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TWO-SIDED A POSTERIORI ERROR BOUNDS FOR ELECTRO-MAGNETOSTATIC PROBLEMS

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This paper is concerned with the derivation of computable and guaranteed upper and lower bounds of the difference between exact and approximate solutions of a boundary value problem for static Maxwell equations. Our analysis is based upon purely functional argumentation and does not invoke specific properties of the approximation method. For this reason, the estimates derived in the paper at hand are applicable to any approximate solution that belongs to the corresponding energy space. Such estimates (also called error majorants of the functional type) have been derived earlier for elliptic problems. Bibliography: 24 titles.

Dedicated to Professor N. Uraltseva

1. Introduction and notation

The main goal of the present paper is to derive guaranteed and computable upper and lower bounds of the difference between the exact solution of an electro-magnetostatic boundary value problem and any approximation from the corresponding energy space. We discuss the method with the paradigm of a prototypical electro-magnetostatic problem. Its generalized statement is given by integral identity (2.6). We show that (as in many other problems of mathematical physics) certain transformations of (2.6) lead to guaranteed and fully computable majorants and minorants of the approximation error. However, the case considered here has specific features, which make (at some points) the derivation procedure different from, e.g., what has earlier been applied to other elliptic type problems. This is because the corresponding differential operator has a nonzero kernel (which contains curl-free vector fields) and the set of trial functions in (2.6) is restricted to a rather special affine manifold. For these reasons, the derivation of estimates is based on the Helmholtz–Weyl decomposition of vector fields, on properties of the operators of orthogonal projection onto subspaces, and on a version of the Poincaré–Friedrichs estimate for the operator curl.

First, we show that the distance between the exact solution E and an approximate solution $\widetilde{E}$ (measured through the norm generated by the operator curl) is equal to the norm of the so-called *residual functional* $\ell_{\widetilde{E}}$ (cf. (3.2)). If $\widetilde{E}$ satisfies the boundary condition exactly, i.e., $\tau_{t,\gamma}\widetilde{E} = G$, then the latter functional vanishes if and only if $\operatorname{curl}\widetilde{E}$ coincides with $\operatorname{curl} E$. Lemma 9 shows that the error majorant can be expressed in terms of a certain norm of $\ell_{\widetilde{E}}$ (cf. (3.2)). However, in general, the computation of this norm is hardly possible, because it requires the supremum over an infinite number of functions.

Theorem 12 provides a computable form of the upper bound. The corresponding estimate (3.12) shows that the error majorant is the sum of five terms, which can be thought of as penalties for possible violations of relations (2.1)–(2.4) and of the prescribed boundary condition. It contains only known functions and global constants depending on geometrical properties of the domain. Moreover, it is easy to see that the upper bound vanishes if and only if $\widetilde{E}$ coincides with the exact solution E and the "free variable" Y occurring in the estimate coincides with $\mu^{-1}\operatorname{curl}\widetilde{E}$. Also, we show that the estimates derived are sharp in the sense that estimates (3.13) and (3.14) have no irremovable gap between the left- and right-hand sides (Remark 14). Finally, in Sec. 4, we derive lower estimates of the difference between exact and approximate solutions. The corresponding result is presented by Theorem 18. This estimate is also computable, guaranteed, and sharp, provided that the approximation exactly satisfy the prescribed boundary conditions.

Throughout this paper, we consider an open bounded domain $\Omega \subset \mathbb{R}^3$ with Lipschitz continuous boundary γ and denote the corresponding outward unit normal vector by n; E and H stand for electric and magnetic vector fields, respectively, while ε and μ denote positive definite, symmetric matrices with measurable, bounded coefficients, which describe properties of media (dielectricity and permeability, respectively). For the sake of brevity, the matrices (matrix-valued functions) with such properties are called "admissible." We note that the

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corresponding inverse matrices are admissible as well. In particular, there exists a constant $c_\mu > 0$ such that for a.a. $x \in \Omega$, we have

$$c_\mu |\xi|^2 \le \mu^{-1}(x)\xi \cdot \xi \quad \text{for any} \quad \xi \in \mathbb{R}^3. \tag{1.1}$$

We denote by $\mathsf{L}^2(\Omega)$ the usual Hilbert space of square integrable functions over Ω and by $\mathsf{H}(\Omega)$ the Hilbert space of L^2-vector fields, i.e., $\mathsf{L}^2(\Omega, \mathbb{R}^3)$. For the sake of simplicity, we restrict our analysis to the case of real-valued functions and vector fields. The generalization to complex-valued spaces is straightforward.

The orthogonality and orthogonal sum with respect to the scalar product of $\mathsf{H}(\Omega)$ is denoted by $\perp$ and $\oplus$, respectively, i.e., $\Phi \perp \Psi$ if

$$\langle \Phi, \Psi \rangle_\Omega := \int_\Omega \Phi \cdot \Psi \, d\lambda = 0,$$

where λ denotes the Lebesgue measure. Moreover, by $\perp_\nu$ ($\oplus_\nu$) we indicate the orthogonality (orthogonal sum, respectively) in terms of the weighted L^2-scalar product $\langle \nu\Phi, \Psi \rangle_\Omega$ generated by an admissible matrix ν.

Throughout the paper we exploit the following functional spaces:

$$\begin{aligned}
\mathsf{H}(\operatorname{curl}, \Omega) &:= \{\Psi \in \mathsf{H}(\Omega) \mid \operatorname{curl}\Psi \in \mathsf{H}(\Omega)\}, \\
\mathsf{H}(\operatorname{curl}_0, \Omega) &:= \{\Psi \in \mathsf{H}(\operatorname{curl}, \Omega) \mid \operatorname{curl}\Psi = 0\}, \\
\mathsf{H}(\overset{\circ}{\operatorname{curl}}, \Omega) &:= \overline{\overset{\circ}{\mathsf{C}}{}^\infty(\Omega)}, \quad \text{closure in } \mathsf{H}(\operatorname{curl}, \Omega), \\
\mathsf{H}(\overset{\circ}{\operatorname{curl}}_0, \Omega) &:= \mathsf{H}(\overset{\circ}{\operatorname{curl}}, \Omega) \cap \mathsf{H}(\operatorname{curl}_0, \Omega).
\end{aligned}$$

We define similarily the spaces associated with the operators div and grad. Furthermore, we introduce the spaces (containing the so-called Dirichlet and Neumann fields)

$$\begin{aligned}
\mathcal{H}_{D,\varepsilon}(\Omega) &:= \mathsf{H}(\overset{\circ}{\operatorname{curl}}_0, \Omega) \cap \varepsilon^{-1}\mathsf{H}(\operatorname{div}_0, \Omega) = \{\Psi \in \mathsf{H}(\Omega) \mid \operatorname{curl}\Psi = 0, \operatorname{div}\varepsilon\Psi = 0, n \times \Psi|_\gamma = 0\}, \\
\mathcal{H}_{N,\mu}(\Omega) &:= \mathsf{H}(\operatorname{curl}_0, \Omega) \cap \mu^{-1}\mathsf{H}(\overset{\circ}{\operatorname{div}}_0, \Omega) = \{\Psi \in \mathsf{H}(\Omega) \mid \operatorname{curl}\Psi = 0, \operatorname{div}\mu\Psi = 0, n \cdot \mu\Psi|_\gamma = 0\}.
\end{aligned}$$

Henceforth we write $E \in \varepsilon^{-1}\mathsf{H}(\operatorname{div}_0, \Omega)$ if $\varepsilon E \in \mathsf{H}(\operatorname{div}_0, \Omega)$. These are finite-dimensional spaces whose dimensions are denoted by d_D and d_N, respectively. In fact, these numbers are equal to the so-called Betti numbers of Ω and depend only on topological properties of the domain (for a detailed presentation, see [10]). A basis of $\mathcal{H}_{D,\varepsilon}(\Omega)$ shall be given by special vector fields $\{H_1, \dots, H_{d_D}\}$.

Finally, we note that, being equipped with proper inner products, all the above-introduced functional spaces are Hilbert spaces.

The classical statement of the electro-magnetostatic problem for a given vector field F (driving force) and given ε and μ reads as follows: Find a magnetic field

$$H \in \mathsf{H}(\operatorname{curl}, \Omega) \cap \mu^{-1}\mathsf{H}(\overset{\circ}{\operatorname{div}}_0, \Omega) \cap \mathcal{H}_{N,\mu}(\Omega)^{\perp_\mu}$$

and the corresponding electric field

$$E \in \mathsf{H}(\overset{\circ}{\operatorname{curl}}, \Omega) \cap \varepsilon^{-1}\mathsf{H}(\operatorname{div}_0, \Omega) \cap \mathcal{H}_{D,\varepsilon}(\Omega)^{\perp_\varepsilon}$$

such that in Ω

$$\operatorname{curl} H = F, \quad \operatorname{curl} E = \mu H.$$

In other words, the problem is to find vector fields

$$H \in \mathsf{H}(\operatorname{curl}, \Omega) \cap \mu^{-1}\mathsf{H}(\operatorname{div}, \Omega) \text{ and } E \in \mathsf{H}(\operatorname{curl}, \Omega) \cap \varepsilon^{-1}\mathsf{H}(\operatorname{div}, \Omega)$$

such that

$$\begin{aligned}
\operatorname{curl} H &= F, & \operatorname{curl} E &= \mu H & &\text{in } \Omega, \\
\operatorname{div}\mu H &= 0, & \operatorname{div}\varepsilon E &= 0 & &\text{in } \Omega, \\
n \cdot \mu H|_\gamma &= 0, & n \times E|_\gamma &= 0 & &\text{on } \gamma, \\
\mu H &\perp \mathcal{H}_{N,\mu}(\Omega), & \varepsilon E &\perp \mathcal{H}_{D,\varepsilon}(\Omega),
\end{aligned}$$

where the homogeneous boundary conditions are to be understood in the weak sense.

This coupled problem is equivalent to an electro-magnetostatic Maxwell problem in second order form, which in classical terms reads as follows: Find an electric field $E \in \mathsf{H}(\mathrm{curl}\,,\Omega) \cap \varepsilon^{-1}\mathsf{H}\,(\mathrm{div}\,,\Omega)$ such that $\mu^{-1}\mathrm{curl}\,E \in \mathsf{H}(\mathrm{curl}\,,\Omega)$ and

$$\mathrm{curl}\,\mu^{-1}\mathrm{curl}\,E = F \qquad \text{in } \Omega, \tag{1.2}$$
$$\mathrm{div}\,\varepsilon E = 0 \qquad \text{in } \Omega, \tag{1.3}$$
$$n \times E|_\gamma = 0 \qquad \text{on } \gamma, \tag{1.4}$$
$$\varepsilon E \perp \mathcal{H}_{D,\varepsilon}(\Omega). \tag{1.5}$$

Once E has been found, the magnetic field is defined by the relation $H = \mu^{-1}\mathrm{curl}\,E$.

We note that the problem

$$\mathrm{curl}\,\mu^{-1}\mathrm{curl}\,E + \kappa^2 E = F \qquad \text{in } \Omega,$$
$$n \times E|_\gamma = 0 \qquad \text{on } \gamma$$

with positive κ was considered in [2] in the context of functional type a posteriori error estimates. From the mathematical point of view, this problem is much simpler than problem (1.2)–(1.5), since the zero order term makes the overall operator positive definite.

2. Variational statement and solution theory

Henceforth, we consider (1.2)–(1.5) assuming that the boundary condition on γ may be nonhomogeneous (physically, such a condition is motivated by the presence of an electric current on the boundary). Hence, we intend to discuss the following prototypical electro-magnetostatic Maxwell problem in second order form: Find an electric field E such that

$$\mathrm{curl}\,\mu^{-1}\mathrm{curl}\,E = F \qquad \text{in } \Omega, \tag{2.1}$$
$$\mathrm{div}\,\varepsilon E = 0 \qquad \text{in } \Omega, \tag{2.2}$$
$$n \times E|_\gamma = G \qquad \text{on } \gamma, \tag{2.3}$$
$$\varepsilon E \perp \mathcal{H}_{D,\varepsilon}(\Omega). \tag{2.4}$$

There are at least two methods for proving the existence of the solution. One is based upon Helmholtz–Weyl decompositions (e.g., see [9, 14, 15, 17, 10, 11]). The second method consists of introducing and studying a suitable generalized statement of problem (2.1)–(2.4). In the present paper, we use the second method, because it provides a natural way of deriving error estimates. Both methods are based on Poincaré–Friedrich estimates (see Remark 1) and (if necessary) exploit suitable extension operators for the boundary data. On this way, we also need a certain version of the Poincaré–Friedrichs estimate, namely,

$$||\Psi||_\Omega \le c_p\,||\mathrm{curl}\,\Psi||_\Omega \quad \text{for any} \quad \Psi \in \mathsf{H}\,(\mathrm{curl}\,^\circ, \mathrm{div}\,_0\varepsilon, \perp_\varepsilon, \Omega), \tag{2.5}$$

where we define

$$\mathsf{H}\,(\mathrm{curl}\,^\circ, \mathrm{div}\,_0\varepsilon, \perp_\varepsilon, \Omega) := \mathsf{H}\,(\mathrm{curl}\,^\circ, \Omega) \cap \varepsilon^{-1}\mathsf{H}\,(\mathrm{div}\,_0, \Omega) \cap \mathcal{H}_{D,\varepsilon}(\Omega)^{\perp_\varepsilon}.$$

Remark 1. More general variants of the Poincaré–Friedrich estimate for vector fields (2.5) are known. For instance, we have

$$||\Psi||_\Omega \le c_p\Big(\,||\mathrm{curl}\,\Psi||_\Omega + ||\mathrm{div}\,\varepsilon\Psi||_\Omega + ||\tau_{t,\gamma}\Psi||_{\mathsf{H}_t^{-1/2}(\mathrm{curl}_s,\gamma)} + \sum_{n=1}^{d} |\langle\varepsilon\Psi, H_n\rangle_\Omega|\,\Big),$$

which holds for all $\Psi \in \mathsf{H}\,(\mathrm{curl}\,,\Omega) \cap \varepsilon^{-1}\mathsf{H}\,(\mathrm{div}\,,\Omega)$. Here $\tau_{t,\gamma}$ is the tangential trace and curl_s is the boundary curl operator. The exact definitions of these two operators are presented in Remark 7. This estimate can be proved by an indirect argument using a "Maxwell compact embedding property" of Ω, which holds true not only for

Lipschitz domains, but also, if the homogeneous boundary condition is considered, for more irregular domains (cone properties) (see [18]). For nonhomogeneous boundary conditions, the Lipschitz assumption cannot be weakened. Actually, it is just the continuity of the solution operator of the corresponding electrostatic boundary value problem (see [5–7]).

Let E_γ be a vector field in

$$\mathsf{H}(\mathrm{curl}, \mathrm{div}_0\varepsilon, \perp_\varepsilon, \Omega) := \mathsf{H}(\mathrm{curl}, \Omega) \cap \varepsilon^{-1}\mathsf{H}(\mathrm{div}_0, \Omega) \cap \mathcal{H}_{D,\varepsilon}(\Omega)^{\perp_\varepsilon}$$

satisfying boundary condition (2.3) (in a generalized sense). The generalized solution

$$E \in \mathsf{H}(\overset{\circ}{\mathrm{curl}}, \mathrm{div}_0\varepsilon, \perp_\varepsilon, \Omega) + E_\gamma \subset \mathsf{H}(\mathrm{curl}, \mathrm{div}_0\varepsilon, \perp_\varepsilon, \Omega)$$

of (2.1)–(2.4) is then defined by the relation

$$\langle \mu^{-1}\mathrm{curl}\, E, \mathrm{curl}\, W\rangle_\Omega = \langle F, W\rangle_\Omega \quad \text{for any} \quad W \in \mathsf{H}(\overset{\circ}{\mathrm{curl}}, \mathrm{div}_0\varepsilon, \perp_\varepsilon, \Omega). \tag{2.6}$$

If $F \in \mathsf{H}(\Omega)$, then, by the Cauchy–Schwarz inequality, the right-hand side of (2.6) is a linear and continuous functional over $\mathsf{H}(\overset{\circ}{\mathrm{curl}}, \mathrm{div}_0\varepsilon, \perp_\varepsilon, \Omega)$. By (2.5), the left-hand side of (2.6) is a strongly coercive bilinear form over $\mathsf{H}(\overset{\circ}{\mathrm{curl}}, \mathrm{div}_0\varepsilon, \perp_\varepsilon, \Omega)$. Thus, under these assumptions problem (2.6) is uniquely solvable in $\mathsf{H}(\overset{\circ}{\mathrm{curl}}, \mathrm{div}_0\varepsilon, \perp_\varepsilon, \Omega) + E_\gamma$ by Lax–Milgram's theorem.

First, we note some Helmholtz–Weyl decompositions of $\mathsf{H}(\Omega)$, i.e., decompositions into solenoidal and curl-free fields, which will be used frequently throughout our analysis.

Lemma 2. *$\mathsf{H}(\Omega)$ can be decomposed as*

$$\begin{aligned}\mathsf{H}(\Omega) &= \varepsilon\mathsf{H}(\overset{\circ}{\mathrm{curl}}_0, \Omega) \oplus_{\varepsilon^{-1}} \overline{\mathrm{curl}\,\mathsf{H}(\mathrm{curl}, \Omega)} = \varepsilon\overline{\mathrm{grad}\,\mathsf{H}(\overset{\circ}{\mathrm{grad}}, \Omega)} \oplus_{\varepsilon^{-1}} \mathsf{H}(\mathrm{div}_0, \Omega)\\ &= \varepsilon\overline{\mathrm{grad}\,\mathsf{H}(\overset{\circ}{\mathrm{grad}}, \Omega)} \oplus_{\varepsilon^{-1}} \varepsilon\mathcal{H}_{D,\varepsilon}(\Omega) \oplus_{\varepsilon^{-1}} \overline{\mathrm{curl}\,\mathsf{H}(\mathrm{curl}, \Omega)}\end{aligned} \tag{2.7}$$

and

$$\begin{aligned}\mathsf{H}(\Omega) &= \mathsf{H}(\overset{\circ}{\mathrm{curl}}_0, \Omega) \oplus_\varepsilon \varepsilon^{-1}\overline{\mathrm{curl}\,\mathsf{H}(\mathrm{curl}, \Omega)} = \overline{\mathrm{grad}\,\mathsf{H}(\overset{\circ}{\mathrm{grad}}, \Omega)} \oplus_\varepsilon \varepsilon^{-1}\mathsf{H}(\mathrm{div}_0, \Omega)\\ &= \overline{\mathrm{grad}\,\mathsf{H}(\overset{\circ}{\mathrm{grad}}, \Omega)} \oplus_\varepsilon \mathcal{H}_{D,\varepsilon}(\Omega) \oplus_\varepsilon \varepsilon^{-1}\overline{\mathrm{curl}\,\mathsf{H}(\mathrm{curl}, \Omega)},\end{aligned} \tag{2.8}$$

where the closures are taken in $\mathsf{H}(\Omega)$ and $\mathsf{H}(\overset{\circ}{\mathrm{grad}}, \Omega) = \overset{\circ}{\mathsf{H}}{}^1(\Omega)$. Moreover,

$$\overline{\mathrm{curl}\,\mathsf{H}(\mathrm{curl}, \Omega)} = \mathsf{H}(\mathrm{div}_0, \perp, \Omega) := \mathsf{H}(\mathrm{div}_0, \Omega) \cap \mathcal{H}_{D,\varepsilon}(\Omega)^\perp.$$

Remark 3. We denote the ε-orthogonal projection onto $\varepsilon^{-1}\overline{\mathrm{curl}\,\mathsf{H}(\mathrm{curl}, \Omega)}$ in (2.8) by π. Then we have

$$\tau_{t,\gamma}\pi\Phi = \tau_{t,\gamma}\Phi, \quad \mathrm{curl}\,\pi\Phi = \mathrm{curl}\,\Phi \tag{2.9}$$

for all $\Phi \in \mathsf{H}(\mathrm{curl}, \Omega)$ and

$$\mathrm{div}\,\varepsilon\pi\Psi = 0, \quad \varepsilon\pi\Psi \perp \mathcal{H}_{D,\varepsilon}(\Omega), \quad \mathrm{curl}\,(1-\pi)\Psi = 0, \quad \tau_{t,\gamma}(1-\pi)\Psi = 0$$

for all $\Psi \in \mathsf{H}(\Omega)$.

The latter line can be written in a more compact and precise form:

$$\begin{aligned}\pi\mathsf{H}(\Omega) &= \varepsilon^{-1}\overline{\mathrm{curl}\,\mathsf{H}(\mathrm{curl}, \Omega)} = \mathsf{H}(\mathrm{div}_0\,\varepsilon, \perp_\varepsilon, \Omega),\\ (1-\pi)\mathsf{H}(\Omega) &= \mathsf{H}(\overset{\circ}{\mathrm{curl}}_0, \Omega).\end{aligned}$$

Remark 4. Note that, by (2.1), F must be solenoidal and perpendicular in $\mathsf{H}(\Omega)$ to $\mathsf{H}(\overset{\circ}{\mathrm{curl}}_0, \Omega)$. Using Helmholtz–Weyl decomposition (2.7), we represent the vector field $F \in \mathsf{H}(\Omega)$ in the form

$$F = \varepsilon F_D + \varepsilon F_{\mathrm{grad}} + F_{\mathrm{curl}}.$$

Then, for any $W \in \mathsf{H}(\overset{\circ}{\mathrm{curl}}, \mathrm{div}_0\varepsilon, \perp_\varepsilon, \Omega)$ we compute

$$\langle F, W\rangle_\Omega = \langle F_{\mathrm{curl}}, W\rangle_\Omega.$$

Hence, the functional on the right-hand side of (2.6) cannot distinguish between F and the projection F_{curl}.

The following theorem provides the main existence result.

Theorem 5. *Let $F \in \mathsf{H}(\mathrm{div}_0, \perp, \Omega)$, and let $E_\gamma \in \mathsf{H}(\mathrm{curl}, \mathrm{div}_0\varepsilon, \perp_\varepsilon, \Omega)$ satisfy boundary condition* (2.3)*. Then boundary value problem* (2.1)–(2.4) *is uniquely weakly solvable in* $\mathsf{H}(\mathrm{curl}^\circ, \mathrm{div}_0\varepsilon, \perp_\varepsilon, \Omega) + E_\gamma$. *The solution operator is continuous.*

Remark 6. The kernel of (2.1)–(2.3) equals $\mathcal{H}_{D,\varepsilon}(\Omega)$. We only need to show that $\mathrm{curl}\, E = 0$, but this follows immediately since $E \in \mathsf{H}(\mathrm{curl}^\circ, \Omega)$ and, thus,

$$0 = \left\langle \mathrm{curl}\, \mu^{-1} \mathrm{curl}\, E, E \right\rangle_\Omega = \left\langle \mu^{-1} \mathrm{curl}\, E, \mathrm{curl}\, E \right\rangle_\Omega .$$

Remark 7. The boundary data G and its extension E_γ can be described in more detail. Owing to papers [1, 3, 4] and the more general paper of Weck [23], we know that even for Lipschitz domains, where the nonscalar trace business is a challenging task, there exist a bounded linear tangential trace operator $\tau_{t,\gamma}$ and the corresponding bounded linear tangential extension operator $\check{\tau}_{t,\gamma}$ (right inverse) that map $\mathsf{H}(\mathrm{curl}, \Omega)$ to special tangential vector fields on the boundary, i.e.,

$$\mathsf{H}_t^{-1/2}(\mathrm{curl}_s, \gamma) := \left\{ \psi \in \mathsf{H}_t^{-1/2}(\gamma) \mid \mathrm{curl}_s \psi \in \mathsf{H}^{-1/2}(\gamma) \right\},$$

and vice versa. Here, curl_s denotes the surface curl. Using Helmholtz–Weyl decomposition (2.8), we even get an improved extension operator. We have

$$\tau_{t,\gamma} : \mathsf{H}(\mathrm{curl}, \Omega) \to \mathsf{H}_t^{-1/2}(\mathrm{curl}_s, \gamma),$$
$$\check{\tau}_{t,\gamma} : \mathsf{H}_t^{-1/2}(\mathrm{curl}_s, \gamma) \to \mathsf{H}(\mathrm{curl}, \mathrm{div}_0\varepsilon, \perp_\varepsilon, \Omega).$$

Applied to smooth vector fields, this yields $\tau_{t,\gamma} = n \times \cdot|_\gamma$. Now, we may specify the boundary data $G \in \mathsf{H}_t^{-1/2}(\mathrm{curl}_s, \gamma)$ and the extension

$$E_\gamma := \check{\tau}_{t,\gamma} G \in \mathsf{H}(\mathrm{curl}, \mathrm{div}_0\varepsilon, \perp_\varepsilon, \Omega),$$

as well as our variational statement for $E = \widehat{E} + \check{\tau}_{t,\gamma} G$: Find $\widehat{E} \in \mathsf{H}(\mathrm{curl}^\circ, \mathrm{div}_0\varepsilon, \perp_\varepsilon, \Omega)$ such that

$$b(\widehat{E}, W) := \left\langle \mu^{-1} \mathrm{curl}\, \widehat{E}, \mathrm{curl}\, W \right\rangle_\Omega = \langle F, W \rangle_\Omega - \left\langle \mu^{-1} \mathrm{curl}\, \check{\tau}_{t,\gamma} G, \mathrm{curl}\, W \right\rangle_\Omega =: \ell(W)$$

holds for all $W \in \mathsf{H}(\mathrm{curl}^\circ, \mathrm{div}_0\varepsilon, \perp_\varepsilon, \Omega)$.

Remark 8. Henceforth, we assume that G is given by a tangential trace of a vector field $T \in \mathsf{H}(\mathrm{curl}, \Omega)$.

3. Upper bounds for the deviation from the exact solution

Let $\widetilde{E}$ be an approximation of

$$E \in \mathsf{H}(\mathrm{curl}^\circ, \mathrm{div}_0\varepsilon, \perp_\varepsilon, \Omega) + E_\gamma \subset \mathsf{H}(\mathrm{curl}, \mathrm{div}_0\varepsilon, \perp_\varepsilon, \Omega).$$

We assume that $\widetilde{E}$ belongs to $\mathsf{H}(\mathrm{curl}, \Omega) \cap \varepsilon^{-1}\mathsf{H}(\mathrm{div}, \Omega)$, which means that, in general, the boundary condition, the divergence-free condition, and the orthogonality to the Dirichlet fields might be violated, i.e., the approximation field may be such that

$$\tau_{t,\gamma} \widetilde{E} \neq G, \quad \mathrm{div}\, \varepsilon \widetilde{E} \neq 0, \quad \left\langle \varepsilon \widetilde{E}, H \right\rangle \neq 0 \quad \text{for some} \quad H \in \mathcal{H}_{D,\varepsilon}(\Omega).$$

Moreover, for subsequent analysis and then also for a numerical application, which is even more important, it is sufficient to assume just $\widetilde{E} \in \mathsf{H}(\mathrm{curl}, \Omega)$.

Our goal is to obtain upper bounds for the difference between $\mathrm{curl}\, E$ and $\mathrm{curl}\, \widetilde{E}$ in terms of the weighted norm

$$||\Psi||_{\mu^{-1},\Omega} := \left|\left| \mu^{-1/2} \Psi \right|\right|_\Omega = \left\langle \mu^{-1} \Psi, \Psi \right\rangle_\Omega^{1/2} .$$

First, we use (2.6) and for all $W \in \mathsf{H}(\mathrm{curl}^\circ, \mathrm{div}_0\varepsilon, \perp_\varepsilon, \Omega)$ we get

$$\left\langle \mu^{-1} \mathrm{curl}\, (E - \widetilde{E}), \mathrm{curl}\, W \right\rangle_\Omega = \langle F, W \rangle_\Omega - \left\langle \mu^{-1} \mathrm{curl}\, \widetilde{E}, \mathrm{curl}\, W \right\rangle_\Omega =: \ell_{\widetilde{E}}(W), \tag{3.1}$$

where ℓ is a linear and continuous functional over $\mathsf{H}(\mathrm{curl}^\circ, \mathrm{div}_0\varepsilon, \perp_\varepsilon, \Omega)$, as well as over $\mathsf{H}(\mathrm{curl}, \Omega)$.

Obviously, $\ell_{\widetilde{E}}$ vanishes if $\mathrm{curl}\, E = \mathrm{curl}\, \widetilde{E}$. Furthermore, if $\widetilde{E}$ satisfies the boundary condition exactly, i.e., $\tau_{t,\gamma} \widetilde{E} = G$, then $\ell_{\widetilde{E}} = 0$ if and only if $\mathrm{curl}\, E = \mathrm{curl}\, \widetilde{E}$ (or, what is equivalent, if and only if $E = \pi \widetilde{E}$). With the help of the Helmholtz–Weyl decomposition, this holds by the following argument: if $\tau_{t,\gamma} \widetilde{E} = G$, then $E - \pi \widetilde{E} \in \mathsf{H}(\mathrm{curl}^\circ, \mathrm{div}_0\varepsilon, \perp_\varepsilon, \Omega)$. Thus, $\mathrm{curl}\, (E - \pi \widetilde{E}) = 0$ by $\ell_{\widetilde{E}} = 0$. But then $E - \pi \widetilde{E}$ is a Dirichlet field and hence must vanish by orthogonality. Finally, $\mathrm{curl}\, \pi \widetilde{E} = \mathrm{curl}\, \widetilde{E}$.

The second step is based upon the following result.

Lemma 9. *Let $E \in \mathsf{H}(\operatorname{curl}, \operatorname{div}_0 \varepsilon, \perp_\varepsilon, \Omega)$, and let $\widetilde{E} \in \mathsf{H}(\operatorname{curl}, \Omega)$ be as above. Moreover, suppose there exists $c_\ell > 0$ such that the relation*

$$\left\langle \mu^{-1} \operatorname{curl}(E - \widetilde{E}), \operatorname{curl} W \right\rangle_\Omega = \ell_{\widetilde{E}}(W) \leq c_\ell \, ||\operatorname{curl} W||_{\mu^{-1},\Omega}$$

holds for all $W \in \mathsf{H}(\operatorname{curl}^\circ, \operatorname{div}_0 \varepsilon, \perp_\varepsilon, \Omega)$. Then the inequality

$$\left|\left|\operatorname{curl}(E - \widetilde{E})\right|\right|_{\mu^{-1},\Omega} \leq c_\ell + 2 \, ||\operatorname{curl} T||_{\mu^{-1},\Omega} \tag{3.2}$$

is valid for all $T \in \mathsf{H}(\operatorname{curl}, \Omega)$ for which the tangential trace coincides with the tangential trace of $E - \widetilde{E}$, i.e., $G - \tau_{t,\gamma}\widetilde{E}$, on the boundary γ. If additionally $\tau_{t,\gamma}\widetilde{E} = G$, then

$$\left|\left|\operatorname{curl}(E - \widetilde{E})\right|\right|_{\mu^{-1},\Omega} \leq c_\ell. \tag{3.3}$$

Proof. We use Helmholtz–Weyl decomposition (2.8) and the projection π from Remark 3. We consider a vector field $T \in \mathsf{H}(\operatorname{curl}, \Omega)$ with $\tau_{t,\gamma} T = G - \tau_{t,\gamma}\widetilde{E}$ and define the vector field

$$W := E - \pi(T + \widetilde{E}) = E - \widetilde{E} + (1 - \pi)\widetilde{E} - \pi T \in \mathsf{H}(\operatorname{curl}^\circ, \operatorname{div}_0 \varepsilon, \perp_\varepsilon, \Omega),$$

which holds by (2.9). Hence, $\operatorname{curl} W = \operatorname{curl}(E - \widetilde{E}) - \operatorname{curl} T$. Using Cauchy–Schwarz' inequality, we obtain

$$||\operatorname{curl} W||^2_{\mu^{-1},\Omega} = \left\langle \mu^{-1} \operatorname{curl}(E - \widetilde{E}), \operatorname{curl} W \right\rangle_\Omega - \langle \mu^{-1} \operatorname{curl} T, \operatorname{curl} W \rangle_\Omega \leq \left(c_\ell + ||\operatorname{curl} T||_{\mu^{-1},\Omega} \right) ||\operatorname{curl} W||_{\mu^{-1},\Omega}$$

and, thus, $||\operatorname{curl} W||_{\mu^{-1},\Omega} \leq c_\ell + ||\operatorname{curl} T||_{\mu^{-1},\Omega}$. By the triangle inequality, we get (3.2); (3.3) is trivial if we set $T := 0$. □

Using the trace and extension operators from Remark 7, we can also represent the estimate in the following form.

Corollary 10. *Let the assumptions of Lemma 9 be satisfied. Then*

$$\left|\left|\operatorname{curl}(E - \widetilde{E})\right|\right|_{\mu^{-1},\Omega} \leq c_\ell + 2 \left|\left|\operatorname{curl} \check{\tau}_{t,\gamma}(G - \tau_{t,\gamma}\widetilde{E})\right|\right|_{\mu^{-1},\Omega} \leq c_\ell + 2c_\gamma \left|\left| G - \tau_{t,\gamma}\widetilde{E} \right|\right|_{\mathsf{H}_t^{-1/2}(\operatorname{curl}_s, \gamma)}. \tag{3.4}$$

Here, $c_\gamma > 0$ is the constant in the inequality

$$||\operatorname{curl} \check{\tau}_{t,\gamma}\psi||_{\mu^{-1},\Omega} \leq c_\gamma \, ||\psi||_{\mathsf{H}_t^{-1/2}(\operatorname{curl}_s,\gamma)} \quad \text{for any} \quad \psi \in \mathsf{H}_t^{-1/2}(\operatorname{curl}_s, \gamma). \tag{3.5}$$

Proof. Setting $T := \check{\tau}_{t,\gamma}(G - \tau_{t,\gamma}\widetilde{E})$ in (3.2) and using (3.5), we have (3.4). We note that (3.3) follows directly from the corollary as well. □

Corollary 10 implies the following result.

Theorem 11. *Let E and $\widetilde{E}$ be as in Lemma 9. Then*

$$\left|\left|\operatorname{curl}(E - \widetilde{E})\right|\right|_{\mu^{-1},\Omega} \leq \frac{c_p}{\sqrt{c_\mu}} \, ||F - \operatorname{curl} Y||_\Omega + \left|\left| Y - \mu^{-1} \operatorname{curl} \widetilde{E} \right|\right|_{\mu,\Omega} + 2c_\gamma \left|\left| G - \tau_{t,\gamma}\widetilde{E} \right|\right|_{\mathsf{H}_t^{-1/2}(\operatorname{curl}_s,\gamma)}, \tag{3.6}$$

where Y is an arbitrary vector field in $\mathsf{H}(\operatorname{curl}, \Omega)$.

Proof. For any $Y \in \mathsf{H}(\operatorname{curl}, \Omega)$ and any $W \in \mathsf{H}(\operatorname{curl}^\circ, \Omega)$, we have

$$-\langle \operatorname{curl} Y, W \rangle_\Omega + \langle Y, \operatorname{curl} W \rangle_\Omega = 0. \tag{3.7}$$

Combining (3.1) and (3.7), for all $W \in \mathsf{H}(\operatorname{curl}^\circ, \operatorname{div}_0 \varepsilon, \perp_\varepsilon, \Omega)$ we obtain

$$\left\langle \mu^{-1} \operatorname{curl}(E - \widetilde{E}), \operatorname{curl} W \right\rangle_\Omega = \langle F - \operatorname{curl} Y, W \rangle_\Omega + \left\langle Y - \mu^{-1} \operatorname{curl} \widetilde{E}, \operatorname{curl} W \right\rangle_\Omega = \ell_{\widetilde{E}}(W). \tag{3.8}$$

By the Cauchy–Schwarz inequality, Poincaré–Friedrichs estimate (2.5), and (1.1), we estimate the right-hand side $\ell_{\widetilde{E}}(W)$ of (3.8):

$$\begin{aligned} |\langle F - \operatorname{curl} Y, W\rangle_\Omega| &\le ||F - \operatorname{curl} Y||_\Omega \, ||W||_\Omega \le c_p \, ||F - \operatorname{curl} Y||_\Omega \, ||\operatorname{curl} W||_\Omega \\ &\le \frac{c_p}{\sqrt{c_\mu}} \, ||F - \operatorname{curl} Y||_\Omega \, ||\operatorname{curl} W||_{\mu^{-1},\Omega} , \end{aligned} \tag{3.9}$$

$$\left| \left\langle Y - \mu^{-1} \operatorname{curl} \widetilde{E}, \operatorname{curl} W \right\rangle_\Omega \right| \le \left\| Y - \mu^{-1} \operatorname{curl} \widetilde{E} \right\|_{\mu,\Omega} ||\operatorname{curl} W||_{\mu^{-1},\Omega} . \tag{3.10}$$

Now, Lemma 9 completes the proof. □

We note that the latter estimate is unable to measure adequately the deviation of the divergence of $\varepsilon\widetilde{E}$ to 0 (this is obvious, since $\varepsilon\widetilde{E}$ even does not need to have any divergence). On the other hand, even if $\operatorname{div} \varepsilon\widetilde{E} \neq 0$, then the semi-norm $||\operatorname{curl} \cdot||_{\mu^{-1},\Omega}$ could not feel the lack of the constraint $\operatorname{div} \varepsilon\widetilde{E} = 0$. However, it is not difficult to transform the estimate into a form in which the estimate is represented in terms of the seminorm

$$|||\Psi|||_\Omega := \|\operatorname{curl} \Psi\|_{\mu^{-1},\Omega} + ||\operatorname{div} \varepsilon\Psi||_\Omega + \sum_{n=1}^{d} |\langle \varepsilon\Psi, H_n\rangle_\Omega| \tag{3.11}$$

on $\mathsf{H}(\operatorname{curl}, \Omega) \cap \varepsilon^{-1}\mathsf{H}(\operatorname{div}, \Omega)$, which obviously is a norm on

$$\mathsf{H}(\overset{\circ}{\operatorname{curl}}, \Omega) \cap \varepsilon^{-1}\mathsf{H}(\operatorname{div}, \Omega).$$

Theorem 12. *Let E be as in Lemma 9 and*

$$\widetilde{E} \in \mathsf{H}(\operatorname{curl}, \Omega) \cap \varepsilon^{-1}\mathsf{H}(\operatorname{div}, \Omega).$$

Then for any $Y \in \mathsf{H}(\operatorname{curl}, \Omega)$,

$$\begin{aligned} \left|\left|\left| E - \widetilde{E} \right|\right|\right|_\Omega \le M_+(\widetilde{E}, Y) &:= \frac{c_p}{\sqrt{c_\mu}} \, ||F - \operatorname{curl} Y||_\Omega + \left\| Y - \mu^{-1} \operatorname{curl} \widetilde{E} \right\|_{\mu,\Omega} \\ &+ 2c_\gamma \left\| G - \tau_{t,\gamma}\widetilde{E} \right\|_{\mathsf{H}_t^{-1/2}(\operatorname{curl}_s,\gamma)} + \left\| \operatorname{div} \varepsilon\widetilde{E} \right\|_\Omega + \sum_{n=1}^{d} \left| \left\langle \varepsilon\widetilde{E}, H_n \right\rangle_\Omega \right| . \end{aligned} \tag{3.12}$$

If $E - \widetilde{E}$ even belongs to $\mathsf{H}(\overset{\circ}{\operatorname{curl}}, \Omega) \cap \varepsilon^{-1}\mathsf{H}(\operatorname{div}, \Omega)$, i.e., if the approximation $\widetilde{E}$ satisfies the boundary condition exactly, then $|||\cdot|||_\Omega$ is a norm for $E - \widetilde{E}$, and we have

$$\left|\left|\left| E - \widetilde{E} \right|\right|\right|_\Omega \le M_+(\widetilde{E}, Y) = \frac{c_p}{\sqrt{c_\mu}} \, ||F - \operatorname{curl} Y||_\Omega + \left\| Y - \mu^{-1} \operatorname{curl} \widetilde{E} \right\|_{\mu,\Omega} + \left\| \operatorname{div} \varepsilon\widetilde{E} \right\|_\Omega + \sum_{n=1}^{d} \left| \left\langle \varepsilon\widetilde{E}, H_n \right\rangle_\Omega \right| \tag{3.13}$$

for all $Y \in \mathsf{H}(\operatorname{curl}, \Omega)$.

Remark 13. If $\widetilde{E}$ satisfies the prescribed boundary condition and $\varepsilon\widetilde{E}$ is solenoidal and perpendicular to Dirichlet fields, then, for all $Y \in \mathsf{H}(\operatorname{curl}, \Omega)$, (3.6) or (3.12), (3.13) imply

$$\left|\left|\left| E - \widetilde{E} \right|\right|\right|_\Omega = \left\| \operatorname{curl}(E - \widetilde{E}) \right\|_{\mu^{-1},\Omega} \le M_+(\widetilde{E}, Y) = \frac{c_p}{\sqrt{c_\mu}} \, ||F - \operatorname{curl} Y||_\Omega + \left\| Y - \mu^{-1} \operatorname{curl} \widetilde{E} \right\|_{\mu,\Omega} \tag{3.14}$$

and the left-hand side is a norm for $E - \widetilde{E}$. Estimates (3.6)–(3.14) show that deviations from exact solutions contain weighted residuals of basic relations with weights given by constants in the corresponding embedding inequalities. These are typical features of the so-called functional a posteriori error estimates.

Remark 14. We see that $M_+(\widetilde{E}, Y) = 0$ if and only if

$$\widetilde{E} := E \in \mathsf{H}(\operatorname{curl}, \operatorname{div}_0 \varepsilon, \perp_\varepsilon, \Omega)$$

and $Y := \mu^{-1}\mathrm{curl}\, E \in \mathsf{H}(\mathrm{curl}\,,\Omega)$ in view of Lemma 17. Moreover, we note that (3.14) is sharp, which can easily be seen by setting

$$Y := \mu^{-1}\mathrm{curl}\, E \in \mathsf{H}(\mathrm{curl}\,,\Omega).$$

In other words, if $\widetilde{E} \in \mathsf{H}(\mathrm{curl}\,,\mathrm{div}_0\varepsilon,\perp_\varepsilon,\Omega)$ and satisfies the boundary condition exactly, then

$$\left|\left|\left|E-\widetilde{E}\right|\right|\right|_\Omega \leq \inf_{Y\in\mathsf{H}(\mathrm{curl}\,,\Omega)} M_+(\widetilde{E},Y).$$

Remark 15. In Theorems 11 and 12, we can replace the boundary term on the right-hand side by $2\,||\mathrm{curl}\, T||_{\mu^{-1},\Omega}$ or $2\left|\left|\mathrm{curl}\,\check{\tau}_{t,\gamma}(G-\tau_{t,\gamma}\widetilde{E})\right|\right|_{\mu^{-1},\Omega}$, using Lemma 9 and Corollary 10.

Remark 16. If the domain is "simple" in terms of the vanishing second Betti number, i.e., there are no "handles," then no Dirichlet fields exist. Thus, for instance, in Theorem 12 the last summand in the respective estimates does not occur.

4. Lower bounds for the error

Now, we proceed to deriving computable lower bounds of the error. First, we present the following subsidiary result.

Lemma 17. *If E satisfies* (2.6), *then $\mu^{-1}\mathrm{curl}\, E \in \mathsf{H}(\mathrm{curl}\,,\Omega)$ and $\mathrm{curl}\,\mu^{-1}\mathrm{curl}\, E = F$.*

Proof. We need to show that

$$\left\langle \mu^{-1}\mathrm{curl}\, E, \mathrm{curl}\,\Phi\right\rangle_\Omega = \langle F,\Phi\rangle_\Omega \quad \text{for any} \quad \Phi\in\overset{\circ}{\mathsf{C}}{}^\infty(\Omega). \tag{4.1}$$

Using π from Remark 3, we get $W = \pi\Phi \in \mathsf{H}(\mathrm{curl}^\circ,\mathrm{div}_0\varepsilon,\perp_\varepsilon,\Omega)$, provided that $\Phi\in\overset{\circ}{\mathsf{C}}{}^\infty(\Omega)$. Thus, by (2.6) and the fact that $\mathrm{curl}\,(1-\pi)\Phi = 0$, we obtain

$$\left\langle \mu^{-1}\mathrm{curl}\, E, \mathrm{curl}\,\Phi\right\rangle_\Omega = \left\langle \mu^{-1}\mathrm{curl}\, E, \mathrm{curl}\,\pi\Phi\right\rangle_\Omega = \langle F,\pi\Phi\rangle_\Omega \quad \text{for any} \quad \Phi\in\overset{\circ}{\mathsf{C}}{}^\infty(\Omega). \tag{4.2}$$

Since $F\in\mathsf{H}(\mathrm{div}_0,\perp,\Omega) = \overline{\mathrm{curl}\,\mathsf{H}(\mathrm{curl}\,,\Omega)}$, we get (by approximation) $\langle F,\pi\Phi\rangle_\Omega = \langle F,\Phi\rangle_\Omega$ and (4.1) follows. To be more precise, we select $F_n\in\mathsf{H}(\mathrm{curl}\,,\Omega)$, for which $(\mathrm{curl}\, F_n)_{n\in\mathbb{N}}$ converges in $\mathsf{H}(\Omega)$ to F, using $\pi\Phi\in\mathsf{H}(\mathrm{curl}^\circ,\Omega)$ and $\mathrm{curl}\,(1-\pi)\Phi=0$. Then

$$\langle \mathrm{curl}\, F_n,\pi\Phi\rangle_\Omega = \langle F_n,\mathrm{curl}\,\pi\Phi\rangle_\Omega = \langle F_n,\mathrm{curl}\,\Phi\rangle_\Omega = \langle \mathrm{curl}\, F_n,\Phi\rangle_\Omega \quad \text{for any} \quad \Phi\in\overset{\circ}{\mathsf{C}}{}^\infty(\Omega). \qquad \square$$

Theorem 18. *Let $\widetilde{E}\in\mathsf{H}(\mathrm{curl}\,,\Omega)$ be an approximation. Then*

$$\left|\left|\mathrm{curl}\,(E-\widetilde{E})\right|\right|^2_{\mu^{-1},\Omega} \geq \sup_W M_-(\widetilde{E},W),$$

where

$$M_-(\widetilde{E},W) := 2\,\langle F,W\rangle_\Omega - \left\langle \mu^{-1}\mathrm{curl}\,(2\widetilde{E}+W),\mathrm{curl}\, W\right\rangle_\Omega$$

and the supremum is taken over $\mathsf{H}(\mathrm{curl}^\circ,\Omega)$. This estimate is sharp if $E-\widetilde{E}$ belongs to the latter space, i.e., if the approximation $\widetilde{E}$ satisfies the boundary condition exactly.

Proof. We begin with the obvious identity

$$\left|\left|\mathrm{curl}\,(E-\widetilde{E})\right|\right|^2_{\mu^{-1},\Omega} = \sup_{Y\in\mathsf{H}(\Omega)}\left(2\left\langle \mu^{-1}\mathrm{curl}\,(E-\widetilde{E}),Y\right\rangle_\Omega - ||Y||^2_{\mu^{-1},\Omega}\right).$$

Thus, for all $W \in \mathsf{H}(\operatorname{curl}, \Omega)$ we obtain the estimate

$$\left|\left|\operatorname{curl}(E-\widetilde{E})\right|\right|^2_{\mu^{-1},\Omega} \geq 2\left\langle \mu^{-1}\operatorname{curl}(E-\widetilde{E}), \operatorname{curl} W\right\rangle_\Omega - ||\operatorname{curl} W||^2_{\mu^{-1},\Omega}$$
$$= 2\left\langle \mu^{-1}\operatorname{curl} E, \operatorname{curl} W\right\rangle_\Omega - \left\langle \mu^{-1}\operatorname{curl}(2\widetilde{E}+W), \operatorname{curl} W\right\rangle_\Omega .$$

Clearly, this estimate is sharp, because we can always put $W = E - \widetilde{E}$. However, to exclude the unknown exact solution E from the right-hand side we need $W \in \mathsf{H}(\operatorname{curl}^\circ, \operatorname{div}_0\varepsilon, \perp_\varepsilon, \Omega)$. Then, by (2.6),

$$\left\langle \mu^{-1}\operatorname{curl} E, \operatorname{curl} W\right\rangle_\Omega = \langle F, W\rangle_\Omega , \tag{4.3}$$

and by Lemma 17, (4.3) even holds for all $W \in \mathsf{H}(\operatorname{curl}^\circ, \Omega)$. Thus, for all $W \in \mathsf{H}(\operatorname{curl}^\circ, \Omega)$

$$\left|\left|\operatorname{curl}(E-\widetilde{E})\right|\right|^2_{\mu^{-1},\Omega} \geq M_-(\widetilde{E}, W).$$

Obviously, this lower bound is sharp if we can set

$$W = E - \widetilde{E} \in \mathsf{H}(\operatorname{curl}^\circ, \operatorname{div}_0\varepsilon, \perp_\varepsilon, \Omega) \quad \text{or} \quad W = E - \widetilde{E} \in \mathsf{H}(\operatorname{curl}^\circ, \Omega). \qquad \square$$

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