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On the Maxwell constants in 3D

Dirk Pauly

Dedicated to Martin Costabel on the occasion of his 65th birthday

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Using tools from functional analysis, we show that for bounded and convex domains in three dimensions, the Maxwell constants are bounded from below and above by Friedrichs' and Poincaré's constants. Copyright © 2014 John Wiley & Sons, Ltd.

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1. Introduction and preliminaries

Throughout this paper, let us fix a bounded domain $\Omega \subset \mathbb{R}^3$ with boundary $\Gamma := \partial\Omega$, which is divided into two relatively open subsets Γ_{t} and its complement $\Gamma_n := \Gamma \setminus \overline{\Gamma_{\mathrm{t}}}$. The letters t and n should remind on homogeneous tangential and normal boundary conditions.

It is well known that the Poincaré (or Friedrichs) inequality, that is, for all $u \in \mathsf{H}^1_{\Gamma_{\mathrm{t}}}(\Omega)$,

$$|u|_{\mathsf{L}^2(\Omega)} \leq c_{\mathrm{p},\Gamma_{\mathrm{t}},\varepsilon} |\nabla u|_{\mathsf{L}^2_\varepsilon(\Omega)} \tag{1.1}$$

holds with some $c_{\mathrm{p},\Gamma_{\mathrm{t}},\varepsilon} > 0$, as long as Rellich's selection theorem is valid, that is, the embedding

$$\mathsf{H}^1(\Omega) \hookrightarrow \mathsf{L}^2(\Omega) \tag{1.2}$$

is compact. Here, $\mathsf{L}^2(\Omega)$ and $\mathsf{H}^1(\Omega)$ denote the usual Lebesgue and Sobolev (Hilbert) spaces, respectively. Moreover, $\varepsilon : \Omega \to \mathbb{R}^{3\times 3}$ denotes a symmetric and uniformly positive definite L^∞ matrix field. We introduce $\mathsf{L}^2_\varepsilon(\Omega)$ as $\mathsf{L}^2(\Omega)$ equipped with the weighted inner product $\langle \cdot, \cdot \rangle_{\mathsf{L}^2_\varepsilon(\Omega)} := \langle \varepsilon \cdot, \cdot \rangle_{\mathsf{L}^2(\Omega)}$.[‡] For $\Gamma_{\mathrm{t}} \neq \emptyset$, the Sobolev space $\mathsf{H}^1_{\Gamma_{\mathrm{t}}}(\Omega)$ is defined as the closure (taken in $\mathsf{H}^1(\Omega)$) of test functions

$$\mathsf{C}^\infty_{\Gamma_{\mathrm{t}}}(\Omega) := \left\{ \varphi|_\Omega : \varphi \in \mathsf{C}^\infty(\mathbb{R}^3), \ \mathrm{dist}(\mathrm{supp}\,\varphi, \Gamma_{\mathrm{t}}) > 0 \right\}.$$

Otherwise, we set $\mathsf{H}^1_\emptyset(\Omega) := \mathsf{H}^1(\Omega) \cap \mathbb{R}^\perp$. Let us assume that we have chosen the best constant in (1.1), this is

$$\frac{1}{c_{\mathrm{p},\Gamma_{\mathrm{t}},\varepsilon}} := \inf_{0 \neq u \in \mathsf{H}^1_{\Gamma_{\mathrm{t}}}(\Omega)} \frac{|\nabla u|_{\mathsf{L}^2_\varepsilon(\Omega)}}{|u|_{\mathsf{L}^2(\Omega)}}.$$

Analogously, it is also well known that the (let's call it) Maxwell inequality, that is, for all $E \in \mathsf{R}_{\Gamma_{\mathrm{t}}}(\Omega) \cap \varepsilon^{-1}\mathsf{D}_{\Gamma_n}(\Omega)$

$$|E - \pi_{\mathrm{DN}} E|_{\mathsf{L}^2_\varepsilon(\Omega)} \leq c_{\mathrm{m},\Gamma_{\mathrm{t}},\varepsilon} \left(|\mathrm{div}\, \varepsilon E|^2_{\mathsf{L}^2(\Omega)} + |\mathrm{rot}\, E|^2_{\mathsf{L}^2(\Omega)} \right)^{1/2}$$

or equivalently for all $E \in \mathsf{R}_{\Gamma_{\mathrm{t}}}(\Omega) \cap \varepsilon^{-1}\mathsf{D}_{\Gamma_n}(\Omega) \cap \mathcal{H}_{\mathrm{DN},\varepsilon}(\Omega)^{\perp_\varepsilon}$

$$|E|_{\mathsf{L}^2_\varepsilon(\Omega)} \leq c_{\mathrm{m},\Gamma_{\mathrm{t}},\varepsilon} \left(|\mathrm{div}\, \varepsilon E|^2_{\mathsf{L}^2(\Omega)} + |\mathrm{rot}\, E|^2_{\mathsf{L}^2(\Omega)} \right)^{1/2}, \tag{1.3}$$

Universität Duisburg-Essen, Campus Essen, Fakultät für Mathematik
** Correspondence to: Dirk Pauly, Universität Duisburg-Essen, Campus Essen, Fakultät für Mathematik.*
[†] *E-mail: dirk.pauly@uni-due.de*
[‡] *Throughout this paper, norms (resp.) scalar products will be denoted by $|\cdot|_X$ (resp.) $\langle \cdot, \cdot \rangle_X$ if X is a normed space or a space featuring a scalar product.*

holds with some $c_{m,\Gamma_t,\varepsilon} > 0$, as long as the Maxwell selection theorem or the Maxwell compactness property is given, that is, the embedding

$$R_{\Gamma_t}(\Omega) \cap \varepsilon^{-1} D_{\Gamma_n}(\Omega) \hookrightarrow L^2(\Omega) \tag{1.4}$$

is compact, see Appendix A.2.1 for details. Here, we introduce the Sobolev (Hilbert) spaces

$$R(\Omega) := \{E \in L^2(\Omega) : \operatorname{rot} E \in L^2(\Omega)\}, \quad D(\Omega) := \{E \in L^2(\Omega) : \operatorname{div} E \in L^2(\Omega)\}$$

in the distributional sense. As mentioned earlier, if $\Gamma_t \neq \emptyset$, we define as closures (taken in $R(\Omega)$ (resp.) $D(\Omega)$) of test vector fields $C^\infty_{\Gamma_t}(\Omega)$ the Sobolev spaces $R_{\Gamma_t}(\Omega)$ and $D_{\Gamma_t}(\Omega)$ (and of course the same for Γ_n). If $\Gamma_t = \emptyset$, we set $R_\emptyset(\Omega) := R(\Omega)$ and $D_\emptyset(\Omega) := D(\Omega)$. Then, for $\Gamma_t \neq \emptyset$ in $H^1_{\Gamma_t}(\Omega)$, $R_{\Gamma_t}(\Omega)$ and $D_{\Gamma_t}(\Omega)$ homogeneous scalar, tangential and normal traces at Γ_t are generalized, respectively. Moreover, we define the closed subspaces

$$R_0(\Omega) := \{E \in L^2(\Omega) : \operatorname{rot} E = 0, \quad D_0(\Omega) := E \in L^2(\Omega) : \operatorname{div} E = 0\}$$

as well as $R_{\Gamma_t,0}(\Omega) := R_{\Gamma_t}(\Omega) \cap R_0(\Omega)$ and $D_{\Gamma_t,0}(\Omega) := D_{\Gamma_t}(\Omega) \cap D_0(\Omega)$. Finally, we have the harmonic Dirichlet–Neumann fields

$$\mathcal{H}_{DN,\varepsilon} := R_{\Gamma_t,0}(\Omega) \cap \varepsilon^{-1} D_{\Gamma_n,0}(\Omega),$$

which are finite dimensional because by (1.4), the unit ball is compact in $\mathcal{H}_{DN,\varepsilon}$. The $L^2_\varepsilon(\Omega)$-orthogonal projector onto them will be denoted by $\pi_{DN} : L^2_\varepsilon(\Omega) \to \mathcal{H}_{DN,\varepsilon}$ and $\perp_\varepsilon$ means orthogonality in $L^2_\varepsilon(\Omega)$. If $\Gamma_t = \Gamma$ (resp.) $\Gamma_n = \Gamma$, we have the classical Dirichlet (resp.) Neumann fields and write $\mathcal{H}_{D,\varepsilon}$ (resp.) $\mathcal{H}_{N,\varepsilon}(\Omega)$. We also need the Neumann–Dirichlet fields $\mathcal{H}_{ND,\varepsilon} := R_{\Gamma_n,0}(\Omega) \cap \varepsilon^{-1} D_{\Gamma_t,0}(\Omega)$. In the case $\varepsilon = \mathrm{id}$, we usually omit ε in our notations. Again, we assume that also in (1.3), the best constant

$$\frac{1}{c_{m,\Gamma_t,\varepsilon}} := \inf_{0 \neq E \in R_{\Gamma_t}(\Omega) \cap \varepsilon^{-1} D_{\Gamma_n}(\Omega) \cap \mathcal{H}_{DN,\varepsilon}(\Omega)^{\perp_\varepsilon}} \frac{\left(|\operatorname{div} \varepsilon E|^2_{L^2(\Omega)} + |\operatorname{rot} E|^2_{L^2(\Omega)}\right)^{1/2}}{|E|_{L^2_\varepsilon(\Omega)}}$$

is taken.

The crucial property for (1.3) to hold is the Maxwell compactness property (1.4), which holds, for example, if Ω has a (strongly) Lipschitz continuous boundary Γ with a (strongly) Lipschitz continuous interface $\gamma := \overline{\Gamma_t} \cap \overline{\Gamma_n}$, see [1] for details. More precisely, the boundary Γ and the interface γ can be described locally as graphs of Lipschitz functions. From now on, we assume this properties of Γ and Γ_t, Γ_n as *general assumption*. Note that then, also, (1.2) and (1.1) hold. Another successful approach proving the Maxwell compactness property using a different technique from [2] has been shown in [3]. For the Maxwell compactness property in the case of full boundary conditions, we refer to [2,4–14].

With the help of the $L^2_\varepsilon(\Omega)$-orthogonal Helmholtz decomposition,

$$L^2_\varepsilon(\Omega) = \nabla H^1_{\Gamma_t}(\Omega) \oplus_\varepsilon \mathcal{H}_{DN,\varepsilon}(\Omega) \oplus_\varepsilon \varepsilon^{-1} \operatorname{rot} R_{\Gamma_n}(\Omega), \tag{1.5}$$

where

$$R_{\Gamma_t,0}(\Omega) = \nabla H^1_{\Gamma_t}(\Omega) \oplus_\varepsilon \mathcal{H}_{DN,\varepsilon}(\Omega), \quad \varepsilon^{-1} D_{\Gamma_n,0}(\Omega) = \varepsilon^{-1} \operatorname{rot} R_{\Gamma_n}(\Omega) \oplus_\varepsilon \mathcal{H}_{DN,\varepsilon}(\Omega),$$

see Appendix A.2.2 for details, we can split the estimate (1.3) into two, namely,

$$\forall\, E \in \varepsilon^{-1} D_{\Gamma_n}(\Omega) \cap \nabla H^1_{\Gamma_t}(\Omega) \qquad |E|_{L^2_\varepsilon(\Omega)} \leq c_{m,\Gamma_n,\operatorname{div},\varepsilon} |\operatorname{div} \varepsilon E|_{L^2(\Omega)}, \tag{1.6}$$

$$\forall\, E \in R_{\Gamma_t}(\Omega) \cap \varepsilon^{-1} \operatorname{rot} R_{\Gamma_n}(\Omega) \qquad |E|_{L^2_\varepsilon(\Omega)} \leq c_{m,\Gamma_t,\operatorname{rot},\varepsilon,\mathrm{id}} |\operatorname{rot} E|_{L^2(\Omega)}, \tag{1.7}$$

where we again assume to use the best constants

$$\frac{1}{c_{m,\Gamma_n,\operatorname{div},\varepsilon}} := \inf_{0 \neq E \in \varepsilon^{-1} D_{\Gamma_n}(\Omega) \cap \nabla H^1_{\Gamma_t}(\Omega)} \frac{|\operatorname{div} \varepsilon E|_{L^2(\Omega)}}{|E|_{L^2_\varepsilon(\Omega)}},$$
$$\frac{1}{c_{m,\Gamma_t,\operatorname{rot},\varepsilon,\mathrm{id}}} := \inf_{0 \neq E \in R_{\Gamma_t}(\Omega) \cap \varepsilon^{-1} \operatorname{rot} R_{\Gamma_n}(\Omega)} \frac{|\operatorname{rot} E|_{L^2(\Omega)}}{|E|_{L^2_\varepsilon(\Omega)}}.$$

By the assumptions on ε, there exist $\underline{\varepsilon}, \overline{\varepsilon} > 0$ such that for all $E \in L^2(\Omega)$,

$$\frac{1}{\underline{\varepsilon}} |E|_{L^2(\Omega)} \leq |E|_{L^2_\varepsilon(\Omega)} \leq \overline{\varepsilon} |E|_{L^2(\Omega)}.$$

We note $|E|_{L^2_\varepsilon(\Omega)} = \left|\varepsilon^{1/2}E\right|_{L^2(\Omega)}$ and $\left|\varepsilon^{1/2}E\right|_{L^2_\varepsilon(\Omega)} = |\varepsilon E|_{L^2(\Omega)}$. Thus, for all $E \in L^2(\Omega)$,

$$\frac{1}{\underline{\varepsilon}}|E|_{L^2_\varepsilon(\Omega)} \leq |\varepsilon E|_{L^2(\Omega)} \leq \overline{\varepsilon}|E|_{L^2_\varepsilon(\Omega)}.$$

The inverse ε^{-1} satisfies for all $E \in L^2(\Omega)$

$$\frac{1}{\overline{\varepsilon}}|E|_{L^2(\Omega)} \leq |E|_{L^2_{\varepsilon^{-1}}(\Omega)} \leq \underline{\varepsilon}|E|_{L^2(\Omega)}, \quad \frac{1}{\overline{\varepsilon}}|E|_{L^2_{\varepsilon^{-1}}(\Omega)} \leq \left|\varepsilon^{-1}E\right|_{L^2(\Omega)} \leq \underline{\varepsilon}|E|_{L^2_{\varepsilon^{-1}}(\Omega)},$$

which immediately follows by

$$|E|_{L^2_{\varepsilon^{-1}}(\Omega)} = \left|\varepsilon^{-1/2}E\right|_{L^2(\Omega)} \begin{cases} \leq \underline{\varepsilon}\left|\varepsilon^{-1/2}E\right|_{L^2_\varepsilon(\Omega)} = \underline{\varepsilon}|E|_\Omega \\ \geq \overline{\varepsilon}^{-1}\left|\varepsilon^{-1/2}E\right|_{L^2_\varepsilon(\Omega)} = \overline{\varepsilon}^{-1}|E|_\Omega \end{cases}.$$

For later purposes, let us also define $\hat{\varepsilon} := \max\{\underline{\varepsilon}, \overline{\varepsilon}\}$.

In this contribution, we will study these different constants $c_{p,\Gamma_t,\varepsilon}$, $c_{m,\Gamma_t,\varepsilon}$, $c_{m,\Gamma_n,\text{div},\varepsilon}$, $c_{m,\Gamma_t,\text{rot},\varepsilon,\text{id}}$ and their relations to each other. It turns out that

$$c_{p,\Gamma_t,\varepsilon} = c_{m,\Gamma_n,\text{div},\varepsilon}, \quad c_{m,\Gamma_t,\text{rot},\varepsilon,\text{id}} = c_{m,\Gamma_n,\text{rot},\text{id},\varepsilon}, \quad c_{m,\Gamma_t,\varepsilon} = \max\{c_{p,\Gamma_t,\varepsilon}, c_{m,\Gamma_t,\text{rot},\varepsilon,\text{id}}\}$$

hold, see Lemmas 3, 10, and 6. The main result of this paper states that in the special case of full boundary conditions, that is, $\Gamma_t = \Gamma$ or $\Gamma_n = \Gamma$, and for bounded and *convex* domains, we have

$$\frac{c_{p,\Gamma}}{\overline{\varepsilon}} \leq c_{m,\Gamma,\varepsilon} \leq \hat{\varepsilon}c_p, \quad \frac{c_p}{\overline{\varepsilon}} \leq c_{m,\emptyset,\varepsilon} \leq \hat{\varepsilon}c_p$$

and especially for $\varepsilon = \text{id}$,

$$\max\{c_{p,\Gamma}, c_{m,\text{rot}}\} = c_{m,\Gamma} \leq c_{m,\emptyset} = c_p,$$

see Theorem 17. Here, we introduce for the special case $\varepsilon = \text{id}$

$$c_{p,\Gamma_t} := c_{p,\Gamma_t,\text{id}}, \quad c_p := c_{p,\emptyset}, \quad c_{m,\Gamma_t} := c_{m,\Gamma_t,\text{id}},$$

and

$$c_{m,\Gamma_t,\text{rot}} := c_{m,\Gamma_t,\text{rot},\text{id},\text{id}} = c_{m,\Gamma_n,\text{rot},\text{id},\text{id}} = c_{m,\Gamma_n,\text{rot}},$$

as well as

$$c_{m,\text{rot}} := c_{m,\Gamma,\text{rot},\text{id},\text{id}} = c_{m,\emptyset,\text{rot},\text{id},\text{id}}.$$

The crucial point in our analysis is that for convex domains,

$$c_{m,\text{rot}} \leq c_p, \quad c_{m,\Gamma,\text{rot},\varepsilon,\text{id}}, c_{m,\emptyset,\text{rot},\varepsilon,\text{id}} \leq \overline{\varepsilon}c_p$$

hold, see Lemma 16. Some of these results have also been obtained recently in [15, 16] utilizing different and more elementary[§] methods. We note that in the convex case, we can estimate the Poincaré constant c_p by the diameter of Ω. More precisely, by the famous paper of Payne and Weinberger [17],[¶] we have

$$c_p \leq \frac{\text{diam}(\Omega)}{\pi}.$$

In [17] also, the optimality of this estimate has been shown. Furthermore, $c_{p,\Gamma} < c_p$ is well known even for non-convex domains, see, for example, [19] and the cited literature, yielding

$$\frac{1}{\sqrt{\lambda_1}} = c_{p,\Gamma} < c_p = \frac{1}{\sqrt{\mu_2}} \leq \frac{\text{diam}(\Omega)}{\pi}, \tag{1.8}$$

where λ_1 (resp.) μ_2 is the first Dirichlet (resp.) second Neumann eigenvalue of the negative Laplacian.

At least some of our results extend in a natural way to bounded domains $\Omega \subset \mathbb{R}^N$ or even to Riemannian manifolds with compact closure, see Remark 5 and Appendix A.1.

Our new estimates have important applications, for example, to numerical analysis, where especially an upper bound for the Maxwell constants is needed, for example, for preconditioning and for functional a posteriori error estimates in the framework of Maxwell's equations.

§ *In the sense that no tools from functional analysis were used.*
¶ *A little mistake or inconsistency in* [17] *has been corrected later in* [18].

2. An abstract setting

Let X and Y be Hilbert spaces and

$$\mathrm{A} : D(\mathrm{A}) \subset \mathrm{X} \to \mathrm{Y}, \quad \mathrm{A}^* : D(\mathrm{A}^*) \subset \mathrm{Y} \to \mathrm{X}$$

be a closed and densely defined linear operator and its adjoint. Here, D denotes the domain of definition and we introduce the kernel N and the range R. Because A is closed, we have $(\mathrm{A}^*)^* = \bar{\mathrm{A}} = \mathrm{A}$, and sometimes, $(\mathrm{A}, \mathrm{A}^*)$ is called a dual pair. The projection theorem yields the orthogonal 'Helmholtz' decompositions

$$\mathrm{X} = N(\mathrm{A}) \oplus \overline{R(\mathrm{A}^*)}, \quad \mathrm{Y} = N(\mathrm{A}^*) \oplus \overline{R(\mathrm{A})}. \tag{2.1}$$

We collect some standard results from functional analysis, see, for example, [8, 20].

$\mathrm{A}^*\mathrm{A}$ and $\mathrm{A}\mathrm{A}^*$ are non-negative and self-adjoint and their spectra coincide if we exclude $\{0\}$, that is,

$$\sigma(\mathrm{A}^*\mathrm{A}) \setminus \{0\} = \sigma(\mathrm{A}\mathrm{A}^*) \setminus \{0\}, \quad \sigma_{\mathrm{p}}(\mathrm{A}^*\mathrm{A}) \setminus \{0\} = \sigma_{\mathrm{p}}(\mathrm{A}\mathrm{A}^*) \setminus \{0\}. \tag{2.2}$$

Let us assume that the embedding (using the graph-norm)

$$D(\mathrm{A}) \cap \overline{R(\mathrm{A}^*)} \hookrightarrow \mathrm{X} \tag{2.3}$$

is compact.

Lemma 1
There exist $c_{\mathrm{A}}, c_{\mathrm{A}^*} > 0$, such that

$$\forall\, x \in D(\mathrm{A}) \cap R(\mathrm{A}^*) \qquad |x|_{\mathrm{X}} \leq c_{\mathrm{A}} |\mathrm{A}x|_{\mathrm{Y}},$$
$$\forall\, y \in D(\mathrm{A}^*) \cap R(\mathrm{A}) \qquad |y|_{\mathrm{Y}} \leq c_{\mathrm{A}^*} |\mathrm{A}^* y|_{\mathrm{X}}.$$

Moreover, $R(\mathrm{A})$ and $R(\mathrm{A}^*)$ are closed and

$$\mathrm{X} = N(\mathrm{A}) \oplus R(\mathrm{A}^*), \quad \mathrm{Y} = N(\mathrm{A}^*) \oplus R(\mathrm{A}).$$

Furthermore, $D(\mathrm{A}^*) \cap R(\mathrm{A}) \hookrightarrow \mathrm{Y}$ is compact as well.

We note that the same lemma can be proved assuming the compactness of the embedding of $D(\mathrm{A}^*) \cap \overline{R(\mathrm{A})} \hookrightarrow \mathrm{Y}$ instead of (2.3). By Lemma 1, the restricted operator

$$\mathcal{A} := \mathrm{A}|_{D(\mathcal{A})} : D(\mathcal{A}) \subset R(\mathrm{A}^*) \to R(\mathrm{A}), \quad D(\mathcal{A}) := D(\mathrm{A}) \cap R(\mathrm{A}^*)$$

has a bounded inverse $\mathcal{A}^{-1} : R(\mathrm{A}) \to D(\mathcal{A})$ with $\left|\mathcal{A}^{-1}\right| \leq \left(1 + c_{\mathrm{A}}^2\right)^{1/2}$, which is compact as an operator from $R(\mathrm{A})$ to $R(\mathrm{A}^*)$. Hence, $\mathrm{A}^*\mathrm{A}$ and $\mathrm{A}\mathrm{A}^*$ have pure point spectra, which can only accumulate at infinity and which coincide by (2.2). Especially, the first positive eigenvalues are equal, and therefore, we conclude the following.

Theorem 2
For the best constants in Lemma 1, it holds $c_{\mathrm{A}} = c_{\mathrm{A}^*}$, this is

$$\frac{1}{c_{\mathrm{A}}} = \min_{0 \neq x \in D(\mathrm{A}) \cap R(\mathrm{A}^*)} \frac{|\mathrm{A}x|_{\mathrm{Y}}}{|x|_{\mathrm{X}}} = \min_{0 \neq y \in D(\mathrm{A}^*) \cap R(\mathrm{A})} \frac{|\mathrm{A}^* y|_{\mathrm{X}}}{|y|_{\mathrm{Y}}} = \frac{1}{c_{\mathrm{A}^*}}.$$

Hence, $c_{\mathrm{A}}^{-2} = c_{\mathrm{A}^*}^{-2}$ is the first positive eigenvalue of $\mathrm{A}^*\mathrm{A}$ as well as of $\mathrm{A}\mathrm{A}^*$.

3. The Maxwell estimates

We remind on Ω and its properties from the introduction.

3.1. *General Lipschitz domains*

In this subsection, we frequently use Lemma 1 and Theorem 2.

3.1.1. Gradient and divergence. Let us consider A as

$$\nabla : \mathrm{H}^1_{\Gamma_\mathrm{t}}(\Omega) \subset \mathrm{L}^2(\Omega) \to \mathrm{L}^2_\varepsilon(\Omega).$$

Then A^* is equal to

$$-\operatorname{div} \varepsilon : \varepsilon^{-1} \mathrm{D}_{\Gamma_\mathrm{n}}(\Omega) \subset \mathrm{L}^2_\varepsilon(\Omega) \to \mathrm{L}^2(\Omega).$$

More precisely, we have the following table:

A	$D(\mathrm{A})$	X	Y	$N(\mathrm{A})$	$R(\mathrm{A})$
∇	$\mathrm{H}^1_{\Gamma_t}(\Omega)$	$\mathrm{L}^2(\Omega)$	$\mathrm{L}^2_\varepsilon(\Omega)$	$\{0\}$	$\nabla\mathrm{H}^1_{\Gamma_t}(\Omega) = \mathrm{R}_{\Gamma_t,0}(\Omega) \cap \mathcal{H}^{\perp}_{\mathrm{DN}}$
A^*	$D(\mathrm{A}^*)$	Y	X	$N(\mathrm{A}^*)$	$R(\mathrm{A}^*)$
$-\operatorname{div}\varepsilon$	$\varepsilon^{-1}\mathrm{D}_{\Gamma_n}(\Omega)$	$\mathrm{L}^2_\varepsilon(\Omega)$	$\mathrm{L}^2(\Omega)$	$\varepsilon^{-1}\mathrm{D}_{\Gamma_n,0}(\Omega)$	$\operatorname{div}\mathrm{D}_{\Gamma_n}(\Omega)$

We note that $\operatorname{div}\mathrm{D}_{\Gamma_n}(\Omega) = \mathrm{L}^2(\Omega)$ if $\Gamma_n \neq \Gamma$ and $\operatorname{div}\mathrm{D}_\Gamma(\Omega) = \mathrm{L}^2(\Omega) \cap \mathbb{R}^\perp$. Moreover, we emphasize that indeed, $D(\mathrm{A}^*) = \varepsilon^{-1}\mathrm{D}_{\Gamma_n}(\Omega)$ holds, see for example, [1]. Note that for this, one has to show the approximation property

$$\mathrm{D}_{\Gamma_n}(\Omega) = \left\{H \in \mathrm{D}(\Omega) : \langle \operatorname{div} H, u\rangle_{\mathrm{L}^2(\Omega)} = -\langle H, \nabla u\rangle_{\mathrm{L}^2(\Omega)}\ \forall\, u \in \mathrm{H}^1_{\Gamma_t}(\Omega)\right\},$$

which is not trivial at all for mixed boundary conditions. Only in the special cases of full boundary conditions this is clear. In fact, by definition $D(\mathrm{A}^*) = \varepsilon^{-1}\mathrm{D}(\Omega)$ holds for $\Gamma_t = \Gamma$ by definition. For $\Gamma_t = \emptyset$, we see that the closed operator

$$\mathrm{B} := -\operatorname{div} : \mathrm{D}_\Gamma(\Omega) \subset \mathrm{L}^2(\Omega) \to \mathrm{L}^2(\Omega)$$

has the adjoint

$$\mathrm{B}^* = \nabla : \mathrm{H}^1(\Omega) \subset \mathrm{L}^2(\Omega) \to \mathrm{L}^2(\Omega)$$

by definition. Because in this case $\mathrm{A} = \mathrm{B}^*$, we have $D(\mathrm{A}^*) = D(\mathrm{B}^{**}) = D(\mathrm{B}) = \mathrm{D}_\Gamma(\Omega)$. The crucial compact embedding (2.3) reads

$$\mathrm{H}^1_{\Gamma_t}(\Omega) \cap \overline{\operatorname{div}\mathrm{D}_{\Gamma_n}(\Omega)} \hookrightarrow \mathrm{L}^2(\Omega)$$

and is just Rellich's selection theorem because

$$\mathrm{H}^1_{\Gamma_t}(\Omega) \cap \overline{\operatorname{div}\mathrm{D}_{\Gamma_n}(\Omega)} \subset \mathrm{H}^1_{\Gamma_t}(\Omega) \subset \mathrm{H}^1(\Omega) \hookrightarrow \mathrm{L}^2(\Omega).$$

Theorem 2 yields

$$0 < \frac{1}{c_{\mathrm{p},\Gamma_t,\varepsilon}} = \min_{0\neq u\in\mathrm{H}^1_{\Gamma_t}(\Omega)} \frac{|\nabla u|_{\mathrm{L}^2_\varepsilon(\Omega)}}{|u|_{\mathrm{L}^2(\Omega)}} = \min_{0\neq E\in\varepsilon^{-1}\mathrm{D}_{\Gamma_n}(\Omega)\cap\nabla\mathrm{H}^1_{\Gamma_t}(\Omega)} \frac{|\operatorname{div}\varepsilon E|_{\mathrm{L}^2(\Omega)}}{|E|_{\mathrm{L}^2_\varepsilon(\Omega)}} = \frac{1}{c_{\mathrm{m},\Gamma_n,\operatorname{div},\varepsilon}}.$$

We note that $\lambda_{\Gamma_t,\varepsilon} := c^{-2}_{\mathrm{p},\Gamma_t,\varepsilon}$ is the first positive Dirichlet–Neumann eigenvalue of the weighted negative Laplacian $-\Delta_\varepsilon := -\operatorname{div}\varepsilon\nabla$. For $\varepsilon = \mathrm{id}$ and $\Gamma_t = \Gamma$ (resp.) $\Gamma_t = \emptyset$, we see that $\lambda_{\Gamma,\mathrm{id}} =: \lambda_1$ (resp.) $\lambda_{\emptyset,\mathrm{id}} =: \mu_2$ is the first Dirichlet (resp.) second Neumann eigenvalue of the negative Laplacian. As $\lambda_{\Gamma_t,\varepsilon} = c^{-2}_{\mathrm{m},\Gamma_n,\operatorname{div},\varepsilon}$ holds too, $\lambda_{\Gamma_t,\varepsilon}$ is also the first positive Neumann–Dirichlet eigenvalue of the weighted negative reduced grad-div-operator $-\nabla\operatorname{div}\varepsilon$, which can also be interpreted as the weighted negative vector Laplacian $-\vec{\Delta}_\varepsilon := -\nabla\operatorname{div}\varepsilon + \operatorname{rot}\operatorname{rot}$ on a subspace of irrotational vector fields.

Lemma 3
The Poincaré constant in $\mathrm{H}^1_{\Gamma_t}(\Omega)$ and the Maxwell divergence constant in $\varepsilon^{-1}\mathrm{D}_{\Gamma_n}(\Omega) \cap \nabla\mathrm{H}^1_{\Gamma_t}(\Omega)$, that is, the best constants in the inequalities

$$\begin{aligned} &\forall\, u \in \mathrm{H}^1_{\Gamma_t}(\Omega) && |u|_{\mathrm{L}^2(\Omega)} \leq c_{\mathrm{p},\Gamma_t,\varepsilon}|\nabla u|_{\mathrm{L}^2_\varepsilon(\Omega)},\\ &\forall\, E \in \varepsilon^{-1}\mathrm{D}_{\Gamma_n}(\Omega) \cap \nabla\mathrm{H}^1_{\Gamma_t}(\Omega) && |E|_{\mathrm{L}^2_\varepsilon(\Omega)} \leq c_{\mathrm{m},\Gamma_n,\operatorname{div},\varepsilon}|\operatorname{div}\varepsilon E|_{\mathrm{L}^2(\Omega)}, \end{aligned}$$

coincide and correspond to the first positive Dirichlet–Neumann eigenvalue of the weighted negative Laplacian $-\Delta_\varepsilon$, more precisely $c_{\mathrm{p},\Gamma_t,\varepsilon} = c_{\mathrm{m},\Gamma_n,\operatorname{div},\varepsilon} = 1/\sqrt{\lambda_{\Gamma_t,\varepsilon}}$.

Lemma 4
It holds $\overline{\varepsilon}^{-1}c_{\mathrm{p},\Gamma_t} \leq c_{\mathrm{p},\Gamma_t,\varepsilon} \leq \underline{\varepsilon}c_{\mathrm{p},\Gamma_t}$ as well as $c_{\mathrm{p},\Gamma} \leq c_{\mathrm{p},\Gamma_t}$ and $c_{\mathrm{p},\Gamma,\varepsilon} \leq c_{\mathrm{p},\Gamma_t,\varepsilon}$.

Proof
For $u \in \mathrm{H}^1_{\Gamma_t}(\Omega)$, we have

$$\begin{aligned} |u|_{\mathrm{L}^2(\Omega)} &\leq c_{\mathrm{p},\Gamma_t}|\nabla u|_{\mathrm{L}^2(\Omega)} \leq \underline{\varepsilon}c_{\mathrm{p},\Gamma_t}|\nabla u|_{\mathrm{L}^2_\varepsilon(\Omega)},\\ |u|_{\mathrm{L}^2(\Omega)} &\leq c_{\mathrm{p},\Gamma_t,\varepsilon}|\nabla u|_{\mathrm{L}^2_\varepsilon(\Omega)} \leq \overline{\varepsilon}c_{\mathrm{p},\Gamma_t,\varepsilon}|\nabla u|_{\mathrm{L}^2(\Omega)}, \end{aligned}$$

which gives $c_{\mathrm{p},\Gamma_t,\varepsilon} \leq \underline{\varepsilon}c_{\mathrm{p},\Gamma_t}$ and $c_{\mathrm{p},\Gamma_t} \leq \overline{\varepsilon}c_{\mathrm{p},\Gamma_t,\varepsilon}$. □

Remark 5
The results of this section extend to bounded domains $\Omega \subset \mathbb{R}^N$, $N \in \mathbb{N}$, having the proper regularity of the boundary.

3.1.2. Rotations. Now, let A be

$$\mu^{-1}\mathrm{rot} : \mathrm{R}_{\Gamma_t}(\Omega) \subset \mathrm{L}^2_\varepsilon(\Omega) \to \mathrm{L}^2_\mu(\Omega).$$

Then A^* is

$$\varepsilon^{-1}\mathrm{rot} : \mathrm{R}_{\Gamma_n}(\Omega) \subset \mathrm{L}^2_\mu(\Omega) \to \mathrm{L}^2_\varepsilon(\Omega),$$

where μ is another matrix field similar to ε. More precisely,

A	$D(\mathrm{A})$	X	Y	$N(\mathrm{A})$	$R(\mathrm{A})$
$\mu^{-1}\mathrm{rot}$	$\mathrm{R}_{\Gamma_t}(\Omega)$	$\mathrm{L}^2_\varepsilon(\Omega)$	$\mathrm{L}^2_\mu(\Omega)$	$\mathrm{R}_{\Gamma_t,0}(\Omega)$	$\mu^{-1}\mathrm{rot}\,\mathrm{R}_{\Gamma_t}(\Omega)$
A^*	$D(\mathrm{A}^*)$	Y	X	$N(\mathrm{A}^*)$	$R(\mathrm{A}^*)$
$\varepsilon^{-1}\mathrm{rot}$	$\mathrm{R}_{\Gamma_n}(\Omega)$	$\mathrm{L}^2_\mu(\Omega)$	$\mathrm{L}^2_\varepsilon(\Omega)$	$\mathrm{R}_{\Gamma_n,0}(\Omega)$	$\varepsilon^{-1}\mathrm{rot}\,\mathrm{R}_{\Gamma_n}(\Omega)$

We note

$$R(\mathrm{A}) = \mu^{-1}\left(\mathrm{D}_{\Gamma_t,0}(\Omega) \cap \mathcal{H}^\perp_{\mathrm{ND}}\right), \quad R(\mathrm{A}^*) = \varepsilon^{-1}\left(\mathrm{D}_{\Gamma_n,0}(\Omega) \cap \mathcal{H}^\perp_{\mathrm{DN}}\right)$$

and that indeed, $D(\mathrm{A}^*) = \mathrm{R}_{\Gamma_n}(\Omega)$ holds, see again for example, [1]. As before, for this, one has to show the approximation property

$$\mathrm{R}_{\Gamma_n}(\Omega) = \left\{H \in \mathrm{R}(\Omega) : \langle \mathrm{rot}\,H, E\rangle_{\mathrm{L}^2(\Omega)} = \langle \mathrm{rot}\,H, E\rangle_{\mathrm{L}^2(\Omega)}\ \forall\ E \in \mathrm{R}_{\Gamma_t}(\Omega)\right\},$$

which is not trivial at all for mixed boundary conditions. Again, only in the special cases of full boundary conditions this is clear. Because $D(\mathrm{A}^*) = \mathrm{R}(\Omega)$ holds for $\Gamma_t = \Gamma$ by definition, we have also $D(\mathrm{B}^*) = D(\mathrm{A}^{**}) = D(\mathrm{A}) = \mathrm{R}_\Gamma(\Omega)$ for $\mathrm{B} = \mathrm{A}^*$, which shows the result for $\Gamma_t = \emptyset$. The crucial compact embedding (2.3) reads

$$\mathrm{R}_{\Gamma_t}(\Omega) \cap \varepsilon^{-1}\overline{\mathrm{rot}\,\mathrm{R}_{\Gamma_n}(\Omega)} \hookrightarrow \mathrm{L}^2_\varepsilon(\Omega)$$

and is just the Maxwell compactness property (1.4) because

$$\mathrm{R}_{\Gamma_t}(\Omega) \cap \varepsilon^{-1}\overline{\mathrm{rot}\,\mathrm{R}_{\Gamma_n}(\Omega)} \subset \mathrm{R}_{\Gamma_t}(\Omega) \cap \varepsilon^{-1}\mathrm{D}_{\Gamma_n,0}(\Omega) \subset \mathrm{R}_{\Gamma_t}(\Omega) \cap \varepsilon^{-1}\mathrm{D}_{\Gamma_n}(\Omega) \hookrightarrow \mathrm{L}^2(\Omega) \subset \mathrm{L}^2_\varepsilon(\Omega).$$

By Theorem 2, we have

$$\begin{aligned} 0 < \frac{1}{c_{\mathrm{m},\Gamma_t,\mathrm{rot},\varepsilon,\mu}} &= \min_{0\neq E\in \mathrm{R}_{\Gamma_t}(\Omega)\cap\varepsilon^{-1}\mathrm{rot}\,\mathrm{R}_{\Gamma_n}(\Omega)} \frac{|\mu^{-1}\mathrm{rot}\,E|_{\mathrm{L}^2_\mu(\Omega)}}{|E|_{\mathrm{L}^2_\varepsilon(\Omega)}} \\ &= \min_{0\neq H\in \mathrm{R}_{\Gamma_n}(\Omega)\cap\mu^{-1}\mathrm{rot}\,\mathrm{R}_{\Gamma_t}(\Omega)} \frac{|\varepsilon^{-1}\mathrm{rot}\,H|_{\mathrm{L}^2_\varepsilon(\Omega)}}{|H|_{\mathrm{L}^2_\mu(\Omega)}} = \frac{1}{c_{\mathrm{m},\Gamma_n,\mathrm{rot},\mu,\varepsilon}}, \end{aligned}$$

which serves also as a definition for the constants $c_{\mathrm{m},\Gamma_t,\mathrm{rot},\varepsilon,\mu}$ and $c_{\mathrm{m},\Gamma_n,\mathrm{rot},\mu,\varepsilon}$. Therefore, $\kappa_{\Gamma_t,\varepsilon,\mu} := c^{-2}_{\mathrm{m},\Gamma_t,\mathrm{rot},\varepsilon,\mu}$ is the first positive Dirichlet–Neumann eigenvalue of the weighted reduced double-rot-operator $\square_{\varepsilon,\mu} := \varepsilon^{-1}\mathrm{rot}\,\mu^{-1}\mathrm{rot}$, which can also be interpreted as the weighted negative vector Laplacian $-\vec{\Delta}_{\varepsilon,\mu} := -\nabla\mathrm{div}\,\varepsilon + \varepsilon^{-1}\mathrm{rot}\,\mu^{-1}\mathrm{rot}$ on a subspace of ε-solenoidal vector fields. Because $\kappa_{\Gamma_t,\varepsilon,\mu} = c^{-2}_{\mathrm{m},\Gamma_n,\mathrm{rot},\mu,\varepsilon}$ holds as well, $\kappa_{\Gamma_t,\varepsilon,\mu}$ is also the first positive Neumann–Dirichlet eigenvalue of the weighted reduced double-rot-operator $\square_{\mu,\varepsilon} = \mu^{-1}\mathrm{rot}\,\varepsilon^{-1}\mathrm{rot}$, which can also be interpreted as the weighted negative vector Laplacian on a subspace of μ-solenoidal vector fields, that is, $-\vec{\Delta}_{\mu,\varepsilon} = -\nabla\mathrm{div}\,\mu + \mu^{-1}\mathrm{rot}\,\varepsilon^{-1}\mathrm{rot}$.

Lemma 6
The tangential-normal and normal-tangential Maxwell rotation constants, that is, the best constants in the inequalities

$$\begin{aligned} &\forall\ E \in \mathrm{R}_{\Gamma_t}(\Omega) \cap \varepsilon^{-1}\mathrm{rot}\,\mathrm{R}_{\Gamma_n}(\Omega) && |E|_{\mathrm{L}^2_\varepsilon(\Omega)} \leq c_{\mathrm{m},\Gamma_t,\mathrm{rot},\varepsilon,\mu}|\mathrm{rot}\,E|_{\mathrm{L}^2_{\mu^{-1}}(\Omega)}, \\ &\forall\ H \in \mathrm{R}_{\Gamma_n}(\Omega) \cap \mu^{-1}\mathrm{rot}\,\mathrm{R}_{\Gamma_t}(\Omega) && |H|_{\mathrm{L}^2_\mu(\Omega)} \leq c_{\mathrm{m},\Gamma_n,\mathrm{rot},\mu,\varepsilon}|\mathrm{rot}\,H|_{\mathrm{L}^2_{\varepsilon^{-1}}(\Omega)}, \end{aligned}$$

coincide and correspond to the first positive Dirichlet–Neumann eigenvalue of the weighted reduced double-rot-operator $\square_{\varepsilon,\mu}$, more precisely, $c_{\mathrm{m},\Gamma_t,\mathrm{rot},\varepsilon,\mu} = c_{\mathrm{m},\Gamma_n,\mathrm{rot},\mu,\varepsilon} = 1/\sqrt{\kappa_{\Gamma_t,\varepsilon,\mu}}$.

Let us define for $\varepsilon = \mu$ and for $\varepsilon = \mu = \mathrm{id}$

$$c_{\mathrm{m},\Gamma_t,\mathrm{rot},\varepsilon} := c_{\mathrm{m},\Gamma_t,\mathrm{rot},\varepsilon,\varepsilon} = c_{\mathrm{m},\Gamma_n,\mathrm{rot},\varepsilon,\varepsilon}$$

and note

$$c_{m,\Gamma_t,rot,\varepsilon} = c_{m,\Gamma_n,rot,\varepsilon}, \quad c_{m,\Gamma_t,rot} = c_{m,\Gamma_n,rot}. \tag{3.1}$$

Corollary 7
For all $E \in \big(R_{\Gamma_t}(\Omega) \cap \varepsilon^{-1}\text{rot}\, R_{\Gamma_n}(\Omega)\big) \cup \big(R_{\Gamma_n}(\Omega) \cap \varepsilon^{-1}\text{rot}\, R_{\Gamma_t}(\Omega)\big)$,

$$|E|_{L^2_\varepsilon(\Omega)} \le c_{m,\Gamma_t,rot,\varepsilon}|\text{rot}\, E|_{L^2_{\varepsilon^{-1}}(\Omega)} \le \underline{\varepsilon}c_{m,\Gamma_t,rot,\varepsilon}|\text{rot}\, E|_{L^2(\Omega)} \tag{3.2}$$

holds with sharp constants.

Moreover, the inequalities

$$\forall\, E \in R_{\Gamma_t}(\Omega) \cap \varepsilon^{-1}\text{rot}\, R_{\Gamma_n}(\Omega) \qquad |E|_{L^2_\varepsilon(\Omega)} \le c_{m,\Gamma_t,rot,\varepsilon,id}|\text{rot}\, E|_{L^2(\Omega)} \tag{3.3}$$

$$\forall\, H \in R_{\Gamma_n}(\Omega) \cap \varepsilon^{-1}\text{rot}\, R_{\Gamma_t}(\Omega) \qquad |H|_{L^2_\varepsilon(\Omega)} \le c_{m,\Gamma_n,rot,\varepsilon,id}|\text{rot}\, H|_{L^2(\Omega)} \tag{3.4}$$

hold, where these sharp constants do not need to coincide if $\varepsilon \neq \text{id}$.

Lemma 8
It holds

(i) $\underline{\varepsilon}^{-2}c_{m,\Gamma_t,rot} \le c_{m,\Gamma_t,rot,\varepsilon} \le \overline{\varepsilon}^2 c_{m,\Gamma_t,rot}$,

(ii) $c_{m,\Gamma_t,rot,\varepsilon,id}, c_{m,\Gamma_n,rot,\varepsilon,id} \begin{cases} \le \min\{\underline{\varepsilon}c_{m,\Gamma_t,rot,\varepsilon}, \overline{\varepsilon}c_{m,\Gamma_t,rot}\} \le \overline{\varepsilon}c_{m,\Gamma_t,rot}, \\ \ge \max\{\overline{\varepsilon}^{-1}c_{m,\Gamma_t,rot,\varepsilon}, \underline{\varepsilon}^{-1}c_{m,\Gamma_t,rot}\} \ge \underline{\varepsilon}^{-1}c_{m,\Gamma_t,rot}. \end{cases}$

Proof
It is clear that $c_{m,\Gamma_t,rot,\varepsilon,id}, c_{m,\Gamma_n,rot,\varepsilon,id} \le \underline{\varepsilon}c_{m,\Gamma_t,rot,\varepsilon}$ holds. To prove the other estimates, let $E \in R_{\Gamma_t}(\Omega) \cap \varepsilon^{-1}\text{rot}\, R_{\Gamma_n}(\Omega)$. We decompose (see Appendix A.2.2)

$$E = E_0 + E_{rot} \in R_{\Gamma_t,0}(\Omega) \oplus \text{rot}\, R_{\Gamma_n}(\Omega).$$

Then $E_{rot} \in R_{\Gamma_t}(\Omega) \cap \text{rot}\, R_{\Gamma_n}(\Omega)$ and $\text{rot}\, E = \text{rot}\, E_{rot}$. Thus by orthogonality

$$|E|^2_{L^2_\varepsilon(\Omega)} = \langle \varepsilon E, E_{rot}\rangle_{L^2(\Omega)} \le c_{m,\Gamma_t,rot} \underbrace{|\varepsilon E|_{L^2(\Omega)}}_{\le \overline{\varepsilon}|E|_{L^2_\varepsilon(\Omega)}} |\text{rot}\, E|_{L^2(\Omega)}$$

and hence

$$|E|_{L^2_\varepsilon(\Omega)} \le \overline{\varepsilon}c_{m,\Gamma_t,rot}|\text{rot}\, E|_{L^2(\Omega)} \le \overline{\varepsilon}^2 c_{m,\Gamma_t,rot}|\text{rot}\, E|_{L^2_{\varepsilon^{-1}}(\Omega)}.$$

This shows $c_{m,\Gamma_t,rot,\varepsilon,id} \le \overline{\varepsilon}c_{m,\Gamma_t,rot}$ and $c_{m,\Gamma_t,rot,\varepsilon} \le \overline{\varepsilon}^2 c_{m,\Gamma_t,rot}$. Interchanging Γ_t and Γ_n proves $c_{m,\Gamma_n,rot,\varepsilon,id} \le \overline{\varepsilon}c_{m,\Gamma_n,rot,id,id} = \overline{\varepsilon}c_{m,\Gamma_t,rot}$. By $\underline{\varepsilon}^{-1}|E|_{L^2(\Omega)} \le |E|_{L^2_\varepsilon(\Omega)}$ and (3.2) (resp.) (3.3) (resp.) (3.4) we see $c_{m,\Gamma_t,rot} \le \underline{\varepsilon}^2 c_{m,\Gamma_t,rot,\varepsilon}$ (resp.) $\underline{\varepsilon}^{-1}c_{m,\Gamma_t,rot} \le c_{m,\Gamma_t,rot,\varepsilon,id}, c_{m,\Gamma_n,rot,\varepsilon,id}$. Using $|\text{rot}\, E|_{L^2(\Omega)} \le \overline{\varepsilon}|\text{rot}\, E|_{L^2_{\varepsilon^{-1}}(\Omega)}$ and (3.3), (3.4) we obtain $\overline{\varepsilon}^{-1}c_{m,\Gamma_t,rot,\varepsilon} \le c_{m,\Gamma_t,rot,\varepsilon,id}, c_{m,\Gamma_n,rot,\varepsilon,id}$, which completes the proof. □

3.1.3. The full maxwell estimates.

Theorem 9
For all $E \in R_{\Gamma_t}(\Omega) \cap \varepsilon^{-1}D_{\Gamma_n}(\Omega)$, the tangential-normal Maxwell estimate

$$|E - \pi_{DN}E|^2_{L^2_\varepsilon(\Omega)} \le c^2_{p,\Gamma_t,\varepsilon}|\text{div}\, \varepsilon E|^2_{L^2(\Omega)} + c^2_{m,\Gamma_t,rot,\varepsilon,id}|\text{rot}\, E|^2_{L^2(\Omega)}$$

holds with sharp constants. Moreover, $c_{p,\Gamma_t,\varepsilon} \le \underline{\varepsilon}c_{p,\Gamma_t}$ and $c_{m,\Gamma_t,rot,\varepsilon,id} \le \overline{\varepsilon}c_{m,\Gamma_t,rot}$.

Here the word 'sharp' is meant with respect to the restrictions of the estimate to the subspaces $R_{\Gamma_t,0}(\Omega) \cap \varepsilon^{-1}D_{\Gamma_n}(\Omega)$ and $R_{\Gamma_t}(\Omega) \cap \varepsilon^{-1}D_{\Gamma_n,0}(\Omega)$.

Proof
By the Helmholtz decomposition (see Appendix A.2.2), we have

$$R_{\Gamma_t}(\Omega) \cap \varepsilon^{-1}D_{\Gamma_n}(\Omega) \cap \mathcal{H}_{DN,\varepsilon}(\Omega)^{\perp_\varepsilon} \ni E - \pi_{DN}E = E_\nabla + E_{rot} \in \nabla H^1_{\Gamma_t}(\Omega) \oplus_\varepsilon \varepsilon^{-1}\text{rot}\, R_{\Gamma_n}(\Omega)$$

with

$$E_\nabla \in \varepsilon^{-1}D_{\Gamma_n}(\Omega) \cap \nabla H^1_{\Gamma_t}(\Omega) = R_{\Gamma_t,0}(\Omega) \cap \varepsilon^{-1}D_{\Gamma_n}(\Omega) \cap \mathcal{H}_{DN,\varepsilon}(\Omega)^{\perp_\varepsilon}, \quad \text{div}\, \varepsilon E_\nabla = \text{div}\, \varepsilon E,$$

$$E_{rot} \in R_{\Gamma_t}(\Omega) \cap \varepsilon^{-1}\text{rot}\, R_{\Gamma_n}(\Omega) = R_{\Gamma_t}(\Omega) \cap \varepsilon^{-1}D_{\Gamma_n,0}(\Omega) \cap \mathcal{H}_{DN,\varepsilon}(\Omega)^{\perp_\varepsilon}, \quad \text{rot}\, E_{rot} = \text{rot}\, E.$$

Thus, by Lemma 3 and Corollary 7 as well as orthogonality, we obtain

$$|E-\pi_{\mathrm{DN}}E|^2_{\mathrm{L}^2_\varepsilon(\Omega)}=|E_\nabla|^2_{\mathrm{L}^2_\varepsilon(\Omega)}+|E_{\mathrm{rot}}|^2_{\mathrm{L}^2_\varepsilon(\Omega)}\leq c^2_{\mathrm{p},\Gamma_\mathrm{t},\varepsilon}|\operatorname{div}\varepsilon E|^2_{\mathrm{L}^2(\Omega)}+c^2_{\mathrm{m},\Gamma_\mathrm{t},\mathrm{rot},\varepsilon,\mathrm{id}}|\operatorname{rot}E|^2_{\mathrm{L}^2(\Omega)}.$$

Lemmas 4 and 8 show the two estimates for the constants, completing the proof. □

Lemma 10
It holds

$$c_{\mathrm{m},\Gamma_\mathrm{t},\varepsilon}=\max\{c_{\mathrm{p},\Gamma_\mathrm{t},\varepsilon},c_{\mathrm{m},\Gamma_\mathrm{t},\mathrm{rot},\varepsilon,\mathrm{id}}\}\begin{cases}\leq\max\{\underline{\varepsilon}c_{\mathrm{p},\Gamma_\mathrm{t}},\overline{\varepsilon}c_{\mathrm{m},\Gamma_\mathrm{t},\mathrm{rot}}\}\leq\hat{\varepsilon}\max\{c_{\mathrm{p},\Gamma_\mathrm{t}},c_{\mathrm{m},\Gamma_\mathrm{t},\mathrm{rot}}\}\\ \geq\max\{\overline{\varepsilon}^{-1}c_{\mathrm{p},\Gamma_\mathrm{t}},\underline{\varepsilon}^{-1}c_{\mathrm{m},\Gamma_\mathrm{t},\mathrm{rot}}\}\geq\hat{\varepsilon}^{-1}\max\{c_{\mathrm{p},\Gamma_\mathrm{t}},c_{\mathrm{m},\Gamma_\mathrm{t},\mathrm{rot}}\}\end{cases}$$

and for $\varepsilon=\mathrm{id}$

$$c_{\mathrm{m},\Gamma_\mathrm{t}}=\max\{c_{\mathrm{p},\Gamma_\mathrm{t}},c_{\mathrm{m},\Gamma_\mathrm{t},\mathrm{rot}}\}.$$

Proof
We have $c_{\mathrm{m},\Gamma_\mathrm{t},\varepsilon}\leq\max\{c_{\mathrm{p},\Gamma_\mathrm{t},\varepsilon},c_{\mathrm{m},\Gamma_\mathrm{t},\mathrm{rot},\varepsilon,\mathrm{id}}\}$. Inserting $E\in\varepsilon^{-1}\mathrm{D}_{\Gamma_\mathrm{n}}(\Omega)\cap\nabla\mathrm{H}^1_{\Gamma_\mathrm{t}}(\Omega)$ (resp.) $E\in\mathrm{R}_{\Gamma_\mathrm{t}}(\Omega)\cap\varepsilon^{-1}\operatorname{rot}\mathrm{R}_{\Gamma_\mathrm{n}}(\Omega)$ into the tangential-normal Maxwell estimate (1.3) shows $c_{\mathrm{p},\Gamma_\mathrm{t},\varepsilon},c_{\mathrm{m},\Gamma_\mathrm{t},\mathrm{rot},\varepsilon,\mathrm{id}}\leq c_{\mathrm{m},\Gamma_\mathrm{t},\varepsilon}$ and the first equation follows. The other estimates are given by Lemmas 4 and 8, completing the proof. □

By the latter theorem and lemma it remains to estimate only the two constants $c_{\mathrm{p},\Gamma_\mathrm{t}}$ and $c_{\mathrm{m},\Gamma_\mathrm{t},\mathrm{rot}}$ for the various Γ_t.

3.2. Full boundary conditions

We summarize our results for the two important extreme cases $\Gamma_\mathrm{t}=\Gamma$ (resp.) $\Gamma_\mathrm{t}=\emptyset$, that is, the full tangential (resp.) the full normal case, and emphasize that in these two cases, the tangential and normal Maxwell rotation constants coincide by (3.1) and hence beside the Poincaré constants, we just have to estimate one constant, namely,

$$c_{\mathrm{m},\mathrm{rot},\varepsilon}:=c_{\mathrm{m},\Gamma,\mathrm{rot},\varepsilon}=c_{\mathrm{m},\emptyset,\mathrm{rot},\varepsilon},\quad c_{\mathrm{m},\mathrm{rot}}=c_{\mathrm{m},\Gamma,\mathrm{rot}}=c_{\mathrm{m},\emptyset,\mathrm{rot}}. \tag{3.5}$$

For the convenience of the reader, let us recall our estimates from the latter sections in these two extreme cases. Lemmas 3 and 4 read

Corollary 11
The Poincaré constant $c_{\mathrm{p},\Gamma,\varepsilon}$ in $\mathrm{H}^1\Gamma(\Omega)$ (resp.) $c_{\mathrm{p},\varepsilon}$ in $\mathrm{H}^1_\emptyset(\Omega)$ and the Maxwell divergence constant $c_{\mathrm{m},\emptyset,\mathrm{div},\varepsilon}$ in $\varepsilon^{-1}\mathrm{D}(\Omega)\cap\nabla\mathrm{H}^1\Gamma(\Omega)$ (resp.) $c_{\mathrm{m},\Gamma,\mathrm{div},\varepsilon}$ in $\varepsilon^{-1}\mathrm{D}_\Gamma(\Omega)\cap\nabla\mathrm{H}^1(\Omega)$ equal, that is, the inequalities

$$\begin{aligned}&\forall\,u\in\mathrm{H}^1\Gamma(\Omega) && |u|_{\mathrm{L}^2(\Omega)}\leq c_{\mathrm{p},\Gamma,\varepsilon}|\nabla u|_{\mathrm{L}^2_\varepsilon(\Omega)}\\ &\forall\,E\in\varepsilon^{-1}\mathrm{D}(\Omega)\cap\nabla\mathrm{H}^1\Gamma(\Omega) && |E|_{\mathrm{L}^2_\varepsilon(\Omega)}\leq c_{\mathrm{p},\Gamma,\varepsilon}|\operatorname{div}\varepsilon E|_{\mathrm{L}^2(\Omega)}\end{aligned}$$

(resp.)

$$\begin{aligned}&\forall\,u\in\mathrm{H}^1(\Omega)\cap\mathbb{R}^\perp && |u|_{\mathrm{L}^2(\Omega)}\leq c_{\mathrm{p},\varepsilon}|\nabla u|_{\mathrm{L}^2_\varepsilon(\Omega)}\\ &\forall\,E\in\varepsilon^{-1}\mathrm{D}_\Gamma(\Omega)\cap\nabla\mathrm{H}^1(\Omega) && |E|_{\mathrm{L}^2_\varepsilon(\Omega)}\leq c_{\mathrm{p},\varepsilon}|\operatorname{div}\varepsilon E|_{\mathrm{L}^2(\Omega)}\end{aligned}$$

hold with sharp constants. Moreover, $\overline{\varepsilon}^{-1}c_{\mathrm{p},\Gamma}\leq c_{\mathrm{p},\Gamma,\varepsilon}\leq\underline{\varepsilon}c_{\mathrm{p},\Gamma}$ and $\overline{\varepsilon}^{-1}c_\mathrm{p}\leq c_{\mathrm{p},\varepsilon}\leq\underline{\varepsilon}c_\mathrm{p}$.

Here, $c_{\mathrm{p},\varepsilon}:=c_{\mathrm{p},\emptyset,\varepsilon}$. Corollary 7 and Lemma 8 read.

Corollary 12
The tangential Maxwell rotation constant $c_{\mathrm{m},\Gamma,\mathrm{rot},\varepsilon}$ in $\mathrm{R}_\Gamma(\Omega)\cap\varepsilon^{-1}\operatorname{rot}\mathrm{R}(\Omega)$ and the normal Maxwell rotation constant $c_{\mathrm{m},\emptyset,\mathrm{rot},\varepsilon}$ in the space $\mathrm{R}(\Omega)\cap\varepsilon^{-1}\operatorname{rot}\mathrm{R}_\Gamma(\Omega)$ is equal, that is, for all $E\in\big(\mathrm{R}_\Gamma(\Omega)\cap\varepsilon^{-1}\operatorname{rot}\mathrm{R}(\Omega)\big)\cup\big(\mathrm{R}(\Omega)\cap\varepsilon^{-1}\operatorname{rot}\mathrm{R}_\Gamma(\Omega)\big)$,

$$|E|_{\mathrm{L}^2_\varepsilon(\Omega)}\leq c_{\mathrm{m},\mathrm{rot},\varepsilon}|\operatorname{rot}E|_{\mathrm{L}^2_{\varepsilon^{-1}}(\Omega)}\leq\underline{\varepsilon}c_{\mathrm{m},\mathrm{rot},\varepsilon}|\operatorname{rot}E|_{\mathrm{L}^2(\Omega)}$$

holds with sharp constants. Moreover, the inequalities

$$\begin{aligned}&\forall\,E\in\mathrm{R}_\Gamma(\Omega)\cap\varepsilon^{-1}\operatorname{rot}\mathrm{R}(\Omega) && |E|_{\mathrm{L}^2_\varepsilon(\Omega)}\leq c_{\mathrm{m},\Gamma,\mathrm{rot},\varepsilon,\mathrm{id}}|\operatorname{rot}E|_{\mathrm{L}^2(\Omega)}\\ &\forall\,H\in\mathrm{R}(\Omega)\cap\varepsilon^{-1}\operatorname{rot}\mathrm{R}_\Gamma(\Omega) && |H|_{\mathrm{L}^2_\varepsilon(\Omega)}\leq c_{\mathrm{m},\emptyset,\mathrm{rot},\varepsilon,\mathrm{id}}|\operatorname{rot}H|_{\mathrm{L}^2(\Omega)}\end{aligned}$$

hold, where these sharp constants do not need to coincide if $\varepsilon\neq\mathrm{id}$. Moreover, it holds $\underline{\varepsilon}^{-2}c_{\mathrm{m},\mathrm{rot}}\leq c_{\mathrm{m},\mathrm{rot},\varepsilon}\leq\overline{\varepsilon}^2c_{\mathrm{m},\mathrm{rot}}$ and

$$\begin{aligned}\underline{\varepsilon}^{-1}c_{\mathrm{m},\mathrm{rot}}\leq\max\{\overline{\varepsilon}^{-1}c_{\mathrm{m},\mathrm{rot},\varepsilon},\underline{\varepsilon}^{-1}c_{\mathrm{m},\mathrm{rot}}\}&\leq c_{\mathrm{m},\Gamma,\mathrm{rot},\varepsilon,\mathrm{id}},c_{\mathrm{m},\emptyset,\mathrm{rot},\varepsilon,\mathrm{id}}\\ &\leq\min\{\underline{\varepsilon}c_{\mathrm{m},\mathrm{rot},\varepsilon},\overline{\varepsilon}c_{\mathrm{m},\mathrm{rot}}\}\leq\overline{\varepsilon}c_{\mathrm{m},\mathrm{rot}}.\end{aligned}$$

Theorem 9 and Lemma 10 read.

Corollary 13
For all $E \in \mathsf{R}_\Gamma(\Omega) \cap \varepsilon^{-1}\mathsf{D}(\Omega)$ and all $H \in \mathsf{R}(\Omega) \cap \varepsilon^{-1}\mathsf{D}_\Gamma(\Omega)$, the tangential and normal Maxwell estimates

$$|E - \pi_{\mathrm{D}} E|^2_{\mathsf{L}^2_\varepsilon(\Omega)} \le c^2_{\mathrm{p},\Gamma,\varepsilon} |\operatorname{div} \varepsilon E|^2_{\mathsf{L}^2(\Omega)} + c^2_{\mathrm{m},\Gamma,\mathrm{rot},\varepsilon,\mathrm{id}} |\operatorname{rot} E|^2_{\mathsf{L}^2(\Omega)},$$
$$|H - \pi_{\mathrm{N}} H|^2_{\mathsf{L}^2_\varepsilon(\Omega)} \le c^2_{\mathrm{p},\varepsilon} |\operatorname{div} \varepsilon H|^2_{\mathsf{L}^2(\Omega)} + c^2_{\mathrm{m},\emptyset,\mathrm{rot},\varepsilon,\mathrm{id}} |\operatorname{rot} H|^2_{\mathsf{L}^2(\Omega)},$$

hold with sharp constants. Furthermore, the estimates $\overline{\varepsilon}^{-1} c_{\mathrm{p},\Gamma} \le c_{\mathrm{p},\Gamma,\varepsilon}, c_{\mathrm{p},\varepsilon} \le \underline{\varepsilon} c_{\mathrm{p}}$ and $\underline{\varepsilon}^{-1} c_{\mathrm{m,rot}} \le c_{\mathrm{m},\Gamma,\mathrm{rot},\varepsilon,\mathrm{id}}, c_{\mathrm{m},\emptyset,\mathrm{rot},\varepsilon,\mathrm{id}} \le \overline{\varepsilon} c_{\mathrm{m,rot}}$ as well as

$$c_{\mathrm{m},\Gamma,\varepsilon} = \max\{c_{\mathrm{p},\Gamma,\varepsilon}, c_{\mathrm{m},\Gamma,\mathrm{rot},\varepsilon,\mathrm{id}}\} \begin{cases} \le \max\{\underline{\varepsilon} c_{\mathrm{p},\Gamma}, \overline{\varepsilon} c_{\mathrm{m,rot}}\} \le \hat{\varepsilon} \max\{c_{\mathrm{p},\Gamma}, c_{\mathrm{m,rot}}\}, \\ \ge \max\{\overline{\varepsilon}^{-1} c_{\mathrm{p},\Gamma}, \underline{\varepsilon}^{-1} c_{\mathrm{m,rot}}\} \ge \hat{\varepsilon}^{-1} \max\{c_{\mathrm{p},\Gamma}, c_{\mathrm{m,rot}}\}, \end{cases}$$

$$c_{\mathrm{m},\emptyset,\varepsilon} = \max\{c_{\mathrm{p},\varepsilon}, c_{\mathrm{m},\emptyset,\mathrm{rot},\varepsilon,\mathrm{id}}\} \begin{cases} \le \max\{\underline{\varepsilon} c_{\mathrm{p}}, \overline{\varepsilon} c_{\mathrm{m,rot}}\} \le \hat{\varepsilon} \max\{c_{\mathrm{p}}, c_{\mathrm{m,rot}}\}, \\ \ge \max\{\overline{\varepsilon}^{-1} c_{\mathrm{p}}, \underline{\varepsilon}^{-1} c_{\mathrm{m,rot}}\} \ge \hat{\varepsilon}^{-1} \max\{c_{\mathrm{p}}, c_{\mathrm{m,rot}}\} \end{cases}$$

hold. Therefore, in both cases,

$$\begin{aligned} \hat{\varepsilon}^{-1} \max\{c_{\mathrm{p},\Gamma}, c_{\mathrm{m,rot}}\} \le \max\{\overline{\varepsilon}^{-1} c_{\mathrm{p},\Gamma}, \underline{\varepsilon}^{-1} c_{\mathrm{m,rot}}\} &\le c_{\mathrm{m},\Gamma,\varepsilon}, c_{\mathrm{m},\emptyset,\varepsilon} \\ &\le \max\{\underline{\varepsilon} c_{\mathrm{p}}, \overline{\varepsilon} c_{\mathrm{m,rot}}\} \le \hat{\varepsilon} \max\{c_{\mathrm{p}}, c_{\mathrm{m,rot}}\}. \end{aligned}$$

For $\varepsilon = \mathrm{id}$, it holds

$$c_{\mathrm{m},\Gamma} = \max\{c_{\mathrm{p},\Gamma}, c_{\mathrm{m,rot}}\}, \quad c_{\mathrm{m},\emptyset} = \max\{c_{\mathrm{p}}, c_{\mathrm{m,rot}}\}.$$

As the two Poincaré constants $c_{\mathrm{p},\Gamma} < c_{\mathrm{p}}$ are more or less well known, by the latter corollaries, it remains only to estimate the Maxwell constant $c_{\mathrm{m,rot}}$.

3.2.1. Convex domains. Now, let $\Omega \subset \mathbb{R}^3$ be a bounded domain and convex domain. Then Ω is strongly Lipschitz, see, for example, [21, Corollary 1.2.2.3]. Moreover, there are no Dirichlet or Neumann fields because Ω is simply connected and has a connected boundary. As noted before in (1.8), in the convex case, we can estimate the Poincaré constant c_{p} by the diameter of Ω, that is,

$$c_{\mathrm{p},\Gamma} < c_{\mathrm{p}} \le \frac{\operatorname{diam}(\Omega)}{\pi}.$$

We show that we can also estimate the Maxwell constant $c_{\mathrm{m,rot}}$ in the two extreme cases $\Gamma_t = \Gamma$ (resp.) $\Gamma_t = \emptyset$ by c_{p}. In [22, Theorem 2.17], the following crucial lemma has been proved, which is the key point in our investigations for convex domains.

Lemma 14
Let E belong to $\mathsf{R}_\Gamma(\Omega) \cap \mathsf{D}(\Omega)$ or $\mathsf{R}(\Omega) \cap \mathsf{D}_\Gamma(\Omega)$. Then $E \in \mathsf{H}^1(\Omega)$ and

$$|\nabla E|^2_{\mathsf{L}^2(\Omega)} \le |\operatorname{rot} E|^2_{\mathsf{L}^2(\Omega)} + |\operatorname{div} E|^2_{\mathsf{L}^2(\Omega)}. \tag{3.6}$$

We note that the latter lemma has already been proved in [13] in the case $\mathsf{R}_\Gamma(\Omega) \cap \mathsf{D}(\Omega)$.

Remark 15
For $E \in \mathsf{H}^1\Gamma(\Omega)$, it is clear that for any domain $\Omega \subset \mathbb{R}^3$ (or even in $\mathbb{R}^N$)

$$|\nabla E|^2_{\mathsf{L}^2(\Omega)} = |\operatorname{rot} E|^2_{\mathsf{L}^2(\Omega)} + |\operatorname{div} E|^2_{\mathsf{L}^2(\Omega)}$$

holds because $-\Delta = \operatorname{rot}\operatorname{rot} - \nabla\operatorname{div}$. In general, this formula is no longer valid if E has just the tangential or normal boundary condition.

With the help of Lemma 14, we can now estimate $c_{\mathrm{m,rot}}$.

Lemma 16
$c_{\mathrm{m,rot}} \le c_{\mathrm{p}}$. More precisely, for all E in $\mathsf{R}_\Gamma(\Omega) \cap \operatorname{rot} \mathsf{R}(\Omega)$ or $\mathsf{R}(\Omega) \cap \operatorname{rot} \mathsf{R}_\Gamma(\Omega)$

$$|E|_{\mathsf{L}^2(\Omega)} \le c_{\mathrm{p}} |\operatorname{rot} E|_{\mathsf{L}^2(\Omega)}.$$

Furthermore, $c_{\mathrm{m},\Gamma,\mathrm{rot},\varepsilon,\mathrm{id}}, c_{\mathrm{m},\emptyset,\mathrm{rot},\varepsilon,\mathrm{id}} \le \overline{\varepsilon} c_{\mathrm{p}}$.

Proof
By (3.5), the boundary condition does not matter. So, let

$$E \in \mathsf{R}(\Omega) \cap \operatorname{rot} \mathsf{R}_\Gamma(\Omega) = \mathsf{R}(\Omega) \cap \mathsf{D}_{\Gamma,0}(\Omega)$$

with $E = \operatorname{rot} H$ for some $H \in \mathsf{R}_\Gamma(\Omega)$. Then, for any constant vector $a \in \mathbb{R}^3$,

$$\langle E, a \rangle_{textL^2(\Omega)} = \langle \operatorname{rot} H, a \rangle_{textL^2(\Omega)} = 0 \tag{3.7}$$

holds. Thus, by Poincaré's estimate and Lemma 14, we obtain $E \in \mathrm{H}^1(\Omega) \cap (\mathbb{R}^3)^\perp$ and

$$|E|_{\mathrm{L}^2(\Omega)} \leq c_\mathrm{p} |\nabla E|_{\mathrm{L}^2(\Omega)} \leq c_\mathrm{p} |\mathrm{rot}\, E|_{\mathrm{L}^2(\Omega)},$$

which shows $c_{\mathrm{m,rot}} = c_{\mathrm{m},\emptyset,\mathrm{rot}} \leq c_\mathrm{p}$. □

We can now formulate the main result for convex domains, which follows immediately from Corollary 13 and Lemma 16.

Theorem 17
For all $E \in \mathrm{R}_\Gamma(\Omega) \cap \varepsilon^{-1}\mathrm{D}(\Omega)$ and all $H \in \mathrm{R}(\Omega) \cap \varepsilon^{-1}\mathrm{D}_\Gamma(\Omega)$, the tangential and normal Maxwell estimates

$$|E|^2_{\mathrm{L}^2_\varepsilon(\Omega)} \leq \underline{\varepsilon}^2 c^2_{\mathrm{p},\Gamma} |\mathrm{div}\, \varepsilon E|^2_{\mathrm{L}^2(\Omega)} + \overline{\varepsilon}^2 c^2_\mathrm{p} |\mathrm{rot}\, E|^2_{\mathrm{L}^2(\Omega)},$$
$$|H|^2_{\mathrm{L}^2_\varepsilon(\Omega)} \leq \underline{\varepsilon}^2 c^2_\mathrm{p} |\mathrm{div}\, \varepsilon H|^2_{\mathrm{L}^2(\Omega)} + \overline{\varepsilon}^2 c^2_\mathrm{p} |\mathrm{rot}\, H|^2_{\mathrm{L}^2(\Omega)}$$

hold. Moreover,

$$\frac{c_{\mathrm{p},\Gamma}}{\overline{\varepsilon}} \leq c_{\mathrm{m},\Gamma,\varepsilon} \leq \hat{\varepsilon} c_\mathrm{p}, \qquad \frac{c_\mathrm{p}}{\overline{\varepsilon}} \leq c_{\mathrm{m},\emptyset,\varepsilon} \leq \hat{\varepsilon} c_\mathrm{p}.$$

Especially, for $\varepsilon = \mathrm{id}$,

$$\max\{c_{\mathrm{p},\Gamma}, c_{\mathrm{m,rot}}\} = c_{\mathrm{m},\Gamma} \leq c_{\mathrm{m},\emptyset} = c_\mathrm{p}.$$

Theorem 18
For all $E \in \big(\mathrm{R}_\Gamma(\Omega) \cap \varepsilon^{-1}\mathrm{D}(\Omega)\big) \cup \big(\mathrm{R}(\Omega) \cap \varepsilon^{-1}\mathrm{D}_\Gamma(\Omega)\big)$,

$$|E|_{\mathrm{L}^2_\varepsilon(\Omega)} \leq \hat{\varepsilon} c_\mathrm{p} \left(|\mathrm{div}\, \varepsilon E|^2_{\mathrm{L}^2(\Omega)} + |\mathrm{rot}\, E|^2_{\mathrm{L}^2(\Omega)} \right)^{1/2}.$$

Appendix A

A.1. More general operators

There are obvious generalizations to differential forms. Let Ω be a smooth Riemannian manifold of dimension $N \geq 2$ with boundary Γ and compact closure. We assume that the boundary manifold Γ is divided into two $(N-1)$-dimensional Riemannian sub-manifolds Γ_t and Γ_n with boundaries. Let us denote by $\mathrm{L}^{2,q}(\Omega)$ the usual Lebesgue (Hilbert) space of q-forms. For the exterior derivative and co-derivative, we define the well-known Sobolev spaces

$$\mathrm{D}^q(\Omega) := \left\{ E \in \mathrm{L}^{2,q}(\Omega) : \mathrm{d}\, E \in \mathrm{L}^{2,q+1}(\Omega) \right\}, \quad \Delta^q(\Omega) := \left\{ E \in \mathrm{L}^{2,q}(\Omega) : \delta E \in \mathrm{L}^{2,q-1}(\Omega) \right\}.$$

As before, we introduce weak homogeneous boundary conditions by closures of respective test forms, yielding the Sobolev spaces

$$\mathrm{D}^q_{\Gamma_\mathrm{t}}(\Omega), \quad \Delta^q_{\Gamma_\mathrm{n}}(\Omega).$$

Let A be

$$\mu^{-1}\mathrm{d} : \mathrm{D}^q_{\Gamma_\mathrm{t}}(\Omega) \subset \mathrm{L}^{2,q}_\varepsilon(\Omega) \to \mathrm{L}^{2,q+1}_\mu(\Omega).$$

Then A* is

$$-\varepsilon^{-1}\delta : \Delta^{q+1}_{\Gamma_\mathrm{n}}(\Omega) \subset \mathrm{L}^{2,q+1}_\mu(\Omega) \to \mathrm{L}^{2,q}_\varepsilon(\Omega),$$

where ε (resp.) μ are bounded, symmetric, real and uniformly positive definite linear transformations on q- (resp.) $(q+1)$-forms. More precisely,

A	$D(\mathrm{A})$	X	Y	$N(\mathrm{A})$	$R(\mathrm{A})$
$\mu^{-1}\mathrm{d}$	$\mathrm{D}^q_{\Gamma_\mathrm{t}}(\Omega)$	$\mathrm{L}^{2,q}_\varepsilon(\Omega)$	$\mathrm{L}^{2,q+1}_\mu(\Omega)$	$\mathrm{D}^q\Gamma_\mathrm{t}, 0(\Omega)$	$\mu^{-1}\mathrm{d}\mathrm{D}^q_{\Gamma_\mathrm{t}}(\Omega)$
A*	$D(\mathrm{A}^*)$	Y	X	$N(\mathrm{A}^*)$	$R(\mathrm{A}^*)$
$-\varepsilon^{-1}\delta$	$\Delta^{q+1}_{\Gamma_\mathrm{n}}(\Omega)$	$\mathrm{L}^{2,q+1}_\mu(\Omega)$	$\mathrm{L}^{2,q}_\varepsilon(\Omega)$	$\Delta^{q+1}_{\Gamma_\mathrm{n},0}(\Omega)$	$\varepsilon^{-1}\delta\Delta^{q+1}_{\Gamma_\mathrm{n}}(\Omega)$

Here,

$$\mathrm{D}^q_{\Gamma_\mathrm{t},0}(\Omega) := \left\{ E \in \mathrm{D}^q_{\Gamma_\mathrm{t}}(\Omega) : \mathrm{d}E = 0 \right\}, \quad \Delta^q_{\Gamma_\mathrm{n},0}(\Omega) := \left\{ E \in \Delta^q_{\Gamma_\mathrm{n}}(\Omega) : \delta E = 0 \right\}$$

and we note

$$R(\mathrm{A})=\mu^{-1}\left(\mathrm{D}^{q+1}_{\Gamma_t,0}(\Omega)\cap\mathcal{H}^{q+1}_{\mathrm{DN}}(\Omega)^{\perp}\right),\quad R(\mathrm{A}^*)=\varepsilon^{-1}\left(\Delta^{q}_{\Gamma_n,0}(\Omega)\cap\mathcal{H}^{q}_{\mathrm{DN}}(\Omega)^{\perp}\right),$$

where $\mathcal{H}^q_{\mathrm{DN}}(\Omega):=\mathrm{D}^q\Gamma_t,0(\Omega)\cap\Delta^q_{\Gamma_n,0}(\Omega)$. Indeed, $D(\mathrm{A}^*)=\Delta^{q+1}_{\Gamma_n}(\Omega)$ holds. We have the same remarks as in Section 3.1.2. Again, for this, one has to show the approximation property

$$\Delta^{q+1}_{\Gamma_n}(\Omega)=\left\{H\in\Delta^{q+1}(\Omega):\langle\delta H,E\rangle_{\mathrm{L}^{2,q}(\Omega)}=-\langle H,\mathrm{d}E\rangle_{\mathrm{L}^{2,q+1}(\Omega)}\ \forall\ E\in\mathrm{D}^q_{\Gamma_t}(\Omega)\right\},$$

which is not trivial at all for mixed boundary conditions. And again, only in the special cases of full boundary conditions this is clear. Because $D(\mathrm{A}^*)=\Delta^{q+1}(\Omega)$ holds for $\Gamma_t=\Gamma$ by definition, we have also $D(\mathrm{B}^*)=D(\mathrm{A}^{**})=D(\mathrm{A})=\mathrm{D}^q_{\Gamma}(\Omega)$ for $\mathrm{B}=\mathrm{A}^*$, which shows the result for $\Gamma_t=\emptyset$. The crucial compact embedding (2.3) is

$$\mathrm{D}^q_{\Gamma_t}(\Omega)\cap\varepsilon^{-1}\overline{\delta\Delta^{q+1}_{\Gamma_n}(\Omega)}\hookrightarrow\mathrm{L}^{2,q}_{\varepsilon}(\Omega).$$

Both latter properties of Ω, that is, the approximation and the compactness property, hold, for example, if the boundary manifolds Γ, Γ_t, Γ_n are Lipschitz and the boundary manifolds Γ_t, Γ_n are separated by a $(N-2)$-dimensional Riemannian and Lipschitz sub-manifold, the interface $\gamma:=\overline{\Gamma_t}\cap\overline{\Gamma_n}$, see [23, 24] for details and proofs. We note that

$$\mathrm{D}^q_{\Gamma_t}(\Omega)\cap\varepsilon^{-1}\overline{\delta\Delta^{q+1}_{\Gamma_n}(\Omega)}\subset\mathrm{D}^q_{\Gamma_t}(\Omega)\cap\varepsilon^{-1}\Delta^q_{\Gamma_n,0}(\Omega)\subset\mathrm{D}^q_{\Gamma_t}(\Omega)\cap\varepsilon^{-1}\Delta^q_{\Gamma_n}(\Omega)$$

holds and that even the compact embedding of the latter space into $\mathrm{L}^{2,q}(\Omega)$.

$$\mathrm{D}^q_{\Gamma_t}(\Omega)\cap\varepsilon^{-1}\Delta^q_{\Gamma_n}(\Omega)\hookrightarrow\mathrm{L}^{2,q}(\Omega)\subset\mathrm{L}^{2,q}_{\varepsilon}(\Omega)$$

has been shown in [24]$^{\|}$. By Theorem 2, we have

$$\kappa:=\min_{0\neq E\in\mathrm{D}^q_{\Gamma_t}(\Omega)\cap\varepsilon^{-1}\delta\Delta^{q+1}_{\Gamma_n}(\Omega)}\frac{|\mu^{-1}\mathrm{d}E|_{\mathrm{L}^{2,q+1}_{\mu}(\Omega)}}{|E|_{\mathrm{L}^{2,q}_{\varepsilon}(\Omega)}}=\min_{0\neq H\in\Delta^{q+1}_{\Gamma_n}(\Omega)\cap\mu^{-1}\mathrm{d}\mathrm{D}^q_{\Gamma_t}(\Omega)}\frac{|\varepsilon^{-1}\delta H|_{\mathrm{L}^{2,q}_{\varepsilon}(\Omega)}}{|H|_{\mathrm{L}^{2,q+1}_{\mu}(\Omega)}},$$

and κ^2 is the first positive Dirichlet–Neumann eigenvalue of the weighted-reduced δ-d-operator $-\varepsilon^{-1}\delta\mu^{-1}\mathrm{d}$. Analogously, κ^2 is also the first positive Neumann–Dirichlet eigenvalue of the weighted reduced d-δ-operator $-\mu^{-1}\mathrm{d}\varepsilon^{-1}\delta$.

Lemma 19

The tangential-normal and normal-tangential generalized Maxwell constants, that is, the best constants in the inequalities

$$\forall\ E\in\mathrm{D}^q_{\Gamma_t}(\Omega)\cap\varepsilon^{-1}\delta\Delta^{q+1}_{\Gamma_n}(\Omega)\qquad |E|_{\mathrm{L}^{2,q}_{\varepsilon}(\Omega)}\leq c_{\mathrm{gm},\Gamma_t,\mathrm{d},\varepsilon,\mu}\,|\mathrm{d}E|_{\mathrm{L}^{2,q+1}_{\mu^{-1}}(\Omega)}$$

$$\forall\ H\in\Delta^{q+1}_{\Gamma_n}(\Omega)\cap\mu^{-1}\mathrm{d}\mathrm{D}^q_{\Gamma_t}(\Omega)\qquad |H|_{\mathrm{L}^{2,q+1}_{\mu}(\Omega)}\leq c_{\mathrm{gm},\Gamma_n,\delta,\mu,\varepsilon}\,|\delta H|_{\mathrm{L}^{2,q}_{\varepsilon^{-1}}(\Omega)}$$

coincide and are equal to $1/\kappa$, that is, $c_{\mathrm{gm},\Gamma_t,\mathrm{d},\varepsilon,\mu}=c_{\mathrm{gm},\Gamma_n,\delta,\mu,\varepsilon}=\kappa^{-1}$.

Remark 20

It is clear that more results of this contribution can be generalized to the differential form setting.

A.2. Maxwell tools

Let the general assumptions from the introduction be satisfied.

A.2.1. The Maxwell estimates

By the Maxwell compactness property, we obtain immediately the Maxwell estimate.

Lemma 21

There exists $c_{\mathrm{m},\Gamma_t,\varepsilon}>0$, such that for all E in $\mathrm{R}_{\Gamma_t}(\Omega)\cap\varepsilon^{-1}\mathrm{D}_{\Gamma_n}(\Omega)\cap\mathcal{H}_{\mathrm{DN},\varepsilon}(\Omega)^{\perp_\varepsilon}$

$$|E|_{\mathrm{L}^2_{\varepsilon}(\Omega)}\leq c_{\mathrm{m},\Gamma_t,\varepsilon}\left(|\mathrm{rot}\,E|^2_{\mathrm{L}^2(\Omega)}+|\mathrm{div}\,\varepsilon E|^2_{\mathrm{L}^2(\Omega)}\right)^{1/2}.$$

$^{\|}$*In [24], it is proved that $D^q_{\Gamma_t}(\Omega)\cap\Delta^q_{\Gamma_n}(\Omega)$ even embeds continuously to $\mathrm{H}^{1/2,q}(\Omega)$ and hence compactly to $L^{2,q}(\Omega)$. We note that the compactness property is independent of ε, see, for example, [3].*

Proof
If the estimate would not hold, there would exist a sequence of vector fields $(E_n) \subset \mathrm{R}_{\Gamma_t}(\Omega) \cap \varepsilon^{-1}\mathrm{D}_{\Gamma_n}(\Omega) \cap \mathcal{H}_{\mathrm{DN},\varepsilon}(\Omega)^{\perp_\varepsilon}$ with the property $|E_n|_{\mathrm{L}^2_\varepsilon(\Omega)} = 1$ and

$$|\operatorname{rot} E_n|_{\mathrm{L}^2(\Omega)} + |\operatorname{div} \varepsilon E_n|_{\mathrm{L}^2(\Omega)} < \frac{1}{n}.$$

By the Maxwell compactness property, we can assume w.l.o.g. that (E_n) converges in $\mathrm{L}^2_\varepsilon(\Omega)$ to some $E \in \mathrm{L}^2_\varepsilon(\Omega)$. By testing, E belongs to $\mathrm{R}_0(\Omega) \cap \varepsilon^{-1}\mathrm{D}_0(\Omega) \cap \mathcal{H}_{\mathrm{DN},\varepsilon}(\Omega)^{\perp_\varepsilon}$, and (E_n) converges to E also in $\mathrm{R}(\Omega) \cap \varepsilon^{-1}\mathrm{D}(\Omega)$. As $\mathrm{R}_{\Gamma_t}(\Omega)$ (resp.) $\mathrm{D}_{\Gamma_n}(\Omega)$ is a closed subspace of $\mathrm{R}(\Omega)$ (resp.) $\mathrm{D}(\Omega)$, E belongs even to $\mathrm{R}_{\Gamma_t,0}(\Omega) \cap \varepsilon^{-1}\mathrm{D}_{\Gamma_n,0}(\Omega) = \mathcal{H}_{\mathrm{DN},\varepsilon}(\Omega)$. Hence, $E = 0$, which contradicts $1 = |E_n|_{\mathrm{L}^2(\Omega)} \to 0$. □

Corollary 22
For all E in $\mathrm{R}_{\Gamma_t}(\Omega) \cap \varepsilon^{-1}\mathrm{D}_{\Gamma_n}(\Omega)$,

$$|(1-\pi_{\mathrm{DN}})E|_{\mathrm{L}^2_\varepsilon(\Omega)} \leq c_{\mathrm{m},\Gamma_t,\varepsilon}\left(|\operatorname{rot} E|^2_{\mathrm{L}^2(\Omega)} + |\operatorname{div} \varepsilon E|^2_{\mathrm{L}^2(\Omega)}\right)^{1/2}.$$

Proof
As $H := (1-\pi_{\mathrm{DN}})E \in \mathrm{R}_{\Gamma_t}(\Omega) \cap \varepsilon^{-1}\mathrm{D}_{\Gamma_n}(\Omega) \cap \mathcal{H}_{\mathrm{DN},\varepsilon}(\Omega)^{\perp_\varepsilon}$ with $\operatorname{div} \varepsilon H = \operatorname{div} \varepsilon E$ and $\operatorname{rot} H = \operatorname{rot} E$, Lemma 21 completes the proof. □

The same arguments show that the Maxwell estimate remains valid in any dimension and even for compact Riemannian manifolds as long as the crucial Maxwell compactness property holds.

Helmholtz–Weyl decompositions

By the projection theorem we have for the operator ∇,

$$\mathrm{L}^2_\varepsilon(\Omega) = \overline{\nabla \mathrm{H}^1_{\Gamma_t}(\Omega)} \oplus_\varepsilon \varepsilon^{-1}\mathrm{D}_{\Gamma_n,0}(\Omega),$$

where indeed, $\left(\nabla \mathrm{H}^1_{\Gamma_t}(\Omega)\right)^{\perp} = \mathrm{D}_{\Gamma_n,0}(\Omega)$ holds by [1]. Note that $\nabla \mathrm{H}^1_{\Gamma_t}(\Omega)$ is already closed by Rellich's selection theorem. Analogously, we obtain for the operator rot

$$\mathrm{L}^2_\varepsilon(\Omega) = \mathrm{R}_{\Gamma_t,0}(\Omega) \oplus_\varepsilon \varepsilon^{-1}\overline{\operatorname{rot} \mathrm{R}_{\Gamma_n}(\Omega)}, \tag{A.1}$$

where again and indeed, $\left(\operatorname{rot} \mathrm{R}_{\Gamma_n}(\Omega)\right)^{\perp} = \mathrm{R}_{\Gamma_t,0}(\Omega)$ holds by [1]. For $\varepsilon = \mathrm{id}$, we obtain by (A.1)

$$\mathrm{R}_{\Gamma_t}(\Omega) = \mathrm{R}_{\Gamma_t,0}(\Omega) \oplus \left(\mathrm{R}_{\Gamma_t}(\Omega) \cap \overline{\operatorname{rot} \mathrm{R}_{\Gamma_n}(\Omega)}\right),$$

and therefore,

$$\operatorname{rot} \mathrm{R}_{\Gamma_t}(\Omega) = \operatorname{rot}\left(\mathrm{R}_{\Gamma_t}(\Omega) \cap \overline{\operatorname{rot} \mathrm{R}_{\Gamma_n}(\Omega)}\right).$$

As $\overline{\operatorname{rot} \mathrm{R}_{\Gamma_n}(\Omega)} \subset \mathrm{D}_{\Gamma_n,0}(\Omega) \cap \mathcal{H}^{\perp}_{\mathrm{DN}}$, the Maxwell estimate Lemma 21 implies that also $\operatorname{rot} \mathrm{R}_{\Gamma_t}(\Omega)$ is already closed. Moreover,

$$\operatorname{rot} \mathrm{R}_{\Gamma_t}(\Omega) = \operatorname{rot} \mathrm{R}_{\Gamma_t}(\Omega), \quad \mathrm{R}_{\Gamma_t}(\Omega) := \mathrm{R}_{\Gamma_t}(\Omega) \cap \operatorname{rot} \mathrm{R}_{\Gamma_n}(\Omega) = \mathrm{R}_{\Gamma_t}(\Omega) \cap \operatorname{rot} \mathrm{R}_{\Gamma_n}(\Omega).$$

Because $\nabla \mathrm{H}^1_{\Gamma_t}(\Omega) \subset \mathrm{R}_{\Gamma_t,0}(\Omega)$ and $\operatorname{rot} \mathrm{R}_{\Gamma_n}(\Omega) \subset \mathrm{D}_{\Gamma_n,0}(\Omega)$, we obtain

$$\mathrm{R}_{\Gamma_t,0}(\Omega) = \nabla \mathrm{H}^1_{\Gamma_t}(\Omega) \oplus_\varepsilon \left(\underbrace{\mathrm{R}_{\Gamma_t,0}(\Omega) \cap \varepsilon^{-1}\mathrm{D}_{\Gamma_n,0}(\Omega)}_{=\mathcal{H}_{\mathrm{DN},\varepsilon}(\Omega)}\right),$$

$$\varepsilon^{-1}\mathrm{D}_{\Gamma_n,0}(\Omega) = \varepsilon^{-1}\operatorname{rot}\mathrm{R}_{\Gamma_n}(\Omega) \oplus_\varepsilon \left(\overbrace{\mathrm{R}_{\Gamma_t,0}(\Omega) \cap \varepsilon^{-1}\mathrm{D}_{\Gamma_n,0}(\Omega)}\right).$$

Finally, we have the well-known Helmholtz decompositions.

Lemma 23
It holds

$$\begin{aligned}\mathrm{L}^2_\varepsilon(\Omega) &= \nabla \mathrm{H}^1_{\Gamma_t}(\Omega) \oplus_\varepsilon \varepsilon^{-1}\mathrm{D}_{\Gamma_n,0}(\Omega) = \mathrm{R}_{\Gamma_t,0}(\Omega) \oplus_\varepsilon \varepsilon^{-1}\operatorname{rot} \mathrm{R}_{\Gamma_n}(\Omega)\\ &= \nabla \mathrm{H}^1_{\Gamma_t}(\Omega) \oplus_\varepsilon \mathcal{H}_{\mathrm{DN},\varepsilon}(\Omega) \oplus_\varepsilon \varepsilon^{-1}\operatorname{rot} \mathrm{R}_{\Gamma_n}(\Omega)\end{aligned}$$

as well as

$$\nabla \mathrm{H}^1_{\Gamma_t}(\Omega) = \mathrm{R}_{\Gamma_t,0}(\Omega) \cap \mathcal{H}_{\mathrm{DN},\varepsilon}(\Omega)^{\perp_\varepsilon}, \quad \varepsilon^{-1}\operatorname{rot} \mathrm{R}_{\Gamma_n}(\Omega) = \varepsilon^{-1}\mathrm{D}_{\Gamma_n,0}(\Omega) \cap \mathcal{H}_{\mathrm{DN},\varepsilon}(\Omega)^{\perp_\varepsilon}$$

and $\mathrm{R}_{\Gamma_t}(\Omega) = \mathrm{R}_{\Gamma_t}(\Omega) \cap \mathrm{D}_{\Gamma_n,0}(\Omega) \cap \mathcal{H}^{\perp}_{\mathrm{DN}}$.

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