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ON CONSTANTS IN MAXWELL INEQUALITIES FOR BOUNDED AND CONVEX DOMAINS

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It is shown that for a bounded and convex domain $\Omega \subset \mathbb{R}^3$, the Maxwell constants are bounded from below and above by the Friedrichs and Poincaré constants of Ω, respectively. Bibliography: 14 titles.

1. Introduction

Throughout this paper, we fix a bounded and convex domain $\Omega \subset \mathbb{R}^3$. Let us recall the well-known Poincaré[1] inequalities

$$\exists\, c_{\mathtt{p},0} > 0 \qquad \forall\, u \in \overset{\circ}{\mathsf{H}}{}^1 \qquad |u| \le c_{\mathtt{p},0} |\nabla u|, \tag{1}$$

$$\exists\, c_{\mathtt{p}} > 0 \qquad \forall\, u \in \mathsf{H}^1 \cap \mathbb{R}^\perp \qquad |u| \le c_{\mathtt{p}} |\nabla u|, \tag{2}$$

which can be deduced from, e.g., Rellich's selection theorem by standard indirect arguments. We assume that these constants are best possible, i.e., that

$$\frac{1}{c_{\mathtt{p},0}} := \inf_{0 \ne u \in \overset{\circ}{\mathsf{H}}{}^1} \frac{|\nabla u|}{|u|}, \quad \frac{1}{c_{\mathtt{p}}} := \inf_{0 \ne u \in \mathsf{H}^1 \cap \mathbb{R}^\perp} \frac{|\nabla u|}{|u|}.$$

Then $c_{\mathtt{p},0}$ and $c_{\mathtt{p}}$ are the well-known Friedrichs and Poincaré constants, respectively, which satisfy the relations

$$0 < c_{\mathtt{p},0}^2 = \frac{1}{\lambda_1} < \frac{1}{\mu_2} = c_{\mathtt{p}}^2,$$

where λ_1 and μ_2 are the first Dirichlet and the second Neumann eigenvalues of the Laplacian, respectively. By $\langle\,\cdot\,,\,\cdot\,\rangle$ and $|\,\cdot\,|$, we denote the standard inner product and the induced norm in L^2; the usual L^2-Sobolev spaces are denoted by H^1 and $\overset{\circ}{\mathsf{H}}{}^1$. The latter is defined to be the closure in H^1 of smooth and compactly supported test functions. All spaces and norms are defined on Ω. Moreover, we introduce the standard Sobolev spaces R for the rotation and D for the divergence by

$$\mathsf{R} := \{E \in \mathsf{L}^2 : \operatorname{rot} E \in \mathsf{L}^2\}, \quad \mathsf{D} := \{E \in \mathsf{L}^2 : \operatorname{div} E \in \mathsf{L}^2\},$$

where rot = curl and div should be understood in the usual distributional or weak sense. As before, we denote the closures of the test vector fields in the respective graph norms by $\overset{\circ}{\mathsf{R}}$ and $\overset{\circ}{\mathsf{D}}$. The subscript zero in this notation indicates a vanishing derivative, e.g.,

$$\mathsf{R}_0 := \{E \in \mathsf{R} : \operatorname{rot} E = 0\}, \quad \overset{\circ}{\mathsf{D}}_0 := \{E \in \overset{\circ}{\mathsf{D}} : \operatorname{div} E = 0\}.$$

Since Ω is convex, it is especially simply connected and its boundary is connected. Hence, the Neumann and Dirichlet fields of Ω vanish, i.e.,

$$\mathcal{H}_{\mathtt{N}} := \mathsf{R}_0 \cap \overset{\circ}{\mathsf{D}}_0 = \{0\} = \overset{\circ}{\mathsf{R}}_0 \cap \mathsf{D}_0 =: \mathcal{H}_{\mathtt{D}}.$$

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[1]The estimate (1) is often called the Friedrichs–Steklov inequality as well.

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By the Maxwell compactness properties (see [5–7, 12–14]), i.e., by the compactness of the two embeddings

$$\mathring{\mathsf{R}} \cap \mathsf{D} \hookrightarrow \mathsf{L}^2, \quad \mathsf{R} \cap \mathring{\mathsf{D}} \hookrightarrow \mathsf{L}^2,$$

and again by a standard indirect argument, the Maxwell inequalities hold:

$$\exists\, c_{\mathtt{m},\mathtt{t}} > 0 \qquad \forall\, E \in \mathring{\mathsf{R}} \cap \mathsf{D} \qquad |E| \le c_{\mathtt{m},\mathtt{t}} \big(|\operatorname{rot} E|^2 + |\operatorname{div} E|^2 \big)^{1/2}, \tag{3}$$

$$\exists\, c_{\mathtt{m},\mathtt{n}} > 0 \qquad \forall\, H \in \mathsf{R} \cap \mathring{\mathsf{D}} \qquad |H| \le c_{\mathtt{m},\mathtt{n}} \big(|\operatorname{rot} H|^2 + |\operatorname{div} H|^2 \big)^{1/2}. \tag{4}$$

Again, we assume that the constants are best possible, i.e.,

$$\frac{1}{c_{\mathtt{m},\mathtt{t}}^2} := \inf_{0 \ne E \in \mathring{\mathsf{R}} \cap \mathsf{D}} \frac{|\operatorname{rot} E|^2 + |\operatorname{div} E|^2}{|E|^2},$$

$$\frac{1}{c_{\mathtt{m},\mathtt{n}}^2} := \inf_{0 \ne H \in \mathsf{R} \cap \mathring{\mathsf{D}}} \frac{|\operatorname{rot} H|^2 + |\operatorname{div} H|^2}{|H|^2}.$$

The notation $c_{\mathtt{m},\mathtt{t}}$ and $c_{\mathtt{m},\mathtt{n}}$ should indicate the homogeneous tangential and normal boundary condition, respectively. To the best of the author's knowledge, there are no general bounds for the Maxwell constants $c_{\mathtt{m},\mathtt{t}}$ and $c_{\mathtt{m},\mathtt{n}}$. On the other hand, estimates for $c_{\mathtt{m},\mathtt{t}}$ and $c_{\mathtt{m},\mathtt{n}}$, at least from above, are very important for applications such as preconditioning or a priori and a posteriori error estimation for numerical methods (see, e.g., [8, 10]).

In the present paper, we will prove that

$$c_{\mathtt{p},0} \le c_{\mathtt{m},\mathtt{t}} \le c_{\mathtt{m},\mathtt{n}} = c_{\mathtt{p}} \le \frac{\operatorname{diam}(\Omega)}{\pi}. \tag{5}$$

We note that (5) has already been well known in two dimensions, where even the inequalities

$$c_{\mathtt{p},0} < c_{\mathtt{m},\mathtt{t}} = c_{\mathtt{m},\mathtt{n}} = c_{\mathtt{p}} \le \frac{\operatorname{diam}(\Omega)}{\pi}$$

hold,[2] see the Appendix. However, (5) is new in three dimensions. Furthermore, the last inequality in (5) has been proved in the famous paper of Payne and Weinberger [9], where the optimality of this estimate was also shown.

2. Results and proofs

We start with an inequality for irrotational fields.

Lemma 1. *For all* $E \in \nabla \mathring{\mathsf{H}}^1 \cap \mathsf{D}$ *and all* $H \in \nabla \mathsf{H}^1 \cap \mathring{\mathsf{D}}$, *we have*

$$|E| \le c_{\mathtt{p},0} |\operatorname{div} E|, \quad |H| \le c_{\mathtt{p}} |\operatorname{div} H|.$$

Proof. Let $\varphi \in \mathring{\mathsf{H}}^1$ with $E = \nabla\varphi$. By (1), we get

$$|E|^2 = \langle E, \nabla\varphi \rangle = -\langle \operatorname{div} E, \varphi \rangle \le |\operatorname{div} E||\varphi| \le c_{\mathtt{p},0} |\operatorname{div} E||\nabla\varphi| = c_{\mathtt{p},0} |\operatorname{div} E||E|.$$

Let $\varphi \in \mathsf{H}^1$ with $H = \nabla\varphi$ and $\varphi \perp \mathbb{R}$. Since $H \in \mathring{\mathsf{D}}$, by (2), we obtain

$$|H|^2 = \langle H, \nabla\varphi \rangle = -\langle \operatorname{div} H, \varphi \rangle \le |\operatorname{div} H||\varphi| \le c_{\mathtt{p}} |\operatorname{div} H||\nabla\varphi| = c_{\mathtt{p}} |\operatorname{div} H||H|. \qquad \square$$

Remark 2. Lemma 1 extends to arbitrary Lipschitz domains $\Omega \subset \mathbb{R}^N$, $N \in \mathbb{N}$.

[2] In 2D, the equality $c_{\mathtt{m},\mathtt{t}} = c_{\mathtt{m},\mathtt{n}} = c_{\mathtt{p}}$ holds even for general Lipschitz domains, see the Appendix.

As usual in the theory of Maxwell's equations, we need another crucial tool, the Helmholtz decompositions of vector fields into irrotational and solenoidal vector fields. For convex domains, these decompositions are very simple. We have

$$\mathsf{L}^2 = \nabla\mathring{\mathsf{H}}^1 \oplus \operatorname{rot}\mathsf{R}, \quad \mathsf{L}^2 = \nabla\mathsf{H}^1 \oplus \operatorname{rot}\mathring{\mathsf{R}}, \tag{6}$$

where $\oplus$ denotes the orthogonal sum in L^2. We note that

$$\mathring{\mathsf{R}}_0 = \nabla\mathring{\mathsf{H}}^1, \quad \mathsf{R}_0 = \nabla\mathsf{H}^1, \quad \mathsf{D}_0 = \operatorname{rot}\mathsf{R}, \quad \mathring{\mathsf{D}}_0 = \operatorname{rot}\mathring{\mathsf{R}}.$$

Moreover, setting

$$\mathring{\mathcal{R}} := \mathring{\mathsf{R}} \cap \operatorname{rot}\mathsf{R}, \quad \mathcal{R} := \mathsf{R} \cap \operatorname{rot}\mathring{\mathsf{R}},$$

we have

$$\mathring{\mathsf{R}} = \nabla\mathring{\mathsf{H}}^1 \oplus \mathring{\mathcal{R}}, \quad \mathsf{R} = \nabla\mathsf{H}^1 \oplus \mathcal{R} \tag{7}$$

and see that

$$\operatorname{rot}\mathring{\mathsf{R}} = \operatorname{rot}\mathring{\mathcal{R}}, \quad \operatorname{rot}\mathsf{R} = \operatorname{rot}\mathcal{R}.$$

We note that all occurring spaces of range type are closed subspaces of L^2: this immediately follows by the estimates (1)-(4). More details about the Helmholtz decompositions can be found, e.g., in [5].

To get similar inequalities for solenoidal vector fields as in Lemma 1, we need a crucial lemma from [1, Theorem 2.17] (see also [2–4, 11] for related partial results).

Lemma 3. *Let E belong to $\mathring{\mathsf{R}} \cap \mathsf{D}$ or $\mathsf{R} \cap \mathring{\mathsf{D}}$. Then $E \in \mathsf{H}^1$ and*

$$|\nabla E|^2 \leq |\operatorname{rot} E|^2 + |\operatorname{div} E|^2. \tag{8}$$

We emphasize that for $E \in \mathring{\mathsf{H}}^1$ and any domain $\Omega \subset \mathbb{R}^3$,

$$|\nabla E|^2 = |\operatorname{rot} E|^2 + |\operatorname{div} E|^2 \tag{9}$$

holds since $-\Delta = \operatorname{rot}\operatorname{rot} - \nabla\operatorname{div}$. This formula is no longer valid if E satisfies just the tangential or normal boundary condition. However, for convex domains the inequality (8) remains true.

Lemma 4. *For all vector fields E in $\mathring{\mathsf{R}} \cap \operatorname{rot}\mathsf{R}$ or $\mathsf{R} \cap \operatorname{rot}\mathring{\mathsf{R}}$, we have*

$$|E| \leq c_{\mathtt{p}} |\operatorname{rot} E|.$$

Proof. Let $E \in \operatorname{rot}\mathsf{R} = \operatorname{rot}\mathcal{R}$, and let $\Phi \in \mathcal{R}$ with $\operatorname{rot}\Phi = E$. Then $\Phi \in \mathsf{H}^1$ by Lemma 3 since $\mathcal{R} = \mathsf{R} \cap \mathring{\mathsf{D}}_0$. Moreover, $\Phi = \operatorname{rot}\Psi$ can be represented by some $\Psi \in \mathring{\mathsf{R}}$. Hence, for any constant vector $a \in \mathbb{R}^3$ we have $\langle \Phi, a \rangle = 0$. Thus, Φ belongs to $\mathsf{H}^1 \cap (\mathbb{R}^3)^\perp$. Then, by Lemma 3, for $E \in \mathring{\mathsf{R}} \cap \operatorname{rot}\mathsf{R}$ we get

$$|E|^2 = \langle E, \operatorname{rot}\Phi \rangle = \langle \operatorname{rot} E, \Phi \rangle \leq |\operatorname{rot} E||\Phi| \leq c_{\mathtt{p}} |\operatorname{rot} E||\nabla\Phi| \leq c_{\mathtt{p}} |\operatorname{rot} E||\underbrace{\operatorname{rot}\Phi}_{=E}|.$$

If $E \in \mathsf{R} \cap \operatorname{rot}\mathring{\mathsf{R}}$, then there exists $\Phi \in \mathring{\mathsf{R}}$ with $\operatorname{rot}\Phi = E$. As before, by Lemma 3, we see that $E \in \mathsf{H}^1 \cap (\mathbb{R}^3)^\perp$ and $|E| \leq c_{\mathtt{p}} |\nabla E| \leq c_{\mathtt{p}} |\operatorname{rot} E|$, which completes the proof. □

Theorem 5. *For all vector fields $E \in \mathring{\mathsf{R}} \cap \mathsf{D}$ and $H \in \mathsf{R} \cap \mathring{\mathsf{D}}$, the inequalities*

$$|E|^2 \leq c_{\mathtt{p},0}^2 |\operatorname{div} E|^2 + c_{\mathtt{p}}^2 |\operatorname{rot} E|^2, \quad |H|^2 \leq c_{\mathtt{p}}^2 |\operatorname{div} H|^2 + c_{\mathtt{p}}^2 |\operatorname{rot} H|^2$$

hold, i.e., $c_{\mathtt{m,t}}, c_{\mathtt{m,n}} \leq c_{\mathtt{p}}$. Moreover, $c_{\mathtt{p},0} \leq c_{\mathtt{m,t}} \leq c_{\mathtt{m,n}} = c_{\mathtt{p}} \leq \operatorname{diam}(\Omega)/\pi$.

Proof. By the Helmholtz decomposition (6), we have

$$\mathring{\mathsf{R}} \cap \mathsf{D} \ni E = E_\nabla + E_{\mathrm{rot}} \in \nabla\mathring{\mathsf{H}}^1 \oplus \operatorname{rot}\mathsf{R}$$

with $E_\nabla \in \nabla\mathring{\mathsf{H}}^1 \cap \mathsf{D}$, $E_{\mathrm{rot}} \in \mathring{\mathsf{R}} \cap \operatorname{rot}\mathsf{R}$, and $\operatorname{div} E_\nabla = \operatorname{div} E$, as well as $\operatorname{rot} E_{\mathrm{rot}} = \operatorname{rot} E$. By Lemma 1, Lemma 4, and the orthogonality, we obtain

$$|E|^2 = |E_\nabla|^2 + |E_{\mathrm{rot}}|^2 \le c_{\mathtt{p},0}^2 |\operatorname{div} E|^2 + c_{\mathtt{p}}^2 |\operatorname{rot} E|^2.$$

Similarly, we have

$$\mathsf{R} \cap \mathring{\mathsf{D}} \ni H = H_\nabla + H_{\mathrm{rot}} \in \nabla\mathsf{H}^1 \oplus \operatorname{rot}\mathring{\mathsf{R}}$$

with $H_\nabla \in \nabla\mathsf{H}^1 \cap \mathring{\mathsf{D}}$, $H_{\mathrm{rot}} \in \mathsf{R} \cap \operatorname{rot}\mathring{\mathsf{R}}$, $\operatorname{div} H_\nabla = \operatorname{div} H$, and $\operatorname{rot} H_{\mathrm{rot}} = \operatorname{rot} H$. As before,

$$|H|^2 = |H_\nabla|^2 + |H_{\mathrm{rot}}|^2 \le c_{\mathtt{p}}^2 |\operatorname{div} H|^2 + c_{\mathtt{p}}^2 |\operatorname{rot} H|^2.$$

This proves the upper bounds. For the lower bounds, let λ_1 be the first Dirichlet eigenvalue of the negative Laplacian $-\Delta$, i.e.,

$$\frac{1}{c_{\mathtt{p},0}^2} = \lambda_1 = \inf_{0 \neq \varphi \in \mathring{\mathsf{H}}^1} \frac{|\nabla\varphi|^2}{|\varphi|^2},$$

and let $u \in \mathring{\mathsf{H}}^1$ be an eigenfunction to λ_1. Note that u satisfies the relation

$$\forall\, \varphi \in \mathring{\mathsf{H}}^1 \quad \langle \nabla u, \nabla\varphi \rangle = \lambda_1 \langle u, \varphi \rangle.$$

Then $0 \neq E := \nabla u \in \nabla\mathring{\mathsf{H}}^1 \cap \mathsf{D} = \mathring{\mathsf{R}}_0 \cap \mathsf{D}$ and $-\operatorname{div} E = -\operatorname{div}\nabla u = \lambda_1 u$. By (3) and (1), we have

$$|E| \le c_{\mathtt{m},\mathtt{t}} |\operatorname{div} E| = c_{\mathtt{m},\mathtt{t}} \lambda_1 |u| \le c_{\mathtt{m},\mathtt{t}} \lambda_1 c_{\mathtt{p},0} |\nabla u| = \frac{c_{\mathtt{m},\mathtt{t}}}{c_{\mathtt{p},0}} |E|,$$

yielding $c_{\mathtt{p},0} \le c_{\mathtt{m},\mathtt{t}}$. Now, let μ_2 be the second Neumann eigenvalue of the negative Laplacian $-\Delta$, i.e.,

$$\frac{1}{c_{\mathtt{p}}^2} = \mu_2 = \inf_{0 \neq \varphi \in \mathsf{H}^1 \cap \mathbb{R}^\perp} \frac{|\nabla\varphi|^2}{|\varphi|^2},$$

and let $u \in \mathsf{H}^1 \cap \mathbb{R}^\perp$ be an eigenfunction to μ_2. Note that u satisfies the relation

$$\forall\, \varphi \in \mathsf{H}^1 \cap \mathbb{R}^\perp \quad \langle \nabla u, \nabla\varphi \rangle = \mu_2 \langle u, \varphi \rangle,$$

which holds even for all $\varphi \in \mathsf{H}^1$. Then $0 \neq H := \nabla u$ belongs to $\nabla\mathsf{H}^1 \cap \mathring{\mathsf{D}} = \mathsf{R}_0 \cap \mathring{\mathsf{D}}$ and $-\operatorname{div} H = -\operatorname{div}\nabla u = \mu_2 u$. By (4) and (2), we have

$$|H| \le c_{\mathtt{m},\mathtt{n}} |\operatorname{div} H| = c_{\mathtt{m},\mathtt{n}} \mu_2 |u| \le c_{\mathtt{m},\mathtt{n}} \mu_2 c_{\mathtt{p}} |\nabla u| = \frac{c_{\mathtt{m},\mathtt{n}}}{c_{\mathtt{p}}} |H|,$$

yielding $c_{\mathtt{p}} \le c_{\mathtt{m},\mathtt{n}}$, which completes the proof. □

Remark 6. It follows from the proof that the lower bounds $c_{\mathtt{p},0} \le c_{\mathtt{m},\mathtt{t}}$, as well as $c_{\mathtt{p}} \le c_{\mathtt{m},\mathtt{n}}$, remain true in a more general situation, i.e., for bounded Lipschitz[3] domains $\Omega \subset \mathbb{R}^N$.

[3]The Lipschitz assumption can also be weakened. It is sufficient to assume that Ω admits the Maxwell compactness properties.

Appendix A. The 2D case

In 2D, there are scalar- and vector-valued rotations: $\operatorname{rot} = \operatorname{div} R$ and $\vec{\operatorname{rot}} = \nabla^\perp = R\nabla$. The scalar-valued rotation is just the divergence div after a 90°-rotation

$$R := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

and the vector-valued one is actually the gradient ∇ followed by the same rotation R. Hence, applying the Poincaré estimates to the potentials generated by the Helmholtz decompositions, we immediately get the desired estimates. Of course, this special trick works only in 2D.

More precisely, let $\Omega \subset \mathbb{R}^2$ be a Lipschitz domain. Then Lemma 1 holds by Remark 2. Moreover, even a stronger version of Lemma 4 is true.

Lemma 7. *For all vector fields $E \in \mathring{\mathsf{R}} \cap \vec{\operatorname{rot}}\,\mathsf{H}^1$ and $H \in \mathsf{R} \cap \vec{\operatorname{rot}}\,\mathring{\mathsf{H}}^1$, we have*

$$|E| \leq c_{\mathtt{p}} |\operatorname{rot} E|, \quad |H| \leq c_{\mathtt{p},0} |\operatorname{rot} H|.$$

This follows immediately from Lemma 1 by the arguments below.

Proof. Let $E \in \mathring{\mathsf{R}} \cap \vec{\operatorname{rot}}\,\mathsf{H}^1 = \mathring{\mathsf{R}} \cap R\,\nabla\mathsf{H}^1$. Then $H := RE \in \mathring{\mathsf{D}} \cap \nabla\mathsf{H}^1$. By Lemma 1, we get

$$|E| = |H| \leq c_{\mathtt{p}} |\operatorname{div} H| = c_{\mathtt{p}} |\operatorname{rot} E|.$$

If $H \in \mathsf{R} \cap \vec{\operatorname{rot}}\,\mathring{\mathsf{H}}^1 = \mathsf{R} \cap R\,\nabla\mathring{\mathsf{H}}^1$, then $E := RH \in \mathsf{D} \cap \nabla\mathring{\mathsf{H}}^1$. By Lemma 1, we obtain

$$|H| = |E| \leq c_{\mathtt{p},0} |\operatorname{div} E| = c_{\mathtt{p},0} |\operatorname{rot} H|. \qquad \square$$

We note that in 2D, the Helmholtz decompositions read as follows:

$$\mathsf{L}^2 = \nabla\mathring{\mathsf{H}}^1 \oplus \mathcal{H}_{\mathtt{D}} \oplus \vec{\operatorname{rot}}\,\mathsf{H}^1, \quad \mathsf{L}^2 = \nabla\mathsf{H}^1 \oplus \mathcal{H}_{\mathtt{N}} \oplus \vec{\operatorname{rot}}\,\mathring{\mathsf{H}}^1,$$

where owing to the possibly nontrivial topology (we do not assume Ω to be convex), nonvanishing Dirichlet or Neumann fields may exist.

Theorem 8. *For all vector fields $E \in \mathring{\mathsf{R}} \cap \mathsf{D} \cap \mathcal{H}_{\mathtt{D}}^\perp$ and $H \in \mathsf{R} \cap \mathring{\mathsf{D}} \cap \mathcal{H}_{\mathtt{N}}^\perp$, we have*

$$|E|^2 \leq c_{\mathtt{p},0}^2 |\operatorname{div} E|^2 + c_{\mathtt{p}}^2 |\operatorname{rot} E|^2, \quad |H|^2 \leq c_{\mathtt{p}}^2 |\operatorname{div} H|^2 + c_{\mathtt{p},0}^2 |\operatorname{rot} H|^2,$$

i.e., $c_{\mathtt{m},\mathtt{t}}, c_{\mathtt{m},\mathtt{n}} \leq c_{\mathtt{p}}$. Moreover, even $c_{\mathtt{p},0} < c_{\mathtt{m},\mathtt{t}} = c_{\mathtt{m},\mathtt{n}} = c_{\mathtt{p}}$.

Proof. Following the proof of Theorem 5, we use the Helmholtz decomposition to show that for $E \in \mathring{\mathsf{R}} \cap \mathsf{D} \cap \mathcal{H}_{\mathtt{D}}^\perp$,

$$E = E_\nabla + E_{\mathrm{rot}} \in \nabla\mathring{\mathsf{H}}^1 \oplus \vec{\operatorname{rot}}\,\mathsf{H}^1$$

with $E_\nabla \in \nabla\mathring{\mathsf{H}}^1 \cap \mathsf{D}$, $E_{\mathrm{rot}} \in \mathring{\mathsf{R}} \cap \vec{\operatorname{rot}}\,\mathsf{H}^1$, $\operatorname{div} E_\nabla = \operatorname{div} E$, and $\operatorname{rot} E_{\mathrm{rot}} = \operatorname{rot} E$. Hence, by Lemma 1, Lemma 7, and orthogonality, we obtain

$$|E|^2 = |E_\nabla|^2 + |E_{\mathrm{rot}}|^2 \leq c_{\mathtt{p},0}^2 |\operatorname{div} E|^2 + c_{\mathtt{p}}^2 |\operatorname{rot} E|^2,$$

and the estimate for H follows analogously. For the lower bounds, we look again at the second Neumann eigenvalue $\mu_2 = 1/c_{\mathtt{p}}^2$ of $-\Delta$ and the corresponding eigenfunction $u \in \mathsf{H}^1 \cap \mathbb{R}^\perp$ with $\nabla u \in \mathring{\mathsf{D}}$ and $-\Delta u = \mu_2 u$. Then, as before, $0 \neq H := \nabla u$ belongs to $\nabla\mathsf{H}^1 \cap \mathring{\mathsf{D}} = \mathsf{R}_0 \cap \mathring{\mathsf{D}} \cap \mathcal{H}_{\mathtt{N}}^\perp$ with $-\operatorname{div} H = -\operatorname{div} \nabla u = \mu_2 u$. By the definition of $c_{\mathtt{m},\mathtt{n}}$ and relation (2) (for nonconvex Ω), we have

$$|H| \leq c_{\mathtt{m},\mathtt{n}} |\operatorname{div} H| = c_{\mathtt{m},\mathtt{n}} \mu_2 |u| \leq c_{\mathtt{m},\mathtt{n}} \mu_2 c_{\mathtt{p}} |\nabla u| = \frac{c_{\mathtt{m},\mathtt{n}}}{c_{\mathtt{p}}} |H|,$$

yielding $c_{\mathtt{p}} \leq c_{\mathtt{m,n}}$. On the other hand, $E := RH \in \mathsf{D}_0 \cap \overset{\circ}{\mathsf{R}} \cap \mathcal{H}_{\mathtt{D}}^{\perp}$ and

$$|E| \leq c_{\mathtt{m,t}} |\operatorname{rot} E| = c_{\mathtt{m,t}} |\operatorname{div} H| = c_{\mathtt{m,t}} \mu_2 |u| \leq c_{\mathtt{m,t}} \mu_2 c_{\mathtt{p}} |\nabla u| = \frac{c_{\mathtt{m,t}}}{c_{\mathtt{p}}} |E|,$$

showing $c_{\mathtt{p}} \leq c_{\mathtt{m,t}}$. □

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