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ON A CANONICAL EXTENSION OF KORN'S FIRST AND POINCARÉ'S INEQUALITIES TO H(CURL)

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We prove a Korn-type inequality in $\overset{\circ}{\mathsf{H}}(\mathrm{Curl};\Omega,\mathbb{R}^{3\times3})$ *for tensor fields* P *mapping* Ω *to* $\mathbb{R}^{3\times3}$*. More precisely, let* $\Omega\subset\mathbb{R}^3$ *be a bounded domain with connected Lipschitz boundary* $\partial\Omega$*. Then there exists a constant* $c>0$ *such that*

$$c\,\|P\|_{\mathsf{L}^2(\Omega,\mathbb{R}^{3\times3})}\le\|\operatorname{sym}P\|_{\mathsf{L}^2(\Omega,\mathbb{R}^{3\times3})}+\|\operatorname{Curl}P\|_{\mathsf{L}^2(\Omega,\mathbb{R}^{3\times3})}\tag{0.1}$$

for all tensor fields $P\in\overset{\circ}{\mathsf{H}}(\mathrm{Curl};\Omega,\mathbb{R}^{3\times3})$*, i.e., all* $P\in\mathsf{H}(\mathrm{Curl};\Omega,\mathbb{R}^{3\times3})$ *with vanishing tangential trace on* $\partial\Omega$*. Here the rotation and tangential trace are defined row-wise. For compatible* P *of the form* $P=\nabla v$*,* $\operatorname{Curl}P=0$*, where* $v\in\mathsf{H}^1(\Omega,\mathbb{R}^3)$ *is a vector field with components* v_n *for which* ∇v_n *are normal at* $\partial\Omega$*, estimate* (0.1) *is reduced to a nonstandard variant of Korn's first inequality:*

$$c\,\|\nabla v\|_{\mathsf{L}^2(\Omega,\mathbb{R}^{3\times3})}\le\|\operatorname{sym}\nabla v\|_{\mathsf{L}^2(\Omega,\mathbb{R}^{3\times3})}\,.$$

For skew-symmetric P *(with* $\operatorname{sym}P=0$*), estimate* (0.1) *generates a nonstandard version of Poincaré's inequality. Therefore, the estimate is a generalization of two classical inequalities of Poincaré and Korn. Bibliography: 24 titles.*

1. Introduction: infinitesimal gradient plasticity

The motivation for our new estimate is a formulation of infinitesimal gradient plasticity [2]. Our model is taken from [9]. Let $\Omega\subset\mathbb{R}^3$ be a bounded domain. The goal is to find a displacement $u:[0,\infty)\times\Omega\mapsto\mathbb{R}^3$ and a possibly nonsymmetric plastic distortion tensor $P:[0,\infty)\times\Omega\mapsto\mathbb{R}^{3\times3}$ such that in $[0,\infty)\times\Omega$ the relations

$$\operatorname{Div}\sigma=f,\quad \sigma=2\mu\operatorname{sym}(\nabla u-P)+\lambda\operatorname{tr}(\nabla u-P)\,\mathrm{id},\tag{1.1}$$

$$\dot P\in\Phi(\Sigma),\quad \Sigma=\sigma-2\mu\operatorname{sym}P-\mu L_c^2\operatorname{Curl}\operatorname{Curl}P\tag{1.2}$$

hold. The system is completed by the boundary conditions

$$u(t,x)=0,\quad \nu(x)\times P(t,x)=0\qquad\text{for all }\ (t,x)\in[0,\infty)\times\partial\Omega\tag{1.3}$$

and the initial condition $P(0,x)=0$ for all $x\in\Omega$. The underlying thermodynamic potential including the plastic gradients in form of the dislocation density tensor $\operatorname{Curl}P$ is

$$\int_\Omega \mu|\operatorname{sym}(\nabla u-P)|^2+\frac{\lambda}{2}|\operatorname{tr}(\nabla u-P)|^2-f\cdot u+\mu|\operatorname{sym}P|^2+\frac{\mu}{2}L_c^2|\operatorname{Curl}P|^2.$$

Here μ,λ are the elastic Lamé moduli and σ is the symmetric Cauchy stress tensor. The system is driven by nonzero body forces denoted by f. The outward normal to the boundary $\partial\Omega$ is denoted by ν, and the plastic distortion P is required to satisfy row-wise the homogeneous tangential boundary condition which means that the boundary $\partial\Omega$ is a perfect conductor regarding the plastic distortion.[1]

Moreover, $\Phi:\mathbb{R}^{3\times3}\mapsto\mathbb{R}^{3\times3}$ is a monotone multivalued flow function with $\Phi(0)=0$ and $\Phi(\mathbb{R}^{3\times3}_{\mathrm{sym}})\subset\mathbb{R}^{3\times3}_{\mathrm{sym}}$. In general, Σ is not symmetric even if P is symmetric. Thus the plastic inhomogeneity is responsible for the plastic spin (the possible nonsymmetry of P). The mathematically suitable space for the symmetric plastic distortion P is the classical space $\mathsf{H}(\mathrm{curl};\Omega)$ for each row of P, see [2,13]. This case appears when choosing $\Phi:\mathbb{R}^{3\times3}\mapsto\mathbb{R}^{3\times3}_{\mathrm{sym}}$.

In the large scale limit $L_c\to0$, we recover a classical elasto-plasticity model with local kinematic hardening and symmetric plastic strain $\varepsilon_P:=\operatorname{sym}P$, since then $\dot P\in\mathbb{R}^{3\times3}_{\mathrm{sym}}$.

The uniqueness of classical solutions for rate-independent and rate-dependent formulations of this model is shown in [9]. A more difficult existence question for the rate-independent model in terms of a weak reformulation is addressed in [9]. First numerical results for a simplified rate-independent irrotational formulation (no plastic spin, i.e., a symmetric plastic distortion P) are presented in [13], cf. [19]. In [3], the model was extended to rate-independent isotropic hardening based on the concept of a dissipation function defined in terms of the equivalent

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[1]This homogeneous tangential boundary condition on P is consistent with the condition $\nu\times\nabla u=0$ on $\partial\Omega$, which follows from the condition $u=0$ on $\partial\Omega$.

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plastic strain. From a modeling point of view, it is strongly preferable to have again only the symmetric (rate) part of the plastic distortion P appear in the dissipation potential.

The existence and uniqueness can be settled by recasting the model as a variational inequality, if it is possible to define a bilinear form that is coercive with respect to appropriate spaces. This program was carried out for other variants of the model in [3]. It had to remain basically open for the above system (1.1)–(1.3). In this case, the appropriate space for the plastic distortion P is the completion $\overset{\circ}{\mathsf{H}}_{\mathrm{sym}}(\mathrm{Curl};\Omega)$ of the linear space

$$\{P \in \mathsf{C}^\infty(\overline{\Omega}, \mathbb{R}^{3\times 3}) \,:\, P_n \text{ normal at } \partial\Omega,\, n = 1,2,3\}$$

with respect to the norm $\|\!|\cdot|\!\|$, where P_n are the columns of P^t and

$$|\!|\!|P|\!|\!|^2 := \|\operatorname{sym} P\|^2_{\mathsf{L}^2(\Omega)} + \|\operatorname{Curl} P\|^2_{\mathsf{L}^2(\Omega)}\,.$$

Despite first appearances, this quadratic form indeed defines a norm on $\overset{\circ}{\mathsf{H}}_{\mathrm{sym}}(\mathrm{Curl};\Omega)$, since for skew-symmetric tensors the operator Curl controls all derivatives, i.e., the full gradient. This was already mentioned in [9]. Thus $\overset{\circ}{\mathsf{H}}_{\mathrm{sym}}(\mathrm{Curl};\Omega)$ is a Hilbert space. However, it is not clear whether this space admits a (linear and bounded) tangential trace operator. Since only $\|\operatorname{sym} P\|_{\mathsf{L}^2(\Omega)}$ appears in $|\!|\!|P|\!|\!|$, it is also not at all clear whether the skew-symmetric part of P is controlled. Therefore, the crucial embedding

$$\overset{\circ}{\mathsf{H}}_{\mathrm{sym}}(\mathrm{Curl};\Omega) \subset \mathsf{L}^2(\Omega)$$

might fail. As a consequence of our main result in this paper, we show that nevertheless the equality

$$\overset{\circ}{\mathsf{H}}_{\mathrm{sym}}(\mathrm{Curl};\Omega) = \overset{\circ}{\mathsf{H}}(\mathrm{Curl};\Omega)$$

holds with equivalent norms in the case where the domain Ω has a connected Lipschitz boundary.

The result of this paper was announced in [10,11], and the forthcoming paper [12] will be devoted to the case $\Omega \subset \mathbb{R}^N$.

For the proof of our main result (0.1), we combine techniques from electro-magnetic and elastic theory, namely, the Helmholtz decomposition, the Maxwell compactness property, and Korn's first inequality. Their basic variants are well-known results, which can be found in many books, e.g., [6] and the literature cited therein. More sophisticated and related versions are presented, e.g., in [14,16–18,23] for Maxwell's equations and in [1,8] for Korn's inequality.

The paper at hand is organized as follows. After this preliminary motivation we introduce our notation, definitions, and provide some background results. In Sec. 3, we give the proof of our main estimate. In the last Section 4, we establish a connection with a related result by Garroni *et al.* [4] for the two-dimensional case.

2. Definitions and preliminaries

Let Ω be a bounded domain in $\mathbb{R}^3$ with connected Lipschitz continuous boundary $\Gamma := \partial\Omega$.

2.1. Functions and vector fields. The usual Lebesgue spaces of square integrable functions, vector fields, or tensor fields on Ω, with values in $\mathbb{R}$, $\mathbb{R}^3$, or $\mathbb{R}^{3\times 3}$, respectively, will be denoted by $\mathsf{L}^2(\Omega)$. Moreover, we introduce the standard Sobolev spaces

$$\begin{aligned}
\mathsf{H}(\mathrm{grad};\Omega) &= \{u \in \mathsf{L}^2(\Omega) \,:\, \operatorname{grad} u \in \mathsf{L}^2(\Omega)\},\\
\|u\|^2_{\mathsf{H}(\mathrm{Grad};\Omega)} &:= \|u\|^2_{\mathsf{L}^2(\Omega)} + \|\operatorname{grad} u\|^2_{\mathsf{L}^2(\Omega)}\,,\\
\mathsf{H}(\mathrm{curl};\Omega) &= \{v \in \mathsf{L}^2(\Omega) \,:\, \operatorname{curl} v \in \mathsf{L}^2(\Omega)\},\\
\|v\|^2_{\mathsf{H}(\mathrm{curl};\Omega)} &:= \|v\|^2_{\mathsf{L}^2(\Omega)} + \|\operatorname{curl} v\|^2_{\mathsf{L}^2(\Omega)}\,,\\
\mathsf{H}(\mathrm{div};\Omega) &= \{v \in \mathsf{L}^2(\Omega) \,:\, \operatorname{div} v \in \mathsf{L}^2(\Omega)\},\\
\|v\|^2_{\mathsf{H}(\mathrm{div};\Omega)} &:= \|v\|^2_{\mathsf{L}^2(\Omega)} + \|\operatorname{div} v\|^2_{\mathsf{L}^2(\Omega)}\,.
\end{aligned}$$

The space $\mathsf{H}(\mathrm{grad};\Omega)$ is often denoted by $\mathsf{H}^1(\Omega)$. Furthermore, we define closed subspaces $\overset{\circ}{\mathsf{H}}(\mathrm{grad};\Omega)$, $\overset{\circ}{\mathsf{H}}(\mathrm{curl};\Omega)$ as the completions under the respective norms of the scalar (respectively, vector-valued) space $\overset{\circ}{\mathsf{C}}{}^\infty(\Omega)$ of compactly supported and smooth test functions (respectively, vector fields). In the latter Sobolev spaces, the usual homogeneous scalar (respectively, tangential) boundary conditions

$$u|_\Gamma = 0, \quad \nu \times v|_\Gamma = 0$$

are generalized, where ν denotes the outward unit normal at Γ. We note in passing that $\nu \times v|_\Gamma = 0$ is equivalent to $\tau \cdot v|_\Gamma = 0$ for all tangential directions τ at Γ, which means that v is normal at Γ. Furthermore, we need the spaces of irrotational or solenoidal vector fields

$$\mathsf{H}(\mathrm{curl}_0;\Omega) := \{v \in \mathsf{H}(\mathrm{curl};\Omega) \,:\, \mathrm{curl}\, v = 0\},$$

$$\overset{\circ}{\mathsf{H}}(\mathrm{curl}_0;\Omega) := \{v \in \overset{\circ}{\mathsf{H}}(\mathrm{curl};\Omega) \,:\, \mathrm{curl}\, v = 0\},$$

$$\mathsf{H}(\mathrm{div}_0;\Omega) := \{v \in \mathsf{H}(\mathrm{div};\Omega) \,:\, \mathrm{div}\, v = 0\},$$

where the subscript 0 indicates the vanishing of curl or div, respectively. All these spaces are Hilbert spaces. For example, in classical terms we have $v \in \overset{\circ}{\mathsf{H}}(\mathrm{curl}_0;\Omega)$ if and only if

$$\mathrm{curl}\, v = 0, \quad \nu \times v|_\Gamma = 0.$$

For an introduction to these spaces, see [6, pp. 11–12, 148] or [5, p. 26]. The most important tool for our analysis is the compact embedding

$$\overset{\circ}{\mathsf{H}}(\mathrm{curl};\Omega) \cap \mathsf{H}(\mathrm{div};\Omega) \hookrightarrow \mathsf{L}^2(\Omega),$$

which is often referred to as the "Maxwell compactness property," see [6, p. 158] and [16,18,21,23,24]. The first immediate consequence is that the space of so-called "harmonic Dirichlet fields"

$$\mathcal{H}(\Omega) := \overset{\circ}{\mathsf{H}}(\mathrm{curl}_0;\Omega) \cap \mathsf{H}(\mathrm{div}_0;\Omega)$$

is finite-dimensional. A vector field v belonging to $\mathcal{H}(\Omega)$ means in classical terms that

$$\mathrm{curl}\, v = 0, \quad \mathrm{div}\, v = 0, \quad \nu \times v|_\Gamma = 0.$$

The dimension of $\mathcal{H}(\Omega)$ equals the second Betti number of Ω, see [6, p. 159] and [15, Theorem 1]. Since we assume the boundary Γ to be connected, there are no Dirichlet fields except the zero one, i.e.,

$$\mathcal{H}(\Omega) = \{0\}.$$

This condition on the domain Ω resp. its boundary Γ is satisfied, e.g., for a ball or a torus.

By a usual indirect argument we achieve another immediate consequence, see [6, p. 158, Theorem 8.9] or [5, Lemma 3.4].

Lemma 1 (Maxwell estimate for vector fields)**.** *There exists a positive constant c_m such that for all vector fields $v \in \overset{\circ}{\mathsf{H}}(\mathrm{curl};\Omega) \cap \mathsf{H}(\mathrm{div};\Omega)$,*

$$\|v\|_{\mathsf{L}^2(\Omega)} \le c_m \big(\|\mathrm{curl}\, v\|^2_{\mathsf{L}^2(\Omega)} + \|\mathrm{div}\, v\|^2_{\mathsf{L}^2(\Omega)} \big)^{1/2}.$$

By the definition of weak divergence, the projection theorem, and Rellich's selection theorem [6, p. 14], we obtain from [6, p. 148, Theorem 8.3] or [22, Lemma 3.5], [7, Theorem 3.45] the following result.

Lemma 2 (Helmholtz decomposition for vector fields)**.** *We have the orthogonal decomposition*

$$\mathsf{L}^2(\Omega) = \mathrm{grad}\, \overset{\circ}{\mathsf{H}}(\mathrm{grad};\Omega) \oplus \mathsf{H}(\mathrm{div}_0;\Omega).$$

2.2. Tensor fields. We extend our calculus to 3×3-tensor (matrix) fields. For vector fields v with components in $\mathsf{H}(\mathrm{grad};\Omega)$ and tensor fields T with rows in $\mathsf{H}(\mathrm{curl};\Omega)$ (respectively, $\mathsf{H}(\mathrm{div};\Omega)$), i.e.,

$$v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}, \quad v_n \in \mathsf{H}(\mathrm{grad};\Omega),$$

$$T^t = [T_1\, T_2\, T_3], \quad T_n \in \mathsf{H}(\mathrm{curl};\Omega) \quad (\text{respectively, } \mathsf{H}(\mathrm{div};\Omega)),$$

we define

$$\mathrm{Grad}\, v := \begin{bmatrix} \mathrm{grad}^t v_1 \\ \mathrm{grad}^t v_2 \\ \mathrm{grad}^t v_3 \end{bmatrix} = J_v = \nabla v, \quad \mathrm{Curl}\, T := \begin{bmatrix} \mathrm{curl}^t T_1 \\ \mathrm{curl}^t T_2 \\ \mathrm{curl}^t T_3 \end{bmatrix}, \quad \mathrm{Div}\, T := \begin{bmatrix} \mathrm{div}\, T_1 \\ \mathrm{div}\, T_2 \\ \mathrm{div}\, T_3 \end{bmatrix},$$

where J_v denotes the Jacobian of v and ^{t} stands for the transpose. We note that v and $\operatorname{Div} T$ are vector fields, whereas T, $\operatorname{Curl} T$, and $\operatorname{Grad} v$ are tensor fields. The corresponding Sobolev spaces will be denoted by $\mathsf{H}(\operatorname{Grad};\Omega)$, $\overset{\circ}{\mathsf{H}}(\operatorname{Grad};\Omega)$, $\mathsf{H}(\operatorname{Curl};\Omega)$, $\overset{\circ}{\mathsf{H}}(\operatorname{Curl};\Omega)$, $\mathsf{H}(\operatorname{Curl}_0;\Omega)$, $\overset{\circ}{\mathsf{H}}(\operatorname{Curl}_0;\Omega)$, $\mathsf{H}(\operatorname{Div};\Omega)$, $\mathsf{H}(\operatorname{Div}_0;\Omega)$. As usual, we denote by $\operatorname{sym} T := 1/2(T + T^t)$ the symmetric part of a tensor T.

Let us now present our three crucial tools to prove the new estimate. First we have the following obvious consequences of Lemmas 1 and 2.

Corollary 3 (Maxwell estimate for tensor fields)**.** *For all* $T \in \overset{\circ}{\mathsf{H}}(\operatorname{Curl};\Omega) \cap \mathsf{H}(\operatorname{Div};\Omega)$,

$$\|T\|_{\mathsf{L}^2(\Omega)} \leq c_m \big(\|\operatorname{Curl} T\|^2_{\mathsf{L}^2(\Omega)} + \|\operatorname{Div} T\|^2_{\mathsf{L}^2(\Omega)} \big)^{1/2}.$$

Corollary 4 (Helmholtz decomposition for tensor fields)**.** *We have the orthogonal decomposition*

$$\mathsf{L}^2(\Omega) = \operatorname{Grad} \overset{\circ}{\mathsf{H}}(\operatorname{Grad};\Omega) \oplus \mathsf{H}(\operatorname{Div}_0;\Omega).$$

The third important tool is Korn's first inequality, see [6, p. 207] or [20, p. 54].

Lemma 5 (Korn's first inequality)**.** *For all* $v \in \overset{\circ}{\mathsf{H}}(\operatorname{Grad};\Omega)$,

$$\|\operatorname{Grad} v\|_{\mathsf{L}^2(\Omega)} \leq \sqrt{2}\, \|\operatorname{sym} \operatorname{Grad} v\|_{\mathsf{L}^2(\Omega)}.$$

3. Main results

For tensor fields $T \in \mathsf{H}(\operatorname{Curl};\Omega)$, we define the seminorm

$$|\!|\!| T |\!|\!| := \big(\|\operatorname{sym} T\|^2_{\mathsf{L}^2(\Omega)} + \|\operatorname{Curl} T\|^2_{\mathsf{L}^2(\Omega)} \big)^{1/2}.$$

Lemma 6. *Let* $\widehat{c} := \max\{2, \sqrt{5} c_m\}$. *Then for all* $T \in \overset{\circ}{\mathsf{H}}(\operatorname{Curl};\Omega)$,

$$\|T\|_{\mathsf{L}^2(\Omega)} \leq \widehat{c}\, |\!|\!| T |\!|\!| .$$

Proof. Let $T \in \overset{\circ}{\mathsf{H}}(\operatorname{Curl};\Omega)$. According to Corollary 4, we orthogonally decompose

$$T = \operatorname{Grad} v + S \in \operatorname{Grad} \overset{\circ}{\mathsf{H}}(\operatorname{Grad};\Omega) \oplus \mathsf{H}(\operatorname{Div}_0;\Omega).$$

Then $\operatorname{Curl} T = \operatorname{Curl} S$, and we observe that $S \in \overset{\circ}{\mathsf{H}}(\operatorname{Curl};\Omega) \cap \mathsf{H}(\operatorname{Div}_0;\Omega)$, since

$$\operatorname{Grad} \overset{\circ}{\mathsf{H}}(\operatorname{Grad};\Omega) \subset \overset{\circ}{\mathsf{H}}(\operatorname{Curl}_0;\Omega). \tag{3.1}$$

By Corollary 3, we have

$$\|S\|_{\mathsf{L}^2(\Omega)} \leq c_m \|\operatorname{Curl} T\|_{\mathsf{L}^2(\Omega)}. \tag{3.2}$$

Then, by Lemma 5 and (3.2), we easily obtain

$$\|T\|^2_{\mathsf{L}^2(\Omega)} = \|\operatorname{Grad} v\|^2_{\mathsf{L}^2(\Omega)} + \|S\|^2_{\mathsf{L}^2(\Omega)} \leq 2 \|\operatorname{sym} \operatorname{Grad} v\|^2_{\mathsf{L}^2(\Omega)} + \|S\|^2_{\mathsf{L}^2(\Omega)} \leq 4 \|\operatorname{sym} T\|^2_{\mathsf{L}^2(\Omega)} + 5 \|S\|^2_{\mathsf{L}^2(\Omega)},$$

which completes the proof. □

An immediate consequence is the following theorem.

Theorem 7. *On* $\overset{\circ}{\mathsf{H}}(\operatorname{Curl};\Omega)$ *the norms* $\|\cdot\|_{\mathsf{H}(\operatorname{Curl};\Omega)}$ *and* $|\!|\!| \cdot |\!|\!|$ *are equivalent. In particular,* $|\!|\!| \cdot |\!|\!|$ *is a norm on* $\overset{\circ}{\mathsf{H}}(\operatorname{Curl};\Omega)$ *and there exists* $c > 0$ *such that*

$$c \|T\|_{\mathsf{H}(\operatorname{Curl};\Omega)} \leq \|\operatorname{sym} T\|_{\mathsf{L}^2(\Omega)} + \|\operatorname{Curl} T\|_{\mathsf{L}^2(\Omega)} \quad \textit{for all} \;\; T \in \overset{\circ}{\mathsf{H}}(\operatorname{Curl};\Omega).$$

3.1. Consequences for irrotational tensors: Korn's first inequality. Picking irrotational tensor fields T or setting $T := \operatorname{Grad} v$, we obtain, by Lemma 6 and (3.1), the following.

Corollary 8 (Korn's first inequality: tangential variants)**.**

(i) $\|T\|_{\mathsf{L}^2(\Omega)} \leq \widehat{c}\,\|\operatorname{sym} T\|_{\mathsf{L}^2(\Omega)}$ *for all tensor fields* $T \in \overset{\circ}{\mathsf{H}}(\operatorname{Curl}_0;\Omega)$.

(ii) $\|\operatorname{Grad} v\|_{\mathsf{L}^2(\Omega)} \leq \widehat{c}\,\|\operatorname{sym}\operatorname{Grad} v\|_{\mathsf{L}^2(\Omega)}$ *for all vector fields* $v \in \mathsf{H}(\operatorname{Grad};\Omega)$ *with* $\operatorname{Grad} v \in \overset{\circ}{\mathsf{H}}(\operatorname{Curl}_0;\Omega)$.

(iii) $\|\operatorname{Grad} v\|_{\mathsf{L}^2(\Omega)} \leq \widehat{c}\,\|\operatorname{sym}\operatorname{Grad} v\|_{\mathsf{L}^2(\Omega)}$ *for all vector fields* $v \in \overset{\circ}{\mathsf{H}}(\operatorname{Grad};\Omega)$.

These are different but equivalent versions of Korn's first inequality from Lemma 5 with a larger constant $\widehat{c}$, since the boundary Γ is connected, i.e., $\mathcal{H}(\Omega) = \{0\}$, and hence

$$\operatorname{Grad} \overset{\circ}{\mathsf{H}}(\operatorname{Grad};\Omega) = \overset{\circ}{\mathsf{H}}(\operatorname{Curl}_0;\Omega).$$

Thus, e.g., (ii) holds, where the boundary condition means that $\operatorname{Grad} v_n$, $n = 1,2,3$, are normal at Γ, which then extends Lemma 5 through the (apparently) weaker boundary condition.

3.2. Consequences for skew-symmetric tensors: Poincaré's inequality. Taking the special skew-symmetric tensor fields

$$T = \begin{bmatrix} 0 & 0 & u \\ 0 & 0 & 0 \\ -u & 0 & 0 \end{bmatrix} \cong u, \quad \operatorname{Curl} T = \begin{bmatrix} \partial_2 u & -\partial_1 u & 0 \\ 0 & 0 & 0 \\ 0 & -\partial_3 u & \partial_2 u \end{bmatrix}$$

with some scalar function u, we obtain by Lemma 6 (since now $\operatorname{Curl} T$ is as good as $\operatorname{grad} u$, and, in view of

$$\nu \times T|_{\partial\Omega} = \begin{bmatrix} \nu_2 u|_{\partial\Omega} & -\nu_1 u|_{\partial\Omega} & 0 \\ 0 & 0 & 0 \\ 0 & -\nu_3 u|_{\partial\Omega} & \nu_2 u|_{\partial\Omega} \end{bmatrix}$$

$\nu \times T|_{\partial\Omega} = 0$ is equivalent to $u|_{\partial\Omega} = 0$) the following result.

Corollary 9 (Poincaré's inequality)**.** *For all special skew-symmetric tensor fields T in* $\overset{\circ}{\mathsf{H}}(\operatorname{Curl};\Omega)$*, i.e., for all functions* $u \in \overset{\circ}{\mathsf{H}}(\operatorname{grad};\Omega)$ *with* $u \cong T$*,*

$$\|u\|_{\mathsf{L}^2(\Omega)} \leq \widehat{c}\,\|\operatorname{grad} u\|_{\mathsf{L}^2(\Omega)}\,.$$

Proof. We have $T \in \overset{\circ}{\mathsf{H}}(\operatorname{Curl};\Omega)$ if and only if $u \in \overset{\circ}{\mathsf{H}}(\operatorname{grad};\Omega)$. Moreover,

$$2\,\|u\|^2_{\mathsf{L}^2(\Omega)} = \|T\|^2_{\mathsf{L}^2(\Omega)} \leq \widehat{c}^2\,\|\operatorname{Curl} T\|^2_{\mathsf{L}^2(\Omega)} \leq 2\widehat{c}^2\,\|\operatorname{grad} u\|^2_{\mathsf{L}^2(\Omega)}\,.$$

□

We note that the latter corollary also remains true for general skew-symmetric tensor fields $T \in \overset{\circ}{\mathsf{H}}(\operatorname{Curl};\Omega)$ and vector fields $v \in \overset{\circ}{\mathsf{H}}(\operatorname{Grad};\Omega)$ with

$$T = \begin{bmatrix} 0 & v_1 & v_2 \\ -v_1 & 0 & v_3 \\ -v_2 & -v_3 & 0 \end{bmatrix} \cong v.$$

4. The two-dimensional case

Let Ω be a bounded domain in $\mathbb{R}^2$ with connected Lipschitz continuous boundary Γ, which is equivalent (in $\mathbb{R}^2$) to the topological property that Ω is simply connected. For tensor fields $T : \Omega \mapsto \mathbb{R}^{2\times 2}$, we analogously define the Curl-operator row-wise by the formula

$$\operatorname{Curl} T = \operatorname{Curl} \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} = \begin{bmatrix} \operatorname{curl}\,[T_{11}\,T_{12}]^t \\ \operatorname{curl}\,[T_{21}\,T_{22}]^t \end{bmatrix} = \begin{bmatrix} \partial_1 T_{12} - \partial_2 T_{11} \\ \partial_1 T_{22} - \partial_2 T_{21} \end{bmatrix},$$

where now curl denotes the two-dimensional scalar rotation and $\operatorname{Curl} T : \Omega \mapsto \mathbb{R}^2$ is a vector. With the appropriate changes, Lemma 6 and Theorem 7 hold as well. In particular, there exists a positive constant c such that

$$c\,\|T\|_{\mathsf{L}^2(\Omega)} \leq \|\operatorname{sym} T\|_{\mathsf{L}^2(\Omega)} + \|\operatorname{Curl} T\|_{\mathsf{L}^2(\Omega)}$$

for all $T \in \overset{\circ}{\mathsf{H}}(\mathrm{Curl};\Omega)$.

During the preparation of our paper, we become aware that a related two-dimensional result may be inferred from the paper [4] by Garroni *et al.* Instead of the tangential boundary conditions $\nu \times P|_\Gamma = 0$, they impose the normalization condition

$$\int_\Omega \operatorname{skew} T = 0. \tag{4.1}$$

Let us define the total variation measure of the distribution $\operatorname{Curl} T$ for $T \in \mathsf{L}^1(\Omega)$ by

$$|\operatorname{Curl} T|_\Omega := \sup_{\substack{v \in \overset{\circ}{\mathsf{C}}{}^1(\Omega) \\ \|v\|_{\mathsf{L}^\infty(\Omega)} \le 1}} \langle T, \operatorname{CoGrad} v\rangle_{\mathsf{L}^2(\Omega)}, \quad \operatorname{CoGrad} v := \begin{bmatrix} \partial_2 v_1 & -\partial_1 v_1 \\ \partial_2 v_2 & -\partial_1 v_2 \end{bmatrix}.$$

We note that

$$\langle T, \operatorname{CoGrad} v\rangle_{\mathsf{L}^2(\Omega)} = \int_\Omega T_{11}\,\partial_2 v_1 - T_{12}\,\partial_1 v_1 + T_{21}\,\partial_2 v_2 - T_{22}\,\partial_1 v_2.$$

Using partial integration, i.e., the relation $\langle T, \operatorname{CoGrad} v\rangle_{\mathsf{L}^2(\Omega)} = \langle \operatorname{Curl} T, v\rangle_{\mathsf{L}^2(\Omega)}$ for $v \in \overset{\circ}{\mathsf{C}}{}^1(\Omega)$, it is easy to see that $|\operatorname{Curl} T|_\Omega = \|\operatorname{Curl} T\|_{\mathsf{L}^1(\Omega)}$ if $\operatorname{Curl} T \in \mathsf{L}^1(\Omega)$. In [4, Theorem 9] it is shown that for Ω having a Lipschitz boundary and a special "slicing" property, there exists a constant $c > 0$ such that

$$c\,\|T\|_{\mathsf{L}^2(\Omega)} \le \|\operatorname{sym} T\|_{\mathsf{L}^2(\Omega)} + |\operatorname{Curl} T|_\Omega$$

for all $T \in \mathsf{L}^1(\Omega)$ with (4.1). Their proof essentially uses the fact that in $\mathbb{R}^2$ the operators curl and div can be exchanged by a simple rotation, i.e., $\operatorname{curl}[v_1, v_2]^t = \operatorname{div}[-v_2, v_1]^t$. Thus, such a strong result may not be true in higher space dimensions $N \ge 3$, and it is an open question whether the normalization condition (4.1) can be exchanged with the more natural tangential boundary condition.

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