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Traces for Hilbert complexes

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ABSTRACT

We study a new notion of trace operators and trace spaces for abstract Hilbert complexes. We introduce trace spaces as quotient spaces/annihilators. We characterize the kernels and images of the related trace operators and discuss duality relationships between trace spaces. We elaborate that many properties of the classical boundary traces associated with the Euclidean de Rham complex on bounded Lipschitz domains are rooted in the general structure of Hilbert complexes. We arrive at abstract trace Hilbert complexes that can be formulated using quotient spaces/annihilators. We show that, if a Hilbert complex admits stable "regular decompositions" with compact lifting operators, then the associated trace Hilbert complex is Fredholm. Incarnations of abstract concepts and results in the concrete case of the de Rham complex in three-dimensional Euclidean space will be discussed throughout.

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1. Introduction

1.1. Starting point: the de Rham complex

In vector-analytic notation, the L^2 de Rham complex in a bounded domain $\Omega \subset \mathbb{R}^3$ reads[1]

$$\mathbb{R} \xhookrightarrow{\imath_{\mathbb{R}}} L^2(\Omega) \xrightarrow{\mathbf{grad}} L^2(\Omega) \xrightarrow{\mathbf{curl}} L^2(\Omega) \xrightarrow{\mathrm{div}} L^2(\Omega) \xrightarrow{\pi_{\{0\}}} \{0\}. \tag{1.1}$$

It involves unbounded first-order differential operators inducing the domain Hilbert complex

$$\mathbb{R} \xhookrightarrow{\imath_{\mathbb{R}}} H^1(\Omega) \xrightarrow{\mathbf{grad}} \mathbf{H}(\mathbf{curl},\Omega) \xrightarrow{\mathbf{curl}} \mathbf{H}(\mathrm{div},\Omega) \xrightarrow{\mathrm{div}} L^2(\Omega) \xrightarrow{\pi_{\{0\}}} \{0\}, \tag{1.2}$$

where customary notation for Sobolev spaces equipped with graph inner products was adopted.[2] Taking the closure of compactly supported functions in these Sobolev spaces and tagging the resulting closed subspaces with '$\circ$' on top, we obtain a subcomplex

$$\{0\} \xhookrightarrow{\imath} \mathring{H}^1(\Omega) \xrightarrow{\mathbf{grad}} \mathring{\mathbf{H}}(\mathbf{curl},\Omega) \xrightarrow{\mathbf{curl}} \mathring{\mathbf{H}}(\mathrm{div},\Omega) \xrightarrow{\mathrm{div}} L^2(\Omega) \xrightarrow{0} \{0\}, \tag{1.3}$$

giving rise to the following structure:

$$\begin{array}{ccccccc} H^1(\Omega) & \xrightarrow{\mathbf{grad}} & \mathbf{H}(\mathbf{curl},\Omega) & \xrightarrow{\mathbf{curl}} & \mathbf{H}(\mathrm{div},\Omega) & \xrightarrow{\mathrm{div}} & L^2(\Omega) \\ \cup & & \cup & & \cup & & \cup \\ \mathring{H}^1(\Omega) & \xrightarrow{\mathbf{grad}} & \mathring{\mathbf{H}}(\mathbf{curl},\Omega) & \xrightarrow{\mathbf{curl}} & \mathring{\mathbf{H}}(\mathrm{div},\Omega) & \xrightarrow{\mathrm{div}} & L^2(\Omega). \end{array} \tag{1.4}$$

1.2. The de Rham complex and trace operators

The focus of this work is on trace operators. For the de Rham complex above, those are usually introduced as linear mappings of functions in Ω to functions on $\Gamma = \partial\Omega$. Let us confine ourselves to Lipschitz boundaries Γ. In this case, the spaces $H^s(\Gamma)$, $0 \leq s \leq 1$, can be defined by localization and pullback under charts [30, Ch. 3] and, subsequently, by duality for $-1 \leq s < 0$: $H^{-s}(\Gamma) = (H^s(\Gamma))'$. Local charts can also be used to (almost everywhere on Γ) introduce surface differential operators, for instance the surface gradient $\mathbf{grad}_\Gamma : H^1(\Gamma) \to \mathbf{L}^2_t(\Gamma)$ mapping into tangential surface vector fields and its rotated version, the vector-valued surface rotation $\mathbf{curl}_\Gamma$ [33, Sect. 2.5]. By duality

[1] Throughout, we use special arrows to indicate properties of mappings: '↠' for surjectivity, '↪' for injectivity and '⇢' for isometry.

[2] For instance, the spaces $H^1(\Omega)$, $\mathbf{H}(\mathbf{curl},\Omega)$ and $\mathbf{H}(\mathrm{div},\Omega)$ are discussed in [21]. They are equipped with the obvious graph norms making the operators involved in the domain Hilbert complex trivially bounded. In the Euclidean setting, we distinguish vector quantities from scalars by using a bold font.

and interpolation these operators can be extended to $H^s(\Gamma)$, $-1 \leq s < 1$, see [14, Sect. 3]. Duality also yields the scalar-valued surface rotation $\mathtt{curl}_\Gamma : \mathbf{L}^2_\mathtt{t}(\Gamma) \to \mathtt{H}^{-1}(\Gamma)$ as the adjoint of $\mathbf{curl}_\Gamma$.

The classical trace operators are obtained by extending the restriction operators[3]

$$\gamma u := u\big|_\Gamma \qquad \text{(pointwise trace)}, \tag{1.5a}$$

$$\gamma_t \mathbf{u} := \mathbf{n} \times (\mathbf{u}\big|_\Gamma \times \mathbf{n}) \qquad \text{(pointwise tangential component trace)}, \tag{1.5b}$$

$$\gamma_n \mathbf{u} := \mathbf{u}\big|_\Gamma \cdot \mathbf{n} \qquad \text{(pointwise normal component trace)}, \tag{1.5c}$$

to continuous and surjective mappings from the Sobolev spaces involved in the domain de Rham complex to so-called trace spaces whose characterization is the main assertion of the standard trace theorems for a Lipschitz domain Ω:

$$\gamma : H^1(\Omega) \to H^{1/2}(\Gamma) \qquad \text{[25, Thm. 4.2.1]}, \tag{1.6a}$$

$$\gamma_t : \mathbf{H}(\mathbf{curl}, \Omega) \to \mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma) \qquad \text{[14, Thm. 4.1]}, \tag{1.6b}$$

$$\gamma_n : \mathbf{H}(\mathrm{div}, \Omega) \to H^{-1/2}(\Gamma) \qquad \text{[21, Thm. 2.5, Cor. 2.8]}. \tag{1.6c}$$

The classical trace spaces can be defined based on surface differential operators as

$$H^{1/2}(\Gamma) := \left\{ \boldsymbol{\phi} \in H^{-1/2}(\Gamma) \mid \mathbf{curl}_\Gamma\, \boldsymbol{\phi} \in \mathbf{H}_t^{-1/2}(\Gamma) \right\}, \tag{1.7a}$$

$$\mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma) := \left\{ \boldsymbol{\phi} \in \mathbf{H}_t^{-1/2}(\Gamma) \mid \mathrm{curl}_\Gamma\, \boldsymbol{\phi} \in H^{-1/2}(\Gamma) \right\}, \tag{1.7b}$$

where $\mathbf{H}_t^{-1/2}(\Gamma)$ designates the dual of the range of the tangential trace applied to $\mathbf{H}^1(\Omega)$. The mathematical theory of the pointwise trace γ is well established, cf. [30, Chap. 3]. That for the normal component trace γ_n is carefully developed in [21, Chap. 1]. Regarding the tangential trace γ_t in (1.6b) and the trace space (1.7b), we refer to the comprehensive and profound analysis of [14], based on the earlier works [1,12,13].

These important results were generalized to arbitrary dimensions by Weck in [45] using the framework of differential forms, where pullback by the boundary's inclusion map provides a unified description and generalization of the traces (1.6). A similar characterization of the range of the boundary restriction operator for Lipschitz subdomains of compact manifolds is given in [31], where a boundary de Rham complex involving surface differential operators is also studied.

One may wonder whether the structures shining through in (1.7a) and (1.7b) hint at a more general pattern governing the structure of trace spaces. Thus, in this article, we are going to elaborate this structure in the abstract framework of Hilbert complexes, of which the de Rham complex is the best-known representative. Since there is no notion of "boundary" in that abstract framework, we have to detach the concept of a trace space

[3] We denote by $\mathbf{n} \in \mathbf{L}^\infty(\Gamma)$ the exterior unit normal vector-field on the boundary Γ.

from the idea of a function space on a boundary. This can be accomplished by adopting a quotient-space view of traces.

Let us sketch this idea for the Euclidean de Rham complex. Since the kernels of the classical trace operators (1.6a)-(1.6c) are[4]

$$\mathcal{N}(\gamma) = \mathring{H}^1(\Omega) := \overline{C_0^\infty(\Omega)}^{H^1(\Omega)} \qquad [30, \text{ Thm. } 3.40], \tag{1.8a}$$

$$\mathcal{N}(\gamma_t) = \mathring{\mathbf{H}}(\mathbf{curl}, \Omega) := \overline{C_0^\infty(\Omega)^3}^{\mathbf{H}(\mathbf{curl},\Omega)} \qquad [32, \text{ Thm. } 3.33], \tag{1.8b}$$

$$\mathcal{N}(\gamma_n) = \mathring{\mathbf{H}}(\mathrm{div}, \Omega) := \overline{C_0^\infty(\Omega)^3}^{\mathbf{H}(\mathrm{div},\Omega)} \qquad [32, \text{ Thm. } 3.25], \tag{1.8c}$$

we immediately conclude that these trace operators induce isomorphisms between the classical trace spaces and the quotient spaces:

$$H^1(\Omega)/\mathring{H}^1(\Omega) \cong H^{1/2}(\Gamma), \tag{1.9a}$$

$$\mathbf{H}(\mathbf{curl}, \Omega)/\mathring{\mathbf{H}}(\mathbf{curl}, \Omega) \cong \mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma), \tag{1.9b}$$

$$\mathbf{H}(\mathrm{div}, \Omega)/\mathring{\mathbf{H}}(\mathrm{div}, \Omega) \cong H^{-1/2}(\Gamma). \tag{1.9c}$$

This paves the way for an alternative characterization of trace spaces independent of the notion of "function space on Γ". We remark that the quotient space approach to the definition of trace spaces has also proved successful for the de Rham complex in order to define traces on sets more complicated than boundaries of Lipschitz domains [16,17].

Classical theory of trace spaces for $H^1(\Omega)$, $\mathbf{H}(\mathbf{curl}, \Omega)$ and $\mathbf{H}(\mathrm{div}, \Omega)$ also addresses duality between trace spaces:

- The $L^2(\Gamma)$ inner product induces a duality between $H^{1/2}(\Gamma)$ and $H^{-1/2}(\Gamma)$; cf. [25, Chap. 4.2] and [30, Chap. 3].
- The skew-symmetric pairing[5]

$$\langle \mathbf{u}, \mathbf{v} \rangle_\times := \int_\Gamma (\mathbf{u} \times \mathbf{n}) \cdot \mathbf{v} \, \mathrm{d}\sigma \tag{1.10}$$

can be extended from $\mathbf{L}^2(\Gamma) \times \mathbf{L}^2(\Gamma)$ to $\mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma) \times \mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)$, allowing the identification of $\mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)$ with its own dual space, cf. [14,15,32].

The following diagram hints that also the possibility to put trace spaces for the 3D de Rham complex into duality is governed by general rules.

[4] We write $\mathcal{N}(\mathsf{T})$ and $\mathcal{R}(\mathsf{T})$ for the kernel/nullspace and range/image space, respectively, of a linear operator T.

[5] We denote by σ the surface measure on the boundary.

$$
\begin{array}{ccccc}
H^1(\Omega) & \xrightarrow{\ \mathbf{grad}\ } & \mathbf{H}(\mathbf{curl},\Omega) & \xrightarrow{\ \mathbf{curl}\ } & \mathbf{H}(\mathrm{div},\Omega) \\
\big\downarrow{\scriptstyle \gamma} & & \big\downarrow{\scriptstyle \gamma_t} & & \big\downarrow{\scriptstyle \gamma_n} \\
\mathbf{H}^{1/2}(\Gamma) & & \mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma,\Gamma) \ \circlearrowleft\ L^2\text{-self duality} & & H^{-1/2}(\Gamma) \\
 & & L^2\text{-duality} \ (\mathbf{H}^{1/2}(\Gamma) \leftrightarrow H^{-1/2}(\Gamma)) & &
\end{array}
\tag{1.11}
$$

1.3. Goals, outline, and main results

There are obvious parallels in the definitions of the different trace spaces and their duality relations. One may wonder if this kind of resemblance between the trace spaces arises only for the de Rham complex or whether it is already manifest in a more basic/general setting, of which the de Rham complex is just a prominent specimen. That setting is the framework of *Hilbert complexes*,[6] first introduced in [11]. Therefore, the guiding question behind this work is:

> *To what extent can results about traces for the de Rham domain complex be transferred to abstract Hilbert complexes?*

Of course, abstract Hilbert complexes know neither domains nor boundaries. Therefore, as already mentioned above, we cannot expect to arrive at a characterization of trace spaces as function spaces on a boundary. Yet, a theory based on the quotient space view of trace spaces is feasible. Its development will be pursued in Section 3. There, we first propose trace operators induced by "generalized integration by parts formulas" and mapping into dual spaces, and then generalize (1.9) to a quotient-space understanding of trace spaces.

Next, in Section 4, we shed light on duality relationships between trace spaces and find that the observation made in (1.11) is a generic pattern; see Theorem 4.8. This even holds in a setting simpler than Hilbert complexes. "Minimal Hilbert complexes" will only enter the stage in Section 5 in order to define so-called "surface operators", which are abstract counterparts of the classical surface differential operators such as $\mathbf{grad}_\Gamma$ and $\mathbf{curl}_\Gamma$. The full structure of Hilbert complexes is exploited starting from Section 6. Augmenting it by assumptions about the existence of so-called stable regular decompositions (Assumptions B and C), we obtain characterizations of traces spaces, in Theorem 6.8 and Theorem 6.9, which reveal that the definitions (1.7a) and (1.7b) of classical trace spaces reflect a more general pattern. This paves the way for the key insight expressed in Theorem 7.1 that trace spaces and surface operators are the building blocks

6 For the functional analytic foundations, we refer to parts of the FA-ToolBox from [37, Sec. 2], which is a compilation of useful functional analysis results that grew from its use in previous works, cf. [35, Sec. 4.1], [36, Sec. 2], [38, Sec. 2.1], [39, Sec. 2.1], [40, 2.2], [37, Sec. 2] and [34, App. 3]. We find the introduction in [6, Chap. 4] to be an accessible resource for readers unacquainted with Hilbert complexes, because it reviews in detail the material more concisely presented in [8, Sec. 3], cf. [7, Sec. 2] and [11].

of what we call a trace Hilbert complex, a full-fledged Hilbert complex of unbounded, densely defined, and closed operators.

Parallel to its development, we will apply our new abstract theory to the de Rham complex in three-dimensional Euclidean space. We hope that this will motivate some of the assumptions made on the abstract spaces. The discussion will take the form of an ongoing specialization of the definitions and results, set apart from the main line of reasoning.

3D de Rham setting I: Traces and integration by parts. The key trace operators and trace spaces associated with the Euclidean de Rham complex in three space dimensions have already been introduced in (1.5) and (1.6). We just want to add the well-known fact that the trace operators (1.6a)-(1.6c) have a close link with Green's formulas

$$\langle \gamma u, \gamma_n \mathbf{v} \rangle_\Gamma = \int_\Omega \mathbf{grad} u \cdot v + u \operatorname{div}(\mathbf{v}) \, \mathrm{d}\mathbf{x} \qquad \forall u \in H^1(\Omega), \forall \mathbf{v} \in \mathbf{H}(\operatorname{div}, \Omega), \tag{1.12a}$$

$$\langle \gamma_t \mathbf{u}, \gamma_t \mathbf{v} \rangle_\times = \int_\Omega \mathbf{curl}\, \mathbf{u} \cdot \mathbf{v} - \mathbf{u} \cdot \mathbf{curl}\, \mathbf{v} \mathrm{d}\mathbf{x} \qquad \forall \mathbf{u}, \mathbf{v} \in \mathbf{H}(\mathbf{curl}, \Omega). \tag{1.12b}$$

On the left, we denoted the duality pairing between $H^{1/2}(\Gamma)$ and $H^{-1/2}(\Gamma)$ by $\langle \cdot, \cdot \rangle_\Gamma$, but wrote $\langle \cdot, \cdot \rangle_\times$ for the skew-symmetric self-duality pairing on $\mathbf{H}^{-1/2}(\operatorname{curl}_\Gamma, \Gamma)$, cf. [14, Lem. 5.6]. ⊿

Finally, we stress that we could have demonstrated the specialization of our results also in the setting of general exterior calculus, but refrained from it in the interest of readability.

List of symbols

A_k	$\widehat{=}$ closed densely defined unbounded operators	Section 2.2, (2.5a)
A_k^*	$\widehat{=}$ Hilbert space adjoint of A_k	Section 2.2, (2.5b)
$\mathring{\mathsf{A}}_k$	$\widehat{=}$ closed densely defined unbounded operator $\mathring{\mathsf{A}}_k \subset \mathsf{A}_k$	Section 2.3, (2.8a)
$\mathring{\mathsf{A}}_k^*$	$\widehat{=}$ Hilbert space adjoint of $\mathring{\mathsf{A}}_k$	Section 2.3, (2.8b)
$\mathsf{R}_{\mathcal{D}(\mathring{\mathsf{A}}_k^*)}$	$\widehat{=}$ Riesz isomorphism $\mathcal{D}(\mathring{\mathsf{A}}_k^*) \to \mathcal{D}(\mathring{\mathsf{A}}_k^*)'$	Section 3.3, (3.12)
T_k^t	$\widehat{=}$ primal Hilbert trace $\mathcal{D}(\mathsf{A}_k) \to \mathcal{D}(\mathring{\mathsf{A}}_k^*)'$	Section 3.1, (3.3)
T_k^n	$\widehat{=}$ dual Hilbert trace $\mathcal{D}(\mathring{\mathsf{A}}_k^*) \to \mathcal{D}(\mathsf{A}_k)'$	Section 4.1, (4.2)
$\mathcal{T}(\mathsf{A}_k)$	$\widehat{=}$ quotient space $\mathcal{D}(\mathsf{A}_k)/\mathcal{D}(\mathring{\mathsf{A}}_k)$	Section 3.2, (3.23)
$\mathcal{T}(\mathring{\mathsf{A}}_k^*)$	$\widehat{=}$ quotient space $\mathcal{D}(\mathring{\mathsf{A}}_k^*)/\mathcal{D}(\mathsf{A}_k^*)$	Section 4.1, (4.8)
I_k^t	$\widehat{=}$ isometric isomorphism $\mathcal{D}(\mathsf{A}_k) \to \mathcal{R}(\mathsf{T}_k^t)$	Section 3.2, (3.39)
I_k^n	$\widehat{=}$ isometric isomorphism $\mathcal{D}(\mathring{\mathsf{A}}_k^*) \to \mathcal{R}(\mathsf{T}_k^n)$	Section 4.1, (4.19)
$\langle\langle \cdot, \cdot \rangle\rangle_k$	$\widehat{=}$ duality pairing	Section 4.2, (4.24b)
K_k	$\widehat{=}$ isometric isomorphism induced by $\langle \cdot, \cdot \rangle_k$	Section 4.2, (4.26)
P_k^t	$\widehat{=}$ orthogonal projection $\mathcal{D}(\mathsf{A}_k) \to \mathcal{D}(\mathring{\mathsf{A}})^\perp$	Section 3.1, (3.28)
P_k^n	$\widehat{=}$ orthogonal projection $\mathcal{D}(\mathring{\mathsf{A}}_k^*) \to \mathcal{D}(\mathsf{A}_k^*)^\perp$	Section 4.1, (4.12)
$\boldsymbol{\pi}_k^t$	$\widehat{=}$ canonical quotient map $\mathcal{D}(\mathsf{A}_k) \to \mathcal{T}(\mathsf{A}_k)$	Section 3.1, (3.28)
$\boldsymbol{\pi}_k^n$	$\widehat{=}$ canonical quotient map $\mathcal{D}(\mathring{\mathsf{A}}_k^*) \to \mathcal{T}(\mathring{\mathsf{A}}_k^*)$	Section 3.1, (4.12)

$\mathbf{W}_k^+$	$\hat{=}$ dense inclusion $\mathbf{W}_k^+ \hookrightarrow \mathcal{D}(\mathsf{A}_k)$ and/or $\mathbf{W}_k^+ \hookrightarrow \mathcal{D}(\mathring{\mathsf{A}}_{k-1}^*)$	Section 6.1, (6.1)
$\mathbf{W}_k^-$	$\hat{=}$ dual space $(\mathbf{W}_k^+)'$	Section 6.1, (6.7)
$\mathring{\mathbf{W}}_k^{n,+}$	$\hat{=}$ intersection space $\mathcal{D}(\mathsf{A}_{k-1}^*) \cap \mathbf{W}_k^+ = \mathcal{N}(\mathsf{T}_{k-1}^n) \cap \mathbf{W}_k^+$	Section 6.3, (6.32)
$\mathring{\mathbf{W}}_k^{t,+}$	$\hat{=}$ intersection space $\mathcal{D}(\mathring{\mathsf{A}}_k) \cap \mathbf{W}_k^+ = \mathcal{N}(\mathsf{T}_k^t) \cap \mathbf{W}_k^+$	Section 6.3, (6.32)
$\mathbf{T}_k^{n,+}$	$\hat{=}$ quotient space $\mathbf{W}_k^+/\mathring{\mathbf{W}}^{n,+}$	Section 6.4, (6.41b)
$\mathbf{T}_k^{t,+}$	$\hat{=}$ quotient space $\mathbf{W}_k^+/\mathring{\mathbf{W}}^{t,+}$	Section 6.4, (6.41a)
$\mathbf{T}_k^{n,-}$	$\hat{=}$ dual space $(\mathbf{T}_k^{n,+})'$	Section 6.4, (6.41b)
$\mathbf{T}_k^{t,-}$	$\hat{=}$ dual space $(\mathbf{T}_k^{t,+})'$	Section 6.4, (6.41a)
D_k^t	$\hat{=}$ surface operator $(\mathring{\mathsf{A}}_{k+1}^*)' : \mathcal{D}(\mathring{\mathsf{A}}_k^*)' \to \mathcal{D}(\mathring{\mathsf{A}}_{k+1}^*)'$	Section 5.1, (5.4a)
D_k^n	$\hat{=}$ surface operator $\mathsf{A}_{k-1}' : \mathcal{D}(\mathsf{A}_k)' \to \mathcal{D}(\mathsf{A}_{k-1})'$	Section 5.1, (5.4b)
S_k^t	$\hat{=}$ surface operator $\boldsymbol{\pi}_{k+1}^t \mathsf{A}_k : \mathcal{T}(\mathsf{A}_k) \to \mathcal{T}(\mathsf{A}_{k+1})$	Section 5.2, (5.23)
S_{k+1}^n	$\hat{=}$ surface operator $\boldsymbol{\pi}_k^n \mathring{\mathsf{A}}_{k+1}^* : \mathcal{T}(\mathring{\mathsf{A}}_k^*) \to \mathcal{T}(\mathring{\mathsf{A}}_{k-1}^*)$	Section 5.2, (5.23)
$\hat{\mathsf{S}}_k^t$	$\hat{=}$ surface operator $\boldsymbol{\pi}_{k+1}^t \mathsf{A}_k : \mathbf{T}_{k+1}^{t,+} \to \mathcal{T}(\mathsf{A}_{k+1})$	Section 6.4 (6.44)
$\hat{\mathsf{S}}_k^n$	$\hat{=}$ surface operator $\boldsymbol{\pi}_k^n \mathring{\mathsf{A}}_{k+1}^* : \mathbf{T}_{k+1}^{n,+} \to \mathcal{T}(\mathring{\mathsf{A}}_{k-1}^*)$	Section 6.4, (6.44)
$\hat{\mathsf{D}}_k^t$	$\hat{=}$ surface operator $(\hat{\mathsf{S}}_{k+1}^n)' : \mathcal{T}(\mathring{\mathsf{A}}_k^*)' \to \mathbf{T}_{k+2}^{n,-}$	Section 6.4, (6.46)
$\hat{\mathsf{D}}_k^n$	$\hat{=}$ surface operator $(\hat{\mathsf{S}}_k^t)' : \mathcal{T}(\mathsf{A}_{k+1})' \to \mathbf{T}_k^{t,-}$	Section 6.4, (6.46)

2. Hilbert complexes

2.1. Operators on Hilbert spaces

In this article, both *bounded* and *unbounded* linear operators take center stage.[7] We distinguish them using the following notation. Let $\mathbf{X}$ and $\mathbf{Y}$ be two Hilbert spaces equipped with the inner products $(\cdot,\cdot)_{\mathbf{X}}$ and $(\cdot,\cdot)_{\mathbf{Y}}$, respectively. We will consistently write $\mathsf{A} : \mathcal{D}(\mathsf{A}) \subset \mathbf{X} \to \mathbf{Y}$ to indicate that A is regarded as an *unbounded* linear operator from $\mathbf{X}$ to $\mathbf{Y}$ with domain $\mathcal{D}(\mathsf{A})$, whereas we mean by $\mathsf{A} : \mathbf{X} \to \mathbf{Y}$ that A is viewed as a *bounded* operator from $\mathbf{X}$ to $\mathbf{Y}$ defined on the whole space $\mathbf{X}$.

Recall that the difference between $\mathsf{A} : \mathcal{D}(\mathsf{A}) \subset \mathbf{X} \to \mathbf{Y}$ and $\mathsf{A} : \mathcal{D}(\mathsf{A}) \to \mathbf{Y}$ comes from whether the topology of the subspace $\mathcal{D}(\mathsf{A}) \subset \mathbf{X}$ is given by the norm of $\mathbf{X}$ or the graph norm induced by the inner product $(\mathbf{x}_1,\mathbf{x}_2)_{\mathcal{D}(\mathsf{A})} := (\mathbf{x}_1,\mathbf{x}_2)_{\mathbf{X}} + (\mathsf{A}\,\mathbf{x}_1,\mathsf{A}\,\mathbf{x}_2)_{\mathbf{Y}}$ $\forall \mathbf{x}_1,\mathbf{x}_2 \in \mathcal{D}(\mathsf{A})$.

An unbounded operator $\mathsf{A} : \mathcal{D}(\mathsf{A}) \subset \mathbf{X} \to \mathbf{Y}$ is said to be *closed* if and only if its domain $\mathcal{D}(\mathsf{A})$ is a Hilbert space when endowed with the graph norm, cf. [6, Prop. 3.1]. It is *densely defined* if $\mathcal{D}(A)$ is a dense subset of $\mathbf{X}$. The kernel and range of A, whether it is bounded or not, will be denoted $\mathcal{N}(\mathsf{A})$ and $\mathcal{R}(\mathsf{A})$, respectively.

Topological dual spaces will be tagged with prime, e.g. $\mathbf{X}'$. We use angle brackets for duality pairings, e.g. $\langle \boldsymbol{\phi}, \mathbf{x}\rangle_{\mathbf{X}'}$, $\boldsymbol{\phi} \in \mathbf{X}'$, $\mathbf{x} \in \mathbf{X}$. Accordingly, the operator dual to a *bounded* linear operator $\mathsf{A} : \mathbf{X} \to \mathbf{Y}$ is a bounded operator $\mathsf{A}' : \mathbf{Y}' \to \mathbf{X}'$.

[7] Standard references concerning bounded and unbounded linear operators are [28, Chap. 3] and [46, Chap. 7]. We also particularly recommend [6, Chap. 3], [10, Chap. 1-6] and [42, Chap. 6-8].

The Hilbert space adjoint of $\mathsf{A} : \mathcal{D}(\mathsf{A}) \subset \mathbf{X} \to \mathbf{Y}$ is written $\mathsf{A}^* : \mathcal{D}(\mathsf{A}^*) \subset \mathbf{Y} \to \mathbf{X}$. Recall that it is the unbounded linear operator satisfying

$$(\mathsf{A}^* \mathbf{y}, \mathbf{x})_{\mathbf{X}} = (\mathbf{y}, \mathsf{A}\,\mathbf{x})_{\mathbf{Y}} \quad \forall \mathbf{y} \in \mathcal{D}(\mathsf{A}^*), \forall \mathbf{x} \in \mathcal{D}(\mathsf{A}), \tag{2.1}$$

whose domain $\mathcal{D}(\mathsf{A}^*)$ consists of all $\mathbf{y} \in \mathbf{Y}$ for which the linear functional $\mathcal{D}(\mathsf{A}) \to \mathbb{R}$ defined by $\mathbf{x} \mapsto (\mathbf{y}, \mathsf{A}\,\mathbf{x})_{\mathbf{Y}}$ is continuous in the $\mathbf{X}$ norm, i.e. for every $\mathbf{y} \in \mathcal{D}(\mathsf{A}^*)$, $\exists C_{\mathbf{y}} > 0$ such that $|(\mathbf{y}, \mathsf{A}\,\mathbf{x})_{\mathbf{Y}}| \leq C_{\mathbf{y}} \|\mathbf{x}\|_{\mathbf{X}}$, $\forall \mathbf{x} \in \mathcal{D}(\mathsf{A})$. If A is closed and densely defined, then A^* is also closed and densely defined [6, Prop. 3.3]—in which case $\mathsf{A}^{**} = \mathsf{A}$.

We write $\mathring{\mathsf{A}} \subset \mathsf{A}$ and say that an unbounded linear operator $\mathsf{A} : \mathcal{D}(\mathsf{A}) \subset \mathbf{X} \to \mathbf{Y}$ is an extension of another unbounded linear operator $\mathring{\mathsf{A}} : \mathcal{D}(\mathring{\mathsf{A}}) \subset \mathbf{X} \to \mathbf{Y}$ when $\mathcal{D}(\mathring{\mathsf{A}}) \subset \mathcal{D}(\mathsf{A})$ and $\mathsf{A}\,\mathbf{x}_\circ = \mathring{\mathsf{A}}\,\mathbf{x}_\circ$ for all $\mathbf{x}_\circ \in \mathcal{D}(\mathring{\mathsf{A}})$.

3D de Rham setting II: Differential operators. We refer to [6, Chap. 3] for the following mappings properties. The linear differential operators

$$\mathbf{grad} : H^1(\Omega) \subset L^2(\Omega) \to \mathbf{L}^2(\Omega), \tag{2.2a}$$

$$\mathbf{curl} : \mathbf{H}(\mathbf{curl}, \Omega) \subset \mathbf{L}^2(\Omega) \to \mathbf{L}^2(\Omega), \tag{2.2b}$$

$$\operatorname{div} : \mathbf{H}(\operatorname{div}, \Omega) \subset \mathbf{L}^2(\Omega) \to L^2(\Omega), \tag{2.2c}$$

are densely defined and closed unbounded linear operators. They are extensions of

$$\mathring{\mathbf{grad}} : \mathring{H}^1(\Omega) \subset L^2(\Omega) \to \mathbf{L}^2(\Omega), \tag{2.3a}$$

$$\mathring{\mathbf{curl}} : \mathring{\mathbf{H}}(\mathbf{curl}, \Omega) \subset \mathbf{L}^2(\Omega) \to \mathbf{L}^2(\Omega), \tag{2.3b}$$

$$\mathring{\operatorname{div}} : \mathring{\mathbf{H}}(\operatorname{div}, \Omega) \subset \mathbf{L}^2(\Omega) \to L^2(\Omega). \tag{2.3c}$$

The L^2 Hilbert space adjoints of (2.2a)-(2.2c) are

$$\mathbf{grad}^* = -\mathring{\operatorname{div}} : \mathring{\mathbf{H}}(\operatorname{div}, \Omega) \subset \mathbf{L}^2(\Omega) \to L^2(\Omega), \tag{2.4a}$$

$$\mathbf{curl}^* = \mathring{\mathbf{curl}} : \mathring{\mathbf{H}}(\mathbf{curl}, \Omega) \subset \mathbf{L}^2(\Omega) \to \mathbf{L}^2(\Omega), \tag{2.4b}$$

$$\operatorname{div}^* = -\mathring{\mathbf{grad}} : \mathring{H}^1(\Omega) \subset L^2(\Omega) \to \mathbf{L}^2(\Omega), \tag{2.4c}$$

respectively. Then, the adjoint operators of (2.3a)-(2.3c) are obtained using the fact that $\mathsf{A}^{**} = \mathsf{A}$ for all densely defined and closed unbounded linear operators between Hilbert spaces.

By abuse of notation, we generally write $\mathbf{grad} = \mathring{\mathbf{grad}}$, $\mathbf{curl} = \mathring{\mathbf{curl}}$ and $\operatorname{div} = \mathring{\operatorname{div}}$. ⊿

2.2. Definition

A *Hilbert complex* is a sequence of Hilbert spaces $\mathbf{W}_k$, $k \in \mathbb{Z}$, together with a sequence of closed and densely defined unbounded linear operators $\mathsf{A}_k : \mathcal{D}(\mathsf{A}_k) \subset \mathbf{W}_k \to \mathbf{W}_{k+1}$ such that $\mathcal{R}(\mathsf{A}_k) \subset \mathcal{N}(\mathsf{A}_{k+1})$, i.e. $\mathsf{A}_{k+1} \circ \mathsf{A}_k \equiv 0$ for all $k \in \mathbb{Z}$. It can be written as

$$\cdots \xrightarrow{\mathsf{A}_{k-2}} \mathcal{D}(\mathsf{A}_{k-1}) \subset \mathbf{W}_{k-1} \xrightarrow{\mathsf{A}_{k-1}} \mathcal{D}(\mathsf{A}_k) \subset \mathbf{W}_k \xrightarrow{\mathsf{A}_k} \mathcal{D}(\mathsf{A}_{k+1}) \subset \mathbf{W}_{k+1} \xrightarrow{\mathsf{A}_{k+1}} \cdots, \tag{2.5a}$$

cf. [6, Def. 4.1]. The associated sequence of adjoint operators spawns the so-called dual Hilbert complex

$$\cdots \xleftarrow[\mathsf{A}^*_{k-2}]{} \mathcal{D}(A^*_{k-2}) \subset \mathbf{W}_{k-1} \xleftarrow[\mathsf{A}^*_{k-1}]{} \mathcal{D}(A^*_{k-1}) \subset \mathbf{W}_{k} \xleftarrow[\mathsf{A}^*_{k}]{} \mathcal{D}(A^*_{k}) \subset \mathbf{W}_{k+1} \xleftarrow[\mathsf{A}^*_{k+1}]{} \cdots, \tag{2.5b}$$

which by (2.1) is itself a Hilbert complex, because $\mathsf{A}^*_{k-1} \circ \mathsf{A}^*_k \equiv 0$ for all $k \in \mathbb{Z}$. “Finite” Hilbert complexes can be embedded into (2.5a) by setting $\mathbf{W}_k = \{0\}$ for all $k \notin \{0, 1, ..., N\}$.

Notice that since $\mathcal{R}(\mathsf{A}_k) \subset \mathcal{D}(\mathsf{A}_{k+1})$ and $\mathcal{R}(\mathsf{A}^*_{k+1}) \subset \mathcal{D}(\mathsf{A}^*_k)$, the sequences of *bounded* operators $\mathsf{A}_k : \mathcal{D}(\mathsf{A}_k) \to \mathbf{W}_{k+1}$ and $\mathsf{A}^*_k : \mathcal{D}(\mathsf{A}^*_k) \to \mathbf{W}_k$ also induce Hilbert complexes themselves:

$$\cdots \xrightarrow{\mathsf{A}_{k-2}} \mathcal{D}(\mathsf{A}_{k-1}) \xrightarrow{\mathsf{A}_{k-1}} \mathcal{D}(\mathsf{A}_{k}) \xrightarrow{\mathsf{A}_{k}} \mathcal{D}(\mathsf{A}_{k+1}) \xrightarrow{\mathsf{A}_{k+1}} \cdots, \tag{2.6a}$$

$$\cdots \xleftarrow[\mathsf{A}^*_{k-2}]{} \mathcal{D}(\mathsf{A}^*_{k-2}) \xleftarrow[\mathsf{A}^*_{k-1}]{} \mathcal{D}(\mathsf{A}^*_{k-1}) \xleftarrow[\mathsf{A}^*_{k}]{} \mathcal{D}(\mathsf{A}^*_{k}) \xleftarrow[\mathsf{A}^*_{k+1}]{} \cdots. \tag{2.6b}$$

These are examples of *bounded* Hilbert complexes in which every operator is continuous. We refer to (2.6a) and (2.6b) as the *domain complexes* of (2.5a) and (2.5b).

If the range $\mathcal{R}(\mathsf{A}_k)$ is a closed subset of $\mathbf{W}_{k+1}$ for all k, we say that the Hilbert complex (2.5a) is *closed*. If this is the case, then $\mathcal{R}(\mathsf{A}^*_k)$ is also closed in $\mathbf{W}_k$ by the closed range theorem [6, Thm. 3.7], rendering the dual complex (2.5b) a closed Hilbert complex too. Furthermore, (2.5a) is said to be *Fredholm* if the codimension of $\mathcal{R}(\mathsf{A}_k)$ is finite in $\mathcal{N}(\mathsf{A}_{k+1})$—in which case it is also closed by [6, Thm. 3.8]. Equivalently, a Hilbert complex is Fredholm if the quotient spaces $\mathcal{N}(\mathsf{A}_{k+1})/\mathcal{R}(\mathsf{A}_k)$ and $\mathcal{N}(\mathsf{A}^*_k)/\mathcal{R}(\mathsf{A}^*_{k+1})$ are finite dimensional, in other words, if the cohomology spaces of (2.5a) and (2.5b) have finite dimension. It is a sufficient condition for a Hilbert complex to be Fredholm to satisfy the *compactness property*, that is, the embedding $\mathcal{D}(\mathsf{A}_k) \cap \mathcal{D}(\mathsf{A}^*_{k-1}) \hookrightarrow \mathbf{W}_k$ is compact for all $k \in \mathbb{Z}$.

3D de Rham setting III: The L^2 de Rham complex in $\mathbb{R}^3$. The L^2 de Rham complex (1.1) is a standard example of a Hilbert complex, where $\mathsf{A}_k \equiv 0$ and $\mathbf{W}_k = \{0\}$ is set for $k \in \mathbb{Z}\backslash\{0, 1, 2, 3\}$. From (2.4) we conclude that its dual complex is represented by the sequence

$$\{0\} \xleftarrow[0]{} L^2(\Omega) \xleftarrow[-\mathrm{div}]{} \mathring{\mathbf{H}}(\mathrm{div}, \Omega) \subset \mathbf{L}^2(\Omega) \xleftarrow[\mathbf{curl}]{} \mathring{\mathbf{H}}(\mathbf{curl}, \Omega) \subset \mathbf{L}^2(\Omega) \xleftarrow[-\mathbf{grad}]{} \mathring{H}^1(\Omega) \subset L^2(\Omega) \xleftarrow[\imath]{} \{0\}, \tag{2.7}$$

cf. [6, Sec. 3.4] and [6, Sec. 4.3], and its embedding into our abstract framework is summarized in the following table:

k	$\mathbf{W}_k$	A_k	$\mathcal{D}(\mathsf{A}_k)$	A^*_k	$\mathcal{D}(\mathsf{A}^*_k)$	$\mathcal{D}(\mathsf{A}_k) \cap \mathcal{D}(\mathsf{A}^*_{k-1})$
0	$L^2(\Omega)$	$\mathbf{grad}$	$H^1(\Omega)$	$-\mathrm{div}$	$\mathring{\mathbf{H}}(\mathrm{div}, \Omega)$	$H^1(\Omega)$
1	$\mathbf{L}^2(\Omega)$	$\mathbf{curl}$	$\mathbf{H}(\mathbf{curl}, \Omega)$	$\mathbf{curl}$	$\mathring{\mathbf{H}}(\mathbf{curl}, \Omega)$	$\mathbf{H}(\mathbf{curl}, \Omega) \cap \mathring{\mathbf{H}}(\mathrm{div}, \Omega)$
2	$\mathbf{L}^2(\Omega)$	div	$\mathbf{H}(\mathrm{div}, \Omega)$	$-\mathbf{grad}$	$\mathring{H}^1(\Omega)$	$\mathbf{H}(\mathrm{div}, \Omega) \cap \mathring{\mathbf{H}}(\mathbf{curl}, \Omega)$
3	$L^2(\Omega)$	0	$L^2(\Omega)$	Id	$\{0\}$	$\mathring{H}^1(\Omega)$

The de Rham complex satisfies the compactness property, and thus it is Fredholm. Indeed, recall that Rellich's compact embedding theorem states that the inclusion of $H^1(\Omega)$ and $\mathring{H}^1(\Omega)$ in $L^2(\Omega)$ is compact. We refer to [41] for a proof that $\mathbf{H}(\mathbf{curl},\Omega)\cap\mathring{\mathbf{H}}(\mathrm{div},\Omega)$ and $\mathbf{H}(\mathrm{div},\Omega)\cap\mathring{\mathbf{H}}(\mathbf{curl},\Omega)$ are compactly embedded in $\mathbf{L}^2(\Omega)$. ⊿

2.3. Basic setting

Now, let a Hilbert complex as in (2.5a) be given and suppose that the unbounded linear operators of a second Hilbert complex

$$\cdots \xrightarrow{\mathring{\mathsf{A}}_{k-2}} \mathcal{D}(\mathring{\mathsf{A}}_{k-1}) \subset \mathbf{W}_{k-1} \xrightarrow{\mathring{\mathsf{A}}_{k-1}} \mathcal{D}(\mathring{\mathsf{A}}_{k}) \subset \mathbf{W}_{k} \xrightarrow{\mathring{\mathsf{A}}_{k}} \mathcal{D}(\mathring{\mathsf{A}}_{k+1}) \subset \mathbf{W}_{k+1} \xrightarrow{\mathring{\mathsf{A}}_{k+1}} \cdots \tag{2.8a}$$

are such that $\mathring{\mathsf{A}}_k \subset \mathsf{A}_k$, i.e. $\mathcal{D}(\mathring{\mathsf{A}}_k) \subset \mathcal{D}(\mathsf{A}_k)$ and $\mathsf{A}_k|_{\mathcal{D}(\mathring{\mathsf{A}}_k)} = \mathring{\mathsf{A}}_k$. In other words, for all $k \in \mathbb{Z}$, A_k is an extension of $\mathring{\mathsf{A}}_k$. It is easy to verify that the adjoint operators $\mathring{\mathsf{A}}_k^* := \mathring{\mathsf{A}}_k^{\,*} : \mathcal{D}(\mathring{\mathsf{A}}_k^{\,*}) \subset \mathbf{W}_{k+1} \to \mathbf{W}_k$ involved in the dual complex

$$\cdots \xleftarrow[\mathring{\mathsf{A}}_{k-2}^*]{} \mathcal{D}(\mathring{\mathsf{A}}_{k-2}^*) \subset \mathbf{W}_{k-1} \xleftarrow[\mathring{\mathsf{A}}_{k-1}^*]{} \mathcal{D}(\mathring{\mathsf{A}}_{k-1}^*) \subset \mathbf{W}_{k} \xleftarrow[\mathring{\mathsf{A}}_{k}^*]{} \mathcal{D}(\mathring{\mathsf{A}}_{k}^*) \subset \mathbf{W}_{k+1} \xleftarrow[\mathring{\mathsf{A}}_{k+1}^*]{} \cdots \tag{2.8b}$$

are such that $\mathsf{A}_k^* \subset \mathring{\mathsf{A}}_k^*$. In particular, the bounded domain complexes

$$\cdots \xrightarrow{\mathring{\mathsf{A}}_{k-2}} \mathcal{D}(\mathring{\mathsf{A}}_{k-1}) \xrightarrow{\mathring{\mathsf{A}}_{k-1}} \mathcal{D}(\mathring{\mathsf{A}}_{k}) \xrightarrow{\mathring{\mathsf{A}}_{k}} \mathcal{D}(\mathring{\mathsf{A}}_{k+1}) \xrightarrow{\mathring{\mathsf{A}}_{k+1}} \cdots, \tag{2.9a}$$

$$\cdots \xleftarrow[\mathsf{A}_{k-2}^*]{} \mathcal{D}(\mathsf{A}_{k-2}^*) \xleftarrow[\mathsf{A}_{k-1}^*]{} \mathcal{D}(\mathsf{A}_{k-1}^*) \xleftarrow[\mathsf{A}_{k}^*]{} \mathcal{D}(\mathsf{A}_{k}^*) \xleftarrow[\mathsf{A}_{k+1}^*]{} \cdots, \tag{2.9b}$$

are examples of Hilbert *subcomplexes* of the domain Hilbert complexes (2.6a) and (2.8b).

For reference, this basic setting is summarized in the following assumption.

Assumption A. For all $k \in \mathbb{Z}$ let $\mathbf{W}_k$ be real Hilbert spaces, and suppose that $\mathsf{A}_k : \mathcal{D}(\mathsf{A}_k) \subset \mathbf{W}_k \to \mathbf{W}_{k+1}$ and $\mathring{\mathsf{A}}_k : \mathcal{D}(\mathring{\mathsf{A}}_k) \subset \mathbf{W}_k \to \mathbf{W}_{k+1}$ are densely defined and closed unbounded linear operators such that $\mathcal{R}(\mathsf{A}_k) \subset \mathcal{N}(\mathsf{A}_{k+1})$, $\mathcal{R}(\mathring{\mathsf{A}}_k) \subset \mathcal{N}(\mathring{\mathsf{A}}_{k+1})$, and A_k is an extension of $\mathring{\mathsf{A}}_k$, i.e. $\mathcal{D}(\mathring{\mathsf{A}}_k) \subset \mathcal{D}(\mathsf{A}_k)$ and $\mathsf{A}_k\,\mathbf{x}_\circ = \mathring{\mathsf{A}}_k\,\mathbf{x}_\circ$ for all $\mathbf{x}_\circ \in \mathcal{D}(\mathring{\mathsf{A}}_k)$.

3D de Rham setting IV: Boundary conditions. The Hilbert complex

$$\{0\} \xhookrightarrow{\imath} \mathring{H}^1(\Omega) \subset L^2(\Omega) \xrightarrow{\mathbf{grad}} \mathring{\mathbf{H}}(\mathbf{curl},\Omega) \subset \mathbf{L}^2(\Omega) \xrightarrow{\mathbf{curl}} \mathring{\mathbf{H}}(\mathrm{div},\Omega) \subset \mathbf{L}^2(\Omega) \xrightarrow{\mathrm{div}} L^2(\Omega) \xrightarrow{0} \{0\}$$

fulfills the hypothesis on (2.8a) for the L^2 de Rham complex (1.1). Owing to (2.4a)-(2.4c), its dual complex is written

$$\{0\} \xleftarrow[0]{} L^2(\Omega) \xleftarrow[-\mathrm{div}]{} \mathbf{H}(\mathrm{div},\Omega) \subset \mathbf{L}^2(\Omega) \xleftarrow[\mathbf{curl}]{} \mathbf{H}(\mathbf{curl},\Omega) \subset \mathbf{L}^2(\Omega) \xleftarrow[-\mathbf{grad}]{} H^1 \subset L^2(\Omega) \xleftarrow[\imath]{} \{0\}. \tag{2.10a}$$

Summing up, the various operators and spaces have the following incarnations for the de Rham complex in three-dimensional Euclidean space:

k	$\mathbf{W}_k$	$\mathring{\mathsf{A}}_k$	$\mathcal{D}(\mathring{\mathsf{A}}_k)$	$\mathring{\mathsf{A}}_k^*$	$\mathcal{D}(\mathring{\mathsf{A}}_k^*)$	$\mathcal{D}(\mathring{\mathsf{A}}_k) \cap \mathcal{D}(\mathring{\mathsf{A}}_{k-1}^*)$
0	$L^2(\Omega)$	$\mathbf{grad}$	$\mathring{H}^1(\Omega)$	$-\operatorname{div}$	$\mathbf{H}(\operatorname{div},\Omega)$	$\mathring{H}^1(\Omega)$
1	$\mathbf{L}^2(\Omega)$	$\mathbf{curl}$	$\mathring{\mathbf{H}}(\mathbf{curl},\Omega)$	$\mathbf{curl}$	$\mathbf{H}(\mathbf{curl},\Omega)$	$\mathring{\mathbf{H}}(\mathbf{curl},\Omega) \cap \mathbf{H}(\operatorname{div},\Omega)$
2	$\mathbf{L}^2(\Omega)$	div	$\mathring{\mathbf{H}}(\operatorname{div},\Omega)$	$-\mathbf{grad}$	$H^1(\Omega)$	$\mathring{\mathbf{H}}(\operatorname{div},\Omega) \cap \mathbf{H}(\mathbf{curl},\Omega)$
3	$L^2(\Omega)$	0	$L^2(\Omega)$	Id	$\{0\}$	$H^1(\Omega)$

◿

3. Trace operators

The following sections lay the foundations of a general quotient-based abstract theory for traces in Hilbert spaces. To that end, we do not require the full structure of Hilbert complexes, but it suffices to focus on the following snippet of the Hilbert complexes (2.5a) and (2.8a):

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{\mathsf{A}_{k-2}} & \mathcal{D}(\mathsf{A}_{k-1}) \subset \mathbf{W}_{k-1} & \xrightarrow{\mathsf{A}_{k-1}} & \mathcal{D}(\mathsf{A}_k) \subset \mathbf{W}_k & \xrightarrow{\mathsf{A}_k} & \mathcal{D}(\mathsf{A}_{k+1}) \subset \mathbf{W}_{k+1} & \xrightarrow{\mathsf{A}_{k+1}} & \cdots, \\
 & & \cup & & \cup & & \cup & & \\
\cdots & \xrightarrow{\mathring{\mathsf{A}}_{k-2}} & \mathcal{D}(\mathring{\mathsf{A}}_{k-1}) \subset \mathbf{W}_{k-1} & \xrightarrow{\mathring{\mathsf{A}}_{k-1}} & \mathcal{D}(\mathring{\mathsf{A}}_k) \subset \mathbf{W}_k & \xrightarrow{\mathring{\mathsf{A}}_k} & \mathcal{D}(\mathring{\mathsf{A}}_{k+1}) \subset \mathbf{W}_{k+1} & \xrightarrow{\mathring{\mathsf{A}}_{k+1}} & \cdots.
\end{array}$$

In the sequel, we fix $k \in \mathbb{Z}$ and take for granted Assumption A.

3.1. Hilbert traces

From the estimate

$$\begin{aligned} |(\mathsf{A}_k\,\mathbf{x},\mathbf{y})_{\mathbf{W}_{k+1}} - (\mathbf{x},\mathring{\mathsf{A}}_k^*\mathbf{y})_{\mathbf{W}_k}| &\le \|\,\mathsf{A}_k\,\mathbf{x}\|_{\mathbf{W}_{k+1}}\|\mathbf{y}\|_{\mathbf{W}_{k+1}} + \|\mathbf{x}\|_{\mathbf{W}_k}\|\mathring{\mathsf{A}}_k^*\mathbf{y}\|_{\mathbf{W}_k} \\ &\le \|\mathbf{x}\|_{\mathcal{D}(\mathsf{A}_k)}\|\mathbf{y}\|_{\mathcal{D}(\mathring{\mathsf{A}}_k^*)} \end{aligned} \tag{3.1}$$

we infer that the following definition of a particular notion of a trace makes sense. mklet -n

Definition 3.1. In the setting of Assumption A, the bounded linear operator

$$\mathsf{T}_k^t : \mathcal{D}(\mathsf{A}_k) \to \mathcal{D}(\mathring{\mathsf{A}}_k^*)' \tag{3.2}$$

defined for all $\mathbf{x} \in \mathcal{D}(\mathsf{A}_k)$ and $\mathbf{y} \in \mathcal{D}(\mathring{\mathsf{A}}_k^*)$ by

$$\langle \mathsf{T}_k^t\mathbf{x},\mathbf{y}\rangle_{\mathcal{D}(\mathring{\mathsf{A}}_k^*)'} := (\mathsf{A}_k\,\mathbf{x},\mathbf{y})_{\mathbf{W}_{k+1}} - (\mathbf{x},\mathring{\mathsf{A}}_k^*\mathbf{y})_{\mathbf{W}_k} \tag{3.3}$$

is called the *(primal) Hilbert trace* associated with the pair of operators A_k and $\mathring{\mathsf{A}}_k$.

It also follows from (3.1) that

$$\|\mathsf{T}_k^t\| = 1, \tag{3.4}$$

where $\|\cdot\|$ is the operator norm.

Remark 3.2. We point out that defining a trace operator as a mapping into a dual space has precedents in the theory of Friedrichs operators, has been pursued in [26, Sect. 1.5], [27, Sect. 2], [20, Sect. 2.2] and [19, Sect. 56.3.2], and is also discussed in [3–5]. In these works, the authors have dubbed "boundary operators" what we have decided to call "Hilbert traces". In fact, the developments of Section 3 and Section 4 can probably be extended to the setting of Friedrichs operators, but this is outside the scope of this work.

3D de Rham setting V: Hilbert traces. Let us motivate the above notion of trace with classical examples. Applying Definition 3.1 in the 3D de Rham setting II, we obtain the Hilbert traces

$$\mathsf{T}_0^t = \mathsf{T}_{\mathbf{grad}}^t : H^1(\Omega) \to \mathbf{H}(\mathrm{div},\Omega)', \tag{3.5a}$$

$$\mathsf{T}_1^t = \mathsf{T}_{\mathbf{curl}}^t : \mathbf{H}(\mathbf{curl},\Omega) \to \mathbf{H}(\mathbf{curl},\Omega)', \tag{3.5b}$$

$$\mathsf{T}_2^t = \mathsf{T}_{\mathrm{div}}^t : \mathbf{H}(\mathrm{div},\Omega) \to H^1(\Omega)', \tag{3.5c}$$

defined by

$$\langle \mathsf{T}_{\mathbf{grad}}^t v, \mathbf{u}\rangle_{\mathbf{H}(\mathrm{div},\Omega)'} := (\mathbf{grad} v, \mathbf{u})_{\mathbf{L}^2(\Omega)} + (v, \mathrm{div}\,\mathbf{u})_{L^2(\Omega)}, \tag{3.6a}$$

$$\langle \mathsf{T}_{\mathbf{curl}}^t \mathbf{z}, \mathbf{w}\rangle_{\mathbf{H}(\mathbf{curl},\Omega)'} := (\mathbf{curl}\,\mathbf{z}, \mathbf{w})_{\mathbf{L}^2(\Omega)} - (\mathbf{z}, \mathbf{curl}\,\mathbf{w})_{\mathbf{L}^2(\Omega)}, \tag{3.6b}$$

$$\langle \mathsf{T}_{\mathrm{div}}^t \mathbf{u}, v\rangle_{H^1(\Omega)'} := (\mathrm{div}\,\mathbf{u}, v)_{\mathbf{L}^2(\Omega)} + (\mathbf{u}, \mathbf{grad}\, v)_{\mathbf{L}^2(\Omega)}, \tag{3.6c}$$

for all $v \in H^1$, $\mathbf{u} \in \mathbf{H}(\mathrm{div},\Omega)$ and $\mathbf{z}, \mathbf{w} \in \mathbf{H}(\mathbf{curl},\Omega)$.

We recognize on the right hand sides of (3.6a)-(3.6c) the continuous bilinear forms occurring in Green's formulas (1.12a) and (1.12b). Introducing the operators

$$\gamma_n' : H^{1/2}(\Gamma) \to \mathbf{H}(\mathrm{div},\Omega)', \qquad \gamma_t' : \mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma,\Gamma) \to \mathbf{H}(\mathbf{curl},\Omega)', \qquad \gamma' : H^{-1/2}(\Gamma) \to H^1(\Omega)', \tag{3.7}$$

dual to the classical traces, where we have identified $H^{-1/2}(\Gamma)$ with $(H^{1/2}(\Gamma))'$ through the $L^2(\Gamma)$-pairing on the boundary and $\mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma,\Gamma)$ with its own dual through the skew-symmetric pairing defined in (1.10), we obtain

$$\mathsf{T}_{\mathbf{grad}}^t = \gamma_n' \circ \gamma, \qquad \mathsf{T}_{\mathbf{curl}}^t = \gamma_t' \circ \gamma_t, \qquad \mathsf{T}_{\mathrm{div}}^t = \gamma' \circ \gamma_n. \tag{3.8}$$

Observe that, when identifying the reflexive space $\mathbf{H}(\mathrm{div},\Omega)$ with its bi-dual $\mathbf{H}(\mathrm{div},\Omega)''$,

$$(\mathsf{T}_{\mathbf{grad}}^t)' = \mathsf{T}_{\mathrm{div}}^t. \tag{3.9}$$

The appeal of definitions (3.6a)-(3.6c) is that they do not explicitly depend on Γ. In fact, notice that they are well-defined for general bounded open sets Ω without any assumption on the regularity of their boundary $\Gamma := \partial\Omega$. ⊿

Proposition 3.3. *Under Assumption A,*

$$\mathcal{N}(\mathsf{T}_k^t) = \mathcal{D}(\mathring{\mathsf{A}}_k). \tag{3.10}$$

Proof. On the one hand, for any $\mathbf{x}_\circ \in \mathcal{D}(\mathring{\mathsf{A}}_k)$, it follows from $\mathring{\mathsf{A}}_k \subset \mathsf{A}_k$ and (2.1) that

$$\begin{aligned}\langle \mathsf{T}_k^t \mathbf{x}_\circ, \mathbf{y}\rangle_{\mathcal{D}(\mathring{\mathsf{A}}_k^*)'} &= (\mathsf{A}_k\, \mathbf{x}_\circ, \mathbf{y})_{\mathbf{W}_{k+1}} - (\mathbf{x}_\circ, \mathring{\mathsf{A}}_k^* \mathbf{y})_{\mathbf{W}_k} = (\mathring{\mathsf{A}}_k\, \mathbf{x}_\circ, \mathbf{y})_{\mathbf{W}_{k+1}} - (\mathbf{x}_\circ, \mathring{\mathsf{A}}_k^* \mathbf{y})_{\mathbf{W}_k} \\ &= (\mathbf{x}_\circ, \mathring{\mathsf{A}}_k^* \mathbf{y})_{\mathbf{W}_k} - (\mathbf{x}_\circ, \mathring{\mathsf{A}}_k^* \mathbf{y})_{\mathbf{W}_k} = 0\end{aligned} \tag{3.11}$$

for all $\mathbf{y} \in \mathcal{D}(\mathring{\mathsf{A}}_k^*)$. This shows that $\mathcal{D}(\mathring{\mathsf{A}}_k) \subset \mathcal{N}(\mathsf{T}_k^t)$.

On the other hand, if $\mathbf{x} \in \mathcal{D}(\mathsf{A}_k)$ is such that $\mathbf{x} \in \mathcal{N}(\mathsf{T}_k^t)$, then

$$0 = \langle \mathsf{T}_k^t \mathbf{x}, \mathbf{y}\rangle_{\mathcal{D}(\mathring{\mathsf{A}}_k^*)'} = (\mathsf{A}_k\, \mathbf{x}, \mathbf{y})_{\mathbf{W}_{k+1}} - (\mathbf{x}, \mathring{\mathsf{A}}_k^* \mathbf{y})_{\mathbf{W}_k} \quad \forall \mathbf{y} \in \mathcal{D}(\mathring{\mathsf{A}}_k^*). \tag{3.12}$$

If we set $C_{\mathbf{x}} := \|\mathbf{x}\|_{\mathcal{D}(\mathsf{A}_k)}$, we see that

$$|(\mathbf{x}, \mathring{\mathsf{A}}_k^* \mathbf{y})_{\mathbf{W}_k}| = |(\mathsf{A}_k\, \mathbf{x}, \mathbf{y})_{\mathbf{W}_{k+1}}| \leq \|\, \mathsf{A}_k\, \mathbf{x}\|_{\mathbf{W}_{k+1}} \|\mathbf{y}\|_{\mathbf{W}_{k+1}} \leq C_{\mathbf{x}} \|\mathbf{y}\|_{\mathbf{W}_{k+1}} \quad \forall \mathbf{y} \in \mathcal{D}(\mathring{\mathsf{A}}_k^*). \tag{3.13}$$

As explained in Section 2.1, this means that $\mathbf{x} \in \mathcal{D}((\mathring{\mathsf{A}}_k^*)^*) = \mathcal{D}(\mathring{\mathsf{A}}_k^{**}) = \mathcal{D}(\mathring{\mathsf{A}}_k)$. □

3D de Rham setting VI: Kernels of classical Hilbert traces. Comparing Proposition 3.3 with (1.8a)-(1.8c), we verify that

$$\mathcal{N}(\mathsf{T}_{\mathbf{grad}}^t) = \mathcal{N}(\gamma), \qquad \mathcal{N}(\mathsf{T}_{\mathbf{curl}}^t) = \mathcal{N}(\gamma_t), \qquad \mathcal{N}(\mathsf{T}_{\mathrm{div}}^t) = \mathcal{N}(\gamma_n). \tag{3.14}$$

⊿

Remark 3.4. Intuitively, we think of a trace operator as a means of imposing "boundary conditions". The idea behind Definition 3.1 is to impose these boundary conditions on the operator itself, which is a common strategy in the analysis of variational problems and related operator equations. In this work, A_k is the operator of interest. We regard $\mathring{\mathsf{A}}_k$ as the operator on which boundary conditions are imposed. From that perspective, the operator $\mathring{\mathsf{A}}_k^*$ does not feature boundary conditions. The right hand side of (3.3) plays a role akin to the bilinear form involved in classical integration by parts formulas.

3.2. Trace spaces

Recall that by hypothesis, $\mathcal{D}(\mathsf{A}_k^*) \subset \mathcal{D}(\mathring{\mathsf{A}}_k^*)$. The next proposition involves the annihilator of $\mathcal{D}(\mathsf{A}_k^*)$ in $\mathcal{D}(\mathring{\mathsf{A}}_k^*)'$:

$$\mathcal{D}(\mathsf{A}_k^*)^\circ := \left\{ \boldsymbol{\phi} \in \mathcal{D}(\mathring{\mathsf{A}}_k^*)' \mid \langle \boldsymbol{\phi}, \mathbf{y}\rangle_{\mathcal{D}(\mathring{\mathsf{A}}_k^*)'} = 0\ \forall \mathbf{y} \in \mathcal{D}(\mathsf{A}_k^*) \right\} \subset \mathcal{D}(\mathring{\mathsf{A}}_k^*)'. \tag{3.15}$$

Proposition 3.5. *Under Assumption A, we find for the ranges of the Hilbert traces*

$$\mathcal{R}(\mathsf{T}_k^t) = \mathcal{D}(\mathsf{A}_k^*)^\circ. \tag{3.16}$$

Proof. Suppose that $\boldsymbol{\phi} \in \mathcal{D}(\mathsf{A}_k^*)^\circ$ and let $\mathbf{w} \in \mathcal{D}(\mathring{\mathsf{A}}_k^*)$ be its Riesz representative in $\mathcal{D}(\mathring{\mathsf{A}}_k^*)$, that is

$$\langle \boldsymbol{\phi}, \mathbf{y} \rangle_{\mathcal{D}(\mathring{\mathsf{A}}_k^*)'} = (\mathbf{w}, \mathbf{y})_{\mathcal{D}(\mathring{\mathsf{A}}_k^*)} \qquad \forall \mathbf{y} \in \mathcal{D}(\mathring{\mathsf{A}}_k^*). \tag{3.17}$$

We claim that $\mathbf{x} := -\mathring{\mathsf{A}}_k^* \mathbf{w} \in \mathcal{D}(\mathsf{A}_k)$. Indeed, (3.17) implies that for all $\mathbf{y}_* \in \mathcal{D}(\mathsf{A}_k^*)$, we have

$$0 = (\mathbf{w}, \mathbf{y}_*)_{\mathcal{D}(\mathring{\mathsf{A}}_k^*)} = (\mathbf{w}, \mathbf{y}_*)_{\mathbf{W}_{k+1}} + (\mathring{\mathsf{A}}_k^* \mathbf{w}, \mathring{\mathsf{A}}_k^* \mathbf{y}_*)_{\mathbf{W}_k} = (\mathbf{w}, \mathbf{y}_*)_{\mathbf{W}_{k+1}} + (\mathring{\mathsf{A}}_k^* \mathbf{w}, \mathsf{A}_k^* \mathbf{y}_*)_{\mathbf{W}_k}. \tag{3.18}$$

This means

$$(\mathbf{w}, \mathbf{y}_*)_{\mathbf{W}_{k+1}} = (\mathbf{x}, \mathsf{A}_k^* \mathbf{y}_*)_{\mathbf{W}_k} \quad \forall \mathbf{y}_* \in \mathcal{D}(\mathsf{A}_k^*) \tag{3.19}$$

Therefore, if we set $C_{\mathbf{x}} := \|\mathbf{w}\|_{\mathbf{W}_{k+1}}$, we find the estimate

$$|(\mathbf{x}, \mathsf{A}_k^* \mathbf{y}_*)_{\mathbf{W}_k}| = |(\mathbf{w}, \mathbf{y}_*)_{\mathbf{W}_{k+1}}| \le \|\mathbf{w}\|_{\mathbf{W}_{k+1}} \|\mathbf{y}_*\|_{\mathbf{W}_{k+1}} = C_{\mathbf{x}} \|\mathbf{y}_*\|_{\mathbf{W}_{k+1}} \qquad \forall \mathbf{y}_* \in \mathcal{D}(\mathsf{A}_k^*), \tag{3.20}$$

which, as explained in Section 2.1, implies that $\mathbf{x} \in \mathcal{D}(\mathsf{A}_k^{**}) = \mathcal{D}(\mathsf{A}_k)$.

Thus appealing to (2.1) we can rewrite (3.19) as $(\mathbf{w}, \mathbf{y}_*)_{\mathbf{W}_{k+1}} = (\mathsf{A}_k \mathbf{x}, \mathbf{y}_*)_{\mathbf{W}_{k+1}}$. Since $\mathcal{D}(\mathsf{A}_k^*)$ is dense in $\mathbf{W}_{k+1}$, we infer $\mathsf{A}_k \mathbf{x} = \mathbf{w}$. Hence, the inclusion $\mathcal{R}(\mathsf{T}_k^t) \supset \mathcal{D}(\mathsf{A}_k^*)^\circ$ is verified by observing that for all $\mathbf{y} \in \mathcal{D}(\mathring{\mathsf{A}}_k^*)$,

$$\begin{aligned} \langle \mathsf{T}_k^t \mathbf{x}, \mathbf{y} \rangle_{\mathcal{D}(\mathring{\mathsf{A}}_k^*)'} &= (\mathsf{A}_k \mathbf{x}, \mathbf{y})_{\mathbf{W}_{k+1}} - (\mathbf{x}, \mathring{\mathsf{A}}_k^* \mathbf{y})_{\mathbf{W}_k} = (\mathbf{w}, \mathbf{y})_{\mathbf{W}_{k+1}} + (\mathring{\mathsf{A}}_k^* \mathbf{w}, \mathring{\mathsf{A}}_k^* \mathbf{y})_{\mathbf{W}_k} \\ &= (\mathbf{w}, \mathbf{y})_{\mathcal{D}(\mathring{\mathsf{A}}_k^*)} = \langle \boldsymbol{\phi}, \mathbf{y} \rangle_{\mathcal{D}(\mathring{\mathsf{A}}_k^*)'}, \end{aligned} \tag{3.21}$$

i.e. $\mathsf{T}_k^t \mathbf{x} = \boldsymbol{\phi}$.

To show that $\mathcal{R}(\mathsf{T}_k^t) \subset \mathcal{D}(\mathsf{A}_k^*)^\circ$, let $\boldsymbol{\phi} = \mathsf{T}_k^t \mathbf{x}$ for some $\mathbf{x} \in \mathcal{D}(\mathsf{A}_k)$. Then, since $\mathsf{A}_k^* \subset \mathring{\mathsf{A}}_k^*$, we obtain by (2.1) that for all $\mathbf{y}_* \in \mathcal{D}(\mathsf{A}_k^*)$

$$\langle \boldsymbol{\phi}, \mathbf{y}_* \rangle_{\mathcal{D}(\mathring{\mathsf{A}}_k^*)'} = (\mathsf{A}_k \mathbf{x}, \mathbf{y}_*)_{\mathbf{W}_{k+1}} - (\mathbf{x}, \mathring{\mathsf{A}}_k^* \mathbf{y}_*)_{\mathbf{W}_k} = (\mathbf{x}, \mathsf{A}_k^* \mathbf{y}_*)_{\mathbf{W}_k} - (\mathbf{x}, \mathsf{A}_k^* \mathbf{y}_*)_{\mathbf{W}_k} = 0, \tag{3.22}$$

i.e. $\boldsymbol{\phi} \in \mathcal{D}(\mathsf{A}_k^*)^\circ$. □

Since $\mathcal{D}(\mathring{\mathsf{A}}_k)$ is a Hilbert subspace of $\mathcal{D}(\mathsf{A}_k)$, it is closed and we can proceed with the next definition.

Definition 3.6. In the setting of Definition 3.1, we call *trace spaces* the quotient spaces

$$\mathcal{T}(\mathsf{A}_k) := \mathcal{D}(\mathsf{A}_k)/\mathcal{D}(\mathring{\mathsf{A}}_k), \tag{3.23}$$

equipped with the quotient norm

$$\|[\mathbf{x}]\|_{\mathcal{T}(\mathsf{A}_k)} := \inf_{\mathring{z}\in\mathcal{D}(\mathring{\mathsf{A}}_k)} \|\mathbf{x}-\mathring{\mathbf{z}}\|_{\mathcal{D}(\mathsf{A}_k)} \quad \forall \mathbf{x}\in\mathcal{D}(\mathsf{A}_k). \tag{3.24}$$

Remark 3.7. Notice that due to Proposition 3.3,

$$\mathcal{T}(\mathsf{A}_k) = \mathcal{D}(\mathsf{A}_k)/\mathcal{N}(\mathsf{T}_k^t). \tag{3.25}$$

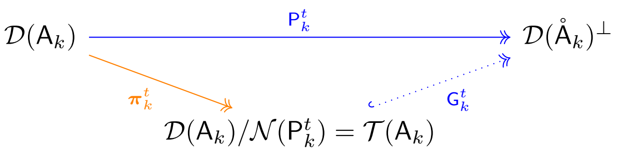
In Definition 3.6, the equivalence class in $\mathcal{T}(\mathsf{A}_k)$ of $\mathbf{x} \in D(\mathsf{A}_k)$ is denoted $[\mathbf{x}] = \{\mathbf{x} + \mathring{\mathbf{z}} \,|\, \mathring{\mathbf{z}} \in \mathcal{D}(\mathring{\mathsf{A}}_k)\}$. Write $\boldsymbol{\pi}_k^t : \mathcal{D}(\mathsf{A}_k) \to \mathcal{T}(\mathsf{A}_k)$ for the canonical projection (also frequently called quotient map), i.e. $\boldsymbol{\pi}_k^t(\mathbf{x}) = [\mathbf{x}]$. It is an application of a classical theorem of functional analysis that there exists a bounded orthogonal projection $\mathsf{P}_k^t : \mathcal{D}(\mathsf{A}_k) \to \mathcal{D}(\mathring{\mathsf{A}}_k)^\perp$ onto the complement space

$$\mathcal{D}(\mathring{\mathsf{A}}_k)^\perp := \left\{ \mathbf{x}\in\mathcal{D}(\mathsf{A}_k) \mid (\mathbf{x},\mathring{\mathbf{z}})_{\mathcal{D}(\mathsf{A}_k)} = 0 \ \forall \mathring{\mathbf{z}}\in\mathcal{D}(\mathring{\mathsf{A}}_k) \right\} \subset \mathcal{D}(\mathsf{A}_k) \tag{3.26}$$

such that

$$\|\mathsf{P}_k^t\mathbf{x}\|_{\mathcal{D}(\mathsf{A}_k)} = \|[\mathbf{x}]\|_{\mathcal{T}(\mathsf{A}_k)} \qquad \forall \mathbf{x}\in\mathcal{D}(\mathsf{A}_k), \tag{3.27}$$

cf. [46, Chap. 3.1] and [10, Chap. 5]. Write $\imath_k^t : \mathcal{D}(\mathring{\mathsf{A}}_k)^\perp \hookrightarrow \mathcal{D}(\mathsf{A}_k)$ for canonical inclusion maps. Since $\mathcal{N}(\mathsf{P}_k^t) = \mathcal{D}(\mathring{\mathsf{A}}_k)$ by (3.27), the bounded linear map $\mathsf{G}_k^t : \mathcal{T}(\mathsf{A}_k) \to \mathcal{D}(\mathring{\mathsf{A}}_k)^\perp$ defined by $\mathsf{G}_k^t[\mathbf{x}] := \mathsf{P}_k^t\mathbf{x}$ and involved in the commutative diagram



(3.28)

as provided by the first isomorphism theorem for modules is a well-defined isometric isomorphism, cf. [18, Chap. 10.2, Thm. 4]. Since $\mathcal{D}(\mathring{\mathsf{A}}_k)^\perp$ is closed [46, Chap. 3.1, Thm. 1], it is a Hilbert space, and therefore so is $\mathcal{T}(\mathsf{A}_k)$. The quotient norm is induced by the inner product

$$([\mathbf{x}],[\mathbf{z}])_{\mathcal{T}(\mathsf{A}_k)} := (\mathsf{P}_k^t\mathbf{x},\mathsf{P}_k^t\mathbf{z})_{\mathcal{D}(\mathsf{A}_k)} \qquad \forall[\mathbf{x}],[\mathbf{z}]\in\mathcal{T}(\mathsf{A}_k). \tag{3.29}$$

Remark 3.8. Notice that $\mathcal{N}(\mathsf{P}_k^t)=\mathcal{D}(\mathring{\mathsf{A}}_k)=\mathcal{N}(\mathsf{T}_k^t)$.

That the projection P_k^t is orthogonal means that $(\mathbf{x}-\mathsf{P}_k^t\mathbf{x},\mathbf{z}_\perp)_{\mathcal{D}(\mathsf{A}_k)}=0$ for all $\mathbf{x}\in\mathcal{D}(\mathsf{A}_k)$ and $\mathbf{z}_\perp\in\mathcal{D}(\mathring{\mathsf{A}}_k)^\perp$. In other words, $(\mathsf{Id}-\mathsf{P}_k^t)\mathbf{x}\in\mathcal{D}(\mathring{\mathsf{A}}_k)$ for all $\mathbf{x}\in\mathcal{D}(\mathsf{A}_k)$. Hence, the simple observation that $\mathsf{Id}=\mathsf{P}_k^t+(\mathsf{Id}-\mathsf{P}_k^t)$ shows that any element $\mathbf{x}\in\mathcal{D}(\mathsf{A}_k)$ can be decomposed as

$$\mathbf{x}=\mathbf{x}_\perp+\mathbf{x}_\circ \tag{3.30}$$

where $\mathbf{x}_\perp\in\mathcal{D}(\mathring{\mathsf{A}}_k)^\perp$ and $\mathbf{x}_\circ\in D(\mathring{\mathsf{A}}_k)$. It is easy to see that the decomposition (3.30) is unique.

3D de Rham setting VII: Trace spaces. In the 3D de Rham setting V, applying Definition 3.6 leads to

$$\mathcal{T}(\mathsf{A}_0)=\mathcal{T}(\mathbf{grad})=H^1(\Omega)/\mathring{H}^1(\Omega), \tag{3.31a}$$

$$\mathcal{T}(\mathsf{A}_1)=\mathcal{T}(\mathbf{curl})=\mathbf{H}(\mathbf{curl},\Omega)/\mathring{\mathbf{H}}(\mathbf{curl},\Omega), \tag{3.31b}$$

$$\mathcal{T}(\mathsf{A}_2)=\mathcal{T}(\mathrm{div})=\mathbf{H}(\mathrm{div},\Omega)/\mathring{\mathbf{H}}(\mathrm{div},\Omega). \tag{3.31c}$$

Based on (1.8) the linear mappings

$$\mathsf{X}_{\mathbf{grad}}: H^1(\Omega)/\mathring{H}^1(\Omega)\to H^{1/2}(\Gamma), \tag{3.32a}$$

$$\mathsf{X}_{\mathbf{curl}}: \mathbf{H}(\mathbf{curl},\Omega)/\mathring{\mathbf{H}}(\mathbf{curl},\Omega)\to \mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma,\Gamma), \tag{3.32b}$$

$$\mathsf{X}_{\mathrm{div}}: \mathbf{H}(\mathrm{div},\Omega)/\mathring{\mathbf{H}}(\mathrm{div},\Omega)\to H^{-1/2}(\Gamma) \tag{3.32c}$$

defined by

$$\mathsf{X}_{\mathbf{grad}}[u]:=\gamma u \qquad \forall u\in H^1(\Omega), \tag{3.33a}$$

$$\mathsf{X}_{\mathbf{curl}}[\mathbf{u}]:=\gamma_t\mathbf{u} \qquad \forall \mathbf{u}\in \mathbf{H}(\mathbf{curl},\Omega), \tag{3.33b}$$

$$\mathsf{X}_{\mathrm{div}}[\mathbf{v}]:=\gamma_n\mathbf{v} \qquad \forall \mathbf{v}\in \mathbf{H}(\mathrm{div},\Omega), \tag{3.33c}$$

are the Hilbert space isomorphisms induced by the canonical projections involved in the following commutative diagrams, in which $\leftrightarrow$ indicates an isomorphism:

$$\begin{array}{ccc} H^1(\Omega) & \xrightarrow{\gamma} & H^{1/2}(\Gamma) \\ {\scriptstyle \boldsymbol{\pi}^t_{\mathbf{grad}}}\downarrow & \nearrow\swarrow {\scriptstyle \mathsf{X}_{\mathbf{grad}}} & \\ \mathcal{T}(\mathbf{grad}) & & \end{array} \qquad \begin{array}{ccc} \mathbf{H}(\mathbf{curl},\Omega) & \xrightarrow{\gamma_t} & \mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma,\Gamma) \\ {\scriptstyle \boldsymbol{\pi}^t_{\mathbf{curl}}}\downarrow & \nearrow\swarrow {\scriptstyle \mathsf{X}_{\mathbf{curl}}} & \\ \mathcal{T}(\mathbf{curl}) & & \end{array} \qquad \begin{array}{ccc} \mathbf{H}(\mathbf{curl},\Omega) & \xrightarrow{\gamma_n} & H^{-1/2}(\Gamma) \\ {\scriptstyle \boldsymbol{\pi}^t_{\mathrm{div}}}\downarrow & \nearrow\swarrow {\scriptstyle \mathsf{X}_{\mathbf{curl}}} & \\ \mathcal{T}(\mathrm{div}) & & \end{array}$$

The trace spaces $H^{1/2}(\Gamma)$, $\mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma,\Gamma)$ and $H^{-1/2}(\Gamma)$ can therefore be identified with the quotient spaces $\mathcal{T}(\mathbf{grad})$, $\mathcal{T}(\mathbf{curl})$ and $\mathcal{T}(\mathrm{div})$, respectively, as we have already observed in (1.9). Under these identifications, the bounded inverse theorem guarantees that the quotient spaces are equipped with equivalent norms. Moreover, due to the Lipschitz regularity of Γ and Sobolev extension theorems, the definitions of $\mathcal{T}(\mathbf{grad})$, $\mathcal{T}(\mathbf{curl})$ and $\mathcal{T}(\mathrm{div})$ are intrinsic, in the sense that the quotient spaces $H^1(\mathbb{R}^3\backslash\overline{\Omega})/\mathring{H}^1(\mathbb{R}^3\backslash\overline{\Omega})$, $\mathbf{H}(\mathbf{curl},\mathbb{R}^3\backslash\overline{\Omega})/\mathring{\mathbf{H}}(\mathbf{curl},\mathbb{R}^3\backslash\overline{\Omega})$ and $\mathbf{H}(\mathrm{div},\mathbb{R}^3\backslash\overline{\Omega})/\mathring{\mathbf{H}}(\mathrm{div},\mathbb{R}^3\backslash\overline{\Omega})$ are also Hilbert spaces with equivalent norms [16]. ⊿

Lemma 3.9. *Under Assumption A, if* $\mathbf{x}_\perp \in \mathcal{D}(\mathring{\mathsf{A}}_k)^\perp$*, then* $\mathsf{A}_k\,\mathbf{x}_\perp \in \mathcal{D}(\mathring{\mathsf{A}}_k^*)$ *and*

$$(\mathring{\mathsf{A}}_k^*\,\mathsf{A}_k + \mathsf{Id})\,\mathbf{x}_\perp = 0. \tag{3.34}$$

Proof. Suppose that $\mathbf{x}_\perp \in \mathcal{D}(\mathring{\mathsf{A}}_k)^\perp$. Since $\mathring{\mathsf{A}}_k \subset \mathsf{A}_k$, we have by definition that

$$\begin{aligned} 0 = (\mathbf{x}_\perp, \mathbf{z}_\circ)_{\mathcal{D}(\mathsf{A}_k)} &= (\mathbf{x}_\perp, \mathbf{z}_\circ)_{\mathbf{W}_k} + (\mathsf{A}_k\,\mathbf{x}_\perp, \mathsf{A}_k\,\mathbf{z}_\circ)_{\mathbf{W}_{k+1}} \\ &= (\mathbf{x}_\perp, \mathbf{z}_\circ)_{\mathbf{W}_k} + (\mathsf{A}_k\,\mathbf{x}_\perp, \mathring{\mathsf{A}}_k\,\mathbf{z}_\circ)_{\mathbf{W}_{k+1}} \end{aligned} \tag{3.35}$$

for all $\mathbf{z}_\circ \in \mathcal{D}(\mathring{\mathsf{A}}_k)$, which means

$$(\mathsf{A}_k\,\mathbf{x}_\perp, \mathring{\mathsf{A}}_k\,\mathbf{z}_\circ)_{\mathbf{W}_{k+1}} = -(\mathbf{x}_\perp, \mathbf{z}_\circ)_{\mathbf{W}_k} \qquad \forall \mathbf{z}_\circ \in \mathcal{D}(\mathring{\mathsf{A}}_k). \tag{3.36}$$

So by setting $C_{\mathbf{x}_\perp} := \|\mathbf{x}_\perp\|_{\mathbf{W}_k}$, we conclude from the estimate

$$|(\mathsf{A}_k\,\mathbf{x}_\perp, \mathring{\mathsf{A}}_k\,\mathbf{z}_\circ)_{\mathbf{W}_{k+1}}| = |(\mathbf{x}_\perp, \mathbf{z}_\circ)_{\mathbf{W}_k}| \le \|\mathbf{x}_\perp\|_{\mathbf{W}_k} \|\mathbf{z}_\circ\|_{\mathbf{W}_k} = C_{\mathbf{x}_\perp} \|\mathbf{z}_\circ\|_{\mathbf{W}_k} \qquad \forall \mathbf{z}_\circ \in \mathcal{D}(\mathring{\mathsf{A}}_k), \tag{3.37}$$

that $\mathsf{A}_k\,\mathbf{x}_\perp \in \mathcal{D}(\mathring{\mathsf{A}}_k^{\,*}) = \mathcal{D}(\mathring{\mathsf{A}}_k^*)$. Then as in (2.1), the identity (3.34) follows from (3.36). □

Corollary 3.10. *Under Assumption A, the linear map* $\mathsf{A}_k : \mathcal{D}(\mathring{\mathsf{A}}_k)^\perp \to \mathcal{D}(\mathring{\mathsf{A}}_k^*)$ *is an isometry.*

Proof. Suppose that $\mathbf{x}_\perp \in \mathcal{D}(\mathring{\mathsf{A}}_k)^\perp$. Then, by Lemma 3.9,

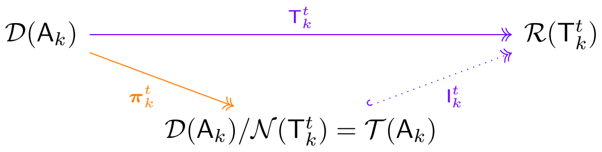
$$\begin{aligned} \|\,\mathsf{A}_k\,\mathbf{x}_\perp\|^2_{\mathcal{D}(\mathring{\mathsf{A}}_k^*)} &= \|\,\mathsf{A}_k\,\mathbf{x}_\perp\|^2_{\mathbf{W}_{k+1}} + \|\mathring{\mathsf{A}}_k^*\,\mathsf{A}_k\,\mathbf{x}_\perp\|^2_{\mathbf{W}_k} = \|\,\mathsf{A}_k\,\mathbf{x}_\perp\|^2_{\mathbf{W}_{k+1}} + \|\mathbf{x}_\perp\|^2_{\mathbf{W}_k} \\ &= \|\mathbf{x}_\perp\|^2_{\mathcal{D}(\mathsf{A}_k)}. \quad \square \end{aligned} \tag{3.38}$$

Theorem 3.11. *Under Assumption A, the linear map*

$$\mathsf{I}_k^t : \begin{cases} \mathcal{T}(\mathsf{A}_k) \to \mathcal{R}(\mathsf{T}_k^t) \\ [\mathbf{x}] \mapsto \mathsf{T}_k^t\mathbf{x} \end{cases} \tag{3.39}$$

is a well-defined isometric isomorphism.

Proof. Since $\mathcal{D}(\mathring{\mathsf{A}}_k) = \mathcal{N}(\mathsf{T}^t)$ by Proposition 3.3, notice that $\mathsf{I}_k^t : \mathcal{T}(\mathsf{A}_k) \to \mathcal{R}(\mathsf{T}_k^t)$ is simply the well-defined induced isomorphism of modules involved in the commutative diagram



provided by the first isomorphism theorem [18, Chap. 10.2, Thm. 4]. It only remains to show that it is an isometry.

Let $\mathbf{x} \in \mathcal{D}(\mathsf{A}_k)$. By Proposition 3.3,

$$\| \mathsf{I}_k^t[\mathbf{x}] \|_{\mathcal{D}(\mathring{\mathsf{A}}_k^*)'} = \|\mathsf{T}_k^t\mathbf{x}\|_{\mathcal{D}(\mathring{\mathsf{A}}_k^*)'} = \|\mathsf{T}_k^t(\mathbf{x}_\perp + \mathbf{x}_\circ)\|_{\mathcal{D}(\mathring{\mathsf{A}}_k^*)'} = \|\mathsf{T}_k^t\mathbf{x}_\perp\|_{\mathcal{D}(\mathring{\mathsf{A}}_k^*)'}. \tag{3.40}$$

Using that $\mathring{\mathsf{A}}_k^* \, \mathsf{A}_k \, \mathbf{x}_\perp = -\mathbf{x}_\perp$ by Lemma 3.9, we can choose $\mathbf{y} = \mathsf{A}_k \, \mathbf{x}_\perp \in D(\mathring{\mathsf{A}}_k^*)$ to obtain

$$\begin{aligned} \|\mathsf{T}_k^t\mathbf{x}_\perp\|_{\mathcal{D}(\mathring{\mathsf{A}}_k^*)'} &= \sup_{0\neq\mathbf{y}\in D(\mathring{\mathsf{A}}_k^*)} \frac{|\langle \mathsf{T}_k^t\mathbf{x}_\perp, \mathbf{y}\rangle|}{\|\mathbf{y}\|_{\mathcal{D}(\mathring{\mathsf{A}}_k^*)}} \geq \frac{|\langle \mathsf{T}_k^t\mathbf{x}_\perp, \mathsf{A}_k \, \mathbf{x}_\perp\rangle|}{\| \mathsf{A}_k \, \mathbf{x}_\perp\|_{\mathcal{D}(\mathring{\mathsf{A}}_k^*)}} \\ &= \frac{|(\mathsf{A}_k \, \mathbf{x}_\perp, \mathsf{A}_k \, \mathbf{x}_\perp)_{\mathbf{W}_{k+1}} - (\mathbf{x}_\perp, \mathring{\mathsf{A}}_k^* \, \mathsf{A}_k \, \mathbf{x}_\perp)_{\mathbf{W}_k}|}{\| \mathsf{A}_k \, \mathbf{x}_\perp\|_{\mathcal{D}(\mathring{\mathsf{A}}_k^*)}} = \frac{\|\mathbf{x}_\perp\|^2_{D(\mathsf{A}_k)}}{\| \mathsf{A}_k \, \mathbf{x}_\perp\|_{\mathcal{D}(\mathring{\mathsf{A}}_k^*)}}. \end{aligned} \tag{3.41}$$

Recalling that $\| \mathsf{A}_k \, \mathbf{x}_\perp\|_{\mathcal{D}(\mathring{\mathsf{A}}_k^*)} = \|\mathbf{x}_\perp\|_{\mathcal{D}(\mathsf{A}_k)}$ by Corollary 3.10, we arrive at the inequality

$$\|\mathsf{T}_k^t\mathbf{x}_\perp\|_{\mathcal{D}(\mathring{\mathsf{A}}_k^*)'} \geq \frac{\|\mathbf{x}_\perp\|^2_{D(\mathsf{A}_k)}}{\|\mathbf{x}_\perp\|_{D(\mathsf{A}_k)}} = \|\mathbf{x}_\perp\|_{D(\mathsf{A}_k)}. \tag{3.42}$$

Therefore, on the one hand, $\| \mathsf{I}_k^t[\mathbf{x}] \|_{\mathcal{D}(\mathring{\mathsf{A}}_k^*)'} \geq \|\mathbf{x}_\perp\|_{\mathcal{D}(\mathsf{A}_k)} = \|[\mathbf{x}]\|_{\mathcal{T}(\mathsf{A}_k)}$ by (3.27).

On the other hand, inserting (3.4) in (3.40) leads to the estimate

$$\| \mathsf{I}_k^t[\mathbf{x}] \|_{D(\mathring{\mathsf{A}}_k^*)'} = \|\mathsf{T}_k^t\mathbf{x}_\perp\|_{\mathcal{D}(\mathring{\mathsf{A}}_k^*)} \leq \|\mathsf{T}_k^t\| \|\mathbf{x}_\perp\|_{\mathcal{D}(\mathsf{A}_k)} = \|\mathbf{x}_\perp\|_{\mathcal{D}(\mathsf{A}_k)} = \|[\mathbf{x}]\|_{\mathcal{T}(\mathsf{A}_k)}, \tag{3.43}$$

which concludes the proof. □

It is natural to think of a trace operator as a bounded linear operator from a domain to a trace space. Therefore, based on the identification provided by Theorem 3.11, we introduce the following perspective: in the setting of Definition 3.1, we call *quotient trace* the canonical projection

$$\pi_k^t : \begin{cases} \mathcal{D}(\mathsf{A}_k) \to \mathcal{T}(\mathsf{A}_k) \\ \mathbf{x} \mapsto [\mathbf{x}] \end{cases}. \tag{3.44}$$

Notice that because I_k^t is an isomorphism, it follows from $\mathsf{I}_k^t(\mathsf{I}_k^t)^{-1}\mathsf{T}_k^t\mathbf{x} = \mathsf{T}_k^t\mathbf{x} = \mathsf{I}_k^t[\mathbf{x}]$ that

$$\pi_k^t\mathbf{x} = (\mathsf{I}_k^t)^{-1}\mathsf{T}_k^t\mathbf{x}. \tag{3.45}$$

3.3. Riesz representatives

Let $\mathsf{R}_{\mathcal{D}(\mathring{\mathsf{A}}_k^*)} : \mathcal{D}(\mathring{\mathsf{A}}_k^*) \to \mathcal{D}(\mathring{\mathsf{A}}_k^*)'$ be the Riesz isomorphism defined by $\mathsf{R}_{\mathcal{D}(\mathring{\mathsf{A}}_k^*)}\mathbf{y} = (\mathbf{y}, \cdot)_{\mathcal{D}(\mathring{\mathsf{A}}_k^*)}$ for all $\mathbf{y} \in \mathcal{D}(\mathring{\mathsf{A}}_k^*)$, cf. [10, Thm. 5.5]. Notice that in the first part of the proof of Proposition 3.5, we have shown that the following result holds with $\mathring{\mathsf{A}}_k^* \mathsf{R}^{-1}_{\mathcal{D}(\mathring{\mathsf{A}}_k^*)}\boldsymbol{\phi} \in \mathcal{D}(\mathsf{A}_k)$.

Lemma 3.12. *Under Assumption A, if $\boldsymbol{\phi} \in \mathcal{D}(\mathsf{A}_k^*)^\circ$, then $\mathring{\mathsf{A}}_k^* \mathsf{R}^{-1}_{\mathcal{D}(\mathring{\mathsf{A}}_k^*)}\boldsymbol{\phi} \in \mathcal{D}(\mathring{\mathsf{A}}_k)^\perp$ with*

$$(\mathsf{A}_k\, \mathring{\mathsf{A}}_k^* + \mathsf{Id})\, \mathsf{R}^{-1}_{\mathcal{D}(\mathring{\mathsf{A}}_k^*)}\boldsymbol{\phi} = 0 \qquad \textit{and} \qquad \mathsf{T}^t_{\mathsf{A}_k}(\mathring{\mathsf{A}}_k^* \mathsf{R}^{-1}_{\mathcal{D}(\mathring{\mathsf{A}}_k^*)}\boldsymbol{\phi}) = -\boldsymbol{\phi}. \tag{3.46}$$

Proof. It only remains to show that in particular $\mathring{\mathsf{A}}_k^* \mathsf{R}^{-1}_{\mathcal{D}(\mathring{\mathsf{A}}_k^*)}\boldsymbol{\phi} \in D(\mathring{\mathsf{A}}_k)^\perp$. Since $\mathsf{A}_k^* \subset \mathring{\mathsf{A}}_k^*$, we find, using $(\mathsf{A}_k\, \mathring{\mathsf{A}}_k^* + \mathsf{Id})\, \mathsf{R}^{-1}_{\mathcal{D}(\mathring{\mathsf{A}}_k^*)}\boldsymbol{\phi} = 0$, that for all $\mathbf{x}_\circ \in \mathcal{D}(\mathring{\mathsf{A}}_k)$,

$$\begin{aligned}(\mathring{\mathsf{A}}_k^* \mathsf{R}^{-1}_{\mathcal{D}(\mathring{\mathsf{A}}_k^*)}\boldsymbol{\phi}, \mathbf{x}_\circ)_{\mathcal{D}(\mathsf{A}_k)} &= (\mathring{\mathsf{A}}_k^* \mathsf{R}^{-1}_{\mathcal{D}(\mathring{\mathsf{A}}_k^*)}\boldsymbol{\phi}, \mathbf{x}_\circ)_{\mathbf{W}_k} + (\mathsf{A}_k\, \mathring{\mathsf{A}}_k^* \mathsf{R}^{-1}_{\mathcal{D}(\mathring{\mathsf{A}}_k^*)}\boldsymbol{\phi}, \mathsf{A}_k\, \mathbf{x}_\circ)_{\mathbf{W}_{k+1}} \\ &= (\mathsf{R}^{-1}_{\mathcal{D}(\mathring{\mathsf{A}}_k^*)}\boldsymbol{\phi}, \mathring{\mathsf{A}}_k\, \mathbf{x}_\circ)_{\mathbf{W}_{k+1}} - (\mathsf{R}^{-1}_{\mathcal{D}(\mathring{\mathsf{A}}_k^*)}\boldsymbol{\phi}, \mathring{\mathsf{A}}_k\, \mathbf{x}_\circ)_{\mathbf{W}_{k+1}} = 0. \quad \square\end{aligned} \tag{3.47}$$

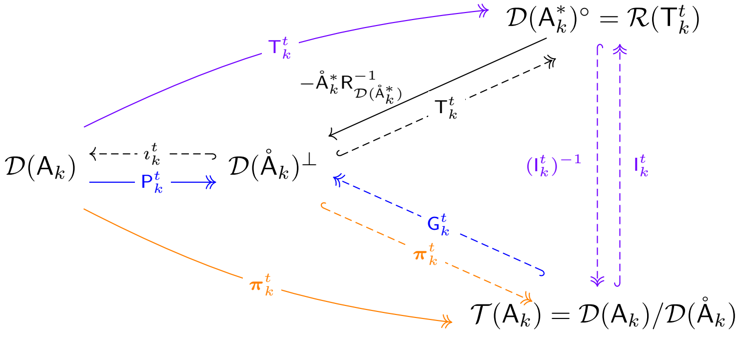
Applying $(\mathsf{I}_k^t)^{-1}$ on both sides of the second identity in Lemma 3.12, we find using (3.45) a slightly more explicit expression of the inverse $(\mathsf{I}_k^t)^{-1}$.

Lemma 3.13. *Under Assumption A, we have*

$$(\mathsf{I}_k^t)^{-1}\boldsymbol{\phi} = -\boldsymbol{\pi}^t_{\mathsf{A}_k}(\mathring{\mathsf{A}}_k^* \mathsf{R}^{-1}_{\mathcal{D}(\mathring{\mathsf{A}}_k^*)}\boldsymbol{\phi}) \qquad \forall \boldsymbol{\phi} \in D(\mathsf{A}_k^*)^\circ = \mathcal{R}(\mathsf{T}_k^t). \tag{3.48}$$

Remark 3.14. The operators $\mathring{\mathsf{A}}_k^* \mathsf{R}^{-1}_{\mathcal{D}(\mathring{\mathsf{A}}_k^*)} : \mathcal{R}(\mathsf{T}_k^t) \to \mathcal{D}(\mathring{\mathsf{A}}_k)^\perp \subset \mathcal{D}(\mathsf{A}_k)$ could be called $\mathcal{D}(\mathsf{A}_k)$-harmonic extension operators.

In summary, we have shown so far in Section 3 that the following diagram is commutative:



4. Duality

In this section, we maintain the setting of Assumption A, and we focus on the following snippet of the dual Hilbert complex (cf. Sections 2.2 and 2.3):

$$
\begin{array}{ccccccccc}
\cdots \xleftarrow[\mathring{\mathsf{A}}^*_{k-2}]{} & \mathcal{D}(\mathring{\mathsf{A}}^*_{k-2}) \subset \mathbf{W}_{k-1} & \xleftarrow[\mathring{\mathsf{A}}^*_{k-1}]{} & \mathcal{D}(\mathring{\mathsf{A}}^*_{k-1}) \subset \mathbf{W}_{k} & \xleftarrow[\mathring{\mathsf{A}}^*_{k}]{} & \mathcal{D}(\mathring{\mathsf{A}}^*_{k}) \subset \mathbf{W}_{k+1} & \xleftarrow[\mathring{\mathsf{A}}^*_{k+1}]{} \cdots \\
& \cup & & \cup & & \cup & \\
\cdots \xleftarrow[\mathsf{A}^*_{k-2}]{} & \mathcal{D}(A^*_{k-2}) \subset \mathbf{W}_{k-1} & \xleftarrow[\mathsf{A}^*_{k-1}]{} & \mathcal{D}(A^*_{k-1}) \subset \mathbf{W}_{k} & \xleftarrow[\mathsf{A}^*_{k}]{} & \mathcal{D}(A^*_{k}) \subset \mathbf{W}_{k+1} & \xleftarrow[\mathsf{A}^*_{k+1}]{} \cdots
\end{array}
$$

Recall the simple though important observation that, because $(\mathring{\mathsf{A}}^*_k)^* = \mathring{\mathsf{A}}^{**}_k = \mathring{\mathsf{A}}_k$, we have $\mathring{\mathsf{A}}_k \subset \mathsf{A}_k \iff \mathsf{A}^*_k \subset \mathring{\mathsf{A}}^*_k$. Given two operators $\mathsf{A}_k : \mathcal{D}(\mathsf{A}_k) \subset \mathbf{W}_k \to \mathbf{W}_{k+1}$ and $\mathring{\mathsf{A}}_k : \mathcal{D}(\mathring{\mathsf{A}}_k) \subset \mathbf{W}_k \to \mathbf{W}_{k+1}$ satisfying Assumption A, the Hilbert space adjoints $\mathring{\mathsf{A}}^*_k : \mathcal{D}(\mathring{\mathsf{A}}^*_k) \subset \mathbf{W}_{k+1} \to \mathbf{W}_k$ and $\mathsf{A}^*_k : \mathcal{D}(\mathsf{A}^*_k) \subset \mathbf{W}_{k+1} \to \mathbf{W}_k$ thus also satisfy Assumption A, but with the roles of $\mathbf{W}_k$ and $\mathbf{W}_{k+1}$ swapped. Indeed, both $\mathring{\mathsf{A}}^*_k$ and A^*_k are densely defined and closed unbounded linear operators between the Hilbert spaces and $\mathring{\mathsf{A}}^*_k$ is an extension of A^*_k, i.e. $\mathcal{D}(\mathsf{A}^*_k) \subset \mathcal{D}(\mathring{\mathsf{A}}^*_k)$ and $\mathsf{A}^*_k\, \mathbf{y}_* = \mathring{\mathsf{A}}^*_k \mathbf{y}_*$ for all $\mathbf{y}_* \in \mathcal{D}(\mathsf{A}^*_k)$.

In Section 4.1, the *dual Hilbert trace* T^n_k will be nothing more than the primal Hilbert trace from Definition 3.1 but associated with the pair of operators $\mathring{\mathsf{A}}^*_k$ and A^*_k. Nevertheless, we state its properties for completeness and to set up notation, because it will be used for the important duality results of Section 4.2.

4.1. Dual traces

As before, it follows from (3.1) that the following operator is well-defined.

Definition 4.1. Under Assumption A, we call *dual Hilbert trace* the bounded operator

$$\mathsf{T}^n_k : \mathcal{D}(\mathring{\mathsf{A}}^*_k) \to \mathcal{D}(\mathsf{A}_k)', \tag{4.1}$$

defined for all $\mathbf{y} \in \mathcal{D}(\mathring{\mathsf{A}}^*_k)$ and $\mathbf{x} \in \mathcal{D}(\mathsf{A}_k)$ by

$$\langle \mathsf{T}^n_k \mathbf{y}, \mathbf{x} \rangle_{\mathcal{D}(\mathring{\mathsf{A}}^*_k)'} := (\mathring{\mathsf{A}}^*_k \mathbf{y}, \mathbf{x})_{\mathbf{W}_k} - (\mathbf{y}, \mathsf{A}_k\, \mathbf{x})_{\mathbf{W}_{k+1}}. \tag{4.2}$$

As in (3.4), we have $\|\mathsf{T}^n_k\| = 1$, where $\|\cdot\|$ is the operator norm. Note that for all $\mathbf{x} \in \mathcal{D}(\mathsf{A}_k)$ and $\mathbf{y} \in \mathcal{D}(\mathring{\mathsf{A}}^*_k)$,

$$\langle \mathsf{T}^t_k \mathbf{x}, \mathbf{y} \rangle_{\mathcal{D}(\mathsf{A}_k)'} = -\langle \mathbf{x}, \mathsf{T}^n_k \mathbf{y} \rangle_{\mathcal{D}(\mathring{\mathsf{A}}^*_k)'}. \tag{4.3}$$

In other formulas,

$$(\mathsf{T}_k^t)' = -\mathsf{T}_k^n \qquad \text{and} \qquad (\mathsf{T}_k^n)' = -\mathsf{T}_k^t. \tag{4.4}$$

The results of Section 3 can be mirrored by interchanging the roles of A_k and $\mathring{\mathsf{A}}_k^*$ (and the roles of $\mathring{\mathsf{A}}_k$ and A_k^* accordingly). We translate a few of them without proof.

Proposition 4.2 *(cf. Proposition 3.3). Under Assumption A, we have*

$$\mathcal{N}(\mathsf{T}_k^n) = \mathcal{D}(\mathsf{A}_k^*). \tag{4.5}$$

The next proposition involves the annihilator of $\mathcal{D}(\mathring{\mathsf{A}}_k)$ in $\mathcal{D}(\mathsf{A}_k)'$:

$$\mathcal{D}(\mathring{\mathsf{A}}_k)^\circ := \{\, \boldsymbol{\phi} \in \mathcal{D}(\mathsf{A}_k)' \mid \langle \boldsymbol{\phi}, \mathbf{x}_\circ \rangle = 0,\ \forall \mathbf{x}_\circ \in \mathcal{D}(\mathring{\mathsf{A}}_k)\,\}. \tag{4.6}$$

Proposition 4.3 *(cf. Proposition 3.5). Under Assumption A, we have*

$$\mathcal{R}(\mathsf{T}_k^n) = \mathcal{D}(\mathring{\mathsf{A}}_k)^\circ. \tag{4.7}$$

Definition 4.4 *(cf. Definition 3.6).* We call *dual trace spaces* the quotient spaces

$$\mathcal{T}(\mathring{\mathsf{A}}_k^*) := \mathcal{D}(\mathring{\mathsf{A}}_k^*)/\mathcal{D}(\mathsf{A}_k^*), \tag{4.8}$$

equipped with the quotient norm

$$\|[\mathbf{y}]\|_{\mathcal{T}(\mathring{\mathsf{A}}_k^*)} := \inf_{\mathbf{z}_* \in \mathcal{D}(\mathsf{A}_k^*)} \|\mathbf{y} - \mathbf{z}_*\|_{\mathcal{D}(\mathring{\mathsf{A}}_k^*)} \qquad \forall \mathbf{y} \in \mathcal{D}(\mathring{\mathsf{A}}_k^*). \tag{4.9}$$

Remark 4.5. Just as in Remark 3.7, notice that due to Proposition 4.2,

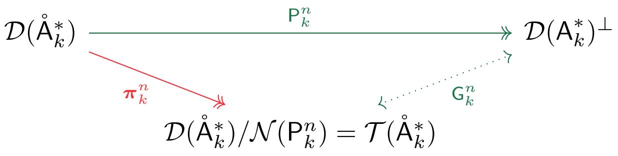
$$\mathcal{T}(\mathring{\mathsf{A}}_k^*) = \mathcal{D}(\mathring{\mathsf{A}}_k^*)/\mathcal{N}(\mathsf{T}_k^n). \tag{4.10}$$

In (4.9), we used square brackets to denote the equivalence class in $\mathcal{T}(\mathring{\mathsf{A}}_k^*)$ of $\mathbf{y} \in \mathcal{D}(\mathring{\mathsf{A}}_k^*)$, i.e. $[\mathbf{y}] = \{\mathbf{y} + \mathbf{z}_* \,|\, \mathbf{z}_* \in \mathcal{D}(\mathsf{A}_k^*)\}$. We will write $\boldsymbol{\pi}_k^n : D(\mathring{\mathsf{A}}_k^*) \to \mathcal{T}(\mathring{\mathsf{A}}_k^*)$ for the associated canonical projection (quotient map), i.e. $\boldsymbol{\pi}_k^n(\mathbf{y}) = [\mathbf{y}]$. Then, as previously detailed in Section 3.2, there exists a bounded orthogonal projection $\mathsf{P}_k^n : \mathcal{D}(\mathring{\mathsf{A}}_k^*) \to \mathcal{D}(\mathsf{A}_k^*)^\perp$ onto the complement space

$$\mathcal{D}(\mathsf{A}_k^*)^\perp := \left\{ \mathbf{y} \in \mathcal{D}(\mathring{\mathsf{A}}_k^*) \mid (\mathbf{y}, \mathbf{z}_*)_{\mathcal{D}(\mathring{\mathsf{A}}_k^*)} = 0,\ \forall \mathbf{z}_* \in \mathcal{D}(\mathsf{A}_k^*) \right\} \tag{4.11}$$

satisfying $\|\mathsf{P}_k^n \mathbf{y}\|_{\mathcal{D}(\mathring{\mathsf{A}}_k^*)} = \|[\mathbf{y}]\|_{\mathcal{T}(\mathring{\mathsf{A}}_k^*)}$ for all $\mathbf{y} \in \mathcal{D}(\mathring{\mathsf{A}}_k^*)$. We denote by $\imath_k^n : \mathcal{D}(\mathsf{A}_k^*)^\perp \hookrightarrow \mathcal{D}(\mathring{\mathsf{A}}_k^*)$ the canonical inclusion maps.

The induced operator $\mathsf{G}_k^n : \mathcal{T}(\mathring{\mathsf{A}}_k^*) \to \mathcal{D}(\mathsf{A}_k^*)^\perp$ involved in the commutative diagram



(4.12)

is an isometric isomorphism. Accordingly, any $\mathbf{y} \in \mathcal{D}(\mathring{\mathsf{A}}_k^*)$ can be uniquely decomposed as

$$\mathbf{y} = \mathsf{P}_k^n \mathbf{y} + \mathbf{y}_*, \qquad \mathbf{y}_* := (\mathsf{Id} - \mathsf{P}_k^n)\mathbf{y} \in \mathcal{N}(\mathsf{P}_k^n) = \mathcal{D}(\mathsf{A}_k^*). \tag{4.13}$$

3D de Rham setting VIII: Classical dual traces. Using (4.4), we find for the de Rham complex that, after identifying spaces and their biduals,

$$\mathsf{T}^n_{\mathbf{grad}} = -\gamma' \circ \gamma_n, \qquad \mathsf{T}^n_{\mathbf{curl}} = \gamma_t' \circ \gamma_t, \qquad \mathsf{T}^n_{\mathrm{div}} = -\gamma_n' \circ \gamma. \tag{4.14}$$

Recalling (1.8a) to (1.8c), we see from the table of the 3D de Rham setting IV that based on Proposition 4.2,

$$\mathcal{N}(\mathsf{T}^n_{\mathbf{grad}}) = \mathcal{N}(\gamma_n), \qquad \mathcal{N}(\mathsf{T}^n_{\mathbf{curl}}) = \mathcal{N}(\gamma_t), \qquad \mathcal{N}(\mathsf{T}^n_{\mathrm{div}}) = \mathcal{N}(\gamma). \tag{4.15}$$

The trace spaces provided by Definition 4.4 in this setting are

$$\mathcal{T}(\mathbf{grad}^*) = \mathcal{T}(\mathrm{div}) = \mathbf{H}(\mathrm{div}, \Omega)/\mathring{\mathbf{H}}(\mathrm{div}, \Omega), \tag{4.16a}$$

$$\mathcal{T}(\mathbf{curl}^{*p}) = \mathcal{T}(\mathbf{curl}) = \mathbf{H}(\mathbf{curl}, \Omega)/\mathring{\mathbf{H}}(\mathbf{curl}, \Omega), \tag{4.16b}$$

$$\mathcal{T}(\mathrm{div}^{*p}) = \mathcal{T}(\mathbf{grad}) = H^1(\Omega)/\mathring{H}^1(\Omega). \tag{4.16c}$$

Notice that from (3.9), we also have

$$(\mathsf{T}^t_{\mathrm{div}})' = \mathsf{T}^t_{\mathbf{grad}} = -\mathsf{T}^n_{\mathrm{div}} = -(\mathsf{T}^n_{\mathbf{grad}})' \qquad \text{and} \qquad (\mathsf{T}^t_{\mathbf{grad}})' = \mathsf{T}^t_{\mathrm{div}} = -\mathsf{T}^n_{\mathbf{grad}} = -(\mathsf{T}^n_{\mathrm{div}})'. \tag{4.17}$$

Moreover, we see that the skew-symmetry behind (1.10) is rooted in the fact that the identity $\mathsf{A}_1 = \mathbf{curl} = \mathring{\mathsf{A}}_1^*$ leads to skew-symmetry of the pairing

$$(\mathbf{x}, \mathbf{y}) \mapsto (\mathbf{curl}\,\mathbf{x}, \mathbf{y})_{\mathbf{L}^2(\Omega)} - (\mathbf{x}, \mathbf{curl}\,\mathbf{y})_{\mathbf{L}^2(\Omega)}. \tag{4.18}$$

This is reflected in the observation that $(\gamma_t' \circ \gamma_t)' = (\mathsf{T}_1^t)' = -\mathsf{T}_1^n = -\gamma_t' \circ \gamma_t$, which indeed occurs when duality is taken with respect to the skew-symmetric pairing (1.10). ⊿

Theorem 4.6 *(cf. Theorem 3.11). Under Assumption A, the linear map*

$$\mathsf{I}_k^n : \begin{cases} \mathcal{T}(\mathring{\mathsf{A}}_k^*) \to \mathcal{R}(\mathsf{T}_k^n) \\ \quad [\mathbf{y}] \mapsto \mathsf{T}_k^n \mathbf{y} \end{cases} \tag{4.19}$$

is a well-defined isometric isomorphism.

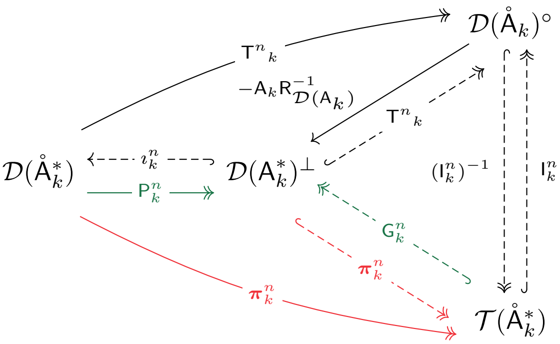
We call *dual quotient trace* the canonical projection (cf. (3.44))

$$\boldsymbol{\pi}_k^n : \begin{cases} \mathcal{D}(\mathring{\mathsf{A}}_k^*) \to \mathcal{T}(\mathring{\mathsf{A}}_k^*) \\ \mathbf{y} \mapsto [\mathbf{y}] \end{cases} . \tag{4.20}$$

Similarly as before, notice that (cf. (3.45))

$$\boldsymbol{\pi}_k^n \mathbf{y} = (\mathsf{I}_k^n)^{-1} \mathsf{T}_k^n \mathbf{y}, \tag{4.21}$$

and the following diagram commutes:



4.2. Duality of trace spaces

In this section, we show that the trace spaces $\mathcal{T}(\mathsf{A}_k)$ and $\mathcal{T}(\mathring{\mathsf{A}}_k^*)$ can be put in duality through an isometry. In fact, this follows immediately from a classical result in functional analysis. Indeed, according to [43, Thm. 4.9], we have the isometric isomorphisms

$$\mathcal{D}(\mathsf{A}_k^*)^\circ \cong \Big(\mathcal{D}(\mathring{\mathsf{A}}_k^*)/\mathcal{D}(\mathsf{A}_k^*)\Big)' \qquad \text{and} \qquad \mathcal{D}(\mathring{\mathsf{A}}_k)^\circ \cong \Big(\mathcal{D}(\mathsf{A}_k)/\mathcal{D}(\mathring{\mathsf{A}}_k)\Big)'. \tag{4.22}$$

Combining these results with Propositions 3.5 and 4.3, along with Theorems 3.11 and 4.6,

$$\mathcal{T}(\mathsf{A}_k) \cong \mathcal{R}(\mathsf{T}_k^t) = \mathcal{D}(\mathsf{A}_k^*)^\circ \cong \Big(\mathcal{D}(\mathring{\mathsf{A}}_k^*)/\mathcal{D}(\mathsf{A}_k^*)\Big)' = (\mathcal{T}(\mathring{\mathsf{A}}_k^*))', \tag{4.23a}$$

$$\mathcal{T}(\mathring{\mathsf{A}}_k^*) \cong \mathcal{R}(\mathsf{T}_k^n) = \mathcal{D}(\mathring{\mathsf{A}}_k)^\circ \cong \Big(\mathcal{D}(\mathsf{A}_k)/\mathcal{D}(\mathring{\mathsf{A}}_k)\Big)' = (\mathcal{T}(\mathsf{A}_k))'. \tag{4.23b}$$

Nevertheless, we provide a detailed proof below, not only for convenience and completeness, but also because the exercise is illuminating. We proceed with the definition of a continuous bilinear form on $\mathcal{T}(\mathsf{A}_k) \times \mathcal{T}(\mathring{\mathsf{A}}_k^*)$ and prove that the associated induced linear operator is an isometry. This pairing will be at the heart of sections 7.2 and 7, where it will be used to prove that Hilbert complexes affording so-called compact regular decompositions spawn Fredholm trace Hilbert complexes.

Lemma 4.7. *Under Assumption A, the bilinear form*

$$\langle\langle \cdot, \cdot \rangle\rangle_k : \mathcal{T}(\mathsf{A}_k) \times \mathcal{T}(\mathring{\mathsf{A}}_k^*) \to \mathbb{R}, \tag{4.24a}$$

defined by

$$\langle\langle [\mathbf{x}], [\mathbf{y}] \rangle\rangle_k := (\mathsf{A}_k\, \mathbf{x}, \mathbf{y})_{\mathbf{W}_{k+1}} - (\mathbf{x}, \mathring{\mathsf{A}}_k^* \mathbf{y})_{\mathbf{W}_k} \quad \forall [\mathbf{x}] \in \mathcal{T}(\mathsf{A}_k), \forall [\mathbf{y}] \in \mathcal{T}(\mathring{\mathsf{A}}_k^*), \tag{4.24b}$$

is well-defined and continuous with norm ≤ 1.

Proof. Since $\langle\langle [\mathbf{x}], [\mathbf{y}] \rangle\rangle_k = \langle \mathsf{T}_k^t \mathbf{x}, \mathbf{y} \rangle_{\mathcal{D}(\mathsf{A}_k)'}$, it is well-defined thanks to Proposition 3.3 and Proposition 3.5. By the same propositions, the orthogonal decompositions (3.30) and (4.13) yield the estimate

$$\begin{aligned} |\langle \mathsf{T}_k^t \mathbf{x}, \mathbf{y} \rangle_{\mathcal{D}(\mathsf{A}_k)'}| &= |\langle \mathsf{T}_k^t \mathsf{P}_k^t \mathbf{x}, \mathsf{P}_k^n \mathbf{y} \rangle_{\mathcal{D}(\mathsf{A}_k)'}| \\ &= |(\mathsf{A}_k\, \mathsf{P}_k^t \mathbf{x}, \mathsf{P}_k^n \mathbf{y})_{\mathbf{W}_{k+1}} - (\mathsf{P}_k^t \mathbf{x}, \mathring{\mathsf{A}}_k^* \mathsf{P}_k^n \mathbf{y})_{\mathbf{W}_k}| \\ &\leq \|\, \mathsf{A}_k\, \mathsf{P}_k^t \mathbf{x}\|_{\mathbf{W}_{k+1}} \|\mathsf{P}_k^n \mathbf{y}\|_{\mathbf{W}_{k+1}} + \|\mathsf{P}_k^t \mathbf{x}\|_{\mathbf{W}_k} \|\mathring{\mathsf{A}}_k^* \mathsf{P}_k^n \mathbf{y}\|_{\mathbf{W}_k} \\ &\leq \|\mathsf{P}_k^t \mathbf{x}\|_{\mathcal{D}(\mathsf{A}_k)} \|\mathsf{P}_k^n \mathbf{y}\|_{\mathcal{D}(\mathring{\mathsf{A}}_k^*)} = \|[\mathbf{x}]\|_{\mathcal{T}(\mathsf{A}_k)} \|[\mathbf{y}]\|_{\mathcal{T}(\mathring{\mathsf{A}}_k^*)}, \end{aligned} \tag{4.25}$$

showing that the bilinear form is continuous with norm ≤ 1. □

The next result shows in particular that $\mathcal{T}(\mathsf{A}_k)$ and $\mathcal{T}(\mathring{\mathsf{A}}_k^*)$ can be put in duality through the bilinear form $\langle\langle \cdot, \cdot \rangle\rangle_k$.

Theorem 4.8. *Under Assumption A, the bounded linear operator*

$$\mathsf{K}_k : \begin{cases} \mathcal{T}(\mathsf{A}_k) \to \mathcal{T}(\mathring{\mathsf{A}}_k^*)' \\ \quad [\mathbf{x}] \mapsto \langle\langle [\mathbf{x}], \cdot \rangle\rangle_k \end{cases} \tag{4.26}$$

induced by the bilinear form defined in Lemma 4.7 is an isometric isomorphism.

Proof. The key to the proof is that (4.24b) permits us to appeal to Theorem 3.11.

Notice that since $\mathcal{R}(\mathsf{T}_k^t) = \mathcal{D}(\mathsf{A}_k^*)^\circ$, it follows from the orthogonal decomposition (4.13) that K_k is the pullback by G_k^n of I_k^t, i.e. $\mathsf{K}_k[\mathbf{x}]([\mathbf{y}]) = \mathsf{I}_k^t[\mathbf{x}](\mathsf{G}_k^n[\mathbf{y}])$. We first show that it is an isomorphism.

If $\mathsf{K}_k[\mathbf{x}] = \mathsf{K}_k[\mathbf{z}]$, then since G_k^n is an isomorphism onto $\mathcal{D}(\mathsf{A}_k^*)^\perp$, it then follows from Proposition 3.5 and decomposition (4.13) that $\mathsf{I}_k^t[\mathbf{x}](\mathbf{y}) = \mathsf{I}_k^t[\mathbf{z}](\mathbf{y})$ for all $\mathbf{y} \in \mathcal{D}(\mathring{\mathsf{A}}_k^*)$. But I_k^t is also an isomorphism, so $\mathsf{I}_k^t[\mathbf{x}] = \mathsf{I}_k^t[\mathbf{z}]$ implies that $\mathbf{x} = \mathbf{z}$ and we conclude that K_k is injective.

Suppose that $\phi \in \mathcal{T}(\mathring{\mathsf{A}}_k^*)'$. Then the pullback of ϕ by the canonical quotient map $\pi_k^n : \mathcal{D}(\mathring{\mathsf{A}}_k^*) \to \mathcal{T}(\mathring{\mathsf{A}}_k^*)$ is a bounded linear functional on $\mathcal{D}(\mathring{\mathsf{A}}_k^*)$, i.e. $\phi \circ \pi_k^n \in \mathcal{D}(\mathring{\mathsf{A}}_k^*)'$. Indeed, this simply holds because

$$|\phi(\pi_k^n \mathbf{y})| \leq \|\phi\| \|\pi_k^n \mathbf{y}\|_{\mathcal{T}(\mathring{\mathsf{A}}_k^*)} \leq \|\phi\| \|\pi_k^n\| \|\mathbf{y}\|_{\mathcal{D}(\mathring{\mathsf{A}}_k^*)} \qquad \forall \mathbf{y} \in \mathcal{D}(\mathring{\mathsf{A}}_k^*). \tag{4.27}$$

Moreover, since $\mathcal{N}(\pi_k^n) = \mathcal{D}(\mathsf{A}_k^*)$, we find in particular that $\phi \circ \pi_k^n \in \mathcal{D}(\mathsf{A}_k^*)^\circ = \mathcal{R}(\mathsf{T}_k^t)$. But I_k^t is an isomorphism onto $\mathcal{R}(\mathsf{T}_k^t)$, so there exists $[\mathbf{x}] \in \mathcal{T}(\mathsf{A}_k)$ such that $\mathsf{I}_k^t[\mathbf{x}] = \phi \circ \pi_k^n$. Evaluating

$$\mathsf{K}_k[\mathbf{x}] = \mathsf{I}_k^t[\mathbf{x}] \circ \mathsf{G}_k^n = \phi \circ \pi_k^n \circ \mathsf{G}_k^n = \phi \tag{4.28}$$

shows that K_k is surjective.

We now prove that K_k is an isometry. Using similar arguments as above, we estimate

$$\|\mathsf{K}_k[\mathbf{x}]\| = \sup_{\substack{[\mathbf{y}] \in \mathcal{T}(\mathring{\mathsf{A}}_k^*), \\ \|[\mathbf{y}]\|_{\mathcal{T}(\mathring{\mathsf{A}}_k^*)} = 1}} |\mathsf{K}_k[\mathbf{x}]([\mathbf{y}])| = \sup_{\substack{\mathbf{y}_\perp \in \mathcal{D}(\mathsf{A}_k^*)^\perp, \\ \|\mathbf{y}_\perp\|_{\mathcal{D}(\mathring{\mathsf{A}}_k^*)} = 1}} |\mathsf{I}_k^t[\mathbf{x}](\mathbf{y}_\perp)| = \|\mathsf{I}_k^t[\mathbf{x}]\| = \|[\mathbf{x}]\|_{\mathcal{T}(\mathsf{A}_k)}. \quad \square \tag{4.29}$$

We have arrived at an integration by parts formula involving the traces from Section 3.1 and Section 4.1: for all $\mathbf{x} \in \mathcal{D}(\mathsf{A}_k)$ and $\mathbf{y} \in \mathcal{D}(\mathring{\mathsf{A}}_k^*)$,

$$(\mathsf{A}_k\, \mathbf{x}, \mathbf{y})_{\mathbf{W}_{k+1}} - (\mathbf{x}, \mathring{\mathsf{A}}_k^* \mathbf{y})_{\mathbf{W}_k} = \langle\!\langle \pi_k^t \mathbf{x}, \pi_k^n \mathbf{y} \rangle\!\rangle_k. \tag{4.30}$$

Theorem 4.8, in combination with (1.12a) and (1.12b), reveals the abstract version of the duality observed for the de Rham complex in Section 1.

5. Operators on trace spaces

Starting from this section, we start exploiting more of the structure of Hilbert complexes by introducing the *minimal* Hilbert complex setting required to define what we will call *surface operators*. We "zoom in" on short snippets of (2.5a) and (2.8a) of the form

$$\begin{array}{ccccccccc}
\cdots \xrightarrow{\mathsf{A}_{k-1}} & \mathcal{D}(\mathsf{A}_k) \subset \mathbf{W}_k & \xrightarrow{\mathsf{A}_k} & \mathcal{D}(\mathsf{A}_{k+1}) \subset \mathbf{W}_{k+1} & \xrightarrow{\mathsf{A}_{k+1}} & \mathcal{D}(\mathsf{A}_{k+2}) \subset \mathbf{W}_{k+2} & \xrightarrow{\mathsf{A}_{k+2}} \cdots \\
 & \cup & & \cup & & & \\
\cdots \xrightarrow{\mathring{\mathsf{A}}_{k-1}} & \mathcal{D}(\mathring{\mathsf{A}}_k) \subset \mathbf{W}_k & \xrightarrow{\mathring{\mathsf{A}}_k} & \mathcal{D}(\mathring{\mathsf{A}}_{k+1}) \subset \mathbf{W}_{k+1} & \xrightarrow{\mathring{\mathsf{A}}_{k+1}} & \mathcal{D}(\mathring{\mathsf{A}}_{k+2}) \subset \mathbf{W}_{k+2} & \xrightarrow{\mathring{\mathsf{A}}_{k+2}} \cdots
\end{array} \tag{5.1}$$

We may call the highlighted sequences "minimal Hilbert complexes". The index k should be considered arbitrary but fixed in this section.

3D de Rham setting IX: Minimal Hilbert complexes. Based on the 3D de Rham setting III and IV, we obtain two minimal complexes such as (5.1). For $k = 0$, we have

$$\begin{aligned} H^1(\Omega) \subset L^2(\Omega) &\xrightarrow{\mathbf{grad}} \mathbf{H}(\mathbf{curl}, \Omega) \subset \mathbf{L}^2(\Omega) \xrightarrow{\mathbf{curl}} \mathbf{L}^2(\Omega), \\ \mathring{H}^1(\Omega) \subset L^2(\Omega) &\xrightarrow{\mathbf{grad}} \mathring{\mathbf{H}}(\mathbf{curl}, \Omega) \subset \mathbf{L}^2(\Omega) \xrightarrow{\mathbf{curl}} \mathbf{L}^2(\Omega). \end{aligned} \tag{5.2}$$

For $k = 1$, we get

$$\begin{aligned} \mathbf{H}(\mathbf{curl}, \Omega) \subset \mathbf{L}^2(\Omega) &\xrightarrow{\mathbf{curl}} \mathbf{H}(\mathrm{div}, \Omega) \subset \mathbf{L}^2(\Omega) \xrightarrow{\mathrm{div}} \subset \mathbf{L}^2(\Omega), \\ \mathring{\mathbf{H}}(\mathbf{curl}, \Omega) \subset \mathbf{L}^2(\Omega) &\xrightarrow{\mathbf{curl}} \mathring{\mathbf{H}}(\mathrm{div}, \Omega) \subset \mathbf{L}^2(\Omega) \xrightarrow{\mathrm{div}} \mathbf{L}^2(\Omega). \end{aligned} \tag{5.3}$$

The associated dual minimal complexes can be excised from (2.7) and (2.10a). ⊿

5.1. *Surface operators in domains*

Notice that due to the complex property, we have in particular that $\mathcal{R}(\mathsf{A}_k) \subset \mathcal{D}(\mathsf{A}_{k+1})$ and $\mathcal{R}(\mathring{\mathsf{A}}^*_{k+1}) \subset \mathcal{D}(\mathring{\mathsf{A}}^*_k)$. The following key operators are thus well-defined.

Definition 5.1. We call *surface operators* the bounded linear maps

$$\mathsf{D}^t_k := (\mathring{\mathsf{A}}^*_{k+1})' : \mathcal{D}(\mathring{\mathsf{A}}^*_k)' \to \mathcal{D}(\mathring{\mathsf{A}}^*_{k+1})', \tag{5.4a}$$

$$\mathsf{D}^n_{k+1} := \mathsf{A}'_k : \mathcal{D}(\mathsf{A}_{k+1})' \to \mathcal{D}(\mathsf{A}_k)', \tag{5.4b}$$

dual to $\mathring{\mathsf{A}}^*_{k+1} : \mathcal{D}(\mathring{\mathsf{A}}^*_{k+1}) \to \mathcal{D}(\mathring{\mathsf{A}}^*_k)$ and $\mathsf{A}_k : \mathcal{D}(\mathsf{A}_k) \to \mathcal{D}(\mathsf{A}_{k+1})$, respectively. Equivalently,

$$\langle \mathsf{D}^t_k \boldsymbol{\phi}, \mathbf{z} \rangle_{\mathcal{D}(\mathring{\mathsf{A}}^*_{k+1})'} = \langle \boldsymbol{\phi}, \mathring{\mathsf{A}}^*_{k+1} \mathbf{z} \rangle_{\mathcal{D}(\mathring{\mathsf{A}}^*_k)'}, \qquad \forall \boldsymbol{\phi} \in \mathcal{D}(\mathring{\mathsf{A}}^*_k)', \forall \mathbf{z} \in \mathcal{D}(\mathring{\mathsf{A}}^*_{k+1}) \subset \mathbf{W}_{k+2}, \tag{5.5a}$$

$$\langle \mathsf{D}^n_{k+1} \boldsymbol{\psi}, \mathbf{x} \rangle_{D(\mathsf{A}_k)'} = \langle \boldsymbol{\psi}, \mathsf{A}_k \, \mathbf{x} \rangle_{\mathcal{D}(\mathsf{A}_{k+1})'}, \qquad \forall \boldsymbol{\psi} \in \mathcal{D}(\mathsf{A}_{k+1})', \forall \mathbf{x} \in \mathcal{D}(\mathsf{A}_k) \subset \mathbf{W}_k. \tag{5.5b}$$

Remark 5.2. Recall the distinction made in Section 2.1 between the notation for bounded and unbounded linear operators. We point out that in Definition 5.1, the operators $\mathring{\mathsf{A}}^*_{k+1} : \mathcal{D}(\mathring{\mathsf{A}}^*_{k+1}) \to \mathcal{D}(\mathring{\mathsf{A}}^*_k)$ and $\mathsf{A}_k : \mathcal{D}(\mathsf{A}_k) \to \mathcal{D}(\mathsf{A}_{k+1})$ are *bounded.*

Remark 5.3. The name 'surface operators' was chosen by analogy with standard surface operators on the boundary of a domain, despite the fact that there is no boundary involved in the above definition. The relation between Definition 5.1 and standard surface operators is made more explicit in the two following 3D de Rham settings X and XI.

3D de Rham setting X: Surface operators in domains. In the 3D de Rham setting IX, we find the surface operators

$$\mathsf{D}_0^t := \mathbf{curl}' : \mathbf{H}(\mathrm{div}, \Omega)' \to \mathbf{H}(\mathbf{curl}, \Omega)', \tag{5.6a}$$

$$\mathsf{D}_1^t := (-\mathbf{grad})' : \mathbf{H}(\mathbf{curl}, \Omega)' \to \widetilde{H}^{-1}(\Omega), \tag{5.6b}$$

dual to the *bounded* operators

$$\mathbf{curl} : \mathbf{H}(\mathbf{curl}, \Omega) \to \mathbf{H}(\mathrm{div}, \Omega) \qquad \text{and} \qquad -\mathbf{grad} : H^1(\Omega) \to \mathbf{H}(\mathbf{curl}, \Omega), \tag{5.7}$$

where we have written $\widetilde{H}^{-1}(\Omega) := H^1(\Omega)'$. In other words,

$$\langle \mathsf{D}_0^t \boldsymbol{\phi}, \mathbf{v}\rangle_{\mathbf{H}(\mathbf{curl},\Omega)'} = \langle \boldsymbol{\phi}, \mathbf{curl}\,\mathbf{v}\rangle_{\mathbf{H}(\mathrm{div},\Omega)'}, \qquad \forall \boldsymbol{\phi} \in \mathbf{H}(\mathrm{div}, \Omega)', \forall \mathbf{v} \in \mathbf{H}(\mathbf{curl}, \Omega), \tag{5.8a}$$

$$\langle \mathsf{D}_1^t \boldsymbol{\phi}, u\rangle_{\widetilde{H}^{-1}(\Omega)} = \langle \boldsymbol{\phi}, -\mathbf{grad} u\rangle_{\mathbf{H}(\mathbf{curl},\Omega)'} \qquad \forall \boldsymbol{\phi} \in \mathbf{H}(\mathbf{curl}, \Omega)', \forall u \in H^1(\Omega). \tag{5.8b}$$

In the adjoint perspective, the bounded linear operators

$$\mathsf{D}_1^n := \mathbf{grad}' : \mathbf{H}(\mathbf{curl}, \Omega)' \to \widetilde{H}^{-1}(\Omega), \tag{5.9a}$$

$$\mathsf{D}_2^n := \mathbf{curl}' : \mathbf{H}(\mathrm{div}, \Omega)' \to \mathbf{H}(\mathbf{curl}, \Omega)' \tag{5.9b}$$

are dual to the bounded linear operators

$$\mathbf{grad} : H^1(\Omega) \to \mathbf{H}(\mathbf{curl}, \Omega) \qquad \text{and} \qquad \mathbf{curl} : \mathbf{H}(\mathbf{curl}, \Omega) \to \mathbf{H}(\mathrm{div}, \Omega). \tag{5.10}$$

That is,

$$\langle \mathsf{D}_1^n \boldsymbol{\psi}, u\rangle_{\widetilde{H}^{-1}(\Omega)} = \langle \boldsymbol{\psi}, \mathbf{grad} u\rangle_{\mathbf{H}(\mathbf{curl},\Omega)'} \qquad \forall \boldsymbol{\psi} \in \mathbf{H}(\mathbf{curl}, \Omega)', \forall u \in H^1(\Omega), \tag{5.11a}$$

$$\langle \mathsf{D}_2^n \boldsymbol{\psi}, \mathbf{v}\rangle_{\mathbf{H}(\mathbf{curl},\Omega)'} = \langle \boldsymbol{\psi}, \mathbf{curl}\,\mathbf{v}\rangle_{\mathbf{H}(\mathrm{div},\Omega)'} \qquad \forall \boldsymbol{\psi} \in \mathbf{H}(\mathrm{div}, \Omega)', \forall \mathbf{v} \in \mathbf{H}(\mathbf{curl}, \Omega). \tag{5.11b}$$

◿

Since

$$\mathcal{R}(\mathsf{A}_k) \subset \mathcal{D}(\mathsf{A}_{k+1}) = \mathcal{D}(\mathsf{T}_{k+1}^t), \qquad \mathcal{R}(\mathsf{T}_k^t) \subset \mathcal{D}(\mathring{\mathsf{A}}_k^*)' = \mathcal{D}(\mathsf{D}_k^t), \tag{5.12a}$$

$$\mathcal{R}(\mathring{\mathsf{A}}_{k+1}^*) \subset \mathcal{D}(\mathring{\mathsf{A}}_k^*) = \mathcal{D}(\mathsf{T}_k^n), \qquad \mathcal{R}(\mathsf{T}_{k+1}^n) \subset \mathcal{D}(\mathsf{A}_{k+1})' = \mathcal{D}(\mathsf{D}_{k+1}^n), \tag{5.12b}$$

the linear operators

$$\mathsf{D}_k^t \circ \mathsf{T}_k^t : \mathcal{D}(\mathsf{A}_k) \to \mathcal{D}(\mathring{\mathsf{A}}_{k+1}^*)', \qquad \mathsf{T}_{k+1}^t \circ \mathsf{A}_k : \mathcal{D}(\mathsf{A}_k) \to \mathcal{D}(\mathring{\mathsf{A}}_{k+1}^*)', \tag{5.13a}$$

$$\mathsf{D}_{k+1}^n \circ \mathsf{T}_{k+1}^n : \mathcal{D}(\mathring{\mathsf{A}}_{k+1}^*) \to \mathcal{D}(\mathsf{A}_k)', \qquad \mathsf{T}_k^n \circ \mathring{\mathsf{A}}_{k+1}^* : \mathcal{D}(\mathring{\mathsf{A}}_{k+1}^*) \to \mathcal{D}(\mathsf{A}_k)', \tag{5.13b}$$

are also well-defined and bounded.

Lemma 5.4. *Assumption A implies the following commuting relations:*

$$-\mathsf{D}_k^t \circ \mathsf{T}_k^t = \mathsf{T}_{k+1}^t \circ \mathsf{A}_k \qquad \textit{and} \qquad -\mathsf{D}_{k+1}^n \circ \mathsf{T}_{k+1}^n = \mathsf{T}_k^n \circ \mathring{\mathsf{A}}_{k+1}^*. \tag{5.14}$$

Proof. By symmetry, we need to verify only one relation. Recall that because of the complex property $\mathsf{A}_{k+1} \circ \mathsf{A}_k = 0$, we also have $\mathring{\mathsf{A}}_k^* \circ \mathring{\mathsf{A}}_{k+1}^* = 0$. Therefore, for all $\mathbf{x} \in \mathcal{D}(\mathsf{A}_k) \subset \mathbf{W}_k$ and $\mathbf{z} \in \mathcal{D}(\mathring{\mathsf{A}}_{k+1}^*) \subset \mathbf{W}_{k+2}$, we have on the one hand that

$$\begin{aligned} \langle \mathsf{D}_k^t \mathsf{T}_k^t \mathbf{x}, \mathbf{z} \rangle_{\mathcal{D}(\mathring{\mathsf{A}}_{k+1}^*)'} &= \langle \mathsf{T}_k^t \mathbf{x}, \mathring{\mathsf{A}}_{k+1}^* \mathbf{z} \rangle_{\mathcal{D}(\mathring{\mathsf{A}}_k^*)'} = (\mathsf{A}_k\, \mathbf{x}, \mathring{\mathsf{A}}_{k+1}^* \mathbf{z})_{\mathbf{W}_{k+1}} - (\mathbf{u}, \mathring{\mathsf{A}}_k^* \mathring{\mathsf{A}}_{k+1}^* \mathbf{z})_{\mathbf{W}_k} \\ &= (\mathsf{A}_k\, \mathbf{x}, \mathring{\mathsf{A}}_{k+1}^* \mathbf{z})_{\mathbf{W}_{k+1}}. \end{aligned} \tag{5.15}$$

On the other hand, we also evaluate

$$\begin{aligned} \langle \mathsf{T}_{k+1}^t\, \mathsf{A}_k\, \mathbf{x}, \mathbf{z} \rangle_{\mathcal{D}(\mathring{\mathsf{A}}_{k+1}^*)'} &= (\mathsf{A}_{k+1}\, \mathsf{A}_k\, \mathbf{x}, \mathbf{z})_{\mathbf{W}_{k+2}} - (\mathsf{A}_k\, \mathbf{x}, \mathring{\mathsf{A}}_{k+1}^* \mathbf{z})_{\mathbf{W}_{k+1}} \\ &= -(\mathsf{A}_k\, \mathbf{x}, \mathring{\mathsf{A}}_{k+1}^* \mathbf{z})_{\mathbf{W}_{k+1}}. \end{aligned} \qquad \square \tag{5.16}$$

Remark 5.5. Consistent with (4.4), $(\mathsf{D}_k^t \circ \mathsf{T}_k^t)' = \mathsf{D}_{k+1}^n \circ \mathsf{T}_{k+1}^n$ and $\mathsf{D}_k^t \circ \mathsf{T}_k^t = (\mathsf{D}_{k+1}^n \circ \mathsf{T}_{k+1}^n)'$.

Lemma 5.4 states that the following diagrams commute:

$$\begin{array}{ccc} \mathcal{D}(\mathsf{A}_k) & \xrightarrow{\mathsf{A}_k} & \mathcal{D}(\mathsf{A}_{k+1}) \\ \mathsf{T}^t{}_k \big\downarrow & & \big\downarrow \mathsf{T}^t{}_{k+1} \\ \mathcal{R}(\mathsf{T}_k^t) & \xrightarrow{-\mathsf{D}_k^t} & \mathcal{R}(\mathsf{T}_{k+1}^t) \end{array} \qquad \begin{array}{ccc} \mathcal{D}(\mathring{\mathsf{A}}_{k+1}^*) & \xrightarrow{\mathring{\mathsf{A}}_{k+1}^*} & \mathcal{D}(\mathring{\mathsf{A}}_k^*) \\ \mathsf{T}^n{}_{k+1} \big\downarrow & & \big\downarrow \mathsf{T}^n{}_k \\ \mathcal{R}(\mathsf{T}_{k+1}^n) & \xrightarrow{-\mathsf{D}_{k+1}^n} & \mathcal{R}(\mathsf{T}_k^n) \end{array} \tag{5.17}$$

An important consequence of this result is that

$$\mathsf{D}_k^t(\mathcal{R}(\mathsf{T}_k^t)) \subset \mathcal{R}(\mathsf{T}_{k+1}^t) = \mathcal{D}(A_{k+1}^*)^\circ, \tag{5.18}$$

an observation that is key to the introduction of trace Hilbert complexes in later sections.

3D de Rham setting XI: Commutative relations. In the 3D de Rham setting, it follows from (4.17) that the four relations obtained from Lemma 5.4 boil down to the single identity

$$\mathbf{grad}'\gamma_t' \circ \gamma_t = \gamma' \circ \gamma_n \mathbf{curl}. \tag{5.19}$$

In particular, (5.19) states that for all $\mathbf{u} \in \mathbf{H}(\mathbf{curl}, \Omega)$ and $v \in H^1(\Omega)$,

$$\int_\Gamma v\, \mathbf{n} \cdot \mathbf{curl}\, \mathbf{u}\, \mathrm{d}\sigma = \int_\Gamma \mathbf{n} \times (\mathbf{u} \times \mathbf{n}) \cdot (\mathbf{grad}\, v \times \mathbf{n})\, \mathrm{d}\sigma. \tag{5.20}$$

Recall that $\mathbf{n}\cdot\mathbf{curl} = \mathrm{curl}_\Gamma \circ \gamma_t$ on $\mathbf{H}(\mathbf{curl},\Omega)$, while the $L^2(\Gamma)$-dual operator $\mathrm{curl}_\Gamma = \mathbf{curl}_\Gamma'$ is such that $\mathbf{grad}\cdot\times\mathbf{n} = \mathbf{curl}_\Gamma \circ \gamma$ on $H^1(\Omega)$. Therefore, (5.20) expresses that

$$\int_\Gamma u\,\mathrm{curl}_\Gamma \mathbf{u}\,\mathrm{d}\sigma = \int_\Gamma \mathbf{curl}_\Gamma \mathbf{u}\cdot u\,\mathrm{d}\sigma \quad \forall u \in H^{1/2}(\Gamma),\ \mathbf{u}\in\mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma,\Gamma). \tag{5.21}$$

We conclude that the duality between the surface operators and their surface vector calculus counterparts in classical trace spaces is indeed captured by the duality in Section 4.2 and Lemma 5.4.

We point out that if one works with the $\mathbf{L}^2(\Gamma)$-pairing instead of the skew-symmetric pairing (1.10) from the start, then the two isometrically isomorphic perspectives of tangential and "rotated" tangential traces from [14] are also captured by the abstract theory. Indeed, by introducing the trace $\gamma_\tau : \cdot \mapsto \cdot\times\mathbf{n}$, one obtains $\mathsf{T}^t_{\mathbf{curl}} = \gamma_t' \circ \gamma_\tau$ and $\mathsf{T}^n_{\mathbf{curl}} = -\gamma_\tau' \circ \gamma_\tau$, which also satisfy (4.4). With these definitions, Lemma 5.4 leads to two identities corresponding to (5.21) and

$$\int_\Gamma v\,\mathrm{div}_\Gamma \mathbf{v}\,\mathrm{d}\sigma = -\int_\Gamma \mathbf{grad}_\Gamma v\cdot\mathbf{v}\mathrm{d}\sigma \quad \forall v \in H^{1/2}(\Gamma),\ \mathbf{v}\in\mathbf{H}^{-1/2}(\mathrm{div}_\Gamma,\Gamma), \tag{5.22}$$

which is a "rotated" version of (5.21), where $\gamma_n\mathbf{curl} = \mathrm{div}_\Gamma\gamma_\tau$ on $\mathbf{H}(\mathbf{curl},\Omega)$ and $\mathbf{H}^{-1/2}(\mathrm{div}_\Gamma,\Gamma)$ is defined by analogy with (1.7b). ⊿

5.2. *Surface operators in quotient spaces*

Let us investigate the properties of the linear operators between trace spaces induced by the surface operators defined in Section 5.1.

Definition 5.6. We call *quotient surface operators* the bounded linear maps

$$\mathsf{S}^t_k : \begin{cases} \mathcal{T}(\mathsf{A}_k) \to \mathcal{T}(\mathsf{A}_{k+1}) \\ \quad [\mathbf{x}] \mapsto \boldsymbol{\pi}^t_{k+1}\,\mathsf{A}_k\,\mathbf{x} \end{cases} \quad\text{and}\quad \mathsf{S}^n_{k+1} : \begin{cases} \mathcal{T}(\mathring{\mathsf{A}}^*_{k+1}) \to \mathcal{T}(\mathring{\mathsf{A}}^*_k) \\ \quad [\mathbf{z}] \mapsto \boldsymbol{\pi}^n_k \mathring{\mathsf{A}}^*_{k+1}\mathbf{z} \end{cases}. \tag{5.23}$$

We verify that S^t_k is well-defined. The analogous result holds for S^n_{k+1} by duality. Suppose that $\mathbf{x}_\circ \in \mathcal{D}(\mathring{\mathsf{A}}_k)$. Harnessing the complex property and the definition (4.24b) of the duality pairing we evaluate

$$\begin{aligned}\langle\!\langle \boldsymbol{\pi}^t_{k+1}\,\mathsf{A}_k\,\mathbf{x}_\circ, [\mathbf{z}]\rangle\!\rangle_{k+1} &= (\mathsf{A}_{k+1}\,\mathsf{A}_k\,\mathbf{x}_\circ, \mathbf{z})_{\mathbf{W}_{k+2}} - (\mathsf{A}_k\,\mathbf{x}_\circ, \mathring{\mathsf{A}}^*_{k+1}\mathbf{z})_{\mathbf{W}_{k+1}} \\ &= -(\mathring{\mathsf{A}}_k\,\mathbf{x}_\circ, \mathring{\mathsf{A}}^*_{k+1}\,\mathbf{z})_{\mathbf{W}_{k+1}} = -(\mathring{\mathsf{A}}_{k+1}\,\mathring{\mathsf{A}}_k\,\mathbf{x}_\circ, \mathbf{z})_{\mathbf{W}_{k+2}} = 0\end{aligned} \tag{5.24}$$

for all $\mathbf{z} \in \mathcal{D}(\mathring{\mathsf{A}}^*_{k+1}) \subset \mathbf{W}_{k+2}$. By the duality of $\mathcal{T}(\mathsf{A}_{k+1})$ and $\mathcal{T}(\mathring{\mathsf{A}}^*_{k+1})$ asserted in Theorem 4.8 we conclude that $\boldsymbol{\pi}^t_{k+1}\,\mathsf{A}_k\,\mathring{\mathbf{x}} = 0$.

From the above, we also find that for all $\mathbf{x}\in\mathcal{D}(\mathsf{A}_k) \subset \mathbf{W}_k$ and $\mathbf{z}\in\mathcal{D}(\mathring{\mathsf{A}}^*_{k+1}) \subset \mathbf{W}_{k+2}$,

$$\langle\!\langle \mathsf{S}^t_k \circ \boldsymbol{\pi}^t_k\,\mathbf{x}, \boldsymbol{\pi}^n_{k+1}\,\mathbf{z}\rangle\!\rangle_{k+1} = -(\mathsf{A}_k\,\mathbf{x}, \mathring{\mathsf{A}}^*_{k+1}\mathbf{z})_{\mathbf{W}_{k+1}} = -\langle\!\langle \boldsymbol{\pi}^t_k\,\mathbf{x}, \mathsf{S}^n_{k+1} \circ \boldsymbol{\pi}^n_{k+1}\,\mathbf{z}\rangle\!\rangle_k. \tag{5.25}$$

We can view the identity

$$\langle\!\langle \mathsf{S}_k^t[\mathbf{x}], [\mathbf{z}] \rangle\!\rangle_{k+1} = -\langle\!\langle [\mathbf{x}], \mathsf{S}_{k+1}^n[\mathbf{z}] \rangle\!\rangle_k \quad \forall [\mathbf{x}] \in \mathcal{T}(\mathsf{A}_k), \forall [\mathbf{z}] \in \mathcal{T}(\mathring{\mathsf{A}}_{k+1}^*), \tag{5.26}$$

as an integration by parts formula in (quotient) trace spaces.

Recalling Section 4.2, we can rewrite (5.26) as

$$\mathsf{K}_{k+1} \circ \mathsf{S}_k^t = -(\mathsf{S}_{k+1}^n)' \circ \mathsf{K}_k, \tag{5.27}$$

which gives rise to the commutative diagram

$$\begin{array}{ccc} \mathcal{T}(\mathring{\mathsf{A}}_k^*)' & \xrightarrow{-(\mathsf{S}_{k+1}^n)'} & \mathcal{T}(\mathring{\mathsf{A}}_{k+1}^*)' \\ {\scriptstyle \mathsf{K}_k}\,\Uparrow & & \Uparrow\,{\scriptstyle \mathsf{K}_{k+1}} \\ \mathcal{T}(\mathsf{A}_k) & \xrightarrow{\mathsf{S}_k^t} & \mathcal{T}(\mathsf{A}_{k+1}) \end{array} \tag{5.28}$$

We end this section by putting the results of the subsections 5.1 and 5.2 together into a single diagram. On the one hand, for all $\mathbf{x} \in \mathcal{D}(\mathsf{A}_k)$ and $\mathbf{z} \in \mathcal{D}(\mathring{\mathsf{A}}_{k+1}^*)$, we find from the proof of Lemma 5.4 that

$$\begin{aligned} \langle \mathsf{D}_k^t \circ \mathsf{T}_k^t\, \mathbf{x}, \mathbf{z} \rangle_{\mathcal{D}(\mathring{\mathsf{A}}_{k+1}^*)'} &= \langle \mathsf{D}_{k+1}^n \circ \mathsf{T}_{k+1}^n \mathbf{z}, \mathbf{x} \rangle_{\mathcal{D}(\mathsf{A}_k)'} \\ &= \langle\!\langle \boldsymbol{\pi}_k^t\, \mathbf{x}, \mathsf{S}_{k+1}^n \circ \boldsymbol{\pi}_{k+1}^n\, \mathbf{z} \rangle\!\rangle_k = -\langle\!\langle \mathsf{S}_k^t \circ \boldsymbol{\pi}_k^t\, \mathbf{x}, \boldsymbol{\pi}_{k+1}^n\, \mathbf{z} \rangle\!\rangle_{k+1}. \end{aligned} \tag{5.29}$$

On the other hand, we have by definition

$$\mathsf{S}_k^t \boldsymbol{\pi}_k^t\, \mathbf{x} = \boldsymbol{\pi}_{k+1}^t\, \mathsf{A}_k\, \mathbf{x} \qquad \text{and} \qquad \mathsf{S}_{k+1}^n \boldsymbol{\pi}_{k+1}^n\, \mathbf{z} = \boldsymbol{\pi}_k^n \mathring{\mathsf{A}}_{k+1}^* \mathbf{z}. \tag{5.30}$$

Also recall (3.45) and (4.21). In summary, the following diagrams commute:

$$\begin{array}{ccccc} & \mathcal{T}(\mathsf{A}_k) & \xrightarrow{\mathsf{S}_k^t} & \mathcal{T}(\mathsf{A}_{k+1}) & \\ & \Uparrow{\scriptstyle \boldsymbol{\pi}_k^t} & & \Uparrow{\scriptstyle \boldsymbol{\pi}_{k+1}^t} & \\ {\scriptstyle \mathsf{I}_k^t} & \mathcal{D}(\mathsf{A}_k) & \xrightarrow{\mathsf{A}_k} & \mathcal{D}(\mathsf{A}_{k+1}) & {\scriptstyle \mathsf{I}_{k+1}^t} \\ & \Downarrow{\scriptstyle \mathsf{T}^t{}_k} & & \Downarrow{\scriptstyle \mathsf{T}^t{}_{k+1}} & \\ & \mathcal{R}(\mathsf{T}_k^t) & \xrightarrow{-\mathsf{D}_k^t} & \mathcal{R}(\mathsf{T}_{k+1}^t) & \end{array} \qquad \begin{array}{ccccc} & \mathcal{T}(\mathring{\mathsf{A}}_{k+1}^*) & \xrightarrow{\mathsf{S}_{k+1}^n} & \mathcal{T}(\mathring{\mathsf{A}}_k^*) & \\ & \Uparrow{\scriptstyle \boldsymbol{\pi}_{k+1}^n} & & \Uparrow{\scriptstyle \boldsymbol{\pi}_k^n} & \\ {\scriptstyle \mathsf{I}_{k+1}^n} & \mathcal{D}(\mathring{\mathsf{A}}_{k+1}^*) & \xrightarrow{\mathring{\mathsf{A}}_{k+1}^*} & \mathcal{D}(\mathring{\mathsf{A}}_k^*) & {\scriptstyle \mathsf{I}_k^n} \\ & \Downarrow{\scriptstyle \mathsf{T}^n{}_{k+1}} & & \Downarrow{\scriptstyle \mathsf{T}^n{}_k} & \\ & \mathcal{R}(\mathsf{T}_{k+1}^n) & \xrightarrow{-\mathsf{D}_{k+1}^n} & \mathcal{R}(\mathsf{T}_k^n) & \end{array} \tag{5.31}$$

6. Trace spaces: characterization by regular subspaces

6.1. Bounded regular decompositions

In this section, we augment Assumption A. We first detail results in the setting of Definition 3.1 for primal Hilbert traces, then formulate their analogs in the dual setting of Definition 4.1. By symmetry, the primal and dual settings are evidently two faces of the same coin. From an abstract point of view, they are identical. Nevertheless, the dual setting is presented for convenience. The two settings are covered independently to avoid loosing sight of the core considerations.

6.1.1. Primal decomposition

Now, we aim at a more detailed characterization of the space $\mathcal{D}(\mathring{\mathsf{A}}_k^*)'$. Recall that by the complex property, $\mathcal{R}(\mathring{\mathsf{A}}_{k+1}^*) \subset \mathcal{D}(\mathring{\mathsf{A}}_k^*)$.

We refer to [37, Def. 2.12] for the next assumption, which introduces additional structure.

Assumption B. For all $k \in \mathbb{Z}$, Assumption A holds along with the following hypotheses:

I The Hilbert spaces $\mathbf{W}_k^+ \subset \mathbf{W}_k$ are such that the inclusion maps spawn continuous and dense embeddings

$$\mathbf{W}_k^+ \hookrightarrow \mathcal{D}(\mathring{\mathsf{A}}_{k-1}^*). \tag{6.1}$$

II There exist bounded operators

$$\mathsf{L}_{k+1}^t : \mathcal{D}(\mathring{\mathsf{A}}_k^*) \to \mathbf{W}_{k+1}^+ \qquad \text{and} \qquad \mathsf{V}_{k+1}^t : \mathcal{D}(\mathring{\mathsf{A}}_k^*) \to \mathbf{W}_{k+2}^+ \tag{6.2}$$

such that

$$\mathbf{y} = (\mathsf{L}_{k+1}^t + \mathring{\mathsf{A}}_{k+1}^* \mathsf{V}_{k+1}^t)\,\mathbf{y} \qquad \forall \mathbf{y} \in \mathcal{D}(\mathring{\mathsf{A}}_k^*). \tag{6.3}$$

III The Hilbert spaces

$$\mathbf{W}_{k+2}^+(\mathring{\mathsf{A}}_{k+1}^*) := \left\{ \mathbf{z} \in \mathbf{W}_{k+2}^+ \mid \mathring{\mathsf{A}}_{k+1}^* \mathbf{z} \in \mathbf{W}_{k+1}^+ \right\}, \tag{6.4}$$

equipped with the graph inner product defined for all $\mathbf{z}_1, \mathbf{z}_z \in \mathbf{W}_{k+2}^+(\mathring{\mathsf{A}}_{k+1}^*)$ by

$$(\mathbf{z}_1, \mathbf{z}_2)_{\mathbf{W}_{k+2}^+(\mathring{\mathsf{A}}_{k+1}^*)} := (\mathbf{z}_1, \mathbf{z}_2)_{\mathbf{W}_{k+2}^+} + (\mathring{\mathsf{A}}_{k+1}^* \mathbf{z}_1, \mathring{\mathsf{A}}_{k+1}^* \mathbf{z}_2)_{\mathbf{W}_{k+1}^+}, \tag{6.5}$$

are such that the inclusions $\mathbf{W}^+_{k+2} \subset \mathbf{W}_{k+2}$ induce continuous and dense embeddings

$$\mathbf{W}^+_{k+2}(\mathring{\mathsf{A}}^*_{k+1}) \hookrightarrow \mathcal{D}(\mathring{\mathsf{A}}^*_{k-1}). \tag{6.6}$$

We adopt a shorter notation for the dual spaces:

$$\mathbf{W}^-_k := (\mathbf{W}^+_k)', \quad k \in \mathbb{Z}. \tag{6.7}$$

Remark 6.1. In Hypothesis ***II***, (6.3) implies a stable regular decomposition of the form

$$\mathcal{D}(\mathring{\mathsf{A}}^*_k) = \mathbf{W}^+_{k+1} + \mathring{\mathsf{A}}^*_{k+1}\mathbf{W}^+_{k+2}, \quad k \in \mathbb{Z}. \tag{6.8}$$

By stable, we mean that the lifting and potential operators in (6.2) are bounded. We call it regular due to Hypothesis ***I***, based on which we can imagine the $\mathbf{W}^+_k$s as subspaces of "extra regularity".

Remark 6.2. The decomposition in (6.3)/(6.8) need not be direct.

Remark 6.3. Assumption B is stated for all $k \in \mathbb{Z}$. Strictly speaking, in the setting of a minimal complex with $k \in \mathbb{Z}$ fixed, to which we adhere in this section, only one stable regular decomposition (the one written in (6.3) and involving the regular spaces $\mathbf{W}^+_{k+1}$ and $\mathbf{W}^+_{k+2}$) is necessary for the characterization of $\mathcal{D}(\mathring{\mathsf{A}}^*_k)'$ and $\mathcal{R}(\mathsf{T}^t_k)$.

Lemma 6.4. *Under Assumption B, the surface operator* $\mathsf{D}^t_k : \mathcal{D}(\mathring{\mathsf{A}}^*_k)' \to \mathcal{D}(\mathring{\mathsf{A}}^*_{k+1})'$ *defined in* (5.4a) *can be extended to a continuous mapping*

$$\mathsf{D}^t_k : \begin{cases} \mathbf{W}^-_{k+1} \to \mathbf{W}^+_{k+2}(\mathring{\mathsf{A}}^*_{k+1})' \\ \quad \boldsymbol{\phi} \mapsto \langle \boldsymbol{\phi}, \mathring{\mathsf{A}}^*_{k+1} \cdot \rangle_{\mathbf{W}^-_{k+1}} \end{cases}, \tag{6.9}$$

still designated by the same notation.

Proof. For all $\boldsymbol{\phi} \in \mathbf{W}^-_{k+1}$, it follows by definition that $\forall \mathbf{z} \in \mathbf{W}^+_{k+2}(\mathring{\mathsf{A}}^*_{k+1})$,

$$|\langle \boldsymbol{\phi}, \mathring{\mathsf{A}}^*_{k+1}\mathbf{z}\rangle_{\mathbf{W}^-_{k+1}}| \le \|\boldsymbol{\phi}\|_{\mathbf{W}^-_{k+1}} \|\mathring{\mathsf{A}}^*_{k+1}\mathbf{z}\|_{\mathbf{W}^+_{k+1}} \le \|\boldsymbol{\phi}\|_{\mathbf{W}^-_{k+1}} \|\mathbf{z}\|_{\mathbf{W}^+_{k+2}(\mathring{\mathsf{A}}^*_{k+1})}. \quad \square \tag{6.10}$$

6.1.2. Dual decomposition

We may also adopt the adjoint perspective. It goes without saying that the development is completely symmetric to Section 6.1.1. We present it for completeness.

Assumption C. *(cf. Assumption B)* For all $k \in \mathbb{Z}$, beside Assumption A we stipulate the following:

I The Hilbert spaces $\mathbf{W}_k^+ \subset \mathbf{W}_k$ are such that the inclusion maps spawn continuous and dense embeddings

$$\mathbf{W}_k^+ \hookrightarrow \mathcal{D}(\mathsf{A}_k). \tag{6.11}$$

II There exist bounded operators

$$\mathsf{L}_{k+1}^n : \mathcal{D}(\mathsf{A}_{k+1}) \to \mathbf{W}_{k+1}^+ \qquad \text{and} \qquad \mathsf{V}_{k+1}^n : \mathcal{D}(\mathsf{A}_{k+1}) \to \mathbf{W}_k^+ \tag{6.12}$$

such that

$$\mathbf{y} = (\mathsf{L}_{k+1}^n + \mathsf{A}_k\, \mathsf{V}_{k+1}^n)\,\mathbf{y} \qquad \forall \mathbf{y} \in \mathcal{D}(\mathsf{A}_{k+1}). \tag{6.13}$$

III The Hilbert spaces

$$\mathbf{W}_k^+(\mathsf{A}_k) := \left\{ \mathbf{x} \in \mathbf{W}_k^+ \;\middle|\; \mathsf{A}_k\,\mathbf{x} \in \mathbf{W}_{k+1}^+ \right\}, \tag{6.14}$$

equipped with the graph inner product defined for all $\mathbf{x}_1, \mathbf{x}_2 \in \mathbf{W}_k^+(\mathsf{A}_k)$ by

$$(\mathbf{x}_1, \mathbf{x}_2)_{\mathbf{W}_k^+(\mathsf{A}_k)} := (\mathbf{x}_1, \mathbf{x}_2)_{\mathbf{W}_k^+} + (\mathsf{A}_k\,\mathbf{x}_1, \mathsf{A}_k\,\mathbf{x}_2)_{\mathbf{W}_{k+1}^+}, \tag{6.15}$$

are such that the inclusions $\mathbf{W}_k^+ \subset \mathbf{W}_k$ induce continuous and dense embeddings

$$\mathbf{W}_k^+(\mathsf{A}_k) \hookrightarrow \mathcal{D}(\mathsf{A}_k). \tag{6.16}$$

Lemma 6.5. *Under Assumption C, the surface operator D_{k+1}^n can be extended to a continuous mapping*

$$\mathsf{D}_k^n : \begin{cases} \mathbf{W}_{k+1}^- \to \mathbf{W}_k^+(\mathsf{A}_k)' \\ \quad \boldsymbol{\psi} \mapsto \langle \boldsymbol{\psi}, \mathsf{A}_k \,\cdot\, \rangle_{\mathbf{W}_{k+1}^-} \end{cases}. \tag{6.17}$$

Proof. Parallel to the proof of Lemma 6.4, it follows by definition that given $\boldsymbol{\psi} \in \mathbf{W}_{k+1}^-$,

$$|\langle \boldsymbol{\psi}, \mathsf{A}_k\,\mathbf{x}\rangle_{\mathbf{W}_{k+1}^-}| \le \|\boldsymbol{\psi}\|_{\mathbf{W}_{k+1}^-} \|\,\mathsf{A}_k\,\mathbf{x}\|_{\mathbf{W}_{k+1}^+} \le \|\boldsymbol{\psi}\|_{\mathbf{W}_{k+1}^-} \|\mathbf{x}\|_{\mathbf{W}_k^+(\mathsf{A}_k)} \qquad \forall \mathbf{x} \in \mathbf{W}_k^+(\mathsf{A}_k). \quad \square \tag{6.18}$$

It is not excluded that both Assumptions B and C hold, in which case the inclusion

$$\mathbf{W}_{k+1}^+ \hookrightarrow \mathcal{D}(\mathring{\mathsf{A}}_k^*) \cap \mathcal{D}(\mathsf{A}_{k+1}) \tag{6.19}$$

is assumed to be a dense embedding.

3D de Rham setting XII: Stable regular decompositions. There is some freedom in choosing the spaces $\mathbf{W}_k^+$, $k \in \mathbb{Z}$. For the de Rham complex though, there are obvious candidates satisfying (6.19) that also satisfy both Assumptions B and C: functions in the Sobolev space $H^1(\Omega)$ and vector-fields with components in $H^1(\Omega)$, which by Rellich's lemma [30, Thm. 3.27] are compactly embedded in the spaces $L^2(\Omega)$ and $\mathbf{L}^2(\Omega)$, respectively.

k	0	1	2	3
$\mathbf{W}_k$	$L^2(\Omega)$	$\mathbf{L}^2(\Omega)$	$\mathbf{L}^2(\Omega)$	$L^2(\Omega)$
$\mathbf{W}_k^+$	$H^1(\Omega)$	$\mathbf{H}^1(\Omega)$	$\mathbf{H}^1(\Omega)$	$H^1(\Omega)$
$\mathcal{D}(\mathsf{A}_k)$	$H^1(\Omega)$	$\mathbf{H}(\mathbf{curl},\Omega)$	$\mathbf{H}(\mathrm{div},\Omega)$	$L^2(\Omega)$
$\mathcal{D}(\mathring{\mathsf{A}}_k^*)$	$\mathbf{H}(\mathrm{div},\Omega)$	$\mathbf{H}(\mathbf{curl},\Omega)$	$H^1(\Omega)$	$\{0\}$

It is well-known (cf. [23, Sec. 2], [22, Lem. 2.4] and [24, Sec. 3]) that the graph spaces $\mathcal{D}(\mathsf{A}_k)$ and $\mathcal{D}(\mathring{\mathsf{A}}_k^*)$ given in the above table admit the stable decompositions

$$\mathcal{D}(\mathsf{A}_2) = \mathcal{D}(\mathring{\mathsf{A}}_0^*) = \mathbf{H}(\mathrm{div},\Omega) = \mathbf{H}^1(\Omega) + \mathbf{curl}\,\mathbf{H}^1(\Omega), \tag{6.20a}$$

$$\mathcal{D}(\mathsf{A}_1) = \mathcal{D}(\mathring{\mathsf{A}}_1^*) = \mathbf{H}(\mathbf{curl},\Omega) = \mathbf{H}^1(\Omega) + \mathbf{grad}\,H^1(\Omega) \tag{6.20b}$$

These satisfy Assumptions B and C. Moreover, you may recall that

$$\mathbf{H}^1(\Omega) \hookrightarrow \mathbf{H}(\mathbf{curl},\Omega) \cap \mathbf{H}(\mathrm{div},\Omega) \tag{6.21}$$

is a dense embedding [2, Prop. 2.3]. ⊿

6.2. Characterization of dual spaces

In light of Lemma 6.4, the Hilbert space

$$\mathbf{W}_{k+1}^-(\mathsf{D}_k^t) := \left\{ \boldsymbol{\phi} \in \mathbf{W}_{k+1}^- \mid \mathsf{D}_k^t \boldsymbol{\phi} \in \mathbf{W}_{k+2}^- \right\}, \tag{6.22}$$

equipped with the graph norm $\|\cdot\|^2_{\mathbf{W}_{k+1}^-(\mathsf{D}_k^t)} := \|\cdot\|^2_{\mathbf{W}_{k+1}^-} + \|\mathsf{D}_k^t\cdot\|^2_{\mathbf{W}_{k+2}^-}$, is well-defined under Assumption B. In this setting, observe that, if $\boldsymbol{\phi} \in \mathbf{W}_{k+1}^-(\mathsf{D}_k^t)$, then based on the decomposition (6.3), the evaluation

$$\boldsymbol{\phi}(\mathbf{y}) = \boldsymbol{\phi}(\mathsf{L}_{k+1}^t\mathbf{y}) + \boldsymbol{\phi}(\mathring{\mathsf{A}}_{k+1}^*\mathsf{V}_{k+1}^t\mathbf{y}) = \boldsymbol{\phi}(\mathsf{L}_{k+1}^t\mathbf{y}) + \mathsf{D}_k^t\boldsymbol{\phi}(\mathsf{V}_{k+1}^t\mathbf{y}) \tag{6.23}$$

is well-defined for all $\mathbf{y} \in \mathcal{D}(\mathring{\mathsf{A}}_k^*)$ thanks to the hypothesis that guarantees $\mathcal{R}(\mathsf{L}_{k+1}^t) \subset \mathbf{W}_{k+1}^+$ and $\mathcal{R}(\mathsf{V}_{k+1}^t) \subset \mathbf{W}_{k+2}^+$.

Theorem 6.6. *Assumption B guarantees the following isomorphism of normed vector spaces,*

$$\mathcal{D}(\mathring{\mathsf{A}}_k^*)' \cong \mathbf{W}_{k+1}^-(\mathsf{D}_k^t). \tag{6.24}$$

Proof. Due to (6.1) from Hypothesis **I** of Assumption B, the restriction of functionals $\mathcal{D}(\mathring{\mathsf{A}}^*_{k+1})' \hookrightarrow \mathbf{W}^-_{k+2}$ is a continuous embedding, so the inclusion $\mathcal{D}(\mathring{\mathsf{A}}^*_k)' \subset \mathbf{W}^-_{k+1}(\mathsf{D}^t_k)$ is immediate from Definition 5.1.

Moreover, for all $\boldsymbol{\phi} \in \mathbf{W}^-_{k+1}(\mathsf{D}^t_k)$, we estimate using (6.23) that

$$\begin{aligned} |\boldsymbol{\phi}(\mathbf{y})| &\leq \|\boldsymbol{\phi}\|_{\mathbf{W}^-_{k+1}} \|\mathsf{L}^t_{k+1}\mathbf{y}\|_{\mathbf{W}^+_{k+1}} + \|\mathsf{D}^t_k\boldsymbol{\phi}\|_{\mathbf{W}^-_{k+2}} \|\mathsf{V}^t_{k+1}\mathbf{y}\|_{\mathbf{W}^+_{k+2}} \\ &\leq C(\|\boldsymbol{\phi}\|_{\mathbf{W}^-_{k+1}} + \|\mathsf{D}^t_k\boldsymbol{\phi}\|_{\mathbf{W}^-_{k+2}})\|\mathbf{y}\|_{\mathcal{D}(\mathring{\mathsf{A}}^*_k)} \end{aligned} \tag{6.25}$$

for all $\mathbf{y} \in \mathcal{D}(\mathring{\mathsf{A}}^*_k)$, where $C > 0$ is a constant of continuity related to the boundedness of the potential and lifting operators in hypothesis **II** of Assumption B. We conclude that

$$\mathbf{W}^-_{k+1}(\mathsf{D}^t_k) \subset \mathcal{D}(\mathring{\mathsf{A}}^*_k)'. \tag{6.26}$$

Notice that it also follows from (6.25) that

$$\|\boldsymbol{\phi}\|_{\mathcal{D}(\mathring{\mathsf{A}}^*_k)'} = \sup_{0\neq\mathbf{y}\in\mathcal{D}(\mathring{\mathsf{A}}^*)} \frac{|\boldsymbol{\phi}(\mathbf{y})|}{\|\mathbf{y}\|_{\mathcal{D}(\mathring{\mathsf{A}}^*_k)}} \leq C(\|\boldsymbol{\phi}\|_{\mathbf{W}^-_{k+1}} + \|\mathsf{D}^t_k\boldsymbol{\phi}\|_{\mathbf{W}^-_{k+2}}) = C\|\boldsymbol{\phi}\|_{\mathbf{W}^-_{k+1}(\mathsf{D}^t_k)} \tag{6.27}$$

for all $\boldsymbol{\phi} \in \mathbf{W}^-_{k+1}(\mathsf{D}^t_k)$. In other words, the identity map is continuous as a mapping

$$\mathbf{W}^-_{k+1}(\mathsf{D}^t_k) \hookrightarrow \mathcal{D}(\mathring{\mathsf{A}}^*_k)'. \tag{6.28}$$

Appealing to the bounded inverse theorem verifies the equivalence of norms. □

Similarly, under Assumption C, Lemma 6.5 ensures that the Hilbert space

$$\mathbf{W}^-_{k+1}(\mathsf{D}^n_{k+1}) := \left\{ \boldsymbol{\psi} \in \mathbf{W}^-_{k+1} \mid \mathsf{D}^n_{k+1}\boldsymbol{\psi} \in \mathbf{W}^-_k \right\}, \tag{6.29}$$

equipped with the graph norm $\|\cdot\|_{\mathbf{W}^-_{k+1}(\mathsf{D}^n_{k+1})} := \|\cdot\|_{\mathbf{W}^-_{k+1}} + \|\mathsf{D}^n_{k+1}\cdot\|_{\mathbf{W}^-_k}$, is well-defined. We obtain the following analogous result.

Theorem 6.7 *(cf. Theorem 6.6). Under Assumption C, we conclude the isomorphism of normed vector spaces*

$$\mathcal{D}(\mathsf{A}_{k+1})' \cong \mathbf{W}^-_{k+1}(\mathsf{D}^n_{k+1}). \tag{6.30}$$

3D de Rham setting XIII: Characterization of dual spaces. Now, we specialize the theoretical results of Section 6.2 to the 3D de Rham setting using the table in Example XII. We obtain the following characterization of the dual spaces:

$$\mathbf{H}(\mathbf{curl},\Omega)' = \mathcal{D}(\mathsf{A}_1)' = \mathcal{D}(\mathring{\mathsf{A}}^*_1)' \cong \left\{ \boldsymbol{\phi} \in \widetilde{\mathbf{H}}^{-1}(\Omega) \mid \mathbf{grad}'\,\boldsymbol{\phi} \in \widetilde{H}^{-1}(\Omega) \right\}, \tag{6.31a}$$

$$\mathbf{H}(\mathrm{div},\Omega)' = \mathcal{D}(\mathsf{A}_2)' = \mathcal{D}(\mathring{\mathsf{A}}^*_0)' \cong \left\{ \boldsymbol{\phi} \in \widetilde{\mathbf{H}}^{-1}(\Omega) \mid \mathbf{curl}'\,\boldsymbol{\phi} \in \widetilde{\mathbf{H}}^{-1}(\Omega) \right\}. \tag{6.31b}$$

Note that these characterizations are interesting in their own right. They do not depend on the theory of traces developed in the previous sections. The take-home message from the de Rham settings XII and XIII is that via the decompositions (6.20a) and (6.20b), the dual spaces of $\mathbf{H}(\mathbf{curl}, \Omega)$ and $\mathbf{H}(\mathrm{div}, \Omega)$ can be characterized using more regular spaces such as $H^1(\Omega)$ and $\mathbf{H}^1(\Omega)$. ⊿

6.3. Characterization of trace spaces

We have almost reached characterizations of the ranges of the Hilbert traces $\mathcal{R}(\mathsf{T}_k^t)$ and $\mathcal{R}(\mathsf{T}_k^n)$ in terms of the spaces of "extra regularity" provided by Assumptions B and C. To achieve these new characterizations, we introduce the following spaces for all $k \in \mathbb{Z}$:

$$\mathring{\mathbf{W}}_k^{n,+} := \mathbf{W}_k^+ \cap \mathcal{D}(\mathsf{A}_{k-1}^*), \qquad \text{and} \qquad \mathring{\mathbf{W}}_k^{t,+} := \mathbf{W}_k^+ \cap \mathcal{D}(\mathring{\mathsf{A}}_k). \tag{6.32}$$

Notice that by Propositions 4.2 and 3.3, we have

$$\mathring{\mathbf{W}}_k^{n,+} = \mathbf{W}_k^+ \cap \mathcal{N}(\mathsf{T}_{k-1}^n), \qquad \text{and} \qquad \mathring{\mathbf{W}}_k^{t,+} = \mathbf{W}_k^+ \cap \mathcal{N}(\mathsf{T}_k^t), \tag{6.33}$$

respectively.

Assumption D. Suppose that Assumption B holds. For all $k \in \mathbb{Z}$, we make the hypothesis that the inclusion map $\mathbf{W}_k^+ \subset \mathcal{D}(\mathring{\mathsf{A}}_{k-1}^*)$ spawns a *continuous* and *dense* embedding

$$\mathring{\mathbf{W}}_k^{n,+} \hookrightarrow \mathcal{D}(\mathsf{A}_{k-1}^*). \tag{6.34}$$

The next result involves the annihilator

$$(\mathring{\mathbf{W}}_{k+1}^{n,+})^\circ := \left\{ \boldsymbol{\phi} \in \mathbf{W}_{k+1}^- \mid \langle \boldsymbol{\phi}, \mathbf{y} \rangle_{\mathbf{W}_{k+1}^-} = 0, \, \forall \mathbf{y} \in \mathring{\mathbf{W}}_{k+1}^{n,+} \right\}. \tag{6.35}$$

Theorem 6.8. *Taking for granted Assumption D we obtain the characterization*

$$\mathcal{R}(\mathsf{T}_k^t) = \mathbf{W}_{k+1}^-(\mathsf{D}_k^t) \cap (\mathring{\mathbf{W}}_{k+1}^{n,+})^\circ = \left\{ \boldsymbol{\psi} \in (\mathring{\mathbf{W}}_{k+1}^{n,+})^\circ \mid \mathsf{D}_k^t \boldsymbol{\psi} \in (\mathring{\mathbf{W}}_{k+2}^{n,+})^\circ \right\}, \tag{6.36}$$

in the sense of equality of functionals in $\mathbf{W}_{k+1}^-$ and with equivalent norms.

Proof. We already know by Proposition 3.5 that $\mathcal{R}(\mathsf{T}_k^t) = \mathcal{D}(\mathsf{A}_k^*)^\circ$. To verify the equality on the right, recall that $\mathsf{D}_k^t(\mathcal{R}(\mathsf{T}_k^t)) \subset \mathcal{R}(\mathsf{T}_{k+1}^t) = \mathcal{D}(\mathsf{A}_{k+1}^*)^\circ$.

"$\subset$": On the one hand, since $\mathcal{D}(\mathsf{A}_k^*)^\circ \subset \mathcal{D}(\mathring{\mathsf{A}}_k^*)'$, it follows immediately from Theorem 6.6 and (6.34) that $\mathcal{R}(\mathsf{T}_k^t) \subset \mathbf{W}_{k+1}^-(\mathsf{D}_k^t)$. Moreover, as $\mathring{\mathbf{W}}_{k+1}^{n,+} \subset \mathcal{D}(\mathsf{A}_k^*)$, any functional in the annihilator of $\mathcal{D}(\mathsf{A}_k^*)$ will, in particular, vanish on $\mathring{\mathbf{W}}_{k+1}^{n,+}$, which implies $\mathcal{D}(\mathsf{A}_k^*)^\circ \subset (\mathring{\mathbf{W}}_{k+1}^{n,+})^\circ$.

Thanks to the continuous embedding of Assumption B*I* and (5.5a) from the definition of the operator D_k^t, we find for every $\boldsymbol{\varphi} \in \mathcal{D}(\mathring{\mathsf{A}}_k^*)'$:

$$\begin{aligned}
\|\boldsymbol{\varphi}\|_{\mathbf{W}_{k+1}^-} + \|\mathsf{D}_k^t\boldsymbol{\varphi}\|_{\mathbf{W}_{k+2}^-} &= \sup_{\mathbf{w}\in\mathbf{W}_{k+1}^+} \frac{|\boldsymbol{\varphi}(\mathbf{w})|}{\|\mathbf{w}\|_{\mathbf{W}_{k+1}^+}} + \sup_{\mathbf{w}\in\mathbf{W}_{k+2}^+} \frac{|\boldsymbol{\varphi}(\mathring{\mathsf{A}}_{k+1}^*\mathbf{w})|}{\|\mathbf{w}\|_{\mathbf{W}_{k+2}^+}} \\
&\leq C \sup_{\mathbf{w}\in\mathcal{D}(\mathring{\mathsf{A}}_k^*)} \frac{|\boldsymbol{\varphi}(\mathbf{w})|}{\|\mathbf{w}\|_{\mathcal{D}(\mathring{\mathsf{A}}_k^*)}} + \sup_{\mathbf{w}\in\mathcal{D}(\mathring{\mathsf{A}}_{k+1}^*)} \frac{|\boldsymbol{\varphi}(\mathring{\mathsf{A}}_{k+1}^*\mathbf{w})|}{\|\mathbf{w}\|_{\mathcal{D}(\mathring{\mathsf{A}}_{k+1}^*)}} \\
&\leq 2C\|\boldsymbol{\varphi}\|_{\mathcal{D}(\mathring{\mathsf{A}}_k^*)'},
\end{aligned}$$

for some constant $C > 0$ independent of $\boldsymbol{\varphi}$.

"$\supset$": On the other hand, it also follows by Theorem 6.6 that any $\boldsymbol{\phi} \in \mathbf{W}_{k+1}^-(\mathsf{D}_k^t) \cap (\mathring{\mathbf{W}}_{k+1}^{n,+})^\circ$ is a continuous functional in $\mathcal{D}(\mathring{\mathsf{A}}_k^*)'$ vanishing on $\mathring{\mathbf{W}}_{k+1}^{n,+}$. By Assumption D $\mathring{\mathbf{W}}_{k+1}^{n,+}$ is densely embedded in $\mathcal{D}(\mathsf{A}_k^*)$. Thus, $\boldsymbol{\phi}$ must also vanish on $\mathcal{D}(\mathsf{A}_k^*)$ by continuity. We conclude that the inclusion $\mathbf{W}_{k+1}^-(\mathsf{D}_k^t) \cap (\mathring{\mathbf{W}}_{k+1}^{n,+})^\circ \subset \mathcal{R}(\mathsf{T}_k^t) = \mathcal{D}(\mathsf{A}_k^*)^\circ$ holds.

Finally, the estimate (6.25) gives us

$$\|\boldsymbol{\phi}\|_{\mathcal{D}(\mathring{\mathsf{A}}_k^*)'} \leq C(\|\boldsymbol{\phi}\|_{\mathbf{W}_{k+1}^-} + \|\mathsf{D}_k^t\boldsymbol{\phi}\|_{\mathbf{W}_{k+2}^-})$$

with $C > 0$ independent of $\boldsymbol{\phi}$. $\square$

Of course, there is a symmetric statement on the dual side.

Assumption E. *(cf. Assumption D)* Suppose that Assumption C holds. For all $k \in \mathbb{Z}$, we make the hypothesis that the inclusion map $\mathbf{W}_k^+ \subset \mathcal{D}(\mathsf{A}_k)$ spawns a continuous and dense embedding

$$\mathring{\mathbf{W}}_k^{t,+} \hookrightarrow \mathcal{D}(\mathring{\mathsf{A}}_k). \tag{6.37}$$

Theorem 6.9 *(cf. Theorem 6.8). Under Assumption E we have equality in $\mathbf{W}_{k+1}^-$ with equivalent norms,*

$$\mathcal{R}(\mathsf{T}_{k+1}^n) = \mathbf{W}_{k+1}^-(\mathsf{D}_{k+1}^n) \cap (\mathring{\mathbf{W}}_{k+1}^{t,+})^\circ = \left\{ \boldsymbol{\psi} \in (\mathring{\mathbf{W}}_{k+1}^{t,+})^\circ \mid \mathsf{D}_{k+1}^n\boldsymbol{\psi} \in (\mathring{\mathbf{W}}_k^{t,+})^\circ \right\}, \tag{6.38}$$

where $(\mathring{\mathbf{W}}_{k+1}^{t,+})^\circ := \left\{ \boldsymbol{\phi} \in \mathbf{W}_{k+1}^- \mid \langle \boldsymbol{\phi}, \mathbf{y}\rangle_{\mathbf{W}_{k+1}^-} = 0, \forall \mathbf{y} \in \mathring{\mathbf{W}}_{k+1}^{t,+} \right\}$ *is defined analogously to (6.35).*

3D de Rham setting XIV: Characterization of trace spaces. We specialize the theoretical results of Section 6.3 to the 3D de Rham setting.

k	0	1	2	3
$\mathbf{W}_k$	$L^2(\Omega)$	$\mathbf{L}^2(\Omega)$	$\mathbf{L}^2(\Omega)$	$L^2(\Omega)$
$\mathbf{W}_k^+$	$H^1(\Omega)$	$\mathbf{H}^1(\Omega)$	$\mathbf{H}^1(\Omega)$	$H^1(\Omega)$
$\mathring{\mathbf{W}}_k^{t,+}$	$\mathring{H}^1(\Omega)$	$\mathbf{H}^1(\Omega) \cap \mathring{\mathbf{H}}(\mathbf{curl}, \Omega)$	$\mathbf{H}^1(\Omega) \cap \mathring{\mathbf{H}}(\mathrm{div}, \Omega)$	$\mathring{H}^1(\Omega)$
$\mathring{\mathbf{W}}_k^{n,+}$	$\mathring{H}^1(\Omega)$	$\mathbf{H}^1(\Omega) \cap \mathring{\mathbf{H}}(\mathrm{div}, \Omega)$	$\mathbf{H}^1(\Omega) \cap \mathring{\mathbf{H}}(\mathbf{curl}, \Omega)$	$\mathring{H}^1(\Omega)$

Loosely speaking, Theorems 6.8 and 6.9 state that the range of the Hilbert trace is a subspace of functionals in the dual of a regular space $\mathbf{W}_k^+$ whose image under the corresponding surface operator also lies in the dual of $\mathbf{W}_{k+1}^+$. Linear functionals in that subspace vanish on a dense subset of the dual trace's kernel:

$$\mathcal{R}(\mathsf{T}^t_{\mathbf{curl}}) = \mathcal{R}(\mathsf{T}^n_{\mathbf{curl}}) = \left\{ \boldsymbol{\phi} \in \widetilde{\mathbf{H}}^{-1}(\Omega) \cap \mathring{\mathbf{H}}(\mathbf{curl}, \Omega)^\circ \mid \mathbf{grad}'\, \phi \in \widetilde{H}^{-1}(\Omega) \cap \mathring{H}^1(\Omega)^\circ \right\}, \tag{6.39a}$$

$$\mathcal{R}(\mathsf{T}^t_{\mathbf{grad}}) = \mathcal{R}(\mathsf{T}^n_{\mathrm{div}}) \ = \left\{ \boldsymbol{\phi} \in \widetilde{\mathbf{H}}^{-1}(\Omega) \cap \mathring{\mathbf{H}}(\mathrm{div}, \Omega)^\circ \mid \mathbf{curl}'\, \boldsymbol{\phi} \in \widetilde{\mathbf{H}}^{-1}(\Omega) \cap \mathring{\mathbf{H}}(\mathbf{curl}, \Omega)^\circ \right\}. \tag{6.39b}$$

One thing immediately apparent is that $\mathcal{R}(\mathsf{T}^n_{\mathbf{curl}}) = \mathcal{R}(\mathsf{T}^t_{\mathbf{curl}})$ and $\mathcal{R}(\mathsf{T}^n_{\mathrm{div}}) = \mathcal{R}(\mathsf{T}^t_{\mathbf{grad}})$, which is expected because we already know from previous sections that

$$\mathcal{R}(\mathsf{T}^n_{\mathbf{curl}}) = \mathcal{D}(\mathring{\mathsf{A}}_1)^\circ = \mathring{\mathbf{H}}(\mathbf{curl}, \Omega)' = \mathcal{D}(\mathsf{A}_1^*)^\circ = \mathcal{R}(\mathsf{T}^t_{\mathbf{curl}}), \tag{6.40a}$$

$$\mathcal{R}(\mathsf{T}^n_{\mathrm{div}}) = \mathcal{D}(\mathring{\mathsf{A}}_2)^\circ = \mathring{\mathbf{H}}(\mathrm{div}, \Omega)' = \mathcal{D}(\mathsf{A}_1^*) = \mathcal{R}(\mathsf{T}^t_{\mathbf{grad}}). \tag{6.40b}$$

Before we compare these characterizations with (1.7a) and (1.7b), we want to reformulate them in terms of quotient spaces in the next section. ⊿

6.4. Characterization of trace spaces in quotient spaces

We can reformulate the characterizations of Section 6.3 in terms of quotient spaces. To proceed, let us set

$$\mathbf{T}_k^{t,+} := \mathbf{W}_k^+ / \mathring{\mathbf{W}}_k^{t,+}, \qquad \mathbf{T}_k^{t,-} := \left(\mathbf{T}_k^{t,+}\right)', \tag{6.41a}$$

$$\mathbf{T}_k^{n,+} := \mathbf{W}_k^+ / \mathring{\mathbf{W}}_k^{n,+}, \qquad \mathbf{T}_k^{n,-} := \left(\mathbf{T}_k^{n,+}\right)'. \tag{6.41b}$$

Under Assumption D (resp. E), it follows by definition of the space $\mathring{\mathbf{W}}_k^{n,+}$ (resp. $\mathring{\mathbf{W}}_k^{t,+}$) that the dense embedding $\mathbf{W}_k^+ \hookrightarrow \mathcal{D}(\mathring{\mathsf{A}}_{k-1}^*)$ (resp. $\mathbf{W}_k^+ \hookrightarrow \mathcal{D}(\mathsf{A}_k)$) induces a well-defined and dense embedding

$$\begin{cases} \mathbf{T}_k^{n,+} \hookrightarrow \mathcal{T}(\mathring{\mathsf{A}}_{k-1}^*) \\ \quad [\mathbf{x}] \mapsto \boldsymbol{\pi}_{k-1}^n \mathbf{x} \end{cases} \quad \left(\text{resp.} \quad \begin{cases} \mathbf{T}_k^{t,+} \hookrightarrow \mathcal{T}(\mathsf{A}_k) \\ \quad [\mathbf{x}] \mapsto \boldsymbol{\pi}_k^t \mathbf{x} \end{cases} \right) \tag{6.42}$$

on the quotient spaces. Accordingly, the associated restriction of functionals

$$\begin{cases} \mathcal{T}(\mathring{\mathsf{A}}_{k-1}^*)' \hookrightarrow \mathbf{T}_k^{n,-} \\ \qquad \boldsymbol{\psi} \mapsto \{ [\mathbf{x}] \mapsto \boldsymbol{\psi}(\boldsymbol{\pi}_{k-1}^n \mathbf{x}) \} \end{cases} \quad \left(\text{resp.} \quad \begin{cases} \mathcal{T}(\mathsf{A}_k)' \hookrightarrow \mathbf{T}_k^{t,-} \\ \qquad \boldsymbol{\phi} \mapsto \{ [\mathbf{x}] \mapsto \boldsymbol{\phi}(\boldsymbol{\pi}_k^t \mathbf{x}) \} \end{cases} \right) \tag{6.43}$$

is also well-defined and gives rise to dense embeddings.

In the next lemma, we make explicit the mappings induced on the quotient spaces by restricting the operators $\mathring{\mathsf{A}}^*_{k-1}$ and A_k to $\mathbf{W}^+_k$. Those are the restrictions of the surface operators S^n_{k-1} and S^t_k to $\mathbf{T}^{n,+}_k$ and $\mathbf{T}^{t,+}_k$, respectively; cf. Definition 5.6.

Lemma 6.10. *Assumptions D and E imply that the mappings*

$$\hat{\mathsf{S}}^n_{k+1} : \begin{cases} \mathbf{T}^{n,+}_{k+2} \to \mathcal{T}(\mathring{\mathsf{A}}^*_k) \\ \quad [\mathbf{z}] \mapsto \boldsymbol{\pi}^n_k \mathring{\mathsf{A}}^*_{k+1}\mathbf{z} \end{cases} \qquad \textit{and} \qquad \hat{\mathsf{S}}^t_k : \begin{cases} \mathbf{T}^{t,+}_k \to \mathcal{T}(\mathsf{A}_{k+1}) \\ \quad [\mathbf{x}] \mapsto \boldsymbol{\pi}^t_{k+1}\,\mathsf{A}_k\,\mathbf{x} \end{cases}, \tag{6.44}$$

respectively, are well-defined and continuous.

Proof. Consider the mapping on the left. We know from the complex property for $\mathring{\mathsf{A}}^*_k$ in Assumption A that $\mathring{\mathsf{A}}^*_{k+1}\mathbf{z} \in \mathcal{D}(\mathring{\mathsf{A}}^*_k)$ for all $\mathbf{z} \in \mathbf{W}^+_{k+1}$. We only need to verify that $\mathring{\mathsf{A}}^*_{k+1}\mathbf{z}_\circ \in \mathcal{D}(\mathsf{A}^*_k)$ for all $\mathbf{z}_\circ \in \mathring{\mathbf{W}}^{n,+}_{k+2} = \mathbf{W}^+_{k+2} \cap \mathcal{D}(\mathsf{A}^*_{k+1})$, but this immediately follows from the complex property for A^*_{k+1}, also provided by Assumption A. The proof is similar for $\hat{\mathsf{S}}^t_k$. □

Using the same strategy as in Lemmas 6.4 and 6.5, the mappings

$$\hat{\mathsf{D}}^t_k := (\hat{\mathsf{S}}^n_{k+1})' : \mathcal{T}(\mathring{\mathsf{A}}^*_k)' \to \mathbf{T}^{n,-}_{k+2} \qquad \text{and} \qquad \hat{\mathsf{D}}^n_k := (\hat{\mathsf{S}}^t_k)' : \mathcal{T}(\mathsf{A}_{k+1})' \to \mathbf{T}^{t,-}_k, \tag{6.45}$$

defined as the bounded operators dual to $\hat{\mathsf{S}}^n_{k+1}$ and $\hat{\mathsf{S}}^t_k$, can be extended, using (6.43), to the continuous mappings

$$\hat{\mathsf{D}}^t_k : \mathbf{T}^{n,-}_{k+1} \to \mathbf{T}^{n,+}_{k+2}(\hat{\mathsf{S}}^n_{k+1})' \qquad \text{and} \qquad \hat{\mathsf{D}}^n_k : \mathbf{T}^{t,-}_{k+1} \to \mathbf{T}^{t,+}_k(\hat{\mathsf{S}}^t_k)', \tag{6.46}$$

involving the dual spaces of the Hilbert spaces

$$\mathbf{T}^{n,+}_{k+2}(\hat{\mathsf{S}}^n_{k+1}) := \left\{ [\mathbf{z}] \in \mathbf{T}^{n,+}_{k+2} \mid \hat{\mathsf{S}}^n_{k+1}[\mathbf{z}] \in \mathbf{T}^{n,+}_{k+1} \right\}, \tag{6.47a}$$

$$\mathbf{T}^{t,+}_k(\hat{\mathsf{S}}^t_k) := \left\{ [\mathbf{x}] \in \mathbf{T}^{t,+}_k \mid \hat{\mathsf{S}}^t_k[\mathbf{x}] \in \mathbf{T}^{t,+}_{k+1} \right\}, \tag{6.47b}$$

equipped with the natural graph inner products.

With the operators (6.46), we can reformulate Theorems 6.8 and 6.9 using the isometric isomorphisms

$$(\mathbf{W}^+_k/\mathring{\mathbf{W}}^{t,+}_k)' \cong (\mathring{\mathbf{W}}^{t,+}_k)^\circ \qquad \text{and} \qquad \mathbf{W}^+_k/\mathring{\mathbf{W}}^{n,+}_k \cong (\mathring{\mathbf{W}}^{n,+}_k)^\circ \tag{6.48}$$

provided by [43, Thm. 4.9].

3D de Rham setting XV: Characterization of trace spaces as quotient spaces. Recall from (1.8b) and (1.8c) that $\mathcal{N}(\gamma_t) = \mathring{\mathbf{H}}(\mathbf{curl}, \Omega)$ and $\mathcal{N}(\gamma_n) = \mathring{\mathbf{H}}(\mathrm{div}, \Omega)$. So let us denote the spaces of $\mathbf{H}^1$-regular vector fields with vanishing tangential and normal traces by

Theorem 6.11. *Under Assumptions D and E we have the isomorphisms of Hilbert spaces*

$$\mathcal{R}(\mathsf{T}^t_k) \cong \left\{ \boldsymbol{\phi} \in \mathbf{T}^{n,-}_{k+1} \mid \hat{\mathsf{D}}^t_k \boldsymbol{\phi} \in \mathbf{T}^{n,-}_{k+2} \right\} \quad \textit{and} \quad \mathcal{R}(\mathsf{T}^n_k) \cong \left\{ \boldsymbol{\phi} \in \mathbf{T}^{t,-}_{k} \mid \hat{\mathsf{D}}^n_k \boldsymbol{\phi} \in \mathbf{T}^{t,-}_{k-1} \right\}, \tag{6.49a}$$

respectively.

$$\mathbf{H}^1_t(\Omega) := \mathcal{N}(\gamma_t|_{\mathbf{H}^1(\Omega)}) = \mathbf{H}^1(\Omega) \cap \mathring{\mathbf{H}}(\mathbf{curl}, \Omega) \tag{6.50a}$$

$$\mathbf{H}^1_n(\Omega) := \mathcal{N}(\gamma_n|_{\mathbf{H}^1(\Omega)}) = \mathbf{H}^1(\Omega) \cap \mathring{\mathbf{H}}(\mathrm{div}, \Omega), \tag{6.50b}$$

respectively.

k	0	1	2	3
$\mathbf{W}_k$	$L^2(\Omega)$	$\mathbf{L}^2(\Omega)$	$\mathbf{L}^2(\Omega)$	$L^2(\Omega)$
$\mathbf{W}^+_k$	$H^1(\Omega)$	$\mathbf{H}^1(\Omega)$	$\mathbf{H}^1(\Omega)$	$H^1(\Omega)$
$\mathbf{T}^{t,+}_k$	$H^1(\Omega)/\mathring{H}^1(\Omega)$	$\mathbf{H}^1(\Omega)/\mathbf{H}^1_t(\Omega)$	$\mathbf{H}^1(\Omega)/\mathbf{H}^1_n(\Omega)$	$H^1(\Omega)/\mathring{H}^1(\Omega)$
$\mathbf{T}^{n,+}_k$	$H^1(\Omega)/\mathring{H}^1(\Omega)$	$\mathbf{H}^1(\Omega)/\mathbf{H}^1_n(\Omega)$	$\mathbf{H}^1(\Omega)/\mathbf{H}^1_t(\Omega)$	$H^1(\Omega)/\mathring{H}^1(\Omega)$

Reformulating (6.39a) and (6.39b), we obtain

$$\mathcal{R}(\mathsf{T}^t_{\mathbf{curl}}) = \mathcal{R}(\mathsf{T}^n_{\mathbf{curl}}) \cong \left\{ \boldsymbol{\phi} \in \left(\mathbf{H}^1(\Omega)/\mathbf{H}^1_t(\Omega)\right)' \mid \mathbf{grad}' \boldsymbol{\phi} \in \left(H^1(\Omega)/\mathring{H}^1(\Omega)\right)' \right\}, \tag{6.51a}$$

$$\mathcal{R}(\mathsf{T}^t_{\mathbf{grad}}) = \mathcal{R}(\mathsf{T}^n_{\mathrm{div}}) \cong \left\{ \boldsymbol{\phi} \in \left(\mathbf{H}^1(\Omega)/\mathbf{H}^1_n(\Omega)\right)' \mid \mathbf{curl}' \boldsymbol{\phi} \in \left(\mathbf{H}^1(\Omega)/\mathbf{H}^1_t(\Omega)\right)' \right\}. \tag{6.51b}$$

These characterizations are to be compared with

$$\mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma) = \left\{ \boldsymbol{\phi} \in \mathbf{H}^{-1/2}_t(\Gamma) \mid \mathrm{curl}_\Gamma \boldsymbol{\phi} \in H^{-1/2}(\Gamma) \right\} = \mathcal{R}(\gamma_t), \tag{6.52a}$$

$$H^{1/2}(\Gamma) = \left\{ \phi \in H^{-1/2}(\Gamma) \mid \mathbf{curl}_\Gamma \phi \in \mathbf{H}^{-1/2}_t(\Gamma) \right\} = \mathcal{R}(\gamma), \tag{6.52b}$$

where as before the two spaces

$$H^{-1/2}(\Gamma) = \left(H^{1/2}(\Gamma)\right)' = (\gamma H^1(\Omega))' \tag{6.53a}$$

$$\mathbf{H}^{-1/2}_t(\Gamma) = \left(\mathbf{H}^{1/2}_t(\Gamma)\right)' = (\gamma_t \mathbf{H}^1(\Omega))' \tag{6.53b}$$

are dual to the more regular spaces $\gamma\, H^1(\Omega)$ and $\gamma_t\, \mathbf{H}^1(\Omega)$, respectively.

In the classical trace spaces, the quotient spaces involved in (6.51a) and (6.51b) are featured implicitly, because as previously stated in (6.50a) and (6.50b), $\mathbf{H}^1_t(\Omega)$ and $\mathbf{H}^1_n(\Omega)$ are kernels which vanish under application of the traces. In fact, since $\gamma : H^1(\Omega) \to H^{1/2}(\Gamma)$ and $\gamma_t : \mathbf{H}^1(\Omega) \to \mathbf{H}^{1/2}_t(\Gamma)$ are surjective, it follows from (6.50a) and (6.50b) that the same argument as in the 3D de Rham setting VII shows that the traces induce the isomorphisms

$$\mathbf{H}^{1/2}_t(\Gamma) \cong \mathbf{H}^1(\Omega)/\mathbf{H}^1_t(\Omega) \qquad \text{and} \qquad H^{1/2}(\Gamma) \cong H^1(\Omega)/\mathring{H}^1(\Omega), \tag{6.54}$$

which in turn imply isomorphisms between the dual spaces.

We would like to draw the reader's attention to the fact that it is an annihilator related to the kernel of the dual trace that is used to characterize the range of the primal trace and vice-versa. This is in agreement with the characterizations provided in [14], where the range of γ_t is characterized using the dual space $(\gamma_\tau \mathbf{H}^1(\Omega))'$, involving the rotated tangential trace γ_τ discussed in the 3D de Rham setting XI. As in [14], recall that if the skew-symmetric pairing (1.10) is replaced with the $\mathbf{L}^2(\Gamma)$-pairing, the dual trace $\mathsf{T}^n_{\mathbf{curl}}$, corresponding with the rotated tangential trace (roughly speaking), arises in the abstract setting of Section 4.1 as dual to $\mathsf{T}^t_{\mathbf{curl}}$, which corresponds to γ_t.

Finally, notice that the surface operators curl_Γ and $\mathbf{curl}_\Gamma$ are dual to the domain operators on which the relevant traces are applied, which is in line with (6.51a) and (6.51b), i.e. (cf. [14])

$$\mathrm{curl}_\Gamma \circ \gamma = (\gamma_t \circ \nabla)' \qquad \text{and} \qquad \mathbf{curl}_\Gamma \circ \gamma_t = (\gamma_n \circ \mathbf{curl})'. \tag{6.55}$$

◿

7. Trace Hilbert complexes

From now on, we make use of the full setting of Hilbert complexes as presented in Section 2.2. Both Assumptions D and E are not required for the mere characterization of the trace Hilbert complexes in Section 7.1: each one of these hypotheses suffices for the corresponding characterization. However, we do rely on *both* decompositions for the upcoming compactness result in Section 7.2, where we must take (6.19) for granted.

7.1. Complexes of quotient spaces

It is easy to verify that $\mathsf{D}^t_{k+1} \circ \mathsf{D}^t_k = 0$, $\mathsf{D}^n_k \circ \mathsf{D}^n_{k+1} = 0$, $\mathsf{S}^t_{k+1} \circ \mathsf{S}^t_k = 0$ and $\mathsf{S}^n_k \circ \mathsf{S}^n_{k+1} = 0$. Therefore, we have already seen from (5.31) that Hilbert complexes give rise to Hilbert complexes in trace spaces. The bounded complexes

$$\cdots \xrightarrow{\mathsf{D}^t_{k-1}} \mathcal{R}(\mathsf{T}^t_k) \xrightarrow{\mathsf{D}^t_k} \mathcal{R}(\mathsf{T}^t_{k+1}) \xrightarrow{\mathsf{D}^t_{k+1}} \mathcal{R}(\mathsf{T}^t_{k+2}) \xrightarrow{\mathsf{D}_{k+2}} \cdots, \tag{7.1a}$$

and

$$\cdots \xleftarrow[\mathsf{D}^n_k]{} \mathcal{R}(\mathsf{T}^n_k) \xleftarrow[\mathsf{D}^n_{k+1}]{} \mathcal{R}(\mathsf{T}^n_{k+1}) \xleftarrow[\mathsf{D}^n_{k+2}]{} \mathcal{R}(\mathsf{T}^n_{k+2}) \xleftarrow[\mathsf{D}_{k+3}]{} \cdots, \tag{7.1b}$$

are isometrically isomorphic to the bounded complexes of quotient spaces

$$\cdots \xrightarrow{\mathsf{S}^t_k} \mathcal{T}(\mathsf{A}_k) \xrightarrow{\mathsf{S}^t_k} \mathcal{T}(\mathsf{A}_{k+1}) \xrightarrow{\mathsf{S}^t_{k+1}} \mathcal{T}(\mathsf{A}_{k+2}) \xrightarrow{\mathsf{S}^t_{k+2}} \cdots, \tag{7.2a}$$

and

$$\cdots \xleftarrow[\mathsf{S}^n_k]{} \mathcal{T}(\mathring{\mathsf{A}}^*_k) \xleftarrow[\mathsf{S}^n_{k+1}]{} \mathcal{T}(\mathring{\mathsf{A}}^*_{k+1}) \xleftarrow[\mathsf{S}^n_{k+2}]{} \mathcal{T}(\mathring{\mathsf{A}}^*_{k+2}) \xleftarrow[\mathsf{S}^n_{k+3}]{} \cdots. \tag{7.2b}$$

While the bounded domain complexes are interesting in their own right, the rich structure of Hilbert complexes reveals itself when closed densely defined unbounded operators are introduced. As stated in [6, Chap. 4], the complex produced by the latter contains more information than the associated domain complexes. It turns out that the characterizations provided in Section 6 shed more light on the structure of (7.1a)-(7.2b). The next theorem provides a first characterization of what we call *trace Hilbert complexes*.

Theorem 7.1. *Under Assumptions D and E respectively, the sequences of unbounded operators*

$$\cdots \xrightarrow{\mathsf{D}^t_{k-1}} \mathcal{R}(\mathsf{T}^t_k) \subset (\mathring{\mathbf{W}}^{n,+}_{k+1})^\circ \xrightarrow{\mathsf{D}^t_k} \mathcal{R}(\mathsf{T}^t_{k+1}) \subset (\mathring{\mathbf{W}}^{n,+}_{k+2})^\circ \xrightarrow{\mathsf{D}^t_{k+2}} \cdots \tag{7.3}$$

and

$$\cdots \xleftarrow[\mathsf{D}^n_k]{} \mathcal{R}(\mathsf{T}^n_k) \subset (\mathring{\mathbf{W}}^{t,+}_{k})^\circ \xleftarrow[\mathsf{D}^n_{k+1}]{} \mathcal{R}(\mathsf{T}^n_{k+1}) \subset (\mathring{\mathbf{W}}^{t,+}_{k+1})^\circ \xleftarrow[\mathsf{D}^n_{k+2}]{} \cdots \tag{7.4}$$

are Hilbert complexes as defined in Section 2.2.

Proof. By symmetry, it is sufficient to verify the claim for (7.3). In light for (7.1a) and Theorem 6.8, we simply need to show that $\mathsf{D}^t_k : \mathcal{R}(\mathsf{T}^t_k) \subset (\mathring{\mathbf{W}}^{n,+}_{k+1})^\circ \to (\mathring{\mathbf{W}}^{n,+}_{k+2})^\circ$ is a densely defined and closed unbounded linear operator.

To begin with, from Proposition 3.5 we know that $\mathcal{D}(\mathsf{D}^t_k) = \mathcal{R}(\mathsf{T}^t_k) = \mathcal{D}(\mathsf{A}^*_k)^\circ \subset \mathcal{D}(\mathring{\mathsf{A}}^*_k)'$ is a Hilbert space. Next, Theorem 6.8 tells us that, indeed, it is a Hilbert space as a subspace of $(\mathring{\mathbf{W}}^{n,+}_{k+1})^\circ$. This implies that the operator D^t_k must be closed on $(\mathring{\mathbf{W}}^{n,+}_{k+1})^\circ$.

It remains to confirm that $\mathcal{R}(\mathsf{T}^t_k)$ is dense in $(\mathring{\mathbf{W}}^{n,+}_{k+1})^\circ$. To that end we employ two key mappings:

(I) Recall that since $\mathbf{W}^+_{k+1}$ is a Hilbert space and Hilbert spaces are reflexive (cf. [43, Sec. 4.5], [10, Thm. 5.5]), the map

$$\boldsymbol{\rho} : \begin{cases} \mathbf{W}^+_{k+1} \longrightarrow (\mathbf{W}^-_{k+1})' \\ \quad \mathbf{y} \;\mapsto \begin{cases} \mathbf{W}^-_{k+1} \to \mathbb{R} \\ \quad \boldsymbol{\phi} \mapsto \boldsymbol{\rho}\mathbf{y}(\boldsymbol{\phi}) = \boldsymbol{\phi}(\mathbf{y}) \end{cases} \end{cases} \tag{7.5}$$

is an isometric isomorphism. Substituting $\boldsymbol{\rho}^{-1}(\widetilde{\boldsymbol{\phi}})$ for $\mathbf{y}$ in the definition $(\boldsymbol{\rho}\mathbf{y})(\boldsymbol{\phi}) = \boldsymbol{\phi}(\mathbf{y})$, we find a useful formula involving the inverse:

$$\widetilde{\boldsymbol{\psi}}(\boldsymbol{\phi}) = \boldsymbol{\phi}(\boldsymbol{\rho}^{-1}\widetilde{\boldsymbol{\psi}}) \tag{7.6}$$

for all $\boldsymbol{\phi} \in \mathbf{W}^-_{k+1}$ and $\widetilde{\boldsymbol{\psi}} \in (\mathbf{W}^-_{k+1})'$.

(II) Since the inclusion $\mathbf{W}^+_{k+1} \hookrightarrow \mathcal{D}(\mathring{\mathsf{A}}^*_k)$ is continuous and dense by Assumption B, the *restriction* of functionals $\mathsf{J} : \mathcal{D}(\mathring{\mathsf{A}}^*_k)' \to \mathbf{W}^-_{k+1}$ is also a continuous and dense embedding. In particular, because $\mathcal{R}(\mathsf{T}^t_k) = \mathcal{D}(\mathsf{A}^*_k)^\circ$ by Proposition 3.5 and $\mathring{\mathbf{W}}^{n,+}_{k+1} \subset \mathcal{D}(\mathsf{A}^*_k)$ by definition, it satisfies the important property that $\mathsf{J}(\mathcal{R}(\mathsf{T}^t_k)) \subset (\mathring{\mathbf{W}}^{n,+}_{k+1})^\circ$.

To prove density, we show that an arbitrary functional $\widetilde{\boldsymbol{\phi}}_\circ \in ((\mathring{\mathbf{W}}^{n,+}_{k+1})^\circ)'$ such that $\widetilde{\boldsymbol{\phi}}_\circ(\mathsf{J}\boldsymbol{\xi}) = 0$ for all $\boldsymbol{\xi} \in \mathcal{R}(\mathsf{T}^t_k)$ vanishes in $((\mathring{\mathbf{W}}^{n,+}_{k+1})^\circ)'$. We proceed in three short steps.

1) First, we use the Hahn–Banach theorem to extend $\widetilde{\boldsymbol{\phi}}_\circ$ to a functional $\widetilde{\boldsymbol{\phi}} \in (\mathbf{W}^-_{k+1})'$. By definition,

$$\widetilde{\boldsymbol{\phi}}(\mathsf{J}\boldsymbol{\xi}) = 0 \qquad \forall \boldsymbol{\xi} \in \mathcal{R}(\mathsf{T}^t_k). \tag{7.7}$$

2) Secondly, we set $\mathbf{y} := \boldsymbol{\rho}^{-1}\widetilde{\boldsymbol{\phi}} \in \mathbf{W}^+_{k+1} \subset \mathcal{D}(\mathring{\mathsf{A}}^*_k)$. Based on (7.6), it follows from (7.7) that

$$\boldsymbol{\xi}(\mathbf{y}) = \mathsf{J}\boldsymbol{\xi}(\mathbf{y}) = \mathsf{J}\boldsymbol{\xi}(\boldsymbol{\rho}^{-1}\widetilde{\boldsymbol{\phi}}) = \widetilde{\boldsymbol{\phi}}(\mathsf{J}\boldsymbol{\xi}) = 0 \qquad \forall \boldsymbol{\xi} \in \mathcal{R}(\mathsf{T}^t_k) = \mathcal{D}(\mathsf{A}^*_k)^\circ. \tag{7.8}$$

In particular, we obtain from (7.8) that $\mathbf{y} \in \mathcal{D}(\mathsf{A}^*_k)$. Thus, under the choice made in (6.32), $\mathbf{y} \in \mathcal{D}(\mathsf{A}^*_k) \cap \mathbf{W}^+_{k+1} = \mathring{\mathbf{W}}^{n,+}_k$.

3) Finally, the previous step implies that

$$\widetilde{\boldsymbol{\phi}}(\boldsymbol{\phi}_\circ) = \boldsymbol{\rho}\mathbf{y}(\boldsymbol{\phi}_\circ) = \boldsymbol{\phi}_\circ(\mathbf{y}) = 0 \qquad \forall \boldsymbol{\phi}_\circ \in (\mathring{\mathbf{W}}^{n,+}_{k+1})^\circ. \tag{7.9}$$

Therefore, $\widetilde{\boldsymbol{\phi}}_\circ = \widetilde{\boldsymbol{\phi}}\big|_{(\mathring{\mathbf{W}}^{n,+}_{k+1})^\circ} = 0$, which concludes the proof. □

Now, rewriting the trace Hilbert complexes (7.3) and (7.4) in terms of the isometrically isomorphic characterizations given in Theorem 6.11, we obtain the Hilbert complexes

$$\cdots \xrightarrow{\hat{\mathsf{D}}^t_{k-1}} \mathbf{T}^{n,-}_{k+1}(\hat{\mathsf{D}}^t_k) \subset \mathbf{T}^{n,-}_{k+1} \xrightarrow{\hat{\mathsf{D}}^t_k} \mathbf{T}^{n,-}_{k+2}(\hat{\mathsf{D}}^t_{k+1}) \subset \mathbf{T}^{n,-}_{k+2} \xrightarrow{\hat{\mathsf{D}}^t_{k+1}} \cdots \tag{7.10a}$$

and

$$\cdots \xleftarrow[\hat{\mathsf{D}}^n_k]{} \mathbf{T}^{t,-}_k(\hat{\mathsf{D}}^n_k) \subset \mathbf{T}^{t,-}_k \xleftarrow[\hat{\mathsf{D}}^n_{k+1}]{} \mathbf{T}^{t,-}_{k+1}(\hat{\mathsf{D}}^n_{k+1}) \subset \mathbf{T}^{t,-}_{k+1} \xleftarrow[\hat{\mathsf{D}}^n_{k+2}]{} \cdots. \tag{7.10b}$$

7.2. Compactness property

It is well-known that compact embeddings of the regular spaces $\mathbf{W}^+_k \subset \mathbf{W}_k$ in the stable decompositions (6.3) and (6.13) lead to the Hilbert complexes (2.5a) and (2.8b) being Fredholm. For convenience, we review this result in the next lemma.

Assumption F. Suppose that the dense inclusions $\imath_k^+ : \mathbf{W}_k^+ \hookrightarrow \mathbf{W}_k$ are compact for all $k \in \mathbb{Z}$.

Lemma 7.2. *Under Assumption F, Assumptions B and C guarantee compactness of the inclusions*

$$\mathcal{D}(\mathring{\mathsf{A}}_k^*) \cap \mathcal{D}(\mathring{\mathsf{A}}_{k+1}) \hookrightarrow \mathbf{W}_{k+1} \qquad \text{and} \qquad \mathcal{D}(\mathsf{A}_{k+1}) \cap \mathcal{D}(\mathsf{A}_k^*) \hookrightarrow \mathbf{W}_{k+1}, \tag{7.11}$$

respectively.

Proof. By symmetry, it is sufficient to prove that, under Assumption F, it follows from Assumption C that the dense inclusion $\mathcal{D}(\mathsf{A}_{k+1}) \cap \mathcal{D}(\mathsf{A}_k^*) \hookrightarrow \mathbf{W}_{k+1}$ is a compact operator. In particular, let $(\mathbf{y}_\ell)_{\ell\in\mathbb{Z}} \subset \mathcal{D}(\mathsf{A}_{k+1}) \cap \mathcal{D}(\mathsf{A}_k^*)$ be an arbitrary sequence that is bounded in $\mathcal{D}(\mathsf{A}_{k+1}) \cap \mathcal{D}(\mathsf{A}_k^*)$. We only need to show that there exists a subsequence $(\mathbf{y}_{\ell_\rho})_{\rho\in\mathbb{Z}}$ that is Cauchy in $\mathbf{W}_k$.

By Assumption C, for all $\ell \in \mathbb{Z}$, there exist $\mathbf{p}_\ell^+ \in \mathbf{W}_{k+1}^+$ and $\mathbf{x}_\ell^+ \in \mathbf{W}_k^+$ such that

$$\mathbf{y}_\ell = \mathbf{p}_\ell^+ + \mathsf{A}_k\, \mathbf{x}_\ell^+ \qquad \Big(\text{in particular, } \mathbf{p}_\ell^+ := \mathsf{L}_{k+1}^n \mathbf{y}_\ell \text{ and } \mathbf{x}_\ell^+ := \mathsf{V}_{k+1}^n \mathbf{y}_\ell\Big). \tag{7.12}$$

The norm in $\mathcal{D}(\mathsf{A}_{k+1}) \cap \mathcal{D}(\mathsf{A}_k^*)$ is stronger than the norm in $\mathcal{D}(\mathsf{A}_{k+1})$. Therefore, since the decomposition is stable by hypothesis **II** from Assumption C, the sequences $(\mathbf{p}_\ell^+)_\ell$ and $(\mathbf{x}_\ell^+)_\ell$ are bounded in $\mathbf{W}_{k+1}^+$ and $\mathbf{W}_k^+$, respectively. Under Assumption F, we can thus find subsequences $(\mathbf{p}_{\ell_\rho}^+)_\rho$ and $(\mathbf{x}_{\ell_\rho}^+)_\rho$ that are Cauchy in $\mathbf{W}_{k+1}$ and $\mathbf{W}_k$, respectively. Evaluating

$$\begin{aligned}
\|\mathbf{y}_{\ell_n} - \mathbf{y}_{\ell_m}\|_{\mathbf{W}_{k+1}}^2 &= \left(\mathbf{p}_{\ell_n}^+ - \mathbf{p}_{\ell_n}^+, \mathbf{y}_{\ell_n} - \mathbf{y}_{\ell_n}\right)_{\mathbf{W}_{k+1}} + \left(\mathsf{A}_k \left(\mathbf{x}_{\ell_n}^+ - \mathbf{x}_{\ell_n}^+\right), \mathbf{y}_{\ell_n} - \mathbf{y}_{\ell_n}\right)_{\mathbf{W}_{k+1}} \\
&\le \|\mathbf{p}_{\ell_n}^+ - \mathbf{p}_{\ell_n}^+\|_{\mathbf{W}_{k+1}} \|\mathbf{y}_{\ell_n} - \mathbf{y}_{\ell_n}\|_{\mathbf{W}_{k+1}} + \left(\mathbf{x}_{\ell_n}^+ - \mathbf{x}_{\ell_n}^+, \mathsf{A}_k^* \left(\mathbf{y}_{\ell_n} - \mathbf{y}_{\ell_n}\right)\right)_{\mathbf{W}_k} \\
&\le \underbrace{\|\mathbf{p}_{\ell_n}^+ - \mathbf{p}_{\ell_n}^+\|_{\mathbf{W}_{k+1}}}_{\to 0 \text{ as } n,m\to 0} \|\mathbf{y}_{\ell_n} - \mathbf{y}_{\ell_n}\|_{\mathbf{W}_{k+1}} + \underbrace{\|\mathbf{x}_{\ell_n}^+ - \mathbf{x}_{\ell_n}^+\|_{\mathbf{W}_k}}_{\to 0 \text{ as } n,m\to 0} \|\, \mathsf{A}_k^* \left(\mathbf{y}_{\ell_n} - \mathbf{y}_{\ell_n}\right)\|_{\mathbf{W}_k},
\end{aligned}$$

we arrive at the conclusion once noticing that $\|\mathbf{y}_{\ell_n} - \mathbf{y}_{\ell_n}\|_{\mathbf{W}_{k+1}}$ and $\|\, \mathsf{A}_k^* \left(\mathbf{y}_{\ell_n} - \mathbf{y}_{\ell_n}\right)\|_{\mathbf{W}_k}$ are also bounded by hypothesis. □

In other words, under Assumption F, the stable decompositions of Section 6.1 imply complex properties, which as stated in Section 2.2, guarantee that the associated Hilbert complexes are Fredholm. The goal of this section is to show that this carries over to the trace spaces. Ultimately, this is because what is essential for Lemma 7.2 to hold is not compactness of the embeddings, but rather that the potential and lifting operators L_{k+1}^n and V_{k+1}^n are compact operators.

In order to obtain the complex properties for the trace Hilbert complexes, we find it most convenient to work with the characterizations provided in Theorem 7.1, because it allows us to harness the theory developed in Section 3.3. By symmetry, we may focus on (7.3).

For any $\mathbf{x} \in \mathcal{D}(\mathsf{A}_k)$, it follows from Assumption C and the commuting relations of Lemma 5.4 that

$$\mathsf{T}_k^t\mathbf{x} = \mathsf{T}_k^t\mathsf{L}_k^n\mathbf{x} + \mathsf{T}_k^t\,\mathsf{A}_{k-1}\,\mathsf{V}_k^n\mathbf{x} = \mathsf{T}_k^t\mathsf{L}_k^n\mathbf{x} - \mathsf{D}_{k-1}^t\mathsf{T}_{k-1}^t\mathsf{V}_k^n\mathbf{x}. \tag{7.13}$$

Recall from Lemma 3.12 that the $\mathcal{D}(\mathsf{A}_k)$-harmonic extension operators $-\mathring{\mathsf{A}}_k^*\mathsf{R}_{\mathcal{D}(\mathring{\mathsf{A}}_k^*)}^{-1} : \mathcal{R}(\mathsf{T}_k^t) \to \mathcal{D}(\mathsf{A}_k)$ satisfy $\mathsf{T}_k^t(-\mathring{\mathsf{A}}_k^*\mathsf{R}_{\mathcal{D}(\mathring{\mathsf{A}}_k^*)}^{-1}\boldsymbol{\phi}) = \boldsymbol{\phi}$ for all $\boldsymbol{\phi} \in \mathcal{R}(\mathsf{T}_k^t)$. Inserting this identity in (7.13) yields the decomposition

$$\boldsymbol{\phi} = \left(-\mathsf{T}_k^t\mathsf{L}_k^n\mathring{\mathsf{A}}_k^*\mathsf{R}_{\mathcal{D}(\mathring{\mathsf{A}}_k^*)}^{-1}\boldsymbol{\phi}\right) + \mathsf{D}_{k-1}^t\left(\mathsf{T}_{k-1}^t\mathsf{V}_k^n\mathring{\mathsf{A}}_k^*\mathsf{R}_{\mathcal{D}(\mathring{\mathsf{A}}_k^*)}^{-1}\boldsymbol{\phi}\right) \tag{7.14}$$

for all $\boldsymbol{\phi} \in \mathcal{R}(\mathsf{T}_k^t)$.

Compare (7.14) with the regular decompositions provided in (6.3) and (6.13). In (7.14), the bounded maps

$$-\mathsf{T}_k^t\mathsf{L}_k^n\mathring{\mathsf{A}}_k^*\mathsf{R}_{\mathcal{D}(\mathring{\mathsf{A}}_k^*)}^{-1} : \mathcal{R}(\mathsf{T}_k^t) \to \mathsf{T}_k^t(\mathbf{W}_k^+) \subset \mathcal{R}(\mathsf{T}_k^t) \tag{7.15}$$

and

$$\mathsf{T}_{k-1}^t\mathsf{V}_k^n\mathring{\mathsf{A}}_k^*\mathsf{R}_{\mathcal{D}(\mathring{\mathsf{A}}_k^*)}^{-1} : \mathcal{R}(\mathsf{T}_k^t) \to \mathsf{T}_{k-1}^t(\mathbf{W}_{k-1}^+) \subset \mathcal{R}(\mathsf{T}_{k-1}^t) \tag{7.16}$$

play the roles of lifting and potential operators. Compactness of these operators as mappings $\mathcal{R}(\mathsf{T}_k^t) \to (\mathring{\mathbf{W}}_{k+1}^{n,+})^\circ$ and $\mathcal{R}(\mathsf{T}_k^t) \to (\mathring{\mathbf{W}}_k^{n,+})^\circ$ follows upon observing that under Assumption F, the map

$$\mathsf{T}_k^t : \mathbf{W}_k^+ \to (\mathring{\mathbf{W}}_{k+1}^{n,+})^\circ \tag{7.17}$$

is a compact operator, because the product of two bounded linear operators between normed spaces is compact if any one of the operands is [29, Thm. 2.16]. To confirm that the operator (7.17) is compact, it is sufficient to recall from Definition 3.1 that it is the operator associated with the compact bilinear form (cf. [44, Chap. 3])

$$\begin{cases} \mathbf{W}_k^+ \times \mathbf{W}_{k+1}^+ \to \mathbb{R} \\ \qquad (\mathbf{x},\mathbf{y}) \mapsto (\mathsf{A}_k\,\mathbf{x}, \imath_{k+1}^+\mathbf{y})_{\mathbf{W}_{k+1}} - (\imath_k^+\mathbf{x}, \mathring{\mathsf{A}}_k^*\mathbf{y})_{\mathbf{W}_k} \end{cases} \tag{7.18}$$

where, for the sake of clarity, we have introduced the compact inclusions $\imath_{k+1}^+$ and $\imath_k^+$ supplied by Assumption F.

In the next theorem, the unbounded linear operators

$$(\mathsf{D}_k^t)^* : \mathcal{D}\left((\mathsf{D}_k^t)^*\right) \subset (\mathring{\mathbf{W}}_{k+2}^{n,+})^\circ \to (\mathring{\mathbf{W}}_{k+1}^{n,+})^\circ, \tag{7.19a}$$

$$(\mathsf{D}_k^n)^* : \mathcal{D}\left((\mathsf{D}_k^n)^*\right) \subset (\mathring{\mathbf{W}}_{k-1}^{t,+})^\circ \to (\mathring{\mathbf{W}}_k^{t,+})^\circ, \tag{7.19b}$$

are the Hilbert space adjoints of the closed densely defined unbounded operators

$$\mathsf{D}_k^t : \mathcal{R}(\mathsf{T}_k^t) \subset (\mathbf{W}_{k+1}^{n,+})^\circ \to (\mathbf{W}_{k+2}^{n,+})^\circ \quad \text{and} \quad \mathsf{D}_k^n : \mathcal{R}(\mathsf{T}_k^n) \subset (\mathring{\mathbf{W}}_k^{t,+})^\circ \to (\mathring{\mathbf{W}}_{k-1}^{t,+})^\circ, \tag{7.20}$$

respectively.

Theorem 7.3. *Under Assumptions D, E and F, the inclusions*

$$\mathcal{R}(\mathsf{T}_k^t) \cap \mathcal{D}\left((\mathsf{D}_{k-1}^t)^*\right) \hookrightarrow (\mathbf{W}_{k+1}^{n,+})^\circ \qquad \text{and} \qquad \mathcal{R}(\mathsf{T}_k^n) \cap \mathcal{D}\left((\mathsf{D}_{k+1}^n)^*\right) \hookrightarrow (\mathbf{W}_k^{t,+})^\circ \tag{7.21}$$

are compact.

Proof. We follow the arguments in the proof of Lemma 7.2. Let $(\boldsymbol{\phi}_\ell)_{\ell\in\mathbb{Z}} \subset \mathcal{R}(\mathsf{T}_k^t) \cap \mathcal{D}\left((\mathsf{D}_{k-1}^t)^*\right)$ be a bounded sequence in $\mathcal{R}(\mathsf{T}_k^t) \cap \mathcal{D}\left((\mathsf{D}_{k-1}^t)^*\right)$.

The goal is to find a subsequence $(\boldsymbol{\phi}_{\ell_\rho})_{\rho\in\mathbb{Z}}$ that is Cauchy in $(\mathbf{W}_{k+1}^{n,+})^\circ$. Similarly to (7.12), we use the stable decomposition in trace spaces (7.14):

$$\boldsymbol{\phi}_\ell = \boldsymbol{\xi}_\ell^+ + \mathsf{D}_{k-1}^t \boldsymbol{\zeta}_\ell^+ \tag{7.22}$$

for all $\ell \in \mathbb{Z}$, where

$$\boldsymbol{\xi}_\ell^+ := -\mathsf{T}_k^t \mathsf{L}_k^n \mathring{\mathsf{A}}_k^* \mathsf{R}_{\mathcal{D}(\mathring{\mathsf{A}}_k^*)}^{-1} \boldsymbol{\phi}_\ell \in \mathsf{T}_k^t(\mathbf{W}_k^+) \quad \text{and} \quad \boldsymbol{\zeta}_\ell := \mathsf{T}_{k-1}^t \mathsf{V}_k^n \mathring{\mathsf{A}}_k^* \mathsf{R}_{\mathcal{D}(\mathring{\mathsf{A}}_k^*)}^{-1} \boldsymbol{\phi}_\ell \in \mathsf{T}_{k-1}^t(\mathbf{W}_{k-1}^+).$$

Since the norm in $\mathcal{R}(\mathsf{T}_k^t) \cap \mathcal{D}\left((\mathsf{D}_{k-1}^t)^*\right)$ is stronger than the norm in $\mathcal{R}(\mathsf{T}_k^t)$, the sequence $(\boldsymbol{\phi}_\ell)_{\ell\in\mathbb{Z}}$ is bounded in the norm of $\mathcal{R}(\mathsf{T}_k^t)$. Hence, by compactness of the operators $-\mathsf{T}_k^t \mathsf{L}_k^n \mathring{\mathsf{A}}_k^* \mathsf{R}_{\mathcal{D}(\mathring{\mathsf{A}}_k^*)}^{-1} : \mathcal{R}(\mathsf{T}_k^t) \to (\mathbf{W}_{k+1}^{n,+})^\circ$ and $\mathsf{T}_{k-1}^t \mathsf{V}_k^n \mathring{\mathsf{A}}_k^* \mathsf{R}_{\mathcal{D}(\mathring{\mathsf{A}}_k^*)}^{-1} : \mathcal{R}(\mathsf{T}_k^t) \to (\mathbf{W}_k^{n,+})^\circ$, there exist subsequences $(\boldsymbol{\xi}_{\ell_\rho}^+)_{\rho\in\mathbb{Z}}$ and $(\boldsymbol{\zeta}_{\ell_\rho}^+)_{\rho\in\mathbb{Z}}$ that are Cauchy in $(\mathbf{W}_{k+1}^{n,+})^\circ$ and $(\mathbf{W}_k^{n,+})^\circ$, respectively.

Now, we verify that $(\boldsymbol{\phi}_{\ell_\rho})_{\rho\in\mathbb{Z}}$ is indeed Cauchy in $(\mathbf{W}_{k+1}^{t,+})^\circ$. We evaluate directly

$$\begin{aligned}
&\|\boldsymbol{\phi}_{\ell_n} - \boldsymbol{\phi}_{\ell_n}\|^2_{(\mathbf{W}_{k+1}^{t,+})^\circ} \\
&\quad = \left(\boldsymbol{\xi}_{\ell_n}^+ - \boldsymbol{\xi}_{\ell_n}^+, \boldsymbol{\phi}_{\ell_n} - \boldsymbol{\phi}_{\ell_n}\right)_{(\mathbf{W}_{k+1}^{n,+})^\circ} + \left(\mathsf{D}_{k-1}^t\left(\boldsymbol{\zeta}_{\ell_n}^+ - \boldsymbol{\zeta}_{\ell_n}^+\right), \boldsymbol{\phi}_{\ell_n} - \boldsymbol{\phi}_{\ell_n}\right)_{(\mathbf{W}_{k+1}^{n,+})^\circ} \\
&\quad \leq \|\boldsymbol{\xi}_{\ell_n}^+ - \boldsymbol{\xi}_{\ell_n}^+\|_{(\mathbf{W}_{k+1}^{n,+})^\circ} \|\boldsymbol{\phi}_{\ell_n} - \boldsymbol{\phi}_{\ell_n}\|_{(\mathbf{W}_{k+1}^{n,+})^\circ} \\
&\qquad + \left(\boldsymbol{\zeta}_{\ell_n}^+ - \boldsymbol{\zeta}_{\ell_n}^+, (\mathsf{D}_{k-1}^t)^*\left(\boldsymbol{\phi}_{\ell_n} - \boldsymbol{\phi}_{\ell_n}\right)\right)_{(\mathbf{W}_k^{n,+})^\circ},
\end{aligned}$$

from which we conclude that

$$\|\boldsymbol{\phi}_{\ell_n} - \boldsymbol{\phi}_{\ell_n}\|_{(\mathbf{W}^{t,+}_{k+1})^\circ} \le \underbrace{\|\boldsymbol{\xi}^+_{\ell_n} - \boldsymbol{\xi}^+_{\ell_n}\|_{(\mathbf{W}^{n,+}_{k+1})^\circ}}_{\to 0 \text{ as } m,n\to 0} \|\boldsymbol{\phi}_{\ell_n} - \boldsymbol{\phi}_{\ell_n}\|_{(\mathbf{W}^{n,+}_{k+1})^\circ}$$
$$+ \underbrace{\|\boldsymbol{\xi}^+_{\ell_n} - \boldsymbol{\xi}^+_{\ell_n}\|_{(\mathbf{W}^{n,+}_{k})^\circ}}_{\to 0 \text{ as } m,n\to 0} \|(\mathsf{D}^t_{k-1})^* \left(\boldsymbol{\phi}_{\ell_n} - \boldsymbol{\phi}_{\ell_n}\right)\|_{(\mathbf{W}^{n,+}_{k})^\circ}.$$

The desired result thus follows because the norms $\|\boldsymbol{\phi}_{\ell_n} - \boldsymbol{\phi}_{\ell_n}\|_{(\mathbf{W}^{n,+}_{k+1})^\circ}$ and $\|(\mathsf{D}^t_{k-1})^*(\boldsymbol{\phi}_{\ell_n} - \boldsymbol{\phi}_{\ell_n})\|_{(\mathbf{W}^{n,+}_{k})^\circ}$ are bounded uniformly in ℓ_n by hypothesis. □

Corollary 7.4. *Under Assumptions D, E and F, the trace Hilbert complexes introduced in Theorem 7.1 are Fredholm.*

It is particularly interesting that while only one decomposition was sufficient to obtain Lemma 7.2, we needed *both* decompositions (Assumptions B and C) to achieve a proof of the compactness property for the trace Hilbert complex: one for the space characterization and the other for the decomposition formula itself. The question whether it is *necessary* to have both remains open.

3D de Rham setting XVI: Trace de Rham complexes. Trace Hilbert complexes for the de Rham complex in 3D arise from the results of XV:

$$\begin{array}{cc}
\{0\} & \{0\} \\
\big\downarrow \imath & \big\downarrow \imath \\
\mathcal{D}(\mathbf{curl}') \subset \widetilde{\mathbf{H}}^{-1}(\Omega) \cap \mathring{\mathbf{H}}(\mathrm{div},\Omega)^\circ & \mathcal{D}(\mathbf{curl}') \subset \left(\mathbf{H}^1(\Omega)/\mathbf{H}^1_n(\Omega)\right)' \\
\big\downarrow \mathbf{curl}' & \big\downarrow \mathbf{curl}' \\
\mathcal{D}(\mathbf{grad}') \subset \widetilde{\mathbf{H}}^{-1}(\Omega) \cap \mathring{\mathbf{H}}(\mathbf{curl},\Omega)^\circ & \mathcal{D}(\mathbf{grad}) \subset \left(\mathbf{H}^1(\Omega)/\mathbf{H}^1_t(\Omega)\right)' \\
\big\downarrow \mathbf{grad}' & \big\downarrow \mathbf{grad}' \\
\widetilde{H}^{-1}(\Omega) \cap \mathring{H}^1(\Omega)^\circ & \left(H^1(\Omega)/\mathring{H}^1(\Omega)\right)' \\
\big\downarrow 0 & \big\downarrow 0 \\
\{0\} & \{0\}
\end{array} \tag{7.23}$$

In light of the de Rham setting XV, they correspond to

$$\{0\} \xrightarrow{\imath} H^{1/2}(\Gamma) \subset H^{-1/2}(\Gamma) \xrightarrow{\mathbf{curl}_\Gamma} \mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma,\Gamma) \subset \mathbf{H}^{-1/2}_t \xrightarrow{\mathrm{curl}_\Gamma} H^{-1/2}(\Gamma) \xrightarrow{0} \{0\} \tag{7.24}$$

or its rotated version.

Since by Rellich's lemma the embeddings $H^1(\Omega) \hookrightarrow L^2(\Omega)$ and $\mathbf{H}^1(\Omega) \hookrightarrow \mathbf{L}^2(\Omega)$ are compact, the de Rham complexes in (1.4) satisfy Assumption F with the regular decompositions presented in the de Rham setting XII. Therefore, the associated trace de Rham complexes are Fredholm. As a consequence, their cohomology spaces are finite-dimensional. ⊿

8. Conclusion

As we have demonstrated in the present article, it takes only a pair of Hilbert complexes linked by the sub-complex relationship of their domain complexes to recover essential aspects of the structures inherent in the trace operators and trace spaces for the de Rham complex. Relying on notions of trace spaces as dual spaces or quotient spaces, we could establish detailed characterizations merely assuming the existence of stable regular decompositions induced by bounded lifting operators. These developments culminated in the discovery of associated trace Hilbert complexes, which are Fredholm under the mild additional assumption that the lifting operators are compact.

Hilbert complexes have recently moved into the focus of applied mathematicians, since they underlie a host of PDE-based mathematical models in areas as diverse as linear elasticity, gravity, and fluid dynamics. The related complexes are known as the elasticity complex, [7, Sect. 11] and [40], conformal complex, or Stokes complex [9, Sect. 4.4]. These and many more complexes [38,39] arise from the de Rham complex through the powerful Bernstein-Gelfand-Gelfand (BGG) construction, as has been shown in [9]. Most likely, many more Hilbert complexes relevant for mathematical modeling still await discovery.

This backdrop lends relevance to our present work. Once the Hilbert complex structure is established, trace operators and trace spaces become available, which can serve as stepping stones towards the study of boundary value problems and the development of integral representations.

Data availability

No data was used for the research described in the article.

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