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FUNCTIONAL A POSTERIORI ERROR ESTIMATES FOR ELLIPTIC PROBLEMS IN EXTERIOR DOMAINS

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This paper is concerned with the derivation of computable and guaranteed upper bounds of the difference between the exact and approximate solutions of an exterior domain boundary value problem for a linear elliptic equation. Our analysis is based upon purely functional argumentation and does not attract specific properties of an approximation method. Therefore, the estimates derived in the paper at hand are applicable to any approximate solution that belongs to the corresponding energy space. Such estimates (*also called error majorants of functional type*) *were derived earlier for problems in bounded domains of* $\mathbb{R}^N$. *Bibliography*: 4 *titles. Illustrations:* 1 *figure.*

1. Introduction

The main focus of our investigations is to suggest a method of deriving guaranteed and computable upper bounds of the difference between the exact solution u of an elliptic exterior domain boundary value problem and any approximation from the corresponding energy space. Note that such estimates (also called error majorants of functional type) were derived for problems in bounded domains of $\mathbb{R}^N$ in [**2, 3**].

We discuss the method with the paradigm of the prototypical elliptic problem

$$-\operatorname{div} A\nabla u = f, \qquad \text{in } \Omega, \tag{1.1}$$

$$u|_\gamma = g, \qquad \text{on } \gamma := \partial\Omega. \tag{1.2}$$

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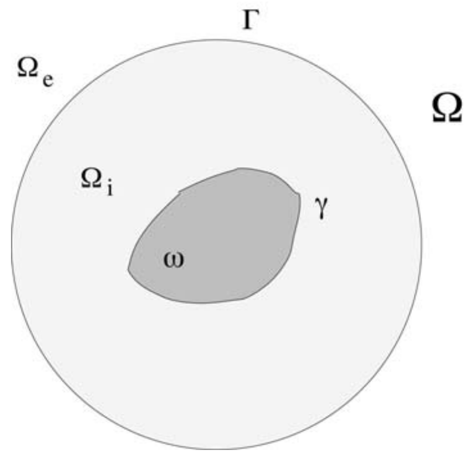

FIGURE 1. The exterior domain Ω with artificial interface Γ.

We assume that $\Omega \subset \mathbb{R}^N$ with $N \geqslant 1$ is an exterior domain, i.e., $\mathbb{R}^N \setminus \Omega$ is compact, with Lipschitz continuous boundary γ (cf. Fig. 1).

Throughout the paper, we use the weighted Lebesgue function spaces

$$L^2_s(\Omega) := \left\{ \varphi \mid \rho^s \varphi \in L^2(\Omega) \right\}, \quad s \in \mathbb{R},$$

where $\rho := (1 + r^2)^{1/2}$ and $r(x) := |x|$ denotes the radius vector. $L^2_s(\Omega)$ is a Hilbert space equipped with the scalar product

$$\langle \varphi, \psi \rangle_{s,\Omega} := \langle \rho^s \varphi, \rho^s \psi \rangle_\Omega := \int_\Omega \rho^{2s} \varphi \psi \, \mathrm{d}\lambda,$$

where φ and ψ belong to $L^2_s(\Omega)$ and λ is the Lebesgue measure. The corresponding norms are denoted by $\|\varphi\|_{s,\Omega} = \|\rho^s \varphi\|_\Omega$. If $s = 0$, then $L^2_s(\Omega)$ coincides with the usual Lebesgue space $L^2(\Omega)$. For the sake of simplicity, we keep the same notation for spaces of vector-valued functions. Moreover, we introduce the weighted Sobolev space

$$H^1_{-1}(\Omega) := \left\{ \varphi \in L^2_{-1}(\Omega) \mid \nabla \varphi \in L^2(\Omega) \right\},$$

which is a Hilbert space as well with respect to the scalar product

$$(\varphi, \psi) \mapsto \langle \varphi, \psi \rangle_{-1,\Omega} + \langle \nabla \varphi, \nabla \psi \rangle_\Omega .$$

We denote by $\overset{\circ}{H}{}^1_{-1}(\Omega)$ the closure of $\overset{\circ}{C}{}^\infty(\Omega)$, the space of compactly supported smooth test functions, in the norm of $H^1_{-1}(\Omega)$. Whenever we consider Sobolev spaces on bounded domains, we use the usual unweighted L^2-scalar products and L^2-norms.

For dimensions $N \geqslant 3$ the solution theory for the problem (1.1)–(1.2) is based on the weighted Poincaré/Friedrichs estimate (cf. Corollary 4.2 (i) and Remark 4.3 of the appendix)

$$\|\varphi\|_{-1,\Omega} \leqslant \frac{2}{N-2} \|\nabla \varphi\|_\Omega \quad \forall \varphi \in \overset{\circ}{H}{}^1_{-1}(\Omega), \tag{1.3}$$

the Lax–Milgram theorem and, if needed, an adequate extension operator for the boundary data. Let u_γ be some function in $H^1_{-1}(\Omega)$ satisfying the boundary condition (1.2). The weak solution $u \in \overset{\circ}{H}{}^1_{-1}(\Omega) + u_\gamma \subset H^1_{-1}(\Omega)$ of (1.1)–(1.2) is then defined by the variational formulation

$$\langle A\nabla u, \nabla w\rangle_\Omega = \langle f, w\rangle_\Omega \quad \forall\, w \in \overset{\circ}{H}{}^1_{-1}(\Omega). \tag{1.4}$$

By (1.3), the left-hand side of (1.4) is a strongly coercitive sesqui-linear form over $\overset{\circ}{H}{}^1_{-1}(\Omega)$ provided that the real matrix-valued function A is measurable, bounded a.e., symmetric, and uniformly strongly elliptic, i.e.,

$$\exists\, c_A > 0 \quad \forall\, \xi \in \mathbb{R}^N \quad \forall\, x \in \Omega \quad c_A|\xi|^2 \leqslant A(x)\xi \cdot \xi. \tag{1.5}$$

If $f \in L^2_1(\Omega)$, then, by the Cauchy–Schwarz inequality, the right-hand side of (1.4) is a linear and continuous functional over $\overset{\circ}{H}{}^1_{-1}(\Omega)$. Thus, under these assumptions, the problem (1.4) is uniquely solvable in $\overset{\circ}{H}{}^1_{-1}(\Omega) + u_\gamma$ by the Lax–Milgram theorem.

If $N = 1, 2$, one can apply the same arguments with the difference that (1.3) has to be modified. For $N = 1$ and, for example, $\Omega \subset \mathbb{R}_+$, by Corollary 4.2 (iii) and Remark 4.3, we have

$$\|\varphi\|_{-1,\Omega} \leqslant 2\,\|\varphi'\|_\Omega \quad \forall\, \varphi \in \overset{\circ}{H}{}^1_{-1}(\Omega). \tag{1.6}$$

Hence we get the same solution theory with tiny restrictions on Ω, which easily can be removed by a translation. For $N = 2$ the singularities are stronger and, in addition, we have to utilize logarithmic terms. By Corollary 4.2 (ii) and Remark 4.3, for domains $\Omega \subset \mathbb{R}^2$ such that the complement $\mathbb{R}^2 \setminus \Omega$ contains the unit ball we have

$$\|\varphi/(r\ln r)\|_\Omega \leqslant 2\,\|\nabla\varphi\|_\Omega \quad \forall\, \varphi \in \overset{\circ}{H}{}^1_{-1,\ln}(\Omega), \tag{1.7}$$

where

$$H^1_{-1,\ln}(\Omega) := \{\varphi \mid \varphi/(r\ln r), \nabla\varphi \in L^2(\Omega)\}$$

is a Hilbert space equipped with the natural scalar product

$$(\varphi, \psi) \mapsto \langle \varphi/(r\ln r), \psi/(r\ln r)\rangle_\Omega + \langle \nabla\varphi, \nabla\psi\rangle_\Omega$$

and again $\overset{\circ}{H}{}^1_{-1,\ln}(\Omega)$ denotes the closure of $\overset{\circ}{C}{}^\infty(\Omega)$ in the norm of $H^1_{-1,\ln}(\Omega)$. Consequently, for all f with $r\ln r f \in L^2(\Omega)$ and u_γ in $H^1_{-1,\ln}(\Omega)$ satisfying the boundary condition (1.2) we obtain a unique solution u belonging to $\overset{\circ}{H}{}^1_{-1,\ln}(\Omega) + u_\gamma$.

We summarize the results in the following theorem.

Theorem 1.1. *Suppose that $N \geqslant 3$ as well as $f \in L^2_1(\Omega)$ and $u_\gamma \in H^1_{-1}(\Omega)$ satisfying the boundary condition* (1.2)*. Then the exterior boundary value problem* (1.1)–(1.2) *is uniquely weakly solvable in $\overset{\circ}{H}{}^1_{-1}(\Omega) + u_\gamma$. The solution operator is continuous.*

By the above discussion, it is clear that for $N = 1, 2$ the existence of weak solutions in suitable spaces can also be proved.

Remark 1.2. The boundary data g and its extension u_γ can be described in more detail. In the bounded domain case, it is well known that there exists a bounded linear trace operator and a corresponding bounded linear extension operator (right inverse) mapping $H^1(\Omega)$ to $H^{1/2}(\gamma)$ and vice verse. Hence, by restriction, we get a bounded linear trace operator

$$\tau_\gamma : H^1_{-1}(\Omega) \to H^{1/2}(\gamma)$$

and, by extension and applying an obvious cutting technique, we obtain a bounded linear extension operator

$$E : H^{1/2}(\gamma) \to H^1_{-1}(\Omega)$$

for our exterior domain Ω, which even maps to functions with (arbitrarily thin) compact support. As in the bounded domain case, E is a right inverse of τ_γ. Then we may specify $g \in H^{1/2}(\gamma)$ and $u_\gamma := Eg \in H^1_{-1}(\Omega)$ as well as our variational formulation for $u = \widetilde{u} + Eg$: Find $\widetilde{u} \in \overset{\circ}{H}{}^1_{-1}(\Omega)$ such that

$$B(\widetilde{u}, w) := \langle A\nabla\widetilde{u}, \nabla w\rangle_\Omega = \langle f, w\rangle_\Omega - \langle A\nabla Eg, \nabla w\rangle_\Omega =: F(w) \quad \forall\, w \in \overset{\circ}{H}{}^1_{-1}(\Omega).$$

Finally, we introduce

$$D(\Omega) := \left\{\varphi \in L^2(\Omega) \mid \operatorname{div}\varphi \in L^2_1(\Omega)\right\},$$

which is a Hilbert space with respect to the canonical scalar product

$$(\varphi, \psi) \mapsto \langle\varphi, \psi\rangle_\Omega + \langle\operatorname{div}\varphi, \operatorname{div}\psi\rangle_{1,\Omega}.$$

2. Upper Bounds for the Deviation from the Exact Solution in Dimensions $N \geqslant 3$

Let v be an approximation of $u \in \overset{\circ}{H}{}^1_{-1}(\Omega) + u_\gamma \subset H^1_{-1}(\Omega)$, where v is assumed just to belong to $H^1_{-1}(\Omega)$ since the boundary condition may not be satisfied exactly. Our goal is to obtain upper bounds for the difference between ∇u and ∇v in terms of the norm

$$\|\varphi\|_{A,\Omega} := \left\|A^{1/2}\varphi\right\|_\Omega = \langle A\varphi, \varphi\rangle_\Omega^{1/2}.$$

Using (1.4), we get for all $w \in \overset{\circ}{H}{}^1_{-1}(\Omega)$

$$\langle A\nabla(u - v), \nabla w\rangle_\Omega = \langle f, w\rangle_\Omega - \langle A\nabla v, \nabla w\rangle_\Omega. \tag{2.1}$$

Before we proceed, we note two useful results.

Theorem 2.1. *Let $u, v \in H^1_{-1}(\Omega)$ be as above. Moreover, let Φ be a linear continuous functional over $\overset{\circ}{H}{}^1_{-1}(\Omega)$, and let $c_\Phi > 0$ be such that for all $w \in \overset{\circ}{H}{}^1_{-1}(\Omega)$*

$$\langle A\nabla(u - v), \nabla w\rangle_\Omega = \Phi(w) \leqslant c_\Phi \|\nabla w\|_{A,\Omega}.$$

Then

$$\|\nabla(u - v)\|_{A,\Omega} \leqslant c_\Phi + 2\|\nabla(\widehat{u} - \widehat{v})\|_{A,\Omega} \tag{2.2}$$

for all $\widehat{u}, \widehat{v} \in H^1_{-1}(\Omega)$ *for which* $\widehat{u} - \widehat{v}$ *coincides with* $u - v$ *on the boundary* γ*. If, in addition,* $u - v$ *belongs to* $\overset{\circ}{H}{}^1_{-1}(\Omega)$*, then*

$$\|\nabla(u - v)\|_{A,\Omega} \leqslant c_\Phi. \tag{2.3}$$

Proof. We consider

$$w := u - v - (\widehat{u} - \widehat{v}) \in \overset{\circ}{H}{}^1_{-1}(\Omega).$$

Using the Cauchy–Schwarz inequality, we obtain

$$\|\nabla w\|^2_{A,\Omega} = \langle A\nabla(u - v), \nabla w\rangle_\Omega - \langle A\nabla(\widehat{u} - \widehat{v}), \nabla w\rangle_\Omega \leqslant \Big(c_\Phi + \|\nabla(\widehat{u} - \widehat{v})\|_{A,\Omega}\Big) \|\nabla w\|_{A,\Omega}$$

and thus

$$\|\nabla w\|_{A,\Omega} \leqslant c_\Phi + \|\nabla(\widehat{u} - \widehat{v})\|_{A,\Omega}.$$

By the triangle inequality, we get (2.2). Note that (2.3) is trivial since we can set $w := u - v$, i.e., $\widehat{u} := \widehat{v} := 0$. □

We may be more specific using the trace and extension operators from Remark 1.2.

Corollary 2.2. *Let the assumptions of Theorem* 2.1 *be satisfied. Then*

$$\|\nabla(u - v)\|_{A,\Omega} \leqslant c_\Phi + 2\,\|\nabla E(g - \tau_\gamma v)\|_{A,\Omega} \leqslant c_\Phi + 2c_\gamma\,\|g - \tau_\gamma v\|_{H^{1/2}(\gamma)},$$

where $c_\gamma > 0$ *is the constant in the inequality*

$$\|\nabla E\varphi\|_{A,\Omega} \leqslant c_\gamma\,\|\varphi\|_{H^{1/2}(\gamma)} \quad \forall \varphi \in H^{1/2}(\gamma). \tag{2.4}$$

Proof. Setting $\widehat{u} := Eg$ and $\widehat{v} := E\tau_\gamma v$ as well as using (2.4), we prove the inequalities. We note that (2.3) follows directly from the corollary as well. □

In the subsequent sections, we introduce and discuss some different functionals Φ and corresponding constants c_Φ.

2.1. First estimate. For any $y \in D(\Omega)$ and $w \in \overset{\circ}{H}{}^1_{-1}(\Omega)$ we have

$$\langle \operatorname{div} y, w\rangle_\Omega + \langle y, \nabla w\rangle_\Omega = 0. \tag{2.5}$$

Combining (2.1) and (2.5), for all $w \in \overset{\circ}{H}{}^1_{-1}(\Omega)$ and $y \in D(\Omega)$ we obtain

$$\langle A\nabla(u - v), \nabla w\rangle_\Omega = \langle f + \operatorname{div} y, w\rangle_\Omega + \langle y - A\nabla v, \nabla w\rangle_\Omega =: \Phi(w). \tag{2.6}$$

By the Cauchy–Schwarz inequality, (1.3) with $c_N := 2/(N - 2)$, and (1.5), we estimate the right-hand side $\Phi(w)$ of (2.6) as follows:

$$
\begin{aligned}
|\langle f + \operatorname{div} y, w\rangle_\Omega| \leqslant \|f + \operatorname{div} y\|_{1,\Omega} \|w\|_{-1,\Omega} &\leqslant c_N \|f + \operatorname{div} y\|_{1,\Omega} \|\nabla w\|_\Omega \\
&\leqslant \frac{c_N}{\sqrt{c_A}} \|f + \operatorname{div} y\|_{1,\Omega} \|\nabla w\|_{A,\Omega}\,, \qquad (2.7)
\end{aligned}
$$

$$
|\langle y - A\nabla v, \nabla w\rangle_\Omega| \leqslant \|y - A\nabla v\|_{A^{-1},\Omega} \|\nabla w\|_{A,\Omega}\,. \qquad (2.8)
$$

By Corollary 2.2, we arrive at the following result.

Proposition 2.3. *Let u and v be the same as in Theorem* 2.1. *Then*

$$
\|\nabla(u - v)\|_{A,\Omega} \leqslant \frac{c_N}{\sqrt{c_A}} \|f + \operatorname{div} y\|_{1,\Omega} + \|y - A\nabla v\|_{A^{-1},\Omega} + 2c_\gamma \|g - \tau_\gamma v\|_{H^{1/2}(\gamma)}\,, \qquad (2.9)
$$

where y is an arbitrary vector field in $D(\Omega)$.

Remark 2.4. If v satisfies the prescribed boundary condition, then (2.9) implies

$$
\|\nabla(u - v)\|_{A,\Omega} \leqslant \frac{c_N}{\sqrt{c_A}} \|f + \operatorname{div} y\|_{1,\Omega} + \|y - A\nabla v\|_{A^{-1},\Omega}\,. \qquad (2.10)
$$

The estimates (2.9) and (2.10)) show that deviations from exact solutions of exterior boundary value problems have the same structure as for problems in bounded domains, namely they contain weighted residuals of basic relations with weights given by constants in the corresponding embedding inequalities.

2.2. Second estimate. Assume that Ω is decomposed into two subdomains Ω_i and Ω_e with interface $\Gamma := \partial\Omega_e$ (cf. Fig. 1) and that the fields $y \in D(\Omega)$ exactly satisfy the relation

$$
\operatorname{div} y + f = 0 \quad \text{in } \Omega_e. \qquad (2.11)
$$

In particular, such a situation may arise if the source term f has compact support and y is represented (in the exterior domain Ω_e) as a linear combination of solenoidal fields having proper decay at infinity. In this case, the estimate of Proposition 2.3 turns trivially to the estimate

$$
\|\nabla(u - v)\|_{A,\Omega} \leqslant c_o \|f + \operatorname{div} y\|_{\Omega_i} + \|y - A\nabla v\|_{A^{-1},\Omega} + 2c_\gamma \|g - \tau_\gamma v\|_{H^{1/2}(\gamma)}\,, \qquad (2.12)
$$

which holds for all $y \in D(\Omega)$ satisfying (2.11), where the weight constant is

$$
c_o := \frac{c_N(1 + \|r\|_{\infty,\Omega_i})}{\sqrt{c_A}}, \qquad (2.13)
$$

which follows directly from

$$
\|f + \operatorname{div} y\|_{1,\Omega} = \|f + \operatorname{div} y\|_{1,\Omega_i} \leqslant |\rho|_{\infty,\Omega_i} \|f + \operatorname{div} y\|_{\Omega_i} \leqslant (1 + |r|_{\infty,\Omega_i}) \|f + \operatorname{div} y\|_{\Omega_i}\,.
$$

But we also may derive another estimate. We rewrite (2.7) and use the Cauchy–Schwarz inequality in Ω_i

$$|\langle f + \operatorname{div} y, w\rangle_\Omega| = \left|\langle f + \operatorname{div} y, w\rangle_{\Omega_i}\right| \leqslant \|f + \operatorname{div} y\|_{\Omega_i} \|w\|_{\Omega_i} \tag{2.14}$$

and estimate

$$\|w\|_{\Omega_i} \leqslant c_{\Omega_i} \|\nabla w\|_{\Omega_i} \leqslant \frac{c_{\Omega_i}}{\sqrt{c_A}} \|\nabla w\|_{A,\Omega}\,, \tag{2.15}$$

where c_{Ω_i} denotes a Poincaré/Friedrichs constant associated with the bounded domain Ω_i, i.e., the best constant of the inequality

$$\|\varphi\|_{\Omega_i} \leqslant c_{\Omega_i} \|\nabla\varphi\|_{\Omega_i} \quad \forall\, \varphi \in \left\{\psi \in H^1(\Omega_i) \;\middle|\; \tau_{\partial\Omega_i}\psi|_\gamma = 0 \text{ on } \gamma\right\},$$

where $\tau_{\partial\Omega_i} : H^1(\Omega_i) \to H^{1/2}(\partial\Omega_i)$ denotes the trace operator. In this case, we have again (2.12), but now with the (optimal) weight constant

$$c_o := \frac{c_{\Omega_i}}{\sqrt{c_A}}. \tag{2.16}$$

We note that the constant (2.13) may also be achieved by (2.7) and the argument (2.14) if we replace the estimate (2.15) by

$$\|w\|_{\Omega_i} \leqslant (1 + |r|_{\infty,\Omega_i}) \|w\|_{-1,\Omega_i} \leqslant (1 + |r|_{\infty,\Omega_i}) \|w\|_{-1,\Omega} \leqslant \frac{c_N}{\sqrt{c_A}}(1 + |r|_{\infty,\Omega_i}) \|\nabla w\|_{A,\Omega}\,.$$

We summarize and get our second a posteriori error estimate.

Proposition 2.5. *For all $y \in D(\Omega)$ with* (2.11) *we have*

$$\|\nabla(u - v)\|_{A,\Omega} \leqslant c_o \|f + \operatorname{div} y\|_{\Omega_i} + \|y - A\nabla v\|_{A^{-1},\Omega} + 2c_\gamma \|g - \tau_\gamma v\|_{H^{1/2}(\gamma)}\,,$$

where c_o is defined either by (2.13) *or by* (2.16)*.*

Remark 2.6. In general, the number c_{Ω_i} will be smaller and thus provides a better bound than $c_N(1 + \|r\|_{\infty,\Omega_i})$. On the other hand, the number $c_N(1 + \|r\|_{\infty,\Omega_i})/\sqrt{c_A}$ is an easily computable upper bound for the best possible constant c_o.

2.3. Third estimate. Let y_i and y_e be the restrictions of some $y \in L^2(\Omega)$ to Ω_i and Ω_e respectively. Assuming $y_i \in D(\Omega_i)$ and $y_e \in D(\Omega_e)$, but not necessarily $y \in D(\Omega)$ we use the equations

$$\langle y_i, \nabla w\rangle_{\Omega_i} + \langle \operatorname{div} y_i, w\rangle_{\Omega_i} = \langle \tau_{n,\Gamma} y_i, \tau_\Gamma w\rangle_\Gamma\,, \tag{2.17}$$

$$\langle y_e, \nabla w\rangle_{\Omega_e} + \langle \operatorname{div} y_e, w\rangle_{\Omega_e} = -\langle \tau_{n,\Gamma} y_e, \tau_\Gamma w\rangle_\Gamma\,, \tag{2.18}$$

which hold for all $w \in \overset{\circ}{H}{}^1_{-1}(\Omega)$ and in the sense of the traces $\tau_\Gamma : H^1_{-1}(\Omega) \to H^{1/2}(\Gamma)$ and $\tau_{n,\Gamma} : D(\Omega_i) \to H^{-1/2}(\Gamma)$ respectively $\tau_{n,\Gamma} : D(\Omega_e) \to H^{-1/2}(\Gamma)$. At this point, we assume that the interface Γ is Lipschitz (in order to guarantee that the traces are well defined). We denote by $\langle\varphi, \psi\rangle_\Gamma$ the duality product of $H^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$. We recall that the normal traces $\tau_{n,\Gamma} y_i$

and $\tau_{n,\Gamma} y_e$ possess weak surface divergences in $H^{-1/2}(\Gamma)$ as well. If $y \in D(\Omega)$, then $\operatorname{div} y_i = \operatorname{div} y$ in Ω_i and $\operatorname{div} y_e = \operatorname{div} y$ in Ω_e. Hence, in this case, adding (2.17) and (2.18), we obtain

$$\langle \tau_{n,\Gamma} y_i - \tau_{n,\Gamma} y_e, \tau_\Gamma w \rangle_\Gamma = \langle y, \nabla w \rangle_{\Omega_i} + \langle \operatorname{div} y, w \rangle_\Omega = 0$$

for all $w \in \overset{\circ}{H}{}^1_{-1}(\Omega)$ in view (2.5). Therefore,

$$\tau_{n,\Gamma} y_i = \tau_{n,\Gamma} y_e$$

for all $y \in D(\Omega)$ since τ_Γ is surjective.

In our way to find Φ like in (2.6), we now insert (2.17), (2.18) instead of (2.5) into (2.1) and obtain

$$\begin{aligned} \langle A\nabla(u - v), \nabla w \rangle_\Omega &= \langle f + \operatorname{div} y_i, w \rangle_{\Omega_i} + \langle f + \operatorname{div} y_e, w \rangle_{\Omega_e} \\ &\quad + \langle y - A\nabla v, \nabla w \rangle_\Omega + \langle \tau_{n,\Gamma} y_e - \tau_{n,\Gamma} y_i, \tau_\Gamma w \rangle_\Gamma =: \Phi(w). \end{aligned} \tag{2.19}$$

The third term of $\Phi(w)$ will be estimated by (2.8) and for the last term we may use the continuity of the trace operator τ_Γ in combination with a Poincaré/Friedrichs estimate, i.e.,

$$\|\tau_\Gamma \varphi\|_{H^{1/2}(\Gamma)} \leqslant c_\Gamma \|\nabla \varphi\|_{A,\Omega} \quad \forall \varphi \in \overset{\circ}{H}{}^1_{-1}(\Omega), \tag{2.20}$$

and obtain

$$\begin{aligned} \left| \langle \tau_{n,\Gamma} y_e - \tau_{n,\Gamma} y_i, \tau_\gamma w \rangle_\Gamma \right| &\leqslant \|\tau_{n,\Gamma} y_e - \tau_{n,\Gamma} y_i\|_{H^{-1/2}(\Gamma)} \|\tau_\Gamma w\|_{H^{1/2}(\Gamma)} \\ &\leqslant c_\Gamma \|\tau_{n,\Gamma} y_e - \tau_{n,\Gamma} y_i\|_{H^{-1/2}(\Gamma)} \|\nabla w\|_{A,\Omega} . \end{aligned} \tag{2.21}$$

To estimate the second term of $\Phi(w)$, we again use (1.3) and (1.5) and obtain

$$\begin{aligned} \left| \langle f + \operatorname{div} y_e, w \rangle_{\Omega_e} \right| &\leqslant \|f + \operatorname{div} y_e\|_{1,\Omega_e} \|w\|_{-1,\Omega_e} \leqslant \|f + \operatorname{div} y_e\|_{1,\Omega_e} \|w\|_{-1,\Omega} \\ &\leqslant \frac{c_N}{\sqrt{c_A}} \|f + \operatorname{div} y_e\|_{1,\Omega_e} \|\nabla w\|_{A,\Omega} . \end{aligned} \tag{2.22}$$

Considering the first (and last) term of $\Phi(w)$, we have once more at least two options as in Section 2.2 to obtain the estimate

$$\left| \langle f + \operatorname{div} y_i, w \rangle_{\Omega_i} \right| \leqslant c_o \|f + \operatorname{div} y_i\|_{\Omega_i} \|\nabla w\|_{A,\Omega} \tag{2.23}$$

with c_o defined either by (2.13) or by (2.16).

Finally, with (2.19) and (2.8), (2.21), (2.22), and (2.23), by Corollary 2.2, we get the third estimate.

Proposition 2.7. *For all $y \in L^2(\Omega)$ with $y_i \in D(\Omega_i)$ and $y_e \in D(\Omega_e)$*

$$\begin{aligned} \|\nabla(u - v)\|_{A,\Omega} &\leqslant c_o \|f + \operatorname{div} y_i\|_{\Omega_i} + \frac{c_N}{\sqrt{c_A}} \|f + \operatorname{div} y_e\|_{1,\Omega_e} + \|y - A\nabla v\|_{A^{-1},\Omega} \\ &\quad + c_\Gamma \|\tau_{n,\Gamma} y_e - \tau_{n,\Gamma} y_i\|_{H^{-1/2}(\Gamma)} + 2c_\gamma \|g - \tau_\gamma v\|_{H^{1/2}(\gamma)} \end{aligned} \tag{2.24}$$

with c_o from Proposition 2.5*. The right-hand side of* (2.24) *vanishes if and only if v coincides with u and y with $A\nabla u$.*

Remark 2.8. There are many ways to deduce (2.20). We just mention that $\tau_\Gamma \varphi$ can be considered as a trace of a function defined in Ω_i or Ω_e or even of a function, which is just defined in a small neighborhood of Γ. Thus, we may adjust the constant c_Γ according to our needs.

Remark 2.9. This estimate suggests even a solution method. We construct approximations using locally supported trial functions in Ω_i, for example, FEM, and utilize global approximations properly behaving at infinity for Ω_e. These two types of approximations are usually difficult to meet together exactly on the artificial boundary Γ. However, Proposition 2.7 shows that this is not required because we can use instead the penalty term with known penalty factor c_Γ. In addition, we have one more parameter, the "radius" of the interface Γ. Since Γ is artificial and arbitrary, we can use this parameter in the algorithm in order to obtain better results.

Remark 2.10. At this point, we note that all our estimates are sharp, which easily can be seen by setting $v := u \in H^1_{-1}(\Omega)$ and $y := A\nabla u \in D(\Omega)$.

Remark 2.11. In Propositions 2.3, 2.5, and 2.7, we can always replace the last summand on the right-hand side with $2\left\|\nabla(\widehat{u} - \widehat{v})\right\|_{A,\Omega}$ or $2\left\|\nabla E(g - \tau_\gamma v)\right\|_{A,\Omega}$ in view of Theorem 2.1 and Corollary 2.2.

3. Upper Bounds in Dimension $N = 2$

Theorem 2.1 holds for $N = 2$ as well and the modifications on the estimates depend just on the Poincaré/Friedrichs estimate and thus they are obvious using the proper Cauchy–Schwarz inequality. Thus, the following assertions hold.

Proposition 3.1. *Let $\Omega \subset \mathbb{R}^2$ be such that $\mathbb{R}^2 \setminus \Omega$ contains the unit ball.*

(i) *For all $y \in D(\Omega)$*

$$\|\nabla(u - v)\|_{A,\Omega} \leqslant \frac{2}{\sqrt{c_A}} \|r \ln r (f + \operatorname{div} y)\|_\Omega + \|y - A\nabla v\|_{A^{-1},\Omega} + 2c_\gamma \|g - \tau_\gamma v\|_{H^{1/2}(\gamma)}.$$

(ii) *For all $y \in D(\Omega)$ with $\operatorname{div} y + f = 0$ in Ω_e*

$$\|\nabla(u - v)\|_{A,\Omega} \leqslant c_o \|f + \operatorname{div} y\|_{\Omega_i} + \|y - A\nabla v\|_{A^{-1},\Omega} + 2c_\gamma \|g - \tau_\gamma v\|_{H^{1/2}(\gamma)},$$

where

$$c_o = \min\left\{2\|r \ln r\|_{\infty,\Omega_i}, c_{\Omega_i}\right\} / \sqrt{c_A}.$$

(iii) *For all $y \in L^2(\Omega)$ with $y_i \in D(\Omega_i)$ and $y_e \in D(\Omega_e)$*

$$\|\nabla(u - v)\|_{A,\Omega} \leqslant c_o \|f + \operatorname{div} y_i\|_{\Omega_i} + \frac{2}{\sqrt{c_A}} \|r \ln r (f + \operatorname{div} y_e)\|_{1,\Omega_e} + \|y - A\nabla v\|_{A^{-1},\Omega}$$
$$+ c_\Gamma \|\tau_{n,\Gamma} y_e - \tau_{n,\Gamma} y_i\|_{H^{-1/2}(\Gamma)} + 2c_\gamma \|g - \tau_\gamma v\|_{H^{1/2}(\gamma)}.$$

Similarly, Remarks 2.6, 2.8–2.11 *are valid.*

4. Appendix

4.1. Lower bounds for the error. We note that, by a standard variational argument,

$$\|\nabla(u-v)\|^2_{A,\Omega} = \sup_{y\in L^2(\Omega)} \Big(2\,\langle A\nabla(u-v), y\rangle_\Omega - \|y\|^2_{A,\Omega}\Big)\,.$$

Thus, for all $w \in H^1_{-1}(\Omega)$ we obtain the estimate

$$\|\nabla(u-v)\|^2_{A,\Omega} \geqslant 2\,\langle A\nabla(u-v), \nabla w\rangle_\Omega - \|\nabla w\|^2_{A,\Omega} = 2\,\langle A\nabla u, \nabla w\rangle_\Omega - \langle A\nabla(2v+w), \nabla w\rangle_\Omega\,,$$

which is sharp since one can put $w = u - v$. But to exclude the unknown exact solution u from the right-hand side, we need $w \in \mathring{H}^1_{-1}(\Omega)$ since then, by (1.4), we have

$$\|\nabla(u-v)\|^2_{A,\Omega} \geqslant 2\,\langle f, w\rangle_\Omega - \langle A\nabla(2v+w), \nabla w\rangle_\Omega\,. \tag{4.1}$$

But this estimate is no longer sharp because we cannot put $w = u - v$ anymore. In fact, with $A\nabla u \in D(\Omega)$ and $\operatorname{div} A\nabla u = -f$ for $w \in H^1_{-1}(\Omega)$ we get

$$\langle A\nabla u, \nabla w\rangle_\Omega = \langle f, w\rangle_\Omega + \langle \tau_{n,\gamma} A\nabla u, \tau_\gamma w\rangle_\gamma\,.$$

Hence we obtain the estimate

$$\|\nabla(u-v)\|^2_{A,\Omega} \geqslant 2\,\langle f, w\rangle_\Omega - \langle A\nabla(2v+w), \nabla w\rangle_\Omega + 2\,\langle \tau_{n,\gamma} A\nabla u, \tau_\gamma w\rangle_\gamma$$

for all $w \in H^1_{-1}(\Omega)$, which is sharp and coincides with (4.1) if $w \in \mathring{H}^1_{-1}(\Omega)$. But the unknown exact solution u still appears on the right-hand side, i.e., the normal trace of $A\nabla u$ on γ. Furthermore, if $\langle \tau_{n,\gamma} A\nabla u, \tau_\gamma w\rangle_\gamma > 0$, then (4.1) cannot be sharp.

4.2. Poincaré type estimates for exterior domains. We introduce the radial derivative $\partial_r := \xi \cdot \nabla$, where $\xi(x) := x/r(x)$. Furthermore, B_ε and S_ε denote the open ball and sphere of radius ε centered at the origin in $\mathbb{R}^N$ respectively. We use the ideas of [**4**, Lemma 4.1] and [**1**, Poincaré's estimate III, p. 57] with some minor useful modifications.

Lemma 4.1. *Let $\Omega \subset \mathbb{R}^N$, $N \geqslant 1$, be a domain, and let $\beta \in \mathbb{R}$. For all $u \in \mathring{C}^\infty(\Omega)$ the following Poincaré estimates hold*:

(i) *If $\beta > 1 - N/2$, then*

$$(2\beta + N - 2)\left\|r^{\beta-1} u\right\|_\Omega \leqslant 2\left\|r^\beta \partial_r u\right\|_\Omega\,.$$

(ii) *Let $B_1 \subset \mathbb{R}^N \setminus \Omega$. If $\beta \geqslant (3-N)/2$ or $\beta \leqslant 1 - N/2$, then*

$$|2\beta + N - 3|\left\|\frac{r^{\beta-1}}{\ln r} u\right\|_\Omega \leqslant 2\left\|r^\beta \partial_r u\right\|_\Omega\,.$$

(iii) *If $N = 1$, then*

$$|2\beta-1|\left\|(1+r)^{\beta-1}u\right\|_{\Omega}\leqslant 2\left\|(1+r)^{\beta}\partial_r u\right\|_{\Omega}+|2\min\{0,2\beta-1\}|^{1/2}\,|u(0)|,$$

where u will be extended by zero to $\mathbb{R}$.

For the estimates derived in this paper it suffices to set $\beta=0$. In this particular case, the above lemma implies the following assertion.

Corollary 4.2. *Let $\Omega\subset\mathbb{R}^N$, $N\geqslant 1$, be a domain. For all $u\in\overset{\circ}{C}{}^{\infty}(\Omega)$ the following Poincaré estimates hold*:

(i) *If $N\geqslant 3$, then*

$$\|u\|_{-1,\Omega}\leqslant\|u/(1+r)\|_{\Omega}\leqslant\|u/r\|_{\Omega}\leqslant\frac{2}{N-2}\|\partial_r u\|_{\Omega}\leqslant\frac{2}{N-2}\|\nabla u\|_{\Omega}.$$

(ii) *If $N=2$ and $B_1\subset\mathbb{R}^2\setminus\Omega$, then*

$$\|u/(r\ln r)\|_{\Omega}\leqslant 2\|\partial_r u\|_{\Omega}\leqslant 2\|\nabla u\|_{\Omega}.$$

(iii) *If $N=1$, then*

$$\|u\|_{-1,\Omega}\leqslant\|u/(1+r)\|_{\Omega}\leqslant 2\|\partial_r u\|_{\Omega}+\sqrt{2}|u(0)|\leqslant 2\left\|u'\right\|_{\Omega}+\sqrt{2}|u(0)|.$$

Hence, if $\Omega\subset\mathbb{R}_{\pm}$, then

$$\|u\|_{-1,\Omega}\leqslant\|u/(1+r)\|_{\Omega}\leqslant 2\|\partial_r u\|_{\Omega}\leqslant 2\left\|u'\right\|_{\Omega}.$$

Remark 4.3. By continuity, all these estimates extend to appropriate weighted H^1-Sobolev spaces.

Proof. Let $\Omega\subset\mathbb{R}^N$, $N\geqslant 1$, be a domain, and let $u\in\overset{\circ}{C}{}^{\infty}(\Omega)$. By partial integration, for all $\alpha\in\mathbb{R}$ and $\varepsilon>0$ we get

$$2\int\limits_{\Omega\setminus B_\varepsilon} r^{\alpha}u\partial_r u\,\mathrm{d}\lambda=\int\limits_{\Omega\setminus B_\varepsilon} r^{\alpha}\partial_r|u|^2\,\mathrm{d}\lambda=-(\alpha+N-1)\int\limits_{\Omega\setminus B_\varepsilon} r^{\alpha-1}|u|^2\,\mathrm{d}\lambda-\varepsilon^{\alpha}\int\limits_{S_\varepsilon}|u|^2\,\mathrm{d}\sigma.$$

Thus, for all $\gamma\in\mathbb{R}$ and $\beta:=(\alpha+1)/2$

$$\begin{aligned}\left\|r^{\beta}\partial_r u+\gamma r^{\beta-1}u\right\|^2_{\Omega\setminus B_\varepsilon}&=\left\|r^{\beta}\partial_r u\right\|^2_{\Omega\setminus B_\varepsilon}+|\gamma|^2\left\|r^{\beta-1}u\right\|^2_{\Omega\setminus B_\varepsilon}+2\gamma\underbrace{\left\langle r^{\beta}\partial_r u,r^{\beta-1}u\right\rangle_{\Omega\setminus B_\varepsilon}}_{=\int\limits_{\Omega\setminus B_\varepsilon} r^{\alpha}u\partial_r u\,\mathrm{d}\lambda}\\&=\left\|r^{\beta}\partial_r u\right\|^2_{\Omega\setminus B_\varepsilon}+\gamma(\gamma-2\beta-N+2)\left\|r^{\beta-1}u\right\|^2_{\Omega\setminus B_\varepsilon}-\gamma\varepsilon^{2\beta-1}\int\limits_{S_\varepsilon}|u|^2\,\mathrm{d}\sigma.\end{aligned}$$

Now the left-hand side of this equality converges by the monotone convergence theorem. Since $r^\nu \in L^1(U_1)$ if and only if $\nu > -N$ and

$$\Big|\int\limits_{S_\varepsilon} |u|^2 \,\mathrm{d}\sigma\Big| \leqslant c\varepsilon^{N-1},$$

the right-hand side converges for $\beta > 1 - N/2$ by the Lebesgue dominated convergence theorem in $\mathbb{R}$. Hence

$$\left\| r^\beta \partial_r u + \gamma r^{\beta-1} u \right\|_\Omega^2 = \left\| r^\beta \partial_r u \right\|_\Omega^2 + \gamma(\gamma - 2\beta - N + 2) \left\| r^{\beta-1} u \right\|_\Omega^2$$

as $\varepsilon \to 0$. Choosing $\gamma := 2\beta + N - 2 > 0$ and using the triangle inequality, we get

$$\gamma \left\| r^{\beta-1} u \right\|_\Omega \leqslant 2 \left\| r^\beta \partial_r u \right\|_\Omega .$$

Since we are interested in the case $\beta = 0$, this estimate is only applicable in dimensions $N \geqslant 3$.

For $N = 1$ we proceed as follows: For all $\alpha \in \mathbb{R}$ we have

$$\begin{aligned} 2 \int\limits_{\mathbb{R}_\pm} (1+r)^\alpha u \partial_r u \,\mathrm{d}\lambda &= \pm 2 \int\limits_{\mathbb{R}_\pm} (1 \pm t)^\alpha u(t) u(t)' \,\mathrm{d}\, t = \pm 2 \int\limits_{\mathbb{R}_\pm} (1 \pm t)^\alpha \big(|u(t)|^2\big)' \,\mathrm{d}\, t \\ &= -\alpha \int\limits_{\mathbb{R}_\pm} (1 \pm t)^{\alpha-1} |u(t)|^2 \,\mathrm{d}\, t - |u(0)|^2 \end{aligned}$$

and thus

$$2 \int\limits_{\mathbb{R}} (1+r)^\alpha u \partial_r u \,\mathrm{d}\lambda = -\alpha \int\limits_{\mathbb{R}} (1+r)^{\alpha-1} |u(t)|^2 \,\mathrm{d}\lambda - 2|u(0)|^2.$$

Hence for all $\gamma \in \mathbb{R}$ and $\beta := (\alpha+1)/2$

$$\begin{aligned} &\left\| (1+r)^\beta \partial_r u + \gamma (1+r)^{\beta-1} u \right\|_\Omega^2 \\ &= \left\| (1+r)^\beta \partial_r u \right\|_\Omega^2 + |\gamma|^2 \left\| (1+r)^{\beta-1} u \right\|_\Omega^2 + 2\gamma \underbrace{\left\langle (1+r)^\beta \partial_r u, (1+r)^{\beta-1} u \right\rangle_\Omega}_{= \int\limits_\Omega (1+r)^\alpha u \partial_r u \,\mathrm{d}\lambda} \\ &= \left\| (1+r)^\beta \partial_r u \right\|_\Omega^2 + \gamma(\gamma - 2\beta + 1) \left\| (1+r)^{\beta-1} u \right\|_\Omega^2 - 2\gamma |u(0)|^2. \end{aligned}$$

As above, the triangle inequality and the choice $\gamma := 2\beta - 1$, but now without any restrictions on β, lead to

$$|\gamma| \left\|(1+r)^{\beta-1}u\right\|_\Omega \leqslant \left\|(1+r)^\beta\partial_r u\right\|_\Omega + \left(\left\|(1+r)^\beta\partial_r u\right\|_\Omega^2 - 2\gamma|u(0)|^2\right)^{1/2}$$
$$\leqslant 2\left\|(1+r)^\beta\partial_r u\right\|_\Omega + |2\min\{0,\gamma\}|^{1/2}\,|u(0)|.$$

The remaining case $N = 2$ requires the use of logarithms. Moreover, the origin is now a problematic singularity, which has to be removed from our domain. Therefore, we may assume $B_1 \subset \mathbb{R}^N \setminus \Omega$ and $N \geqslant 1$ having $N = 2$ in mind. We start once more for all $\alpha \in \mathbb{R}$ with

$$2\int_\Omega \frac{r^\alpha}{\ln r} u\partial_r u \,\mathrm{d}\lambda = \int_\Omega \frac{r^\alpha}{\ln r}\partial_r |u|^2 \,\mathrm{d}\lambda = -(\alpha+N-1)\int_\Omega \frac{r^{\alpha-1}}{\ln r}|u|^2\,\mathrm{d}\lambda + \int_\Omega \frac{r^{\alpha-1}}{\ln^2 r}|u|^2\,\mathrm{d}\lambda.$$

Now, our usual procedure gives for $\gamma \in \mathbb{R}$ and $\beta := (\alpha+1)/2 \geqslant 0$

$$\left\|r^\beta\partial_r u + \gamma\frac{r^{\beta-1}}{\ln r}u\right\|_\Omega^2 = \left\|r^\beta\partial_r u\right\|_\Omega^2 + |\gamma|^2\left\|\frac{r^{\beta-1}}{\ln r}u\right\|_\Omega^2 + 2\gamma\underbrace{\left\langle r^\beta\partial_r u, \frac{r^{\beta-1}}{\ln r}u\right\rangle_\Omega}_{=\int_\Omega \frac{r^\alpha}{\ln r}u\partial_r u\,\mathrm{d}\lambda}$$
$$= \left\|r^\beta\partial_r u\right\|_\Omega^2 + \gamma(\gamma+1)\left\|\frac{r^{\beta-1}}{\ln r}u\right\|_\Omega^2 - \gamma(N+2\beta-2)\left\|\frac{r^{\beta-1}}{\sqrt{\ln r}}u\right\|_\Omega^2.$$

Thus, for $\gamma(N+2\beta-2) \geqslant 0$ we can estimate

$$\left\|r^\beta\partial_r u + \gamma\frac{r^{\beta-1}}{\ln r}u\right\|_\Omega^2 \leqslant \left\|r^\beta\partial_r u\right\|_\Omega^2 + \gamma(\gamma-2\beta-N+3)\left\|\frac{r^{\beta-1}}{\ln r}u\right\|_\Omega^2,$$

which leads to the estimate

$$\left\|r^\beta\partial_r u + \gamma\frac{r^{\beta-1}}{\ln r}u\right\|_\Omega^2 \leqslant \left\|r^\beta\partial_r u\right\|_\Omega^2$$

if we set $\gamma := 2\beta + N - 3$ with the additional constraint $\gamma(\gamma+1) \geqslant 0$, i.e., $\gamma \geqslant 0$ or $\gamma \leqslant -1$. Finally, again by the triangle inequality,

$$|\gamma|\left\|\frac{r^{\beta-1}}{\ln r}u\right\|_\Omega \leqslant 2\left\|r^\beta\partial_r u\right\|_\Omega$$

follows for all $\beta \geqslant (3-N)/2$ or $\beta \leqslant (2-N)/2$. □

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