilbert complex is already closed.

Note that the cohomology groups at both ends are trivial, that is,

$$\begin{aligned} N_{-1,0} &= N(\mathrm{A}_0) \cap N(\mathrm{A}_{-1}^*) = N(\mathrm{A}_0) \cap N(\mathrm{A}_0)^{\perp_{\mathsf{H}_0}} = \{0\}, \\ N_{2,3} &= N(\mathrm{A}_3) \cap N(\mathrm{A}_2^*) = N(\mathrm{A}_2^*)^{\perp_{\mathsf{H}_3}} \cap N(\mathrm{A}_2^*) = \{0\}. \end{aligned} \tag{13}$$

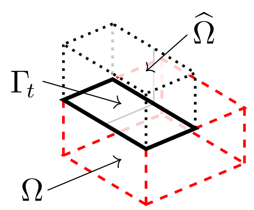
3 | NOTATIONS AND PRELIMINARIES

3.1 | Domains

Throughout this paper, let $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}$, be a bounded strong Lipschitz domain (locally Ω lies above a graph of some Lipschitz function). Moreover, let the boundary Γ of Ω be decomposed into two strong Lipschitz subsets Γ_t and $\Gamma_n := \Gamma \setminus \overline{\Gamma_t}$ forming the interface $\overline{\Gamma_t} \cap \overline{\Gamma_n}$ for the mixed boundary conditions (tangential and normal). See other works[9,17,18] for exact definitions. We call (Ω, Γ_t) a bounded strong Lipschitz pair.

We also recall the notion of an extendable strong Lipschitz domain through either one of the boundary parts Γ_t or Γ_n; see Bauer et al.[18, Section 5.4] and Bauer et al.[17, Section 7] for a definition. Roughly speaking, a bounded strong Lipschitz pair (Ω, Γ_t) is called *extendable*, if

- Ω and Γ_t are *topologically trivial*, and
- Ω can be *extended* through Γ_t to some topologically trivial and bounded strong Lipschitz domain $\widehat{\Omega}$, resulting in a new topologically trivial and bounded strong Lipschitz domain $\widetilde{\Omega} = \operatorname{int}(\overline{\Omega} \cup \overline{\widehat{\Omega}})$, cf. the figure on the right or Bauer et al.[18, Figure 3.2] for more details.



Lemma 3.1. *Any bounded strong Lipschitz pair (Ω, Γ_t) can be decomposed into a finite union of extendable bounded strong Lipschitz pairs $(\Omega_\ell, \Gamma_{t,\ell})$ together with a subordinate partition of unity.*

3.2 | Sobolev spaces of scalar, vector and tensor fields

In this subsection, let $d = 3$. The usual Lebesgue and Sobolev Hilbert spaces (of scalar, vector, or tensor valued fields) are denoted by $\mathsf{L}^2(\Omega)$, $\mathsf{H}^k(\Omega)$, $\mathsf{H}(\mathrm{rot}, \Omega)$, $\mathsf{H}(\mathrm{div}, \Omega)$ for $k \in \mathbb{Z}$ and by $\mathsf{H}_0(\mathrm{rot}, \Omega)$ and $\mathsf{H}_0(\mathrm{div}, \Omega)$ we indicate the spaces with vanishing rot and div, respectively. Homogeneous boundary conditions for these standard differential operators grad, rot and div are introduced in the *strong sense* as closures of respective test fields from

$$\mathsf{C}^\infty_{\Gamma_t}(\Omega) := \left\{ \phi|_\Omega \,:\, \phi \in \mathsf{C}^\infty(\mathbb{R}^d),\ \operatorname{supp}\phi \text{ compact},\ \operatorname{dist}(\operatorname{supp}\phi, \Gamma_t) > 0 \right\},$$

that is, for $k \in \mathbb{N}_0$

$$\mathsf{H}^k_{\Gamma_t}(\Omega) := \overline{\mathsf{C}^\infty_{\Gamma_t}(\Omega)}^{\mathsf{H}^k(\Omega)},\ \mathsf{H}_{\Gamma_t}(\mathrm{rot}, \Omega) := \overline{\mathsf{C}^\infty_{\Gamma_t}(\Omega)}^{\mathsf{H}(\mathrm{rot},\Omega)},\ \mathsf{H}_{\Gamma_t}(\mathrm{div}, \Omega) := \overline{\mathsf{C}^\infty_{\Gamma_t}(\Omega)}^{\mathsf{H}(\mathrm{div},\Omega)},$$

and we have $\mathsf{H}^k_\emptyset(\Omega) = \mathsf{H}^k(\Omega)$, $\mathsf{H}_\emptyset(\mathrm{rot},\Omega) = \mathsf{H}(\mathrm{rot},\Omega)$ and $\mathsf{H}_\emptyset(\mathrm{div},\Omega) = \mathsf{H}(\mathrm{div},\Omega)$, which are well known density results and incorporated into the notation by purpose. Spaces with vanishing rot and div are again denoted by $\mathsf{H}_{\Gamma_t,0}(\mathrm{rot},\Omega)$ and $\mathsf{H}_{\Gamma_t,0}(\mathrm{div},\Omega)$, respectively. Note that for $k=0$, we have $\mathsf{H}^0_{\Gamma_t}(\Omega) = \mathsf{L}^2(\Omega)$ and for the gradient we can also write $\mathsf{H}^1_{\Gamma_t}(\Omega) = \mathsf{H}_{\Gamma_t}(\mathrm{grad},\Omega)$. Moreover, we introduce for $k \in \mathbb{N}_0$ the nonstandard Sobolev spaces

$$\begin{aligned}
\mathsf{H}^k(\mathrm{rot},\Omega) &:= \left\{ v \in \mathsf{H}^k(\Omega) : \mathrm{rot}\ v \in \mathsf{H}^k(\Omega) \right\},\\
\mathsf{H}^k_{\Gamma_t}(\mathrm{rot},\Omega) &:= \left\{ v \in \mathsf{H}^k_{\Gamma_t}(\Omega) \cap \mathsf{H}_{\Gamma_t}(\mathrm{rot},\Omega) : \mathrm{rot}\ v \in \mathsf{H}^k_{\Gamma_t}(\Omega) \right\},\\
\mathsf{H}^k(\mathrm{div},\Omega) &:= \left\{ v \in \mathsf{H}^k(\Omega) : \mathrm{div}\ v \in \mathsf{H}^k(\Omega) \right\},\\
\mathsf{H}^k_{\Gamma_t}(\mathrm{div},\Omega) &:= \left\{ v \in \mathsf{H}^k_{\Gamma_t}(\Omega) \cap \mathsf{H}_{\Gamma_t}(\mathrm{div},\Omega) : \mathrm{div}\ v \in \mathsf{H}^k_{\Gamma_t}(\Omega) \right\}.
\end{aligned}$$

We see $\mathsf{H}^k_\emptyset(\mathrm{rot},\Omega) = \mathsf{H}^k(\mathrm{rot},\Omega)$ and for $k=0$ we have $\mathsf{H}^0_\emptyset(\mathrm{rot},\Omega) = \mathsf{H}^0(\mathrm{rot},\Omega) = \mathsf{H}(\mathrm{rot},\Omega)$ and $\mathsf{H}^0_{\Gamma_t}(\mathrm{rot},\Omega) = \mathsf{H}_{\Gamma_t}(\mathrm{rot},\Omega)$. Note that for $\Gamma_t \neq \emptyset$ and $k \geq 1$, it holds

$$\mathsf{H}^k_{\Gamma_t}(\mathrm{rot},\Omega) = \left\{ v \in \mathsf{H}^k_{\Gamma_t}(\Omega) : \mathrm{rot}\ v \in \mathsf{H}^k_{\Gamma_t}(\Omega) \right\},$$

but for $\Gamma_t \neq \emptyset$ and $k = 0$ $\left(\text{as } \mathsf{H}^0_{\Gamma_t}(\Omega) = \mathsf{L}^2(\Omega)\right)$,

$$\begin{aligned}
\mathsf{H}^0_{\Gamma_t}(\mathrm{rot},\Omega) &= \left\{ v \in \mathsf{H}^0_{\Gamma_t}(\Omega) \cap \mathsf{H}_{\Gamma_t}(\mathrm{rot},\Omega) : \mathrm{rot}\ v \in \mathsf{H}^0_{\Gamma_t}(\Omega) \right\} = \mathsf{H}_{\Gamma_t}(\mathrm{rot},\Omega)\\
&\subsetneq \left\{ v \in \mathsf{H}^0_{\Gamma_t}(\Omega) : \mathrm{rot}\ v \in \mathsf{H}^0_{\Gamma_t}(\Omega) \right\} = \mathsf{H}^0_\emptyset(\mathrm{rot},\Omega) = \mathsf{H}(\mathrm{rot},\Omega).
\end{aligned}$$

As before,

$$\mathsf{H}^k_{\Gamma_t,0}(\mathrm{rot},\Omega) := \mathsf{H}^k_{\Gamma_t}(\Omega) \cap \mathsf{H}_{\Gamma_t,0}(\mathrm{rot},\Omega) = \mathsf{H}^k_{\Gamma_t}(\mathrm{rot},\Omega) \cap \mathsf{H}_0(\mathrm{rot},\Omega) = \left\{ v \in \mathsf{H}^k_{\Gamma_t}(\mathrm{rot},\Omega) : \mathrm{rot}\ v = 0 \right\}.$$

The corresponding remarks and definitions extend to the $\mathsf{H}^k_{\Gamma_t}(\mathrm{div},\Omega)$-spaces as well.

At this point, let us note that boundary conditions can also be defined in the *weak sense* by

$$\begin{aligned}
\mathbf{H}^k_{\Gamma_t}(\Omega) &:= \left\{ u \in \mathsf{H}^k(\Omega) : \langle \partial^\alpha u, \phi \rangle_{\mathsf{L}^2(\Omega)} = (-1)^{|\alpha|} \langle u, \partial^\alpha \phi \rangle_{\mathsf{L}^2(\Omega)} \quad \forall \phi \in \mathsf{C}^\infty_{\Gamma_n}(\Omega) \quad \forall |\alpha| \leq k \right\},\\
\mathbf{H}_{\Gamma_t}(\mathrm{rot},\Omega) &:= \left\{ v \in \mathsf{H}(\mathrm{rot},\Omega) : \langle \mathrm{rot}\ v, \psi \rangle_{\mathsf{L}^2(\Omega)} = \langle v, \mathrm{rot}\ \psi \rangle_{\mathsf{L}^2(\Omega)} \quad \forall \psi \in \mathsf{C}^\infty_{\Gamma_n}(\Omega) \right\},\\
\mathbf{H}_{\Gamma_t}(\mathrm{div},\Omega) &:= \left\{ v \in \mathsf{H}(\mathrm{div},\Omega) : \langle \mathrm{div}\ v, \phi \rangle_{\mathsf{L}^2(\Omega)} = -\langle v, \mathrm{grad}\phi \rangle_{\mathsf{L}^2(\Omega)} \quad \forall \phi \in \mathsf{C}^\infty_{\Gamma_n}(\Omega) \right\}.
\end{aligned}$$

Analogously, we define the Sobolev spaces $\mathbf{H}^k_{\Gamma_t}(\mathrm{rot},\Omega)$, $\mathbf{H}^k_{\Gamma_t}(\mathrm{div},\Omega)$ and $\mathbf{H}^k_{\Gamma_t,0}(\mathrm{rot},\Omega)$, $\mathbf{H}^k_{\Gamma_t,0}(\mathrm{div},\Omega)$ using the respective Sobolev spaces with weak boundary conditions. Note that '*strong* $\subset$ *weak*' holds, for example,

$$\mathsf{H}^k_{\Gamma_t}(\Omega) \subset \mathbf{H}^k_{\Gamma_t}(\Omega), \qquad \mathsf{H}_{\Gamma_t}(\mathrm{rot},\Omega) \subset \mathbf{H}_{\Gamma_t}(\mathrm{rot},\Omega), \qquad \mathsf{H}^k_{\Gamma_t}(\mathrm{div},\Omega) \subset \mathbf{H}^k_{\Gamma_t}(\mathrm{div},\Omega),$$

and that the complex properties hold in both the strong and the weak case, for example,

$$\mathrm{grad}\ \mathsf{H}^{k+1}_{\Gamma_t}(\Omega) \subset \mathsf{H}^k_{\Gamma_t,0}(\mathrm{rot},\Omega), \qquad \mathrm{rot}\ \mathbf{H}^k_{\Gamma_t}(\mathrm{rot},\Omega) \subset \mathbf{H}^k_{\Gamma_t,0}(\mathrm{div},\Omega),$$

which follows immediately by the definitions. The next lemma shows that indeed '*strong* = *weak*' holds.

Lemma 3.2 (Bauer et al.[9, Theorem 4.5]). *The Sobolev spaces defined by weak and strong boundary conditions coincide, for example,* $\mathbf{H}^k_{\Gamma_t}(\Omega) = \mathsf{H}^k_{\Gamma_t}(\Omega)$, $\mathbf{H}_{\Gamma_t}(\mathrm{rot},\Omega) = \mathsf{H}_{\Gamma_t}(\mathrm{rot},\Omega)$ and $\mathbf{H}^k_{\Gamma_t}(\mathrm{div},\Omega) = \mathsf{H}^k_{\Gamma_t}(\mathrm{div},\Omega)$, cf. Lemma 3.3.

Finally, we introduce the cohomology space of Dirichlet/Neumann fields (generalised harmonic fields)

$$\mathcal{H}_{\Gamma_t,\Gamma_n,\varepsilon}(\Omega) := \mathsf{H}_{\Gamma_t,0}(\mathrm{rot},\Omega) \cap \varepsilon^{-1}\mathsf{H}_{\Gamma_n,0}(\mathrm{div},\Omega).$$

The classical Dirichlet and Neumann fields are then given by $\mathcal{H}_{\Gamma,\emptyset,\varepsilon}(\Omega)$ anf $\mathcal{H}_{\emptyset,\Gamma,\varepsilon}(\Omega)$, respectively. Here, $\varepsilon : \mathsf{L}^2(\Omega) \to \mathsf{L}^2(\Omega)$ is a symmetric and positive topological isomorphism (symmetric and positive bijective bounded linear operator), which introduces a new inner product

$$\langle \cdot, \cdot \rangle_{\mathsf{L}^2_\varepsilon(\Omega)} := \langle \varepsilon \cdot, \cdot \rangle_{\mathsf{L}^2(\Omega)},$$

where $\mathsf{L}^2_\varepsilon(\Omega) := \mathsf{L}^2(\Omega)$ (as linear space) equipped with the inner product $\langle \cdot, \cdot \rangle_{\mathsf{L}^2_\varepsilon(\Omega)}$. Such *weights* ε shall be called *admissible*, and a typical example is given by a symmetric, L^∞-bounded and uniformly positive definite tensor (matrix) field $\varepsilon : \Omega \to \mathbb{R}^{3\times3}$.

3.3 | Sobolev spaces of differential forms

For spaces of differential forms, we follow the same rationale. Instead of the differential operators grad, rot and div, we now have only the exterior derivative d and the co-derivative $\delta = \pm * \mathrm{d} *$, given by d and the Hodge star operator $*$. The standard Lebesgue and Sobolev Hilbert spaces are denoted by $\mathsf{L}^{q,2}(\Omega)$, $\mathsf{H}^{q,k}(\Omega)$, $\mathsf{H}^q(\mathrm{d},\Omega)$, $\mathsf{H}^q(\delta,\Omega)$ for $k \in \mathbb{Z}$, and by $\mathsf{H}^q_0(\mathrm{d},\Omega)$ and $\mathsf{H}^q_0(\delta,\Omega)$, we indicate the spaces with vanishing d and δ, respectively. Here, $q \in \mathbb{Z}$ marks the rank of the respective differential forms. As before, homogeneous boundary conditions for d and δ are introduced in the *strong sense* as closures of respective test forms from

$$\mathsf{C}^{q,\infty}_{\Gamma_t}(\Omega) := \left\{\Phi|_\Omega \,:\, \Phi \in \mathsf{C}^{q,\infty}(\mathbb{R}^d),\ \operatorname{supp}\Phi \text{ compact},\ \operatorname{dist}(\operatorname{supp}\Phi,\Gamma_t) > 0\right\},$$

that is, for $k \in \mathbb{N}_0$

$$\mathsf{H}^{q,k}_{\Gamma_t}(\Omega) := \overline{\mathsf{C}^{q,\infty}_{\Gamma_t}(\Omega)}^{\mathsf{H}^{q,k}(\Omega)},\ \mathsf{H}^q_{\Gamma_t}(\mathrm{d},\Omega) := \overline{\mathsf{C}^{q,\infty}_{\Gamma_t}(\Omega)}^{\mathsf{H}^q(\mathrm{d},\Omega)},\ \mathsf{H}^q_{\Gamma_t}(\delta,\Omega) := \overline{\mathsf{C}^{q,\infty}_{\Gamma_t}(\Omega)}^{\mathsf{H}^q(\delta,\Omega)},$$

and we have $\mathsf{H}^{q,k}_{\emptyset}(\Omega) = \mathsf{H}^{q,k}(\Omega)$, $\mathsf{H}^q_{\emptyset}(\mathrm{d},\Omega) = \mathsf{H}^q(\mathrm{d},\Omega)$ and $\mathsf{H}^q_{\emptyset}(\delta,\Omega) = \mathsf{H}^q(\delta,\Omega)$, which are well known density results and incorporated into the notation by purpose. Spaces with vanishing d and δ are again denoted by $\mathsf{H}^q_{\Gamma_t,0}(\mathrm{d},\Omega)$ and $\mathsf{H}^q_{\Gamma_t,0}(\delta,\Omega)$, respectively. Note that for $k = 0$, we have $\mathsf{H}^{q,0}_{\Gamma_t}(\Omega) = \mathsf{L}^{q,2}(\Omega)$, and for $q = 0$, we can also write $\mathsf{H}^{0,1}_{\Gamma_t}(\Omega) = \mathsf{H}^0_{\Gamma_t}(\mathrm{d},\Omega) \cong \mathsf{H}^d_{\Gamma_t}(\delta,\Omega)$. Moreover, we introduce for $k \in \mathbb{N}_0$ the nonstandard Sobolev spaces of q-forms

$$\begin{aligned}
\mathsf{H}^{q,k}(\mathrm{d},\Omega) &:= \left\{E \in \mathsf{H}^{q,k}(\Omega) \,:\, \mathrm{d}E \in \mathsf{H}^{q+1,k}(\Omega)\right\},\\
\mathsf{H}^{q,k}_{\Gamma_t}(\mathrm{d},\Omega) &:= \left\{E \in \mathsf{H}^{q,k}_{\Gamma_t}(\Omega) \cap \mathsf{H}^q_{\Gamma_t}(\mathrm{d},\Omega) \,:\, \mathrm{d}E \in \mathsf{H}^{q+1,k}_{\Gamma_t}(\Omega)\right\},\\
\mathsf{H}^{q,k}(\delta,\Omega) &:= \left\{E \in \mathsf{H}^{q,k}(\Omega) \,:\, \delta E \in \mathsf{H}^{q-1,k}(\Omega)\right\},\\
\mathsf{H}^{q,k}_{\Gamma_t}(\delta,\Omega) &:= \left\{E \in \mathsf{H}^{q,k}_{\Gamma_t}(\Omega) \cap \mathsf{H}^q_{\Gamma_t}(\delta,\Omega) \,:\, \delta E \in \mathsf{H}^{q-1,k}_{\Gamma_t}(\Omega)\right\}.
\end{aligned}$$

We see $\mathsf{H}^{q,k}_{\emptyset}(\mathrm{d},\Omega) = \mathsf{H}^{q,k}(\mathrm{d},\Omega)$, and for $k = 0$, we have $\mathsf{H}^{q,0}_{\emptyset}(\mathrm{d},\Omega) = \mathsf{H}^{q,0}(\mathrm{d},\Omega) = \mathsf{H}^q(\mathrm{d},\Omega)$ and $\mathsf{H}^{q,0}_{\Gamma_t}(\mathrm{d},\Omega) = \mathsf{H}^q_{\Gamma_t}(\mathrm{d},\Omega)$. Note that for $\Gamma_t \neq \emptyset$ and $k \geq 1$, it holds

$$\mathsf{H}^{q,k}_{\Gamma_t}(\mathrm{d},\Omega) = \left\{E \in \mathsf{H}^{q,k}_{\Gamma_t}(\Omega) \,:\, \mathrm{d}E \in \mathsf{H}^{q+1,k}_{\Gamma_t}(\Omega)\right\},$$

but for $\Gamma_t \neq \emptyset$ and $k = 0$ $\left(\text{as } \mathsf{H}^{q,0}_{\Gamma_t}(\Omega) = \mathsf{L}^{q,2}(\Omega)\right)$,

$$\begin{aligned}
\mathsf{H}^{q,0}_{\Gamma_t}(\mathrm{d},\Omega) &= \left\{E \in \mathsf{H}^{q,0}_{\Gamma_t}(\Omega) \cap \mathsf{H}^q_{\Gamma_t}(\mathrm{d},\Omega) \,:\, \mathrm{d}E \in \mathsf{H}^{q+1,0}_{\Gamma_t}(\Omega)\right\} = \mathsf{H}^q_{\Gamma_t}(\mathrm{d},\Omega)\\
&\subsetneq \left\{E \in \mathsf{H}^{q,0}_{\Gamma_t}(\Omega) \,:\, \mathrm{d}E \in \mathsf{H}^{q+1,0}_{\Gamma_t}(\Omega)\right\} = \mathsf{H}^{q,0}_{\emptyset}(\mathrm{d},\Omega) = \mathsf{H}^q(\mathrm{d},\Omega).
\end{aligned}$$

As before,

$$\mathsf{H}^{q,k}_{\Gamma_t,0}(\mathrm{d},\Omega) := \mathsf{H}^{q,k}_{\Gamma_t}(\Omega)\cap\mathsf{H}^{q}_{\Gamma_t,0}(\mathrm{d},\Omega) = \mathsf{H}^{q,k}_{\Gamma_t}(\mathrm{d},\Omega)\cap\mathsf{H}^{q}_{0}(\mathrm{d},\Omega) = \left\{E\in\mathsf{H}^{q,k}_{\Gamma_t}(\mathrm{d},\Omega)\,:\,\mathrm{d}E=0\right\}.$$

The corresponding remarks hold for the $\mathsf{H}^{q,k}_{\Gamma_t}(\delta,\Omega)$-spaces as well.

Again, let us note that boundary conditions can also be defined in the *weak sense* by

$$\begin{aligned}
\mathbf{H}^{q,k}_{\Gamma_t}(\Omega) &:= \left\{E\in\mathsf{H}^{q,k}(\Omega)\,:\,\langle\partial^\alpha E,\Phi\rangle_{\mathsf{L}^{q,2}(\Omega)} = (-1)^{|\alpha|}\langle E,\partial^\alpha\Phi\rangle_{\mathsf{L}^{q,2}(\Omega)}\ \forall\Phi\in\mathsf{C}^{q,\infty}_{\Gamma_n}(\Omega)\ \forall|\alpha|\le k\right\},\\
\mathbf{H}^{q}_{\Gamma_t}(\mathrm{d},\Omega) &:= \left\{E\in\mathsf{H}^{q}(\mathrm{d},\Omega)\,:\,\langle\mathrm{d}E,\Phi\rangle_{\mathsf{L}^{q+1,2}(\Omega)} = -\langle E,\delta\Phi\rangle_{\mathsf{L}^{q,2}(\Omega)}\quad\forall\Phi\in\mathsf{C}^{q+1,\infty}_{\Gamma_n}(\Omega)\right\},\\
\mathbf{H}^{q}_{\Gamma_t}(\delta,\Omega) &:= \left\{E\in\mathsf{H}^{q}(\delta,\Omega)\,:\,\langle\delta E,\Phi\rangle_{\mathsf{L}^{q-1,2}(\Omega)} = -\langle E,\mathrm{d}\Phi\rangle_{\mathsf{L}^{q,2}(\Omega)}\quad\forall\Phi\in\mathsf{C}^{q-1,\infty}_{\Gamma_n}(\Omega)\right\}.
\end{aligned}$$

Analogously, we define the Sobolev spaces $\mathbf{H}^{q,k}_{\Gamma_t}(\mathrm{d},\Omega)$, $\mathbf{H}^{q,k}_{\Gamma_t}(\delta,\Omega)$ and $\mathbf{H}^{q,k}_{\Gamma_t,0}(\mathrm{d},\Omega)$, $\mathbf{H}^{q,k}_{\Gamma_t,0}(\delta,\Omega)$ using the respective Sobolev spaces with weak boundary conditions. Note that '*strong* $\subset$ *weak*' holds, for example,

$$\mathsf{H}^{q,k}_{\Gamma_t}(\Omega)\subset\mathbf{H}^{q,k}_{\Gamma_t}(\Omega),\qquad \mathsf{H}^{q}_{\Gamma_t}(\mathrm{d},\Omega)\subset\mathbf{H}^{q}_{\Gamma_t}(\mathrm{d},\Omega),\qquad \mathsf{H}^{q,k}_{\Gamma_t}(\delta,\Omega)\subset\mathbf{H}^{q,k}_{\Gamma_t}(\delta,\Omega),$$

and that the complex properties hold in both the strong and the weak case, for example,

$$\mathrm{d}\mathsf{H}^{q,k}_{\Gamma_t}(\mathrm{d},\Omega)\subset\mathsf{H}^{q+1,k}_{\Gamma_t,0}(\mathrm{d},\Omega),\qquad \delta\mathbf{H}^{q,k}_{\Gamma_t}(\delta,\Omega)\subset\mathbf{H}^{q-1,k}_{\Gamma_t,0}(\delta,\Omega),$$

which follows immediately by the definitions. The next lemma shows that indeed '*strong* = *weak*' holds.

Lemma 3.3 (Bauer et al.[18, Theorem 4.7]). *The Sobolev spaces defined by weak and strong boundary conditions coincide, for example,* $\mathbf{H}^{q,k}_{\Gamma_t}(\Omega)=\mathsf{H}^{q,k}_{\Gamma_t}(\Omega)$, $\mathbf{H}^{q}_{\Gamma_t}(\mathrm{d},\Omega)=\mathsf{H}^{q}_{\Gamma_t}(\mathrm{d},\Omega)$ and $\mathbf{H}^{q,k}_{\Gamma_t}(\delta,\Omega)=\mathsf{H}^{q,k}_{\Gamma_t}(\delta,\Omega)$.

For convenience, a self-contained proof of Lemma 3.3 (and hence also of Lemma 3.2) is given as a part of Lemma 4.6, cf. Lemma 4.4 and Lemma 4.5.

Lemma 3.4 (Schwarz' lemma). *Let* $|\alpha|\le k$.

(i) For $E\in\mathsf{H}^{q,k}_{\Gamma_t}(\mathrm{d},\Omega)$, *it holds* $\partial^\alpha E\in\mathsf{H}^{q,0}_{\Gamma_t}(\mathrm{d},\Omega)$ *and* $\mathrm{d}\partial^\alpha E=\partial^\alpha\mathrm{d}E$.

(ii) For $H\in\mathsf{H}^{q,k}_{\Gamma_t}(\delta,\Omega)$, *it holds* $\partial^\alpha H\in\mathsf{H}^{q,0}_{\Gamma_t}(\delta,\Omega)$ *and* $\delta\partial^\alpha H=\partial^\alpha\delta H$.

Proof. (i) can be seen as follows: For $\Phi\in\mathsf{C}^{q+1,\infty}_{\Gamma_n}(\Omega)$, we have

$$\begin{aligned}
\langle\partial^\alpha E,\delta\Phi\rangle_{\mathsf{L}^{q,2}(\Omega)} &= (-1)^{|\alpha|}\langle E,\delta\partial^\alpha\Phi\rangle_{\mathsf{L}^{q,2}(\Omega)}\\
&= (-1)^{|\alpha|+1}\langle\mathrm{d}E,\partial^\alpha\Phi\rangle_{\mathsf{L}^{q+1,2}(\Omega)} = -\langle\partial^\alpha\mathrm{d}E,\Phi\rangle_{\mathsf{L}^{q+1,2}(\Omega)}
\end{aligned}$$

as $E\in\mathsf{H}^{q,k}_{\Gamma_t}(\Omega)\cap\mathsf{H}^{q,0}_{\Gamma_t}(\mathrm{d},\Omega)$ and $\mathrm{d}E\in\mathsf{H}^{q+1,k}_{\Gamma_t}(\Omega)$. Hence, $\partial^\alpha E\in\mathbf{H}^{q,0}_{\Gamma_t}(\mathrm{d},\Omega)=\mathsf{H}^{q,0}_{\Gamma_t}(\mathrm{d},\Omega)$ by Lemma 3.3 and $\mathrm{d}\partial^\alpha E=\partial^\alpha\mathrm{d}E$. (ii) follows analogously or by the Hodge $\star$-operator. □

Finally, we introduce the cohomology space of Dirichlet/Neumann forms (generalised harmonic forms)

$$\mathcal{H}^{q}_{\Gamma_t,\Gamma_n,\varepsilon}(\Omega) := \mathsf{H}^{q}_{\Gamma_t,0}(\mathrm{d},\Omega)\cap\varepsilon^{-1}\mathsf{H}^{q}_{\Gamma_n,0}(\delta,\Omega). \tag{14}$$

The classical Dirichlet and Neumann fields are then given by $\mathcal{H}^{q}_{\Gamma,\emptyset,\varepsilon}(\Omega)$ anf $\mathcal{H}^{q}_{\emptyset,\Gamma,\varepsilon}(\Omega)$, respectively. Here, $\varepsilon=\varepsilon_q:\mathsf{L}^{q,2}(\Omega)\to\mathsf{L}^{q,2}(\Omega)$ is a symmetric and positive topological isomorphism (symmetric and positive bijective bounded linear operator), which introduces a new inner product

$$\langle\,\cdot\,,\cdot\,\rangle_{\mathsf{L}^{q,2}_{\varepsilon}(\Omega)} := \langle\varepsilon\,\cdot\,,\cdot\,\rangle_{\mathsf{L}^{q,2}(\Omega)},$$

where $\mathsf{L}^{q,2}_{\varepsilon}(\Omega) := \mathsf{L}^{q,2}(\Omega)$ (as linear space) equipped with the inner product $\langle\cdot,\cdot\rangle_{\mathsf{L}^{q,2}_{\varepsilon}(\Omega)}$. Such *weights* ε shall be called *admissible*, and a typical example is given by a symmetric, L^{∞}-bounded and uniformly positive definite tensor (matrix) field $\varepsilon : \Omega \to \mathbb{R}^{\binom{N}{q}\times\binom{N}{q}}$.

3.4 | Some useful and important results

In Hiptmair et al,[19] the existence of a crucial universal extension operator for the Sobolev spaces $\mathsf{H}^{q,k}(\mathrm{d},\Omega)$ has been shown, which is based on the universal extension operator from Stein's book.[20]

Lemma 3.5 (universal Stein extension operator,[19, Theorem 3.6] cf. Bauer et al.[18, Lemma 2.15]). *Let $\Omega \subset \mathbb{R}^d$ be a bounded strong Lipschitz domain. For all $k \in \mathbb{N}_0$ and all q, there exists a (universal) bounded linear extension operator*

$$\mathcal{E} = \mathcal{E}^{q,k} : \mathsf{H}^{q,k}(\mathrm{d},\Omega) \to \mathsf{H}^{q,k}(\mathrm{d},\mathbb{R}^d).$$

More precisely, there exists $c>0$ such that for all $E \in \mathsf{H}^{q,k}(\mathrm{d},\Omega)$, it holds $\mathcal{E}E \in \mathsf{H}^{q,k}(\mathrm{d},\mathbb{R}^d)$ and $\mathcal{E}E = E$ in Ω as well as $|\mathcal{E}E|_{\mathsf{H}^{q,k}(\mathrm{d},\mathbb{R}^d)} \leq c|E|_{\mathsf{H}^{q,k}(\mathrm{d},\Omega)}$. Furthermore, $\mathcal{E}$ can be chosen such that $\mathcal{E}E$ has fixed compact support in $\mathbb{R}^d$ for all $E \in \mathsf{H}^{q,k}(\mathrm{d},\Omega)$.

From Bauer et al,[18, Theorem 5.2] we have the following Helmholtz decompositions.

Lemma 3.6 (Helmholtz decompositions). *Let $\Omega \subset \mathbb{R}^d$ be a bounded strong Lipschitz domain. For all q, the orthonormal Helmholtz decompositions*

$$\begin{aligned}
\mathsf{L}^{q,2}_{\varepsilon}(\Omega) &= \mathrm{d}\mathsf{H}^{q-1,0}_{\Gamma_t}(\mathrm{d},\Omega) \ \oplus_{\mathsf{L}^{q,2}_{\varepsilon}(\Omega)} \varepsilon^{-1}\mathsf{H}^{q,0}_{\Gamma_n,0}(\delta,\Omega) \\
&= \mathsf{H}^{q,0}_{\Gamma_t,0}(\mathrm{d},\Omega) \ \oplus_{\mathsf{L}^{q,2}_{\varepsilon}(\Omega)} \varepsilon^{-1}\delta\mathsf{H}^{q+1,0}_{\Gamma_n}(\delta,\Omega) \\
&= \mathrm{d}\mathsf{H}^{q-1,0}_{\Gamma_t}(\mathrm{d},\Omega) \ \oplus_{\mathsf{L}^{q,2}_{\varepsilon}(\Omega)} \mathcal{H}^{q}_{\Gamma_t,\Gamma_n,\varepsilon}(\Omega) \ \oplus_{\mathsf{L}^{q,2}_{\varepsilon}(\Omega)} \varepsilon^{-1}\delta\mathsf{H}^{q+1,0}_{\Gamma_n}(\delta,\Omega)
\end{aligned}$$

hold. In particular, the ranges

$$\begin{aligned}
\mathrm{d}\mathsf{H}^{q-1,0}_{\Gamma_t}(\mathrm{d},\Omega) &= \mathsf{H}^{q,0}_{\Gamma_t,0}(\mathrm{d},\Omega) \cap \mathcal{H}^{q}_{\Gamma_t,\Gamma_n,\varepsilon}(\Omega)^{\perp_{\mathsf{L}^{q,2}_{\varepsilon}(\Omega)}}, \\
\delta\mathsf{H}^{q+1,0}_{\Gamma_n}(\delta,\Omega) &= \mathsf{H}^{q,0}_{\Gamma_n,0}(\delta,\Omega) \cap \mathcal{H}^{q}_{\Gamma_t,\Gamma_n,\varepsilon}(\Omega)^{\perp_{\mathsf{L}^{q,2}(\Omega)}}
\end{aligned}$$

are closed subspaces of $\mathsf{L}^{q,2}_{\varepsilon}(\Omega)$, and the potentials can be chosen such that they depend continuously on the data.

Note that Lemma 3.6 even holds for bounded weak Lipschitz domains $\Omega \subset \mathbb{R}^d$. From Picard,[21] cf. Bauer et al,[18, Lemma 2.19] we have the following Helmholtz decompositions for the special case $\Omega = \mathbb{R}^d$.

Lemma 3.7 (Helmholtz decompositions in the whole space). *For all q*

$$\begin{aligned}
\mathsf{L}^{q,2}(\mathbb{R}^d) &= \mathsf{H}^{q}_{0}(\mathrm{d},\mathbb{R}^d) \ \oplus_{\mathsf{L}^{q,2}(\mathbb{R}^d)} \mathsf{H}^{q}_{0}(\delta,\mathbb{R}^d), \\
\mathsf{H}^{q}(\mathrm{d},\mathbb{R}^d) &= \mathsf{H}^{q}_{0}(\mathrm{d},\mathbb{R}^d) \ \oplus_{\mathsf{L}^{q,2}(\mathbb{R}^d)} \left(\mathsf{H}^{q}(\mathrm{d},\mathbb{R}^d) \cap \mathsf{H}^{q}_{0}(\delta,\mathbb{R}^d)\right).
\end{aligned}$$

Let $\pi_{q,\mathbb{R}^d} : \mathsf{L}^{q,2}(\mathbb{R}^d) \to \mathsf{H}^{q}_{0}(\delta,\mathbb{R}^d)$ denote the orthonormal projector onto $\mathsf{H}^{q}_{0}(\delta,\mathbb{R}^d)$. Then for all $E \in \mathsf{H}^{q}(\mathrm{d},\mathbb{R}^d)$, it holds $\pi_{q,\mathbb{R}^d}E \in \mathsf{H}^{q}(\mathrm{d},\mathbb{R}^d) \cap \mathsf{H}^{q}_{0}(\delta,\mathbb{R}^d)$ and $\mathrm{d}\pi_{q,\mathbb{R}^d}E = \mathrm{d}E$ as well as $|\pi_{q,\mathbb{R}^d}E|_{\mathsf{H}^{q}(\mathrm{d},\mathbb{R}^d)} \leq |E|_{\mathsf{H}^{q}(\mathrm{d},\mathbb{R}^d)}$.

From Kuhn and Pauly,[22, Lemma 4.2(i)] cf. Bauer et al,[18, Lemma 2.20] we have the following regularity result.

Lemma 3.8 (regularity in the whole space). *For $k \in \mathbb{N}_0$ and all q, it holds*

$$\left\{E \in \mathsf{H}^{q}(\mathrm{d},\mathbb{R}^d) \cap \mathsf{H}^{q}(\delta,\mathbb{R}^d) : \mathrm{d}E \in \mathsf{H}^{q+1,k}(\mathbb{R}^d) \wedge \delta E \in \mathsf{H}^{q-1,k}(\mathbb{R}^d)\right\} = \mathsf{H}^{q,k+1}(\mathbb{R}^d).$$

More precisely, $E \in \mathsf{H}^q(\mathrm{d}, \mathbb{R}^d) \cap \mathsf{H}^q(\delta, \mathbb{R}^d)$ *with* $\mathrm{d}E \in \mathsf{H}^{q+1,k}(\mathbb{R}^d)$ *and* $\delta E \in \mathsf{H}^{q-1,k}(\mathbb{R}^d)$*, if and only if* $E \in \mathsf{H}^{q,k+1}(\mathbb{R}^d)$ *and*

$$\frac{1}{c}|E|_{\mathsf{H}^{q,k+1}(\mathbb{R}^d)} \leq |E|_{\mathsf{L}^{q,2}(\mathbb{R}^d)} + |\mathrm{d}E|_{\mathsf{H}^{q+1,k}(\mathbb{R}^d)} + |\delta E|_{\mathsf{H}^{q-1,k}(\mathbb{R}^d)} \leq c|E|_{\mathsf{H}^{q,k+1}(\mathbb{R}^d)}$$

with some $c > 0$ *independent of* E.

In Bauer et al,[18, Lemma 3.1] see also Bauer et al[9,17] for more details, the following lemma about the existence of regular potentials without boundary conditions has been shown.

Lemma 3.9 (regular potential for d without boundary condition). *Let* $\Omega \subset \mathbb{R}^d$ *be a bounded strong Lipschitz domain. For all* $q \in \{1, \dots, d\}$*, there exists a bounded linear potential operator*

$$\mathcal{P}^{q,0}_{\mathrm{d},\emptyset} : \mathsf{H}^{q,0}_{\emptyset,0}(\mathrm{d}, \Omega) \cap \mathcal{H}^q_{\emptyset,\Gamma,\mathrm{id}}(\Omega)^{\perp_{\mathsf{L}^{q,2}(\Omega)}} \to \mathsf{H}^{q-1,1}_0(\delta, \mathbb{R}^d),$$

such that $\mathrm{d}\mathcal{P}^{q,0}_{\mathrm{d},\emptyset} = \mathrm{id}|_{\mathsf{H}^{q,0}_{\emptyset,0}(\mathrm{d},\Omega)\cap\mathcal{H}^q_{\emptyset,\Gamma,\mathrm{id}}(\Omega)^{\perp_{\mathsf{L}^{q,2}(\Omega)}}}$*, that is, for all* $E \in \mathsf{H}^{q,0}_{\emptyset,0}(\mathrm{d}, \Omega) \cap \mathcal{H}^q_{\emptyset,\Gamma,\mathrm{id}}(\Omega)^{\perp_{\mathsf{L}^{q,2}(\Omega)}}$

$$\mathrm{d}\mathcal{P}^{q,0}_{\mathrm{d},\emptyset}E = E \ \textit{in} \ \ \Omega.$$

In particular,

$$\mathsf{H}^{q,0}_{\emptyset,0}(\mathrm{d}, \Omega) \cap \mathcal{H}^q_{\emptyset,\Gamma,\mathrm{id}}(\Omega)^{\perp_{\mathsf{L}^{q,2}(\Omega)}} = \mathrm{d}\mathsf{H}^{q-1,0}_{\emptyset}(\delta, \Omega) = \mathrm{d}\mathsf{H}^{q-1,1}_{\emptyset}(\Omega) = \mathrm{d}\mathsf{H}^{q-1,1}_{\emptyset,0}(\delta, \Omega)$$

and the potentials can be chosen such that they depend continuously on the data. Especially, these are closed subspaces of $\mathsf{L}^{q,2}(\Omega)$, and $\mathcal{P}^{q,0}_{\mathrm{d},\emptyset}$ is a right inverse to d.

4 | DE RHAM COMPLEX

In this section, we shall apply the FA-ToolBox from Section 2 to the de Rham complex.

4.1 | Zero-order de Rham complex

Let the exterior derivatives be realised as densely defined (unbounded) linear operators

$$\mathring{\mathrm{d}}^q_{\Gamma_t} : D(\mathring{\mathrm{d}}^q_{\Gamma_t}) \subset \mathsf{L}^{q,2}(\Omega) \to \mathsf{L}^{q+1,2}(\Omega); E \mapsto \mathrm{d}E, \qquad D(\mathring{\mathrm{d}}^q_{\Gamma_t}) := \mathsf{C}^{q,\infty}_{\Gamma_t}(\Omega), \qquad q = 0, \dots, d-1,$$

satisfying the complex properties

$$\mathring{\mathrm{d}}^q_{\Gamma_t}\mathring{\mathrm{d}}^{q-1}_{\Gamma_t} \subset 0.$$

Then the closures $\mathrm{d}^q_{\Gamma_t} := \overline{\mathring{\mathrm{d}}^q_{\Gamma_t}}$ and Hilbert space adjoints $(\mathrm{d}^q_{\Gamma_t})^* = (\mathring{\mathrm{d}}^q_{\Gamma_t})^*$ are given by

$$\mathrm{d}^q_{\Gamma_t} : D(\mathrm{d}^q_{\Gamma_t}) \subset \mathsf{L}^{q,2}(\Omega) \to \mathsf{L}^{q+1,2}(\Omega); E \mapsto \mathrm{d}E, \qquad D(\mathrm{d}^q_{\Gamma_t}) = \mathsf{H}^{q,0}_{\Gamma_t}(\mathrm{d}, \Omega),$$

and

$$(\mathrm{d}^q_{\Gamma_t})^* = -\delta^{q+1}_{\Gamma_n} : D(\delta^{q+1}_{\Gamma_n}) \subset \mathsf{L}^{q+1,2}(\Omega) \to \mathsf{L}^{q,2}(\Omega); H \mapsto -\delta H, \qquad D(\delta^{q+1}_{\Gamma_n}) = \mathsf{H}^{q+1,0}_{\Gamma_n}(\delta, \Omega),$$

where indeed $D(\delta^{q+1}_{\Gamma_n}) = \mathsf{H}^{q+1,0}_{\Gamma_n}(\delta, \Omega)$ holds by Lemma 3.3, cf. Bauer et al,[18, Section 5.2] (weak and strong boundary conditions coincide).

Remark 4.1. Note that by definition, the adjoints are given by

$$(\mathrm{d}^q_{\Gamma_t})^* = (\mathring{\mathrm{d}}^q_{\Gamma_t})^* = -\boldsymbol{\delta}^{q+1}_{\Gamma_n} : D(\boldsymbol{\delta}^{q+1}_{\Gamma_n}) \subset \mathsf{L}^{q+1,2}(\Omega) \to \mathsf{L}^{q,2}(\Omega); H \mapsto -\delta H,$$

with $D(\boldsymbol{\delta}^{q+1}_{\Gamma_n}) = \mathbf{H}^{q+1,0}_{\Gamma_n}(\delta,\Omega)$. Lemma 3.3 (weak and strong boundary conditions coincide) shows indeed $D(\boldsymbol{\delta}^{q+1}_{\Gamma_n}) = \mathbf{H}^{q+1,0}_{\Gamma_n}(\delta,\Omega) = \mathsf{H}^{q+1,0}_{\Gamma_n}(\delta,\Omega) = D(\delta^{q+1}_{\Gamma_n})$, that is, $\boldsymbol{\delta}^{q+1}_{\Gamma_n} = \delta^{q+1}_{\Gamma_n}$.

By definition, the densely defined and closed (unbounded) linear operators

$$\mathrm{A}_q := \mathrm{d}^q_{\Gamma_t}, \qquad \mathrm{A}^*_q = -\delta^{q+1}_{\Gamma_n}, \qquad q = 0, \dots, d-1,$$

define dual pairs $\left(\mathrm{d}^q_{\Gamma_t}, (\mathrm{d}^q_{\Gamma_t})^*\right) = (\mathrm{d}^q_{\Gamma_t}, -\delta^{q+1}_{\Gamma_n})$. Remarks 2.5 and 2.6 show the complex properties $R(\mathrm{d}^{q-1}_{\Gamma_t}) \subset N(\mathrm{d}^q_{\Gamma_t})$ and $R(\delta^{q+1}_{\Gamma_n}) \subset N(\delta^q_{\Gamma_n})$, that is, the complex properties

$$\mathrm{d}^q_{\Gamma_t}\mathrm{d}^{q-1}_{\Gamma_t} \subset 0, \qquad \delta^q_{\Gamma_n}\delta^{q+1}_{\Gamma_n} \subset 0.$$

Note that with $\mathrm{A}_0 = \mathrm{d}^0_{\Gamma_t}$ and $\mathrm{A}^*_{d-1} = (\mathrm{d}^{d-1}_{\Gamma_t})^* = -\delta^d_{\Gamma_n}$ as well as

$$\mathrm{A}_{-1} := \iota_{N(\mathrm{A}_0)},\ \mathrm{A}^*_{-1} = \pi_{N(\mathrm{A}_0)}, \qquad \mathrm{A}^*_d := \iota_{N(\mathrm{A}^*_{d-1})},\ \mathrm{A}_d = \pi_{N(\mathrm{A}^*_{d-1})}$$

(actually, $\mathrm{A}_{-1}\mathrm{A}^*_{-1} = \pi_{N(\mathrm{A}_0)}$ and $\mathrm{A}^*_d\mathrm{A}_d = \pi_{N(\mathrm{A}^*_{d-1})}$, cf. Remark 2.25), we have

$$N(\mathrm{A}_0) = N(\mathrm{d}^0_{\Gamma_t}) = \mathbb{R}_{\Gamma_t}, \qquad N(\mathrm{A}^*_{d-1}) = N(\delta^d_{\Gamma_n}) = *\mathbb{R}_{\Gamma_n}, \qquad \mathbb{R}_\Sigma := \begin{cases} \mathbb{R} & \text{if } \Sigma = \emptyset, \\ \{0\} & \text{otherwise,} \end{cases}$$

and that the long (here even longer) primal and dual de Rham Hilbert complex (12) reads

$$\begin{array}{c}
\mathbb{R}_{\Gamma_t} \underset{\pi_{\mathbb{R}_{\Gamma_t}}}{\overset{\iota_{\mathbb{R}_{\Gamma_t}}}{\rightleftarrows}} \mathsf{L}^{0,2}(\Omega) \underset{-\delta^1_{\Gamma_n}}{\overset{\mathrm{d}^0_{\Gamma_t}}{\rightleftarrows}} \mathsf{L}^{1,2}(\Omega) \underset{-\delta^2_{\Gamma_n}}{\overset{\mathrm{d}^1_{\Gamma_t}}{\rightleftarrows}} \mathsf{L}^{2,2}(\Omega) \underset{\cdots}{\overset{\cdots}{\rightleftarrows}} \cdots \\
\cdots \underset{\cdots}{\overset{\cdots}{\rightleftarrows}} \mathsf{L}^{q-1,2}(\Omega) \underset{-\delta^q_{\Gamma_n}}{\overset{\mathrm{d}^{q-1}_{\Gamma_t}}{\rightleftarrows}} \mathsf{L}^{q,2}(\Omega) \underset{-\delta^{q+1}_{\Gamma_n}}{\overset{\mathrm{d}^q_{\Gamma_t}}{\rightleftarrows}} \mathsf{L}^{q+1,2}(\Omega) \underset{\cdots}{\overset{\cdots}{\rightleftarrows}} \cdots \\
\cdots \underset{\cdots}{\overset{\cdots}{\rightleftarrows}} \mathsf{L}^{d-2,2}(\Omega) \underset{-\delta^{d-1}_{\Gamma_n}}{\overset{\mathrm{d}^{d-2}_{\Gamma_t}}{\rightleftarrows}} \mathsf{L}^{d-1,2}(\Omega) \underset{-\delta^d_{\Gamma_n}}{\overset{\mathrm{d}^{d-1}_{\Gamma_t}}{\rightleftarrows}} \mathsf{L}^{d,2}(\Omega) \underset{\iota_{*\mathbb{R}_{\Gamma_n}}}{\overset{\pi_{*\mathbb{R}_{\Gamma_n}}}{\rightleftarrows}} *\mathbb{R}_{\Gamma_n}
\end{array} \tag{15}$$

with the complex properties

$$R(\mathrm{d}^{q-1}_{\Gamma_t}) \subset N(\mathrm{d}^q_{\Gamma_t}), \qquad R(\delta^{q+1}_{\Gamma_n}) \subset N(\delta^q_{\Gamma_n}), \qquad q = 1, \dots, d-1,$$

and

$$R(\iota_{\mathbb{R}_{\Gamma_t}}) = N(\mathrm{d}^0_{\Gamma_t}) = \mathbb{R}_{\Gamma_t}, \qquad \overline{R(\mathrm{d}^{d-1}_{\Gamma_t})} = N(\pi_{*\mathbb{R}_{\Gamma_n}}) = (*\mathbb{R}_{\Gamma_n})^{\perp_{\mathsf{L}^{d,2}(\Omega)}},$$
$$\overline{R(\delta^1_{\Gamma_n})} = N(\pi_{\mathbb{R}_{\Gamma_t}}) = (\mathbb{R}_{\Gamma_t})^{\perp_{\mathsf{L}^{0,2}(\Omega)}}, \qquad R(\iota_{*\mathbb{R}_{\Gamma_n}}) = N(\delta^d_{\Gamma_n}) = *\mathbb{R}_{\Gamma_n}.$$

We emphasise that the definition of the Dirichlet/Neumann forms (14) is consistent with the definition of the cohomology groups $N_{q-1,q} = N(\mathrm{A}_q) \cap N(\mathrm{A}^*_{q-1})$ as long as $1 \le q \le d-1$. For $q = 0$ and $q = d$, we have the deviations

$$\{0\} = N_{-1,0} \subset N(\mathrm{A}_0) = \mathsf{H}^0_{\Gamma_t,0}(\mathrm{d},\Omega) = \mathcal{H}^0_{\Gamma_t,\Gamma_n,\varepsilon}(\Omega) = \mathbb{R}_{\Gamma_t},$$
$$\{0\} = N_{d-1,d} \subset N(\mathrm{A}^*_{d-1}) = \varepsilon^{-1}\mathsf{H}^d_{\Gamma_n,0}(\delta,\Omega) = \mathcal{H}^d_{\Gamma_t,\Gamma_n,\varepsilon}(\Omega) = \varepsilon^{-1} *\mathbb{R}_{\Gamma_n},$$

cf. (13), which is intended and usefull for latter formulations.

4.2 Higher-order de Rham complex

Similar to (15), we can also investigate the higher Sobolev order primal de Rham complex

$$\cdots \xrightarrow{\cdots} \mathsf{H}^{q-1,k}_{\Gamma_t}(\Omega) \xrightarrow{\mathrm{d}^{q-1,k}_{\Gamma_t}} \mathsf{H}^{q,k}_{\Gamma_t}(\Omega) \xrightarrow{\mathrm{d}^{q,k}_{\Gamma_t}} \mathsf{H}^{q+1,k}_{\Gamma_t}(\Omega) \xrightarrow{\cdots} \cdots$$

together with its *formal* adjoint, the higher Sobolev order dual de Rham complex

$$\cdots \xrightarrow{\cdots} \mathsf{H}^{q-1,k}_{\Gamma_n}(\Omega) \xleftarrow{-\delta^{q,k}_{\Gamma_n}} \mathsf{H}^{q,k}_{\Gamma_n}(\Omega) \xleftarrow{-\delta^{q+1,k}_{\Gamma_n}} \mathsf{H}^{q+1,k}_{\Gamma_n}(\Omega) \xleftarrow{\cdots} \cdots.$$

More precisely, we consider

$$\mathrm{d}^{q,k}_{\Gamma_t} : D(\mathrm{d}^{q,k}_{\Gamma_t}) \subset \mathsf{H}^{q,k}_{\Gamma_t}(\Omega) \to \mathsf{H}^{q+1,k}_{\Gamma_t}(\Omega); E \mapsto \mathrm{d}E, \quad D(\mathrm{d}^{q,k}_{\Gamma_t}) := \mathsf{H}^{q,k}_{\Gamma_t}(\mathrm{d},\Omega),$$

with formal adjoints

$$-\delta^{q+1,k}_{\Gamma_n} : D(\delta^{q+1,k}_{\Gamma_n}) \subset \mathsf{H}^{q+1,k}_{\Gamma_n}(\Omega) \to \mathsf{H}^{q,k}_{\Gamma_n}(\Omega); H \mapsto -\delta H, \quad D(\delta^{q+1,k}_{\Gamma_n}) := \mathsf{H}^{q+1,k}_{\Gamma_n}(\delta,\Omega).$$

Note that $\mathrm{d}^{q,k}_{\Gamma_t}$ and $\delta^{q+1,k}_{\Gamma_n}$ are densely defined and closed as, for example,

$$\mathsf{C}^{q,\infty}_{\Gamma_t}(\Omega) \subset \mathsf{H}^{q,k}_{\Gamma_t}(\mathrm{d},\Omega) \subset \mathsf{H}^{q,k}_{\Gamma_t}(\Omega) = \overline{\mathsf{C}^{q,\infty}_{\Gamma_t}(\Omega)}^{\mathsf{H}^{q,k}(\Omega)},$$

and that indeed the complex properties $R(\mathrm{d}^{q-1,k}_{\Gamma_t}) \subset N(\mathrm{d}^{q,k}_{\Gamma_t})$ and $R(\delta^{q+1,k}_{\Gamma_n}) \subset N(\delta^{q,k}_{\Gamma_n})$ hold.

Unfortunately, the respectively adjoints

$$(\mathrm{d}^{q,k}_{\Gamma_t})^* : D\left((\mathrm{d}^{q,k}_{\Gamma_t})^*\right) \subset \mathsf{H}^{q+1,k}_{\Gamma_t}(\Omega) \to \mathsf{H}^{q,k}_{\Gamma_t}(\Omega),$$
$$-(\delta^{q+1,k}_{\Gamma_n})^* : D\left((\delta^{q+1,k}_{\Gamma_n})^*\right) \subset \mathsf{H}^{q,k}_{\Gamma_n}(\Omega) \to \mathsf{H}^{q+1,k}_{\Gamma_n}(\Omega)$$

are hard to compute. Therefore, only some parts of the FA-ToolBox from Section 2 apply to the higher-order de Rham complex, and a few results have to proved in a less general setting.

Note that for $E \in D(\mathrm{d}^{q,k}_{\Gamma_t})$ and for $H \in D(\delta^{q+1,k}_{\Gamma}) \subset \mathsf{H}^{q+1,k}_{\Gamma_t}(\delta,\Omega) \cap \mathsf{H}^{q+1,k}_{\Gamma_n}(\delta,\Omega)$, we have

$$\langle \mathrm{d}E, H\rangle_{\mathsf{H}^{q+1,k}_{\Gamma_t}(\Omega)} = \sum_{|\alpha|\le k} \langle \partial^\alpha \mathrm{d}E, \partial^\alpha H\rangle_{\mathsf{L}^{q+1,2}(\Omega)} = -\sum_{|\alpha|\le k} \langle \partial^\alpha E, \partial^\alpha \delta H\rangle_{\mathsf{L}^{q,2}(\Omega)} = -\langle E, \delta H\rangle_{\mathsf{H}^{q,k}_{\Gamma_t}(\Omega)}$$

by Lemma 3.4.

Remark 4.2 (Higher-order adjoints for the de Rham complex). It holds $-\delta^{q+1,k}_{\Gamma} \subset (\mathrm{d}^{q,k}_{\Gamma_t})^*$ and $-\mathrm{d}^{q-1,k}_{\Gamma} \subset (\delta^{q,k}_{\Gamma_n})^*$, that is,

$$D(\delta^{q+1,k}_{\Gamma}) \subset D\left((\mathrm{d}^{q,k}_{\Gamma_t})^*\right) \quad \text{and} \quad (\mathrm{d}^{q,k}_{\Gamma_t})^*|_{D(\delta^{q+1,k}_{\Gamma})} = -\delta^{q+1,k}_{\Gamma},$$
$$D(\mathrm{d}^{q-1,k}_{\Gamma}) \subset D\left((\delta^{q,k}_{\Gamma_n})^*\right) \quad \text{and} \quad (\delta^{q,k}_{\Gamma_n})^*|_{D(\mathrm{d}^{q-1,k}_{\Gamma})} = -\mathrm{d}^{q-1,k}_{\Gamma}.$$

Note that, here, we identify $-\delta^{q+1,k}_{\Gamma}$ with $-\delta^{q+1,k}_{\Gamma} : D(\delta^{q+1,k}_{\Gamma}) \subset \mathsf{H}^{q+1,k}_{\Gamma_t}(\Omega) \to \mathsf{H}^{q,k}_{\Gamma_t}(\Omega)$, which is not densely defined. The same holds for $-\mathrm{d}^{q-1,k}_{\Gamma}$.

4.3 Regular potentials without boundary conditions

The next lemma generalises Lemma 3.9 and ensures the existence of regular $\mathsf{H}^{q,k}_{\emptyset}(\Omega)$-potentials without boundary conditions for strong Lipschitz domains.

Lemma 4.3 (regular potential for d without boundary condition). *Let $\Omega \subset \mathbb{R}^d$ be a bounded strong Lipschitz domain and let $k \geq 0$ and $q \in \{1, \dots, d\}$. Then there exists a bounded linear regular potential operator*

$$\mathcal{P}^{q,k}_{\mathrm{d},\emptyset} : \mathsf{H}^{q,k}_{\emptyset,0}(\mathrm{d},\Omega) \cap \mathcal{H}^{q}_{\emptyset,\Gamma,\mathrm{id}}(\Omega)^{\perp_{\mathsf{L}^{q,2}(\Omega)}} \to \mathsf{H}^{q-1,k+1}_{0}(\delta,\mathbb{R}^d),$$

such that $\mathrm{d}\mathcal{P}^{q,k}_{\mathrm{d},\emptyset} = \mathrm{id}|_{\mathsf{H}^{q,k}_{\emptyset,0}(\mathrm{d},\Omega)\cap\mathcal{H}^{q}_{\emptyset,\Gamma,\mathrm{id}}(\Omega)^{\perp_{\mathsf{L}^{q,2}(\Omega)}}}$, that is, for all $E \in \mathsf{H}^{q,k}_{\emptyset,0}(\mathrm{d},\Omega) \cap \mathcal{H}^{q}_{\emptyset,\Gamma,\mathrm{id}}(\Omega)^{\perp_{\mathsf{L}^{q,2}(\Omega)}}$

$$\mathrm{d}\mathcal{P}^{q,k}_{\mathrm{d},\emptyset}E = E \text{ in } \Omega.$$

In particular, the bounded regular potential representations

$$R(\mathrm{d}^{q-1,k}_{\emptyset}) = \mathsf{H}^{q,k}_{\emptyset,0}(\mathrm{d},\Omega) \cap \mathcal{H}^{q}_{\emptyset,\Gamma,\mathrm{id}}(\Omega)^{\perp_{\mathsf{L}^{q,2}(\Omega)}} = \mathrm{d}\mathsf{H}^{q-1,k}_{\emptyset}(\mathrm{d},\Omega) = \mathrm{d}\mathsf{H}^{q-1,k+1}_{\emptyset}(\Omega) = \mathrm{d}\mathsf{H}^{q-1,k+1}_{\emptyset,0}(\delta,\Omega)$$

hold, and the potentials can be chosen such that they depend continuously on the data. Especially, these are closed subspaces of $\mathsf{H}^{q,k}_{\emptyset}(\Omega) = \mathsf{H}^{q,k}(\Omega)$, and $\mathcal{P}^{q,k}_{\mathrm{d},\emptyset}$ is a right inverse to d. By a simple cut-off technique, $\mathcal{P}^{q,k}_{\mathrm{d},\emptyset}$ may be modified to

$$\mathcal{P}^{q,k}_{\mathrm{d},\emptyset} : \mathsf{H}^{q,k}_{\emptyset,0}(\mathrm{d},\Omega) \cap \mathcal{H}^{q}_{\emptyset,\Gamma,\mathrm{id}}(\Omega)^{\perp_{\mathsf{L}^{q,2}(\Omega)}} \to \mathsf{H}^{q-1,k+1}(\delta,\mathbb{R}^d)$$

such that $\mathcal{P}^{q,k}_{\mathrm{d},\emptyset}E$ has a fixed compact support in $\mathbb{R}^d$ for all $E \in \mathsf{H}^{q,k}_{\emptyset,0}(\mathrm{d},\Omega) \cap \mathcal{H}^{q}_{\emptyset,\Gamma,\mathrm{id}}(\Omega)^{\perp_{\mathsf{L}^{q,2}(\Omega)}}$.

Proof. Lemma 3.9 shows the assertions for $k = 0$ and $\mathcal{P}^{q,0}_{\mathrm{d},\emptyset}$. Moreover, the inclusions

$$\mathrm{d}\mathsf{H}^{q-1,k+1}_{\emptyset,0}(\delta,\Omega) \subset \mathrm{d}\mathsf{H}^{q-1,k+1}_{\emptyset}(\Omega) \subset \mathrm{d}\mathsf{H}^{q-1,k}_{\emptyset}(\mathrm{d},\Omega) \subset \mathsf{H}^{q,k}_{\emptyset,0}(\mathrm{d},\Omega) \cap \mathcal{H}^{q}_{\emptyset,\Gamma,\mathrm{id}}(\Omega)^{\perp_{\mathsf{L}^{q,2}(\Omega)}}$$

hold. Suppose $E \in \mathsf{H}^{q,k}_{\emptyset,0}(\mathrm{d},\Omega) \cap \mathcal{H}^{q}_{\emptyset,\Gamma,\mathrm{id}}(\Omega)^{\perp_{\mathsf{L}^{q,2}(\Omega)}}$, $k \geq 1$. Then $E \in \mathsf{H}^{q,k-1}_{\emptyset,0}(\mathrm{d},\Omega) \cap \mathcal{H}^{q}_{\emptyset,\Gamma,\mathrm{id}}(\Omega)^{\perp_{\mathsf{L}^{q,2}(\Omega)}}$. By assumption of induction, there exists $\mathcal{P}^{q,k-1}_{\mathrm{d},\emptyset}E \in \mathsf{H}^{q-1,k}_{\emptyset}(\Omega)$ with $\mathrm{d}\mathcal{P}^{q,k-1}_{\mathrm{d},\emptyset}E = E$ in Ω and

$$|\mathcal{P}^{q,k-1}_{\mathrm{d},\emptyset}E|_{\mathsf{H}^{q-1,k}(\Omega)} \leq c|E|_{\mathsf{H}^{q,k-1}(\Omega)}.$$

Hence, $\mathcal{P}^{q,k-1}_{\mathrm{d},\emptyset}E \in \mathsf{H}^{q-1,k}_{\emptyset}(\mathrm{d},\Omega)$, and by Lemma 3.5, we have $\mathcal{E}\mathcal{P}^{q,k-1}_{\mathrm{d},\emptyset}E \in \mathsf{H}^{q-1,k}(\mathrm{d},\mathbb{R}^d)$ with compact support and

$$|\mathcal{E}\mathcal{P}^{q,k-1}_{\mathrm{d},\emptyset}E|_{\mathsf{H}^{q-1,k}(\mathrm{d},\mathbb{R}^d)} \leq c|\mathcal{P}^{q,k-1}_{\mathrm{d},\emptyset}E|_{\mathsf{H}^{q-1,k}(\mathrm{d},\Omega)} \leq c\left(|\mathcal{P}^{q,k-1}_{\mathrm{d},\emptyset}E|_{\mathsf{H}^{q-1,k}(\Omega)} + |E|_{\mathsf{H}^{q,k}(\Omega)}\right).$$

Using Lemma 3.7, we obtain a uniquely determined

$$\mathcal{P}^{q,k}_{\mathrm{d},\emptyset}E := \pi_{q-1,\mathbb{R}^d}\mathcal{E}\mathcal{P}^{q,k-1}_{\mathrm{d},\emptyset}E \in \mathsf{H}^{q-1,0}(\mathrm{d},\mathbb{R}^d) \cap \mathsf{H}^{q-1,0}_{0}(\delta,\mathbb{R}^d)$$

with $\mathrm{d}\mathcal{P}^{q,k}_{\mathrm{d},\emptyset}E = \mathrm{d}\mathcal{E}\mathcal{P}^{q,k-1}_{\mathrm{d},\emptyset}E \in \mathsf{H}^{q,k}(\mathbb{R}^d)$. Lemma 3.8 shows $\mathcal{P}^{q,k}_{\mathrm{d},\emptyset}E \in \mathsf{H}^{q-1,k+1}(\mathbb{R}^d)$ with

$$|\mathcal{P}^{q,k}_{\mathrm{d},\emptyset}E|_{\mathsf{H}^{q-1,k+1}(\mathbb{R}^d)} \leq c\left(|\mathcal{P}^{q,k}_{\mathrm{d},\emptyset}E|_{\mathsf{L}^{q-1,2}(\mathbb{R}^d)} + |\mathrm{d}\mathcal{E}\mathcal{P}^{q,k-1}_{\mathrm{d},\emptyset}E|_{\mathsf{H}^{q,k}(\mathbb{R}^d)}\right) \leq c|\mathcal{E}\mathcal{P}^{q,k-1}_{\mathrm{d},\emptyset}E|_{\mathsf{H}^{q-1,k}(\mathrm{d},\mathbb{R}^d)}.$$

Finally, $\mathcal{P}^{q,k}_{\mathrm{d},\emptyset}E \in \mathsf{H}^{q-1,k+1}_{0}(\delta,\mathbb{R}^d)$ meets our needs as it holds $|\mathcal{P}^{q,k}_{\mathrm{d},\emptyset}E|_{\mathsf{H}^{q-1,k+1}(\mathbb{R}^d)} \leq c|E|_{\mathsf{H}^{q,k}(\Omega)}$ and $\mathrm{d}\mathcal{P}^{q,k}_{\mathrm{d},\emptyset}E = \mathrm{d}\mathcal{E}\mathcal{P}^{q,k-1}_{\mathrm{d},\emptyset}E = \mathrm{d}\mathcal{P}^{q,k-1}_{\mathrm{d},\emptyset}E = E$ in Ω. □

By Hodge ⋆-duality, we get a corresponding result for the δ-operator, cf. Lemma 4.7.

4.4 Regular potentials and decompositions with boundary conditions

Now we construct regular $\mathsf{H}^{q,k}(\Omega)$-potentials with (partial) boundary conditions. Recall the definitions of Section 3.1 for the different assumptions on the domain $\Omega \subset \mathbb{R}^d$.

4.4.1 Extendable domains

Lemma 4.4 (regular potential for d with partial boundary condition for extendable domains)**.** *Let (Ω, Γ_t) be an extendable bounded strong Lipschitz pair and let $1 \leq q \leq d-1$ as well as $k \geq 0$. Then there exists a bounded linear regular potential operator*

$$\mathcal{P}^{q,k}_{\mathrm{d},\Gamma_t} : \boldsymbol{H}^{q,k}_{\Gamma_t,0}(\mathrm{d},\Omega) \to \mathsf{H}^{q-1,k+1}(\mathbb{R}^d) \cap \mathsf{H}^{q-1,k+1}_{\Gamma_t}(\Omega),$$

such that $\mathrm{d}\mathcal{P}^{q,k}_{\mathrm{d},\Gamma_t} = \mathrm{id}|_{\boldsymbol{H}^{q,k}_{\Gamma_t,0}(\mathrm{d},\Omega)}$*, that is, for all* $E \in \mathbf{H}^{q,k}_{\Gamma_t,0}(\mathrm{d},\Omega)$

$$\mathrm{d}\mathcal{P}^{q,k}_{\mathrm{d},\Gamma_t} E = E \text{ in } \Omega.$$

In particular, the bounded regular potential representation

$$\mathbf{H}^{q,k}_{\Gamma_t,0}(\mathrm{d},\Omega) = \mathsf{H}^{q,k}_{\Gamma_t,0}(\mathrm{d},\Omega) = \mathrm{d}\mathsf{H}^{q-1,k+1}_{\Gamma_t}(\Omega) = \mathrm{d}\mathsf{H}^{q-1,k}_{\Gamma_t}(\mathrm{d},\Omega) = R(\mathrm{d}^{q-1,k}_{\Gamma_t})$$

holds, and the potentials can be chosen such that they depend continuously on the data. Especially, these spaces are closed subspaces of $\mathsf{H}^{q,k}_{\emptyset}(\Omega) = \mathsf{H}^{q,k}(\Omega)$*, and* $\mathcal{P}^{q,k}_{\mathrm{d},\Gamma_t}$ *is a right inverse to d. Without loss of generality,* $\mathcal{P}^{q,k}_{\mathrm{d},\Gamma_t}$ *maps to forms with a fixed compact support in* $\mathbb{R}^d$.

The results extend literally to the case $q=d$ if $\Gamma_t \neq \Gamma$, and the case $q=0$ is trivial since $\mathbf{H}^{0,k}_{\Gamma_t,0}(\mathrm{d},\Omega) = \mathbb{R}_{\Gamma_t}$. In the special case $q=d$ and $\Gamma_t = \Gamma$, the results still remain valid if

$$\mathbf{H}^{d,k}_{\Gamma,0}(\mathrm{d},\Omega) = \mathbf{H}^{d,k}_{\Gamma}(\Omega), \qquad \mathsf{H}^{d,k}_{\Gamma,0}(\mathrm{d},\Omega) = \mathsf{H}^{d,k}_{\Gamma}(\Omega)$$

are replaced by the slightly smaller spaces

$$\mathbf{H}^{d,k}_{\Gamma}(\Omega) \cap (*\ \mathbb{R})^{\perp_{\mathsf{L}^{d,2}(\Omega)}}, \qquad \mathsf{H}^{d,k}_{\Gamma}(\Omega) \cap (*\ \mathbb{R})^{\perp_{\mathsf{L}^{d,2}(\Omega)}},$$

respectively.

Proof. The case $\Gamma_t = \emptyset$ is done in Lemma 4.3. For $\Gamma_t \neq \emptyset$, suppose $E \in \mathbf{H}^{q,k}_{\Gamma_t,0}(\mathrm{d},\Omega)$ and define $\widetilde{E} \in \mathsf{L}^{q,2}(\widetilde{\Omega})$ as extension of E by zero to $\widehat{\Omega}$. By definition, we see $\widetilde{E} \in \mathsf{H}^{q,k}_{\emptyset,0}(\mathrm{d},\widetilde{\Omega})$. Since $\widetilde{\Omega}$ is bounded, strong Lipschitz, and topologically trivial, in particular $\mathcal{H}^q_{\emptyset,\widetilde{\Gamma},\mathrm{id}}(\widetilde{\Omega}) = \{0\}$, Lemma 4.3 yields a regular potential $\mathcal{P}^{q,k}_{\mathrm{d},\emptyset}\widetilde{E} \in \mathsf{H}^{q-1,k+1}_0(\delta,\mathbb{R}^d) \subset \mathsf{H}^{q-1,k+1}(\mathbb{R}^d)$ with $\mathrm{d}\mathcal{P}^{q,k}_{\mathrm{d},\emptyset}\widetilde{E} = \widetilde{E}$ in $\widetilde{\Omega}$ and

$$c|\mathcal{P}^{q,k}_{\mathrm{d},\emptyset}\widetilde{E}|_{\mathsf{H}^{q-1,k+1}(\mathbb{R}^d)} \leq |\widetilde{E}|_{\mathsf{H}^{q,k}(\widetilde{\Omega})} = |E|_{\mathsf{H}^{q,k}(\Omega)}.$$

Let $\iota_{\widehat{\Omega}}$ denote the restriction to $\widehat{\Omega}$. Then $\iota_{\widehat{\Omega}}\mathcal{P}^{q,k}_{\mathrm{d},\emptyset}\widetilde{E} \in \mathsf{H}^{q-1,k+1}_{\emptyset}(\widehat{\Omega})$ and $\mathrm{d}\iota_{\widehat{\Omega}}\mathcal{P}^{q,k}_{\mathrm{d},\emptyset}\widetilde{E} = \iota_{\widehat{\Omega}}\widetilde{E} = 0$ in $\widehat{\Omega}$, that is, $\iota_{\widehat{\Omega}}\mathcal{P}^{q,k}_{\mathrm{d},\emptyset}\widetilde{E} \in \mathsf{H}^{q-1,k+1}_{\emptyset,0}(\mathrm{d},\widehat{\Omega})$. Using Lemma 4.3 again, this time in $\widehat{\Omega}$, which is bounded, strong Lipschitz, and topologically trivial as well, we obtain $\mathcal{P}^{q-1,k+1}_{\mathrm{d},\emptyset}\iota_{\widehat{\Omega}}\mathcal{P}^{q,k}_{\mathrm{d},\emptyset}\widetilde{E} \in \mathsf{H}^{q-2,k+2}(\mathbb{R}^d)$ with $\mathrm{d}\mathcal{P}^{q-1,k+1}_{\mathrm{d},\emptyset}\iota_{\widehat{\Omega}}\mathcal{P}^{q,k}_{\mathrm{d},\emptyset}\widetilde{E} = \iota_{\widehat{\Omega}}\mathcal{P}^{q,k}_{\mathrm{d},\emptyset}\widetilde{E}$ in $\widehat{\Omega}$ and

$$|\mathcal{P}^{q-1,k+1}_{\mathrm{d},\emptyset}\iota_{\widehat{\Omega}}\mathcal{P}^{q,k}_{\mathrm{d},\emptyset}\widetilde{E}|_{\mathsf{H}^{q-2,k+2}(\mathbb{R}^d)} \leq c|\mathcal{P}^{q,k}_{\mathrm{d},\emptyset}\widetilde{E}|_{\mathsf{H}^{q-1,k+1}(\widehat{\Omega})}.$$

Then

$$\begin{aligned} \mathcal{P}^{q,k}_{\mathrm{d},\Gamma_t} : \mathsf{H}^{q,k}_{\Gamma_t,0}(\mathrm{d},\Omega) &\to \mathsf{H}^{q-1,k+1}(\mathbb{R}^d) \\ E &\mapsto \mathcal{P}^{q,k}_{\mathrm{d},\emptyset}\widetilde{E} - \mathrm{d}(\mathcal{P}^{q-1,k+1}_{\mathrm{d},\emptyset}\iota_{\widehat{\Omega}}\mathcal{P}^{q,k}_{\mathrm{d},\emptyset}\widetilde{E}) \end{aligned}$$

is linear and bounded since

$$|\mathcal{P}^{q,k}_{\mathrm{d},\Gamma_t}E|_{\mathsf{H}^{q-1,k+1}(\mathbb{R}^d)} \leq |\mathcal{P}^{q,k}_{\mathrm{d},\emptyset}\widetilde{E}|_{\mathsf{H}^{q-1,k+1}(\mathbb{R}^d)} + |\mathcal{P}^{q-1,k+1}_{\mathrm{d},\emptyset}\iota_{\widehat{\Omega}}\mathcal{P}^{q,k}_{\mathrm{d},\emptyset}\widetilde{E}|_{\mathsf{H}^{q-2,k+2}(\mathbb{R}^d)} \leq c|E|_{\mathsf{H}^{q,k}(\Omega)}.$$

Since $\mathcal{P}^{q,k}_{\mathrm{d},\Gamma_t}E = 0$ in $\widehat{\Omega}$, we obtain by standard arguments for Sobolev spaces $\mathcal{P}^{q,k}_{\mathrm{d},\Gamma_t}E \in \mathsf{H}^{q-1,k+1}_{\Gamma_t}(\Omega)$, cf. Bauer et al.[18, Lemma 2.14] (weak and strong boundary conditions coincide for $\mathsf{H}^{q,k}(\Omega)$). Moreover, it holds $\mathrm{d}\mathcal{P}^{q,k}_{\mathrm{d},\Gamma_t}E = \mathrm{d}\mathcal{P}^{q,k}_{\mathrm{d},\emptyset}\widetilde{E} = \widetilde{E}$ in $\widetilde{\Omega}$, in particular, $\mathrm{d}\mathcal{P}^{q,k}_{\mathrm{d},\Gamma_t}E = E$ in Ω. Finally,

$$\mathrm{d}\mathsf{H}^{q-1,k+1}_{\Gamma_t}(\Omega) \subset \mathrm{d}\mathsf{H}^{q-1,k}_{\Gamma_t}(\mathrm{d},\Omega) \subset \mathsf{H}^{q,k}_{\Gamma_t,0}(\mathrm{d},\Omega) \subset \boldsymbol{\mathsf{H}}^{q,k}_{\Gamma_t,0}(\mathrm{d},\Omega) \subset \mathrm{d}\mathsf{H}^{q-1,k+1}_{\Gamma_t}(\Omega),$$

completing the proof of the main part. In the special case $q = d$ and $\Gamma_t = \Gamma$, we also have to take care of the constant d-forms in $* \mathbb{R}$. □

Hodge $\star$-duality yields a corresponding result for the δ-operator, cf. Lemma 4.8 (i).

Lemma 4.5 (regular decompositions for d with partial boundary condition for extendable domains)**.** *Let (Ω, Γ_t) be an extendable bounded strong Lipschitz pair and let $k \geq 0$. Then the bounded regular decompositions*

$$\begin{aligned}\boldsymbol{H}^{q,k}_{\Gamma_t}(\mathrm{d},\Omega) = \mathsf{H}^{q,k}_{\Gamma_t}(\mathrm{d},\Omega) &= \mathsf{H}^{q,k+1}_{\Gamma_t}(\Omega) + \mathrm{d}\mathsf{H}^{q-1,k+1}_{\Gamma_t}(\Omega)\\ &= \mathcal{Q}^{q,k}_{\mathrm{d},\Gamma_t,1}\mathsf{H}^{q,k}_{\Gamma_t}(\mathrm{d},\Omega) \dotplus \mathrm{d}\mathcal{Q}^{q,k}_{\mathrm{d},\Gamma_t,0}\mathsf{H}^{q,k}_{\Gamma_t}(\mathrm{d},\Omega)\\ &= \mathcal{Q}^{q,k}_{\mathrm{d},\Gamma_t,1}\mathsf{H}^{q,k}_{\Gamma_t}(\mathrm{d},\Omega) \dotplus \mathrm{d}\mathsf{H}^{q-1,k+1}_{\Gamma_t}(\Omega)\\ &= \mathcal{Q}^{q,k}_{\mathrm{d},\Gamma_t,1}\mathsf{H}^{q,k}_{\Gamma_t}(\mathrm{d},\Omega) \dotplus \mathsf{H}^{q,k}_{\Gamma_t,0}(\mathrm{d},\Omega)\end{aligned}$$

hold with bounded linear regular decomposition operators

$$\begin{aligned}\mathcal{Q}^{q,k}_{\mathrm{d},\Gamma_t,1} &:= \mathcal{P}^{q+1,k}_{\mathrm{d},\Gamma_t}\mathrm{d} : \mathsf{H}^{q,k}_{\Gamma_t}(\mathrm{d},\Omega) \to \mathsf{H}^{q,k+1}_{\Gamma_t}(\Omega),\\ \mathcal{Q}^{q,k}_{\mathrm{d},\Gamma_t,0} &:= \mathcal{P}^{q,k}_{\mathrm{d},\Gamma_t}(1 - \mathcal{P}^{q+1,k}_{\mathrm{d},\Gamma_t}\mathrm{d}) : \mathsf{H}^{q,k}_{\Gamma_t}(\mathrm{d},\Omega) \to \mathsf{H}^{q-1,k+1}_{\Gamma_t}(\Omega).\end{aligned}$$

More precisely, it holds $\boldsymbol{H}^{q,k}_{\Gamma_t}(\mathrm{d},\Omega) = \mathsf{H}^{q,k}_{\Gamma_t}(\mathrm{d},\Omega)$ and $\mathcal{Q}^{q,k}_{\mathrm{d},\Gamma_t,1} + \mathrm{d}\mathcal{Q}^{q,k}_{\mathrm{d},\Gamma_t,0} = \mathrm{id}|_{\mathsf{H}^{q,k}_{\Gamma_t}(\mathrm{d},\Omega)}$, that is,

$$E = \mathcal{Q}^{q,k}_{\mathrm{d},\Gamma_t,1}E + \mathrm{d}\mathcal{Q}^{q,k}_{\mathrm{d},\Gamma_t,0}E \in \mathsf{H}^{q,k+1}_{\Gamma_t}(\Omega) + \mathrm{d}\mathsf{H}^{q-1,k+1}_{\Gamma_t}(\Omega)$$

for all $E \in \mathsf{H}^{q,k}_{\Gamma_t}(\mathrm{d},\Omega)$. Moreover, it holds $\mathrm{d}\mathcal{Q}^{q,k}_{\mathrm{d},\Gamma_t,1} = \mathrm{d}^{q,k}_{\Gamma_t}$, and thus, $\mathsf{H}^{q,k}_{\Gamma_t,0}(\mathrm{d},\Omega)$ is invariant under $\mathcal{Q}^{q,k}_{\mathrm{d},\Gamma_t,1}$. Note that for the ranges $\mathcal{Q}^{q,k}_{\mathrm{d},\Gamma_t,1}\mathsf{H}^{q,k}_{\Gamma_t}(\mathrm{d},\Omega) = R(\mathcal{Q}^{q,k}_{\mathrm{d},\Gamma_t,1}) = R(\mathcal{P}^{q+1,k}_{\mathrm{d},\Gamma_t})$ as well as $\mathcal{Q}^{q,k}_{\mathrm{d},\Gamma_t,0}\mathsf{H}^{q,k}_{\Gamma_t}(\mathrm{d},\Omega) = R(\mathcal{Q}^{q,k}_{\mathrm{d},\Gamma_t,0}) = R(\mathcal{P}^{q,k}_{\mathrm{d},\Gamma_t})$ hold.

The proof follows by Corollary 2.20 and Lemma 4.4. For convenience, we give a self-contained proof here.

Proof. Let $E \in \boldsymbol{\mathsf{H}}^{q,k}_{\Gamma_t}(\mathrm{d},\Omega)$. Then $\mathrm{d}E \in \boldsymbol{\mathsf{H}}^{q+1,k}_{\Gamma_t,0}(\mathrm{d},\Omega)$, and we see $\mathcal{P}^{q+1,k}_{\mathrm{d},\Gamma_t}\mathrm{d}E \in \mathsf{H}^{q,k+1}_{\Gamma_t}(\Omega)$ with $\mathrm{d}\mathcal{P}^{q+1,k}_{\mathrm{d},\Gamma_t}\mathrm{d}E = \mathrm{d}E$ by Lemma 4.4. Thus, $E - \mathcal{P}^{q+1,k}_{\mathrm{d},\Gamma_t}\mathrm{d}E \in \boldsymbol{\mathsf{H}}^{q,k}_{\Gamma_t,0}(\mathrm{d},\Omega) = \mathrm{d}\mathsf{H}^{q-1,k+1}_{\Gamma_t}(\Omega)$ and $\mathcal{P}^{q,k}_{\mathrm{d},\Gamma_t}(E - \mathcal{P}^{q+1,k}_{\mathrm{d},\Gamma_t}\mathrm{d}E) \in \mathsf{H}^{q-1,k+1}_{\Gamma_t}(\Omega)$ with $\mathrm{d}\mathcal{P}^{q,k}_{\mathrm{d},\Gamma_t}(E - \mathcal{P}^{q+1,k}_{\mathrm{d},\Gamma_t}\mathrm{d}E) = E - \mathcal{P}^{q+1,k}_{\mathrm{d},\Gamma_t}\mathrm{d}E$ by Lemma 4.4. This yields

$$E = \mathcal{P}^{q+1,k}_{\mathrm{d},\Gamma_t}\mathrm{d}E + \mathrm{d}\mathcal{P}^{q,k}_{\mathrm{d},\Gamma_t}(1 - \mathcal{P}^{q+1,k}_{\mathrm{d},\Gamma_t}\mathrm{d})E \in \mathsf{H}^{q,k+1}_{\Gamma_t}(\Omega) + \mathrm{d}\mathsf{H}^{q-1,k+1}_{\Gamma_t}(\Omega) \subset \mathsf{H}^{q,k}_{\Gamma_t}(\mathrm{d},\Omega),$$

which proves the regular decompositions and also the assertions about the bounded linear regular decomposition operators. To show the directness of the sums, let $H = \mathcal{P}^{q+1,k}_{\mathrm{d},\Gamma_t}\mathrm{d}E \in \mathsf{H}^{q,0}_{\Gamma_t,0}(\mathrm{d},\Omega)$ with some $E \in \mathsf{H}^{q,k}_{\Gamma_t}(\mathrm{d},\Omega)$. Then $0 = \mathrm{d}H = \mathrm{d}E$ as $\mathrm{d}E \in \mathsf{H}^{q+1,k}_{\Gamma_t,0}(\mathrm{d},\Omega)$ and thus $H = 0$. □

Again, by Hodge $\star$-duality, we get a corresponding result for the δ-operator, cf. Lemma 4.8 (ii).

4.4.2 General Lipschitz domains

Lemma 4.6 (regular decompositions for d with partial boundary condition). *Let (Ω, Γ_t) be a bounded strong Lipschitz pair and let $k \geq 0$. Then the bounded regular decompositions*

$$\boldsymbol{H}^{q,k}_{\Gamma_t}(\mathrm{d},\Omega) = \mathsf{H}^{q,k}_{\Gamma_t}(\mathrm{d},\Omega) = \mathsf{H}^{q,k+1}_{\Gamma_t}(\Omega) + \mathrm{d}\mathsf{H}^{q-1,k+1}_{\Gamma_t}(\Omega)$$

hold with bounded linear regular decomposition operators

$$\mathcal{Q}^{q,k}_{\mathrm{d},\Gamma_t,1} : \mathsf{H}^{q,k}_{\Gamma_t}(\mathrm{d},\Omega) \to \mathsf{H}^{q,k+1}_{\Gamma_t}(\Omega), \qquad \mathcal{Q}^{q,k}_{\mathrm{d},\Gamma_t,0} : \mathsf{H}^{q,k}_{\Gamma_t}(\mathrm{d},\Omega) \to \mathsf{H}^{q-1,k+1}_{\Gamma_t}(\Omega)$$

satisfying $\mathcal{Q}^{q,k}_{\mathrm{d},\Gamma_t,1} + \mathrm{d}\mathcal{Q}^{q,k}_{\mathrm{d},\Gamma_t,0} = \mathrm{id}_{\mathsf{H}^{q,k}_{\Gamma_t}(\mathrm{d},\Omega)}$. In particular, weak and strong boundary conditions coincide. Moreover, it holds $\mathrm{d}\mathcal{Q}^{q,k}_{\mathrm{d},\Gamma_t,1} = \mathrm{d}^{q,k}_{\Gamma_t}$, and thus, $\mathsf{H}^{q,k}_{\Gamma_t,0}(\mathrm{d},\Omega)$ is invariant under $\mathcal{Q}^{q,k}_{\mathrm{d},\Gamma_t,1}$.

Proof. According to Lemma 3.1, let us introduce a partition of unity (U_ℓ, χ_ℓ) as in Bauer et al.[18, Section 4.2] or Bauer et al,[17, Section 4.2] such that $(\Omega_\ell, \widehat{\Gamma}_{t,\ell})$ is an extendable bounded strong Lipschitz pair for all $l = 1, \dots, L_+$. Using the notations from Bauer et al,[18] we have

$$\Omega_\ell = \Omega \cap U_\ell, \qquad \Sigma_\ell = \partial\Omega_\ell \setminus \Gamma, \qquad \Gamma_{t,\ell} = \Gamma_t \cap U_\ell, \qquad \widehat{\Gamma}_{t,\ell} = \mathrm{int}(\overline{\Gamma_{t,\ell} \cup \Sigma_\ell}).$$

Maybe $U_0 = \Omega$ has to be replaced by more neighbourhoods $U_{-L_-}, \dots, U_0$ to ensure that all pairs $(\Omega_\ell, \widehat{\Gamma}_{t,\ell})$, $\ell = -L_-, \dots, L_+$, are topologically trivial. Note that for all 'inner' indices $\ell = -L_-, \dots, 0$ we have $\Omega_\ell = U_\ell$ as well as $\widehat{\Gamma}_{t,\ell} = \Sigma_\ell = \partial\Omega_\ell = \partial U_\ell$.

Then for $E \in \mathbf{H}^{q,k}_{\Gamma_t}(\mathrm{d},\Omega)$, we have $\chi_\ell E \in \mathbf{H}^{q,k}_{\widehat{\Gamma}_{t,\ell}}(\mathrm{d},\Omega_\ell) = \mathsf{H}^{q,k}_{\widehat{\Gamma}_{t,\ell}}(\mathrm{d},\Omega_\ell)$ for all ℓ, and Lemma 4.5 shows the bounded regular decompositions

$$\chi_\ell E = E_\ell + \mathrm{d}H_\ell \in \mathsf{H}^{q,k+1}_{\widehat{\Gamma}_{t,\ell}}(\Omega_\ell) + \mathrm{d}\mathsf{H}^{q-1,k+1}_{\widehat{\Gamma}_{t,\ell}}(\Omega_\ell)$$

with E_ℓ and H_ℓ depending continuously on $\chi_\ell E$. Extending E_ℓ and H_ℓ by zero to Ω yields forms $\widetilde{E}_\ell \in \mathsf{H}^{q,k+1}_{\Gamma_t}(\Omega)$ and $\widetilde{H}_\ell \in \mathsf{H}^{q-1,k+1}_{\Gamma_t}(\Omega)$ as well as the representation

$$\mathbf{H}^{q,k}_{\Gamma_t}(\mathrm{d},\Omega) \ni E = \sum_\ell \chi_k E = \sum_\ell \widetilde{E}_\ell + \mathrm{d}\sum_\ell \widetilde{H}_\ell \in \mathsf{H}^{q,k+1}_{\Gamma_t}(\Omega) + \mathrm{d}\mathsf{H}^{q-1,k+1}_{\Gamma_t}(\Omega) \subset \mathsf{H}^{q,k}_{\Gamma_t}(\mathrm{d},\Omega).$$

As all operations have been linear and continuous we set

$$\mathcal{Q}^{q,k}_{\mathrm{d},\Gamma_t,1} E := \sum_\ell \widetilde{E}_\ell \in \mathsf{H}^{q,k+1}_{\Gamma_t}(\Omega), \qquad \mathcal{Q}^{q,k}_{\mathrm{d},\Gamma_t,0} E := \sum_\ell \widetilde{H}_\ell \in \mathsf{H}^{q-1,k+1}_{\Gamma_t}(\Omega),$$

and obtain the assertions. □

Hodge $\star$-duality shows a corresponding result for the δ-operator, cf. Lemma 4.9.

Corollary 4.7 (regular decompositions for d with partial boundary condition). *Let (Ω, Γ_t) be a bounded strong Lipschitz pair and let $k \geq 0$. Then the regular potential representations*

$$R(\mathrm{d}^{q-1,k}_{\Gamma_t}) = \mathrm{d}\mathsf{H}^{q-1,k}_{\Gamma_t}(\mathrm{d},\Omega) = \mathrm{d}\mathsf{H}^{q-1,k+1}_{\Gamma_t}(\Omega) = \mathsf{H}^{q,k}_{\Gamma_t,0}(\mathrm{d},\Omega) \cap \mathcal{H}^q_{\Gamma_t,\Gamma_n,\varepsilon}(\Omega)^{\perp_{\mathsf{L}^{q,2}_\varepsilon(\Omega)}},$$
$$R(\delta^{q+1,k}_{\Gamma_n}) = \delta\mathsf{H}^{q+1,k}_{\Gamma_n}(\delta,\Omega) = \delta\mathsf{H}^{q+1,k+1}_{\Gamma_n}(\Omega) = \mathsf{H}^{q,k}_{\Gamma_n,0}(\delta,\Omega) \cap \mathcal{H}^q_{\Gamma_t,\Gamma_n,\varepsilon}(\Omega)^{\perp_{\mathsf{L}^{q,2}(\Omega)}}$$

hold. In particular, these spaces are closed subspaces of $\mathsf{H}^{q,k}_{\emptyset}(\Omega) = \mathsf{H}^{q,k}(\Omega)$.

Proof. Lemma 4.6 yields

$$R(\mathrm{d}^{q-1,k}_{\Gamma_t}) = \mathrm{d}\mathsf{H}^{q-1,k}_{\Gamma_t}(\mathrm{d},\Omega) = \mathrm{d}\mathsf{H}^{q-1,k+1}_{\Gamma_t}(\Omega) \subset \mathsf{H}^{q,k}_{\Gamma_t,0}(\mathrm{d},\Omega) \cap \mathcal{H}^q_{\Gamma_t,\Gamma_n,\varepsilon}(\Omega)^{\perp_{\mathsf{L}^{q,2}_\varepsilon(\Omega)}}. \tag{16}$$

For $k = 0$, we get by (16) and Lemma 3.6

$$\mathrm{d}\mathsf{H}^{q-1,1}_{\Gamma_t}(\Omega) = \mathrm{d}\mathsf{H}^{q-1,0}_{\Gamma_t}(\mathrm{d},\Omega) = \mathsf{H}^{q,0}_{\Gamma_t,0}(\mathrm{d},\Omega) \cap \mathcal{H}^{q}_{\Gamma_t,\Gamma_n,\varepsilon}(\Omega)^{\perp_{\mathsf{L}^{q,2}_{\varepsilon}(\Omega)}}. \tag{17}$$

Let $E \in \mathsf{H}^{q,k}_{\Gamma_t,0}(\mathrm{d},\Omega) \cap \mathcal{H}^{q}_{\Gamma_t,\Gamma_n,\varepsilon}(\Omega)^{\perp_{\mathsf{L}^{q,2}_{\varepsilon}(\Omega)}}$. By (17), we observe $E \in \mathsf{H}^{q,k}_{\Gamma_t}(\Omega) \cap \mathrm{d}\mathsf{H}^{q-1,1}_{\Gamma_t}(\Omega)$, that is, $E = \mathrm{d}E_1 \in \mathsf{H}^{q,k}_{\Gamma_t}(\Omega)$ with $E_1 \in \mathsf{H}^{q-1,1}_{\Gamma_t}(\Omega)$. Thus, $E_1 \in \mathsf{H}^{q-1,1}_{\Gamma_t}(\mathrm{d},\Omega)$ and $E \in \mathrm{d}\mathsf{H}^{q-1,1}_{\Gamma_t}(\mathrm{d},\Omega)$. By (16), there is $E_2 \in \mathsf{H}^{q-1,2}_{\Gamma_t}(\Omega)$ with $E = \mathrm{d}E_2 \in \mathrm{d}\mathsf{H}^{q-1,k}_{\Gamma_t}(\Omega)$, that is, $E_2 \in \mathsf{H}^{q-1,2}_{\Gamma_t}(\mathrm{d},\Omega)$ as well as $E \in \mathrm{d}\mathsf{H}^{q-1,2}_{\Gamma_t}(\mathrm{d},\Omega)$. After k induction steps, we obtain $E \in \mathrm{d}\mathsf{H}^{q-1,k}_{\Gamma_t}(\mathrm{d},\Omega)$. Hodge $\star$-duality shows the assertions for δ. □

Note that in Corollary 4.7, we claim nothing about bounded regular potential operators, leaving the question of bounded potentials to the next sections.

4.5 | Zero-order mini FA-ToolBox

We shall apply Theorem 2.23 from the FA-ToolBox to the zero-order de Rham complex. In Section 4.1, we have seen that

$$\begin{aligned}
\mathrm{A}_0 &:= \mathrm{d}^{q-1}_{\Gamma_t} : \mathsf{H}^{q-1,0}_{\Gamma_t}(\mathrm{d},\Omega) \subset \mathsf{L}^{q-1,2}(\Omega) \to \mathsf{L}^{q,2}(\Omega),\\
\mathrm{A}_1 &:= \mathrm{d}^{q}_{\Gamma_t} : \mathsf{H}^{q,0}_{\Gamma_t}(\mathrm{d},\Omega) \subset \mathsf{L}^{q,2}(\Omega) \to \mathsf{L}^{q+1,2}(\Omega),\\
\mathrm{A}_0^* &= -\delta^{q}_{\Gamma_n} : \mathsf{H}^{q,0}_{\Gamma_n}(\delta,\Omega) \subset \mathsf{L}^{q,2}(\Omega) \to \mathsf{L}^{q-1,2}(\Omega),\\
\mathrm{A}_1^* &= -\delta^{q+1}_{\Gamma_n} : \mathsf{H}^{q+1,0}_{\Gamma_n}(\delta,\Omega) \subset \mathsf{L}^{q+1,2}(\Omega) \to \mathsf{L}^{q,2}(\Omega)
\end{aligned}$$

are densely defined and closed and form a Hilbert complex of dual pairs, that is, the long primal and dual Hilbert complex (15). Recall also (12) and Definition 2.26 are well as Remark 2.27.

Lemma 4.6 for $k = 0$ yields the bounded regular decomposition

$$D(\mathrm{A}_1) = \mathsf{H}^{q,0}_{\Gamma_t}(\mathrm{d},\Omega) = \mathsf{H}^{q,1}_{\Gamma_t}(\Omega) + \mathrm{d}\mathsf{H}^{q-1,1}_{\Gamma_t}(\Omega) = \mathsf{H}^{+}_{1} + \mathrm{A}_0\mathsf{H}^{+}_{0}$$

with $\mathsf{H}^{+}_{1} := \mathsf{H}^{q,1}_{\Gamma_t}(\Omega)$ and $\mathsf{H}^{+}_{0} := \mathsf{H}^{q-1,1}_{\Gamma_t}(\Omega)$ and $\mathsf{H}_1 := \mathsf{L}^{q,2}(\Omega)$ and $\mathsf{H}_0 := \mathsf{L}^{q-1,2}(\Omega)$. Rellich's selection theorem shows that the assumptions of Lemma 2.22 (i) and Theorem 2.23 as satisfied. Note that it holds $D(\mathrm{d}^{0}_{\Gamma_t}) = \mathsf{H}^{0,1}_{\Gamma_t}(\Omega)$ and $D(\delta^{d}_{\Gamma_n}) = \mathsf{H}^{d,1}_{\Gamma_n}(\Omega)$.

Theorem 4.8 (compact embedding for the de Rham complex)**.** *Let (Ω,Γ_t) be a bounded strong Lipschitz pair. Then for all q, the embedding*

$$D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) = D(\mathrm{d}^{q}_{\Gamma_t}) \cap D(\delta^{q}_{\Gamma_n}) = \mathsf{H}^{q,0}_{\Gamma_t}(\mathrm{d},\Omega) \cap \mathsf{H}^{q,0}_{\Gamma_n}(\delta,\Omega) \hookrightarrow \mathsf{L}^{q,2}(\Omega)$$

is compact. Moreover, the long primal and dual de Rham Hilbert complex (15) is compact. In particular, the complex is closed.

Proof. Apply Theorem 2.23 (i). □

Theorem 4.9 (mini FA-ToolBox for the de Rham complex)**.** *Let (Ω,Γ_t) be a bounded strong Lipschitz pair. Then for all q,*

(i) the ranges $R(\mathrm{d}^{q}_{\Gamma_t})$ and $R(\delta^{q}_{\Gamma_n})$ are closed,
(ii) the inverse operators $(\mathrm{d}^{q}_{\Gamma_t})^{-1}_{\perp}$ and $(\delta^{q}_{\Gamma_n})^{-1}_{\perp}$ are compact,
(iii) the cohomology group $\mathcal{H}^{q}_{\Gamma_t,\Gamma_n,\mathrm{id}}(\Omega) = \mathsf{H}^{q}_{\Gamma_t,0}(\mathrm{d},\Omega) \cap \mathsf{H}^{q}_{\Gamma_n,0}(\delta,\Omega)$ has finite dimension,
(iv) the orthogonal Helmholtz-type decomposition

$$\mathsf{L}^{q,2}(\Omega) = \mathrm{d}\mathsf{H}^{q-1,0}_{\Gamma_t}(\mathrm{d},\Omega) \oplus_{\mathsf{L}^{q,2}(\Omega)} \mathcal{H}^{q}_{\Gamma_t,\Gamma_n,\mathrm{id}}(\Omega) \oplus_{\mathsf{L}^{q,2}(\Omega)} \delta\mathsf{H}^{q+1,0}_{\Gamma_n}(\delta,\Omega)$$

holds,

(v) *there exists $c_q > 0$ such that*

$$\forall E \in D\left((\mathrm{d}^q_{\Gamma_t})_\perp\right) \qquad |E|_{\mathsf{L}^{q,2}(\Omega)} \leq c_q |\mathrm{d}E|_{\mathsf{L}^{q+1,2}(\Omega)},$$
$$\forall H \in D\left((\delta^{q+1}_{\Gamma_n})_\perp\right) \qquad |H|_{\mathsf{L}^{q+1,2}(\Omega)} \leq c_q |\delta H|_{\mathsf{L}^{q,2}(\Omega)},$$

where

$$D\left((\mathrm{d}^q_{\Gamma_t})_\perp\right) = D(\mathrm{d}^q_{\Gamma_t}) \cap N(\mathrm{d}^q_{\Gamma_t})^{\perp_{\mathsf{L}^{q,2}(\Omega)}} = D(\mathrm{d}^q_{\Gamma_t}) \cap R(\delta^{q+1}_{\Gamma_n}),$$
$$D\left((\delta^{q+1}_{\Gamma_n})_\perp\right) = D(\delta^{q+1}_{\Gamma_n}) \cap N(\delta^{q+1}_{\Gamma_n})^{\perp_{\mathsf{L}^{q+1,2}(\Omega)}} = D(\delta^{q+1}_{\Gamma_n}) \cap R(\mathrm{d}^q_{\Gamma_t}),$$

(v') *with c_q from (v) it holds for all $E \in D(\mathrm{d}^q_{\Gamma_t}) \cap D(\delta^q_{\Gamma_n}) \cap \mathcal{H}^q_{\Gamma_t,\Gamma_n,\mathrm{id}}(\Omega)^{\perp_{\mathsf{L}^{q,2}(\Omega)}}$*

$$|E|^2_{\mathsf{L}^{q,2}(\Omega)} \leq c_q^2 |\mathrm{d}E|^2_{\mathsf{L}^{q+1,2}(\Omega)} + c_{q-1}^2 |\delta E|^2_{\mathsf{L}^{q-1,2}(\Omega)},$$

(vi) $\mathcal{H}^q_{\Gamma_t,\Gamma_n,\mathrm{id}}(\Omega) = \{0\}$, *if (Ω, Γ_t) is additionally extendable.*

Proof. Apply Theorem 2.23 (ii), that is, Theoren 4.8 and Theorem 2.9 show (i)–(v'). For $k = 0$, Lemma 4.4 and Lemma 3.6 imply $\mathrm{d}\mathsf{H}^{q-1,0}_{\Gamma_t}(\mathrm{d},\Omega) = \mathsf{H}^{q,0}_{\Gamma_t,0}(\mathrm{d},\Omega) = \mathrm{d}\mathsf{H}^{q-1,0}_{\Gamma_t}(\mathrm{d},\Omega) \oplus_{\mathsf{L}^{q,2}(\Omega)} \mathcal{H}^q_{\Gamma_t,\Gamma_n,\mathrm{id}}(\Omega)$, that is, (vi). □

Remark 4.10 (mini FA-ToolBox for the de Rham complex). Recall the admissible weights ε from Section 3.3. In Pauly and Waurick,[23, Lemma 5.1, Lemma 5.2] we have shown that the compactness in Theoren 4.8, and the dimensions of the cohomology groups do not depend on the particular ε. Hence, for all q

(i) the embedding $\mathsf{H}^{q,0}_{\Gamma_t}(\mathrm{d},\Omega) \cap \varepsilon^{-1}\mathsf{H}^{q,0}_{\Gamma_n}(\delta,\Omega) \hookrightarrow \mathsf{L}^{q,2}(\Omega)$ is compact,
(ii) $d^q_{\Omega,\Gamma_t} := \dim \mathcal{H}^q_{\Gamma_t,\Gamma_n,\varepsilon}(\Omega) = \dim \mathcal{H}^q_{\Gamma_t,\Gamma_n,\mathrm{id}}(\Omega)$.
(iii) Theorem 4.9 holds with appropriate modifications for including ε.

Compare to the more explicit formulations from Section 5 for the vector de Rham complex. All these results carry over literally. In particular, cf. Theorem 4.9 (v'), we have with c_q (now depending also on ε and μ) for all $E \in D(\mu^{-1}\mathrm{d}^q_{\Gamma_t}) \cap D(\delta^q_{\Gamma_n}\varepsilon) \cap \mathcal{H}^q_{\Gamma_t,\Gamma_n,\varepsilon}(\Omega)^{\perp_{\mathsf{L}^{q,2}_\varepsilon(\Omega)}}$

$$|E|^2_{\mathsf{L}^{q,2}_\varepsilon(\Omega)} \leq c_q^2 |\mu^{-1}\mathrm{d}E|^2_{\mathsf{L}^{q+1,2}_\mu(\Omega)} + c_{q-1}^2 |\delta\varepsilon E|^2_{\mathsf{L}^{q-1,2}(\Omega)}.$$

Moreover,

(iv) Theorem 4.8 and hence Theorem 4.9 and (i)–(iii) of this remark hold more generally for bounded weak Lipschitz pairs (Ω, Γ_t); see Bauer et al.[17,18]

Theorem 4.11 (bounded regular potentials for the de Rham complex). *Let (Ω, Γ_t) be a bounded strong Lipschitz pair and let $\mathcal{Q}^{q,0}_{\mathrm{d},\Gamma_t,1}$ be given from Lemma 4.6. Then for all $q \in \{1, \ldots, d\}$, there exists a bounded linear regular potential operator*

$$\mathcal{P}^{q,0}_{\mathrm{d},\Gamma_t} := \mathcal{Q}^{q-1,0}_{\mathrm{d},\Gamma_t,1}(\mathrm{d}^{q-1}_{\Gamma_t})^{-1}_\perp : \mathsf{H}^{q,0}_{\Gamma_t,0}(\mathrm{d},\Omega) \cap \mathcal{H}^q_{\Gamma_t,\Gamma_n,\varepsilon}(\Omega)^{\perp_{\mathsf{L}^{q,2}_\varepsilon(\Omega)}} \to \mathsf{H}^{q-1,1}_{\Gamma_t}(\Omega),$$

such that $\mathrm{d}\mathcal{P}^{q,0}_{\mathrm{d},\Gamma_t} = \mathrm{id}|_{\mathsf{H}^{q,0}_{\Gamma_t,0}(\mathrm{d},\Omega) \cap \mathcal{H}^q_{\Gamma_t,\Gamma_n,\varepsilon}(\Omega)^{\perp_{\mathsf{L}^{q,2}_\varepsilon(\Omega)}}}$. *In particular, the bounded regular potential representations*

$$R(\mathrm{d}^{q-1}_{\Gamma_t}) = \mathsf{H}^{q,0}_{\Gamma_t,0}(\mathrm{d},\Omega) \cap \mathcal{H}^q_{\Gamma_t,\Gamma_n,\varepsilon}(\Omega)^{\perp_{\mathsf{L}^{q,2}_\varepsilon(\Omega)}} = \mathrm{d}\mathsf{H}^{q-1,0}_{\Gamma_t}(\mathrm{d},\Omega) = \mathrm{d}\mathsf{H}^{q-1,1}_{\Gamma_t}(\Omega)$$

hold, and the potentials can be chosen such that they depend continuously on the data.

Proof. Apply Theorem 2.23 (iii). Note that $R(\mathrm{d}^{q-1}_{\Gamma_t})$ is closed by Theorem 4.9, and hence,

$$R(\mathrm{d}^{q-1}_{\Gamma_t}) = \mathrm{d}\mathsf{H}^{q-1,0}_{\Gamma_t}(\mathrm{d},\Omega) = \mathsf{H}^{q,0}_{\Gamma_t,0}(\mathrm{d},\Omega) \cap \mathcal{H}^{q}_{\Gamma_t,\Gamma_n,\varepsilon}(\Omega)^{\perp_{\mathsf{L}^{q,2}_{\varepsilon}(\Omega)}}$$

holds by Lemma 3.6. □

Remark 4.12 (Dirichlet/Neumann forms). Note that $\mathcal{H}^{d}_{\Gamma_t,\Gamma_n,\varepsilon}(\Omega) = \varepsilon^{-1}\mathsf{H}^{d}_{\Gamma_n,0}(\delta,\Omega) = \varepsilon^{-1} * \mathbb{R}_{\Gamma_n}$ and $\mathcal{H}^{d}_{\Gamma_t,\Gamma_n,\varepsilon}(\Omega)^{\perp_{\mathsf{L}^{d,2}_{\varepsilon}(\Omega)}} = (*\mathbb{R}_{\Gamma_n})^{\perp_{\mathsf{L}^{d,2}(\Omega)}}$ holds in the special case $q = d$.

Theorem 4.13 (bounded regular decompositions for the de Rham complex). *Let (Ω, Γ_t) be a bounded strong Lipschitz pair and let $\mathcal{P}^{q,0}_{\mathrm{d},\Gamma_t}$ and $\mathcal{Q}^{q,0}_{\mathrm{d},\Gamma_t,1}$ be given from Theorem 4.11 and from Lemma 4.6, respectively. Then the bounded regular decompositions*

$$\begin{aligned}\mathsf{H}^{q}_{\Gamma_t}(\mathrm{d},\Omega) = \mathsf{H}^{q,0}_{\Gamma_t}(\mathrm{d},\Omega) &= \mathsf{H}^{q,1}_{\Gamma_t}(\Omega) + \mathsf{H}^{q,0}_{\Gamma_t,0}(\mathrm{d},\Omega) = \mathsf{H}^{q,1}_{\Gamma_t}(\Omega) + \mathrm{d}\mathsf{H}^{q-1,1}_{\Gamma_t}(\Omega)\\ &= R(\widetilde{\mathcal{Q}}^{q,0}_{\mathrm{d},\Gamma_t,1}) \dotplus \mathsf{H}^{q,0}_{\Gamma_t,0}(\mathrm{d},\Omega) = R(\widetilde{\mathcal{Q}}^{q,0}_{\mathrm{d},\Gamma_t,1}) \dotplus R(\widetilde{\mathcal{N}}^{q,0}_{\mathrm{d},\Gamma_t})\end{aligned}$$

hold with bounded linear regular decomposition operators

$$\widetilde{\mathcal{Q}}^{q,0}_{\mathrm{d},\Gamma_t,1} := \mathcal{P}^{q+1,0}_{\mathrm{d},\Gamma_t}\mathrm{d}^{q}_{\Gamma_t} : \mathsf{H}^{q,0}_{\Gamma_t}(\mathrm{d},\Omega) \to \mathsf{H}^{q,1}_{\Gamma_t}(\Omega), \qquad \widetilde{\mathcal{N}}^{q,0}_{\mathrm{d},\Gamma_t} : \mathsf{H}^{q,0}_{\Gamma_t}(\mathrm{d},\Omega) \to \mathsf{H}^{q,0}_{\Gamma_t,0}(\mathrm{d},\Omega)$$

satisfying $\widetilde{\mathcal{Q}}^{q,0}_{\mathrm{d},\Gamma_t,1} + \widetilde{\mathcal{N}}^{q,0}_{\mathrm{d},\Gamma_t} = \mathrm{id}_{\mathsf{H}^{q,0}_{\Gamma_t}(\mathrm{d},\Omega)}$. Moreover, it holds $\mathrm{d}\widetilde{\mathcal{Q}}^{q,0}_{\mathrm{d},\Gamma_t,1} = \mathrm{d}\mathcal{Q}^{q,0}_{\mathrm{d},\Gamma_t,1} = \mathrm{d}^{q}_{\Gamma_t}$, and thus, $\mathsf{H}^{q,0}_{\Gamma_t,0}(\mathrm{d},\Omega)$ is invariant under $\mathcal{Q}^{q,0}_{\mathrm{d},\Gamma_t,1}$ and $\widetilde{\mathcal{Q}}^{q,0}_{\mathrm{d},\Gamma_t,1}$. Furthermore, $R(\widetilde{\mathcal{Q}}^{q,0}_{\mathrm{d},\Gamma_t,1}) = R(\mathcal{P}^{q+1,0}_{\mathrm{d},\Gamma_t})$ and $\widetilde{\mathcal{Q}}^{q,0}_{\mathrm{d},\Gamma_t,1} = \mathcal{P}^{q+1,0}_{\mathrm{d},\Gamma_t}\mathrm{d}^{q}_{\Gamma_t} = \mathcal{Q}^{q,0}_{\mathrm{d},\Gamma_t,1}(\mathrm{d}^{q}_{\Gamma_t})^{-1}_{\perp}\mathrm{d}^{q}_{\Gamma_t}$. Hence, $\widetilde{\mathcal{Q}}^{q,0}_{\mathrm{d},\Gamma_t,1}|_{D((\mathrm{d}^{q}_{\Gamma_t})_{\perp})} = \mathcal{Q}^{q,0}_{\mathrm{d},\Gamma_t,1}|_{D((\mathrm{d}^{q}_{\Gamma_t})_{\perp})}$, and thus, $\widetilde{\mathcal{Q}}^{q,0}_{\mathrm{d},\Gamma_t,1}$ may differ from $\mathcal{Q}^{q,0}_{\mathrm{d},\Gamma_t,1}$ only on $\mathsf{H}^{q,0}_{\Gamma_t,0}(\mathrm{d},\Omega)$.

Proof. Apply Theorem 2.23 (iv) and (iv'). □

Again, Theorem 4.11 and Theorem 4.13 have dual versions for the δ-operator by Hodge $\star$-duality, cf. Theorem 5.13 for $k = 0$.

4.6 | Higher-order mini FA-ToolBox

Some results from the latter section hold even for higher Sobolev orders. As pointed out in Section 4.2, the adjoints are much more complicated. Hence, Lemma 2.22 and Theorem 2.23 from the FA-ToolBox are not directly applicable, so that some detours and modifications are needed.

In Section 4.2, we have introduced the higher-order primal and dual de Rham Hilbert complex composed of the densely defined and closed linear operators

$$\begin{aligned}&\mathrm{d}^{q,k}_{\Gamma_t} : D(\mathrm{d}^{q,k}_{\Gamma_t}) \subset \mathsf{H}^{q,k}_{\Gamma_t}(\Omega) \to \mathsf{H}^{q+1,k}_{\Gamma_t}(\Omega), &\qquad D(\mathrm{d}^{q,k}_{\Gamma_t}) = \mathsf{H}^{q,k}_{\Gamma_t}(\mathrm{d},\Omega),\\ &\delta^{q,k}_{\Gamma_n} : D(\delta^{q,k}_{\Gamma_n}) \subset \mathsf{H}^{q,k}_{\Gamma_n}(\Omega) \to \mathsf{H}^{q-1,k}_{\Gamma_n}(\Omega), &\qquad D(\delta^{q,k}_{\Gamma_n}) = \mathsf{H}^{q,k}_{\Gamma_n}(\delta,\Omega).\end{aligned}$$

By Corollary 4.7, see the following:

Theorem 4.14 (higher-order closed ranges for the de Rham complex). *Let (Ω, Γ_t) be a bounded strong Lipschitz pair. Then for all q and for all $k \in \mathbb{N}_0$, the ranges*

$$\begin{aligned}R(\mathrm{d}^{q-1,k}_{\Gamma_t}) &= \mathrm{d}\mathsf{H}^{q-1,k}_{\Gamma_t}(\mathrm{d},\Omega) = \mathrm{d}\mathsf{H}^{q-1,k+1}_{\Gamma_t}(\Omega) = \mathsf{H}^{q,k}_{\Gamma_t,0}(\mathrm{d},\Omega) \cap \mathcal{H}^{q}_{\Gamma_t,\Gamma_n,\mathrm{id}}(\Omega)^{\perp_{\mathsf{L}^{q,2}(\Omega)}},\\ R(\delta^{q+1,k}_{\Gamma_n}) &= \delta\mathsf{H}^{q+1,k}_{\Gamma_n}(\delta,\Omega) = \delta\mathsf{H}^{q+1,k+1}_{\Gamma_n}(\Omega) = \mathsf{H}^{q,k}_{\Gamma_n,0}(\delta,\Omega) \cap \mathcal{H}^{q}_{\Gamma_t,\Gamma_n,\mathrm{id}}(\Omega)^{\perp_{\mathsf{L}^{q,2}(\Omega)}}\end{aligned}$$

are closed, that is, closed subspaces of $\mathsf{H}^{q,k}(\Omega)$. In particular, the higher-order long primal and dual de Rham complex from Section 4.2 is closed.

The corresponding reduced operators read

$$(\mathrm{d}^{q,k}_{\Gamma_t})_\perp : D\left((\mathrm{d}^{q,k}_{\Gamma_t})_\perp\right) \subset \mathsf{H}^{q,k}_{\Gamma_t,0}(\mathrm{d},\Omega)^{\perp_{\mathsf{H}^{q,k}_{\Gamma_t}(\Omega)}} \to \mathrm{d}\mathsf{H}^{q,k}_{\Gamma_t}(\mathrm{d},\Omega), \qquad N(\mathrm{d}^{q,k}_{\Gamma_t}) = \mathsf{H}^{q,k}_{\Gamma_t,0}(\mathrm{d},\Omega),$$
$$-(\delta^{q,k}_{\Gamma_n})_\perp : D\left((\delta^{q,k}_{\Gamma_n})_\perp\right) \subset \mathsf{H}^{q,k}_{\Gamma_n,0}(\delta,\Omega)^{\perp_{\mathsf{H}^{q,k}_{\Gamma_n}(\Omega)}} \to \delta\mathsf{H}^{q,k}_{\Gamma_n}(\delta,\Omega), \qquad N(\delta^{q,k}_{\Gamma_n}) = \mathsf{H}^{q,k}_{\Gamma_n,0}(\delta,\Omega),$$

with

$$D\left((\mathrm{d}^{q,k}_{\Gamma_t})_\perp\right) = \mathsf{H}^{q,k}_{\Gamma_t}(\mathrm{d},\Omega) \cap \mathsf{H}^{q,k}_{\Gamma_t,0}(\mathrm{d},\Omega)^{\perp_{\mathsf{H}^{q,k}_{\Gamma_t}(\Omega)}} = \mathsf{H}^{q,k}_{\Gamma_t}(\mathrm{d},\Omega) \cap R\left((\mathrm{d}^{q,k}_{\Gamma_t})^*\right),$$
$$D\left((\delta^{q,k}_{\Gamma_n})_\perp\right) = \mathsf{H}^{q,k}_{\Gamma_n}(\delta,\Omega) \cap \mathsf{H}^{q,k}_{\Gamma_n,0}(\delta,\Omega)^{\perp_{\mathsf{H}^{q,k}_{\Gamma_n}(\Omega)}} = \mathsf{H}^{q,k}_{\Gamma_n}(\delta,\Omega) \cap R\left((\delta^{q,k}_{\Gamma_n})^*\right),$$

and we have by Lemma 2.1 and Theorem 4.14:

Theorem 4.15 (higher-order fundamental lemma 1 for the de Rham complex)**.** *Let (Ω, Γ_t) be a bounded strong Lipschitz pair. Then for all q and for all $k \in \mathbb{N}_0$, the following assertions hold and are equivalent:*

- *(i)* $\exists\, c > 0 \quad \forall E \in D\left((\mathrm{d}^{q,k}_{\Gamma_t})_\perp\right) \qquad |E|_{\mathsf{H}^{q,k}(\Omega)} \le c|\mathrm{d}E|_{\mathsf{H}^{q+1,k}(\Omega)}$
- *(ii)* $R(\mathrm{d}^{q,k}_{\Gamma_t}) = R\left((\mathrm{d}^{q,k}_{\Gamma_t})_\perp\right) = \mathrm{d}\mathsf{H}^{q,k}_{\Gamma_t}(\mathrm{d},\Omega)$ *is closed.*
- *(iii)* $(\mathrm{d}^{q,k}_{\Gamma_t})^{-1}_\perp : R(\mathrm{d}^{q,k}_{\Gamma_t}) \to D\left((\mathrm{d}^{q,k}_{\Gamma_t})_\perp\right)$ *is bounded.*
- *(iii′)* $(\mathrm{d}^{q,k}_{\Gamma_t})^{-1}_\perp : R(\mathrm{d}^{q,k}_{\Gamma_t}) \to D(\mathrm{d}^{q,k}_{\Gamma_t})$ *is bounded.*

The corresponding results hold for the $\delta^{q,k}_{\Gamma_n}$ as well.

The higher-order version of Theorem 4.8 reads as follows:

Theorem 4.16 (higher-order compact embedding for the de Rham complex)**.** *Let (Ω, Γ_t) be a bounded strong Lipschitz pair. Then for all q and for all $k \in \mathbb{N}_0$, the embedding*

$$D(\mathrm{d}^{q,k}_{\Gamma_t}) \cap D(\delta^{q,k}_{\Gamma_n}) = \mathsf{H}^{q,k}_{\Gamma_t}(\mathrm{d},\Omega) \cap \mathsf{H}^{q,k}_{\Gamma_n}(\delta,\Omega) \hookrightarrow \mathsf{H}^{q,k}_{\Gamma}(\Omega)$$

is compact.

Proof. We follow in close lines the proof of Pauly and Zulehner[8, Theorem 4.11] using induction. The case $k = 0$ is given by Theorem 4.8. Let $k \ge 1$ and let (E_n) be a bounded sequence in $\mathsf{H}^{q,k}_{\Gamma_t}(\mathrm{d},\Omega) \cap \mathsf{H}^{q,k}_{\Gamma_n}(\delta,\Omega)$. Note that

$$\mathsf{H}^{q,k}_{\Gamma_t}(\mathrm{d},\Omega) \cap \mathsf{H}^{q,k}_{\Gamma_n}(\delta,\Omega) \subset \mathsf{H}^{q,k}_{\Gamma_t}(\Omega) \cap \mathsf{H}^{q,k}_{\Gamma_n}(\Omega) = \mathsf{H}^{q,k}_{\Gamma}(\Omega).$$

By assumption and w.l.o.g., we have that (E_n) is a Cauchy sequence in $\mathsf{H}^{q,k-1}_{\Gamma}(\Omega)$. Moreover, for all $|\alpha| = k$, we have $\partial^\alpha E_n \in \mathsf{H}^{q,0}_{\Gamma_t}(\mathrm{d},\Omega) \cap \mathsf{H}^{q,0}_{\Gamma_n}(\delta,\Omega)$ with $\mathrm{d}\partial^\alpha E_n = \partial^\alpha \mathrm{d}E_n$ and $\delta\partial^\alpha E_n = \partial^\alpha \delta E_n$ by Lemma 3.4. Hence, $(\partial^\alpha E_n)$ is a bounded sequence in $\mathsf{H}^{q,0}_{\Gamma_t}(\mathrm{d},\Omega) \cap \mathsf{H}^{q,0}_{\Gamma_n}(\delta,\Omega)$. Thus, w.l.o.g. $(\partial^\alpha E_n)$ is a Cauchy sequence in $\mathsf{L}^{q,2}(\Omega)$ by Theorem 4.8. Finally, (E_n) is a Cauchy sequence in $\mathsf{H}^{q,k}_{\Gamma}(\Omega)$, finishing the proof. □

Higher-order analogues of Theorem 4.9 and Remark 4.10 hold. Some of these results are formulated in the following theorem.

Theorem 4.17 (higher-order Friedrichs/Poincaré type estimates for the de Rham complex)**.** *Let (Ω, Γ_t) be a bounded strong Lipschitz pair. Then for all q and for all $k \ge 0$, there exists $\widetilde{c}_{q,k} > 0$ such that for all $E \in \mathsf{H}^{q,k}_{\Gamma_t}(\mathrm{d},\Omega) \cap \mathsf{H}^{q,k}_{\Gamma_n}(\delta,\Omega) \cap \mathcal{H}^q_{\Gamma_t,\Gamma_n,\mathrm{id}}(\Omega)^{\perp_{\mathsf{L}^{q,2}(\Omega)}}$*

$$|E|_{\mathsf{H}^{q,k}(\Omega)} \le \widetilde{c}_{q,k}\left(|\mathrm{d}E|_{\mathsf{H}^{q+1,k}(\Omega)} + |\delta E|_{\mathsf{H}^{q-1,k}(\Omega)}\right).$$

The condition $\mathcal{H}^{q}_{\Gamma_t,\Gamma_n,\mathrm{id}}(\Omega)^{\perp_{\mathsf{L}^{q,2}(\Omega)}}$ *can be replaced by the weaker conditions* $\mathcal{H}^{q,k}_{\Gamma_t,\Gamma_n,\mathrm{id}}(\Omega)^{\perp_{\mathsf{L}^{q,2}(\Omega)}}$ *or* $\mathcal{H}^{q,k}_{\Gamma_t,\Gamma_n,\mathrm{id}}(\Omega)^{\perp_{\mathsf{H}^{q,k}(\Omega)}}$*. In particular, it holds*

$$\forall\, E \in \mathsf{H}^{q,k}_{\Gamma_t}(\mathrm{d},\Omega) \cap R(\delta^{q+1,k}_{\Gamma_n}) \qquad |E|_{\mathsf{H}^{q,k}(\Omega)} \leq \widetilde{c}_{q,k} |\mathrm{d}E|_{\mathsf{H}^{q+1,k}(\Omega)},$$
$$\forall\, E \in \mathsf{H}^{q,k}_{\Gamma_n}(\delta,\Omega) \cap R(\mathrm{d}^{q-1,k}_{\Gamma_t}) \qquad |E|_{\mathsf{H}^{q,k}(\Omega)} \leq \widetilde{c}_{q,k} |\delta E|_{\mathsf{H}^{q-1,k}(\Omega)}$$

with

$$R(\delta^{q+1,k}_{\Gamma_n}) = \mathsf{H}^{q,k}_{\Gamma_n,0}(\delta,\Omega) \cap \mathcal{H}^{q}_{\Gamma_t,\Gamma_n,\mathrm{id}}(\Omega)^{\perp_{\mathsf{L}^{q,2}(\Omega)}},$$
$$R(\mathrm{d}^{q-1,k}_{\Gamma_t}) = \mathsf{H}^{q,k}_{\Gamma_t,0}(\mathrm{d},\Omega) \cap \mathcal{H}^{q}_{\Gamma_t,\Gamma_n,\mathrm{id}}(\Omega)^{\perp_{\mathsf{L}^{q,2}(\Omega)}}.$$

Proof. To show the first estimate, we use a standard strategy and assume the contrary. Then there is a sequence

$$(E_n) \subset \mathsf{H}^{q,k}_{\Gamma_t}(\mathrm{d},\Omega) \cap \mathsf{H}^{q,k}_{\Gamma_n}(\delta,\Omega) \cap \mathcal{H}^{q}_{\Gamma_t,\Gamma_n,\mathrm{id}}(\Omega)^{\perp_{\mathsf{L}^{q,2}(\Omega)}}$$

with $|E_n|_{\mathsf{H}^{q,k}(\Omega)} = 1$ and $|\mathrm{d}E_n|_{\mathsf{H}^{q+1,k}(\Omega)} + |\delta E_n|_{\mathsf{H}^{q-1,k}(\Omega)} \to 0$. Hence, we may assume that E_n converges weakly to some E in $\mathsf{H}^{q,k}(\Omega) \cap \mathcal{H}^{q}_{\Gamma_t,\Gamma_n,\mathrm{id}}(\Omega) \cap \mathcal{H}^{q}_{\Gamma_t,\Gamma_n,\mathrm{id}}(\Omega)^{\perp_{\mathsf{L}^{q,2}(\Omega)}}$. Thus, $E = 0$. By Theorem 4.16, (E_n) converges strongly to 0 in $\mathsf{H}^{q,k}(\Omega)$, in contradiction to $|E_n|_{\mathsf{H}^{q,k}(\Omega)} = 1$.

The other two estimates follow with Theorem 4.14 by restriction. □

Note that by Theorem 4.15,

$$(\mathrm{d}^{q,k}_{\Gamma_t})^{-1}_{\perp} : R(\mathrm{d}^{q,k}_{\Gamma_t}) \to D(\mathrm{d}^{q,k}_{\Gamma_t}), \qquad (\delta^{q,k}_{\Gamma_n})^{-1}_{\perp} : R(\delta^{q,k}_{\Gamma_n}) \to D(\delta^{q,k}_{\Gamma_n})$$

are bounded. The higher-order versions of Theorem 4.11 and Theorem 4.13 read as follows:

Theorem 4.18 (higher-order bounded regular potentials and decompositions for the de Rham complex). *Let* (Ω, Γ_t) *be a bounded strong Lipschitz pair and let* $k \geq 0$*. Moreover, let* $\mathcal{Q}^{q,k}_{\mathrm{d},\Gamma_t,1}$ *be given from Lemma 4.6. Then:*

(i) For all $q \in \{1, \ldots, d\}$*, there exists a bounded linear regular potential operator*

$$\mathcal{P}^{q,k}_{\mathrm{d},\Gamma_t} := \mathcal{Q}^{q-1,k}_{\mathrm{d},\Gamma_t,1} (\mathrm{d}^{q-1,k}_{\Gamma_t})^{-1}_{\perp} : \mathsf{H}^{q,k}_{\Gamma_t,0}(\mathrm{d},\Omega) \cap \mathcal{H}^{q}_{\Gamma_t,\Gamma_n,\varepsilon}(\Omega)^{\perp_{\mathsf{L}^{q,2}_{\varepsilon}(\Omega)}} \to \mathsf{H}^{q-1,k+1}_{\Gamma_t}(\Omega),$$

such that $\mathrm{d}\mathcal{P}^{q,k}_{\mathrm{d},\Gamma_t} = \mathrm{id}|_{\mathsf{H}^{q,k}_{\Gamma_t,0}(\mathrm{d},\Omega) \cap \mathcal{H}^{q}_{\Gamma_t,\Gamma_n,\varepsilon}(\Omega)^{\perp_{\mathsf{L}^{q,2}_{\varepsilon}(\Omega)}}}$*. In particular, the bounded regular representations*

$$R(\mathrm{d}^{q-1,k}_{\Gamma_t}) = \mathsf{H}^{q,k}_{\Gamma_t,0}(\mathrm{d},\Omega) \cap \mathcal{H}^{q}_{\Gamma_t,\Gamma_n,\varepsilon}(\Omega)^{\perp_{\mathsf{L}^{q,2}_{\varepsilon}(\Omega)}}$$
$$= \mathsf{H}^{q,k}_{\Gamma_t}(\Omega) \cap \mathrm{d}\mathsf{H}^{q-1}_{\Gamma_t}(\mathrm{d},\Omega) = \mathrm{d}\mathsf{H}^{q-1,k}_{\Gamma_t}(\mathrm{d},\Omega) = \mathrm{d}\mathsf{H}^{q-1,k+1}_{\Gamma_t}(\Omega)$$

hold, and the potentials can be chosen such that they depend continuously on the data.

(ii) The bounded regular decompositions

$$\mathsf{H}^{q,k}_{\Gamma_t}(\mathrm{d},\Omega) = \mathsf{H}^{q,k+1}_{\Gamma_t}(\Omega) + \mathsf{H}^{q,k}_{\Gamma_t,0}(\mathrm{d},\Omega) = \mathsf{H}^{q,k+1}_{\Gamma_t}(\Omega) + \mathrm{d}\mathsf{H}^{q-1,k+1}_{\Gamma_t}(\Omega)$$
$$= R(\widetilde{\mathcal{Q}}^{q,k}_{\mathrm{d},\Gamma_t,1}) \dotplus \mathsf{H}^{q,k}_{\Gamma_t,0}(\mathrm{d},\Omega) = R(\widetilde{\mathcal{Q}}^{q,k}_{\mathrm{d},\Gamma_t,1}) \dotplus R(\widetilde{\mathcal{N}}^{q,k}_{\mathrm{d},\Gamma_t})$$

hold with bounded linear regular decomposition operators

$$\widetilde{\mathcal{Q}}^{q,k}_{\mathrm{d},\Gamma_t,1} := \mathcal{P}^{q+1,k}_{\mathrm{d},\Gamma_t} \mathrm{d}^{q,k}_{\Gamma_t} : \mathsf{H}^{q,k}_{\Gamma_t}(\mathrm{d},\Omega) \to \mathsf{H}^{q,k+1}_{\Gamma_t}(\Omega), \qquad \widetilde{\mathcal{N}}^{q,k}_{\mathrm{d},\Gamma_t} : \mathsf{H}^{q,k}_{\Gamma_t}(\mathrm{d},\Omega) \to \mathsf{H}^{q,k}_{\Gamma_t,0}(\mathrm{d},\Omega)$$

satisfying $\widetilde{\mathcal{Q}}^{q,k}_{\mathrm{d},\Gamma_t,1} + \widetilde{\mathcal{N}}^{q,k}_{\mathrm{d},\Gamma_t} = \mathrm{id}_{\mathsf{H}^{q,k}_{\Gamma_t}(\mathrm{d},\Omega)}$*. Moreover,* $\mathrm{d}\widetilde{\mathcal{Q}}^{q,k}_{\mathrm{d},\Gamma_t,1} = \mathrm{d}\mathcal{Q}^{q,k}_{\mathrm{d},\Gamma_t,1} = \mathrm{d}^{q,k}_{\Gamma_t}$*, and thus,* $\mathsf{H}^{q,k}_{\Gamma_t,0}(\mathrm{d},\Omega)$ *is invariant under* $\mathcal{Q}^{q,k}_{\mathrm{d},\Gamma_t,1}$ *and* $\widetilde{\mathcal{Q}}^{q,k}_{\mathrm{d},\Gamma_t,1}$*. It holds* $R(\widetilde{\mathcal{Q}}^{q,k}_{\mathrm{d},\Gamma_t,1}) = R(\mathcal{P}^{q+1,k}_{\mathrm{d},\Gamma_t})$ *and* $\widetilde{\mathcal{Q}}^{q,k}_{\mathrm{d},\Gamma_t,1} = \mathcal{P}^{q+1,k}_{\mathrm{d},\Gamma_t}\mathrm{d}^{q,k}_{\Gamma_t} = \mathcal{Q}^{q,k}_{\mathrm{d},\Gamma_t,1}(\mathrm{d}^{q,k}_{\Gamma_t})^{-1}_{\perp}\mathrm{d}^{q,k}_{\Gamma_t}$*. Hence,* $\widetilde{\mathcal{Q}}^{q,k}_{\mathrm{d},\Gamma_t,1}|_{D((\mathrm{d}^{q,k}_{\Gamma_t})_\perp)} = \mathcal{Q}^{q,k}_{\mathrm{d},\Gamma_t,1}|_{D((\mathrm{d}^{q,k}_{\Gamma_t})_\perp)}$ *and thus* $\widetilde{\mathcal{Q}}^{q,k}_{\mathrm{d},\Gamma_t,1}$ *may differ from* $\mathcal{Q}^{q,k}_{\mathrm{d},\Gamma_t,1}$ *only on* $\mathsf{H}^{q,k}_{\Gamma_t,0}(\mathrm{d},\Omega)$*.*

(ii') The bounded regular kernel decomposition $\mathsf{H}^{q,k}_{\Gamma_t,0}(\mathrm{d},\Omega) = \mathsf{H}^{q,k+1}_{\Gamma_t,0}(\mathrm{d},\Omega) + \mathrm{d}\mathsf{H}^{q-1,k+1}_{\Gamma_t}(\Omega)$ *holds.*

Proof. Lemma 4.6 yields the bounded regular decomposition

$$D(\mathrm{d}^{q,k}_{\Gamma_t}) = \mathsf{H}^{q,k}_{\Gamma_t}(\mathrm{d},\Omega) = \mathsf{H}^{q,k+1}_{\Gamma_t}(\Omega) + \mathrm{d}\mathsf{H}^{q-1,k+1}_{\Gamma_t}(\Omega) = \mathsf{H}^+_1 + \mathrm{d}^{q-1,k}_{\Gamma_t}\mathsf{H}^+_0$$

with $\mathsf{H}^+_1 := \mathsf{H}^{q,k+1}_{\Gamma_t}(\Omega)$ and $\mathsf{H}^+_0 := \mathsf{H}^{q-1,k+1}_{\Gamma_t}(\Omega)$ and $\mathsf{H}_1 := \mathsf{H}^{q,k}_{\Gamma_t}(\Omega)$ and $\mathsf{H}_0 := \mathsf{H}^{q-1,k}_{\Gamma_t}(\Omega)$. Rellich's selection theorem shows that the assumptions of Lemma 2.22 (i) and Theorem 2.23 are satisfied. Note that it holds $D(\mathrm{d}^{0,k}_{\Gamma_t}) = \mathsf{H}^{0,k+1}_{\Gamma_t}(\Omega)$ and $D(\delta^{d,k}_{\Gamma_n}) = \mathsf{H}^{d,k+1}_{\Gamma_n}(\Omega)$. Theorem 2.23 (iii)–(iv') and Theorem 4.14 show the assertions (i) and (ii). (ii') follows directly by (ii). □

Hodge ⋆-duality yields the corresponding results for the co-derivative as well, cf. Theorem 5.13.

Remark 4.19. Let us recall the bounded regular decompositions from Theorem 4.18 (ii), for example,

$$\mathsf{H}^{q,k}_{\Gamma_t}(\mathrm{d},\Omega) = R(\widetilde{\mathcal{Q}}^{q,k}_{\mathrm{d},\Gamma_t,1}) \dotplus R(\widetilde{\mathcal{N}}^{q,k}_{\mathrm{d},\Gamma_t}).$$

By Remark 2.19, we emphasise:

(i) $\widetilde{\mathcal{Q}}^{q,k}_{\mathrm{d},\Gamma_t,1}$ and $\widetilde{\mathcal{N}}^{q,k}_{\mathrm{d},\Gamma_t} = 1 - \widetilde{\mathcal{Q}}^{q,k}_{\mathrm{d},\Gamma_t,1}$ are projections with $\widetilde{\mathcal{Q}}^{q,k}_{\mathrm{d},\Gamma_t,1}\widetilde{\mathcal{N}}^{q,k}_{\mathrm{d},\Gamma_t} = \widetilde{\mathcal{N}}^{q,k}_{\mathrm{d},\Gamma_t}\widetilde{\mathcal{Q}}^{q,k}_{\mathrm{d},\Gamma_t,1} = 0$.

(ii) For $I_\pm := \widetilde{\mathcal{Q}}^{q,k}_{\mathrm{d},\Gamma_t,1} \pm \widetilde{\mathcal{N}}^{q,k}_{\mathrm{d},\Gamma_t}$, it holds $I_+ = I_-^2 = \mathrm{id}_{\mathsf{H}^{q,k}_{\Gamma_t}(\mathrm{d},\Omega)}$. Therefore, I_+, I_-^2, as well as $I_- = 2\widetilde{\mathcal{Q}}^{q,k}_{\mathrm{d},\Gamma_t,1} - \mathrm{id}_{\mathsf{H}^{q,k}_{\Gamma_t}(\mathrm{d},\Omega)}$ are topological isomorphisms on $\mathsf{H}^{q,k}_{\Gamma_t}(\mathrm{d},\Omega)$.

(iii) There exists $c > 0$ such that for all $E \in \mathsf{H}^{q,k}_{\Gamma_t}(\mathrm{d},\Omega)$

$$\begin{aligned} c|\widetilde{\mathcal{Q}}^{q,k}_{\mathrm{d},\Gamma_t,1}E|_{\mathsf{H}^{q,k+1}(\Omega)} &\leq |\mathrm{d}E|_{\mathsf{H}^{q+1,k}(\Omega)} \leq |E|_{\mathsf{H}^{q,k}(\mathrm{d},\Omega)}, \\ |\widetilde{\mathcal{N}}^{q,k}_{\mathrm{d},\Gamma_t}E|_{\mathsf{H}^{q,k}(\Omega)} &\leq |E|_{\mathsf{H}^{q,k}(\Omega)} + |\widetilde{\mathcal{Q}}^{q,k}_{\mathrm{d},\Gamma_t,1}E|_{\mathsf{H}^{q,k}(\Omega)}. \end{aligned}$$

(iii') For $E \in \mathsf{H}^{q,k}_{\Gamma_t,0}(\mathrm{d},\Omega)$, we have $\widetilde{\mathcal{Q}}^{q,k}_{\mathrm{d},\Gamma_t,1}E = 0$ and $\widetilde{\mathcal{N}}^{q,k}_{\mathrm{d},\Gamma_t}E = E$, that is, $\widetilde{\mathcal{Q}}^{q,k}_{\mathrm{d},\Gamma_t,1}|_{\mathsf{H}^{q,k}_{\Gamma_t,0}(\mathrm{d},\Omega)} = 0$ and $\widetilde{\mathcal{N}}^{q,k}_{\mathrm{d},\Gamma_t}|_{\mathsf{H}^{q,k}_{\Gamma_t,0}(\mathrm{d},\Omega)} = \mathrm{id}_{\mathsf{H}^{q,k}_{\Gamma_t,0}(\mathrm{d},\Omega)}$. In particular, $\widetilde{\mathcal{N}}^{q,k}_{\mathrm{d},\Gamma_t}$ is onto.

Theorem 4.18 (ii') shows by induction and by Hodge ⋆-duality:

Corollary 4.20 (higher-order kernels for the de Rham complex)**.** *Let (Ω, Γ_t) be a bounded strong Lipschitz pair and let k, ℓ ≥ 0. Then the bounded regular kernel decompositions*

$$\mathsf{H}^{q,k}_{\Gamma_t,0}(\mathrm{d},\Omega) = \mathsf{H}^{q,\ell}_{\Gamma_t,0}(\mathrm{d},\Omega) + \mathrm{d}\mathsf{H}^{q-1,k+1}_{\Gamma_t}(\Omega),\ \ \mathsf{H}^{q,k}_{\Gamma_n,0}(\delta,\Omega) = \mathsf{H}^{q,\ell}_{\Gamma_n,0}(\delta,\Omega) + \delta\mathsf{H}^{q+1,k+1}_{\Gamma_n}(\Omega)$$

hold. In particular, for k $= 0$ *and all* $\ell \geq 0$

$$\mathsf{H}^{q,0}_{\Gamma_t,0}(\mathrm{d},\Omega) = \mathsf{H}^{q,\ell}_{\Gamma_t,0}(\mathrm{d},\Omega) + \mathrm{d}\mathsf{H}^{q-1,1}_{\Gamma_t}(\Omega),\ \ \mathsf{H}^{q,0}_{\Gamma_n,0}(\delta,\Omega) = \mathsf{H}^{q,\ell}_{\Gamma_n,0}(\delta,\Omega) + \delta\mathsf{H}^{q+1,1}_{\Gamma_n}(\Omega).$$

4.7 | Dirichlet/Neumann forms

By Lemma 3.6, we recall the orthonormal Helmholtz decompositions

$$\begin{aligned}
\mathsf{L}^{q,2}_{\varepsilon}(\Omega) &= \mathrm{d}\mathsf{H}^{q-1,0}_{\Gamma_t}(\mathrm{d},\Omega) \oplus_{\mathsf{L}^{q,2}_{\varepsilon}(\Omega)} \varepsilon^{-1}\mathsf{H}^{q,0}_{\Gamma_n,0}(\delta,\Omega)\\
&= \mathsf{H}^{q,0}_{\Gamma_t,0}(\mathrm{d},\Omega) \oplus_{\mathsf{L}^{q,2}_{\varepsilon}(\Omega)} \varepsilon^{-1}\delta\mathsf{H}^{q+1,0}_{\Gamma_n}(\delta,\Omega)\\
&= \mathrm{d}\mathsf{H}^{q-1,0}_{\Gamma_t}(\mathrm{d},\Omega) \oplus_{\mathsf{L}^{q,2}_{\varepsilon}(\Omega)} \mathcal{H}^{q}_{\Gamma_t,\Gamma_n,\varepsilon}(\Omega) \oplus_{\mathsf{L}^{q,2}_{\varepsilon}(\Omega)} \varepsilon^{-1}\delta\mathsf{H}^{q+1,0}_{\Gamma_n}(\delta,\Omega),\\
\mathsf{H}^{q,0}_{\Gamma_t,0}(\mathrm{d},\Omega) &= \mathrm{d}\mathsf{H}^{q-1,0}_{\Gamma_t}(\mathrm{d},\Omega) \oplus_{\mathsf{L}^{q,2}_{\varepsilon}(\Omega)} \mathcal{H}^{q}_{\Gamma_t,\Gamma_n,\varepsilon}(\Omega),\\
\varepsilon^{-1}\mathsf{H}^{q,0}_{\Gamma_n,0}(\delta,\Omega) &= \mathcal{H}^{q}_{\Gamma_t,\Gamma_n,\varepsilon}(\Omega) \oplus_{\mathsf{L}^{q,2}_{\varepsilon}(\Omega)} \varepsilon^{-1}\delta\mathsf{H}^{q+1,0}_{\Gamma_n}(\delta,\Omega).
\end{aligned} \tag{18}$$

Let us denote the $\mathsf{L}^{q,2}_{\varepsilon}(\Omega)$-orthonormal projector onto $\varepsilon^{-1}\mathsf{H}^{q,0}_{\Gamma_n,0}(\delta,\Omega)$ and $\mathsf{H}^{q,0}_{\Gamma_t,0}(\mathrm{d},\Omega)$ by

$$\pi_\delta : \mathsf{L}^{q,2}_{\varepsilon}(\Omega) \to \varepsilon^{-1}\mathsf{H}^{q,0}_{\Gamma_n,0}(\delta,\Omega), \qquad \pi_{\mathrm{d}} : \mathsf{L}^{q,2}_{\varepsilon}(\Omega) \to \mathsf{H}^{q,0}_{\Gamma_t,0}(\mathrm{d},\Omega),$$

respectively. Then

$$\pi_\delta|_{\mathsf{H}^{q,0}_{\Gamma_t,0}(\mathrm{d},\Omega)} : \mathsf{H}^{q,0}_{\Gamma_t,0}(\mathrm{d},\Omega) \to \mathcal{H}^{q}_{\Gamma_t,\Gamma_n,\varepsilon}(\Omega), \qquad \pi_{\mathrm{d}}|_{\varepsilon^{-1}\mathsf{H}^{q,0}_{\Gamma_n,0}(\delta,\Omega)} : \varepsilon^{-1}\mathsf{H}^{q,0}_{\Gamma_n,0}(\delta,\Omega) \to \mathcal{H}^{q}_{\Gamma_t,\Gamma_n,\varepsilon}(\Omega)$$

are onto. Moreover,

$$\begin{aligned}
\pi_\delta|_{\mathrm{d}\mathsf{H}^{q-1,0}_{\Gamma_t}(\mathrm{d},\Omega)} &= 0, & \pi_{\mathrm{d}}|_{\varepsilon^{-1}\delta\mathsf{H}^{q+1,0}_{\Gamma_n}(\delta,\Omega)} &= 0,\\
\pi_\delta|_{\mathcal{H}^{q}_{\Gamma_t,\Gamma_n,\varepsilon}(\Omega)} &= \mathrm{id}_{\mathcal{H}^{q}_{\Gamma_t,\Gamma_n,\varepsilon}(\Omega)}, & \pi_{\mathrm{d}}|_{\mathcal{H}^{q}_{\Gamma_t,\Gamma_n,\varepsilon}(\Omega)} &= \mathrm{id}_{\mathcal{H}^{q}_{\Gamma_t,\Gamma_n,\varepsilon}(\Omega)}.
\end{aligned}$$

Therefore, by Corollary 4.20 and for all $\ell \geq 0$

$$\begin{aligned}
\mathcal{H}^{q}_{\Gamma_t,\Gamma_n,\varepsilon}(\Omega) &= \pi_\delta\mathsf{H}^{q,0}_{\Gamma_t,0}(\mathrm{d},\Omega) = \pi_\delta\mathsf{H}^{q,\ell}_{\Gamma_t,0}(\mathrm{d},\Omega),\\
\mathcal{H}^{q}_{\Gamma_t,\Gamma_n,\varepsilon}(\Omega) &= \pi_{\mathrm{d}}\varepsilon^{-1}\mathsf{H}^{q,0}_{\Gamma_n,0}(\delta,\Omega) = \pi_{\mathrm{d}}\varepsilon^{-1}\mathsf{H}^{q,\ell}_{\Gamma_n,0}(\delta,\Omega).
\end{aligned}$$

Hence with

$$\mathsf{H}^{q,\infty}_{\Gamma_t,0}(\mathrm{d},\Omega) := \bigcap_{\ell\geq 0}\mathsf{H}^{q,\ell}_{\Gamma_t,0}(\mathrm{d},\Omega), \qquad \mathsf{H}^{q,\infty}_{\Gamma_n,0}(\delta,\Omega) := \bigcap_{\ell\geq 0}\mathsf{H}^{q,\ell}_{\Gamma_n,0}(\delta,\Omega)$$

we get by the monotonicity of the Sobolev spaces the following result:

Theorem 4.21 (smooth prebases of Dirichlet/Neumann forms for the de Rham complex)**.** *Let (Ω,Γ_t) be a bounded strong Lipschitz pair and recall d^q_{Ω,Γ_t} from Remark 4.10. Then*

$$\pi_\delta\mathsf{H}^{q,\infty}_{\Gamma_t,0}(\mathrm{d},\Omega) = \mathcal{H}^{q}_{\Gamma_t,\Gamma_n,\varepsilon}(\Omega) = \pi_{\mathrm{d}}\varepsilon^{-1}\mathsf{H}^{q,\infty}_{\Gamma_n,0}(\delta,\Omega).$$

Moreover, there exists a smooth d-prebasis and a smooth δ-prebasis of $\mathcal{H}^{q}_{\Gamma_t,\Gamma_n,\varepsilon}(\Omega)$; that is, there are linear independent smooth forms

$$\mathcal{B}^q_{\mathrm{d},\Gamma_t}(\Omega) := \{B^q_{\mathrm{d},\Gamma_t,\ell}\}_{\ell=1}^{d^q_{\Omega,\Gamma_t}} \subset \mathsf{H}^{q,\infty}_{\Gamma_t,0}(\mathrm{d},\Omega), \mathcal{B}^q_{\delta,\Gamma_n}(\Omega) := \{B^q_{\delta,\Gamma_n,\ell}\}_{\ell=1}^{d^q_{\Omega,\Gamma_t}} \subset \mathsf{H}^{q,\infty}_{\Gamma_n,0}(\delta,\Omega)$$

such that $\pi_\delta\mathcal{B}^q_{\mathrm{d},\Gamma_t}(\Omega)$ and $\pi_{\mathrm{d}}\varepsilon^{-1}\mathcal{B}^q_{\delta,\Gamma_n}(\Omega)$ are both bases of $\mathcal{H}^{q}_{\Gamma_t,\Gamma_n,\varepsilon}(\Omega)$. In particular,

$$\mathrm{Lin}\,\pi_\delta\mathcal{B}^q_{\mathrm{d},\Gamma_t}(\Omega) = \mathcal{H}^{q}_{\Gamma_t,\Gamma_n,\varepsilon}(\Omega) = \mathrm{Lin}\,\pi_{\mathrm{d}}\varepsilon^{-1}\mathcal{B}^q_{\delta,\Gamma_n}(\Omega).$$

Note that $(1-\pi_\delta)$ and $(1-\pi_{\mathrm{d}})$ are the $\mathsf{L}^{q,2}_{\varepsilon}(\Omega)$-orthonormal projectors onto $\mathrm{d}\mathsf{H}^{q-1,0}_{\Gamma_t}(\mathrm{d},\Omega)$ and $\varepsilon^{-1}\delta\mathsf{H}^{q+1,0}_{\Gamma_n}(\delta,\Omega)$, respectively, that is,

$$(1-\pi_\delta) : \mathsf{L}^{q,2}_{\varepsilon}(\Omega) \to \mathrm{d}\mathsf{H}^{q-1,0}_{\Gamma_t}(\mathrm{d},\Omega), \quad (1-\pi_{\mathrm{d}}) : \mathsf{L}^{q,2}_{\varepsilon}(\Omega) \to \varepsilon^{-1}\delta\mathsf{H}^{q+1,0}_{\Gamma_n}(\delta,\Omega).$$

Then by (18) and Corollary 4.7, cf. Theorem 4.18 (i), we have

$$
\begin{aligned}
\mathsf{H}^{q,0}_{\Gamma_t,0}(\mathrm{d},\Omega) &= \mathrm{d}\mathsf{H}^{q-1,0}_{\Gamma_t}(\mathrm{d},\Omega) \ \oplus_{\mathsf{L}^{q,2}_{\varepsilon}(\Omega)} \mathcal{H}^{q}_{\Gamma_t,\Gamma_n,\varepsilon}(\Omega)\\
&= \mathrm{d}\mathsf{H}^{q-1,0}_{\Gamma_t}(\mathrm{d},\Omega) \ \oplus_{\mathsf{L}^{q,2}_{\varepsilon}(\Omega)} \mathrm{Lin}\,\pi_\delta \mathcal{B}^{q}_{\mathrm{d},\Gamma_t}(\Omega)\\
&= \mathrm{d}\mathsf{H}^{q-1,0}_{\Gamma_t}(\mathrm{d},\Omega) + (\pi_\delta - 1)\mathrm{Lin}\ \mathcal{B}^{q}_{\mathrm{d},\Gamma_t}(\Omega) + \mathrm{Lin}\ \mathcal{B}^{q}_{\mathrm{d},\Gamma_t}(\Omega)\\
&= \mathrm{d}\mathsf{H}^{q-1,0}_{\Gamma_t}(\mathrm{d},\Omega) + \mathrm{Lin}\ \mathcal{B}^{q}_{\mathrm{d},\Gamma_t}(\Omega),\\
\mathsf{H}^{q,k}_{\Gamma_t,0}(\mathrm{d},\Omega) &= \mathrm{d}\mathsf{H}^{q-1,0}_{\Gamma_t}(\mathrm{d},\Omega) \cap \mathsf{H}^{q,k}_{\Gamma_t,0}(\mathrm{d},\Omega) + \mathrm{Lin}\ \mathcal{B}^{q}_{\mathrm{d},\Gamma_t}(\Omega),\\
&= \mathrm{d}\mathsf{H}^{q-1,k+1}_{\Gamma_t}(\Omega) + \mathrm{Lin}\ \mathcal{B}^{q}_{\mathrm{d},\Gamma_t}(\Omega).
\end{aligned} \tag{19}
$$

Theorem 4.22 (higher-order bounded regular direct decompositions for the de Rham complex)**.** *Let (Ω,Γ_t) be a bounded strong Lipschitz pair and let $k \geq 0$. Then the bounded regular direct decompositions*

$$
\mathsf{H}^{q,k}_{\Gamma_t}(\mathrm{d},\Omega) = R(\widetilde{\mathcal{Q}}^{q,k}_{\mathrm{d},\Gamma_t,1}) \dotplus \mathsf{H}^{q,k}_{\Gamma_t,0}(\mathrm{d},\Omega), \qquad \mathsf{H}^{q,k}_{\Gamma_t,0}(\mathrm{d},\Omega) = \mathrm{d}\mathsf{H}^{q-1,k+1}_{\Gamma_t}(\Omega) \dotplus \mathit{Lin}\ \mathcal{B}^{q}_{\mathrm{d},\Gamma_t}(\Omega),
$$

$$
\mathsf{H}^{q,k}_{\Gamma_n}(\delta,\Omega) = R(\widetilde{\mathcal{Q}}^{q,k}_{\delta,\Gamma_n,1}) \dotplus \mathsf{H}^{q,k}_{\Gamma_n,0}(\delta,\Omega), \qquad \mathsf{H}^{q,k}_{\Gamma_n,0}(\delta,\Omega) = \delta\mathsf{H}^{q+1,k+1}_{\Gamma_n}(\Omega) \dotplus \mathit{Lin}\ \mathcal{B}^{q}_{\delta,\Gamma_n}(\Omega)
$$

hold. Note that $R(\widetilde{\mathcal{Q}}^{q,k}_{\mathrm{d},\Gamma_t,1}) \subset \mathsf{H}^{q,k+1}_{\Gamma_t}(\Omega)$ and $R(\widetilde{\mathcal{Q}}^{q,k}_{\delta,\Gamma_n,1}) \subset \mathsf{H}^{q,k+1}_{\Gamma_n}(\Omega)$. In particular, for $k = 0$

$$
\mathsf{H}^{q,0}_{\Gamma_t}(\mathrm{d},\Omega) = R(\widetilde{\mathcal{Q}}^{q,0}_{\mathrm{d},\Gamma_t,1}) \dotplus \mathsf{H}^{q,0}_{\Gamma_t,0}(\mathrm{d},\Omega), \qquad
\begin{aligned}
\mathsf{H}^{q,0}_{\Gamma_t,0}(\mathrm{d},\Omega) &= \mathrm{d}\mathsf{H}^{q-1,1}_{\Gamma_t}(\Omega) \dotplus \mathit{Lin}\ \mathcal{B}^{q}_{\mathrm{d},\Gamma_t}(\Omega)\\
&= \mathrm{d}\mathsf{H}^{q-1,1}_{\Gamma_t}(\Omega) \ \oplus_{\mathsf{L}^{q,2}_{\varepsilon}(\Omega)} \mathcal{H}^{q}_{\Gamma_t,\Gamma_n,\varepsilon}(\Omega),
\end{aligned}
$$

$$
\mathsf{H}^{q,0}_{\Gamma_n}(\delta,\Omega) = R(\widetilde{\mathcal{Q}}^{q,0}_{\delta,\Gamma_n,1}) \dotplus \mathsf{H}^{q,0}_{\Gamma_n,0}(\delta,\Omega), \qquad
\begin{aligned}
\varepsilon^{-1}\mathsf{H}^{q,0}_{\Gamma_n,0}(\delta,\Omega) &= \varepsilon^{-1}\delta\mathsf{H}^{q+1,1}_{\Gamma_n}(\Omega) \dotplus \varepsilon^{-1}\mathit{Lin}\ \mathcal{B}^{q}_{\delta,\Gamma_n}(\Omega)\\
&= \varepsilon^{-1}\delta\mathsf{H}^{q+1,1}_{\Gamma_n}(\Omega) \ \oplus_{\mathsf{L}^{q,2}_{\varepsilon}(\Omega)} \mathcal{H}^{q}_{\Gamma_t,\Gamma_n,\varepsilon}(\Omega)
\end{aligned}
$$

as well as

$$
\begin{aligned}
\mathsf{L}^{q,2}_{\varepsilon}(\Omega) &= \mathsf{H}^{q,0}_{\Gamma_t,0}(\mathrm{d},\Omega) \ \oplus_{\mathsf{L}^{q,2}_{\varepsilon}(\Omega)} \varepsilon^{-1}\delta\mathsf{H}^{q+1,1}_{\Gamma_n}(\Omega)\\
&= \mathrm{d}\mathsf{H}^{q-1,1}_{\Gamma_t}(\Omega) \ \oplus_{\mathsf{L}^{q,2}_{\varepsilon}(\Omega)} \varepsilon^{-1}\mathsf{H}^{q,0}_{\Gamma_n,0}(\delta,\Omega).
\end{aligned}
$$

Proof. Theorem 4.18 (ii) and (19) show

$$
\mathsf{H}^{q,k}_{\Gamma_t}(\mathrm{d},\Omega) = R(\widetilde{\mathcal{Q}}^{q,k}_{\mathrm{d},\Gamma_t,1}) \dotplus \mathsf{H}^{q,k}_{\Gamma_t,0}(\mathrm{d},\Omega), \qquad \mathsf{H}^{q,k}_{\Gamma_t,0}(\mathrm{d},\Omega) = \mathrm{d}\mathsf{H}^{q-1,k+1}_{\Gamma_t}(\Omega) + \mathrm{Lin}\ \mathcal{B}^{q}_{\mathrm{d},\Gamma_t}(\Omega).
$$

To prove the directness, let

$$
\sum_{\ell=1}^{d^q_{\Omega,\Gamma_t}} \lambda_\ell B^q_{\mathrm{d},\Gamma_t,\ell} \in \mathrm{d}\mathsf{H}^{q-1,k+1}_{\Gamma_t}(\Omega) \cap \mathrm{Lin}\ \mathcal{B}^{q}_{\mathrm{d},\Gamma_t}(\Omega).
$$

Then $0 = \sum_\ell \lambda_\ell \pi_\delta B^q_{\mathrm{d},\Gamma_t,\ell} \in \mathrm{Lin}\pi_\delta \mathcal{B}^q_{\mathrm{d},\Gamma_t}(\Omega)$ and hence $\lambda_\ell = 0$ for all ℓ as $\pi_\delta \mathcal{B}^q_{\mathrm{d},\Gamma_t}(\Omega)$ is a basis of $\mathcal{H}^q_{\Gamma_t,\Gamma_n,\varepsilon}(\Omega)$ by Theorem 4.21. Concerning the boundedness of the decompositions, let

$$
\mathsf{H}^{q,k}_{\Gamma_t,0}(\mathrm{d},\Omega) \ni E = \mathrm{d}H + B, \qquad H \in \mathsf{H}^{q-1,k+1}_{\Gamma_t}(\Omega),\ B \in \mathrm{Lin}\ \mathcal{B}^{q}_{\mathrm{d},\Gamma_t}(\Omega).
$$

Then we have by Theorem 4.18 (i) $\mathrm{d}H \in R(\mathrm{d}^{q-1,k}_{\Gamma_t})$ and $E_\mathrm{d} := \mathcal{P}^{q,k}_{\mathrm{d},\Gamma_t}\mathrm{d}H \in \mathsf{H}^{q-1,k+1}_{\Gamma_t}(\Omega)$ solves $\mathrm{d}E_\mathrm{d} = \mathrm{d}H$ with $|E_\mathrm{d}|_{\mathsf{H}^{q-1,k+1}(\Omega)} \leq c|\mathrm{d}H|_{\mathsf{H}^{q,k}(\Omega)}$. Therefore,

$$
|E_\mathrm{d}|_{\mathsf{H}^{q-1,k+1}(\Omega)} + |B|_{\mathsf{H}^{q,k}(\Omega)} \leq c\left(|\mathrm{d}H|_{\mathsf{H}^{q,k}(\Omega)} + |B|_{\mathsf{H}^{q,k}(\Omega)}\right) \leq c\left(|E|_{\mathsf{H}^{q,k}(\Omega)} + |B|_{\mathsf{H}^{q,k}(\Omega)}\right).
$$

Note that the mapping

$$I_{\mathcal{H}}\,:\,\operatorname{Lin}\,\mathcal{B}^q_{\mathrm{d},\Gamma_t}(\Omega)\to\operatorname{Lin}\,\pi_\delta\mathcal{B}^q_{\mathrm{d},\Gamma_t}(\Omega)=\mathcal{H}^q_{\Gamma_t,\Gamma_n,\varepsilon}(\Omega);\,B^q_{\mathrm{d},\Gamma_t,\ell}\mapsto\pi_\delta B^q_{\mathrm{d},\Gamma_t,\ell}$$

is a topological isomorphism (between finite dimensional spaces and with arbitrary norms). Thus,

$$|B|_{\mathsf{H}^{q,k}(\Omega)}\le c|B|_{\mathsf{L}^{q,2}(\Omega)}\le c|\pi_\delta B|_{\mathsf{L}^{q,2}(\Omega)}=c|\pi_\delta E|_{\mathsf{L}^{q,2}(\Omega)}\le c|E|_{\mathsf{L}^{q,2}(\Omega)}\le c|E|_{\mathsf{H}^{q,k}(\Omega)}.$$

Finally, we see $E=\mathrm{d}E_\mathrm{d}+B\in\mathrm{d}\mathsf{H}^{q-1,k+1}_{\Gamma_t}(\Omega)+\operatorname{Lin}\,\mathcal{B}^q_{\mathrm{d},\Gamma_t}(\Omega)$ and

$$|E_\mathrm{d}|_{\mathsf{H}^{q-1,k+1}(\Omega)}+|B|_{\mathsf{H}^{q,k}(\Omega)}\le c|E|_{\mathsf{H}^{q,k}(\Omega)}.$$

Hodge $\star$-duality yields the other assertions. □

Remark 4.23. (higher-order bounded regular direct decompositions for the de Rham complex) Note that by Theorem 4.22, we have, for example,

$$\mathsf{H}^{q,k}_{\Gamma_t}(\mathrm{d},\Omega)=R(\widetilde{\mathcal{Q}}^{q,k}_{\mathrm{d},\Gamma_t,1})\dotplus\operatorname{Lin}\,\mathcal{B}^q_{\mathrm{d},\Gamma_t}(\Omega)\dotplus\mathrm{d}\mathsf{H}^{q-1,k+1}_{\Gamma_t}(\Omega)=\mathsf{H}^{q,k+1}_{\Gamma_t}(\Omega)+\mathrm{d}\mathsf{H}^{q-1,k+1}_{\Gamma_t}(\Omega)$$

with bounded linear regular direct decomposition operators

$$\begin{aligned}\widehat{\mathcal{Q}}^{q,k}_{\mathrm{d},\Gamma_t,1}&:\mathsf{H}^{q,k}_{\Gamma_t}(\mathrm{d},\Omega)\to R(\widetilde{\mathcal{Q}}^{q,k}_{\mathrm{d},\Gamma_t,1}), & R(\widetilde{\mathcal{Q}}^{q,k}_{\mathrm{d},\Gamma_t,1})&\subset\mathsf{H}^{q,k+1}_{\Gamma_t}(\Omega),\\ \widehat{\mathcal{Q}}^{q,k}_{\mathrm{d},\Gamma_t,\infty}&:\mathsf{H}^{q,k}_{\Gamma_t}(\mathrm{d},\Omega)\to\operatorname{Lin}\,\mathcal{B}^q_{\mathrm{d},\Gamma_t}(\Omega), & \mathcal{B}^q_{\mathrm{d},\Gamma_t}(\Omega)&\subset\mathsf{H}^{q,\infty}_{\Gamma_t,0}(\mathrm{d},\Omega)\subset\mathsf{H}^{q,k+1}_{\Gamma_t}(\Omega),\\ \widehat{\mathcal{Q}}^{q,k}_{\mathrm{d},\Gamma_t,0}&:\mathsf{H}^{q,k}_{\Gamma_t}(\mathrm{d},\Omega)\to\mathsf{H}^{q-1,k+1}_{\Gamma_t}(\Omega)\end{aligned}$$

satisfying $\widehat{\mathcal{Q}}^{q,k}_{\mathrm{d},\Gamma_t,1}+\widehat{\mathcal{Q}}^{q,k}_{\mathrm{d},\Gamma_t,\infty}+\mathrm{d}\widehat{\mathcal{Q}}^{q,k}_{\mathrm{d},\Gamma_t,0}=\mathrm{id}_{\mathsf{H}^{q,k}_{\Gamma_t}(\mathrm{d},\Omega)}$. A closer inspection of the latter proof allows for a more precise description of these bounded decomposition operators.

For this, let $E\in\mathsf{H}^{q,k}_{\Gamma_t}(\mathrm{d},\Omega)$. According to Theorem 4.18 and Remark 4.19, we decompose

$$E=E_R+E_N\in R(\widetilde{\mathcal{Q}}^{q,k}_{\mathrm{d},\Gamma_t,1})\dotplus R(\widetilde{\mathcal{N}}^{q,k}_{\mathrm{d},\Gamma_t}),\,R(\widetilde{\mathcal{N}}^{q,k}_{\mathrm{d},\Gamma_t})=\mathsf{H}^{q,k}_{\Gamma_t,0}(\mathrm{d},\Omega)=N(\mathrm{d}^{q,k}_{\Gamma_t}),$$

with $E_R=\widetilde{\mathcal{Q}}^{q,k}_{\mathrm{d},\Gamma_t,1}E$ and $E_N=\widetilde{\mathcal{N}}^{q,k}_{\mathrm{d},\Gamma_t}E$. By Theorem 4.22, we further decompose

$$\mathsf{H}^{q,k}_{\Gamma_t,0}(\mathrm{d},\Omega)\ni E_N=\mathrm{d}E_\mathrm{d}+B\in\mathrm{d}\mathsf{H}^{q-1,k+1}_{\Gamma_t}(\Omega)\dotplus\operatorname{Lin}\,\mathcal{B}^q_{\mathrm{d},\Gamma_t}(\Omega).$$

Then $\pi_\delta E_N=\pi_\delta B\in\mathcal{H}^q_{\Gamma_t,\Gamma_n,\varepsilon}(\Omega)$, and thus, $B=I^{-1}_{\mathcal{H}}\pi_\delta B=I^{-1}_{\mathcal{H}}\pi_\delta E_N\in\operatorname{Lin}\,\mathcal{B}^q_{\mathrm{d},\Gamma_t}(\Omega)$. Therefore, $E_\mathrm{d}=\mathcal{P}^{q,k}_{\mathrm{d},\Gamma_t}\mathrm{d}E_\mathrm{d}=\mathcal{P}^{q,k}_{\mathrm{d},\Gamma_t}(E_N-B)=\mathcal{P}^{q,k}_{\mathrm{d},\Gamma_t}(1-I^{-1}_{\mathcal{H}}\pi_\delta)E_N$. Finally, we see

$$\begin{aligned}\widehat{\mathcal{Q}}^{q,k}_{\mathrm{d},\Gamma_t,1}&=\widetilde{\mathcal{Q}}^{q,k}_{\mathrm{d},\Gamma_t,1}=\mathcal{P}^{q+1,k}_{\mathrm{d},\Gamma_t}\mathrm{d}^{q,k}_{\Gamma_t}=\mathcal{Q}^{q,k}_{\mathrm{d},\Gamma_t,1}(\mathrm{d}^{q,k}_{\Gamma_t})^{-1}_\perp\mathrm{d}^{q,k}_{\Gamma_t},\\ \widehat{\mathcal{Q}}^{q,k}_{\mathrm{d},\Gamma_t,\infty}&=I^{-1}_{\mathcal{H}}\pi_\delta\widetilde{\mathcal{N}}^{q,k}_{\mathrm{d},\Gamma_t}=I^{-1}_{\mathcal{H}}\pi_\delta(1-\widetilde{\mathcal{Q}}^{q,k}_{\mathrm{d},\Gamma_t,1}),\\ \widehat{\mathcal{Q}}^{q,k}_{\mathrm{d},\Gamma_t,0}&=\mathcal{P}^{q,k}_{\mathrm{d},\Gamma_t}(1-I^{-1}_{\mathcal{H}}\pi_\delta)\widetilde{\mathcal{N}}^{q,k}_{\mathrm{d},\Gamma_t}=\mathcal{P}^{q,k}_{\mathrm{d},\Gamma_t}(1-I^{-1}_{\mathcal{H}}\pi_\delta)(1-\widetilde{\mathcal{Q}}^{q,k}_{\mathrm{d},\Gamma_t,1}).\end{aligned}$$

Theorem 4.24 (alternative Dirichlet/Neumann projections for the de Rham complex)**.** *Let (Ω,Γ_t) be a bounded strong Lipschitz pair. Then*

$$\begin{aligned}\mathcal{H}^q_{\Gamma_t,\Gamma_n,\varepsilon}(\Omega)\cap\mathcal{B}^q_{\mathrm{d},\Gamma_t}(\Omega)^{\perp_{\mathsf{L}^{q,2}_\varepsilon(\Omega)}}&=\{0\}, & \varepsilon^{-1}\mathsf{H}^{q,0}_{\Gamma_n,0}(\delta,\Omega)\cap\mathcal{B}^q_{\mathrm{d},\Gamma_t}(\Omega)^{\perp_{\mathsf{L}^{q,2}_\varepsilon(\Omega)}}&=\varepsilon^{-1}\delta\mathsf{H}^{q+1,0}_{\Gamma_n}(\delta,\Omega),\\ \mathcal{H}^q_{\Gamma_t,\Gamma_n,\varepsilon}(\Omega)\cap\mathcal{B}^q_{\delta,\Gamma_n}(\Omega)^{\perp_{\mathsf{L}^{q,2}(\Omega)}}&=\{0\}, & \mathsf{H}^{q,0}_{\Gamma_t,0}(\mathrm{d},\Omega)\cap\mathcal{B}^q_{\delta,\Gamma_n}(\Omega)^{\perp_{\mathsf{L}^{q,2}(\Omega)}}&=\mathrm{d}\mathsf{H}^{q-1,0}_{\Gamma_t}(\mathrm{d},\Omega).\end{aligned}$$

Proof. For $H \in \mathcal{H}^q_{\Gamma_t,\Gamma_n,\varepsilon}(\Omega) \cap \mathcal{B}^q_{\mathrm{d},\Gamma_t}(\Omega)^{\perp_{\mathsf{L}^{q,2}_\varepsilon(\Omega)}}$, we have

$$0 = \langle H, B^q_{\mathrm{d},\Gamma_t,\ell}\rangle_{\mathsf{L}^{q,2}_\varepsilon(\Omega)} = \langle \pi_\delta H, B^q_{\mathrm{d},\Gamma_t,\ell}\rangle_{\mathsf{L}^{q,2}_\varepsilon(\Omega)} = \langle H, \pi_\delta B^q_{\mathrm{d},\Gamma_t,\ell}\rangle_{\mathsf{L}^{q,2}_\varepsilon(\Omega)}$$

and hence $H = 0$ by Theorem 4.21. Analogously, we see for $H \in \mathcal{H}^q_{\Gamma_t,\Gamma_n,\varepsilon}(\Omega) \cap \mathcal{B}^q_{\delta,\Gamma_n}(\Omega)^{\perp_{\mathsf{L}^{q,2}(\Omega)}}$

$$0 = \langle H, B^q_{\delta,\Gamma_n,\ell}\rangle_{\mathsf{L}^{q,2}(\Omega)} = \langle \pi_{\mathrm{d}} H, \varepsilon^{-1} B^q_{\delta,\Gamma_n,\ell}\rangle_{\mathsf{L}^{q,2}_\varepsilon(\Omega)} = \langle H, \pi_{\mathrm{d}}\varepsilon^{-1} B^q_{\delta,\Gamma_n,\ell}\rangle_{\mathsf{L}^{q,2}_\varepsilon(\Omega)}$$

and thus $H = 0$. It holds

$$\varepsilon^{-1}\delta\mathsf{H}^{q+1,0}_{\Gamma_n}(\delta,\Omega) \perp_{\mathsf{L}^{q,2}_\varepsilon(\Omega)} \mathcal{B}^q_{\mathrm{d},\Gamma_t}(\Omega), \qquad \mathrm{d}\mathsf{H}^{q-1,0}_{\Gamma_t}(\mathrm{d},\Omega) \perp_{\mathsf{L}^{q,2}(\Omega)} \mathcal{B}^q_{\delta,\Gamma_n}(\Omega). \tag{20}$$

According to (18), we can decompose

$$\begin{aligned} \varepsilon^{-1}\mathsf{H}^{q,0}_{\Gamma_n,0}(\delta,\Omega) &= \varepsilon^{-1}\delta\mathsf{H}^{q+1,0}_{\Gamma_n}(\delta,\Omega) \oplus_{\mathsf{L}^{q,2}_\varepsilon(\Omega)} \mathcal{H}^q_{\Gamma_t,\Gamma_n,\varepsilon}(\Omega), \\ \mathsf{H}^{q,0}_{\Gamma_t,0}(\mathrm{d},\Omega) &= \mathrm{d}\mathsf{H}^{q-1,0}_{\Gamma_t}(\mathrm{d},\Omega) \oplus_{\mathsf{L}^{q,2}_\varepsilon(\Omega)} \mathcal{H}^q_{\Gamma_t,\Gamma_n,\varepsilon}(\Omega), \end{aligned}$$

which shows by (20) the other two assertions. □

Corollary 4.25 (alternative Dirichlet/Neumann projections for the de Rham complex). *Let (Ω, Γ_t) be a bounded strong Lipschitz pair and let $k \geq 0$. Then*

$$\begin{aligned} \varepsilon^{-1}\mathsf{H}^{q,k}_{\Gamma_n,0}(\delta,\Omega) \cap \mathcal{B}^q_{\mathrm{d},\Gamma_t}(\Omega)^{\perp_{\mathsf{L}^{q,2}_\varepsilon(\Omega)}} &= \varepsilon^{-1}\delta\mathsf{H}^{q+1,k}_{\Gamma_n}(\delta,\Omega) = \varepsilon^{-1}\delta\mathsf{H}^{q+1,k+1}_{\Gamma_n}(\Omega), \\ \mathsf{H}^{q,k}_{\Gamma_t,0}(\mathrm{d},\Omega) \cap \mathcal{B}^q_{\delta,\Gamma_n}(\Omega)^{\perp_{\mathsf{L}^{q,2}(\Omega)}} &= \mathrm{d}\mathsf{H}^{q-1,k}_{\Gamma_t}(\mathrm{d},\Omega) = \mathrm{d}\mathsf{H}^{q-1,k+1}_{\Gamma_t}(\Omega). \end{aligned}$$

Proof. We have by Theorem 4.24 and Theorem 4.18 (i)

$$\begin{aligned} \mathsf{H}^{q,k}_{\Gamma_t,0}(\mathrm{d},\Omega) \cap \mathcal{B}^q_{\delta,\Gamma_n}(\Omega)^{\perp_{\mathsf{L}^{q,2}(\Omega)}} &= \mathsf{H}^{q,k}_{\Gamma_t}(\Omega) \cap \mathsf{H}^{q,0}_{\Gamma_t,0}(\mathrm{d},\Omega) \cap \mathcal{B}^q_{\delta,\Gamma_n}(\Omega)^{\perp_{\mathsf{L}^{q,2}(\Omega)}} \\ &= \mathsf{H}^{q,k}_{\Gamma_t}(\Omega) \cap \mathrm{d}\mathsf{H}^{q-1,0}_{\Gamma_t}(\mathrm{d},\Omega) \\ &= \mathrm{d}\mathsf{H}^{q-1,k}_{\Gamma_t}(\mathrm{d},\Omega) = \mathrm{d}\mathsf{H}^{q-1,k+1}_{\Gamma_t}(\Omega). \end{aligned}$$

Analogously,

$$\begin{aligned} \varepsilon^{-1}\mathsf{H}^{q,k}_{\Gamma_n,0}(\delta,\Omega) \cap \mathcal{B}^q_{\mathrm{d},\Gamma_t}(\Omega)^{\perp_{\mathsf{L}^{q,2}_\varepsilon(\Omega)}} &= \varepsilon^{-1}\mathsf{H}^{q,k}_{\Gamma_n}(\Omega) \cap \varepsilon^{-1}\mathsf{H}^{q,0}_{\Gamma_n,0}(\delta,\Omega) \cap \mathcal{B}^q_{\mathrm{d},\Gamma_t}(\Omega)^{\perp_{\mathsf{L}^{q,2}_\varepsilon(\Omega)}} \\ &= \varepsilon^{-1}\mathsf{H}^{q,k}_{\Gamma_n}(\Omega) \cap \varepsilon^{-1}\delta\mathsf{H}^{q+1,0}_{\Gamma_n}(\delta,\Omega) \\ &= \varepsilon^{-1}\delta\mathsf{H}^{q+1,k}_{\Gamma_n}(\delta,\Omega) = \varepsilon^{-1}\delta\mathsf{H}^{q+1,k+1}_{\Gamma_n}(\Omega), \end{aligned}$$

completing the proof. □

Theorem 4.22 and $\star\mathcal{H}^q_{\Gamma_t,\Gamma_n,\mathrm{id}}(\Omega) = \mathcal{H}^{d-q}_{\Gamma_n,\Gamma_t,\mathrm{id}}(\Omega)$ shows the following result:

Theorem 4.26 (cohomology groups of the de Rham complex). *Let (Ω, Γ_t) be a bounded strong Lipschitz pair. Then ($\cong$ means isomorphic)*

$$N(\mathrm{d}^{q,k}_{\Gamma_t})/R(\mathrm{d}^{q-1,k}_{\Gamma_t}) \cong \mathit{Lin}\ \mathcal{B}^q_{\mathrm{d},\Gamma_t}(\Omega) \cong \mathcal{H}^q_{\Gamma_t,\Gamma_n,\varepsilon}(\Omega) \cong \mathit{Lin}\ \mathcal{B}^q_{\delta,\Gamma_n}(\Omega) \cong N(\delta^{q,k}_{\Gamma_n})/R(\delta^{q+1,k}_{\Gamma_n}).$$

In particular, the dimensions of the cohomology groups (Dirichlet/Neumann forms) are independent of k and ε, and it holds

$$d^q_{\Omega,\Gamma_t} = \dim\left(N(\mathrm{d}^{q,k}_{\Gamma_t})/R(\mathrm{d}^{q-1,k}_{\Gamma_t})\right) = \dim\left(N(\delta^{q,k}_{\Gamma_n})/R(\delta^{q+1,k}_{\Gamma_n})\right).$$

Moroever, $d^{q}_{\Omega,\Gamma_t} = d^{d-q}_{\Omega,\Gamma_n}$.

Remark 4.27. For the case of either no or full boundary conditions, that is, $\Gamma_t = \emptyset$ or $\Gamma_t = \Gamma$, related results on regular potentials, regular decompositions, as well as cohomology groups and their dimensions, even for real Sobolev exponents $k \in \mathbb{R}$, have been proved in Costabel and McIntosh[24] using integral equation representations and methods. In particular, we refer to Costabel and McIntosh.[24, Theorem 1.1, Theorem 4.9]

5 | VECTOR DE RHAM COMPLEX

We reformulate the results from Section 4 in the special case $d = 3$ and $q \in \{0, 1, 2, 3\}$ using vector proxies. Recall Section 3.2 and let ε and μ be admissible weights. To apply the FA-ToolBox from Section 2 for the vector de Rham complex, let grad, rot and div be realised as densely defined (unbounded) linear operators

$$\begin{aligned}
\mathring{\operatorname{grad}}_{\Gamma_t} &: D(\mathring{\operatorname{grad}}_{\Gamma_t}) \subset \mathsf{L}^2(\Omega) \to \mathsf{L}^2_\varepsilon(\Omega); && u \mapsto \operatorname{grad} u,\\
\mu^{-1}\mathring{\operatorname{rot}}_{\Gamma_t} &: D(\mu^{-1}\mathring{\operatorname{rot}}_{\Gamma_t}) \subset \mathsf{L}^2_\varepsilon(\Omega) \to \mathsf{L}^2_\mu(\Omega); && E \mapsto \mu^{-1}\operatorname{rot} E,\\
\mathring{\operatorname{div}}_{\Gamma_t}\mu &: D(\mathring{\operatorname{div}}_{\Gamma_t}\mu) \subset \mathsf{L}^2_\mu(\Omega) \to \mathsf{L}^2(\Omega); && H \mapsto \operatorname{div} \mu H
\end{aligned}$$

with domains of definition

$$D(\mathring{\operatorname{grad}}_{\Gamma_t}) := \mathsf{C}^\infty_{\Gamma_t}(\Omega), \qquad D(\mu^{-1}\mathring{\operatorname{rot}}_{\Gamma_t}) := \mathsf{C}^\infty_{\Gamma_t}(\Omega), \qquad D(\mathring{\operatorname{div}}_{\Gamma_t}\mu) := \mu^{-1}\mathsf{C}^\infty_{\Gamma_t}(\Omega)$$

satisfying the complex properties

$$\mu^{-1}\mathring{\operatorname{rot}}_{\Gamma_t}\mathring{\operatorname{grad}}_{\Gamma_t} \subset 0, \qquad \mathring{\operatorname{div}}_{\Gamma_t}\mu\,\mu^{-1}\mathring{\operatorname{rot}}_{\Gamma_t} = \mathring{\operatorname{div}}_{\Gamma_t}\mathring{\operatorname{rot}}_{\Gamma_t} \subset 0.$$

Then the closures

$$\operatorname{grad}_{\Gamma_t} := \overline{\mathring{\operatorname{grad}}_{\Gamma_t}}, \qquad \mu^{-1}\operatorname{rot}_{\Gamma_t} := \overline{\mu^{-1}\mathring{\operatorname{rot}}_{\Gamma_t}}, \qquad \operatorname{div}_{\Gamma_t}\mu := \overline{\mathring{\operatorname{div}}_{\Gamma_t}\mu}$$

and Hilbert space adjoints

$$\operatorname{grad}^*_{\Gamma_t} = \mathring{\operatorname{grad}}^*_{\Gamma_t}, \qquad (\mu^{-1}\operatorname{rot}_{\Gamma_t})^* = (\mu^{-1}\mathring{\operatorname{rot}}_{\Gamma_t})^*, \qquad (\operatorname{div}_{\Gamma_t}\mu)^* = (\mathring{\operatorname{div}}_{\Gamma_t}\mu)^*$$

are given by

$$\begin{aligned}
\mathrm{A}_0 := \operatorname{grad}_{\Gamma_t} &: D(\operatorname{grad}_{\Gamma_t}) \subset \mathsf{L}^2(\Omega) \to \mathsf{L}^2_\varepsilon(\Omega); && u \mapsto \operatorname{grad} u,\\
\mathrm{A}_1 := \mu^{-1}\operatorname{rot}_{\Gamma_t} &: D(\mu^{-1}\operatorname{rot}_{\Gamma_t}) \subset \mathsf{L}^2_\varepsilon(\Omega) \to \mathsf{L}^2_\mu(\Omega); && E \mapsto \mu^{-1}\operatorname{rot} E,\\
\mathrm{A}_2 := \operatorname{div}_{\Gamma_t}\mu &: D(\operatorname{div}_{\Gamma_t}\mu) \subset \mathsf{L}^2_\mu(\Omega) \to \mathsf{L}^2(\Omega); && H \mapsto \operatorname{div} \mu H,\\
\mathrm{A}_0^* = \operatorname{grad}^*_{\Gamma_t} = -\operatorname{div}_{\Gamma_n}\varepsilon &: D(\operatorname{div}_{\Gamma_n}\varepsilon) \subset \mathsf{L}^2_\varepsilon(\Omega) \to \mathsf{L}^2(\Omega); && E \mapsto -\operatorname{div} \varepsilon E,\\
\mathrm{A}_1^* = (\mu^{-1}\operatorname{rot}_{\Gamma_t})^* = \varepsilon^{-1}\operatorname{rot}_{\Gamma_n} &: D(\varepsilon^{-1}\operatorname{rot}_{\Gamma_n}) \subset \mathsf{L}^2_\mu(\Omega) \to \mathsf{L}^2_\varepsilon(\Omega); && H \mapsto \varepsilon^{-1}\operatorname{rot} H,\\
\mathrm{A}_2^* = (\operatorname{div}_{\Gamma_t}\mu)^* = -\operatorname{grad}_{\Gamma_n} &: D(\operatorname{grad}_{\Gamma_n}) \subset \mathsf{L}^2(\Omega) \to \mathsf{L}^2_\mu(\Omega); && u \mapsto -\operatorname{grad} u
\end{aligned}$$

with domains of definition

$$\begin{aligned}
D(\mathrm{A}_0) &= D(\operatorname{grad}_{\Gamma_t}) = \mathsf{H}^1_{\Gamma_t}(\Omega), & D(\mathrm{A}_0^*) &= D(\operatorname{div}_{\Gamma_n}\varepsilon) = \varepsilon^{-1}\mathsf{H}_{\Gamma_n}(\operatorname{div},\Omega),\\
D(\mathrm{A}_1) &= D(\mu^{-1}\operatorname{rot}_{\Gamma_t}) = \mathsf{H}_{\Gamma_t}(\operatorname{rot},\Omega), & D(\mathrm{A}_1^*) &= D(\varepsilon^{-1}\operatorname{rot}_{\Gamma_n}) = \mathsf{H}_{\Gamma_n}(\operatorname{rot},\Omega),\\
D(\mathrm{A}_2) &= D(\operatorname{div}_{\Gamma_t}\mu) = \mu^{-1}\mathsf{H}_{\Gamma_t}(\operatorname{div},\Omega), & D(\mathrm{A}_2^*) &= D(\operatorname{grad}_{\Gamma_n}) = \mathsf{H}^1_{\Gamma_n}(\Omega).
\end{aligned}$$

As in Section 4, indeed the domains of definition of the adjoints are given as stated.

Remark 5.1. Note that by definition, the adjoints are given by

$$\begin{aligned}
\operatorname{grad}_{\Gamma_t}^* = \mathring{\operatorname{grad}}_{\Gamma_t}^{\,*} &= -\mathbf{div}_{\Gamma_n}\varepsilon \,:\, D(\mathbf{div}_{\Gamma_n}\varepsilon) \subset \mathsf{L}^2_\varepsilon(\Omega) \to \mathsf{L}^2(\Omega),\\
(\mu^{-1}\operatorname{rot}_{\Gamma_t})^* = (\mu^{-1}\mathring{\operatorname{rot}}_{\Gamma_t})^* &= \varepsilon^{-1}\mathbf{rot}_{\Gamma_n} \,:\, D(\varepsilon^{-1}\mathbf{rot}_{\Gamma_n}) \subset \mathsf{L}^2_\mu(\Omega) \to \mathsf{L}^2_\varepsilon(\Omega),\\
(\operatorname{div}_{\Gamma_t}\mu)^* = (\mathring{\operatorname{div}}_{\Gamma_t}\mu)^* &= -\mathbf{grad}_{\Gamma_n} \,:\, D(\mathbf{grad}_{\Gamma_n}) \subset \mathsf{L}^2(\Omega) \to \mathsf{L}^2_\mu(\Omega)
\end{aligned}$$

with domains of definition

$$D(\mathbf{div}_{\Gamma_n}\varepsilon) = \varepsilon^{-1}\mathbf{H}_{\Gamma_n}(\operatorname{div},\Omega), \qquad D(\varepsilon^{-1}\mathbf{rot}_{\Gamma_n}) = \mathbf{H}_{\Gamma_n}(\operatorname{rot},\Omega), \qquad D(\mathbf{grad}_{\Gamma_n}) = \mathbf{H}^1_{\Gamma_n}(\Omega).$$

Lemma 3.2 (weak and strong boundary conditions coincide) shows indeed that $\mathbf{div}_{\Gamma_n}\varepsilon = \operatorname{div}_{\Gamma_n}\varepsilon$, $\varepsilon^{-1}\mathbf{rot}_{\Gamma_n} = \varepsilon^{-1}\operatorname{rot}_{\Gamma_n}$, and $\mathbf{grad}_{\Gamma_n} = \operatorname{grad}_{\Gamma_n}$, in particular

$$\begin{aligned}
D(\mathbf{div}_{\Gamma_n}\varepsilon) = \varepsilon^{-1}\mathbf{H}_{\Gamma_n}(\operatorname{div},\Omega) &= \varepsilon^{-1}\mathsf{H}_{\Gamma_n}(\operatorname{div},\Omega) = D(\operatorname{div}_{\Gamma_n}\varepsilon),\\
D(\varepsilon^{-1}\mathbf{rot}_{\Gamma_n}) = \mathbf{H}_{\Gamma_n}(\operatorname{rot},\Omega) &= \mathsf{H}_{\Gamma_n}(\operatorname{rot},\Omega) = D(\varepsilon^{-1}\operatorname{rot}_{\Gamma_n}),\\
D(\mathbf{grad}_{\Gamma_n}) = \mathbf{H}^1_{\Gamma_n}(\Omega) &= \mathsf{H}^1_{\Gamma_n}(\Omega) = D(\operatorname{grad}_{\Gamma_n}).
\end{aligned}$$

By definition, we have densely defined and closed (unbounded) linear operators defining three dual pairs

$$\begin{aligned}
\big(\operatorname{grad}_{\Gamma_t}, (\operatorname{grad}_{\Gamma_t})^*\big) &= (\operatorname{grad}_{\Gamma_t}, -\operatorname{div}_{\Gamma_n}\varepsilon),\\
\big(\mu^{-1}\operatorname{rot}_{\Gamma_t}, (\mu^{-1}\operatorname{rot}_{\Gamma_t})^*\big) &= (\mu^{-1}\operatorname{rot}_{\Gamma_t}, \varepsilon^{-1}\operatorname{rot}_{\Gamma_n}),\\
\big(\operatorname{div}_{\Gamma_t}\mu, (\operatorname{div}_{\Gamma_t}\mu)^*\big) &= (\operatorname{div}_{\Gamma_t}\mu, -\operatorname{grad}_{\Gamma_n}).
\end{aligned}$$

Remarks 2.5 and 2.6 show the complex properties

$$\begin{aligned}
\mu^{-1}\operatorname{rot}_{\Gamma_t}\operatorname{grad}_{\Gamma_t} &\subset 0, & \operatorname{div}_{\Gamma_t}\mu\,\mu^{-1}\operatorname{rot}_{\Gamma_t} = \operatorname{div}_{\Gamma_t}\operatorname{rot}_{\Gamma_t} &\subset 0,\\
-\operatorname{div}_{\Gamma_n}\varepsilon\,\varepsilon^{-1}\operatorname{rot}_{\Gamma_n} = -\operatorname{div}_{\Gamma_n}\operatorname{rot}_{\Gamma_n} &\subset 0, & -\varepsilon^{-1}\operatorname{rot}_{\Gamma_n}\operatorname{grad}_{\Gamma_n} &\subset 0.
\end{aligned}$$

The long primal and dual vector de Rham Hilbert complex (12), cf. (15), reads

$$\mathbb{R}_{\Gamma_t} \underset{\pi_{\mathbb{R}_{\Gamma_t}}}{\overset{\iota_{\mathbb{R}_{\Gamma_t}}}{\rightleftarrows}} \mathsf{L}^2(\Omega) \underset{-\operatorname{div}_{\Gamma_n}\varepsilon}{\overset{\operatorname{grad}_{\Gamma_t}}{\rightleftarrows}} \mathsf{L}^2_\varepsilon(\Omega) \underset{\varepsilon^{-1}\operatorname{rot}_{\Gamma_n}}{\overset{\mu^{-1}\operatorname{rot}_{\Gamma_t}}{\rightleftarrows}} \mathsf{L}^2_\mu(\Omega) \underset{-\operatorname{grad}_{\Gamma_n}}{\overset{\operatorname{div}_{\Gamma_t}\mu}{\rightleftarrows}} \mathsf{L}^2(\Omega) \underset{\iota_{\mathbb{R}_{\Gamma_n}}}{\overset{\pi_{\mathbb{R}_{\Gamma_n}}}{\rightleftarrows}} \mathbb{R}_{\Gamma_n} \tag{21}$$

with the complex properties

$$\begin{aligned}
R(\iota_{\mathbb{R}_{\Gamma_t}}) = N(\operatorname{grad}_{\Gamma_t}) &= \mathbb{R}_{\Gamma_t}, & \overline{R(\operatorname{div}_{\Gamma_n}\varepsilon)} &= (\mathbb{R}_{\Gamma_t})^{\perp_{\mathsf{L}^2(\Omega)}},\\
R(\operatorname{grad}_{\Gamma_t}) &\subset N(\mu^{-1}\operatorname{rot}_{\Gamma_t}), & R(\varepsilon^{-1}\operatorname{rot}_{\Gamma_n}) &\subset N(\operatorname{div}_{\Gamma_n}\varepsilon),\\
R(\mu^{-1}\operatorname{rot}_{\Gamma_t}) &\subset N(\operatorname{div}_{\Gamma_t}\mu), & R(\operatorname{grad}_{\Gamma_n}) &\subset N(\varepsilon^{-1}\operatorname{rot}_{\Gamma_n}),\\
\overline{R(\operatorname{div}_{\Gamma_t}\mu)} &= (\mathbb{R}_{\Gamma_n})^{\perp_{\mathsf{L}^2(\Omega)}}, & R(\iota_{\mathbb{R}_{\Gamma_n}}) = N(\operatorname{grad}_{\Gamma_n}) &= \mathbb{R}_{\Gamma_n}.
\end{aligned}$$

Recalling Remark 2.25, we note that actually $\iota_{\mathbb{R}_{\Gamma_t}}\iota^*_{\mathbb{R}_{\Gamma_t}} = \pi_{\mathbb{R}_{\Gamma_t}}$ and $\iota_{\mathbb{R}_{\Gamma_n}}\iota^*_{\mathbb{R}_{\Gamma_n}} = \pi_{\mathbb{R}_{\Gamma_n}}$ as self-adjoint projections on $\mathsf{L}^2(\Omega)$. Similar to (21) (for simplicity let $\varepsilon = \mu = 1$), we investigate the higher-order de Rham complex

$$\mathbb{R}_{\Gamma_t} \xrightarrow{\iota_{\mathbb{R}_{\Gamma_t}}} \mathsf{H}^k_{\Gamma_t}(\Omega) \xrightarrow{\operatorname{grad}^k_{\Gamma_t}} \mathsf{H}^k_{\Gamma_t}(\Omega) \xrightarrow{\operatorname{rot}^k_{\Gamma_t}} \mathsf{H}^k_{\Gamma_t}(\Omega) \xrightarrow{\operatorname{div}^k_{\Gamma_t}} \mathsf{H}^k_{\Gamma_t}(\Omega) \xrightarrow{\pi_{\mathbb{R}_{\Gamma_n}}} \mathbb{R}_{\Gamma_n}$$

as well. More precisely, we consider the densely defined and closed linear operators

$$
\begin{aligned}
\operatorname{grad}^k_{\Gamma_t} &: D(\operatorname{grad}^k_{\Gamma_t}) \subset \mathsf{H}^k_{\Gamma_t}(\Omega) \to \mathsf{H}^k_{\Gamma_t}(\Omega);\ u \mapsto \operatorname{grad} u, && D(\operatorname{grad}^k_{\Gamma_t}) := \mathsf{H}^k_{\Gamma_t}(\operatorname{grad},\Omega) = \mathsf{H}^{k+1}_{\Gamma_t}(\Omega),\\
\operatorname{rot}^k_{\Gamma_t} &: D(\operatorname{rot}^k_{\Gamma_t}) \subset \mathsf{H}^k_{\Gamma_t}(\Omega) \to \mathsf{H}^k_{\Gamma_t}(\Omega);\ E \mapsto \operatorname{rot} E, && D(\operatorname{rot}^k_{\Gamma_t}) := \mathsf{H}^k_{\Gamma_t}(\operatorname{rot},\Omega),\\
\operatorname{div}^k_{\Gamma_t} &: D(\operatorname{div}^k_{\Gamma_t}) \subset \mathsf{H}^k_{\Gamma_t}(\Omega) \to \mathsf{H}^k_{\Gamma_t}(\Omega);\ H \mapsto \operatorname{div} H, && D(\operatorname{div}^k_{\Gamma_t}) := \mathsf{H}^k_{\Gamma_t}(\operatorname{div},\Omega).
\end{aligned}
$$

Note that the complex properties $R(\operatorname{grad}^k_{\Gamma_t}) \subset N(\operatorname{rot}^k_{\Gamma_t})$ and $R(\operatorname{rot}^k_{\Gamma_t}) \subset N(\operatorname{div}^k_{\Gamma_t})$ hold.

5.1 | Regular potentials and decompositions

For d ∈ {grad, rot, div} Lemma 4.6, Corollary 4.7, Theorem 4.18 and Remark 4.19 read as follows.

Theorem 5.2 (higher-order bounded regular potentials and decompositions for the vector de Rham complex with partial boundary condition). *Let (Ω, Γ_t) be a bounded strong Lipschitz pair and let $k \geq 0$. Then:*

(i) The bounded regular decompositions

$$
\begin{aligned}
\mathbf{H}^k_{\Gamma_t}(\operatorname{rot},\Omega) = \mathsf{H}^k_{\Gamma_t}(\operatorname{rot},\Omega) &= \mathsf{H}^{k+1}_{\Gamma_t}(\Omega) + \operatorname{grad}\mathsf{H}^{k+1}_{\Gamma_t}(\Omega),\\
\mathbf{H}^k_{\Gamma_t}(\operatorname{div},\Omega) = \mathsf{H}^k_{\Gamma_t}(\operatorname{div},\Omega) &= \mathsf{H}^{k+1}_{\Gamma_t}(\Omega) + \operatorname{rot}\mathsf{H}^{k+1}_{\Gamma_t}(\Omega)
\end{aligned}
$$

hold with bounded linear regular decomposition operators

$$
\begin{aligned}
\mathcal{Q}^k_{\operatorname{rot},\Gamma_t,1} &: \mathsf{H}^k_{\Gamma_t}(\operatorname{rot},\Omega) \to \mathsf{H}^{k+1}_{\Gamma_t}(\Omega), & \mathcal{Q}^k_{\operatorname{rot},\Gamma_t,0} &: \mathsf{H}^k_{\Gamma_t}(\operatorname{rot},\Omega) \to \mathsf{H}^{k+1}_{\Gamma_t}(\Omega),\\
\mathcal{Q}^k_{\operatorname{div},\Gamma_t,1} &: \mathsf{H}^k_{\Gamma_t}(\operatorname{div},\Omega) \to \mathsf{H}^{k+1}_{\Gamma_t}(\Omega), & \mathcal{Q}^k_{\operatorname{div},\Gamma_t,0} &: \mathsf{H}^k_{\Gamma_t}(\operatorname{div},\Omega) \to \mathsf{H}^{k+1}_{\Gamma_t}(\Omega)
\end{aligned}
$$

satisfying $\mathcal{Q}^k_{\operatorname{rot},\Gamma_t,1} + \operatorname{grad}\mathcal{Q}^k_{\operatorname{rot},\Gamma_t,0} = \operatorname{id}_{\mathsf{H}^k_{\Gamma_t}(\operatorname{rot},\Omega)}$ *and* $\mathcal{Q}^k_{\operatorname{div},\Gamma_t,1} + \operatorname{rot}\mathcal{Q}^k_{\operatorname{div},\Gamma_t,0} = \operatorname{id}_{\mathsf{H}^k_{\Gamma_t}(\operatorname{div},\Omega)}$. *In particular, weak and strong boundary conditions coincide. It holds* $\operatorname{rot}\mathcal{Q}^k_{\operatorname{rot},\Gamma_t,1} = \operatorname{rot}^k_{\Gamma_t}$, *and thus,* $\mathsf{H}^k_{\Gamma_t,0}(\operatorname{rot},\Omega)$ *is invariant under* $\mathcal{Q}^k_{\operatorname{rot},\Gamma_t,1}$. *Analogously,* $\operatorname{div}\mathcal{Q}^k_{\operatorname{div},\Gamma_t,1} = \operatorname{div}^k_{\Gamma_t}$, *and thus,* $\mathsf{H}^k_{\Gamma_t,0}(\operatorname{div},\Omega)$ *is invariant under* $\mathcal{Q}^k_{\operatorname{div},\Gamma_t,1}$.

(ii) The regular potential representations

$$
\begin{aligned}
R(\operatorname{grad}^k_{\Gamma_t}) &= \operatorname{grad}\mathsf{H}^{k+1}_{\Gamma_t}(\Omega) = \mathsf{H}^k_{\Gamma_t,0}(\operatorname{rot},\Omega) \cap \mathcal{H}_{\Gamma_t,\Gamma_n,\varepsilon}(\Omega)^{\perp_{\mathsf{L}^2_\varepsilon(\Omega)}} = \mathsf{H}^k_{\Gamma_t}(\Omega) \cap R(\operatorname{grad}_{\Gamma_t}),\\
R(\operatorname{rot}^k_{\Gamma_t}) = \operatorname{rot}\mathsf{H}^k_{\Gamma_t}(\operatorname{rot},\Omega) &= \operatorname{rot}\mathsf{H}^{k+1}_{\Gamma_t}(\Omega) = \mathsf{H}^k_{\Gamma_t,0}(\operatorname{div},\Omega) \cap \mathcal{H}_{\Gamma_n,\Gamma_t,\varepsilon}(\Omega)^{\perp_{\mathsf{L}^2(\Omega)}} = \mathsf{H}^k_{\Gamma_t}(\Omega) \cap R(\operatorname{rot}_{\Gamma_t}),\\
R(\operatorname{div}^k_{\Gamma_t}) = \operatorname{div}\mathsf{H}^k_{\Gamma_t}(\operatorname{div},\Omega) &= \operatorname{div}\mathsf{H}^{k+1}_{\Gamma_t}(\Omega) = \mathsf{H}^k_{\Gamma_t}(\Omega) \cap (\mathbb{R}_{\Gamma_n})^{\perp_{\mathsf{L}^2(\Omega)}} = \mathsf{H}^k_{\Gamma_t}(\Omega) \cap R(\operatorname{div}_{\Gamma_t})
\end{aligned}
$$

hold. In particular, these spaces are closed subspaces of $\mathsf{H}^k_{\emptyset}(\Omega) = \mathsf{H}^k(\Omega)$.

(iii) There exist bounded linear regular potential operators

$$
\begin{aligned}
\mathcal{P}^k_{\operatorname{grad},\Gamma_t} &:= (\operatorname{grad}^k_{\Gamma_t})^{-1}_{\perp} : \mathsf{H}^k_{\Gamma_t,0}(\operatorname{rot},\Omega) \cap \mathcal{H}_{\Gamma_t,\Gamma_n,\varepsilon}(\Omega)^{\perp_{\mathsf{L}^2_\varepsilon(\Omega)}} \to \mathsf{H}^{k+1}_{\Gamma_t}(\Omega),\\
\mathcal{P}^k_{\operatorname{rot},\Gamma_t} &:= \mathcal{Q}^k_{\operatorname{rot},\Gamma_t,1}(\operatorname{rot}^k_{\Gamma_t})^{-1}_{\perp} : \mathsf{H}^k_{\Gamma_t,0}(\operatorname{div},\Omega) \cap \mathcal{H}_{\Gamma_n,\Gamma_t,\varepsilon}(\Omega)^{\perp_{\mathsf{L}^2(\Omega)}} \to \mathsf{H}^{k+1}_{\Gamma_t}(\Omega),\\
\mathcal{P}^k_{\operatorname{div},\Gamma_t} &:= \mathcal{Q}^k_{\operatorname{div},\Gamma_t,1}(\operatorname{div}^k_{\Gamma_t})^{-1}_{\perp} : \mathsf{H}^k_{\Gamma_t}(\Omega) \cap (\mathbb{R}_{\Gamma_n})^{\perp_{\mathsf{L}^2(\Omega)}} \to \mathsf{H}^{k+1}_{\Gamma_t}(\Omega),
\end{aligned}
$$

such that

$$
\begin{aligned}
\operatorname{grad}\mathcal{P}^k_{\operatorname{grad},\Gamma_t} &= \operatorname{id}|_{\mathsf{H}^k_{\Gamma_t,0}(\operatorname{rot},\Omega)\cap\mathcal{H}_{\Gamma_t,\Gamma_n,\varepsilon}(\Omega)^{\perp_{\mathsf{L}^2_\varepsilon(\Omega)}}},\\
\operatorname{rot}\mathcal{P}^k_{\operatorname{rot},\Gamma_t} &= \operatorname{id}|_{\mathsf{H}^k_{\Gamma_t,0}(\operatorname{div},\Omega)\cap\mathcal{H}_{\Gamma_n,\Gamma_t,\varepsilon}(\Omega)^{\perp_{\mathsf{L}^2(\Omega)}}},\\
\operatorname{div}\mathcal{P}^k_{\operatorname{div},\Gamma_t} &= \operatorname{id}|_{\mathsf{H}^k_{\Gamma_t}(\Omega)\cap(\mathbb{R}_{\Gamma_n})^{\perp_{\mathsf{L}^2(\Omega)}}}.
\end{aligned}
$$

In particular, all potentials in (ii) can be chosen such that they depend continuously on the data. $\mathcal{P}^k_{\text{grad},\Gamma_t}$, $\mathcal{P}^k_{\text{rot},\Gamma_t}$ *and* $\mathcal{P}^k_{\text{div},\Gamma_t}$ *are right inverses of* grad, rot *and* div, *respectively.*

(iv) *The bounded regular decompositions*

$$\begin{aligned}\mathsf{H}^k_{\Gamma_t}(\text{rot},\Omega) &= \mathsf{H}^{k+1}_{\Gamma_t}(\Omega) + \mathsf{H}^k_{\Gamma_t,0}(\text{rot},\Omega) = \mathsf{H}^{k+1}_{\Gamma_t}(\Omega) + \text{grad}\,\mathsf{H}^{k+1}_{\Gamma_t}(\Omega)\\ &= R(\widetilde{\mathcal{Q}}^k_{\text{rot},\Gamma_t,1}) \dotplus \mathsf{H}^k_{\Gamma_t,0}(\text{rot},\Omega) = R(\widetilde{\mathcal{Q}}^k_{\text{rot},\Gamma_t,1}) \dotplus R(\widetilde{\mathcal{N}}^k_{\text{rot},\Gamma_t}),\\ \mathsf{H}^k_{\Gamma_t}(\text{div},\Omega) &= \mathsf{H}^{k+1}_{\Gamma_t}(\Omega) + \mathsf{H}^k_{\Gamma_t,0}(\text{div},\Omega) = \mathsf{H}^{k+1}_{\Gamma_t}(\Omega) + \text{rot}\,\mathsf{H}^{k+1}_{\Gamma_t}(\Omega)\\ &= R(\widetilde{\mathcal{Q}}^k_{\text{div},\Gamma_t,1}) \dotplus \mathsf{H}^k_{\Gamma_t,0}(\text{div},\Omega) = R(\widetilde{\mathcal{Q}}^k_{\text{div},\Gamma_t,1}) \dotplus R(\widetilde{\mathcal{N}}^k_{\text{div},\Gamma_t})\end{aligned}$$

hold with bounded linear regular decomposition operators

$$\begin{aligned}\widetilde{\mathcal{Q}}^k_{\text{rot},\Gamma_t,1} &:= \mathcal{P}^k_{\text{rot},\Gamma_t}\text{rot}^k_{\Gamma_t} : \mathsf{H}^k_{\Gamma_t}(\text{rot},\Omega) \to \mathsf{H}^{k+1}_{\Gamma_t}(\Omega), & \widetilde{\mathcal{N}}^k_{\text{rot},\Gamma_t} &: \mathsf{H}^k_{\Gamma_t}(\text{rot},\Omega) \to \mathsf{H}^k_{\Gamma_t,0}(\text{rot},\Omega),\\ \widetilde{\mathcal{Q}}^k_{\text{div},\Gamma_t,1} &:= \mathcal{P}^k_{\text{div},\Gamma_t}\text{div}^k_{\Gamma_t} : \mathsf{H}^k_{\Gamma_t}(\text{div},\Omega) \to \mathsf{H}^{k+1}_{\Gamma_t}(\Omega), & \widetilde{\mathcal{N}}^k_{\text{div},\Gamma_t} &: \mathsf{H}^k_{\Gamma_t}(\text{div},\Omega) \to \mathsf{H}^k_{\Gamma_t,0}(\text{div},\Omega)\end{aligned}$$

satisfying $\widetilde{\mathcal{Q}}^k_{\text{rot},\Gamma_t,1} + \widetilde{\mathcal{N}}^k_{\text{rot},\Gamma_t} = \text{id}_{\mathsf{H}^k_{\Gamma_t}(\text{rot},\Omega)}$ *and* $\widetilde{\mathcal{Q}}^k_{\text{div},\Gamma_t,1} + \widetilde{\mathcal{N}}^k_{\text{div},\Gamma_t} = \text{id}_{\mathsf{H}^k_{\Gamma_t}(\text{div},\Omega)}$. *It holds* $\text{rot}\,\widetilde{\mathcal{Q}}^k_{\text{rot},\Gamma_t,1} = \text{rot}\,\mathcal{Q}^k_{\text{rot},\Gamma_t,1} = \text{rot}^k_{\Gamma_t}$, *and thus,* $\mathsf{H}^k_{\Gamma_t,0}(\text{rot},\Omega)$ *is invariant under* $\mathcal{Q}^k_{\text{rot},\Gamma_t,1}$ *and* $\widetilde{\mathcal{Q}}^k_{\text{rot},\Gamma_t,1}$. *Analogously,* $\text{div}\widetilde{\mathcal{Q}}^k_{\text{div},\Gamma_t,1} = \text{div}\mathcal{Q}^k_{\text{div},\Gamma_t,1} = \text{div}^k_{\Gamma_t}$, *and thus,* $\mathsf{H}^k_{\Gamma_t,0}(\text{div},\Omega)$ *is invariant under* $\mathcal{Q}^k_{\text{div},\Gamma_t,1}$ *and* $\widetilde{\mathcal{Q}}^k_{\text{div},\Gamma_t,1}$. *Moreover, we have* $R(\widetilde{\mathcal{Q}}^k_{\text{rot},\Gamma_t,1}) = R(\mathcal{P}^k_{\text{rot},\Gamma_t})$ *and* $\widetilde{\mathcal{Q}}^k_{\text{rot},\Gamma_t,1} = \mathcal{Q}^k_{\text{rot},\Gamma_t,1}(\text{rot}^k_{\Gamma_t})^{-1}_{\perp}\text{rot}^k_{\Gamma_t}$. *Hence,* $\widetilde{\mathcal{Q}}^k_{\text{rot},\Gamma_t,1}|_{D((\text{rot}^k_{\Gamma_t})_{\perp})} = \mathcal{Q}^k_{\text{rot},\Gamma_t,1}|_{D((\text{rot}^k_{\Gamma_t})_{\perp})}$, *and thus,* $\widetilde{\mathcal{Q}}^k_{\text{rot},\Gamma_t,1}$ *may differ from* $\mathcal{Q}^k_{\text{rot},\Gamma_t,1}$ *only on* $\mathsf{H}^k_{\Gamma_t,0}(\text{rot},\Omega)$. *Analogously, it holds* $R(\widetilde{\mathcal{Q}}^k_{\text{div},\Gamma_t,1}) = R(\mathcal{P}^k_{\text{div},\Gamma_t})$ *and* $\widetilde{\mathcal{Q}}^k_{\text{div},\Gamma_t,1} = \mathcal{Q}^k_{\text{div},\Gamma_t,1}(\text{div}^k_{\Gamma_t})^{-1}_{\perp}\text{div}^k_{\Gamma_t}$. *Hence, we have that* $\widetilde{\mathcal{Q}}^k_{\text{div},\Gamma_t,1}|_{D((\text{div}^k_{\Gamma_t})_{\perp})} = \mathcal{Q}^k_{\text{div},\Gamma_t,1}|_{D((\text{div}^k_{\Gamma_t})_{\perp})}$, *and thus,* $\widetilde{\mathcal{Q}}^k_{\text{div},\Gamma_t,1}$ *may differ from* $\mathcal{Q}^k_{\text{div},\Gamma_t,1}$ *only on* $\mathsf{H}^k_{\Gamma_t,0}(\text{div},\Omega)$.

(iv') *The bounded regular kernel decompositions* $\mathsf{H}^k_{\Gamma_t,0}(\text{rot},\Omega) = \mathsf{H}^{k+1}_{\Gamma_t,0}(\text{rot},\Omega) + \text{grad}\,\mathsf{H}^{k+1}_{\Gamma_t}(\Omega)$ *and* $\mathsf{H}^k_{\Gamma_t,0}(\text{div},\Omega) = \mathsf{H}^{k+1}_{\Gamma_t,0}(\text{div},\Omega) + \text{rot}\,\mathsf{H}^{k+1}_{\Gamma_t}(\Omega)$ *hold.*

Remark 5.2. Let us recall the bounded regular decompositions from Theorem 5.2 (iv), for example,

$$\mathsf{H}^k_{\Gamma_t}(\text{rot},\Omega) = R(\widetilde{\mathcal{Q}}^k_{\text{rot},\Gamma_t,1}) \dotplus R(\widetilde{\mathcal{N}}^k_{\text{rot},\Gamma_t}).$$

(i) $\widetilde{\mathcal{Q}}^k_{\text{rot},\Gamma_t,1}$, $\widetilde{\mathcal{N}}^k_{\text{rot},\Gamma_t} = 1 - \widetilde{\mathcal{Q}}^k_{\text{rot},\Gamma_t,1}$ are projections with $\widetilde{\mathcal{Q}}^k_{\text{rot},\Gamma_t,1}\widetilde{\mathcal{N}}^k_{\text{rot},\Gamma_t} = \widetilde{\mathcal{N}}^k_{\text{rot},\Gamma_t}\widetilde{\mathcal{Q}}^k_{\text{rot},\Gamma_t,1} = 0$.

(ii) For $I_{\pm} := \widetilde{\mathcal{Q}}^k_{\text{rot},\Gamma_t,1} \pm \widetilde{\mathcal{N}}^k_{\text{rot},\Gamma_t}$, it holds $I_+ = I_-^2 = \text{id}_{\mathsf{H}^k_{\Gamma_t}(\text{rot},\Omega)}$. Therefore, I_+, I_-^2, as well as $I_- = 2\widetilde{\mathcal{Q}}^k_{\text{rot},\Gamma_t,1} - \text{id}_{\mathsf{H}^k_{\Gamma_t}(\text{rot},\Omega)}$ are topological isomorphisms on $\mathsf{H}^k_{\Gamma_t}(\text{rot},\Omega)$.

(iii) There exists $c > 0$ such that for all $E \in \mathsf{H}^k_{\Gamma_t}(\text{rot},\Omega)$

$$\begin{aligned}c|\widetilde{\mathcal{Q}}^k_{\text{rot},\Gamma_t,1}E|_{\mathsf{H}^{k+1}(\Omega)} &\le |\text{rot}\,E|_{\mathsf{H}^k(\Omega)} \le |E|_{\mathsf{H}^k(\text{rot},\Omega)},\\ |\widetilde{\mathcal{N}}^k_{\text{rot},\Gamma_t}E|_{\mathsf{H}^k(\Omega)} &\le |E|_{\mathsf{H}^k(\Omega)} + |\widetilde{\mathcal{Q}}^k_{\text{rot},\Gamma_t,1}E|_{\mathsf{H}^k(\Omega)}.\end{aligned}$$

(iii') For $E \in \mathsf{H}^k_{\Gamma_t,0}(\text{rot},\Omega)$, we have $\widetilde{\mathcal{Q}}^k_{\text{rot},\Gamma_t,1}E = 0$ and $\widetilde{\mathcal{N}}^k_{\text{rot},\Gamma_t}E = E$. In particular, $\widetilde{\mathcal{N}}^k_{\text{rot},\Gamma_t}$ is onto.

(iv) Literally, (i)–(iii') hold for div as well.

5.2 | Zero-order mini FA-ToolBox

Theorem 4.8, Theorem 4.9 and Remark 4.10 translate to the following results, cf. (12) and Definition 2.26 as well as Pauly and Waurick.[23, Lemma 5.1, Lemma 5.2]

Theorem 5.4 (compact embedding for the vector de Rham complex). *Let (Ω, Γ_t) be a bounded strong Lipschitz pair. Then the embeddings*

$$
\begin{aligned}
D(\mathrm{A}_0) &= \mathsf{H}^1_{\Gamma_t}(\Omega) \hookrightarrow \mathsf{L}^2(\Omega),\\
D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) &= \mathsf{H}_{\Gamma_t}(\mathrm{rot}, \Omega) \cap \varepsilon^{-1}\mathsf{H}_{\Gamma_n}(\mathrm{div}, \Omega) \hookrightarrow \mathsf{L}^2_\varepsilon(\Omega),\\
D(\mathrm{A}_2) \cap D(\mathrm{A}_1^*) &= \mu^{-1}\mathsf{H}_{\Gamma_t}(\mathrm{div}, \Omega) \cap \mathsf{H}_{\Gamma_n}(\mathrm{rot}, \Omega) \hookrightarrow \mathsf{L}^2_\mu(\Omega),\\
D(\mathrm{A}_2^*) &= \mathsf{H}^1_{\Gamma_n}(\Omega) \hookrightarrow \mathsf{L}^2(\Omega)
\end{aligned}
$$

are compact; that is, the long primal and dual vector de Rham Hilbert complex is compact. In particular, the complex is closed. Moreover, the compactness of the embeddings is independent of ε and μ.

Theorem 5.5 (mini FA-ToolBox for the vector de Rham complex). *Let (Ω, Γ_t) be a bounded strong Lipschitz pair. Then*

(i) the ranges $R(\mathrm{grad}_{\Gamma_t})$, $R(\mathrm{rot}_{\Gamma_t})$, and $R(\mathrm{div}_{\Gamma_t}) = (\mathbb{R}_{\Gamma_n})^{\perp_{\mathsf{L}^2(\Omega)}}$ are closed,
(ii) the inverse operators $(\mathrm{grad}_{\Gamma_t})_\perp^{-1}$, $(\mu^{-1}\mathrm{rot}_{\Gamma_t})_\perp^{-1}$ and $(\mathrm{div}_{\Gamma_t}\mu)_\perp^{-1}$ are compact,
(iii) the cohomology group $\mathcal{H}_{\Gamma_t,\Gamma_n,\varepsilon}(\Omega) = \mathsf{H}_{\Gamma_t,0}(\mathrm{rot}, \Omega) \cap \varepsilon^{-1}\mathsf{H}_{\Gamma_n,0}(\mathrm{div}, \Omega)$ has finite dimension, which is independent of ε,
(iv) the orthogonal Helmholtz-type decomposition

$$\mathsf{L}^2_\varepsilon(\Omega) = \mathrm{grad}\ \mathsf{H}^1_{\Gamma_t}(\Omega)\ \oplus_{\mathsf{L}^2_\varepsilon(\Omega)} \mathcal{H}_{\Gamma_t,\Gamma_n,\varepsilon}(\Omega)\ \oplus_{\mathsf{L}^2_\varepsilon(\Omega)} \varepsilon^{-1}\mathrm{rot}\ \mathsf{H}_{\Gamma_n}(\mathrm{rot}, \Omega)$$

holds,
(v) there exist $c_{\mathrm{grad},\Gamma_t}, c_{\mathrm{rot},\Gamma_t}, c_{\mathrm{div},\Gamma_t} > 0$ such that

$$
\begin{aligned}
&\forall u \in D\left((\mathrm{grad}_{\Gamma_t})_\perp\right) && |u|_{\mathsf{L}^2(\Omega)} \le c_{\mathrm{grad},\Gamma_t} |\mathrm{grad}\ u|_{\mathsf{L}^2_\varepsilon(\Omega)},\\
&\forall E \in D\left((\mathrm{div}_{\Gamma_n}\varepsilon)_\perp\right) && |E|_{\mathsf{L}^2_\varepsilon(\Omega)} \le c_{\mathrm{grad},\Gamma_t} |\mathrm{div}\ \varepsilon E|_{\mathsf{L}^2(\Omega)},\\
&\forall E \in D\left((\mu^{-1}\mathrm{rot}_{\Gamma_t})_\perp\right) && |E|_{\mathsf{L}^2_\varepsilon(\Omega)} \le c_{\mathrm{rot},\Gamma_t} |\mu^{-1}\mathrm{rot}\ E|_{\mathsf{L}^2_\mu(\Omega)},\\
&\forall H \in D\left((\varepsilon^{-1}\mathrm{rot}_{\Gamma_n})_\perp\right) && |H|_{\mathsf{L}^2_\mu(\Omega)} \le c_{\mathrm{rot},\Gamma_t} |\varepsilon^{-1}\mathrm{rot}\ E|_{\mathsf{L}^2_\varepsilon(\Omega)},\\
&\forall H \in D\left((\mathrm{div}_{\Gamma_t}\mu)_\perp\right) && |H|_{\mathsf{L}^2_\mu(\Omega)} \le c_{\mathrm{div},\Gamma_t} |\mathrm{div}\ \mu H|_{\mathsf{L}^2(\Omega)},\\
&\forall u \in D\left((\mathrm{grad}_{\Gamma_n})_\perp\right) && |u|_{\mathsf{L}^2(\Omega)} \le c_{\mathrm{div},\Gamma_t} |\mathrm{grad}\ u|_{\mathsf{L}^2_\mu(\Omega)},
\end{aligned}
$$

where

$$
\begin{aligned}
D\left((\mathrm{grad}_{\Gamma_t})_\perp\right) &= D(\mathrm{grad}_{\Gamma_t}) \cap N(\mathrm{grad}_{\Gamma_t})^{\perp_{\mathsf{L}^2(\Omega)}} = D(\mathrm{grad}_{\Gamma_t}) \cap R(\mathrm{div}_{\Gamma_n}\varepsilon),\\
D\left((\mathrm{div}_{\Gamma_n}\varepsilon)_\perp\right) &= D(\mathrm{div}_{\Gamma_n}\varepsilon) \cap N(\mathrm{div}_{\Gamma_n}\varepsilon)^{\perp_{\mathsf{L}^2_\varepsilon(\Omega)}} = D(\mathrm{div}_{\Gamma_n}\varepsilon) \cap R(\mathrm{grad}_{\Gamma_t}),\\
D\left((\mu^{-1}\mathrm{rot}_{\Gamma_t})_\perp\right) &= D(\mu^{-1}\mathrm{rot}_{\Gamma_t}) \cap N(\mu^{-1}\mathrm{rot}_{\Gamma_t})^{\perp_{\mathsf{L}^2_\varepsilon(\Omega)}} = D(\mu^{-1}\mathrm{rot}_{\Gamma_t}) \cap R(\varepsilon^{-1}\mathrm{rot}_{\Gamma_n}),
\end{aligned}
$$

which also gives $D\left((\varepsilon^{-1}\mathrm{rot}_{\Gamma_n})_\perp\right)$, $D\left((\mathrm{div}_{\Gamma_t}\mu)_\perp\right)$, and $D\left((\mathrm{grad}_{\Gamma_n})_\perp\right)$ by interchanging ε, μ and Γ_t, Γ_n,
(v′) it holds for all $E \in D(\mu^{-1}\mathrm{rot}_{\Gamma_t}) \cap D(\mathrm{div}_{\Gamma_n}\varepsilon) \cap \mathcal{H}_{\Gamma_t,\Gamma_n,\varepsilon}(\Omega)^{\perp_{\mathsf{L}^2_\varepsilon(\Omega)}}$

$$|E|^2_{\mathsf{L}^2_\varepsilon(\Omega)} \le c^2_{\mathrm{rot},\Gamma_t} |\mu^{-1}\mathrm{rot}\ E|^2_{\mathsf{L}^2_\mu(\Omega)} + c^2_{\mathrm{grad},\Gamma_t} |\mathrm{div}\ \varepsilon E|^2_{\mathsf{L}^2(\Omega)},$$

(vi) $\mathcal{H}_{\Gamma_t,\Gamma_n,\varepsilon}(\Omega) = \{0\}$, if Ω is additionally extendable.

Remark 5.6. Theorems 5.4 and 5.5 hold more generally for bounded weak Lipschitz pairs (Ω, Γ_t); see previous studies.[9,17,18]

5.3 Higher-order mini FA-ToolBox and Dirichlet/Neumann fields

Theorem 5.4 holds even for higher Sobolev orders, cf. Theorem 4.16.

Theorem 5.7 (higher-order compact embedding for the vector de Rham complex). *Let (Ω, Γ_t) be a bounded strong Lipschitz pair. Then for all $k \in \mathbb{N}_0$, the embeddings*

$$\begin{aligned} \mathsf{H}^{k+1}_{\Gamma_t}(\Omega) \cap \mathsf{H}^{k}_{\Gamma_n}(\Omega) &\hookrightarrow \mathsf{H}^{k}_{\Gamma}(\Omega), \\ \mathsf{H}^{k}_{\Gamma_t}(\mathrm{rot}, \Omega) \cap \mathsf{H}^{k}_{\Gamma_n}(\mathrm{div}, \Omega) &\hookrightarrow \mathsf{H}^{k}_{\Gamma}(\Omega), \\ \mathsf{H}^{k}_{\Gamma_t}(\mathrm{div}, \Omega) \cap \mathsf{H}^{k}_{\Gamma_n}(\mathrm{rot}, \Omega) &\hookrightarrow \mathsf{H}^{k}_{\Gamma}(\Omega), \\ \mathsf{H}^{k}_{\Gamma_t}(\Omega) \cap \mathsf{H}^{k+1}_{\Gamma_n}(\Omega) &\hookrightarrow \mathsf{H}^{k}_{\Gamma}(\Omega) \end{aligned}$$

are compact.

Remark 5.8. (higher-order Friedrichs/Poincaré type estimates for the vector de Rham complex). Analogues of Theorems 4.15 and 4.17 hold. In particular, for all $k \geq 0$, there exists $\widetilde{c}_k > 0$ such that for all $E \in \mathsf{H}^{k}_{\Gamma_t}(\mathrm{rot}, \Omega) \cap \mathsf{H}^{k}_{\Gamma_n}(\mathrm{div}, \Omega) \cap \mathcal{H}_{\Gamma_t, \Gamma_n, \mathrm{id}}(\Omega)^{\perp_{\mathsf{L}^2(\Omega)}}$

$$|E|^2_{\mathsf{H}^k(\Omega)} \leq \widetilde{c}_k^2 \left(|\mathrm{rot}\ E|^2_{\mathsf{H}^k(\Omega)} + |\mathrm{div}\ E|^2_{\mathsf{H}^k(\Omega)} \right).$$

Theorem 5.2 (iv'), cf. Corollary 4.20, shows by induction for all $k, \ell \geq 0$

$$\mathsf{H}^{k}_{\Gamma_t,0}(\mathrm{rot}, \Omega) = \mathsf{H}^{\ell}_{\Gamma_t,0}(\mathrm{rot}, \Omega) + \mathrm{grad}\ \mathsf{H}^{k+1}_{\Gamma_t}(\Omega), \mathsf{H}^{k}_{\Gamma_t,0}(\mathrm{div}, \Omega) = \mathsf{H}^{\ell}_{\Gamma_t,0}(\mathrm{div}, \Omega) + \mathrm{rot}\ \mathsf{H}^{k+1}_{\Gamma_t}(\Omega). \tag{22}$$

By Theorem 5.5 (iv), we have the orthonormal Helmholtz decompositions

$$\begin{aligned} \mathsf{L}^2_\varepsilon(\Omega) &= \mathrm{grad}\ \mathsf{H}^1_{\Gamma_t}(\Omega) \oplus_{\mathsf{L}^2_\varepsilon(\Omega)} \varepsilon^{-1}\mathsf{H}_{\Gamma_n,0}(\mathrm{div}, \Omega) \\ &= \mathsf{H}_{\Gamma_t,0}(\mathrm{rot}, \Omega) \oplus_{\mathsf{L}^2_\varepsilon(\Omega)} \varepsilon^{-1}\mathrm{rot}\ \mathsf{H}_{\Gamma_n}(\mathrm{rot}, \Omega) \\ &= \mathrm{grad}\ \mathsf{H}^1_{\Gamma_t}(\Omega) \oplus_{\mathsf{L}^2_\varepsilon(\Omega)} \mathcal{H}_{\Gamma_t,\Gamma_n,\varepsilon}(\Omega) \oplus_{\mathsf{L}^2_\varepsilon(\Omega)} \varepsilon^{-1}\mathrm{rot}\ \mathsf{H}_{\Gamma_n}(\mathrm{rot}, \Omega), \\ \mathsf{H}_{\Gamma_t,0}(\mathrm{rot}, \Omega) &= \mathrm{grad}\ \mathsf{H}^1_{\Gamma_t}(\Omega) \oplus_{\mathsf{L}^2_\varepsilon(\Omega)} \mathcal{H}_{\Gamma_t,\Gamma_n,\varepsilon}(\Omega), \\ \varepsilon^{-1}\mathsf{H}_{\Gamma_n,0}(\mathrm{div}, \Omega) &= \mathcal{H}_{\Gamma_t,\Gamma_n,\varepsilon}(\Omega) \oplus_{\mathsf{L}^2_\varepsilon(\Omega)} \varepsilon^{-1}\mathrm{rot}\ \mathsf{H}_{\Gamma_n}(\mathrm{rot}, \Omega). \end{aligned} \tag{23}$$

Let us denote the $\mathsf{L}^2_\varepsilon(\Omega)$-orthonormal projector onto $\varepsilon^{-1}\mathsf{H}_{\Gamma_n,0}(\mathrm{div}, \Omega)$ and $\mathsf{H}_{\Gamma_t,0}(\mathrm{rot}, \Omega)$ by

$$\pi_{\mathrm{div}} : \mathsf{L}^2_\varepsilon(\Omega) \to \varepsilon^{-1}\mathsf{H}_{\Gamma_n,0}(\mathrm{div}, \Omega), \qquad \pi_{\mathrm{rot}} : \mathsf{L}^2_\varepsilon(\Omega) \to \mathsf{H}_{\Gamma_t,0}(\mathrm{rot}, \Omega)$$

respectively. Then

$$\begin{aligned} \pi_{\mathrm{div}}|_{\mathsf{H}_{\Gamma_t,0}(\mathrm{rot},\Omega)} &: \mathsf{H}_{\Gamma_t,0}(\mathrm{rot}, \Omega) \to \mathcal{H}_{\Gamma_t,\Gamma_n,\varepsilon}(\Omega), \\ \pi_{\mathrm{rot}}|_{\varepsilon^{-1}\mathsf{H}_{\Gamma_n,0}(\mathrm{div},\Omega)} &: \varepsilon^{-1}\mathsf{H}_{\Gamma_n,0}(\mathrm{div}, \Omega) \to \mathcal{H}_{\Gamma_t,\Gamma_n,\varepsilon}(\Omega) \end{aligned}$$

are onto. Moreover,

$$\begin{aligned} \pi_{\mathrm{div}}|_{\mathrm{grad}\ \mathsf{H}^1_{\Gamma_t}(\Omega)} &= 0, & \pi_{\mathrm{rot}}|_{\varepsilon^{-1}\mathrm{rot}\ \mathsf{H}_{\Gamma_n}(\mathrm{rot},\Omega)} &= 0, \\ \pi_{\mathrm{div}}|_{\mathcal{H}_{\Gamma_t,\Gamma_n,\varepsilon}(\Omega)} &= \mathrm{id}_{\mathcal{H}_{\Gamma_t,\Gamma_n,\varepsilon}(\Omega)}, & \pi_{\mathrm{rot}}|_{\mathcal{H}_{\Gamma_t,\Gamma_n,\varepsilon}(\Omega)} &= \mathrm{id}_{\mathcal{H}_{\Gamma_t,\Gamma_n,\varepsilon}(\Omega)}. \end{aligned}$$

Therefore, by (22) and for all $\ell \geq 0$,

$$\begin{aligned} \mathcal{H}_{\Gamma_t,\Gamma_n,\varepsilon}(\Omega) &= \pi_{\mathrm{div}}\mathsf{H}_{\Gamma_t,0}(\mathrm{rot}, \Omega) = \pi_{\mathrm{div}}\mathsf{H}^{\ell}_{\Gamma_t,0}(\mathrm{rot}, \Omega), \\ \mathcal{H}_{\Gamma_t,\Gamma_n,\varepsilon}(\Omega) &= \pi_{\mathrm{rot}}\varepsilon^{-1}\mathsf{H}_{\Gamma_n,0}(\mathrm{div}, \Omega) = \pi_{\mathrm{rot}}\varepsilon^{-1}\mathsf{H}^{\ell}_{\Gamma_n,0}(\mathrm{div}, \Omega). \end{aligned}$$

Hence with

$$\mathsf{H}^{\infty}_{\Gamma_t,0}(\mathrm{rot},\Omega) := \bigcap_{k\geq 0}\mathsf{H}^{k}_{\Gamma_t,0}(\mathrm{rot},\Omega), \qquad \mathsf{H}^{\infty}_{\Gamma_t,0}(\mathrm{div},\Omega) := \bigcap_{k\geq 0}\mathsf{H}^{k}_{\Gamma_t,0}(\mathrm{div},\Omega)$$

we have the following result:

Theorem 5.9 (smooth prebases of Dirichlet/Neumann fields for the vector de Rham complex). *Let (Ω,Γ_t) be a bounded strong Lipschitz pair and let $d_{\Omega,\Gamma_t} := \dim \mathcal{H}_{\Gamma_t,\Gamma_n,\varepsilon}(\Omega)$. Then*

$$\pi_{\mathrm{div}}\mathsf{H}^{\infty}_{\Gamma_t,0}(\mathrm{rot},\Omega) = \mathcal{H}_{\Gamma_t,\Gamma_n,\varepsilon}(\Omega) = \pi_{\mathrm{rot}}\varepsilon^{-1}\mathsf{H}^{\infty}_{\Gamma_n,0}(\mathrm{div},\Omega).$$

Moreover, there exists a smooth rot*-prebasis and a smooth* div*-prebasis of $\mathcal{H}_{\Gamma_t,\Gamma_n,\varepsilon}(\Omega)$; that is, there are linear independent smooth fields*

$$\mathcal{B}_{\mathrm{rot},\Gamma_t}(\Omega) := \{\mathcal{B}_{\mathrm{rot},\Gamma_t,\ell}\}_{\ell=1}^{d_{\Omega,\Gamma_t}} \subset \mathsf{H}^{\infty}_{\Gamma_t,0}(\mathrm{rot},\Omega), \mathcal{B}_{\mathrm{div},\Gamma_n}(\Omega) := \{\mathcal{B}_{\mathrm{div},\Gamma_n,\ell}\}_{\ell=1}^{d_{\Omega,\Gamma_t}} \subset \mathsf{H}^{\infty}_{\Gamma_n,0}(\mathrm{div},\Omega)$$

such that $\pi_{\mathrm{div}}\mathcal{B}_{\mathrm{rot},\Gamma_t}(\Omega)$ and $\pi_{\mathrm{rot}}\varepsilon^{-1}\mathcal{B}_{\mathrm{div},\Gamma_n}(\Omega)$ are both bases of $\mathcal{H}_{\Gamma_t,\Gamma_n,\varepsilon}(\Omega)$. In particular,

$$\mathrm{Lin}\ \pi_{\mathrm{div}}\mathcal{B}_{\mathrm{rot},\Gamma_t}(\Omega) = \mathcal{H}_{\Gamma_t,\Gamma_n,\varepsilon}(\Omega) = \mathrm{Lin}\ \pi_{\mathrm{rot}}\varepsilon^{-1}\mathcal{B}_{\mathrm{div},\Gamma_n}(\Omega).$$

Note that $(1-\pi_{\mathrm{div}})$ and $(1-\pi_{\mathrm{rot}})$ are the $\mathsf{L}^2_\varepsilon(\Omega)$-orthonormal projectors onto grad $\mathsf{H}^1_{\Gamma_t}(\Omega)$ and $\varepsilon^{-1}\mathrm{rot}\ \mathsf{H}_{\Gamma_n}(\mathrm{rot},\Omega)$, respectively, that is,

$$(1-\pi_{\mathrm{div}}) : \mathsf{L}^2_\varepsilon(\Omega) \to \mathrm{grad}\ \mathsf{H}^1_{\Gamma_t}(\Omega), (1-\pi_{\mathrm{rot}}) : \mathsf{L}^2_\varepsilon(\Omega) \to \varepsilon^{-1}\mathrm{rot}\ \mathsf{H}_{\Gamma_n}(\mathrm{rot},\Omega).$$

Then by (23) and Theorem 5.2 (ii), we have, for example,

$$\begin{aligned}
\mathsf{H}_{\Gamma_t,0}(\mathrm{rot},\Omega) &= \mathrm{grad}\ \mathsf{H}^1_{\Gamma_t}(\Omega)\ \oplus_{\mathsf{L}^2_\varepsilon(\Omega)}\mathcal{H}_{\Gamma_t,\Gamma_n,\varepsilon}(\Omega)\\
&= \mathrm{grad}\ \mathsf{H}^1_{\Gamma_t}(\Omega)\ \oplus_{\mathsf{L}^2_\varepsilon(\Omega)}\mathrm{Lin}\ \pi_{\mathrm{div}}\mathcal{B}_{\mathrm{rot},\Gamma_t}(\Omega)\\
&= \mathrm{grad}\ \mathsf{H}^1_{\Gamma_t}(\Omega) + (\pi_{\mathrm{div}}-1)\mathrm{Lin}\ \mathcal{B}_{\mathrm{rot},\Gamma_t}(\Omega) + \mathrm{Lin}\ \mathcal{B}_{\mathrm{rot},\Gamma_t}(\Omega)\\
&= \mathrm{grad}\ \mathsf{H}^1_{\Gamma_t}(\Omega) + \mathrm{Lin}\ \mathcal{B}_{\mathrm{rot},\Gamma_t}(\Omega),\\
\mathsf{H}^k_{\Gamma_t,0}(\mathrm{rot},\Omega) &= \mathrm{grad}\ \mathsf{H}^1_{\Gamma_t}(\Omega)\cap \mathsf{H}^k_{\Gamma_t,0}(\mathrm{rot},\Omega) + \mathrm{Lin}\ \mathcal{B}_{\mathrm{rot},\Gamma_t}(\Omega),\\
&= \mathrm{grad}\ \mathsf{H}^{k+1}_{\Gamma_t}(\Omega) + \mathrm{Lin}\ \mathcal{B}_{\mathrm{rot},\Gamma_t}(\Omega).
\end{aligned} \tag{24}$$

Similar to Theorem 4.22, we get:

Theorem 5.10 (higher-order bounded regular direct decompositions for the vector de Rham complex). *Let (Ω,Γ_t) be a bounded strong Lipschitz pair and let $k \geq 0$. Then the bounded regular direct decompositions*

$$\mathsf{H}^k_{\Gamma_t}(\mathrm{rot},\Omega) = R(\widetilde{Q}^k_{\mathrm{rot},\Gamma_t,1}) \dotplus \mathsf{H}^k_{\Gamma_t,0}(\mathrm{rot},\Omega), \qquad \mathsf{H}^k_{\Gamma_t,0}(\mathrm{rot},\Omega) = \mathrm{grad}\ \mathsf{H}^{k+1}_{\Gamma_t}(\Omega) \dotplus \mathrm{Lin}\ \mathcal{B}_{\mathrm{rot},\Gamma_t}(\Omega),$$

$$\mathsf{H}^k_{\Gamma_n}(\mathrm{div},\Omega) = R(\widetilde{Q}^k_{\mathrm{div},\Gamma_n,1}) \dotplus \mathsf{H}^k_{\Gamma_n,0}(\mathrm{div},\Omega), \qquad \mathsf{H}^k_{\Gamma_n,0}(\mathrm{div},\Omega) = \mathrm{rot}\ \mathsf{H}^{k+1}_{\Gamma_n}(\Omega) \dotplus \mathrm{Lin}\ \mathcal{B}_{\mathrm{div},\Gamma_n}(\Omega)$$

hold. Note that $R(\widetilde{Q}^k_{\mathrm{rot},\Gamma_t,1}) \subset \mathsf{H}^{k+1}_{\Gamma_t}(\Omega)$ and $R(\widetilde{Q}^k_{\mathrm{div},\Gamma_n,1}) \subset \mathsf{H}^{k+1}_{\Gamma_n}(\Omega)$. In particular, for $k=0$

$$\mathsf{H}_{\Gamma_t}(\mathrm{rot},\Omega) = R(\widetilde{Q}^0_{\mathrm{rot},\Gamma_t,1}) \dotplus \mathsf{H}_{\Gamma_t,0}(\mathrm{rot},\Omega), \qquad \begin{aligned}\mathsf{H}_{\Gamma_t,0}(\mathrm{rot},\Omega) &= \mathrm{grad}\ \mathsf{H}^1_{\Gamma_t}(\Omega) \dotplus \mathrm{Lin}\ \mathcal{B}_{\mathrm{rot},\Gamma_t}(\Omega),\\ &= \mathrm{grad}\ \mathsf{H}^1_{\Gamma_t}(\Omega)\ \oplus_{\mathsf{L}^2_\varepsilon(\Omega)}\mathcal{H}_{\Gamma_t,\Gamma_n,\varepsilon}(\Omega),\end{aligned}$$

$$\mathsf{H}_{\Gamma_n}(\mathrm{div},\Omega) = R(\widetilde{Q}^0_{\mathrm{div},\Gamma_n,1}) \dotplus \mathsf{H}_{\Gamma_n,0}(\mathrm{div},\Omega), \qquad \begin{aligned}\varepsilon^{-1}\mathsf{H}_{\Gamma_n,0}(\mathrm{div},\Omega) &= \varepsilon^{-1}\mathrm{rot}\ \mathsf{H}^1_{\Gamma_n}(\Omega) \dotplus \varepsilon^{-1}\mathrm{Lin}\ \mathcal{B}_{\mathrm{div},\Gamma_n}(\Omega)\\ &= \varepsilon^{-1}\mathrm{rot}\ \mathsf{H}^1_{\Gamma_n}(\Omega)\ \oplus_{\mathsf{L}^2_\varepsilon(\Omega)}\mathcal{H}_{\Gamma_t,\Gamma_n,\varepsilon}(\Omega)\end{aligned}$$

as well as

$$\mathsf{L}^2_\varepsilon(\Omega) = \mathsf{H}_{\Gamma_t,0}(\mathrm{rot},\Omega)\ \oplus_{\mathsf{L}^2_\varepsilon(\Omega)}\varepsilon^{-1}\mathrm{rot}\ \mathsf{H}^1_{\Gamma_n}(\Omega) = \mathrm{grad}\ \mathsf{H}^1_{\Gamma_t}(\Omega)\ \oplus_{\mathsf{L}^2_\varepsilon(\Omega)}\varepsilon^{-1}\mathsf{H}_{\Gamma_n,0}(\mathrm{div},\Omega).$$

Remark 4.23 holds here as well. Noting

$$\varepsilon^{-1}\operatorname{rot} \mathsf{H}_{\Gamma_n}(\operatorname{rot},\Omega) \perp_{\mathsf{L}^2_\varepsilon(\Omega)} \mathcal{B}_{\operatorname{rot},\Gamma_t}(\Omega), \qquad \operatorname{grad} \mathsf{H}^1_{\Gamma_t}(\Omega) \perp_{\mathsf{L}^2(\Omega)} \mathcal{B}_{\operatorname{div},\Gamma_n}(\Omega) \tag{25}$$

we see:

Theorem 5.11 (alternative Dirichlet/Neumann projections for the vector de Rham complex)**.** *Let (Ω, Γ_t) be a bounded strong Lipschitz pair. Then*

$$\begin{aligned} \mathcal{H}_{\Gamma_t,\Gamma_n,\varepsilon}(\Omega) \cap \mathcal{B}_{\operatorname{rot},\Gamma_t}(\Omega)^{\perp_{\mathsf{L}^2_\varepsilon(\Omega)}} &= \{0\}, & \varepsilon^{-1}\mathsf{H}_{\Gamma_n,0}(\operatorname{div},\Omega) \cap \mathcal{B}_{\operatorname{rot},\Gamma_t}(\Omega)^{\perp_{\mathsf{L}^2_\varepsilon(\Omega)}} &= \varepsilon^{-1}\operatorname{rot} \mathsf{H}_{\Gamma_n}(\operatorname{rot},\Omega), \\ \mathcal{H}_{\Gamma_t,\Gamma_n,\varepsilon}(\Omega) \cap \mathcal{B}_{\operatorname{div},\Gamma_n}(\Omega)^{\perp_{\mathsf{L}^2(\Omega)}} &= \{0\}, & \mathsf{H}_{\Gamma_t,0}(\operatorname{rot},\Omega) \cap \mathcal{B}_{\operatorname{div},\Gamma_n}(\Omega)^{\perp_{\mathsf{L}^2(\Omega)}} &= \operatorname{grad} \mathsf{H}^1_{\Gamma_t}(\Omega). \end{aligned}$$

Moreover, for all $k \geq 0$,

$$\begin{aligned} \varepsilon^{-1}\mathsf{H}^k_{\Gamma_n,0}(\operatorname{div},\Omega) \cap \mathcal{B}_{\operatorname{rot},\Gamma_t}(\Omega)^{\perp_{\mathsf{L}^2_\varepsilon(\Omega)}} &= \varepsilon^{-1}\operatorname{rot} \mathsf{H}^k_{\Gamma_n}(\operatorname{rot},\Omega) = \varepsilon^{-1}\operatorname{rot} \mathsf{H}^{k+1}_{\Gamma_n}(\Omega), \\ \mathsf{H}^k_{\Gamma_t,0}(\operatorname{rot},\Omega) \cap \mathcal{B}_{\operatorname{div},\Gamma_n}(\Omega)^{\perp_{\mathsf{L}^2(\Omega)}} &= \operatorname{grad} \mathsf{H}^{k+1}_{\Gamma_t}(\Omega). \end{aligned}$$

Theorem 5.12 (cohomology groups of the vector de Rham complex)**.** *Let (Ω, Γ_t) be a bounded strong Lipschitz pair. Then*

$$N(\operatorname{rot}^k_{\Gamma_t})/R(\operatorname{grad}^k_{\Gamma_t}) \cong \operatorname{Lin} \mathcal{B}_{\operatorname{rot},\Gamma_t}(\Omega) \cong \mathcal{H}_{\Gamma_t,\Gamma_n,\varepsilon}(\Omega) \cong \operatorname{Lin} \mathcal{B}_{\operatorname{div},\Gamma_n}(\Omega) \cong N(\operatorname{div}^k_{\Gamma_n})/R(\operatorname{rot}^k_{\Gamma_n}).$$

In particular, the dimensions of the cohomology groups (Dirichlet/Neumann fields) are independent of k and ε, and it holds

$$d_{\Omega,\Gamma_t} = \dim\left(N(\operatorname{rot}^k_{\Gamma_t})/R(\operatorname{grad}^k_{\Gamma_t})\right) = \dim\left(N(\operatorname{div}^k_{\Gamma_n})/R(\operatorname{rot}^k_{\Gamma_n})\right).$$

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CONFLICT OF INTEREST

This work does not have any conflicts of interest.

ORCID

Dirk Pauly https://orcid.org/0000-0003-4155-7297

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APPENDIX A: RESULTS FOR THE CO-DERIVATIVE

By Hodge $\star$-duality, we get the corresponding dual results from Section 4 for the δ-operator.

Lemma 4.7 (regular potential for δ without boundary condition). *Let $\Omega \subset \mathbb{R}^d$ be a bounded strong Lipschitz domain and let $k \geq 0$ and $q \in \{0, \ldots, d-1\}$. Then there exists a bounded linear regular potential operator*

$$\mathcal{P}^{q,k}_{\delta,\emptyset} : \mathsf{H}^{q,k}_{\emptyset,0}(\delta,\Omega) \cap \mathcal{H}^{q}_{\Gamma,\emptyset,\mathrm{id}}(\Omega)^{\perp_{\mathsf{L}^{q,2}(\Omega)}} \to \mathsf{H}^{q+1,k+1}_{0}(\mathrm{d},\mathbb{R}^d),$$

such that $\delta\mathcal{P}^{q,k}_{\delta,\emptyset} = \mathrm{id}|_{\mathsf{H}^{q,k}_{\emptyset,0}(\delta,\Omega)\cap\mathcal{H}^{q}_{\Gamma,\emptyset,\mathrm{id}}(\Omega)^{\perp_{\mathsf{L}^{q,2}(\Omega)}}}$*, i.e., for all* $E \in \mathsf{H}^{q,k}_{\emptyset,0}(\delta,\Omega) \cap \mathcal{H}^{q}_{\Gamma,\emptyset,\mathrm{id}}(\Omega)^{\perp_{\mathsf{L}^{q,2}(\Omega)}}$

$$\delta\mathcal{P}^{q,k}_{\delta,\emptyset}E = E \text{ in } \Omega.$$

In particular, the bounded regular potential representations

$$R(\delta^{q+1,k}_{\emptyset}) = \mathsf{H}^{q,k}_{\emptyset,0}(\delta,\Omega) \cap \mathcal{H}^{q}_{\Gamma,\emptyset,\mathrm{id}}(\Omega)^{\perp_{\mathsf{L}^{q,2}(\Omega)}} = \delta\mathsf{H}^{q+1,k}_{\emptyset}(\delta,\Omega) = \delta\mathsf{H}^{q+1,k+1}_{\emptyset}(\Omega) = \delta\mathsf{H}^{q+1,k+1}_{\emptyset,0}(\mathrm{d},\Omega)$$

hold, and the potentials can be chosen such that they depend continuously on the data. Especially, these are closed subspaces of $\mathsf{H}^{q,k}_{\emptyset}(\Omega) = \mathsf{H}^{q,k}(\Omega)$*, and* $\mathcal{P}^{q,k}_{\delta,\emptyset}$ *is a right inverse to* δ*. By a simple cut-off technique,* $\mathcal{P}^{q,k}_{\delta,\emptyset}$ *may be modified to*

$$\mathcal{P}^{q,k}_{\delta,\emptyset} : \mathsf{H}^{q,k}_{\emptyset,0}(\delta,\Omega) \cap \mathcal{H}^{q}_{\Gamma,\emptyset,\mathrm{id}}(\Omega)^{\perp_{\mathsf{L}^{q,2}(\Omega)}} \to \mathsf{H}^{q+1,k+1}(\mathrm{d},\mathbb{R}^d)$$

such that $\mathcal{P}^{q,k}_{\delta,\emptyset}E$ *has a fixed compact support in* $\mathbb{R}^d$ *for all* $E \in \mathsf{H}^{q,k}_{\emptyset,0}(\delta,\Omega) \cap \mathcal{H}^{q}_{\Gamma,\emptyset,\mathrm{id}}(\Omega)^{\perp_{\mathsf{L}^{q,2}(\Omega)}}$.

Lemma 4.8 (regular potentials and decompostions for δ with partial boundary condition for extendable domains). *Let (Ω, Γ_n) be an extendable bounded strong Lipschitz pair and let $k \geq 0$.*

(i) *For $1 \leq q \leq d-1$, there exists a bounded linear regular potential operator*

$$\mathcal{P}^{q,k}_{\delta,\Gamma_n} : \mathbf{H}^{q,k}_{\Gamma_n,0}(\delta,\Omega) \to \mathsf{H}^{q+1,k+1}(\mathbb{R}^d) \cap \mathsf{H}^{q+1,k+1}_{\Gamma_n}(\Omega),$$

such that $\delta\mathcal{P}^{q,k}_{\mathrm{d},\Gamma_n} = \mathrm{id}|_{\mathbf{H}^{q,k}_{\Gamma_n,0}(\delta,\Omega)}$, that is, for all $E \in \mathbf{H}^{q,k}_{\Gamma_n,0}(\delta,\Omega)$

$$\delta\mathcal{P}^{q,k}_{\delta,\Gamma_n}E = E \text{ in } \Omega.$$

In particular, the bounded regular potential representations

$$\mathbf{H}^{q,k}_{\Gamma_n,0}(\delta,\Omega) = \mathsf{H}^{q,k}_{\Gamma_n,0}(\delta,\Omega) = \delta\mathsf{H}^{q+1,k+1}_{\Gamma_n}(\Omega) = \delta\mathsf{H}^{q+1,k}_{\Gamma_n}(\delta,\Omega)$$

hold, and the potentials can be chosen such that they depend continuously on the data. Especially, these are closed subspaces of $\mathsf{H}^{q,k}_{\emptyset}(\Omega) = \mathsf{H}^{q,k}(\Omega)$, and $\mathcal{P}^{q,k}_{\delta,\Gamma_n}$ is a right inverse to δ. The results extend literally to the case $q = 0$ if $\Gamma_n \neq \Gamma$, and the case $q = d$ is trivial since $\mathbf{H}^{d,k}_{\Gamma_n,0}(\delta,\Omega) = \mathbb{R}_{\Gamma_n}$. For $q = 0$ and $\Gamma_n = \Gamma$, the results still remain valid if $\mathbf{H}^{0,k}_{\Gamma,0}(\delta,\Omega) = \mathbf{H}^{0,k}_{\Gamma}(\Omega)$ and $\mathsf{H}^{0,k}_{\Gamma,0}(\delta,\Omega) = \mathsf{H}^{0,k}_{\Gamma}(\Omega)$ are replaced by the slightly smaller spaces $\mathbf{H}^{0,k}_{\Gamma}(\Omega) \cap \mathbb{R}^{\perp_{\mathsf{L}^{0,2}(\Omega)}}$ and $\mathsf{H}^{0,k}_{\Gamma}(\Omega) \cap \mathbb{R}^{\perp_{\mathsf{L}^{0,2}(\Omega)}}$, respectively.

(ii) *For all $0 \leq q \leq d$, the regular decompositions*

$$\begin{aligned}\mathbf{H}^{q,k}_{\Gamma_n}(\delta,\Omega) = \mathsf{H}^{q,k}_{\Gamma_n}(\delta,\Omega) &= \mathsf{H}^{q,k+1}_{\Gamma_n}(\Omega) + \delta\mathsf{H}^{q+1,k+1}_{\Gamma_n}(\Omega)\\ &= \mathcal{Q}^{q,k}_{\delta,\Gamma_n,1}\mathsf{H}^{q,k}_{\Gamma_n}(\delta,\Omega) \dotplus \delta\mathcal{Q}^{q,k}_{\delta,\Gamma_n,0}\mathsf{H}^{q,k}_{\Gamma_n}(\delta,\Omega)\\ &= \mathcal{Q}^{q,k}_{\delta,\Gamma_n,1}\mathsf{H}^{q,k}_{\Gamma_n}(\delta,\Omega) \dotplus \delta\mathsf{H}^{q+1,k+1}_{\Gamma_n}(\Omega)\\ &= \mathcal{Q}^{q,k}_{\delta,\Gamma_n,1}\mathsf{H}^{q,k}_{\Gamma_n}(\delta,\Omega) \dotplus \mathsf{H}^{q,k}_{\Gamma_n,0}(\delta,\Omega)\end{aligned}$$

hold with bounded linear regular decomposition operators

$$\begin{aligned}\mathcal{Q}^{q,k}_{\delta,\Gamma_n,1} &:= \mathcal{P}^{q-1,k}_{\delta,\Gamma_n}\delta : \mathsf{H}^{q,k}_{\Gamma_n}(\delta,\Omega) \to \mathsf{H}^{q,k+1}_{\Gamma_n}(\Omega),\\ \mathcal{Q}^{q,k}_{\delta,\Gamma_n,0} &:= \mathcal{P}^{q,k}_{\delta,\Gamma_n}(1 - \mathcal{P}^{q-1,k}_{\delta,\Gamma_n}\delta) : \mathsf{H}^{q,k}_{\Gamma_n}(\delta,\Omega) \to \mathsf{H}^{q+1,k+1}_{\Gamma_n}(\Omega)\end{aligned}$$

satisfying $\mathcal{Q}^{q,k}_{\delta,\Gamma_n,1} + \delta\mathcal{Q}^{q,k}_{\delta,\Gamma_n,0} = \mathrm{id}|_{\mathsf{H}^{q,k}_{\Gamma_n}(\delta,\Omega)}$. Moreover, it holds $\delta\mathcal{Q}^{q,k}_{\delta,\Gamma_n,1} = \delta^{q,k}_{\Gamma_n}$, and thus, $\mathsf{H}^{q,k}_{\Gamma_n,0}(\delta,\Omega)$ is invariant under $\mathcal{Q}^{q,k}_{\delta,\Gamma_n,1}$. $\mathcal{Q}^{q,k}_{\delta,\Gamma_n,1}\mathsf{H}^{q,k}_{\Gamma_n}(\delta,\Omega) = R(\mathcal{Q}^{q,k}_{\delta,\Gamma_n,1}) = R(\mathcal{P}^{q-1,k}_{\delta,\Gamma_n})$ as well as $\mathcal{Q}^{q,k}_{\delta,\Gamma_n,0}\mathsf{H}^{q,k}_{\Gamma_n}(\delta,\Omega) = R(\mathcal{Q}^{q,k}_{\delta,\Gamma_n,0}) = R(\mathcal{P}^{q,k}_{\delta,\Gamma_n})$ hold.

Lemma 4.9 (regular decompositions for δ with partial boundary condition)**.** *Let (Ω, Γ_n) be a bounded strong Lipschitz pair and let $k \geq 0$. Then the bounded regular decompositions*

$$\boldsymbol{H}^{q,k}_{\Gamma_n}(\delta,\Omega) = \mathsf{H}^{q,k}_{\Gamma_n}(\delta,\Omega) = \mathsf{H}^{q,k+1}_{\Gamma_n}(\Omega) + \delta\mathsf{H}^{q+1,k+1}_{\Gamma_n}(\Omega)$$

hold with bounded linear regular decomposition operators

$$\mathcal{Q}^{q,k}_{\delta,\Gamma_n,1} : \mathsf{H}^{q,k}_{\Gamma_n}(\delta,\Omega) \to \mathsf{H}^{q,k+1}_{\Gamma_n}(\Omega), \qquad \mathcal{Q}^{q,k}_{\delta,\Gamma_n,0} : \mathsf{H}^{q,k}_{\Gamma_n}(\delta,\Omega) \to \mathsf{H}^{q+1,k+1}_{\Gamma_n}(\Omega)$$

satisfying $\mathcal{Q}^{q,k}_{\delta,\Gamma_n,1} + \delta\mathcal{Q}^{q,k}_{\delta,\Gamma_n,0} = \mathrm{id}_{\mathsf{H}^{q,k}_{\Gamma_n}(\delta,\Omega)}$. In particular, weak and strong boundary conditions coincide. Moreover, it holds $\delta\mathcal{Q}^{q,k}_{\delta,\Gamma_n,1} = \delta^{q,k}_{\Gamma_n}$, and thus, $\mathsf{H}^{q,k}_{\Gamma_n,0}(\delta,\Omega)$ is invariant under $\mathcal{Q}^{q,k}_{\delta,\Gamma_n,1}$.

Theorem 5.13 (higher-order bounded regular potentials and decompositions for δ with partial boundary condition)**.** *Let (Ω, Γ_n) be a bounded strong Lipschitz pair and let $k \geq 0$. Moreover, let $\mathcal{Q}^{q,k}_{\delta,\Gamma_n,1}$ be given from Lemma 4.9. Then:*

(i) *For all* $q \in \{0, \dots, d-1\}$*, there exists a bounded linear regular potential operator*

$$\mathcal{P}^{q,k}_{\delta,\Gamma_n} := \mathcal{Q}^{q+1,k}_{\delta,\Gamma_n,1}(\delta^{q+1,k}_{\Gamma_n})^{-1}_{\perp} : \mathsf{H}^{q,k}_{\Gamma_n,0}(\delta,\Omega) \cap \mathcal{H}^{q}_{\Gamma_t,\Gamma_n,\varepsilon}(\Omega)^{\perp_{\mathsf{L}^{q,2}(\Omega)}} \to \mathsf{H}^{q+1,k+1}_{\Gamma_n}(\Omega),$$

such that $\delta\mathcal{P}^{q,k}_{\delta,\Gamma_n} = \mathrm{id}|_{\mathsf{H}^{q,k}_{\Gamma_n,0}(\delta,\Omega)\cap\mathcal{H}^{q}_{\Gamma_t,\Gamma_n,\varepsilon}(\Omega)^{\perp_{\mathsf{L}^{q,2}(\Omega)}}}$*. In particular, the bounded regular representations*

$$\begin{aligned} R(\delta^{q+1,k}_{\Gamma_n}) &= \mathsf{H}^{q,k}_{\Gamma_n,0}(\delta,\Omega) \cap \mathcal{H}^{q}_{\Gamma_t,\Gamma_n,\varepsilon}(\Omega)^{\perp_{\mathsf{L}^{q,2}(\Omega)}} \\ &= \mathsf{H}^{q,k}_{\Gamma_n}(\Omega) \cap \delta\mathsf{H}^{q+1}_{\Gamma_n}(\delta,\Omega) = \delta\mathsf{H}^{q+1,k}_{\Gamma_n}(\delta,\Omega) = \delta\mathsf{H}^{q+1,k+1}_{\Gamma_n}(\Omega) \end{aligned}$$

hold, and the potentials can be chosen such that they depend continuously on the data.

(ii) *The bounded regular decompositions*

$$\begin{aligned} \mathsf{H}^{q,k}_{\Gamma_n}(\delta,\Omega) &= \mathsf{H}^{q,k+1}_{\Gamma_n}(\Omega) + \mathsf{H}^{q,k}_{\Gamma_n,0}(\delta,\Omega) = \mathsf{H}^{q,k+1}_{\Gamma_n}(\Omega) + \delta\mathsf{H}^{q+1,k+1}_{\Gamma_n}(\Omega) \\ &= R(\widetilde{\mathcal{Q}}^{q,k}_{\delta,\Gamma_n,1}) \dotplus \mathsf{H}^{q,k}_{\Gamma_n,0}(\delta,\Omega) = R(\widetilde{\mathcal{Q}}^{q,k}_{\delta,\Gamma_n,1}) \dotplus R(\widetilde{\mathcal{N}}^{q,k}_{\delta,\Gamma_n}) \end{aligned}$$

hold with bounded linear regular decomposition operators

$$\widetilde{\mathcal{Q}}^{q,k}_{\delta,\Gamma_n,1} := \mathcal{P}^{q-1,k}_{\delta,\Gamma_n}\delta^{q,k}_{\Gamma_n} : \mathsf{H}^{q,k}_{\Gamma_n}(\delta,\Omega) \to \mathsf{H}^{q,k+1}_{\Gamma_n}(\Omega), \qquad \widetilde{\mathcal{N}}^{q,k}_{\delta,\Gamma_n} : \mathsf{H}^{q,k}_{\Gamma_n}(\delta,\Omega) \to \mathsf{H}^{q,k}_{\Gamma_n,0}(\delta,\Omega)$$

satisfying $\widetilde{\mathcal{Q}}^{q,k}_{\delta,\Gamma_n,1} + \widetilde{\mathcal{N}}^{q,k}_{\delta,\Gamma_n} = \mathrm{id}_{\mathsf{H}^{q,k}_{\Gamma_n}(\delta,\Omega)}$*. Moreover,* $\delta\widetilde{\mathcal{Q}}^{q,k}_{\delta,\Gamma_n,1} = \delta\mathcal{Q}^{q,k}_{\delta,\Gamma_n,1} = \delta^{q,k}_{\Gamma_n}$*, and thus,* $\mathsf{H}^{q,k}_{\Gamma_n,0}(\delta,\Omega)$ *is invariant under* $\mathcal{Q}^{q,k}_{\delta,\Gamma_n,1}$ *and* $\widetilde{\mathcal{Q}}^{q,k}_{\delta,\Gamma_n,1}$*. It holds* $R(\widetilde{\mathcal{Q}}^{q,k}_{\delta,\Gamma_n,1}) = R(\mathcal{P}^{q-1,k}_{\delta,\Gamma_n})$ *and* $\widetilde{\mathcal{Q}}^{q,k}_{\delta,\Gamma_n,1} = \mathcal{P}^{q-1,k}_{\delta,\Gamma_n}\delta^{q,k}_{\Gamma_n} = \mathcal{Q}^{q,k}_{\delta,\Gamma_n,1}(\delta^{q,k}_{\Gamma_n})^{-1}_{\perp}\delta^{q,k}_{\Gamma_n}$*. Hence,* $\widetilde{\mathcal{Q}}^{q,k}_{\delta,\Gamma_n,1}|_{(\delta^{q,k}_{\Gamma_n})_{\perp}} = \mathcal{Q}^{q,k}_{\delta,\Gamma_n,1}|_{(\delta^{q,k}_{\Gamma_n})_{\perp}}$*, and thus,* $\widetilde{\mathcal{Q}}^{q,k}_{\delta,\Gamma_n,1}$ *may differ from* $\mathcal{Q}^{q,k}_{\delta,\Gamma_n,1}$ *only on* $\mathsf{H}^{q,k}_{\Gamma_n,0}(\delta,\Omega)$*.*

(ii′) *The bounded regular kernel decomposition* $\mathsf{H}^{q,k}_{\Gamma_n,0}(\delta,\Omega) = \mathsf{H}^{q,k+1}_{\Gamma_n,0}(\delta,\Omega) + \delta\mathsf{H}^{q+1,k+1}_{\Gamma_n}(\Omega)$ *holds.*

Note that Remarks 4.12 and 4.19 hold with obvious modifications.