Research Article

Mathematical Methods in the Applied Sciences

Received 25 May 2011

Published online in Wiley Online Library

(wileyonlinelibrary.com) DOI: 10.1002/mma.1534
MOS subject classification: 35A23; 35Q61; 74C05; 78A25; 78A30

Maxwell meets Korn: A new coercive inequality for tensor fields in $\mathbb{R}^{N\times N}$ with square-integrable exterior derivative

Patrizio Neff*†, Dirk Pauly and Karl-Josef Witsch

Communicated by Hans-Dieter Alber

For a bounded domain $\Omega \subset \mathbb{R}^N$ with connected Lipschitz boundary, we prove the existence of some $c > 0$, such that

$$c\,\|T\|_{\mathrm{L}^2(\Omega,\mathbb{R}^{N\times N})} \le \|\mathrm{sym}\,T\|_{\mathrm{L}^2(\Omega,\mathbb{R}^{N\times N})} + \|\mathrm{Curl}\,T\|_{\mathrm{L}^2(\Omega,\mathbb{R}^{N\times(N-1)N/2})}$$

holds for all square-integrable tensor fields $T : \Omega \longrightarrow \mathbb{R}^{N\times N}$, having square-integrable generalized "rotation" tensor fields $\mathrm{Curl}\,T : \Omega \longrightarrow \mathbb{R}^{N\times(N-1)N/2}$ and vanishing tangential trace on $\partial\Omega$, where both operations are to be understood row-wise. Here, in each row, the operator curl is the vector analytical reincarnation of the exterior derivative d in $\mathbb{R}^N$. For compatible tensor fields T, that is, $T = \nabla v$, the latter estimate reduces to a non-standard variant of Korn's first inequality in $\mathbb{R}^N$, namely

$$c\,\|\nabla v\|_{\mathrm{L}^2(\Omega,\mathbb{R}^{N\times N})} \le \|\mathrm{sym}\,\nabla v\|_{\mathrm{L}^2(\Omega,\mathbb{R}^{N\times N})}$$

for all vector fields $v \in \mathrm{H}^1(\Omega,\mathbb{R}^N)$, for which $\nabla v_n, n = 1,\ldots,N$, are normal at $\partial\Omega$.

Keywords: Korn's inequality; theory of Maxwell equations in $\mathbb{R}^N$; Helmholtz decomposition; Poincaré/Friedrichs type estimates

1. Introduction and preliminaries

We extend the results from [1, 2], which have been announced in [3], to the N-dimensional case following in close lines, the arguments presented there. Let $N \in \mathbb{N}$ and Ω be a bounded domain in $\mathbb{R}^N$ with connected Lipschitz boundary $\Gamma := \partial\Omega$. We prove a Korn-type inequality in $\mathring{\mathrm{H}}(\mathrm{Curl};\Omega)$ for eventually non-symmetric tensor fields T mapping Ω to $\mathbb{R}^{N\times N}$. More precisely, there exists a positive constant c, such that

$$c\,\|T\|_{\mathrm{L}^2(\Omega)} \le \|\mathrm{sym}\,T\|_{\mathrm{L}^2(\Omega)} + \|\mathrm{Curl}\,T\|_{\mathrm{L}^2(\Omega)}$$

holds for all tensor fields $T \in \mathring{\mathrm{H}}(\mathrm{Curl};\Omega)$, where T belongs to $\mathring{\mathrm{H}}(\mathrm{Curl};\Omega)$, if $T \in \mathrm{H}(\mathrm{Curl};\Omega)$ has vanishing tangential trace on Γ. Thereby, the generalized Curl and tangential trace are defined as row-wise operations. For compatible tensor fields $T = \nabla v$ with vector fields $v \in \mathrm{H}^1(\Omega)$, for which $\nabla v_n, n = 1,\ldots,N$, are normal at $\partial\Omega$, the latter estimate reduces to a non-standard variant of the well known Korn's first inequality in $\mathbb{R}^N$

$$c\,\|\nabla v\|_{\mathrm{L}^2(\Omega)} \le \|\mathrm{sym}\,\nabla v\|_{\mathrm{L}^2(\Omega)}.$$

Our proof relies on three essential tools, namely

1. Maxwell estimate (Poincaré-type estimate),
2. Helmholtz' decomposition,
3. Korn's first inequality.

Faculty of Mathematics, University of Duisburg Essen, Essen, 45141, Germany
**Correspondence to: Patrizio Neff, Faculty of Mathematics, University of Duisburg Essen, Essen, 45141, Germany.*
†E-mail: patrizio.neff@uni-due.de

In [1], we already pointed out the importance of the Maxwell estimate and the related question of the Maxwell compactness property‡. Here, we mention the papers [4–10]. Results for the Helmholtz decomposition can be found in [6,8,10–16]. Nowadays, differential forms find prominent applications in numerical methods like Finite Element Exterior Calculus [17,18] or Discrete Exterior Calculus [19].

1.1. Differential forms

We may look at Ω as a smooth Riemannian manifold of dimension N with compact closure and connected Lipschitz continuous boundary Γ. The alternating differential forms of rank $q \in \{0,\dots,N\}$ on Ω, briefly q-forms, with square-integrable coefficients will be denoted by $\mathrm{L}^{2,q}(\Omega)$. The exterior derivative d and the co-derivative $\delta = \pm * \mathrm{d} *$ ($*$: Hodge's star operator) are formally skew-adjoint to each other, that is,

$$\forall\, E \in \overset{\circ}{\mathrm{C}}{}^{\infty,q}(\Omega) \quad H \in \overset{\circ}{\mathrm{C}}{}^{\infty,q+1}(\Omega) \qquad \langle \mathrm{d}E, H\rangle_{\mathrm{L}^{2,q+1}(\Omega)} = -\langle E, \delta H\rangle_{\mathrm{L}^{2,q}(\Omega)},$$

where the $\mathrm{L}^{2,q}(\Omega)$-scalar product is given by

$$\forall\, E, H \in \mathrm{L}^{2,q}(\Omega) \qquad \langle E, H\rangle_{\mathrm{L}^{2,q}(\Omega)} := \int_\Omega E \wedge *H.$$

Here, $\overset{\circ}{\mathrm{C}}{}^{\infty,q}(\Omega)$ denotes the space of compactly supported and smooth q-forms on Ω. Using this duality, we can define weak versions of d and δ. The corresponding standard Sobolev spaces are denoted by

$$\mathrm{D}^q(\Omega) := \left\{E \in \mathrm{L}^{2,q}(\Omega) \ : \ \mathrm{d}E \in \mathrm{L}^{2,q+1}(\Omega)\right\},$$
$$\Delta^q(\Omega) := \left\{H \in \mathrm{L}^{2,q}(\Omega) \ : \ \delta H \in \mathrm{L}^{2,q-1}(\Omega)\right\}.$$

The homogeneous tangential boundary condition $\tau_\Gamma E = 0$, where τ_Γ denotes the tangential trace, is generalized in the space

$$\overset{\circ}{\mathrm{D}}{}^q(\Omega) := \overline{\overset{\circ}{\mathrm{C}}{}^{\infty,q}(\Omega)},$$

where the closure is taken in $\mathrm{D}^q(\Omega)$. In classical terms, we have for smooth q-forms $\tau_\Gamma = \iota^*$ with the canonical embedding $\iota : \Gamma \hookrightarrow \overline{\Omega}$. An index 0 at the lower right position indicates vanishing derivatives, that is,

$$\overset{\circ}{\mathrm{D}}{}^q_0(\Omega) = \left\{E \in \overset{\circ}{\mathrm{D}}{}^q(\Omega) \ : \ \mathrm{d}E = 0\right\}, \quad \Delta^q_0(\Omega) = \left\{H \in \Delta^q(\Omega) \ : \ \delta H = 0\right\}.$$

By definition and density, we have

$$\Delta^q_0(\Omega) = (\mathrm{d}\overset{\circ}{\mathrm{D}}{}^{q-1}(\Omega))^\perp, \quad \Delta^q_0(\Omega)^\perp = \overline{\mathrm{d}\overset{\circ}{\mathrm{D}}{}^{q-1}(\Omega)},$$

where $\perp$ denotes the orthogonal complement with respect to the $\mathrm{L}^{2,q}(\Omega)$-scalar product and the closure is taken in $\mathrm{L}^{2,q}(\Omega)$. Hence, we obtain the $\mathrm{L}^{2,q}(\Omega)$-orthogonal decomposition, usually called Hodge–Helmholtz decomposition,

$$\mathrm{L}^{2,q}(\Omega) = \overline{\mathrm{d}\overset{\circ}{\mathrm{D}}{}^{q-1}(\Omega)} \oplus \Delta^q_0(\Omega), \tag{1.1}$$

where $\oplus$ denotes the orthogonal sum with respect to the $\mathrm{L}^{2,q}(\Omega)$-scalar product. In [7,10], the following crucial tool has been proved:

Lemma 1 (Maxwell compactness property)
For all q the embeddings

$$\overset{\circ}{\mathrm{D}}{}^q(\Omega) \cap \Delta^q(\Omega) \hookrightarrow \mathrm{L}^{2,q}(\Omega)$$

are compact.

As the first immediate consequence, the spaces of so called "harmonic Dirichlet forms"

$$\mathcal{H}^q(\Omega) := \overset{\circ}{\mathrm{D}}{}^q_0(\Omega) \cap \Delta^q_0(\Omega)$$

are finite dimensional. In classical terms, a q-form E belongs to $\mathcal{H}^q(\Omega)$, if

$$\mathrm{d}E = 0, \quad \delta E = 0, \quad \iota^* E = 0.$$

‡ *By "Maxwell estimate" and "Maxwell compactness property", we mean the estimates and compact embedding results used in the theory of Maxwell's equations.*

The dimension of $\mathcal{H}^q(\Omega)$ equals the $(N-q)$th Betti number of Ω. Because we assume the boundary Γ to be connected, the $(N-1)$th Betti number of Ω vanishes and therefore there are no Dirichlet forms of rank 1 besides zero, for example,

$$\mathcal{H}^1(\Omega) = \{0\}. \tag{1.2}$$

This condition on the domain Ω respectively its boundary Γ is satisfied, for example, for a ball or a torus.

By a usual indirect argument, we achieve another immediate consequence:

Lemma 2 (Poincaré estimate for differential forms)
For all q there exist positive constants $c_{p,q}$, such that for all $E \in \overset{\circ}{\mathrm{D}}{}^q(\Omega) \cap \Delta^q(\Omega) \cap \mathcal{H}^q(\Omega)^{\perp}$

$$\|E\|_{\mathrm{L}^{2,q}(\Omega)} \leq c_{p,q} \left(\|\mathrm{d}E\|^2_{\mathrm{L}^{2,q+1}(\Omega)} + \|\delta E\|^2_{\mathrm{L}^{2,q-1}(\Omega)} \right)^{1/2}.$$

Because

$$\mathrm{d}\overset{\circ}{\mathrm{D}}{}^{q-1}(\Omega) \subset \overset{\circ}{\mathrm{D}}{}^q_0(\Omega)$$

(note that $\mathrm{dd} = 0$ and $\delta\delta = 0$ hold even in the weak sense) we get by (1.1)

$$\mathrm{d}\overset{\circ}{\mathrm{D}}{}^{q-1}(\Omega) = \mathrm{d}\left(\overset{\circ}{\mathrm{D}}{}^{q-1}(\Omega) \cap \Delta_0^{q-1}(\Omega)\right) = \mathrm{d}\left(\overset{\circ}{\mathrm{D}}{}^{q-1}(\Omega) \cap \Delta_0^{q-1}(\Omega) \cap \mathcal{H}^{q-1}(\Omega)^{\perp}\right).$$

Now, Lemma 2 shows that $\mathrm{d}\overset{\circ}{\mathrm{D}}{}^{q-1}(\Omega)$ is already closed. Hence, we obtain a refinement of (1.1)

Lemma 3 (Hodge–Helmholtz decomposition for differential forms)
The decomposition

$$\mathrm{L}^{2,q}(\Omega) = \mathrm{d}\overset{\circ}{\mathrm{D}}{}^{q-1}(\Omega) \oplus \Delta_0^q(\Omega)$$

holds.

1.2. Functions and vector fields

Let us turn to the special case $q = 1$. In this case, we choose, for example, the identity as single global chart for Ω and use the canonical identification isomorphism for 1-forms (i.e., Riesz' representation theorem) with vector fields $\mathrm{d}x_n \cong e^n$, namely,

$$\sum_{n=1}^{N} v_n(x)\mathrm{d}x_n \cong v(x) = \begin{bmatrix} v_1(x) \\ \vdots \\ v_N(x) \end{bmatrix}, \quad x \in \Omega.$$

0-forms will be isomorphically identified with functions on Ω. Then, $\mathrm{d} \cong \mathrm{grad} = \nabla$ for 0-forms (functions) and $\delta \cong \mathrm{div} = \nabla\cdot$ for 1-forms (vector fields). Hence, the well known first order differential operators from vector analysis occur. Moreover, on 1-forms, we define a new operator $\mathrm{curl} :\cong \mathrm{d}$, which turns into the usual curl if $N = 3$ or $N = 2$. $\mathrm{L}^{2,q}(\Omega)$ equals the usual Lebesgue spaces of square integrable functions or vector fields on Ω with values in $\mathbb{R}^n$, $n := n_{N,q} := \binom{N}{q}$, which will be denoted by $\mathrm{L}^2(\Omega) := \mathrm{L}^2(\Omega, \mathbb{R}^n)$. $\mathrm{D}^0(\Omega)$ and $\Delta^1(\Omega)$ are identified with the standard Sobolev spaces

$$\mathrm{H}(\mathrm{grad}; \Omega) := \left\{ u \in \mathrm{L}^2(\Omega, \mathbb{R}) : \mathrm{grad}\, u \in \mathrm{L}^2(\Omega, \mathbb{R}^N) \right\} = \mathrm{H}^1(\Omega),$$
$$\mathrm{H}(\mathrm{div}; \Omega) := \left\{ v \in \mathrm{L}^2(\Omega, \mathbb{R}^N) : \mathrm{div}\, v \in \mathrm{L}^2(\Omega, \mathbb{R}) \right\},$$

respectively. Moreover, we may now identify $\mathrm{D}^1(\Omega)$ with

$$\mathrm{H}(\mathrm{curl}; \Omega) := \left\{ v \in \mathrm{L}^2(\Omega, \mathbb{R}^N) : \mathrm{curl}\, v \in \mathrm{L}^2(\Omega, \mathbb{R}^{(N-1)N/2}) \right\},$$

which is the well-known $\mathrm{H}(\mathrm{curl}; \Omega)$ for $N = 2, 3$. For example, for $N = 4$ we have

$$\mathrm{curl}\, v = \begin{bmatrix} \partial_1 v_2 - \partial_2 v_1 \\ \partial_1 v_3 - \partial_3 v_1 \\ \partial_1 v_4 - \partial_4 v_1 \\ \partial_2 v_3 - \partial_3 v_2 \\ \partial_2 v_4 - \partial_4 v_2 \\ \partial_3 v_4 - \partial_4 v_3 \end{bmatrix} \in \mathbb{R}^6$$

and for $N = 5$, we get $\operatorname{curl} v \in \mathbb{R}^{10}$. In general, the entries of the $(N-1)N/2$-vector $\operatorname{curl} v$ consist of all possible combinations of

$$\partial_n v_m - \partial_m v_n, \quad 1 \le n < m \le N.$$

Similarly, we obtain the closed subspaces

$$\overset{\circ}{\mathrm{H}}(\operatorname{grad};\Omega) = \overset{\circ}{\mathrm{H}}{}^1(\Omega), \quad \overset{\circ}{\mathrm{H}}(\operatorname{curl};\Omega)$$

as reincarnations of $\overset{\circ}{\mathrm{D}}{}^0(\Omega)$ and $\overset{\circ}{\mathrm{D}}{}^1(\Omega)$, respectively. We note

$$\overset{\circ}{\mathrm{H}}(\operatorname{grad};\Omega) = \overline{\overset{\circ}{\mathrm{C}}{}^\infty(\Omega)}, \quad \overset{\circ}{\mathrm{H}}(\operatorname{curl};\Omega) = \overline{\overset{\circ}{\mathrm{C}}{}^\infty(\Omega)},$$

where the closures are taken in the respective graph norms, and that in these Sobolev spaces the classical homogeneous scalar and tangential (compare with $N = 3$) boundary conditions

$$u|_\Gamma = 0, \quad \nu \times v|_\Gamma = 0$$

are generalized. Here, ν denotes the outward unit normal for Γ. Furthermore, we have the spaces of irrotational or solenoidal vector fields

$$\begin{aligned}
\mathrm{H}(\operatorname{curl}_0;\Omega) &= \{v \in \mathrm{H}(\operatorname{curl};\Omega) : \operatorname{curl} v = 0\}, \\
\overset{\circ}{\mathrm{H}}(\operatorname{curl}_0;\Omega) &= \left\{v \in \overset{\circ}{\mathrm{H}}(\operatorname{curl};\Omega) : \operatorname{curl} v = 0\right\}, \\
\mathrm{H}(\operatorname{div}_0;\Omega) &= \{v \in \mathrm{H}(\operatorname{div};\Omega) : \operatorname{div} v = 0\}.
\end{aligned}$$

Again, all these spaces are Hilbert spaces. Now, we have two compact embeddings

$$\overset{\circ}{\mathrm{H}}(\operatorname{grad};\Omega) \hookrightarrow \mathrm{L}^2(\Omega), \quad \overset{\circ}{\mathrm{H}}(\operatorname{curl};\Omega) \cap \mathrm{H}(\operatorname{div};\Omega) \hookrightarrow \mathrm{L}^2(\Omega),$$

that is, Rellich's selection theorem and the Maxwell compactness property. Moreover, the following Poincaré and Maxwell estimates hold:

Corollary 4 (Poincaré estimate for functions)
Let $c_p := c_{p,0}$. Then, for all functions $u \in \overset{\circ}{\mathrm{H}}(\operatorname{grad};\Omega)$

$$\|u\|_{\mathrm{L}^2(\Omega)} \le c_p \|\operatorname{grad} u\|_{\mathrm{L}^2(\Omega)}.$$

Corollary 5 (Maxwell estimate for vector fields)
Let $c_m := c_{p,1}$. Then, for all vector fields $v \in \overset{\circ}{\mathrm{H}}(\operatorname{curl};\Omega) \cap \mathrm{H}(\operatorname{div};\Omega)$

$$\|v\|_{\mathrm{L}^2(\Omega)} \le c_m \left(\|\operatorname{curl} v\|^2_{\mathrm{L}^2(\Omega)} + \|\operatorname{div} v\|^2_{\mathrm{L}^2(\Omega)}\right)^{1/2}.$$

We note that generally $\mathcal{H}^0(\Omega) = \{0\}$ and by (1.2) also $\mathcal{H}^1(\Omega) = \{0\}$. The appropriate Helmholtz decomposition for our needs is

Corollary 6 (Helmholtz decomposition for vector fields)

$$\mathrm{L}^2(\Omega) = \operatorname{grad} \overset{\circ}{\mathrm{H}}(\operatorname{grad};\Omega) \oplus \mathrm{H}(\operatorname{div}_0;\Omega)$$

1.3. Tensor fields

We extend our calculus to $(N \times N)$-tensor (matrix) fields. For vector fields v with components in $\mathrm{H}(\operatorname{grad};\Omega)$ and tensor fields T with rows in $\mathrm{H}(\operatorname{curl};\Omega)$ resp. $\mathrm{H}(\operatorname{div};\Omega)$, that is,

$$v = \begin{bmatrix} v_1 \\ \vdots \\ v_N \end{bmatrix}, \quad v_n \in \mathrm{H}(\operatorname{grad};\Omega), \quad T = \begin{bmatrix} {T_1}^t \\ \vdots \\ {T_N}^t \end{bmatrix}, \quad T_n \in \mathrm{H}(\operatorname{curl};\Omega) \text{ resp. } \mathrm{H}(\operatorname{div};\Omega)$$

for $n = 1, \ldots, N$, we define

$$\operatorname{Grad} v := \begin{bmatrix} \operatorname{grad}^t v_1 \\ \vdots \\ \operatorname{grad}^t v_N \end{bmatrix} = J_v = \nabla v, \quad \operatorname{Curl} T := \begin{bmatrix} \operatorname{curl}^t T_1 \\ \vdots \\ \operatorname{curl}^t T_N \end{bmatrix}, \quad \operatorname{Div} T := \begin{bmatrix} \operatorname{div} T_1 \\ \vdots \\ \operatorname{div} T_N \end{bmatrix},$$

where J_v denotes the Jacobian of v and ^{t} the transpose. We note that v and $\operatorname{Div} T$ are N-vector fields, T and $\operatorname{Grad} v$ are $(N \times N)$-tensor fields, whereas $\operatorname{Curl} T$ is a $(N \times (N-1)N/2)$-tensor field that may also be viewed as a totally anti-symmetric third order tensor field with entries

$$(\operatorname{Curl} T)_{ijk} = \partial_j T_{ik} - \partial_k T_{ij}.$$

The corresponding Sobolev spaces will be denoted by

$$\mathrm{H}(\mathrm{Grad};\Omega), \qquad \mathring{\mathrm{H}}(\mathrm{Grad};\Omega), \qquad \mathrm{H}(\mathrm{Div};\Omega), \qquad \mathrm{H}(\mathrm{Div}_0;\Omega),$$
$$\mathrm{H}(\mathrm{Curl};\Omega), \qquad \mathring{\mathrm{H}}(\mathrm{Curl};\Omega), \qquad \mathrm{H}(\mathrm{Curl}_0;\Omega), \qquad \mathring{\mathrm{H}}(\mathrm{Curl}_0;\Omega).$$

There are three crucial tools to prove our estimate. First, we have obvious consequences from Corollaries 4, 5, and 6:

Corollary 7 (Poincaré estimate for vector fields)
For all $v \in \mathring{\mathrm{H}}(\mathrm{Grad};\Omega)$

$$\|v\|_{\mathrm{L}^2(\Omega)} \leq c_p \, \|\operatorname{Grad} v\|_{\mathrm{L}^2(\Omega)}.$$

Corollary 8 (Maxwell estimate for tensor fields)
The estimate

$$\|T\|_{\mathrm{L}^2(\Omega)} \leq c_m \left(\|\operatorname{Curl} T\|^2_{\mathrm{L}^2(\Omega)} + \|\operatorname{Div} T\|^2_{\mathrm{L}^2(\Omega)} \right)^{1/2}$$

holds for all tensor fields $T \in \mathring{\mathrm{H}}(\mathrm{Curl};\Omega) \cap \mathrm{H}(\mathrm{Div};\Omega)$.

Corollary 9 (Helmholtz decomposition for tensor fields)

$$\mathrm{L}^2(\Omega) = \operatorname{Grad} \mathring{\mathrm{H}}(\mathrm{Grad};\Omega) \oplus \mathrm{H}(\mathrm{Div}_0;\Omega)$$

The last important tool is Korn's first inequality.

Lemma 10 (Korn's first inequality)
For all vector fields $v \in \mathring{\mathrm{H}}(\mathrm{Grad};\Omega)$

$$\|\operatorname{Grad} v\|_{\mathrm{L}^2(\Omega)} \leq \sqrt{2} \, \|\operatorname{sym} \operatorname{Grad} v\|_{\mathrm{L}^2(\Omega)}.$$

Here, we introduce the symmetric and skew-symmetric parts

$$\operatorname{sym} T := \frac{1}{2}(T + T^t), \quad \operatorname{skew} T := \frac{1}{2}(T - T^t)$$

of a $(N \times N)$-tensor $T = \operatorname{sym} T + \operatorname{skew} T$.

Remark 11
We note that the proof including the value of the constant is simple. By density, we may assume $v \in \mathring{\mathrm{C}}^\infty(\Omega)$. Twofold partial integration yields

$$\langle \partial_n v_m, \partial_m v_n \rangle_{\mathrm{L}^2(\Omega)} = \langle \partial_m v_m, \partial_n v_n \rangle_{\mathrm{L}^2(\Omega)}$$

and hence

$$2\,\|\operatorname{sym} \operatorname{Grad} v\|^2_{\mathrm{L}^2(\Omega)} = \frac{1}{2} \sum_{n,m=1}^{N} \|\partial_n v_m + \partial_m v_n\|^2_{\mathrm{L}^2(\Omega)} = \sum_{n,m=1}^{N} \left(\|\partial_n v_m\|^2_{\mathrm{L}^2(\Omega)} + \langle \partial_n v_m, \partial_m v_n \rangle_{\mathrm{L}^2(\Omega)} \right)$$
$$= \|\operatorname{Grad} v\|^2_{\mathrm{L}^2(\Omega)} + \|\operatorname{div} v\|^2_{\mathrm{L}^2(\Omega)} \geq \|\operatorname{Grad} v\|^2_{\mathrm{L}^2(\Omega)}.$$

More on Korn's first inequality can be found, for example, in [20].

2. Results

For tensor fields $T \in \mathrm{H}(\mathrm{Curl};\Omega)$, we define the semi-norm

$$|||T||| := \left(\|\operatorname{sym} T\|^2_{\mathrm{L}^2(\Omega)} + \|\operatorname{Curl} T\|^2_{\mathrm{L}^2(\Omega)} \right)^{1/2}.$$

The main step is to prove the following.

Lemma 12
Let $\hat{c} := \max\left\{2, \sqrt{5}c_m\right\}$. Then, for all $T \in \mathring{\mathrm{H}}(\mathrm{Curl};\Omega)$

$$\|T\|_{\mathrm{L}^2(\Omega)} \leq \hat{c}\,|\!|\!|T|\!|\!|.$$

Proof
Let $T \in \mathring{\mathrm{H}}(\mathrm{Curl};\Omega)$. According to Corollary 9, we orthogonally decompose

$$T = \mathrm{Grad}\, v + S \in \mathrm{Grad}\, \mathring{\mathrm{H}}\,(\mathrm{Grad};\Omega) \oplus \mathrm{H}(\mathrm{Div}_0;\Omega).$$

Then, $\mathrm{Curl}\, T = \mathrm{Curl}\, S$ and we observe $S \in \mathring{\mathrm{H}}(\mathrm{Curl};\Omega) \cap \mathrm{H}(\mathrm{Div}_0;\Omega)$ because

$$\mathrm{Grad}\, \mathring{\mathrm{H}}(\mathrm{Grad};\Omega) \subset \mathring{\mathrm{H}}\,(\mathrm{Curl}_0;\Omega). \tag{2.1}$$

By Corollary 8, we have

$$\|S\|_{\mathrm{L}^2(\Omega)} \leq c_m \,\|\mathrm{Curl}\, T\|_{\mathrm{L}^2(\Omega)}. \tag{2.2}$$

Then, by Lemma 10 and (2.2), we obtain

$$\|T\|^2_{\mathrm{L}^2(\Omega)} = \|\mathrm{Grad}\, v\|^2_{\mathrm{L}^2(\Omega)} + \|S\|^2_{\mathrm{L}^2(\Omega)} \leq 2\,\|\mathrm{sym}\,\mathrm{Grad}\, v\|^2_{\mathrm{L}^2(\Omega)} + \|S\|^2_{\mathrm{L}^2(\Omega)} \leq 4\,\|\mathrm{sym}\, T\|^2_{\mathrm{L}^2(\Omega)} + 5\,\|S\|^2_{\mathrm{L}^2(\Omega)},$$

which completes the proof. □

The immediate consequence is our main result.

Theorem 13
On $\mathring{\mathrm{H}}(\mathrm{Curl};\Omega)$ the norms $\|\cdot\|_{\mathrm{H}(\mathrm{Curl};\Omega)}$ and $|\!|\!|\cdot|\!|\!|$ are equivalent. In particular, $|\!|\!|\cdot|\!|\!|$ is a norm on $\mathring{\mathrm{H}}(\mathrm{Curl};\Omega)$ and there exists a positive constant c, such that

$$c\,\|T\|^2_{\mathrm{H}(\mathrm{Curl};\Omega)} \leq |\!|\!|T|\!|\!|^2 = \|\mathrm{sym}\, T\|^2_{\mathrm{L}^2(\Omega)} + \|\mathrm{Curl}\, T\|^2_{\mathrm{L}^2(\Omega)}$$

holds for all $T \in \mathring{\mathrm{H}}(\mathrm{Curl};\Omega)$.

Remark 14
For a skew-symmetric tensor field $T : \Omega \to \mathfrak{so}(N)$, our estimate reduces to a Poincaré inequality in disguise, because Curl T controls all partial derivatives of T (compare with [21]) and the homogeneous tangential boundary condition for T is implied by $T|_\Gamma = 0$.

Setting $T := \mathrm{Grad}\, v$, we obtain the following.

Remark 15 (Korn's first inequality: tangential-variant)
For all $v \in \mathring{\mathrm{H}}(\mathrm{Grad};\Omega)$

$$\|\mathrm{Grad}\, v\|_{\mathrm{L}^2(\Omega)} \leq \hat{c}\,\|\mathrm{sym}\,\mathrm{Grad}\, v\|_{\mathrm{L}^2(\Omega)} \tag{2.3}$$

holds by Lemma 12 and (2.1). This is just Korn's first inequality from Lemma 10 with a larger constant $\hat{c}$. Because Γ is connected, that is, $\mathcal{H}^1(\Omega) = \{0\}$, we even have

$$\mathrm{Grad}\, \mathring{\mathrm{H}}(\mathrm{Grad};\Omega) = \mathring{\mathrm{H}}(\mathrm{Curl}_0;\Omega).$$

Thus, (2.3) holds for all $v \in \mathrm{H}\,(\mathrm{Grad};\Omega)$ with $\mathrm{Grad}\, v \in \mathring{\mathrm{H}}(\mathrm{Curl}_0;\Omega)$, that is, with grad v_n, $n = 1, \ldots, N$, normal at Γ, which then extends Lemma 10 through the (apparently) weaker boundary condition.

The elementary arguments above apply certainly to much more general situations, for example, to not necessarily connected boundaries Γ and to tangential boundary conditions that are imposed only on parts of Γ. These discussions are left to forthcoming papers.

Acknowledgements

We thank the referee for pointing out a missing assumption in a preliminary version of the paper.

References

1. Neff P, Pauly D, Witsch K-J. A Canonical Extension of Korn's First Inequality to H(Curl) motivated by Gradient Plasticity with Plastic Spin. *Comptes Rendus Mathématique, Académie des Sciences Paris, Ser. I* 2011; **349**:1251–1254.
2. Neff P, Pauly D, Witsch K-J. On a Canonical Extension of Korn's First and poincare's Inequality to H(Curl), *Zapiski POMI*. *Journal of Mathematical Sciences (Springer New York)* 2011; **349**:115–125.
3. Neff P, Pauly D, Witsch K-J. A Korn's inequality for incompatible tensor fields. *Proceedings in Applied Mathematics and Mechanics (PAMM),* 11, 2011; 683–684.
4. Costabel M. A remark on the regularity of solutions of Maxwell's equations on Lipschitz domains. *Mathematical Methods in the Applied Sciences* 1990; **12**(4):365–368.
5. Kuhn P. Die Maxwellgleichung mit wechselnden Randbedingungen. *Dissertation*, Universität Essen, Fachbereich Mathematik, *Shaker*, 1999.
6. Picard R. Randwertaufgaben der verallgemeinerten Potentialtheorie. *Mathematical Methods in the Applied Sciences* 1981; **3**:218–228.
7. Picard R. An elementary proof for a compact imbedding result in generalized electromagnetic theory. *Mathematische Zeitschrift* 1984; **187**:151–164.
8. Picard R. Some decomposition theorems and their applications to non-linear potential theory and Hodge theory. *Mathematical Methods in the Applied Sciences* 1990; **12**:35–53.
9. Picard R, Weck N, Witsch K-J. Time-harmonic Maxwell equations in the exterior of perfectly conducting, irregular obstacles. *Analysis (Munich)* 2001; **21**:231–263.
10. Weck N. Maxwell's boundary value problems on Riemannian manifolds with nonsmooth boundaries. *Journal of Mathematical Analysis and Applications* 1974; **46**:410–437.
11. Friedrichs KO. Differential forms on Riemannian manifolds. *Communications on Pure and Applied Mathematics* 1955; **8**:551–590.
12. Pauly D. Hodge-Helmholtz decompositions of weighted Sobolev spaces in irregular exterior domains with inhomogeneous and anisotropic media. *Mathematical Methods in the Applied Sciences* 2008; **31**:1509–1543.
13. Sprößig W. On Helmholtz decompositions and their generalizations - An overview. *Mathematical Methods in the Applied Sciences* 2010; **33**:374–383.
14. Mitrea D, Mitrea M. Finite energy solutions of Maxwell's equations and constructive Hodge decompositions on nonsmooth Riemannian manifolds. *Indiana University Mathematics Journal* 2008; **57**(5):2061–2095.
15. Mitrea D, Mitrea M, Shaw M-C. Traces of differential forms on Lipschitz domains, the boundary de Rham complex, and Hodge decompositions. *Journal of Functional Analysis* 2002; **190**(2):339417.
16. Mitrea M. Sharp Hodge decompositions, Maxwell's equations and vector Poisson problems on nonsmooth, three-dimensional Riemannian manifolds. *Duke Mathematical Journal* 2004; **125**(3):467–547.
17. Arnold DN, Falk RS, Winther R. Finite element exterior calculus, homological techniques, and applications. *Acta Numerica* 2006; **15**:1–155.
18. Hiptmair R. Finite elements in computational electromagnetism. *Acta Numerica* 2002; **11**:237–339.
19. Hirani AN. Discrete exterior calculus. *Dissertation*, California Institute of Technology, 2003. http://thesis.library.caltech.edu/1885.
20. Neff P. On Korn's first inequality with nonconstant coefficients. *Proceedings of the Royal Society of Edinburg Section A* 2002; **132**:221–243.
21. Neff P, Münch I. Curl bounds Grad on SO(3). *ESAIM: Control, Optimisation and Calculus of Variations* 2008; **14**(1):148–159. DOI: 101051/cocv:2007050. Preprint 2455 http://www3.mathematik.tu-darmstadt.de/fb/mathe/bibliothek/preprints.html