

# On Korn's first inequality for tangential or normal boundary conditions with explicit constants

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Communicated by R. Picard

**We will prove that for piecewise  $C^2$ -concave domains in  $\mathbb{R}^N$  Korn's first inequality holds for vector fields satisfying homogeneous normal or tangential boundary conditions with explicit Korn constant  $\sqrt{2}$ . Copyright © 2016 John Wiley & Sons, Ltd.**

**Keywords:** Korn inequality; tangential and normal boundary conditions; variational methods including variational inequalities; kinetic theory of gases; rarefied gas flows; Boltzmann equation

## 1. Introduction

In [1], Desvillettes and Villani proved a non-standard version of Korn's first inequality

$$|\nabla v|_{L^2(\Omega)} \leq c_{k,n} |\text{sym} \nabla v|_{L^2(\Omega)} \quad (1)$$

on non-axisymmetric sufficiently smooth and bounded domains in  $\mathbb{R}^N$  for vector fields being tangential at the boundary. Here,  $c_{k,n} > 0$  denotes the best available constant, and the indices  $k$  and  $n$  refer to 'Korn' and 'homogenous normal boundary condition'. As pointed out in [2], this Korn inequality has an important application in statistical physics, more precisely in the study of relaxation to equilibrium of rarefied gases modeled by Boltzmann's equation.

In the paper at hand, we will show that for piecewise  $C^2$ -domains<sup>‡</sup> in  $\mathbb{R}^N$  with concave or even polyhedral boundary parts (Definition 2), Korn's first inequality holds for vector fields satisfying (possibly mixed) homogeneous normal or homogenous tangential boundary conditions; see (3) and (4) for a definition of the relevant spaces. In every case, the Korn constant can be estimated by  $\sqrt{2}$ :

$$|\nabla v|_{L^2(\Omega)} \leq \sqrt{2} |\text{sym} \nabla v|_{L^2(\Omega)};$$

see Theorem 9 for a precise statement. The proof of our main theorem consists of a simple combination of two pointwise equalities of the gradient of a vector field, (5) and (6), and an integration by parts formula derived, for example, by Grisvard in [3, Theorem 3.1.1.2] see Proposition 4. But before going into details of the proof, we shall discuss some disturbing consequences of Theorem 9 seriously questioning at least the physical justification of full normal boundary conditions.

It is well known that Korn's first inequality with full normal boundary condition does not hold if  $\Omega$  is axisymmetric. We illustrate this fact with a simple example: Let  $\Omega \subset \mathbb{R}^3$  be a bounded body of rotation with axis of symmetry  $x_1 = x_2 = 0$ , for example,  $\Omega$  could be a ball, a cylinder, or a cone. Then the vector field  $v$  defined by  $v(x) := (x_2, -x_1, 0)^\top$  belongs to  $H^1(\Omega)$  and is tangential to  $\partial\Omega$ . Hence,  $v \in \overset{\circ}{H}_n^1(\Omega)$  (for a precise definition of  $\overset{\circ}{H}_n^1(\Omega)$ , see (3)) and

$$\nabla v = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \text{sym} \nabla v = 0, \quad \text{div} v = 0.$$

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‡ Throughout this contribution,  $C^2$  can always be replaced by  $C^{1,1}$ .

Thus, Korn's first inequality with full normal boundary condition (1) fails for these special domains  $\Omega$ , that is,  $c_{k,n} = c_{k,n}(\Omega) = \infty$ . On the other hand, Theorem 9 applies for every polyhedral approximation  $\Omega_p$  of  $\Omega$ , and we have the following:

$$\forall v \in \mathring{H}_n^1(\Omega_p) \quad |\nabla v|_{L^2(\Omega_p)} \leq \sqrt{2} |\text{sym} \nabla v|_{L^2(\Omega_p)}.$$

This means the (first) Korn constant can jump from  $\sqrt{2}$  to  $\infty$  caused by an arbitrary small deformation of the domain. Many numerical schemes work on polyhedral domains of computation. The Korn constants of all these domains are bounded from above by  $\sqrt{2}$ . But in many applications, the domain of computation is just an approximation of some 'real' domain, whose Korn constant could be much larger.

Furthermore, we shall discuss some conjectures on the meaning of Korn's first constant made in [1] and [2]. In [1], Korn's first inequality with normal boundary condition, that is, (1), is proved for a bounded  $C^1$ -domain  $\Omega \subset \mathbb{R}^N$ , which is not axisymmetric. An upper bound for the first Korn constant is presented by the following<sup>§</sup>:

$$c_{k,n}^2 \leq 2N(1 + c_{m,n}^2)(1 + c_k^2)(1 + c_g^{-1}), \quad (2)$$

where  $c_k$  denotes the first Korn constant for vector fields in  $H^1(\Omega)$  without boundary conditions,  $c_{m,n}$  a special Gaffney constant for tangential vector fields in  $H^1(\Omega)$ , and  $c_g$  the so-called Grad's number defined by the following:

$$c_g := \frac{1}{|\Omega|} \inf_{|\sigma|=1} \inf_{v_\sigma \in V_{n,\sigma}(\Omega)} |\text{sym} \nabla v_\sigma|_{L^2(\Omega)}^2$$

with the finite dimensional set

$$V_{n,\sigma}(\Omega) := \{v \in \mathring{H}_n^1(\Omega) : \text{div } v = 0 \wedge \text{rot } v = \sigma\}, \quad \sigma \in S^{(N-1)N/2-1}.$$

For a precise definition of and more comments on these constants, see Section 4.

It is now conjectured in [1, pages 607f] and [2, pages 285, 306 and (48)] that the constant  $c_{k,n}$  quantifies the deviation of  $\Omega \subset \mathbb{R}^N$  from being axisymmetric in the sense that  $c_{k,n} = c_{k,n}(\Omega)$  tends to infinity, if  $\Omega$  is approaching axial symmetry. For such a statement, it would be necessary to bound  $c_{k,n}$  from below, while in [1], only the bound (2) from above is proved. However, Theorem 9 clearly shows that this conjecture becomes false at least if polyhedra are allowed to compete. In [1] and [2, page 609], it is also conjectured that it is Grad's number  $c_g = c_g(\Omega)$  steering this blow-up of the Korn constant. It is conjectured that for smooth domains, Grad's number tends to 0, while the domain  $\Omega$  is approaching axial symmetry. The following is actually stated in [1, Proposition 5]:  $c_g(\Omega) = 0$  if and only if  $\Omega$  is axisymmetric. Moreover, there is a lower bound on  $c_g(\Omega)$  that depends on the shape of  $\Omega$ . But in order to prove the conjecture, it would be necessary to give an upper bound on  $c_g(\Omega)$  tending to 0 if the domain is approaching axial symmetry. However, this conjecture gets wrong, too, if we allow for polyhedra: In Section 4, we will show that

$$c_g(\Omega_p) = \frac{1}{2}$$

holds for every convex-bounded polyhedron  $\Omega_p$ . Therefore, for any sequence of bounded and convex polyhedra tending to any axisymmetric domain, Grad's number equals  $\frac{1}{2}$ .

The remaining part of the paper is organized as follows: In Section 2, we give the relevant definitions on the spaces and domains used and establish some equalities and inequalities used in the sequel. In Section 3, we state our main theorem in detail and give the proof. In Section 4, we discuss the constants  $c_{k,n}$ ,  $c_k$ ,  $c_{m,n}$ , and  $c_g$  and give some further comments on the regularity of the boundary needed in the proof of (1) in [1]. In the last section, we provide some more results estimating the gradient of a vector field.

## 2. Preliminaries

Let  $\Omega$  be an open subset of  $\mathbb{R}^N$  with  $2 \leq N \in \mathbb{N}$  and boundary  $\Gamma := \partial\Omega$ . We introduce the standard scalar-valued Lebesgue and Sobolev spaces by  $L^2(\Omega)$  and  $H^1(\Omega)$ , respectively. Moreover, we define  $\mathring{H}^1(\Omega)$  as closure in  $H^1(\Omega)$  of smooth and compactly supported test functions  $\mathring{C}^\infty(\Omega)$ . These definitions extend componentwise to vector or matrix fields, and we will use the same notations for these spaces throughout the paper. Moreover, we will consistently denote functions by  $u$  and vector fields by  $v$ . If  $\Omega$  is Lipschitz, we define the vector-valued Sobolev space  $\mathring{H}_t^1(\Omega)$  resp.  $\mathring{H}_n^1(\Omega)$  as closure in  $H^1(\Omega)$  of the set of test vector fields

$$\mathring{C}_t^\infty(\Omega) := \{v \in \mathring{C}^\infty(\bar{\Omega}) : v_t = 0\}, \quad \mathring{C}_n^\infty(\Omega) := \{v \in \mathring{C}^\infty(\bar{\Omega}) : v_n = 0\}, \quad (3)$$

respectively, generalizing homogeneous tangential resp. normal boundary conditions. Here,  $\nu$  denotes the a.e. defined outer unit normal at  $\Gamma$  giving a.e. the tangential resp. normal component

<sup>§</sup>In [1], the notations are different. For the constants, we have  $c_{k,n}^{-2} = K(\Omega)$ ,  $c_k^{-2} = \bar{K}(\Omega)$ ,  $c_{m,n}^2 = C_H(\Omega)$ , and  $c_g = G(\Omega)$ .

$$v_t := v|_\Gamma - v_n \nu, \quad v_n := \nu \cdot v|_\Gamma$$

of  $v$  on  $\Gamma$ . Here, we denote as usual

$$\mathring{C}^\infty(\bar{\Omega}) := \{v|_\Omega : v \in \mathring{C}^\infty(\mathbb{R}^N)\}.$$

For smooth functions or vector fields  $v$  in  $H^1(\Omega)$ , we have the following<sup>¶</sup>:

$$v \in \mathring{H}^1(\Omega) \Leftrightarrow v|_\Gamma = 0, \quad v \in \mathring{H}_t^1(\Omega) \Leftrightarrow \nu \times v|_\Gamma = 0, \quad v \in \mathring{H}_n^1(\Omega) \Leftrightarrow \nu \cdot v|_\Gamma = 0.$$

If  $\Gamma$  is decomposed into two relatively open subsets  $\Gamma_t$  and  $\Gamma_n := \Gamma \setminus \bar{\Gamma}_t$ , we define the vector-valued  $H^1$ -Sobolev space of mixed boundary conditions  $\mathring{H}_{t,n}^1(\Omega)$  as closure in  $H^1(\Omega)$  of the set of test vector fields

$$\mathring{C}_{t,n}^\infty(\Omega) := \{v \in \mathring{C}^\infty(\bar{\Omega}) : v_t|_{\Gamma_t} = 0 \wedge v_n|_{\Gamma_n} = 0\}, \tag{4}$$

generalizing  $\nu \times v|_{\Gamma_t} = 0$  and  $\nu \cdot v|_{\Gamma_n} = 0$  for  $v \in \mathring{H}_{t,n}^1(\Omega)$ , respectively. For matrices  $A \in \mathbb{R}^{N \times N}$ , we recall the notations

$$\text{sym } A := \frac{1}{2}(A + A^T), \quad \text{skw } A := \frac{1}{2}(A - A^T), \quad \text{dev } A := A - \text{id}_A, \quad \text{id}_A := \frac{\text{tr } A}{N} \text{id}$$

with  $\text{tr } A := A \cdot \text{id}$  using the pointwise scalar product. By pointwise orthogonality, we have the following:

$$|A|^2 = |\text{dev } A|^2 + \frac{1}{N} |\text{tr } A|^2, \quad |A|^2 = |\text{sym } A|^2 + |\text{skw } A|^2, \quad |\text{sym } A|^2 = |\text{dev sym } A|^2 + \frac{1}{N} |\text{tr } A|^2$$

and hence  $|\text{dev } A|, N^{-\frac{1}{2}} |\text{tr } A|, |\text{sym } A|, |\text{skw } A| \leq |A|$ . Especially for  $A := \nabla v := J_v^T$ , where  $J_v$  denotes the Jacobian of  $v \in H^1(\Omega)$ , we see pointwise a.e.<sup>||</sup>

$$|\text{skw } \nabla v|^2 = \frac{1}{2} |\text{rot } v|^2, \quad \text{tr } \nabla v = \text{div } v$$

and

$$|\nabla v|^2 = |\text{dev sym } \nabla v|^2 + \frac{1}{N} |\text{div } v|^2 + \frac{1}{2} |\text{rot } v|^2. \tag{5}$$

Moreover, we have the following:

$$|\nabla v|^2 = |\text{rot } v|^2 + \langle \nabla v, (\nabla v)^T \rangle, \tag{6}$$

because

$$2|\text{skw } \nabla v|^2 = \frac{1}{2} |\nabla v - (\nabla v)^T|^2 = |\nabla v|^2 - \langle \nabla v, (\nabla v)^T \rangle.$$

The simplest version of Korn's first inequality is the following.

**Lemma 1** (Korn's first inequality:  $\mathring{H}^1$ -version)

For all  $v \in \mathring{H}^1(\Omega)$ ,

$$|\nabla v|_{L^2(\Omega)}^2 = 2|\text{dev sym } \nabla v|_{L^2(\Omega)}^2 + \frac{2-N}{N} |\text{div } v|_{L^2(\Omega)}^2 \leq 2|\text{dev sym } \nabla v|_{L^2(\Omega)}^2,$$

and equality holds if and only if  $\text{div } v = 0$  or  $N = 2$ .

Although the proof is very simple, we present it here, because we will use the underlying idea later.

<sup>¶</sup>The cross-product notation needs an explanation. If we identify vector fields  $a, b$  in  $\mathbb{R}^N$  with 1-forms  $\alpha, \beta$ , we have for  $|\beta| = 1$  the identity  $\alpha = \beta \wedge * \beta \wedge * \alpha + (-1)^N * \beta \wedge * \beta \wedge \alpha$ , where the wedge and Hodge star operations are executed from right to left. Especially in  $\mathbb{R}^3$ , we have  $* \beta \wedge \alpha \cong b \times a$ ,  $* \beta \wedge * \beta \wedge \alpha \cong b \times b \times a$ , and  $* \beta \wedge * \alpha \cong b \cdot a$ . Hence,  $a = (b \cdot a)b - b \times b \times a$ . For  $b := \nu$  and  $a := v|_\Gamma$ , we get  $v_t = v|_\Gamma - \nu_n \nu = -\nu \times \nu \times v|_\Gamma$ , and we see  $v_t = 0$  if and only if  $\nu \times v|_\Gamma = 0$ . Now, in this spirit, the cross-product in  $\mathbb{R}^N$  for vector fields is generally defined by  $b \times a := * \beta \wedge \alpha$ , the latter being an  $(N-2)$ -form. This yields, for example,  $b \times a = b_1 a_2 - b_2 a_1$  in  $\mathbb{R}^2$  or generally  $b \times a \in \mathbb{R}^{(N-1)N/2}$  in  $\mathbb{R}^N$ .

<sup>||</sup>The rot-operator can be defined as follows: We identify smooth vector fields  $a$  in  $\mathbb{R}^N$  with smooth 1-forms  $\alpha$ . Then  $\text{rot } a := d\alpha$ , where  $d$  denotes the exterior derivative and  $d\alpha$  is a 2-form. For  $N = 2$ , we obtain the scalar-valued rotation  $\text{rot } a = \partial_1 a_2 - \partial_2 a_1$ , and for  $N = 3$ , the classical rotation  $\text{rot } a$  appears, whereas generally,  $\text{rot } a(x) \in \mathbb{R}^{(N-1)N/2}$  holds.

*Proof*

For all vector fields  $v \in \mathring{C}^\infty(\Omega)$ , using  $-\Delta = \text{rot}^* \text{rot} - \nabla \text{div}$  we have Gaffney's equality:

$$|\nabla v|_{L^2(\Omega)}^2 = |\text{rot } v|_{L^2(\Omega)}^2 + |\text{div } v|_{L^2(\Omega)}^2, \tag{7}$$

which extends to all  $v \in \mathring{H}^1(\Omega)$  by continuity. Hence, with (5)

$$|\nabla v|_{L^2(\Omega)}^2 = |\text{dev sym } \nabla v|_{L^2(\Omega)}^2 + \frac{1}{2} |\nabla v|_{L^2(\Omega)}^2 + \frac{2-N}{2N} |\text{div } v|_{L^2(\Omega)}^2, \tag{8}$$

and the assertion follows immediately. □

Recalling that we work with exterior unit normals at the boundaries, we now introduce our admissible domains.

**Definition 2**

We call  $\Omega$  'piecewise  $C^2$ ', if

- (i)  $\Gamma$  is strongly Lipschitz, that is, locally, a graph of a Lipschitz function,
- (ii)  $\Gamma = \Gamma_0 \cup \Gamma_1$ , where  $\Gamma_0$  has  $(N - 1)$ -dimensional Lebesgue measure zero, and  $\Gamma_1$  is relatively open in  $\Gamma$  and locally, a graph of a  $C^2$ -function.

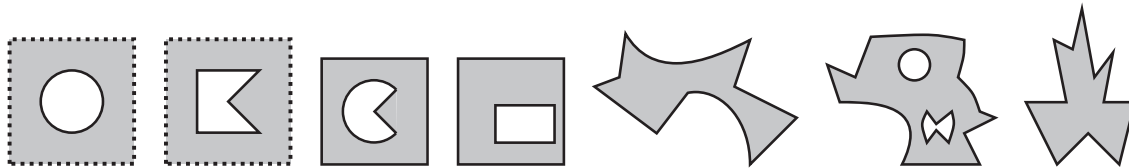
We call  $\Omega$  'piecewise  $C^2$ -convex' resp. 'piecewise  $C^2$ -concave' if  $\Omega$  is piecewise  $C^2$  and

- (iii) the second fundamental form on  $\Gamma_1$  induced by  $\nabla v$  is positive resp. negative semi-definite.

By assumptions, the exterior unit normal  $\nu$  can be extended into a neighborhood of  $\Gamma_1$  such that the second fundamental form, that is, the gradient  $\nabla \nu$ , and its trace  $\text{tr} \nabla \nu = \text{div } \nu = 2H$ , where  $H$  denotes the mean curvature, are well defined. For precise definitions, see, for example, [3, Section 3.1.1].

**Example 3**

The following domains in  $\mathbb{R}^2$  are piecewise  $C^2$ -concave, where the dotted lines indicate an exterior domain:



Our main result is an easy consequence of the pointwise equalities (5) and (6) and the following crucial proposition from Grisvard, [3, Theorem 3.1.1.2].

**Proposition 4** (Integration by parts)

Let  $\Omega$  be piecewise  $C^2$ . Then for all  $v \in \mathring{C}^\infty(\overline{\Omega})$ ,

$$|\text{div } v|_{L^2(\Omega)}^2 - \langle \nabla v, (\nabla v)^T \rangle_{L^2(\Omega)} = \int_{\Gamma_1} (\text{div}_\Gamma(v_n \nu_t) - 2\nu_t \cdot \nabla_\Gamma v_n) + \int_{\Gamma_1} (\text{div } v |v_n|^2 + ((\nabla v) \nu_t) \cdot \nu_t),$$

and for  $v \in \mathring{C}_{\text{tn}}^\infty(\Omega)$ ,

$$|\text{div } v|_{L^2(\Omega)}^2 - \langle \nabla v, (\nabla v)^T \rangle_{L^2(\Omega)} = \int_{\Gamma_1} (\text{div } v |v_n|^2 + ((\nabla v) \nu_t) \cdot \nu_t).$$

Here,  $\text{div}_\Gamma$  and  $\nabla_\Gamma$  are the usual surface differential operators on  $\Gamma_1$ , which may be identified with the co-derivative  $*d^*$  on 1-forms and the exterior derivative  $d$  on 0-forms on  $\Gamma_1$ , respectively. Actually, in [3], it is assumed that  $\Omega$  is bounded. But because we assume that  $v$  has compact support, the asserted formulas hold for unbounded domains as well.

**Remark 5**

We note that in [3], Grisvard uses  $-\nabla v$  to define the second fundamental form, which implies a negative sign for the curvature term, that is, the integral

$$- \int_{\Gamma_1} (\text{div } v |v_n|^2 + ((\nabla v) \nu_t) \cdot \nu_t)$$

\*\*For smooth 1-forms in  $\mathbb{R}^N$ , we have  $-\Delta \alpha = d^* d^* \alpha + (-1)^N * d^* d \alpha$ . This means for a corresponding smooth vector proxy in  $\mathbb{R}^N$  that  $-\Delta a = -\nabla \text{div } a + \text{rot}^* \text{rot } a$ , where  $\text{rot}^* \cong (-1)^N * d^*$ , the latter mapping 2-forms to 1-forms, denotes the (formal) adjoint of  $\text{rot} \cong d$ . Hence,  $\text{rot}^*$  maps smooth vector fields in  $\mathbb{R}^{(N-1)N/2}$  to vector fields in  $\mathbb{R}^N$ . Especially in  $\mathbb{R}^3$ , we have  $\text{rot}^* = \text{rot}$  and hence  $-\Delta a = -\nabla \text{div } a + \text{rot } \text{rot } a$ . In  $\mathbb{R}^2$ , it holds  $\text{rot}^* = R \nabla$ , where  $R$  is the  $90^\circ$ -rotation matrix, and hence  $-\Delta a = -\nabla \text{div } a + R \nabla \text{rot } a$ .

appears in [3, Theorem 3.1.1.2]. Moreover, by  $\operatorname{div}_\Gamma(v_n v_t) = v_n \operatorname{div}_\Gamma v_t + v_t \cdot \nabla_\Gamma v_n$  on  $\Gamma_1$ , we have the following:

$$\operatorname{div}_\Gamma(v_n v_t) - 2v_t \cdot \nabla_\Gamma v_n = v_n \operatorname{div}_\Gamma v_t - v_t \cdot \nabla_\Gamma v_n.$$

An immediate corollary of Proposition 4 is the following.

*Corollary 6* (Gaffney's inequalities)

Let  $\Omega$  be piecewise  $C^2$ -convex resp.  $C^2$ -concave and  $v \in \mathring{H}_{t,n}^1(\Omega)$ . Then

$$|\nabla v|_{L^2(\Omega)}^2 \leq |\operatorname{rot} v|_{L^2(\Omega)}^2 + |\operatorname{div} v|_{L^2(\Omega)}^2 \quad \text{resp.} \quad |\nabla v|_{L^2(\Omega)}^2 \geq |\operatorname{rot} v|_{L^2(\Omega)}^2 + |\operatorname{div} v|_{L^2(\Omega)}^2.$$

If  $\Omega$  is even a polyhedron, equality holds, that is,

$$|\nabla v|_{L^2(\Omega)}^2 = |\operatorname{rot} v|_{L^2(\Omega)}^2 + |\operatorname{div} v|_{L^2(\Omega)}^2. \tag{9}$$

*Proof*

By continuity, it is sufficient to consider  $v \in \mathring{C}_{t,n}^\infty(\Omega)$  instead of  $v \in \mathring{H}_{t,n}^1(\Omega)$ . Using Proposition 4 together with (6), we have the following:

$$|\operatorname{div} v|_{L^2(\Omega)}^2 + |\operatorname{rot} v|_{L^2(\Omega)}^2 = |\nabla v|_{L^2(\Omega)}^2 + \int_{\Gamma_1} (\operatorname{div} v |v_n|^2 + ((\nabla v) v_t) \cdot v_t). \tag{10}$$

Due to the positive resp. negative semi-definiteness of the second fundamental form, the surface integral is non-negative resp. non-positive resp. vanishes.  $\square$

*Remark 7*

For  $N = 3$ , formula (9) has already been proved in [4, Theorem 4.1].

*Remark 8*

By defining the Sobolev spaces with boundary conditions as closures of suitable test vector fields, we avoid discussions about density or approximation arguments and properties. We note that we do not claim the following:

$$\mathring{H}_{t,n}^1(\Omega) = \{v \in H^1(\Omega) : v \times v|_{\Gamma_t} = 0 \wedge v \cdot v|_{\Gamma_n} = 0\},$$

although this equality seems to be reasonable. On the other hand, it is known that at least for polyhedra or curved polyhedra<sup>††</sup> in  $\mathbb{R}^2$  or  $\mathbb{R}^3$  and either full tangential or full normal boundary condition,

$$\mathring{H}_t^1(\Omega) = \{v \in H^1(\Omega) : v \times v|_\Gamma = 0\}, \quad \mathring{H}_n^1(\Omega) = \{v \in H^1(\Omega) : v \cdot v|_\Gamma = 0\}$$

hold; see [5, Theorem 2.1, Lemma 2.6 ( $J = 1$ )] and for the curved case [6, Theorem 2.3] and the corresponding proofs. To the best of the authors knowledge, there are no proofs (yet) for general Lipschitz domains or mixed boundary conditions showing these density properties. We also want to point out that Proposition 4 and formula (10) (and Remark 5) for the special case of  $\Omega \subset \mathbb{R}^3$  have been used, for example, in [6, Lemma 2.1, Lemma 2.2] or [7, Lemma 2.11] as well.

### 3. Results

*Theorem 9* (Korn's first inequality: tangential/normal version)

Let  $\Omega \subset \mathbb{R}^N$  be piecewise  $C^2$ -concave and  $v \in \mathring{H}_{t,n}^1(\Omega)$ . Then Korn's first inequality

$$|\nabla v|_{L^2(\Omega)} \leq \sqrt{2} |\operatorname{dev} \operatorname{sym} \nabla v|_{L^2(\Omega)}$$

holds. If  $\Omega$  is a polyhedron, even

$$|\nabla v|_{L^2(\Omega)}^2 = 2 |\operatorname{dev} \operatorname{sym} \nabla v|_{L^2(\Omega)}^2 + \frac{2-N}{N} |\operatorname{div} v|_{L^2(\Omega)}^2 \leq 2 |\operatorname{dev} \operatorname{sym} \nabla v|_{L^2(\Omega)}^2$$

is true and equality holds if and only if  $\operatorname{div} v = 0$  or  $N = 2$ .

*Proof*

We use (5) in combination with Corollary 6 to see

$$|\nabla v|_{L^2(\Omega)}^2 \leq |\operatorname{dev} \operatorname{sym} \nabla v|_{L^2(\Omega)}^2 + \frac{1}{2} |\nabla v|_{L^2(\Omega)}^2 + \frac{2-N}{2N} |\operatorname{div} v|_{L^2(\Omega)}^2,$$

which shows the first estimate. If  $\Omega$  is a polyhedron, we see by Corollary 6 that equality holds in the latter estimate, which proves the other assertions.  $\square$

<sup>††</sup>In our notation, a so-called curved polyhedron has got a piecewise  $C^\infty$ -boundary.

*Remark 10* (Unbounded domains)

All our results remain true for slightly weaker Sobolev spaces. In exterior domains, that is, domains with compact complement, it is common to work in weighted Sobolev spaces like

$$\begin{aligned} H_{-1}^1(\Omega) &:= \{u \in L_{-1}^2(\Omega) : \nabla u \in L^2(\Omega)\}, \\ L_{-1}^2(\Omega) &:= \{u \in L_{\text{loc}}^2(\Omega) : \rho^{-1}u \in L^2(\Omega)\}, \quad \rho := (1 + r^2)^{\frac{1}{2}}, \quad r(x) := |x|. \end{aligned}$$

If  $N = 2$ , we have to replace  $L_{-1}^2(\Omega)$  and  $H_{-1}^1(\Omega)$  by  $L_{-1,\ln}^2(\Omega)$  and  $H_{-1,\ln}^1(\Omega)$ , respectively, where  $u$  belongs to  $L_{-1,\ln}^2(\Omega)$  if  $(\ln(e+r)\rho)^{-1}u \in L^2(\Omega)$ . The Sobolev spaces generalizing the different boundary conditions are defined as before as closures in  $H_{-1}^1(\Omega)$  resp.  $H_{-1,\ln}^1(\Omega)$  of respective test functions. For bounded domains, these weighted Sobolev spaces coincide with the standard ones equipped with equivalent scalar products. The reason for working in weighted Sobolev spaces is that the standard Poincaré inequalities do not hold in exterior domains. As a proper replacement, we have weighted Poincaré inequalities, that is, for  $N \geq 3$ ,

$$\forall u \in H_{-1}^1(\Omega) \quad |u|_{L_{-1}^2(\Omega)} \leq c_p |\nabla u|_{L^2(\Omega)},$$

and we note that the best Poincaré constant for  $H_{-1}^1(\Omega)$  satisfies  $c_p \leq 2/(N - 2)$ ; see, for example, [8, Poincaré's estimate III, p. 57], [9, Lemma 4.1], or [10, Appendix A.2]. Because our arguments only involve derivatives, it is clear that all our results, mainly Theorem 9 but also the preceding lemmas and corollary, extend easily to the family of Sobolev spaces in  $H_{-1}^1(\Omega)$  resp.  $H_{-1,\ln}^1(\Omega)$ .

From the latter remark, the following clearly holds true.

*Corollary 11* (Korn's first inequality: weighted tangential/normal version)

Theorem 9 extends to all  $v$  in  $\mathring{H}_{-1,t,n}^1(\Omega)$  resp.  $\mathring{H}_{-1,\ln,t,n}^1(\Omega)$  if  $N = 2$ .

*Corollary 12*

Corollary 6 extends to all  $v$  in  $\mathring{H}_{-1,t,n}^1(\Omega)$  resp.  $\mathring{H}_{-1,\ln,t,n}^1(\Omega)$  if  $N = 2$ . Also, Lemma 1 extends to all  $v$  in  $\mathring{H}_{-1}^1(\Omega)$  resp.  $\mathring{H}_{-1,\ln}^1(\Omega)$  if  $N = 2$ .

#### 4. Some remarks on the constants $c_k$ , $c_{m,n}$ , and $c_g$

In this section, we want to discuss in detail some constants and inequalities used in [1]. In [1], Korn's first inequality with normal boundary condition, that is, (1), is proved for a bounded  $C^1$ -domain  $\Omega \subset \mathbb{R}^N$ , which is not axisymmetric. As already mentioned in Section 1, an upper bound for the first Korn constant is presented by (2), that is,

$$c_{k,n}^2 \leq 2N(1 + c_{m,n}^2)(1 + c_k^2)(1 + c_g^{-1}),$$

which we repeat here for the convenience of the reader. All these constants depend on  $\Omega$ , and we always assume to deal with best possible ones.

##### 4.1. Korn constant without boundary condition $c_k$

This constant belongs to the standard first Korn inequality without boundary conditions, that is,

$$\exists c_k > 0 \quad \forall v \in H^1(\Omega) \quad \exists r_v \in \mathcal{R} \quad |\nabla(v - r_v)|_{L^2(\Omega)} \leq c_k |\text{sym} \nabla v|_{L^2(\Omega)}, \quad (11)$$

where  $\mathcal{R}$  is the finite dimensional space of rigid motions and  $r_v$  the  $L^2(\Omega)$ -orthonormal projection onto  $\mathcal{R}$ . Especially, (11) holds for any bounded Lipschitz domain  $\Omega \subset \mathbb{R}^N$ .

##### 4.2. Normal Gaffney constant $c_{m,n}$

Whereas the literature on  $c_k$  is well known, it seems that the knowledge on the normal Gaffney constant  $c_{m,n}$  is more restricted to the community dealing with Maxwell's equations as it is explicitly noted in [1]. For this reason, we examine it here in more detail. In [1], this constant appears in a special Gaffney inequality for tangential vector fields in  $H^1(\Omega)$ , that is, there exists  $c_{m,n} > 0$ , such that for all  $v \in \mathring{H}_n^1(\Omega)$ , there exists  $n_v \in V_{n,0}(\Omega)$  with

$$|\nabla(v - n_v)|_{L^2(\Omega)} \leq c_{m,n} \left( |\text{skw} \nabla v|_{L^2(\Omega)}^2 + |\text{tr} \nabla v|_{L^2(\Omega)}^2 \right)^{\frac{1}{2}}$$

or equivalently (with slightly different  $c_{m,n}$ )

$$|\nabla(v - n_v)|_{L^2(\Omega)} \leq c_{m,n} \left( |\text{rot} v|_{L^2(\Omega)}^2 + |\text{div} v|_{L^2(\Omega)}^2 \right)^{\frac{1}{2}}, \quad (12)$$

where

$$V_{n,0}(\Omega) := \{v \in \mathring{H}_n^1(\Omega) : \text{div} v = 0 \wedge \text{rot} v = 0\}$$

is the finite dimensional<sup>‡‡</sup> subspace of  $H^1(\Omega)$ -Neumann fields and  $n_\nu$  the  $L^2(\Omega)$ -orthonormal projection onto  $V_{n,0}(\Omega)$ . Inequality (12) can be derived by a Maxwell regularity result; see, for example, [11, 12] stating the following: Let

$$X_n(\Omega) := \{v \in L^2(\Omega) : \operatorname{rot} v \in L^2(\Omega) \wedge \operatorname{div} v \in L^2(\Omega) \wedge v \cdot \nu|_\Gamma = 0\},$$

where the vanishing normal trace has to be understood in the weak sense<sup>§§</sup>. If  $\Omega \subset \mathbb{R}^N$  is a bounded domain and either  $C^2$  or convex, then any vector field  $v$  in  $X_n(\Omega)$  already belongs to  $H^1(\Omega)$ , that is,  $v \in \mathring{H}_n^1(\Omega)$ . Because  $X_n(\Omega)$  together with the norm

$$|v|_{X_n(\Omega)}^2 := |v|_{L^2(\Omega)}^2 + |\operatorname{rot} v|_{L^2(\Omega)}^2 + |\operatorname{div} v|_{L^2(\Omega)}^2$$

is a Hilbert space, we can apply the closed graph theorem to the identity mapping  $X_n(\Omega)$  to  $H^1(\Omega)$ . Therefore, there exists  $c_{m,n,\operatorname{reg}} > 0$ , such that for all  $v \in X_n(\Omega) \cap H^1(\Omega) = \mathring{H}_n^1(\Omega)$ ,

$$|\nabla v|_{L^2(\Omega)} \leq c_{m,n,\operatorname{reg}} \left( |v|_{L^2(\Omega)}^2 + |\operatorname{rot} v|_{L^2(\Omega)}^2 + |\operatorname{div} v|_{L^2(\Omega)}^2 \right)^{\frac{1}{2}} \quad (13)$$

holds. Because the embedding of  $X_n(\Omega)$  into  $L^2(\Omega)$  is compact even for bounded Lipschitz (or weaker) domains  $\Omega$ , see [13–17], we also have the so-called normal Maxwell estimate, that is, there exists  $c_{m,n,\operatorname{est}} > 0$ , such that for all  $v \in X_n(\Omega)$ , there exists  $n_\nu \in X_{n,0}(\Omega)$  with

$$|v - n_\nu|_{L^2(\Omega)} \leq c_{m,n,\operatorname{est}} \left( |\operatorname{rot} v|_{L^2(\Omega)}^2 + |\operatorname{div} v|_{L^2(\Omega)}^2 \right)^{\frac{1}{2}}, \quad (14)$$

where

$$X_{n,0}(\Omega) := \{v \in X_n(\Omega) : \operatorname{div} v = 0 \wedge \operatorname{rot} v = 0\}$$

is the finite dimensional<sup>¶¶</sup> subspace of Neumann fields and  $n_\nu$  the  $L^2(\Omega)$ -orthonormal projection onto  $X_{n,0}(\Omega)$ . Now, (12) follows immediately by combining (13) and (14) if  $\Omega$  is bounded and either  $C^2$  or convex with

$$c_{m,n} \leq c_{m,n,\operatorname{reg}} \sqrt{c_{m,n,\operatorname{est}}^2 + 1},$$

because in this case,  $X_n(\Omega) = \mathring{H}_n^1(\Omega)$  and  $X_{n,0}(\Omega) = V_{n,0}(\Omega)$ . In the bounded and convex case, there are even no Neumann fields, that is,  $X_{n,0}(\Omega) = \{0\}$ , and  $c_{m,n} \leq 1$  holds; see, for example, [4, 7, 18–21] for the cases  $N = 2$  or  $N = 3$ , which follows essentially by Corollary 6 and uniform approximation of a convex domain  $\Omega$  by a sequence of smooth and convex domains. The Neumann fields generally vanish if and only if  $\Omega$  is simply connected. On the other hand, we note that (12) also holds in some non-smooth and non-convex situations as well. For example, by Corollary 15, later, we see that (12) is valid if  $\Omega$  is bounded and piecewise  $C^2$ . Especially, for piecewise  $C^2$ -convex domains, we have  $c_{m,n} \leq 1$  by Corollary 6. For polyhedra, it even holds  $c_{m,n} = 1$ . We note that in the latter piecewise  $C^2$ -convex case, we can choose  $n_\nu = 0$  even if  $\Omega$  is not simply connected, that is, even if Neumann fields exist in  $X_n(\Omega)$ . These possible Neumann fields must vanish by Corollary 6 as soon as they belong to  $\mathring{H}_n^1(\Omega)$ . Therefore, there are domains, for example, a polyhedron with a reentrant edge, where  $\mathring{H}_n^1(\Omega)$  is a closed subspace of  $X_n(\Omega)$  in the  $X_n(\Omega)$ -topology, but neither  $X_n(\Omega) \not\subset \mathring{H}_n^1(\Omega)$  nor  $\mathring{H}_n^1(\Omega)$  is dense in  $X_n(\Omega)$ . To the best knowledge of the authors, it is unknown, whether or not (12) holds for general bounded Lipschitz domains or even for general bounded  $C^1$ -domains.

### 4.3. Grad's number $c_g$

Let  $\Omega \subset \mathbb{R}^N$  be a bounded Lipschitz domain. From the introduction, we recall Grad's number

$$c_g = \frac{1}{|\Omega|} \inf_{|\sigma|=1} \inf_{v_\sigma \in V_{n,\sigma}(\Omega)} |\operatorname{sym} \nabla v_\sigma|_{L^2(\Omega)}^2$$

and the finite dimensional set

$$V_{n,\sigma}(\Omega) = \{v \in \mathring{H}_n^1(\Omega) : \operatorname{div} v = 0 \wedge \operatorname{rot} v = \sigma\}, \quad \sigma \in S^{(N-1)N/2-1}.$$

We emphasize that  $V_{n,\sigma}(\Omega)$  might be empty, if  $\Omega$  is not smooth enough, because generally, a solution of

$$\operatorname{div} v_\sigma = 0, \quad \operatorname{rot} v_\sigma = \sigma \text{ in } \Omega, \quad v_\sigma \cdot \nu = 0 \text{ on } \Gamma \quad (15)$$

<sup>‡‡</sup>We remark  $V_{n,0}(\Omega) \subset X_{n,0}(\Omega)$  and that even  $X_{n,0}(\Omega)$  is finite dimensional.

<sup>§§</sup>The vanishing normal trace is realized by the closure of test vector fields  $\mathring{C}^\infty(\Omega)$  under the graph norm of  $\operatorname{div}$  viewed as an unbounded operator acting on  $L^2(\Omega)$ .

<sup>¶¶</sup>We note that by the compact embedding  $X_n(\Omega) \hookrightarrow L^2(\Omega)$ , the unit ball in  $X_{n,0}(\Omega)$  is compact.

does not belong to  $H^1(\Omega)$ . More precisely, (15) admits a solution  $v_\sigma$

$$v_\sigma \in X_{n,\sigma}(\Omega) := \{v \in X_n(\Omega) : \operatorname{div} v = 0 \wedge \operatorname{rot} v = \sigma\}$$

for any  $\sigma \in S^{(N-1)N/2-1}$  because  $\sigma$  belongs to the range of the rotation. This follows by the simple fact that  $\sigma = \operatorname{rot} \hat{\sigma} \in \operatorname{rot} H^1(\Omega)$  or equivalently  $\Sigma = -\operatorname{skw} \nabla \hat{\Sigma} \in \operatorname{skw} \nabla H^1(\Omega)$  holds with  $\hat{\Sigma}(x) := \Sigma x$ , where the skew-symmetric matrix  $\Sigma \in \mathbb{R}^{N \times N}$  corresponds to  $\sigma \in \mathbb{R}^{(N-1)N/2}$  and the vector field  $\hat{\Sigma}$  to the vector field  $\hat{\sigma}$ . An adequate solution theory for these electro-magneto static problems can be found in [22–26]. In fact,  $v_\sigma = \pi \hat{\sigma}$  is the Helmholtz projection  $\pi$  of  $\hat{\sigma}$  onto solenoidal vector fields with homogeneous normal boundary condition. Generally,  $v_\sigma \notin V_{n,\sigma}(\Omega)$ , and thus,  $V_{n,\sigma}(\Omega) = \emptyset$ , that is,  $c_g = +\infty$ , is possible even for  $C^1$ -domains<sup>|||</sup>. On the other hand, if  $\Omega$  is  $C^2$  or convex, the aforementioned regularity theory for Maxwell's equations shows  $v_\sigma \in V_{n,\sigma}(\Omega)$ . Moreover, if  $\Omega$  is convex or simply connected and  $C^2$ , there are even no Neumann fields, which implies in these cases the uniqueness of the solution  $v_\sigma$ , and we simply have the following:

$$c_g = \frac{1}{|\Omega|} \inf_{|\sigma|=1} |\operatorname{sym} \nabla v_\sigma|_{L^2(\Omega)}^2.$$

As announced in Section 1, we now show that  $c_g(\Omega_p) = \frac{1}{2}$  holds for any bounded and convex polyhedron  $\Omega_p \subset \mathbb{R}^N$ . For every  $\sigma \in S^{(N-1)N/2-1}$ , problem (15) has a unique solution  $v_\sigma \in \mathring{H}_n^1(\Omega_p)$ , that is,  $v_\sigma \in V_{n,\sigma}(\Omega_p)$ , (by regularity for static Maxwell's equations in convex domains, see, for example, [18, Theorem 3.1] or [7, Theorem 2.17] for the case  $N = 3$ ) with

$$|\operatorname{rot} v_\sigma|_{L^2(\Omega_p)}^2 = |\sigma|_{L^2(\Omega_p)}^2 = |\Omega_p|.$$

On the other hand, by Corollary 6 and Theorem 9, we also have the following:

$$|\operatorname{rot} v_\sigma|_{L^2(\Omega_p)}^2 = |\nabla v_\sigma|_{L^2(\Omega_p)}^2 = 2|\operatorname{dev} \operatorname{sym} \nabla v_\sigma|_{L^2(\Omega_p)}^2 = 2|\operatorname{sym} \nabla v_\sigma|_{L^2(\Omega_p)}^2,$$

and hence,

$$c_g = \frac{1}{|\Omega_p|} \inf_{|\sigma|=1} |\operatorname{sym} \nabla v_\sigma|_{L^2(\Omega_p)}^2 = \frac{1}{2}.$$

## 5. Some more estimates on the gradient

In this section, we shall combine some more pointwise formulas and estimates on matrices and Jacobians with the integration formula from Proposition 4 in order to get some more equalities and estimates on the norm of gradients.

### 5.1. Matrices

Let us note a few simple and well-known facts about matrices and Jacobians extending the formulas presented in Section 2. The pointwise orthogonal sums

$$A = \operatorname{dev} A \oplus \operatorname{id}_A, \quad A = \operatorname{sym} A \oplus \operatorname{skw} A, \quad \operatorname{sym} A = \operatorname{dev} \operatorname{sym} A \oplus \operatorname{id}_A$$

translate to the pointwise equations

$$|A|^2 = |\operatorname{dev} A|^2 + \frac{1}{N} |\operatorname{tr} A|^2, \quad |A|^2 = |\operatorname{sym} A|^2 + |\operatorname{skw} A|^2, \quad |\operatorname{sym} A|^2 = |\operatorname{dev} \operatorname{sym} A|^2 + \frac{1}{N} |\operatorname{tr} A|^2$$

and the pointwise estimates

$$|\operatorname{dev} A|, \frac{1}{\sqrt{N}} |\operatorname{tr} A|, |\operatorname{sym} A|, |\operatorname{skw} A| \leq |A|.$$

For  $A = \nabla v$  with  $v \in H^1(\Omega)$ , we see pointwise a.e.

$$|\operatorname{skw} \nabla v|^2 = \frac{1}{2} |\operatorname{rot} v|^2, \quad \operatorname{tr} \nabla v = \operatorname{div} v \tag{16}$$

and

$$\begin{aligned} |\nabla v|^2 &= |\operatorname{sym} \nabla v|^2 + |\operatorname{skw} \nabla v|^2 = |\operatorname{sym} \nabla v|^2 + \frac{1}{2} |\operatorname{rot} v|^2, \\ |\nabla v|^2 &= |\operatorname{dev} \operatorname{sym} \nabla v|^2 + \frac{1}{N} |\operatorname{div} v|^2 + |\operatorname{skw} \nabla v|^2 = |\operatorname{dev} \operatorname{sym} \nabla v|^2 + \frac{1}{N} |\operatorname{div} v|^2 + \frac{1}{2} |\operatorname{rot} v|^2. \end{aligned}$$

<sup>|||</sup>In [1, Lemma 4],  $v_\sigma$  is found by solving the Neumann problem,  $\Delta \varphi = 0$  in  $\Omega$ ,  $\nabla \varphi \cdot \nu = -\hat{\Sigma} \nu$  on  $\Gamma$ , and setting  $v_\sigma = \nabla \varphi + \hat{\Sigma}$ . But in order to guarantee  $v_\sigma \in H^1(\Omega)$ , one needs to have  $\varphi \in H^2(\Omega)$ , which itself is only ensured if  $\Omega$  is  $C^2$  or convex. Moreover, as pointed out previously, it seems to be unclear whether [1, (10), (13)], that is, (12), hold for general  $C^1$ -domains.



Especially, we see

$$|\operatorname{div} v|^2 + |\operatorname{rot} v|^2 \leq N|\nabla v|^2.$$

Moreover, by

$$2|\operatorname{sym} \nabla v|^2 = \frac{1}{2} |\nabla v \pm (\nabla v)^T|^2 = |\nabla v|^2 \pm \langle \nabla v, (\nabla v)^T \rangle,$$

we get

$$|\nabla v|^2 = 2|\operatorname{sym} \nabla v|^2 - \langle \nabla v, (\nabla v)^T \rangle, \tag{17}$$

$$|\nabla v|^2 = 2|\operatorname{skw} \nabla v|^2 + \langle \nabla v, (\nabla v)^T \rangle = |\operatorname{rot} v|^2 + \langle \nabla v, (\nabla v)^T \rangle. \tag{18}$$

### 5.2. Integration by parts

Defining

$$I_b := \int_{\Gamma_1} (v_n \operatorname{div}_{\Gamma} v_t - v_t \cdot \nabla_{\Gamma} v_n), \quad I_c := \int_{\Gamma_1} (\operatorname{div} v |v_n|^2 + ((\nabla v) v_t) \cdot v_t),$$

Proposition 4 as well as (17), (18) and (16) show the following:

*Lemma 13* (Integration by parts)

Let  $\Omega \subset \mathbb{R}^N$  be piecewise  $C^2$ . Then for all  $v \in \mathring{C}^\infty(\overline{\Omega})$ ,

$$\begin{aligned} |\nabla v|_{L^2(\Omega)}^2 - I_b - I_c &= 2|\operatorname{sym} \nabla v|_{L^2(\Omega)}^2 - |\operatorname{div} v|_{L^2(\Omega)}^2 = 2|\operatorname{dev} \operatorname{sym} \nabla v|_{L^2(\Omega)}^2 + \frac{2-N}{N} |\operatorname{div} v|_{L^2(\Omega)}^2, \\ |\nabla v|_{L^2(\Omega)}^2 + I_b + I_c &= 2|\operatorname{skw} \nabla v|_{L^2(\Omega)}^2 + |\operatorname{div} v|_{L^2(\Omega)}^2 = |\operatorname{rot} v|_{L^2(\Omega)}^2 + |\operatorname{div} v|_{L^2(\Omega)}^2, \end{aligned}$$

which extend by continuity to all  $v \in H^2(\Omega)$ . For  $v \in \mathring{C}_{t,n}^\infty(\Omega)$ , the integral  $I_b$  containing the boundary differential operators vanishes, and the formulas (without  $I_b$ ) extend by continuity to all  $v \in \mathring{H}_{t,n}^1(\Omega)$ .

For  $N = 2, 3$  these results have already been presented in [6, Lemma 2.2, Theorem 2.3, Remark 2.4].

### 5.3. Gradient estimates

By Lemma 13, we get for all  $v \in \mathring{H}_{t,n}^1(\Omega)$

$$I_c = \begin{cases} |\nabla v|_{L^2(\Omega)}^2 - 2|\operatorname{dev} \operatorname{sym} \nabla v|_{L^2(\Omega)}^2 - \frac{2-N}{N} |\operatorname{div} v|_{L^2(\Omega)}^2, & I_c \leq c \int_{\Gamma_1} |v|^2, \\ |\operatorname{rot} v|_{L^2(\Omega)}^2 + |\operatorname{div} v|_{L^2(\Omega)}^2 - |\nabla v|_{L^2(\Omega)}^2 \end{cases}$$

where  $c > 0$  just depends on the derivatives of  $v$  and  $\Gamma_1$ . In combination with [3, Theorem 1.5.1.10], we obtain the following:

*Corollary 14*

Let  $\Omega \subset \mathbb{R}^N$  be piecewise  $C^2$ . Then there exists  $c > 0$ , such that for all  $v \in \mathring{H}_{t,n}^1(\Omega)$  and for all  $\epsilon > 0$ ,

$$\begin{aligned} (1 - \epsilon) |\nabla v|_{L^2(\Omega)}^2 &\leq 2|\operatorname{dev} \operatorname{sym} \nabla v|_{L^2(\Omega)}^2 + \frac{2-N}{N} |\operatorname{div} v|_{L^2(\Omega)}^2 + \frac{c}{\epsilon} |v|_{L^2(\Omega)}^2, \\ (1 - \epsilon) |\nabla v|_{L^2(\Omega)}^2 &\leq |\operatorname{rot} v|_{L^2(\Omega)}^2 + |\operatorname{div} v|_{L^2(\Omega)}^2 + \frac{c}{\epsilon} |v|_{L^2(\Omega)}^2. \end{aligned}$$

The latter lemma and corollary clearly show that Korn's inequalities and the Maxwell gradient estimate (12) share the same origin. We can also get rid of the  $L^2(\Omega)$ -norm of  $v$  on the right hand sides. Let us focus on the second inequality and assume that  $\Omega$  is a bounded and piecewise  $C^2$ -domain. We introduce

$$V_{t,n,0}(\Omega) := \{v \in \mathring{H}_{t,n}^1(\Omega) : \operatorname{div} v = 0 \wedge \operatorname{rot} v = 0\},$$

which is a finite dimensional (and hence closed) subspace of  $L^2(\Omega)$  because its unit ball is compact by Corollary 14 and Rellich's selection theorem.

*Corollary 15* (Gaffney's inequality)

Let  $\Omega \subset \mathbb{R}^N$  be a bounded and piecewise  $C^2$ -domain. Then there exists  $c > 0$ , such that for all  $v \in \mathring{H}_{t,n}^1(\Omega)$ , there exists  $n_v \in V_{t,n,0}(\Omega)$  with

$$|v - n_v|_{H^1(\Omega)} \leq c (|\operatorname{rot} v|_{L^2(\Omega)} + |\operatorname{div} v|_{L^2(\Omega)}),$$

and  $n_v$  is the  $L^2(\Omega)$ -orthonormal projection of  $v$  onto  $V_{t,n,0}(\Omega)$ .

*Proof*

Because  $v - n_v \in \dot{H}_{t,n}^1(\Omega) \cap V_{t,n,0}(\Omega)^\perp$  as well as  $\operatorname{rot}(v - n_v) = \operatorname{rot} v$  and  $\operatorname{div}(v - n_v) = \operatorname{div} v$ , it is sufficient to show

$$\exists c > 0 \quad \forall v \in \dot{H}_{t,n}^1(\Omega) \cap V_{t,n,0}(\Omega)^\perp |v|_{H^1(\Omega)} \leq c (|\operatorname{rot} v|_{L^2(\Omega)} + |\operatorname{div} v|_{L^2(\Omega)}). \quad (19)$$

If (19) is wrong, there exists a sequence  $(v_n) \subset \dot{H}_{t,n}^1(\Omega) \cap V_{t,n,0}(\Omega)^\perp$  with

$$|v_n|_{H^1(\Omega)} = 1, \quad |\operatorname{rot} v_n|_{L^2(\Omega)} + |\operatorname{div} v_n|_{L^2(\Omega)} \rightarrow 0.$$

As  $(v_n)$  is bounded in  $H^1(\Omega)$ , there exists a subsequence  $(v_{\pi n})$  converging to some  $v$  in  $L^2(\Omega)$  by Rellich's selection theorem. By Corollary 14,  $(v_{\pi n})$  is a Cauchy sequence in  $H^1(\Omega)$ , and thus,

$$v_{\pi n} \rightarrow v \in \dot{H}_{t,n}^1(\Omega) \cap V_{t,n,0}(\Omega)^\perp \quad \text{in } H^1(\Omega).$$

Because  $v$  belongs to  $V_{t,n,0}(\Omega)$  as well, we have  $v = 0$  in contradiction to  $1 = |v_n|_{H^1(\Omega)} \rightarrow |v|_{H^1(\Omega)}$ , which proves (19).  $\square$

*Remark 16*

As in Remark 10 and Corollaries 11 and 12, there are also versions of Corollary 15 for the case of, for example, a piecewise  $C^2$  exterior domain using polynomially weighted Sobolev spaces.

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