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On a class of degenerate abstract parabolic problems and applications to some eddy current models

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We present an abstract framework for parabolic type equations which possibly degenerate on certain spatial regions. The degeneracies are such that the equations under investigation may admit a type change ranging from parabolic to elliptic type problems. The approach is an adaptation of the concept of so-called evolutionary equations in Hilbert spaces and is eventually applied to a degenerate eddy current type model. The functional analytic setting requires quite minimal assumptions on the boundary and interface regularity. The degenerate eddy current model is justified as a limit model of non-degenerate hyperbolic models of Maxwell's equations.

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1. Introduction

The dynamics of electromagnetic fields is described by Maxwell's equations, which for classical materials take the form

$$\partial_0 \varepsilon \mathrm{E} + \sigma \mathrm{E} - \operatorname{curl} \mathrm{H} = -\mathrm{J},$$
$$\partial_0 \mu \mathrm{H} + \operatorname{curl} \mathrm{E} = 0,$$

where ∂_0 denotes time-differentiation, E the electric field, H the magnetic field. The term J summarises external current densities exciting the field, ε and μ describe dielectricity and permeability of the medium, σ its conductivity. Here, we consider Maxwell's equations subject to the electric boundary condition; that is, we ask the electric field to have a vanishing tangential component at the boundary of the underlying domain $\Omega \subseteq \mathbb{R}^3$. If Ω is regular enough to allow for a well-defined unit outward normal field $n\colon \partial\Omega \to \mathbb{R}^3$, the strong form of the mentioned boundary condition reads

$$\mathrm{E} \times n = 0 \text{ on } \partial\Omega.$$

It is possible to generalise this condition in a similar way to homogeneous Dirichlet boundary conditions also to Ω lacking the regularity for a well-defined unit outward normal. This will be detailed later in the text. For the time being we shall use $\mathring{\operatorname{curl}}$ to denote the curl operator with the additional constraint of the appropriate generalisation of vanishing tangential component at the boundary. Consequently, the above mentioned Maxwell's equations subject to the boundary condition read

$$\left(\partial_0 \begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} + \begin{pmatrix} \sigma & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -\operatorname{curl} \\ \mathring{\operatorname{curl}} & 0 \end{pmatrix}\right) \begin{pmatrix} \mathrm{E} \\ \mathrm{H} \end{pmatrix} = \begin{pmatrix} -\mathrm{J} \\ 0 \end{pmatrix}.$$

There is a suitable abstract framework, see [25], extended, for example in [26,30,36,37,41,42], to incorporate dissipative, non-autonomous, and nonlinear systems. If for example ε, μ, σ are all selfadjoint, non-negative, given e.g. by non-negative, real scalar L_∞-multiplication-operators, then this abstract framework yields – with well-chosen boundary conditions – well-posedness of the problem, if we assume that μ and $\varepsilon + \sigma$ are both strictly positive. This allows for a type change by having $\varepsilon = 0$ in some regions (eddy current case) and ε strictly positive in others. This eddy current problem is well-understood and well-justified, see [21] or [42, Section 5.3]. The problem we want to investigate here goes, however, one step further. We assume $\varepsilon = 0$ everywhere and σ may still vanish in some regions, as e.g. suggested in [5].

In the case $\varepsilon = 0$, we eliminate H and obtain

$$\partial_0 \sigma \mathrm{E} + \operatorname{curl} \mu^{-1} \mathring{\operatorname{curl}} \mathrm{E} = -\partial_0 \mathrm{J} \tag{1}$$

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as a degenerate eddy current problem, which formally has parabolic regions, where σ is strictly positive, and elliptic regions, where σ vanishes. Note that this indeed represents a particularly degenerate situation for if σ vanishes in some regions, the resulting problem still has a null-space, stemming from the infinite-dimensional null-space of the curl-operator. In the derivation to be carried out below this is in fact the crucial observation.

In a sense the problems discussed in this manuscript can also be regarded as the parabolic extension of the framework provided for elliptic type problems presented in [38], where nonlinear differential inclusions in divergence form have been discussed.

The extended abstract framework of [26] still allows us to incorporate the degenerate situation, where σ is only supported in a bounded subset Ω_c of the underlying open set Ω (with positive distance to the boundary of Ω).

Although electromagnetic fields are generally accepted to be controlled by Maxwell's equations, it is still well established with engineers, see e.g. [1,11], to discard Maxwell's correction; that is, the displacement current term. It appears that the rigorous justification of the above degenerate eddy current problem, where $\varepsilon = 0$ and σ vanishes in some region, is still open or rather unattainable.

For a survey concerning the eddy current problem the reader may consult [7, Chapter 8] and for various variants [17]. We shall furthermore refer to [3,2,4,32] for the eddy current problem particularly considered in the time harmonic case. A convergence result relating the non-vanishing dielectricity case to the eddy current version of Maxwell's equations is also presented in [32]. We connect this convergence statement to the one derived in the concluding section of the paper at hand at the end of this manuscript, see Remark 6.3. We refer to [35] for a mathematical treatment of eddy current type problems and a selection of applications.

For a treatment of the full time-dependent problem with nowhere vanishing σ, we refer to the recent paper [12]. This treatment prerequisites more assumptions on the smoothness of the boundary of the underlying domain (as well as on the magnetic permeability), which we wish to avoid here.

More specifically, our investigation is inspired by a series of papers by S. Nicaise et al., [20,18,19]. Among other things the so-called A-φ approach is addressed in these references. We shall comment on this approach, when we present the complete solution theory[1] for the eddy current problem discussed here, see Remark 4.22.

We will employ the theory of evolutionary equations as laid out in Section 1, see [28,26], to analyse the structure of the degenerate eddy current problem. It will prove to be beneficial to embed the degenerate eddy current problem into an abstract class of degenerate parabolic systems in order to understand the mechanism of well-posedness

[1] A (linear) solution theory (for a linear operator B) comprises not just a description of a class of right-hand sides f for which a solution u of $Bu = f$ can be found, but also to identify a complete linear space, in which the solution can be found. Furthermore, one needs to ensure that for every right-hand side f produced by an element u in the way that $Bu = f$, we actually can recover the original u from this right-hand side by applying the proposed solution procedure. Indeed, here we consider providing a solution theory as establishing that the operator is a continuous bijection between its domain and its range as complete linear spaces (well-posedness).

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more deeply. After a brief introduction into the theory of a problem class (Section 2), which we will refer to as evolutionary equations or evo-systems, we shall investigate the mentioned abstract class of degenerate parabolic problems as a special case more closely in Section 3.

The application to the degenerate eddy current problem is then given in the concluding Section 4. In particular, having reformulated and solved the degenerate eddy current type problem, we shall address the validity of the equations one started out with. It appears that this a posteriori justification of the original equation has not been addressed in the literature as of yet. The application to the eddy current type model is discussed further in the concluding 2 sections. There we present an alternative saddle-point formulation for the problem at hand, which might be useful for numerical considerations. In fact a similar strategy has led to an efficient numerical treatment of Maxwell's equations (see [34]). Moreover, we shall justify the degenerate eddy current model as a regular limit case of non-degenerate problems. In the framework presented here, we are thus mathematically justifying that the degenerate eddy current problem is indeed approachable by regular problems so that the maybe-easier-to-solve degenerate parabolic problem leads to an appropriate approximation of the full hyperbolic Maxwell's equations.

2. A brief introduction to evo-systems

In this section we shall introduce the general abstract problem class we like to use as the underlying structure of the derivations to come.

More precisely, we will discuss *evolutionary equations*, *evo-systems* for short, in the following. These terms are chosen deliberately in order to distinguish from classical (explicit) evolution equations, which turn out to be just a special case of the class of evo-systems. For convenience of the reader, we gather some necessary information as follows.

The starting idea of the evo-system approach is to realise that the time-differentiation can be established as a normal operator in a real, weighted L^2-type Hilbert space $H_{\varrho,0}(\mathbb{R};H)$, $\varrho\in\,]0,\infty[$, see e.g. [28], characterised by

$$H_{\varrho,0}\left(\mathbb{R},H\right)=\left\{f\in L^{2,\mathrm{loc}}\left(\mathbb{R},H\right)\,|\,\left|f\right|_{\varrho,0,0}:=\sqrt{\int_{\mathbb{R}}\left|f\left(t\right)\right|_H^2\exp\left(-2\varrho t\right)\,dt}<\infty\right\},$$

where $|\cdot|_H$ denotes the norm in the underlying *real* Hilbert space H. Our choice of a real Hilbert space is no important constraint, it merely is an adjustment to account for mostly real physical quantities. Note that every complex Hilbert space is in fact a real Hilbert space if we restrict scalar multipliers to $\mathbb{R}$ and take the real part of the inner product as the real inner product.

The inner product $\langle \cdot | \cdot \rangle_{\varrho,0,0}$ of $H_{\varrho,0}(\mathbb{R}, H)$ is given by

$$(\phi, \psi) \mapsto \int_{\mathbb{R}} \langle \phi(t) | \psi(t) \rangle_H \exp(-2\varrho t) \, dt,$$

where $\langle \cdot | \cdot \rangle_H$ denotes the inner product of H. We define the time-derivative $\partial_{0,\varrho}$ (or just ∂_0, if ϱ is clear from the context) to be the distributional derivative with respect to the first variable in $H_{\varrho,0}(\mathbb{R}, H)$ with maximal domain. We also put $H_{\varrho,1}(\mathbb{R}, H) := D(\partial_0)$ endowed with $\langle \partial_0 \cdot | \partial_0 \cdot \rangle_{\varrho,0,0}$ as scalar product. This is a scalar product the induced norm of which being equivalent to the graph norm of $D(\partial_0)$. Indeed, for this ∂_0 needs to be continuously invertible. This property on the other hand follows from maximal accretivity of ∂_0. In fact, a simple integration-by-parts procedure shows that

$$\frac{1}{2}\overline{(\partial_0 + \partial_0^*)} =: \operatorname{sym}(\partial_0) \supseteq \frac{1}{2}(\partial_0 + \partial_0^*) = \varrho,$$

where ϱ is a short-hand for the operator of multiplying by the scalar value ϱ. So ∂_0 is (real) strictly positive definite (or accretive). This observation can be lifted to obtain a solution theory for systems (evo-systems) of the form

$$\left(\partial_0 M\left(\partial_0^{-1}\right) + A\right) U = F,$$

where here we focus on simple, so-called 'material law' operators of the form

$$M\left(\partial_0^{-1}\right) = M_0 + \partial_0^{-1} M_1,$$

where M_k, $k \in \{0, 1\}$, are certain continuous, linear operators in H. The operator A is densely defined and closed in the Hilbert space H. All the operators M_0, M_1, and A are (canonically) lifted to the H-valued space $H_{\varrho,0}(\mathbb{R}; H)$ by being applied pointwise with maximal domain. Re-using the notation for these lifted operators, we easily verify that M_0 and M_1 are still bounded linear operator in the extended space $H_{\varrho,0}(\mathbb{R}; H)$ even commuting with ∂_0, that is,

$$M_k \partial_0 \subseteq \partial_0 M_k \quad (k \in \{0, 1\}).$$

A acting in $H_{\varrho,0}(\mathbb{R}; H)$ will still be densely defined and closed; the adjoint of the lifted A is the lift of the adjoint of A having acted in H. Focusing on the simple material law mentioned above, we want to solve evo-systems of the form

$$\left(\overline{\partial_0 M_0 + M_1 + A}\right) U = F. \tag{2}$$

By solving this evo-system, we mean to show that for all $F \in H_{\varrho,0}(\mathbb{R}; H)$ there exists a unique $U \in H_{\varrho,0}(\mathbb{R}; H)$ satisfying (2). In other words, $\left(\overline{\partial_0 M_0 + M_1 + A}\right)$ needs to be shown to be continuously invertible.

Furthermore, in order to render (2) 'physically meaningful', we shall show that (2) also leads to a *causal* solution operator, which will be quantified in the next theorem and roughly means that there is 'no reaction' U, if there is 'no action' F. We shall furthermore refer to [40] and to [42, Chapter 2] for a more detailed account on causality.

The issue in the context of well-posedness of (2); that is, continuous invertibility of $\left(\overline{\partial_0 M_0 + M_1 + A}\right)$ is, see e.g. [26,42], to establish estimates of the form

$$\langle U | \left(\partial_0 M_0 + M_1 + A\right) U \rangle_{\varrho,0,0} \geq c_0 \langle U|U \rangle_{\varrho,0,0} \quad (U \in D(A) \cap D(\partial_0)), \tag{3}$$

$$\left\langle V \right| \left(\partial_0 M_0 + M_1 + A\right)^* V \right\rangle_{\varrho,0,0} \geq c_0 \langle V|V \rangle_{\varrho,0,0} \quad (V \in D((\partial_0 M_0 + M_1 + A)^*)) \tag{4}$$

for some $c_0 > 0$.

In the following we shall employ the convention to denote by $D(C)$, $R(C)$, $N(C)$ the domain, range and kernel of a linear operator C.

We record the following variant of [26, Theorem 2.3] or [42, Theorem 3.4.6]. For this we briefly emphasise that in contrast to earlier treatments of this theorem, we shall focus on the real Hilbert space case, only. In this way the real parts used for the positive definiteness estimates in the mentioned theorems can entirely be dispensed with.

Theorem 2.1. *Let $M_0, M_1 \in L(H)$ with $M_0 = M_0^*$. Moreover, let $A : D(A) \subseteq H \to H$ be a closed, densely defined linear operator such that*

$$\langle W | \left(\varrho M_0 + M_1 + A\right) W \rangle_H \geq c_0 \langle W|W \rangle_H \tag{5}$$

$$\langle V | \left(\varrho M_0 + M_1^* + A^*\right) V \rangle_H \geq c_0 \langle V|V \rangle_H \tag{6}$$

for some $c_0, \varrho_0 \in]0, \infty[$ and all $W \in D(A)$, $V \in D(A^*)$ and $\varrho \in [\varrho_0, \infty[$. Then, equation (2) has for every $F \in H_{\varrho,0}(\mathbb{R}, H)$ a unique solution $U \in H_{\varrho,0}(\mathbb{R}, H)$. Moreover, we have for the corresponding solution operator the estimate

$$\left| \chi_{]-\infty,a]} \left(\overline{\partial_0 M_0 + M_1 + A}\right)^{-1} F \right|_{\varrho,0,0} \leq \frac{1}{c_0} \left| \chi_{]-\infty,a]} F \right|_{\varrho,0,0}$$

for all $a \in \mathbb{R}$ and $F \in H_{\varrho,0}(\mathbb{R}, H)$, that is, we have continuous and causal dependence on the data.

Proof. The result largely follows with the general results in [26] and is a special case of [42, Theorem 3.4.6] or of [37, Theorem 3.1, Theorem 4.4]. Since, however, the material law is more elementary here, we outline – for sake of transparency and to remain self-contained – a more straightforward independent proof. By density of $D(A)$ in H, we obtain that $D(A)$-valued continuously differentiable functions with compact support are dense in $H_{\varrho,0}(\mathbb{R}; H)$.

Thus, letting $U \in \mathring{C}_1(\mathbb{R}; D(A))$ and using the Cauchy–Schwarz inequality as well as integration by parts, we obtain

$$
\begin{aligned}
&\left|\chi_{]-\infty,a]}U\right|_{\varrho,0,0}\left|\chi_{]-\infty,a]}\left(\partial_0 M_0+M_1+A\right)U\right|_{\varrho,0,0}\\
&\geq\left\langle\chi_{]-\infty,a]}U\right|\left(\partial_0 M_0+M_1+A\right)U\right\rangle_{\varrho,0,0}\\
&=\int_{-\infty}^{a}\left\langle U\right|\left(\partial_0 M_0+M_1+A\right)U\right\rangle_H(t)\ \exp\left(-2\varrho t\right)\ dt\\
&=\int_{-\infty}^{a}\frac{1}{2}\left\langle U|M_0U\right\rangle_H'(t)\ \exp\left(-2\varrho t\right)\ dt+\int_{-\infty}^{a}\left\langle U|M_1U\right\rangle_H(t)\ \exp\left(-2\varrho t\right)\ dt\\
&\quad+\int_{-\infty}^{a}\left\langle U|AU\right\rangle_H(t)\ \exp\left(-2\varrho t\right)\ dt\\
&=\frac{1}{2}\left\langle U|M_0U\right\rangle_H(a)\ \exp\left(-2\varrho a\right)+\varrho\int_{-\infty}^{a}\left\langle U|M_0U\right\rangle_H(t)\ \exp\left(-2\varrho t\right)\ dt\\
&\quad+\int_{-\infty}^{a}\left\langle U|M_1U\right\rangle_H(t)\ \exp\left(-2\varrho t\right)\ dt+\int_{-\infty}^{a}\left\langle U|AU\right\rangle_H(t)\ \exp\left(-2\varrho t\right)\ dt\\
&\geq\varrho\int_{-\infty}^{a}\left\langle U|M_0U\right\rangle_H(t)\ \exp\left(-2\varrho t\right)\ dt+\int_{-\infty}^{a}\left\langle U|M_1U\right\rangle_H(t)\ \exp\left(-2\varrho t\right)\ dt\\
&\quad+\int_{-\infty}^{a}\left\langle U|AU\right\rangle_H(t)\ \exp\left(-2\varrho t\right)\ dt\\
&=\int_{-\infty}^{a}\left\langle U\right|\left(\varrho M_0+M_1+A\right)U\right\rangle_H(t)\ \exp\left(-2\varrho t\right)\ dt\\
&\geq c_0\left\langle\chi_{]-\infty,a]}U|\chi_{]-\infty,a]}U\right\rangle_{\varrho,0,0}.
\end{aligned}
\tag{7}
$$

Letting $a\to\infty$ in (7) we get (3) with a density argument. Similarly, we obtain (4) by re-doing the above estimate for $a=\infty$ and A replaced by A^* (in which case there is no point-evaluation at the upper time boundary value and we need to confirm that $\left(\partial_0 M_0+M_1+A\right)^*=\overline{\partial_0^* M_0+M_1^*+A^*}$, which in turn follows using suitable density arguments as for instance in [30, the proof of Theorem 2.13]). Thus $\left(\overline{\partial_0 M_0+M_1+A}\right)^{-1}$ is continuous. Hence, from $N\left(\left(\partial_0 M_0+M_1+A\right)^*\right)=N\left(\overline{\partial_0^* M_0+M_1^*+A^*}\right)=\{0\}$, we infer that $\left(\overline{\partial_0 M_0+M_1+A}\right)^{-1}$ is also everywhere defined. Moreover, the above estimate (7) shows

$$
\left|\chi_{]-\infty,a]}\left(\overline{\partial_0 M_0+M_1+A}\right)^{-1}F\right|_{\varrho,0,0}\leq\frac{1}{c_0}\ \left|\chi_{]-\infty,a]}F\right|_{\varrho,0,0}
\tag{8}
$$

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for all $a \in \mathbb{R}$ and $F \in H_{\varrho,0}(\mathbb{R}, H)$. If $F = 0$ on the time interval $]-\infty, a]$ then we read off that also the solution U must vanish on this time-interval; that is, we have causality. Letting $a \to \infty$ in (8) shows continuous dependence in the form

$$\left\| \overline{(\partial_0 M_0 + M_1 + A)}^{-1} \right\| \leq \frac{1}{c_0}. \qquad \square$$

Remark 2.2. We identify the dual spaces

$$H = H',$$
$$H_{\varrho,0}(\mathbb{R}) = H_{\varrho,0}(\mathbb{R})',$$

and so we have

$$H_{\varrho,0}(\mathbb{R}, H) = H_{\varrho,0}(\mathbb{R}, H)'.$$

Moreover, the dual $(\partial_0^*)^\diamond$ of the — by choice of inner product — unitary operator $\partial_0^* \iota_{H_{\varrho,1}(\mathbb{R},H)} : H_{\varrho,1}(\mathbb{R}, H) \to H_{\varrho,0}(\mathbb{R}, H)$ — has an extension to a continuous operator for which we keep the notation ∂_0 and so

$$\partial_0 : H_{\varrho,0}(\mathbb{R}, H) \to H_{\varrho,-1}(\mathbb{R}, H) := H_{\varrho,1}(\mathbb{R}, H)'.$$

Similarly, the continuous mapping

$$A^* \iota_{H_{\varrho,0}(\mathbb{R},D(A^*))} : H_{\varrho,0}(\mathbb{R}, D(A^*)) \to H_{\varrho,0}(\mathbb{R}, H)$$

has as dual

$$(A^*)^\diamond = \left(A^* \iota_{H_{\varrho,0}(\mathbb{R},D(A^*))}\right)^\diamond : H_{\varrho,0}(\mathbb{R}, H) \to H_{\varrho,0}\left(\mathbb{R}, D(A^*)'\right),$$

which may be considered as a continuous extension of A and so justifies (with some care) to keep A as a notation for $(A^*)^\diamond$.[2]

Indeed, for $\Psi \in H_{\varrho,0}(\mathbb{R}, D(A^*))$ we compute

$$\left((A^*)^\diamond \Phi\right)(\Psi) := \left\langle \Psi \middle| (A^*)^\diamond \Phi \right\rangle_{\varrho,0,0} = \langle A^* \Psi | \Phi \rangle_{\varrho,0,0}$$

for all $\Phi \in H_{\varrho,0}(\mathbb{R}, D(A))$, in which case $A\Phi = (A^*)^\diamond \Phi$ and by continuous extension also to $\Phi \in H_{\varrho,0}(\mathbb{R}, H)$. We have for a solution of the evo-system (2) that

$$\partial_0 M_0 U + M_1 U + AU = F$$

[2] Note that we routinely use $D(A)$ for the domain of A also for the corresponding Hilbert space with respect to the graph inner product of A. In this sense $D(A^*)'$ denotes the dual Hilbert space of the Hilbert space $D(A^*)$.

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holds in the space $H_{\varrho,-1}\left(\mathbb{R}, D\left(A^*\right)'\right)$. Note that

$$H_{\varrho,-1}\left(\mathbb{R}, D\left(A^*\right)'\right) \supseteq H_{\varrho,-1}\left(\mathbb{R}, H\right) \cap H_{\varrho,0}\left(\mathbb{R}, D\left(A^*\right)'\right).$$

We shall use this observation to conveniently drop the closure bar in equations of the form (2).

Remark 2.3. (a) In the case of a simple material law as used here it is interesting to note that the result easily carries over to a local-in-time formulation. Indeed, the time-derivative restricted to a finite time-interval $[0,T]$, $T \in \,]0,\infty[$, given as the closure $\partial_{0,\varrho,]0,T]}$ of ∂_0 restricted to $\mathring{C}_1\left(]0,T], H\right)$ in $H_{\varrho,0}\left(]0,T\left[\right., H\right)$ loses the skew-selfadjointness, keeps, however, the maximal accretivity. We emphasise the parentheses of the interval in the index of the time-derivative operator: $\partial_{0,\varrho,]0,T]}$ has a zero boundary condition at 0, and no boundary condition at T; whereas $\partial_{0,\varrho,[0,T[}$ (defined as $\partial_{0,\varrho,]0,T]}$ with $]0,T]$ being interchanged by $[0,T[$) has no boundary condition at 0 and a zero boundary condition at T. In classical terms, we have

$$\begin{aligned} D(\partial_{0,\varrho,]0,T]}) &= \{\phi \in H^1(0,T); \phi(0) = 0\}, \\ D(\partial_{0,\varrho,[0,T[}) &= \{\phi \in H^1(0,T); \phi(T) = 0\}. \end{aligned}$$

For the closure $\partial_{0,\varrho,[0,T[}$ we still have $\partial^*_{0,\varrho,]0,T]} = -\partial_{0,\varrho,[0,T[} + 2\varrho$. Thus, it is rather straightforward to see

$$\partial_{0,\varrho,]0,T]}, \partial^*_{0,\varrho,]0,T]} \geq \varrho,$$

which allows the solution theory of

$$\partial_{0,\varrho} M_0 + M_1 + A$$

to be carried over to

$$\partial_{0,\varrho,]0,T]} M_0 + M_1 + A.$$

In this sense the above solution strategy also carries over to problems with finite time horizon. For this, we also refer to [14] for a numerical treatment of evo-systems. Regarding numerics, we shall furthermore refer to the Section 5.

(b) It is also possible to use the above derived solution theory for incorporating initial value problems. For this there are at least two possibilities. One is to require that the initial datum U_0 is in the domain of A. Then one can show that for the unique solution V of

$$\overline{(\partial_0 M_0 + M_1 + A)} V = -\chi_{[0,\infty)} M_1 U_0 - \chi_{[0,\infty)} A U_0,$$

it follows that $U = V + \chi_{[0,\infty)}U_0$ satisfies the initial value problem

$$\begin{cases} (\partial_0 M_0 + M_1 + A)U = 0 & \text{on } (0,\infty) \\ (M_0 U)(0+) = M_0 U_0 \end{cases}$$

in an appropriate sense. It is also possible to extend the solution operator $(\overline{\partial_0 M_0 + M_1 + A})^{-1}$ to a continuous linear operator S from $H_{\varrho,-1}(\mathbb{R}; H)$ into itself. It can then be shown that the solution U of the just introduced initial value problem satisfies

$$U = S\delta_0 M_0 U_0,$$

where δ_0 is the Dirac delta-distribution. Interestingly, the latter formulation is also well-defined for $U_0 \notin D(A)$ and, thus, serves as a generalisation for the initial value problem for less regular initial data; we refer to [28, Chapter 6], [33, Lecture 9] for the details.

Our focus in the following will be on a rather particular subclass, where $M_1 = 0$ and $A = C^*C$ for a closed, densely defined operator C with closed range. The coefficient M_0 may have a non-trivial null space but, as we shall see, that 0 is in the resolvent set of the reduction of C^*C to the subspace $R(C^*)$, which is also closed, can be used to compensate for this short-coming. Recall that for elliptic problems; that is, for $M_0 = 0$, the strategy of projecting onto $R(C^*)$ has been successfully applied also to non-linear (abstract) differential inclusions, see [38]. Also in [38], the crucial assumption for the well-posedness was a closed range condition.

3. A class of degenerate abstract parabolic equations

In this whole section, we let H and X be Hilbert spaces and let $\eta \in L(H)$ be a bounded, selfadjoint, non-negative operator. Furthermore, let

$$C : D(C) \subseteq H \to X$$

be closed and densely defined; throughout assume C to have a closed range.

Abstractly speaking, we want to consider

$$\left(\overline{\partial_0 \eta + C^*C}\right) U = F. \tag{9}$$

Remark 3.1. Note that the equation holds in the form

$$\partial_0 \eta U + C^*C\, U = F$$

if considered in the space

$$H_{\varrho,-1}\left(\mathbb{R}, D\left(C^*C\right)'\right).$$

This is clear from Remark 2.2. Henceforth, we shall therefore dispose of the closure bar in equations of the form (9) unless it is needed for sake of clarity.

Without having looked at this equation in detail, it is immediately clear, where degeneracies might arise. Indeed, if U attains non-zero values in $N(\eta) \cap N(C)$; that is, if $U \in H_{\varrho,0}\left(\mathbb{R}, N\left(\eta\right) \cap N\left(C\right)\right)$ we have

$$\partial_0 \eta U + C^*CU = 0,$$

and so if $N\left(\eta\right) \cap N\left(C\right)$ is not trivial, well-posedness for (9) is out of reach. Hence, the term 'degenerate'. We shall come back to this issue in a moment's time. Following the solution strategy for evo-systems as it has been sketched in the previous section, we realise that the issue in the context of well-posedness is to establish estimates of the form

$$\begin{aligned}
\left\langle U|\left(\partial_0\eta + C^*C\right)U\right\rangle_{\varrho,0,0} &= \left\langle \eta^{1/2}U|\partial_0\eta^{1/2}U\right\rangle_{\varrho,0,0} + \left\langle CU|CU\right\rangle_{\varrho,0,0} \\
&\geq c_0 \left\langle U|U\right\rangle_{\varrho,0,0}, \\
\left\langle U|\left(\partial_0\eta + C^*C\right)^*U\right\rangle_{\varrho,0,0} &= \left\langle \eta^{1/2}U|\partial_0^*\eta^{1/2}U\right\rangle_{\varrho,0,0} + \left\langle CU|CU\right\rangle_{\varrho,0,0} \\
&\geq c_0 \left\langle U|U\right\rangle_{\varrho,0,0}.
\end{aligned}$$

Since, due to the density of elements with compact time support in $D\left(\partial_0\right)$,

$$\left\langle \eta^{1/2}U|\partial_0^*\eta^{1/2}U\right\rangle_{\varrho,0,0} = \left\langle \eta^{1/2}U|\partial_0\eta^{1/2}U\right\rangle_{\varrho,0,0} = \varrho\left|\eta^{1/2}U\right|_{\varrho,0,0}^2$$

we only need to consider one of the estimates, thus we need to have

$$\varrho\left|\eta^{1/2}U\right|_{\varrho,0,0}^2 + \left|CU\right|_{\varrho,0,0}^2 \geq c_0\left|U\right|_{\varrho,0,0}^2, \tag{10}$$

which again emphasises that the Hilbert space we choose U from cannot contain the space $H_{\varrho,0}\left(\mathbb{R}, N\left(\eta\right) \cap N\left(C\right)\right)$.

It is the aim of this section to show that restricting our attention to the orthogonal complement of $N\left(\eta\right) \cap N\left(C\right)$ as well as assuming an estimate of the type (10) for U attaining values in

$$H_0 := \left(N\left(\eta\right) \cap N\left(C\right)\right)^{\perp} \subseteq H$$

leads to well-posedness and causality with state space H_0. Since both η and C are operators acting on the 'spatial' Hilbert space, only, it is possible to provide an equivalent formulation, which only uses the spatial scalar product.

Proposition 3.2. *Let C and η be as above. Then the following conditions are equivalent:*

1. *There exists $\varrho > 0$ and $c_0 > 0$ such that for all $U \in H_{\varrho,0}(\mathbb{R}, H_0 \cap D(C))$ we have*

$$\varrho \left|\eta^{1/2}U\right|^2_{\varrho,0,0} + |CU|^2_{\varrho,0,0} \geq c_0 |U|^2_{\varrho,0,0}.$$

2. *There exists $c_0 > 0$ such that for all $U \in H_0 \cap D(C)$ we have*

$$\left|\eta^{1/2}U\right|^2_H + |CU|^2_X \geq c_0 |U|^2_H.$$

Proof. An easy density argument implies that the second inequality implies the first one with $\varrho = 1$ and the same $c_0 > 0$. Thus, it remains to show the converse implication. For this, note that with $\varrho_* := \max\{\varrho, 1\}$ we have for all $U \in H_{\varrho,0}(\mathbb{R}, H_0 \cap D(C))$

$$\left|\eta^{1/2}U\right|^2_{\varrho,0,0} + |CU|^2_{\varrho,0,0} \geq \frac{c_0}{\varrho_*} |U|^2_{\varrho,0,0}.$$

Let $x \in H_0 \cap D(C)$. Using the latter inequality for $U(t) := \exp(\varrho t)x$ for $t \in [0, 1]$ and $U(t) = 0$ for $t < 0$ and $t > 1$, we infer the desired inequality. □

Next, note that, since elements in $N(\eta) \cap N(C)$ are orthogonal to $R(C^*)$ and $R(\eta)$ and if C and consequently C^* are operators with closed range we may reduce the operator C to $H_0 := (N(\eta) \cap N(C))^{\perp}$. Indeed, as we shall see next, the operator

$$\begin{aligned} C_0 : D(C) \cap H_0 \subseteq H_0 &\to X \\ u &\mapsto Cu \end{aligned}$$

retain the closedness of the range and is also still densely defined. With $\iota_{H_0 \to H}$ denoting the canonical isometric embedding of H_0 as a subspace of H, we have

$$C_0 = C\iota_{H_0 \to H}.$$

The mentioned properties of C_0 are proved next.

Lemma 3.3. *The operator C_0 is closed, densely defined and has a closed range.*

Proof. It is

$$H = H_0 \oplus H_0^{\perp}$$

and

$$H_0^{\perp} = N(\eta) \cap N(C) \subseteq N(C) \subseteq D(C)$$

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and so

$$D(C) = (D(C) \cap H_0) \oplus H_0^{\perp}.$$

The density of $D(C_0) = D(C) \cap H_0$ in H_0 now follows from the continuity of the orthogonal projector P_{H_0} onto H_0. Indeed, let $x_\infty \in H_0$. Then we find a sequence $(x_n)_n$ in $D(C)$ such that $x_n \to x_\infty$. Thus, also $P_{H_0}x_n \to P_{H_0}x_\infty = x_\infty$. Since, $(1-P_{H_0})x_n \in D(C)$ for all $n \in \mathbb{N}$ by the argument above, we infer that $(P_{H_0}x_n)_{n\in\mathbb{N}}$ is, in fact, a sequence in $D(C_0)$ showing that C_0 is densely defined.

Since $C_0 = C \cap (H_0 \oplus X)$, where we identify the operators with their graphs, the closedness of C_0 follows.

We are left with showing the closedness of the range of C_0. For this, let z be a sequence in H_0 such that $C_0 z = Cz \to w_\infty$ for some $w_\infty \in X$. Then by the closedness of the range of C we have

$$Cx_* = w_\infty$$

for some $x_* \in D(C)$. Since

$$w_\infty = Cx_* = CP_{H_0}x_* = C_0\left(P_{H_0}x_*\right),$$

we confirm that $w_\infty \in R(C_0)$ finally proving that indeed closedness of the range is preserved. □

Lemma 3.4. *We have*

$$C_0^* = \overline{\iota_{H_0\to H}^* C^*}.$$

Proof. Since C_0 is densely defined we obtain the assertion with [27, Theorem 1.2]. □

Thus, we are led to study the reduced – by construction injective – operator

$$\partial_0\eta_0 + C_0^*C_0 = \iota_{H_0\to H}^*\left(\partial_0\eta + C^*C\right)\iota_{H_0\to H}$$

with $\eta_0 := \iota_{H_0\to H}^*\eta\iota_{H_0\to H}$ now being selfadjoint in H_0.

To proceed with our approach we need to assume moreover for some $c_1 > 0$

$$\left|\eta_0^{1/2}U\right|_{H_0}^2 + |C_0U|_X^2 \geq c_1 |U|_{H_0}^2 \tag{11}$$

for all $U \in D(C_0)$.

Remark 3.5. Note that (11) is equivalent to the inequalities asserted in Proposition 3.2. For this, we observe that for all $U \in H_0 \cap D(C) = D(C_0)$ we have $C_0U = CU$. Moreover, for $U \in H_0$ we compute

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$$
\begin{aligned}
\left|\eta_0^{1/2}U\right|_{H_0}^2 &= \langle \eta_0^{1/2}U|\eta_0^{1/2}U\rangle_{H_0} \\
&= \langle U|\eta_0 U\rangle_{H_0} \\
&= \langle U|\iota_{H_0\to H}^*\eta\iota_{H_0\to H}U\rangle_{H_0} \\
&= \langle \iota_{H_0\to H}U|\eta\iota_{H_0\to H}U\rangle_{H_0} \\
&= \langle U|\eta U\rangle_H \\
&= \left|\eta^{1/2}U\right|_H^2,
\end{aligned}
$$

which yields the desired equivalence.

The latter assumption leads to well-posedness of the evo-system under consideration in the state space H_0.

Proposition 3.6. *Assume* (11). *Then* $\overline{\partial_0\eta_0 + C_0^*C_0}$ *is continuously invertible in* $H_{\varrho,0}(\mathbb{R}, H_0)$ *for all* $\varrho \geq 1$.

Proof. In order to prove this theorem, it suffices to apply Theorem 2.1 to $M_0 = \eta_0$ and $A = C_0^*C_0$ note that it is easy to see that the positive definiteness conditions of Theorem 2.1 are then satisfied due to assumption (11). □

The next result relates the solution U of

$$\left(\overline{\partial_0\eta_0 + C_0^*C_0}\right) U = f \tag{12}$$

or

$$\left(\eta + C_0^*C_0\partial_0^{-1}\right) U = \partial_0^{-1} f \tag{13}$$

in H_0 to the equation (9).

Proposition 3.7. *Assume* (11). *Let* $U := \left(\overline{\partial_0\eta_0 + C_0^*C_0}\right)^{-1} F$ *for some* $F \in H_{\varrho,0}(\mathbb{R}, H_0)$ *for some* $\varrho \geq 1$. *Then* U *satisfies* (9).

Proof. Since ∂_0^{-1} commutes with $\left(\overline{\partial_0\eta_0 + C_0^*C_0}\right)^{-1}$, we infer that (13) is in fact a consequence of (12). Moreover, we read off that

$$\partial_0^{-1}U \in D\left(C_0^*C_0\right)$$

and so in particular

$$C_0^*C_0\partial_0^{-1}U = C^*C\partial_0^{-1}U \tag{14}$$

and

$$\eta_0 U = \eta U. \tag{15}$$

Indeed, since

$$\begin{aligned}\left\langle \phi | C_0^* C_0 \partial_0^{-1} U \right\rangle_H &= \left\langle C_0 \phi | C_0 \partial_0^{-1} U \right\rangle_X \\ &= \left\langle C \phi | C \partial_0^{-1} U \right\rangle_X\end{aligned}$$

and

$$\langle \phi | \eta_0 U \rangle_H = \langle \phi | \eta U \rangle_H$$

for all $\phi \in D(C_0) = D(C) \cap H_0$, as well as

$$\left\langle \psi | C_0^* C_0 \partial_0^{-1} U \right\rangle_H = \left\langle C \psi | C \partial_0^{-1} U \right\rangle_X = 0$$

and

$$\langle \psi | \eta_0 U \rangle_H = \langle \psi | \eta U \rangle_H = 0$$

for $\psi \in H_0^{\perp} = N(C) \cap N(\eta)$, we have

$$\left\langle V | C_0^* C_0 \partial_0^{-1} U \right\rangle_H = \left\langle C V | C \partial_0^{-1} U \right\rangle_X$$

and

$$\langle V | \eta_0 U \rangle_H = \langle V | \eta U \rangle_H$$

for all $V \in D(C)$. Thus, we read off (15) and

$$C \partial_0^{-1} U \in D(C^*)$$

as well as

$$C_0^* C_0 \partial_0^{-1} U = C^* C \partial_0^{-1} U,$$

that is, (14). Letting now

$$V := -C \partial_0^{-1} U$$

we obtain

$$\begin{aligned} V + C \partial_0^{-1} U &= 0, \\ \eta U - C^* V &= \partial_0^{-1} F. \end{aligned} \tag{16}$$

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Thus, we find that

$$\begin{aligned} \partial_0 V + CU &= 0, \\ \eta U - C^* V &= \partial_0^{-1} F, \end{aligned} \tag{17}$$

and so also

$$\partial_0 \eta U + C^* C U = F$$

hold in a distributional sense. In particular, this confirms that we have indeed solved the original equation (9). □

Remark 3.8. For $F \in H_{\varrho,0}(\mathbb{R}; H_0)$ we set $U := (\overline{\partial_0 \eta_0 + C_0^* C_0})^{-1} F \in H_{\varrho,0}(\mathbb{R}; H)$. Then, there exists a sequence $(U_n)_{n\in\mathbb{N}}$ in $H_{\varrho,1}(\mathbb{R}; D(C_0^* C_0))$ such $U_n \to U$ and $(\partial_0 \eta_0 + C_0^* C_0) U_n \to F$. For $n \in \mathbb{N}$ we estimate

$$\begin{aligned} |C_0 U_n|_{\varrho,0,0}^2 &\leq \varrho \langle \eta_0 U_n | U_n \rangle_{\varrho,0,0} + \langle C_0^* C_0 U_n | U_n \rangle_{\varrho,0,0} \\ &= \langle (\partial_0 \eta_0 + C_0^* C_0) \, U_n | U_n \rangle_{\varrho,0,0} \end{aligned} \tag{18}$$

and since the right-hand side is bounded, we infer that (up to a subsequence) $C_0 U_n \rightharpoonup w$ for some $w \in H_{\varrho,0}(\mathbb{R}; X)$. By the closedness (and hence, weak closedness) of C_0, we derive that $U \in D(C_0)$ and $w = C_0 u$. In particular $|C_0 U|_{\varrho,0,0} \leq \liminf_{n\to\infty} |C_0 U_n|_{\varrho,0,0}$. Letting n tend to infinity in (18) we get

$$\begin{aligned} \langle F | U \rangle_{\varrho,0,0} &= \varrho \langle \eta_0 U | U \rangle_{\varrho,0,0} + \lim_{n\to\infty} \langle C_0 U_n | C_0 U_n \rangle_{\varrho,0,0} \\ &\geq \varrho \langle \eta_0 U | U \rangle_{\varrho,0,0} + \langle C_0 U | C_0 U \rangle_{\varrho,0,0} \\ &= \frac{1}{2} \left(2\varrho \langle \eta_0 U | U \rangle_{\varrho,0,0} + \langle C_0 U | C_0 U \rangle_{\varrho,0,0} \right) + \frac{1}{2} \langle C_0 U | C_0 U \rangle_{\varrho,0,0} \\ &\geq \frac{1}{2} c_1 |U|_{\varrho,0,0}^2 + \frac{1}{2} |C_0 U|_{\varrho,0,0}^2 \\ &\geq \widetilde{c}_1 |U|_{\varrho,0,1}^2 \end{aligned}$$

with $\widetilde{c}_1 := \frac{1}{2} \min\{1, c_1\}$. Estimating the left hand side by $|F|_{\varrho,0,-1} |U|_{\varrho,0,1}$ we end up with

$$|U|_{\varrho,0,1} \leq \frac{1}{\widetilde{c}_1} |F|_{\varrho,0,-1}.$$

Thus, the solution operator S attains values in $H_{\varrho,0}(\mathbb{R}; D(C_0))$ and can be extended continuously to $H_{\varrho,0}(\mathbb{R}; D(C_0^*)')$. This is a refinement of the earlier observation in the general case, see Remarks 2.2 and 3.1.

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Example 3.9. As a quick example, it might be illustrative to apply the observations in the previous remark to the (non-degenerate) case of the heat equation. So, take $\eta = 1$ to be the identity in $H = L^2(\Omega)$ and $C = \mathring{\text{grad}}$ with $D(C) = H_0^1(\Omega)$. Then the previous remark confirmed a solution theory for the heat equation $(\partial_0 - \Delta)\, U = F$ for right-hand sides F taking values in $H^{-1}(\Omega)$. By the general theory developed here, we obtain that U assumes values even in $H_0^1(\Omega)$.

For sake of later reference let us summarise the core of the above observations in the following theorem.

Theorem 3.10. *Let $C : D(C) \subseteq H \to X$ be a closed densely defined linear operator with closed range and such that (11) holds. Then, for every $F \in H_{\varrho,0}(\mathbb{R}, D(C_0^*)')$ there is a unique (weak) solution $U \in H_{\varrho,0}(\mathbb{R}, D(C_0))$ of (13) or equivalently of the system (16). Moreover the solution operator $S : H_{\varrho,0}(\mathbb{R}, H_0) \to H_{\varrho,0}(\mathbb{R}, D(C_0))$ is continuous ($|\cdot|_{\varrho,0,1}$ denotes the norm of $H_{\varrho,0}(\mathbb{R}, D(C_0))$ and causal in the sense that*

$$\left|\chi_{]-\infty,a]} SF\right|_{\varrho,0,1} \le C_1 \left|\chi_{]-\infty,a]} F\right|_{\varrho,0,-1}$$

for some positive C_1 uniformly in $a \in \mathbb{R}$ and $F \in H_{\varrho,0}(\mathbb{R}, H_0)$ as long as $\varrho \in\,]0, \infty[$ is sufficiently large.

Proof. The result largely follows from our previous considerations. The sharper regularity statement $U \in H_{\varrho,0}(\mathbb{R}, D(C_0))$ and the sharper continuous dependence statement follows by Remark 3.8. The claim of causality follows from a slight refinement of the estimates along the reasoning of Remark 3.8. Indeed, we have for all sufficiently large $\varrho \in\,]0, \infty[$

$$\begin{aligned}
&\left|\chi_{]-\infty,a]} U\right|_{\varrho,0,1} \left|\chi_{]-\infty,a]} \left(\partial_0 \eta_0 + C_0^* C_0\right) U\right|_{\varrho,0,-1} \\
&\ge \left\langle \chi_{]-\infty,a]} U \middle| \left(\partial_0 \eta_0 + C_0^* C_0\right) U \right\rangle_{\varrho,0,0} \\
&= \varrho \left|\eta_0^{1/2} \chi_{]-\infty,a]} U\right|_{\varrho,0,0}^2 + \frac{1}{2} \left|\eta_0^{1/2} U(a)\right|_0^2 \exp(-2\varrho a) + \left|C_0 \chi_{]-\infty,a]} U\right|_{\varrho,0,0}^2 \\
&\ge \frac{1}{2} c_1 \left|\chi_{]-\infty,a]} U\right|_{\varrho,0,0}^2 + \frac{1}{2} \left|C_0 \chi_{]-\infty,a]} U\right|_{\varrho,0,0}^2 \\
&\ge \frac{1}{2} c_1 \left(\left|\chi_{]-\infty,a]} U\right|_{\varrho,0,0}^2 + \left|C_0 \chi_{]-\infty,a]} U\right|_{\varrho,0,0}^2 \right) = \frac{1}{2} c_1 \left|\chi_{]-\infty,a]} U\right|_{\varrho,0,1}^2
\end{aligned}$$

for $U \in H_{\varrho,1}(\mathbb{R}; D(C_0^* C_0))$ from which

$$\left|\chi_{]-\infty,a]} U\right|_{\varrho,0,1} \le \frac{2}{c_1} \left|\chi_{]-\infty,a]} \left(\partial_0 \eta_0 + C_0^* C_0\right) U\right|_{\varrho,0,-1}$$

follows. The result then follows by continuous extension. □

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Note that the estimate obtained here is a slightly stronger causality estimate than available in the general case of Theorem 2.1.

Remark 3.11. Of course we also have (since $|\phi|_{\varrho,0,0} \leq |\phi|_{\varrho,0,1}$ for $\phi \in H_{\varrho,0}(\mathbb{R}, D(C_0))$)

$$|SF|_{D\left(\overline{\partial_0 \eta_0 + C_0^* C_0}\right)} \leq \sqrt{1 + C_1^2}\, |F|_{\varrho,0,0}\,.$$

We also note the resulting energy balance law for solutions of (9).

Theorem 3.12. *(Energy balance law) For a right-hand side $F \in H_{\varrho,1}(\mathbb{R}, H_0)$ with $F = 0$ on $[T_0, T_1]$ we have for the solution $U \in H_{\varrho,1}(\mathbb{R}, D(C_0))$*

$$\begin{aligned} &\frac{1}{2}\langle U|\eta U\rangle_H (T_1) + \int\limits_{[T_0,T_1]} \langle CU|CU\rangle_H = \\ &= \frac{1}{2}\langle U|\eta U\rangle_H (T_0)\,. \end{aligned}$$

Proof. For $F = 0$ on $[T_0, T_1]$ we have

$$\begin{aligned} 0 &= \int\limits_{[T_0,T_1]} \langle U|\partial_0 \eta U\rangle_H + \int\limits_{[T_0,T_1]} \langle CU|CU\rangle_H \\ &= \frac{1}{2}\langle U|\eta U\rangle_H (T_1) - \frac{1}{2}\langle U|\eta U\rangle_H (T_0) + \\ &+ \int\limits_{[T_0,T_1]} \langle CU|CU\rangle_H\,, \end{aligned}$$

where we have used the Sobolev embedding theorem to justify the integration by parts. Furthermore note that the time-derivative commutes with the solution operator. □

For later purposes we analyse the underlying Hilbert spaces

$$\begin{aligned} H &= H_0 \oplus H_0^\perp \\ H_0 &= (N(C) \cap N(\eta))^\perp \end{aligned}$$

further. For a Hilbert space K and a subspace $L \subseteq K$, we define

$$K \ominus L := K \cap L^\perp.$$

Lemma 3.13. *We have*

$$\begin{aligned} H_0 &= R(C^*) \oplus (N(C) \cap H_0) \\ &= R(C^*) \oplus \left(N(C) \cap \overline{R(\eta)}\right) \oplus \left((N(C) \cap H_0) \ominus \left(N(C) \cap \overline{R(\eta)}\right)\right). \end{aligned}$$

Proof. By the projection theorem we have

$$H = R(C^*) \oplus N(C).$$

Intersecting both sides with H_0 and using that $R(C^*) = N(C)^\perp \subseteq H_0$ we obtain the first decomposition. For the second one, we observe that $N(C) \cap \overline{R(\eta)}$ is a closed subspace of $N(C) \cap H_0$, since $\overline{R(\eta)} = N(\eta)^\perp \subseteq H_0$. Hence, by the projection theorem

$$N(C) \cap H_0 = \left(N(C) \cap \overline{R(\eta)}\right) \oplus \left((N(C) \cap H_0) \ominus \left(N(C) \cap \overline{R(\eta)}\right)\right)$$

yielding the second decomposition. □

Example 3.14. As a more elaborate illustrational example let us consider the solution to the linear part of the so-called "bidomain model"[3] used in cardiac electrophysiology, see [8]. For this let $\Omega \subseteq \mathbb{R}^d$ be open, bounded and connected satisfying the segment property. The equation in question is given by

$$\left(\partial_0 \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + C^* C\right) U = F$$

with some given data F taking values in the state space

$$H = L^2(\Omega) \oplus L^2(\Omega)$$

and

$$C = \begin{pmatrix} \sqrt{\sigma_1} & 0 \\ 0 & \sqrt{\sigma_2} \end{pmatrix} \begin{pmatrix} \operatorname{grad} & 0 \\ 0 & \operatorname{grad} \end{pmatrix}$$

with $\sigma_k \in L(L^2(\Omega, \mathbb{R}^d))$, $k \in \{1, 2\}$, selfadjoint and strictly positive definite with $D(C) = H^1(\Omega) \oplus H^1(\Omega)$ and $X = L^2(\Omega)^d \oplus L^2(\Omega)^d$ as well as

$$\eta = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Note that grad (and therefore also C) has closed range, as a standard contradiction argument using the compactness of the embedding $H^1(\Omega) \hookrightarrow L^2(\Omega)$ eventually proving a Poincare-type estimate shows; in fact we have

$$|u|_{L^2(\Omega)} \leq k\, |\operatorname{grad} u|_{L^2(\Omega, \mathbb{R}^d)} \tag{19}$$

[3] We are indebted to Ralph Chill for drawing our attention to this model.

for all $u \in D(\text{grad})$ with $\int_\Omega u = 0$ and some $k \geq 0$.

Next, we aim at applying our abstract findings. In particular, we need to establish the estimate in (11). For this, let us describe the reduced state space, H_0, first. We have

$$\begin{aligned} H_0 &= (N(\eta) \cap N(C))^\perp \\ &= \left\{ V \in L^2(\Omega) \oplus L^2(\Omega) \middle| V = \begin{pmatrix} u \\ -u \end{pmatrix} \text{ for some } u \in N(\text{grad}) \right\}^\perp \\ &= \left\{ V \in L^2(\Omega) \oplus L^2(\Omega) \middle| V = \alpha \begin{pmatrix} \chi_\Omega \\ -\chi_\Omega \end{pmatrix} \text{ for some } \alpha \in \mathbb{R} \right\}^\perp \\ &= \left\{ (W_1, W_2) \in L^2(\Omega) \oplus L^2(\Omega) \middle| \int_\Omega W_1(x)dx = \int_\Omega W_2(x)dx \right\}, \end{aligned}$$

where in the second last equality we have used that Ω is connected in order to have that $N(\text{grad}) = \operatorname{span} \chi_\Omega$. According to our abstract theory we need an estimate of the form

$$\left|P_{R(\eta)} U\right|_H^2 + |C_0 U|_X^2 \geq c_* \, |U|_H^2 \, ,$$

holding for all

$$U \in D(C_0) \subseteq H_0 = R(C^*) \oplus (N(C) \cap R(\eta)) \oplus ((N(C) \cap H_0) \ominus (N(C) \cap R(\eta)))$$

for some $c_* > 0$ and where $P_{R(\eta)}$ denotes the projection onto the range $R(\eta) = \overline{R(\eta)}$ of η. Take $U = U_0 + U_1 + U_2$ in the sense of this orthogonal decomposition. First we note that

$$N(C) \cap R(\eta) = \left\{ \alpha \begin{pmatrix} \chi_\Omega \\ \chi_\Omega \end{pmatrix} \middle| \alpha \in \mathbb{R} \right\}$$

and

$$N(C) \cap H_0 = \left\{ \alpha \begin{pmatrix} \chi_\Omega \\ -\chi_\Omega \end{pmatrix} \middle| \alpha \in \mathbb{R} \right\},$$

so

$$U_2 = 0.$$

Thus, we infer that

$$H_0 = R(C^*) \oplus (N(C) \cap R(\eta)) \, .$$

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Moreover, by (19) and the assumptions on σ_k, we find $c > 0$ satisfying

$$c\,|U_0|_H \leq |CU_0|_H \quad (U_0 \in R(C^*) \cap D(C)).$$

Hence, for all $U \in D(C_0)$ we have with $U = U_0 + U_1$ for uniquely determined $U_0 \in R(C^*) \cap D(C)$ and $U_1 \in N(C) \cap R(\eta)$ that

$$\begin{aligned}
\left|P_{R(\eta)}U\right|_H^2 + |C_0U|_X^2 &= \left|P_{R(\eta)}U_0 + U_1\right|_H^2 + |C_0U_0|_X^2 \\
&= \left|P_{R(\eta)}U_0\right|_H^2 + |U_1|_H^2 + 2\langle P_{R(\eta)}U_0, U_1\rangle_H + |C_0U_0|_X^2 \\
&= \left|P_{R(\eta)}U_0\right|_H^2 + |U_1|_H^2 + 2\langle U_0, U_1\rangle_H + |C_0U_0|_X^2 \\
&= \left|P_{R(\eta)}U_0\right|_H^2 + |U_1|_H^2 + |C_0U_0|_X^2 \\
&\geq |U_1|_H^2 + c^2\,|U_0|_H^2 .
\end{aligned}$$

Thus, we found as desired

$$\left|P_{R(\eta)}U\right|_H^2 + |C_0U|_X^2 \geq \min\left\{1, c^2\right\}|U|_H^2 .$$

Therefore, well-posedness of the evo-system is implied by Theorem 3.10. Moreover, since $\eta\left[R\left(C^*\right)\right] \subseteq R\left(C^*\right)$ the problem can be further reduced to an evo-system in the subspace $R\left(C^*\right)$ and an ordinary differential equation in $N\left(C\right) \cap R\left(C\right)$. This insight might be useful, when dealing with problems in the light of homogenisation, see e.g. [39, Theorem 4.7] for this.

4. Application to a degenerate evo-system associated with the eddy current problem

In this section, we shall now turn to our main application. Consider the system

$$\begin{aligned}
\sigma\mathrm{E} - \operatorname{curl}\mathrm{H} &= -\mathrm{J} \\
\partial_0\mu\mathrm{H} + \mathring{\operatorname{curl}}\mathrm{E} &= K
\end{aligned} \tag{20}$$

in an arbitrary non-empty open bounded set $\Omega \subseteq \mathbb{R}^3$ with connected boundary. We will require more regularity properties for Ω, in the following.

After having specified the constituents of this system of two equations, we shall reformulate the system in order to be in a position to apply our general well-posedness theorem. This reformulation will then be studied and related to the system (20). We shall show that the solution for the reformulation yields a solution for the equation, we started out with. In view of the particular situation of the eddy current model at hand, though this being a natural property to ask for, it appears to have been overlooked in the literature so far.

We specify the operators occurring in (20) next. The operator $\mathring{\operatorname{curl}}$ denotes the closure of the classical vector analytic operation curl defined on C_∞-vector fields with compact support in Ω considered as a mapping in $L^2\left(\Omega, \mathbb{R}^3\right)$; that is,

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$$\mathring{\text{curl}}\colon D(\mathring{\text{curl}}) \subseteq L^2(\Omega, \mathbb{R}^3) \to L^2(\Omega, \mathbb{R}^3)$$

given by

$$\begin{aligned} &\phi \in D(\mathring{\text{curl}}), \psi = \mathring{\text{curl}}\phi \\ \iff &\text{There exists a sequence } (\phi_n)_n \text{ in } \mathring{C}_\infty(\Omega, \mathbb{R}^3) \text{ such that} \\ &\phi_n \to \phi \text{ and } \begin{pmatrix} \partial_2\phi_{n,3} - \partial_3\phi_{n,2} \\ \partial_3\phi_{n,1} - \partial_1\phi_{n,3} \\ \partial_1\phi_{n,2} - \partial_2\phi_{n,1} \end{pmatrix} \to \psi \text{ in } L^2(\Omega, \mathbb{R}^3) \text{ as } n \to \infty. \end{aligned}$$

We emphasise that for smooth Ω belonging to the domain of $D(\mathring{\text{curl}})$ is equivalent to the (classical) vanishing of tangential component at the boundary. We define

$$\text{curl } := \left(\mathring{\text{curl}}\right)^*,$$

which is the so-called weak curl-derivative in $L^2\left(\Omega, \mathbb{R}^3\right)$. The equations can now be written as a block operator matrix system as

$$\left(\partial_0 \begin{pmatrix} 0 & 0 \\ 0 & \mu \end{pmatrix} + \begin{pmatrix} \sigma & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -\text{curl} \\ \mathring{\text{curl}} & 0 \end{pmatrix}\right) \begin{pmatrix} \text{E} \\ \text{H} \end{pmatrix} = \begin{pmatrix} -\text{J} \\ \text{K} \end{pmatrix}. \tag{21}$$

Remark 4.1. It might seem unphysical to assume a non-zero source term K on the right-hand side. In the formulation of evolutionary equations in particular concerning the reformulation of appropriate initial value problems as evolutionary equations with particular right-hand side it so happens that K might be non-zero. We refer to Remark 2.3(b) and to [33, Example 9.44] for the details.

Furthermore, assume that

$$\mu : L^2(\Omega) \to L^2(\Omega)$$

is selfadjoint and strictly positive definite. The assumption on $\sigma : L^2(\Omega) \to L^2(\Omega)$ is less standard. We shall assume a certain degree of degeneracy, which is specified in the following assumption. For convenience of the reader we denote the vector analytical operators defined on the whole of Ω by curl, grad, and div (and the respective ones with full homogeneous boundary conditions by $\mathring{\text{curl}}$, $\mathring{\text{grad}}$, and $\mathring{\text{div}}$). For operators defined on other domains Ω_c, we shall use this domain as an index to refer to these operators such as for example grad_{Ω_c} (the operator $\mathring{\text{grad}}_{\Omega_c}$ is the operator acting as grad_{Ω_c} with domain restricted to $H_0^1(\Omega_c)$).

Remark 4.2. As Ω is bounded, we have that $R(\mathring{\text{grad}})$ is closed by Poincaré's inequality. Moreover, $R(\mathring{\text{grad}}) \subseteq N(\mathring{\text{curl}})$ and thus, the projection theorem gives

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$$N(\mathring{\text{curl}}) = R(\mathring{\text{grad}}) \oplus \Big(R(\mathring{\text{grad}})^{\perp} \cap N(\mathring{\text{curl}}) \Big)$$
$$= R(\mathring{\text{grad}}) \oplus \Big(N(\text{div}) \cap N(\mathring{\text{curl}}) \Big) .$$

The space

$$\mathcal{H}_{D,\Omega} := N(\text{div}) \cap N(\mathring{\text{curl}})$$

is known as the space of harmonic Dirichlet fields. Since the boundary of Ω is connected, it follows that $\mathcal{H}_{D,\Omega} = \{0\}$ by [23, Theorem 1] and thus,

$$N(\mathring{\text{curl}}) = R(\mathring{\text{grad}}). \tag{22}$$

Assumption 4.3. Let $\Omega_c \subseteq \Omega$ be open. Moreover, assume that $\overline{\Omega}_c \subseteq \Omega$ and that Ω_c has a (3-dimensional) Lebesgue null set as topological boundary and is such that Ω_c has finitely many connected components and the connected components of Ω_c have disjoint closures. We also assume that Ω_c is such that

$$D(\text{grad}_{\Omega_c}) = \chi_{\Omega_c} \Big[D(\mathring{\text{grad}}) \Big] . \tag{23}$$

Let

$$\widetilde{\sigma} : L^2 \left(\Omega_c, \mathbb{R}^3 \right) \to L^2 \left(\Omega_c, \mathbb{R}^3 \right)$$

such that $\widetilde{\sigma}$ is strictly positive definite. We shall assume that σ is degenerate in the sense that[4]

$$\sigma = \iota_{\Omega_c} \widetilde{\sigma}\, \iota_{\Omega_c}^* .$$

We note here that (23) indeed is a regularity requirement for Ω_c. In maybe more familiar terms, this requirement equivalently reads as

$$H^1(\Omega_c) = \{\phi \in L^2(\Omega_c) |\text{ there is } \widetilde{\phi} \in H_0^1(\Omega) \text{ such that } \chi_{\Omega_c} \widetilde{\phi} = \phi\}.$$

Remark 4.4. We comment some more on the condition (23). Since for every open set $\Omega_c \subseteq \mathbb{R}^3$, a function $u \in H_0^1(\Omega_c)$ if and only if

[4] In this case

$$\chi_{\Omega_c} := \iota_{\Omega_c} \iota_{\Omega_c}^*$$

is the orthogonal projector $P_{R(\sigma)}$ from $H = L^2 \left(\Omega, \mathbb{R}^3 \right)$ onto the closed linear subspace $R(\sigma) = \iota_{\Omega_c} \left[L^2 \left(\Omega_c, \mathbb{R}^3 \right) \right]$ (the canonical embedding ι_{Ω_c} of $L^2 \left(\Omega_c, \mathbb{R}^3 \right)$ into $L^2 \left(\Omega, \mathbb{R}^3 \right)$ is via "extension by zero").

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$$\widetilde{u} := \begin{cases} u & \text{on } \Omega_c \\ 0 & \text{else} \end{cases} \in H^1(\mathbb{R}^3),$$

the requirement (23) is equivalent of Ω_c being an H^1-extension domain (see [15] for the definition). The Calderon–Stein theorem asserts that Lipschitz domains are H^1-extension domains. An improvement of this result can be found in [16], which holds for so-called uniform domains allowing the boundary of Ω_c to have any Hausdorff dimension strictly less than 3. A necessary criterion, however, is due to [15, Theorem 2] the *measure density condition*; that is, there exists $c > 0$ such that for all $x \in \Omega_c$ and $0 < r \leq 1$ we have

$$\lambda(B(x,r) \cap \Omega_c) \geq cr^3,$$

where $\lambda(\cdot)$ denotes the Lebesgue measure in $\mathbb{R}^3$. Thus, all domains Ω_c failing this condition are no H^1-extension domains. Furthermore, if Ω_c has cracks of big enough Hausdorff-dimension (see e.g. [18] for a two-dimensional setting), Ω_c is no H^1-extension domain.

We record an elementary consequence of the assumptions on σ.

Proposition 4.5. *Assume Assumption 4.3 to be in effect. Then*

$$\begin{aligned} R(\sigma) &= R(\chi_{\Omega_c}) = L^2(\Omega_c, \mathbb{R}^3), \\ N(\sigma) &= R(1 - \chi_{\Omega_c}) = L^2(\Omega, \mathbb{R}^3) \ominus L^2(\Omega_c, \mathbb{R}^3), \\ &= L^2(\Omega \setminus \overline{\Omega_c}, \mathbb{R}^3), \end{aligned}$$

where $L^2(\Omega_c, \mathbb{R}^3)$, $L^2(\Omega \setminus \overline{\Omega_c}, \mathbb{R}^3)$ *are considered as subspaces of* $L^2(\Omega, \mathbb{R}^3)$ *via extension by zero.*

For the transcription of (20) into a problem of the form (9), we need to warrant the closed range condition first. This, in turn, is a regularity requirement for Ω:

Assumption 4.6. Let Ω be such that $\mathring{\text{curl}}$ and consequently its adjoint curl have closed range:

$$R\left(\mathring{\text{curl}}\right), \ R(\text{curl}) \ \text{closed}. \tag{24}$$

Remark 4.7. A closed range requirement is the fundamental property of linear equation theory (see e.g. [38] for a corresponding result in elliptic theory) and linear operator equations with an operator having closed range are therefore, since the beginning of last century also referred to as *normally solvable*. That for exterior domains or for $\mathbb{R}^3$ the differential operators (without or with associated homogeneous boundary condition) grad, $\mathring{\text{grad}}$, curl, $\mathring{\text{curl}}$, div, $\mathring{\text{div}}$ have no closed range in an L^2-setting, can be shown by approximations of the regularised fundamental solution of the scalar or vectorial Laplacian.

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Note that for grad, $\mathring{\text{grad}}$, the closedness of the range is equivalent (this equivalence is due to the closed graph theorem, see e.g. [38, Remarks 3.2]) to Poincaré's inequality, which holds for Ω of bounded width, in particular for pipes and slabs, where Rellich's selection theorem fails. For $R(\text{curl})$ and $R\left(\mathring{\text{curl}}\right)$ closedness has so far only been obtained via a compact embedding result. Open subsets of Riemannian manifold allowing for such a compact embedding result have been described in [29], asking for Ω to satisfy only rather mild conditions (e.g., strictly weaker than $C^{1,1}$-domains and particularly not allowing for Gaffney's inequality to hold). We shall particularly refer to [6] for other boundary conditions.

For later use, we shall further record the last two remaining regularity properties needed for our well-posedness theorem to apply.

Assumption 4.8. Assume Assumption 4.3 to be in effect. We shall assume that

$$N(\mathring{\text{curl}}) \cap L^2(\Omega \setminus \overline{\Omega}_c, \mathbb{R}^3) = N(\mathring{\text{curl}}_{\Omega\setminus\overline{\Omega}_c}) \text{ and}$$
$$N(\mathring{\text{curl}}) \cap L^2(\Omega_c, \mathbb{R}^3) = N(\mathring{\text{curl}}_{\Omega_c}).$$

Moreover, we suppose that

$$R(\text{grad}_{\Omega_c}) \text{ is closed.}$$

Remark 4.9. Assumption 4.8 is another (boundary) regularity property. For this to confirm, we realise that any $\phi \in D(\mathring{\text{curl}}_{\Omega\setminus\overline{\Omega}_c})$ extended by zero to the whole of Ω satisfies $\phi \in D(\mathring{\text{curl}})$. Thus, in this sense, $N(\mathring{\text{curl}}) \cap L^2(\Omega \setminus \overline{\Omega}_c, \mathbb{R}^3) \supseteq N(\mathring{\text{curl}}_{\Omega\setminus\overline{\Omega}_c})$. For the other inclusion the equality

$$D(\mathring{\text{curl}}) \cap L^2(\Omega \setminus \overline{\Omega}_c, \mathbb{R}^3) = \{\phi \in D(\mathring{\text{curl}}) | \phi = 0 \text{ on } \overline{\Omega}_c\} = D(\mathring{\text{curl}}_{\Omega\setminus\overline{\Omega}_c})$$

is sufficient. If for instance, $\Omega \setminus \overline{\Omega}_c$ satisfies the segment property, the desired equality holds. The second equation and the third property in the assumptions are fulfilled, if, for instance, Ω_c has the segment property. We refer to Remark 4.7 for the limitations of the closed range requirement.

We are now in the position to state the setting for the application of Theorem 3.10. We put

$$\begin{aligned} H = X &= L^2\left(\Omega, \mathbb{R}^3\right), \\ C\colon D(C) \subseteq H &\to X, \\ E &\mapsto \mu^{-1/2}\mathring{\text{curl}} E, \\ D(C) &= D(\mathring{\text{curl}}), \\ \eta &= \sigma. \end{aligned} \tag{25}$$

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Proposition 4.10. *Let $\Omega \subseteq \mathbb{R}^3$ be open with connected boundary. Assume Assumption 4.3, 4.6, 4.8, to be in effect. Then C and η as given in (25) satisfy the assumptions in Theorem 3.10.*

The proof of Proposition 4.10 requires a lot of preparations. The main issue is of course to prove inequality (11) under the current assumptions. Indeed, note that since μ is selfadjoint and a topological isomorphism, we easily realise that C is densely defined and closed. Moreover, we obtain $C^* = \operatorname{curl} \mu^{-1/2}$ from which we read off that

$$R(C^*) = R(\operatorname{curl}),$$

which is assumed to be closed by Assumption 4.6. Thus, we are left with showing (11). Before, however, doing so, we reason, why it makes sense to look at the setting (25) for solving (20).

Remark 4.11. Using the assumptions of Proposition 4.10 and using the notation introduced in the previous section, we are led to the evo-system

$$\partial_0 \eta_0 u + C_0^* C_0 u = -j \in H_{\varrho,0}\left(\mathbb{R}, H_0\right),$$

with

$$j := \mathrm{J} - C_0^* \partial_0^{-1} \mu^{-1/2} \mathrm{K},$$

where

$$\mathrm{J} \in H_{\varrho,0}\left(\mathbb{R}, H_0\right).$$

Hence, with

$$\mathrm{E} := \partial_0 u$$

and

$$\mathrm{H} := -\partial_0^{-1} \mu^{-1/2} \left(C_0 \mathrm{E} - \mu^{-1/2} \mathrm{K} \right)$$

we recover

$$\sigma \mathrm{E} - C^* \mu^{1/2} \mathrm{H} = -\mathrm{J}$$
$$\partial_0 \mu^{1/2} \mathrm{H} + C \mathrm{E} = \mu^{-1/2} \mathrm{K}$$

or

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$$\begin{aligned}\sigma \mathrm{E} - \operatorname{curl} \mathrm{H} &= -\mathrm{J},\\ \partial_0 \mu \mathrm{H} + \mathring{\operatorname{curl}}\, \mathrm{E} &= \mathrm{K},\end{aligned}$$

which is (20). Note that the argument just presented is an incarnation of Proposition 3.7, which in turn yields the solvability of the system, we started out with.

To demonstrate (11) we first recall Lemma 3.13. In particular, we have

$$\begin{aligned} H_0 &= (N(C) \cap N(\eta))^{\perp} \\ &= R(C^*) \oplus H_1 \oplus H_2, \text{ where} \\ H_1 &= N(C) \cap R(\eta) \text{ and} \\ H_2 &= (N(C) \cap H_0) \ominus (N(C) \cap R(\eta)). \end{aligned} \tag{26}$$

In the following, we describe these spaces more explicitly. Throughout, we shall assume that the assumptions of Proposition 4.10 are in effect. For the formulation of Lemma 4.14, we recall for an open set $\mathcal{O} \subseteq \mathbb{R}^3$

$$\mathcal{H}_{D,\mathcal{O}} = N(\operatorname{div}_{\mathcal{O}}) \cap N(\mathring{\operatorname{curl}}_{\mathcal{O}}),$$

the space of harmonic Dirichlet fields in $\mathcal{O}$. In the following we will use the projection theorem in different spaces. For the sake of readability, we will use indices at the orthogonal complements in order to clarify, in which space we take the orthogonal complement.

In order to illustrate the following findings, we recall one of the main results in [23], namely the computation of the dimension of the harmonic Dirichlet fields. For this let $\mathcal{O} \subseteq \mathbb{R}^3$ be open bounded with continuous boundary. We denote

$$\mathrm{CC}_{(b)}(\mathcal{O}) := \{z \subseteq \mathbb{R}^3 \setminus \overline{\mathcal{O}}; z \text{ (bounded) connected component}\}.$$

For $z \in \mathrm{CC}_b(\mathcal{O})$ let $\psi_z \in C_c^{\infty}(\mathbb{R}^3)$ such that

$$\psi_z(x) = \begin{cases} 1, & x \in z, \\ 0, & x \in \bigcup_{z' \in \mathrm{CC}(\mathcal{O}) \setminus \{z\}} z'. \end{cases}$$

Define $q_z := \operatorname{grad} \psi_z$ and $\phi_z := \pi_{\mathcal{H}_{D,\mathcal{O}}} q_z|_{\mathcal{O}}$, where $\pi_{\mathcal{H}_{D,\mathcal{O}}}$ denotes the $L^2(\mathcal{O})^3$-orthogonal projection onto $\mathcal{H}_{D,\mathcal{O}}$.

Theorem 4.12 *([23, Theorem 1]). Assume $\Omega \subseteq \mathbb{R}^3$ to be bounded with continuous boundary. Let $m \in \mathbb{N}$ be the number of connected components of $\mathbb{R}^3 \setminus \overline{\Omega}$. Then*

$$\dim \mathcal{H}_{D,\Omega} = m - 1.$$

More precisely, $(\phi_z)_{z \in \mathrm{CC}_b(\mathcal{O})}$ constitutes a basis for $\mathcal{H}_{D,\Omega}$.

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Remark 4.13. (a) In the particular case that Ω is a ball and Ω_c is an inscribed ball, the number of connected components of $\mathbb{R}^3 \setminus \left(\Omega \setminus \overline{\Omega_c}\right)$ is 2. Thus,

$$\dim \mathcal{H}_{D,\Omega\setminus\overline{\Omega_c}} = 1.$$

It is possible to compute this function by appropriately projecting the gradient of a function, which is identically 1 on Ω_c and 0 outside Ω. It is possible to compute such a solution numerically, by solving a variational problem. We refer to [22] for the details. In the situation of Ω_c being a ball, we also have that

$$\mathcal{H}_{D,\Omega_c} = \{0\}.$$

(b) Note that the construction principle to obtain a basis for the space of harmonic Dirichlet fields extends to other differential operators. For this, we also refer to [22] for the details using the machinery of Hilbert complexes.

Lemma 4.14. *The following equalities hold:*

$$\begin{aligned}
H_0 &= \left(N\left(\operatorname{div}_{\Omega\setminus\overline{\Omega_c}}\right) \cap \mathcal{H}_{D,\Omega\setminus\overline{\Omega_c}}^{\perp_{L^2(\Omega\setminus\overline{\Omega_c})}}\right) \oplus L^2\left(\Omega_c, \mathbb{R}^3\right), \\
H_0^{\perp_{L^2(\Omega)}} &= N(\mathring{\operatorname{curl}}_{\Omega\setminus\overline{\Omega_c}}), \\
H_1 &= N\left(\mathring{\operatorname{curl}}_{\Omega_c}\right) \\
&= \left(N\left(\operatorname{div}_{\Omega_c}\right) \cap \mathcal{H}_{D,\Omega_c}^{\perp_{L^2(\Omega_c)}}\right)^{\perp_{L^2(\Omega_c)}}, \qquad (27) \\
H_1^{\perp_{L^2(\Omega)}} &= \left(N\left(\operatorname{div}_{\Omega_c}\right) \cap \mathcal{H}_{D,\Omega_c}^{\perp_{L^2(\Omega_c)}}\right) \oplus L^2\left(\Omega \setminus \overline{\Omega_c}, \mathbb{R}^3\right), \\
H_2 &= N\left(\mathring{\operatorname{curl}}\right) \cap \left(\left(N\left(\operatorname{div}_{\Omega_c}\right) \cap \mathcal{H}_{D,\Omega_c}^{\perp_{L^2(\Omega_c)}}\right) \oplus \left(N\left(\operatorname{div}_{\Omega\setminus\overline{\Omega_c}}\right) \cap \mathcal{H}_{D,\Omega\setminus\overline{\Omega_c}}^{\perp_{L^2(\Omega\setminus\overline{\Omega_c})}}\right)\right).
\end{aligned}$$

Proof. Using Assumptions 4.8 and 4.3, we obtain

$$\begin{aligned}
H_0^{\perp_{L^2(\Omega)}} &= N(C) \cap N(\eta) \\
&= N(\mathring{\operatorname{curl}}) \cap L^2(\Omega \setminus \overline{\Omega}_c, \mathbb{R}^3) \\
&= N(\mathring{\operatorname{curl}}_{\Omega\setminus\overline{\Omega}_c}).
\end{aligned}$$

Since $\mathring{\operatorname{grad}}^*_{\Omega\setminus\overline{\Omega}_c} = -\operatorname{div}_{\Omega\setminus\overline{\Omega}_c}$ with adjoint computed in $L^2(\Omega\setminus\overline{\Omega}_c, \mathbb{R}^3)$ and $R(\mathring{\operatorname{grad}}_{\Omega\setminus\overline{\Omega}_c}) \subseteq N(\mathring{\operatorname{curl}}_{\Omega\setminus\overline{\Omega}_c})$, we thus obtain

$$\begin{aligned}
H_0 &= N(\mathring{\operatorname{curl}}_{\Omega\setminus\overline{\Omega_c}})^{\perp_{L^2(\Omega)}} \\
&= N(\mathring{\operatorname{curl}}_{\Omega\setminus\overline{\Omega_c}})^{\perp_{L^2(\Omega\setminus\overline{\Omega_c})}} \oplus L^2(\Omega_c, \mathbb{R}^3) \\
&= \left(R(\mathring{\operatorname{grad}}_{\Omega\setminus\overline{\Omega}_c}) \oplus \mathcal{H}_{D,\Omega\setminus\overline{\Omega}_c}\right)^{\perp_{L^2(\Omega\setminus\overline{\Omega_c})}} \oplus L^2(\Omega_c, \mathbb{R}^3)
\end{aligned}$$

$$= \left(N(\operatorname{div}_{\Omega\setminus\overline{\Omega}_c}) \cap \mathcal{H}_{D,\Omega\setminus\overline{\Omega}_c}^{\perp_{L^2(\Omega\setminus\overline{\Omega_c})}} \right) \oplus L^2(\Omega_c, \mathbb{R}^3).$$

Next, we have by Assumption 4.8

$$\begin{aligned} H_1 &= N(C) \cap R(\eta) \\ &= N(\mathring{\operatorname{curl}}) \cap L^2(\Omega_c, \mathbb{R}^3) \\ &= N(\mathring{\operatorname{curl}}_{\Omega_c}). \end{aligned}$$

An analogous argument as already done for H_0 implies the asserted equation for $H_1^{\perp_{L^2(\Omega)}}$, which in turn implies the second expression for H_1. Finally, from $R(C^*) = R(\operatorname{curl})$ and the already derived expression for H_0, we deduce

$$N(C) \cap H_0 = N\left(\mathring{\operatorname{curl}}\right) \cap \left(\left(N\left(\operatorname{div}_{\Omega\setminus\overline{\Omega_c}}\right) \cap \mathcal{H}_{D,\Omega\setminus\overline{\Omega_c}}^{\perp_{L^2(\Omega\setminus\overline{\Omega_c})}} \right) \oplus L^2\left(\Omega_c, \mathbb{R}^3\right) \right)$$

and therefore

$$\begin{aligned} H_2 &= (N(C) \cap H_0) \ominus (N(C) \cap R(\eta)) \\ &= (N(C) \cap H_0) \cap H_1^{\perp_{L^2(\Omega)}} \\ &= N\left(\mathring{\operatorname{curl}}\right) \cap \left(\left(N\left(\operatorname{div}_{\Omega\setminus\overline{\Omega_c}}\right) \cap \mathcal{H}_{D,\Omega\setminus\overline{\Omega_c}}^{\perp_{L^2(\Omega\setminus\overline{\Omega_c})}} \right) \oplus L^2\left(\Omega_c, \mathbb{R}^3\right) \right) \cap \\ &\quad \cap \left(\left(N(\operatorname{div}_{\Omega_c}) \cap \mathcal{H}_{D,\Omega_c}^{\perp_{L^2(\Omega_c)}} \right) \oplus L^2\left(\Omega \setminus \overline{\Omega_c}, \mathbb{R}^3\right) \right) \\ &= N\left(\mathring{\operatorname{curl}}\right) \cap \left(\left(N(\operatorname{div}_{\Omega_c}) \cap \mathcal{H}_{D,\Omega_c}^{\perp_{L^2(\Omega_c)}} \right) \oplus \left(N\left(\operatorname{div}_{\Omega\setminus\overline{\Omega_c}}\right) \cap \mathcal{H}_{D,\Omega\setminus\overline{\Omega_c}}^{\perp_{L^2(\Omega\setminus\overline{\Omega_c})}} \right) \right). \quad \square \end{aligned}$$

A next step towards the desired proof of Proposition 4.10 is provided next.

Lemma 4.15. *We have for $U_k \in H_k$, $k \in \{1, 2\}$,*

$$\left|\chi_{\Omega_c}(U_1 + U_2)\right|^2 = |U_1|^2 + \left|\chi_{\Omega_c} U_2\right|^2.$$

Proof. By Lemma 4.14, we obtain that

$$\chi_{\Omega_c} U_2 \in N(\operatorname{div}_{\Omega_c}) \cap \mathcal{H}_{D,\Omega_c}^{\perp_{L^2(\Omega_c)}}$$

Hence, with (27) we deduce

$$\begin{aligned} \left|\chi_{\Omega_c}(U_1 + U_2)\right|^2 &= \left|\chi_{\Omega_c} U_1\right|^2 + \langle \chi_{\Omega_c} U_1 | \chi_{\Omega_c} U_2 \rangle + \left|\chi_{\Omega_c} U_2\right|^2 \\ &= |U_1|^2 + \langle U_1 | \chi_{\Omega_c} U_2 \rangle + \left|\chi_{\Omega_c} U_2\right|^2 \\ &= |U_1|^2 + \left|\chi_{\Omega_c} U_2\right|^2. \quad \square \end{aligned}$$

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By Assumption 4.6, we deduce with an application of the closed graph theorem, that there exists $k_0 \geq 0$ such that

$$|U| \leq k_0 \left| \mathring{\text{curl}}\, U \right| \tag{28}$$

for all $U \in D\left(\mathring{\text{curl}}\right) \cap R\left(C^*\right)$. Finally we need a more subtle result, which is the key step towards showing the desired inequality (11) in the present context.

Proposition 4.16. *There exists $k_1 \geq 0$ so that*

$$|U| \leq k_1 \left|\chi_{\Omega_c} U\right| \tag{29}$$

for all $U \in H_2$

In order to prove this proposition we need some preparations. We start with the following observation.

Lemma 4.17. *Define*

$$H_3 := \text{grad}_{\Omega_c} \left[N(\text{div}_{\Omega_c} \text{grad}_{\Omega_c}) \cap \{\phi \mid \text{grad}_{\Omega_c} \phi \in \mathcal{H}_{D,\Omega_c}^{\perp_{L^2(\Omega_c)}}\} \right] \subseteq L^2(\Omega_c)^3.$$

Then H_3 is a closed subspace of $L^2(\Omega_c)^3$ and for $U \in H_2$ we have that $\chi_{\Omega_c} U \in H_3$.

Proof. Obviously, H_3 is a subspace of $L^2(\Omega_c)^3$. For proving the closedness of H_3, let $(\phi_n)_{n\in\mathbb{N}}$ be a sequence in $N(\text{div}_{\Omega_c} \text{grad}_{\Omega_c}) \cap \left\{\phi \mid \text{grad}_{\Omega_c} \phi \in \mathcal{H}_{D,\Omega_c}^{\perp_{L^2(\Omega_c)}}\right\}$ such that $\text{grad}_{\Omega_c} \phi_n \to u$ for some $u \in L^2(\Omega_c)^3$. Since $R(\text{grad}_{\Omega_c})$ is closed by Assumption 4.8 we infer that $u = \text{grad}_{\Omega_c} \phi$ for some $\phi \in D(\text{grad}_{\Omega_c})$. Since $\text{grad}_{\Omega_c} \phi_n \in N(\text{div}_{\Omega_c})$ for each $n \in \mathbb{N}$ it follows by the closedness of $N(\text{div}_{\Omega_c})$ that also $u = \text{grad}_{\Omega_c} \phi \in N(\text{div}_{\Omega_c})$; that is, $\phi \in N(\text{div}_{\Omega_c} \text{grad}_{\Omega_c})$. Finally, since $\mathcal{H}_{D,\Omega_c}^{\perp}$ is closed and $\text{grad}_{\Omega_c} \phi_n \in \mathcal{H}_{D,\Omega_c}^{\perp_{L^2(\Omega_c)}}$ for each $n \in \mathbb{N}$, the same holds for $u = \text{grad}_{\Omega_c} \phi$. Summarising, we have shown that $u \in H_3$ and thus, H_3 is closed.
Take now $U \in H_2$. In particular, $U \in N(\mathring{\text{curl}}) = R(\mathring{\text{grad}})$ by (22), and hence, $U = \mathring{\text{grad}}\, \psi$ for some $\psi \in D(\mathring{\text{grad}})$. By Assumption 4.3 it follows that $\phi := \chi_{\Omega_c} \psi \in D(\text{grad}_{\Omega_c})$ and

$$\text{grad}_{\Omega_c} \phi = \chi_{\Omega_c} \mathring{\text{grad}} \psi = \chi_{\Omega_c} U.$$

Moreover, since $U \in H_2$, it follows by Lemma 4.14 that $\text{grad}_{\Omega_c} \phi = \chi_{\Omega_c} U \in N(\text{div}_{\Omega_c}) \cap \mathcal{H}_{D,\Omega_c}^{\perp_{L^2(\Omega_c)}}$ which shows that $\chi_{\Omega_c} U \in H_3$. □

In the following, we consider the operator

$$Z : H_2 \to H_3$$
$$U \mapsto \chi_{\Omega_c} U.$$

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Lemma 4.18. *The operator Z is one-to-one.*

Proof. Let $U \in H_2$ with $ZU = \chi_{\Omega_c} U = 0$. Since $U = 0$ on Ω_c and $U \in N(\mathring{\operatorname{curl}})$, we infer by Assumption 4.8 that $U \in N(\mathring{\operatorname{curl}}_{\Omega\setminus\overline{\Omega_c}})$. Moreover, by the definition of H_2 we get that $U \in N(\operatorname{div}_{\Omega\setminus\overline{\Omega_c}}) \cap \mathcal{H}_{D,\Omega\setminus\overline{\Omega_c}}^{\perp_{L^2(\Omega\setminus\overline{\Omega_c})}}$ and thus,

$$U \in N(\mathring{\operatorname{curl}}_{\Omega\setminus\overline{\Omega_c}}) \cap N(\operatorname{div}_{\Omega\setminus\overline{\Omega_c}}) \cap \mathcal{H}_{D,\Omega\setminus\overline{\Omega_c}}^{\perp_{L^2(\Omega\setminus\overline{\Omega_c})}} = \mathcal{H}_{D,\Omega\setminus\overline{\Omega_c}} \cap \mathcal{H}_{D,\Omega\setminus\overline{\Omega_c}}^{\perp_{L^2(\Omega\setminus\overline{\Omega_c})}} = \{0\}. \qquad \square$$

Lemma 4.19. *The operator Z is onto.*

Proof. Let $W \in H_3$. Then, by definition, $W \in N(\operatorname{div}_{\Omega_c}) \cap \mathcal{H}_{D,\Omega_c}^{\perp_{L^2(\Omega_c)}}$ and there is $\phi \in D(\operatorname{grad}_{\Omega_c})$ with

$$W = \operatorname{grad}_{\Omega_c} \phi.$$

By (23) there is $\psi \in D(\mathring{\operatorname{grad}})$ such that $\phi = \chi_{\Omega_c}\psi$. Note that by Poincaré's inequality, $R(\mathring{\operatorname{grad}}_{\Omega\setminus\overline{\Omega_c}})$ is a closed subspace of $L^2(\Omega\setminus\overline{\Omega_c})$. Denoting the orthogonal projector onto $R(\mathring{\operatorname{grad}}_{\Omega\setminus\overline{\Omega_c}})$ by $P_{R(\mathring{\operatorname{grad}}_{\Omega\setminus\overline{\Omega_c}})}$, we consider

$$-P_{R(\mathring{\operatorname{grad}}_{\Omega\setminus\overline{\Omega_c}})}\chi_{\Omega\setminus\overline{\Omega_c}}\mathring{\operatorname{grad}}\psi \in R(\mathring{\operatorname{grad}}_{\Omega\setminus\overline{\Omega_c}}),$$

and thus, we find $\theta \in D(\mathring{\operatorname{grad}}_{\Omega\setminus\overline{\Omega_c}})$ with

$$\mathring{\operatorname{grad}}_{\Omega\setminus\overline{\Omega_c}}\theta = -P_{R(\mathring{\operatorname{grad}}_{\Omega\setminus\overline{\Omega_c}})}\chi_{\Omega\setminus\overline{\Omega_c}}\mathring{\operatorname{grad}}\psi.$$

We set

$$\widetilde{\psi} := \psi + \theta \in D(\mathring{\operatorname{grad}})$$

and obtain

$$\begin{aligned}\chi_{\Omega\setminus\overline{\Omega_c}}\mathring{\operatorname{grad}}\widetilde{\psi} &= \chi_{\Omega\setminus\overline{\Omega_c}}\mathring{\operatorname{grad}}\psi + \mathring{\operatorname{grad}}_{\Omega\setminus\overline{\Omega_c}}\theta \\ &= (1 - P_{R(\mathring{\operatorname{grad}}_{\Omega\setminus\overline{\Omega_c}})})\chi_{\Omega\setminus\overline{\Omega_c}}\mathring{\operatorname{grad}}\psi \\ &\in R(\mathring{\operatorname{grad}}_{\Omega\setminus\overline{\Omega_c}})^{\perp_{L^2(\Omega\setminus\overline{\Omega_c})}} = N(\operatorname{div}_{\Omega\setminus\overline{\Omega_c}}).\end{aligned} \tag{30}$$

Finally, we note that

$$\mathcal{H}_{D,\Omega\setminus\overline{\Omega_c}} \subseteq N(\mathring{\operatorname{curl}}_{\Omega\setminus\overline{\Omega_c}}) = N(\mathring{\operatorname{curl}}) \cap L^2(\Omega \setminus \overline{\Omega_c}) = R(\mathring{\operatorname{grad}}) \cap L^2(\Omega \setminus \overline{\Omega_c}) \subseteq R(\mathring{\operatorname{grad}}),$$

where we have used Assumption 4.8 for the first equality and (22) for the second equality. Hence, $\mathcal{H}_{D,\Omega\setminus\overline{\Omega_c}}$ is a closed subspace of $R(\mathring{\operatorname{grad}})$. We now define

$$U := P_{\mathcal{H}_{D,\Omega\setminus\overline{\Omega_c}}^{\perp_{R(\mathring{\mathrm{grad}})}}} \mathring{\mathrm{grad}}\widetilde{\psi}$$

and obtain

$$U \in \mathcal{H}_{D,\Omega\setminus\overline{\Omega_c}}^{\perp_{R(\mathring{\mathrm{grad}})}} = \mathcal{H}_{D,\Omega\setminus\overline{\Omega_c}}^{\perp_{L^2(\Omega)}} \cap R(\mathring{\mathrm{grad}}) = \left(\mathcal{H}_{D,\Omega\setminus\overline{\Omega_c}}^{\perp_{L^2(\Omega\setminus\overline{\Omega_c})}} \oplus L^2(\Omega_c)\right) \cap R(\mathring{\mathrm{grad}}).$$

Thus, in particular, $U \in N(\mathring{\mathrm{curl}})$ and $\chi_{\Omega\setminus\overline{\Omega_c}} U \in \mathcal{H}_{D,\Omega\setminus\overline{\Omega_c}}^{\perp_{L^2(\Omega\setminus\overline{\Omega_c})}}$. Moreover,

$$U - \mathring{\mathrm{grad}}\widetilde{\psi} \in \mathcal{H}_{D,\Omega\setminus\overline{\Omega_c}} \subseteq N(\mathrm{div}_{\Omega\setminus\Omega_c})$$

and thus, in particular $U - \mathring{\mathrm{grad}}\widetilde{\psi} = 0$ on $L^2(\Omega_c)$ and

$$\chi_{\Omega\setminus\overline{\Omega_c}} U = \chi_{\Omega\setminus\overline{\Omega_c}}(U - \mathring{\mathrm{grad}}\widetilde{\psi}) + \chi_{\Omega\setminus\overline{\Omega_c}} \mathring{\mathrm{grad}}\widetilde{\psi} \in N(\mathrm{div}_{\Omega\setminus\overline{\Omega_c}}),$$

where we have used (30). On the other hand, we have

$$\begin{aligned}
\chi_{\Omega_c} U &= \chi_{\Omega_c} \mathring{\mathrm{grad}}\widetilde{\psi} \\
&= \chi_{\Omega_c} \mathring{\mathrm{grad}}\psi + \chi_{\Omega_c} \mathring{\mathrm{grad}}\theta \\
&= \mathrm{grad}_{\Omega_c} \chi_{\Omega_c} \psi \\
&= \mathrm{grad}_{\Omega_c} \phi \\
&= W \in N(\mathrm{div}_{\Omega_c}) \cap \mathcal{H}_{D,\Omega_c}^{\perp_{L^2(\Omega_c)}},
\end{aligned}$$

and thus, $U \in H_2$ with $ZU = W$. This completes the proof. □

Now we are able to prove Proposition 4.16.

Proof of Proposition 4.16. Since $Z : H_2 \to H_3$ is continuous, one-to-one and onto, it follows that $Z^{-1} : H_3 \to H_2$ is continuous as well by the closed graph theorem. Thus, the assertion follows with $k_1 := \|Z^{-1}\|$. □

We are finally in the position to prove inequality (11) and, therefore, to complete the proof of Proposition 4.10.

Lemma 4.20. *There is a positive constant c_0 such that we have*

$$c_0 |U|^2 \leq \left|\sigma^{1/2} U\right|^2 + \left|\mathring{\mathrm{curl}} U\right|^2 \tag{31}$$

for all $U \in D\left(\mathring{\mathrm{curl}}\right) \cap H_0$.

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Proof. By the positive definiteness of $\widetilde{\sigma}$, see Assumption 4.3, we obtain for all $U \in D(\mathring{\mathrm{curl}}) \cap H_0$

$$c_* \left|\chi_{\Omega_c} U\right|^2 + \left|\mathring{\mathrm{curl}} U\right|^2 \leq \left|\sigma^{1/2} U\right|^2 + \left|\mathring{\mathrm{curl}} U\right|^2$$

for some $c_* > 0$. Thus, the desired estimate follows if we can show that there is $c > 0$ such that for all $U \in D(\mathring{\mathrm{curl}}) \cap H_0$

$$c\left|U\right|^2 \leq \left|\chi_{\Omega_c} U\right|^2 + \left|\mathring{\mathrm{curl}} U\right|^2 .$$

We shall employ the above decomposition (26) so that $U = U_0 + U_1 + U_2$ with $U_0 \in R\left(\mathrm{curl}\,\mu^{-\frac{1}{2}}\right)$, $U_k \in H_k$, $k \in \{1,2\}$. We compute using (28), Lemma 4.16, and Lemma 4.15

$$\begin{aligned}
\left|U\right|^2 &= \left|U_0\right|^2 + \left|U_1\right|^2 + \left|U_2\right|^2 \\
&\leq k_0^2 \left|\mathring{\mathrm{curl}} U_0\right|^2 + k_1^2 \left|\chi_{\Omega_c} U_2\right|^2 + \left|U_1\right|^2 \\
&\leq k_0^2 \left|\mathring{\mathrm{curl}} U_0\right|^2 + \max\left\{1, k_1^2\right\} \left|\chi_{\Omega_c} \left(U_1 + U_2\right)\right|^2 \\
&\leq k_0^2 \left|\mathring{\mathrm{curl}} U_0\right|^2 + 2\max\left\{1, k_1^2\right\} \left|\chi_{\Omega_c} \left(U_0 + U_1 + U_2\right)\right|^2 + \\
&\quad +2\max\left\{1, k_1^2\right\} \left|\chi_{\Omega_c} U_0\right|^2 , \\
&\leq k_0^2 \left|\mathring{\mathrm{curl}} U_0\right|^2 + 2\max\left\{1, k_1^2\right\} \left|\chi_{\Omega_c} \left(U_0 + U_1 + U_2\right)\right|^2 + \\
&\quad +2\max\left\{1, k_1^2\right\} \left|U_0\right|^2 , \\
&\leq k_0^2 \left(1 + 2\max\left\{1, k_1^2\right\}\right) \left|\mathring{\mathrm{curl}} U_0\right|^2 + \\
&\quad +2\max\left\{1, k_1^2\right\} \left|\chi_{\Omega_c} \left(U_0 + U_1 + U_2\right)\right|^2 , \\
&\leq \max\left\{2, 2k_1^2, k_0^2\left(1 + 2\max\left\{1, k_1^2\right\}\right)\right\} \left(\left|\mathring{\mathrm{curl}} U\right|^2 + \left|\chi_{\Omega_c} U\right|^2\right)
\end{aligned}$$

Thus we see that the estimate (31) holds for

$$c_0 = \min\left\{1, c_*\right\}$$

with

$$c_* = \frac{1}{\max\left\{2, 2k_1^2, k_0^2\left(1 + 2\max\left\{1, k_1^2\right\}\right)\right\}}. \qquad \square$$

We shall summarise the findings of this section as follows.

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Theorem 4.21. *Let $\Omega \subseteq \mathbb{R}^3$ be open with connected boundary. Assume Assumptions 4.3, 4.6, 4.8 to be in effect. Then for every $F \in H_{\varrho,0}(\mathbb{R}, D(C_0^*)')$ (with $C_0 := \mu^{-1/2}\mathring{\operatorname{curl}}|_{H_0}$) there is a unique (weak) solution $U \in H_{\varrho,0}\left(\mathbb{R}, D\left(\mathring{\operatorname{curl}}\right)\right) \cap H_{\varrho,0}(\mathbb{R}, H_0)$ of*

$$\left(\overline{\partial_0 \sigma + \operatorname{curl} \mu^{-1} \mathring{\operatorname{curl}}}\right) U = F.$$

Moreover the solution operator $S : H_{\varrho,0}(\mathbb{R}, D(C_0^*)') \to H_{\varrho,0}\left(\mathbb{R}, D\left(\mathring{\operatorname{curl}}\right)\right)$ is continuous ($|\cdot|_{\varrho,0,1}$ denotes the norm of $H_{\varrho,0}\left(\mathbb{R}, D\left(\mathring{\operatorname{curl}}\right)\right)$) and causal in the sense that

$$\left|\chi_{]-\infty,a]} SF\right|_{\varrho,0,1} \leq C_1 \left|\chi_{]-\infty,a]} F\right|_{\varrho,0,-1}$$

for some positive C_1 uniformly in $a \in \mathbb{R}$ and $F \in H_{\varrho,0}(\mathbb{R}, D(C_0^*)')$ as long as $\varrho \in \left]0, \infty\right[$ is sufficiently large.

Proof. The result follows from Theorem 3.10 in conjunction with Proposition 4.10. □

Remark 4.22. There are two famous engineering type approaches, which have inspired a number of mathematical investigations, see e.g. [19,18,5,1] and the literature quoted there. In engineering lingo they are frequently referred to (by a slight abuse of language, turning adhoc names of variables into constant names) as the A-φ approach and the T-Ω approach, where two variants of a vector potential construction come into play. Our approach is designed precisely to avoid these constructions, which are actually adding complexity to an already sufficiently complex topic. A crucial assumption in the application of these approaches is that the current density source term J is supposed to be divergence-free,[5] which, apart from requiring additional regularity of J, excludes perfectly reasonable current densities, say $J = Ie_3$, if I is not completely constant in direction e_3. In contrast, we are here considering the eddy current problem directly by solving

$$\partial_0 \sigma e + \operatorname{curl} \mu^{-1} \operatorname{curl} e = -J$$

with a general square-integrable right-hand side (with an exponential weight in the time direction) with only the obvious constraint that J is required to be in the closure of the range of $\partial_0 \sigma + \operatorname{curl} \mu^{-1} \operatorname{curl}$. We emphasise that the present approach allows to recover the original unknowns, see Remark 4.11.

[5] A divergence-free condition is also imposed in the existence result in [32]. We detail the potential formulation in the discussion here. What is said on the divergence condition, however, also applies to the time-harmonic setting focused on in [32].

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5. An extended system formulation for the pre-Maxwell system

For numerical purposes the construction of H_0 is not particularly comfortable. We therefore want to propose an alternative formulation in the spirit of the extended Maxwell system [24,34,31], which in the context of numerical investigations is of so-called saddle-point form. In fact, the key is to formulate belonging to $H_0^{\perp}$ with the help of belonging to the kernel of certain differential operators. We therefore hope that the proposed reformulation might shed some light on possible numerical implementations of the considered model. Quite recently, this method has been applied to homogenisation problems, see [43].

Throughout this section, we assume Ω to be open and bounded with connected boundary. Moreover, let the Assumptions 4.3, 4.6, 4.8 be in effect. Moreover, we shall rather focus on $\mu = 1$. We need to impose an additional assumption for this section:

Assumption 5.1. Assume that

$$D\left(\mathring{\operatorname{grad}}\right) = \left\{\psi \in D\left(\operatorname{grad}_{\mathbb{R}^3}\right) | \psi = 0 \text{ on } \mathbb{R}^3 \setminus \overline{\Omega}\right\}$$

as well as

$$D(\mathring{\operatorname{grad}}_{\Omega\setminus\overline{\Omega_c}}) = \{\psi \in D(\mathring{\operatorname{grad}}) \,|\, \psi = 0 \text{ on } \Omega_c\}.$$

Remark 5.2. The latter assumption holds for instance, if Ω and Ω_c satisfy the segment property.

Amending the system in question by an equation in $H_0^{\perp_{L^2(\Omega)}}$ suitably leads to

$$\begin{pmatrix} \begin{pmatrix} \partial_0\sigma + \operatorname{curl}\mathring{\operatorname{curl}} & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 \\ \overset{\diamond}{\operatorname{grad}}_{\Omega\setminus\overline{\Omega_c}} \end{pmatrix} \\ \begin{pmatrix} 0 & \overset{\diamond}{\operatorname{div}}_{\Omega\setminus\overline{\Omega_c}} \end{pmatrix} & 0 \end{pmatrix}$$

with $\left(H_0 \oplus H_0^{\perp_{L^2(\Omega)}}\right) \oplus L^2(\Omega \setminus \overline{\Omega_c}, \mathbb{R})$ as underlying Hilbert space. Here we have

$$\begin{aligned} \overset{\diamond}{\operatorname{grad}}_{\Omega\setminus\overline{\Omega_c}} : D(\overset{\diamond}{\operatorname{grad}}_{\Omega\setminus\overline{\Omega_c}}) \subseteq L^2(\Omega \setminus \overline{\Omega_c}, \mathbb{R}) &\to H_0^{\perp_{L^2(\Omega)}} \\ \varphi &\mapsto \mathring{\operatorname{grad}}\varphi \end{aligned}$$

with

$$D\left(\overset{\diamond}{\operatorname{grad}}_{\Omega\setminus\overline{\Omega_c}}\right) = \left\{\chi_{\Omega\setminus\overline{\Omega_c}}\varphi \,|\, \varphi \in D\left(\mathring{\operatorname{grad}}\right), \varphi \text{ constant on } \Omega_c\right\}.$$

To fit our scheme we let here

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$$\overset{\diamond}{\operatorname{div}}_{\Omega\setminus\overline{\Omega_c}} := -\overset{\diamond}{\operatorname{grad}}{}^{*}_{\Omega\setminus\overline{\Omega_c}}.$$

A reason for the introduction of these new operators is the following lemma.

Lemma 5.3. *It is*

$$\mathcal{H}_{D,\Omega\setminus\overline{\Omega_c}} = R\left(\overset{\diamond}{\operatorname{grad}}_{\Omega\setminus\overline{\Omega_c}}\right) \ominus R\left(\overset{\circ}{\operatorname{grad}}_{\Omega\setminus\overline{\Omega_c}}\right).$$

Proof. Let

$$\Phi \in \mathcal{H}_{D,\Omega\setminus\overline{\Omega_c}} \subseteq N(\overset{\circ}{\operatorname{curl}}_{\Omega\setminus\overline{\Omega_c}})$$

and by extension by zero $\Phi \in N\left(\operatorname{curl}_{\mathbb{R}^3}\right)$. Thus

$$\Phi = \operatorname{grad}\psi$$

in $L^{2,\mathrm{loc}}\left(\mathbb{R}^3, \mathbb{R}^3\right)$ for some weakly differentiable ψ. Since $\Phi = 0$ on $\mathbb{R}^3 \setminus \left(\Omega \setminus \overline{\Omega_c}\right)$ we have that ψ is constant on each component of $\mathbb{R}^3 \setminus \left(\Omega \setminus \overline{\Omega_c}\right)$. Adjusting this constant to be zero on the unbounded part $\mathbb{R}^3 \setminus \Omega$ of $\mathbb{R}^3 \setminus \left(\Omega \setminus \overline{\Omega_c}\right)$ we get a $\widehat{\psi} \in D\left(\operatorname{grad}_{\mathbb{R}^3}\right)$ with $\widehat{\psi}$ constant on Ω_c, $\widehat{\psi} = 0$ on $\mathbb{R}^3 \setminus \overline{\Omega}$ and

$$\Phi = \operatorname{grad}_{\mathbb{R}^3}\widehat{\psi}.$$

By Assumption 5.1 we know that

$$D\left(\overset{\circ}{\operatorname{grad}}\right) = \left\{\psi \in D\left(\operatorname{grad}_{\mathbb{R}^3}\right) \mid \psi = 0 \text{ on } \mathbb{R}^3 \setminus \overline{\Omega}\right\}.$$

Thus,

$$\Phi = \overset{\diamond}{\operatorname{grad}}_{\Omega\setminus\overline{\Omega_c}}\widehat{\psi}.$$

Since also $\operatorname{div}_{\Omega\setminus\overline{\Omega_c}}\Phi = 0$ we have indeed shown that

$$\begin{aligned}\mathcal{H}_{D,\Omega\setminus\overline{\Omega_c}} &\subseteq R\left(\overset{\diamond}{\operatorname{grad}}_{\Omega\setminus\overline{\Omega_c}}\right) \cap N(\operatorname{div}_{\Omega\setminus\overline{\Omega_c}}) \\ &= R\left(\overset{\diamond}{\operatorname{grad}}_{\Omega\setminus\overline{\Omega_c}}\right) \cap R\left(\overset{\circ}{\operatorname{grad}}_{\Omega\setminus\overline{\Omega_c}}\right)^{\perp_{L^2(\Omega\setminus\overline{\Omega_c})}} = R\left(\overset{\diamond}{\operatorname{grad}}_{\Omega\setminus\overline{\Omega_c}}\right) \ominus R\left(\overset{\circ}{\operatorname{grad}}_{\Omega\setminus\overline{\Omega_c}}\right).\end{aligned}$$

Let now $\Phi = \overset{\diamond}{\operatorname{grad}}_{\Omega\setminus\overline{\Omega_c}}\widehat{\psi}$ for some $\widehat{\psi} \in D(\overset{\diamond}{\operatorname{grad}}_{\Omega\setminus\overline{\Omega_c}})$ and $\operatorname{div}_{\Omega\setminus\overline{\Omega_c}}\Phi = 0$. Let $\psi_0 \in D(\overset{\circ}{\operatorname{grad}})$ an extension of $\widehat{\psi}|_{\Omega_c}$ such that ψ_0 is constant in a neighbourhood of Ω_c. Then, in particular, $\widehat{\psi} - \psi_0$ vanishes on Ω_c, and so by Assumption 5.1

$$\widehat{\psi} - \psi_0 \in D\left(\overset{\circ}{\operatorname{grad}}_{\Omega\setminus\overline{\Omega_c}}\right).$$

We have by construction that

$$\operatorname{div}_{\Omega\setminus\overline{\Omega_c}} \overset{\diamond}{\operatorname{grad}}_{\Omega\setminus\overline{\Omega_c}} \widehat{\psi} = \operatorname{div}_{\Omega\setminus\overline{\Omega_c}} \Phi = 0$$

and so

$$\operatorname{div}_{\Omega\setminus\overline{\Omega_c}} \overset{\circ}{\operatorname{grad}}_{\Omega\setminus\overline{\Omega_c}} \left(\widehat{\psi} - \psi_0\right) = -\operatorname{div}_{\Omega\setminus\overline{\Omega_c}} \overset{\diamond}{\operatorname{grad}}_{\Omega\setminus\overline{\Omega_c}} \psi_0.$$

Next, we first note that

$$\overset{\circ}{\operatorname{grad}}_{\Omega\setminus\overline{\Omega_c}} \left(\widehat{\psi} - \psi_0\right) \in N\left(\overset{\circ}{\operatorname{curl}}_{\Omega\setminus\overline{\Omega_c}}\right).$$

Since also $\overset{\diamond}{\operatorname{grad}}_{\Omega\setminus\overline{\Omega_c}} \psi_0 \in N\left(\operatorname{curl}_{\Omega\setminus\overline{\Omega_c}}\right) \cap N(\overset{\circ}{\operatorname{curl}})$ and since $\overset{\diamond}{\operatorname{grad}}_{\Omega\setminus\overline{\Omega_c}} \psi_0$ actually vanishes in a neighbourhood of Ω_c we also have

$$\overset{\diamond}{\operatorname{grad}}_{\Omega\setminus\overline{\Omega_c}} \psi_0 \in N\left(\overset{\circ}{\operatorname{curl}}_{\Omega\setminus\overline{\Omega_c}}\right).$$

Thus,

$$\Phi = \overset{\diamond}{\operatorname{grad}}_{\Omega\setminus\overline{\Omega_c}} \left(\widehat{\psi}\right) = \overset{\circ}{\operatorname{grad}}_{\Omega\setminus\overline{\Omega_c}} \left(\widehat{\psi} - \psi_0\right) + \overset{\diamond}{\operatorname{grad}}_{\Omega\setminus\overline{\Omega_c}} \psi_0 \in N\left(\overset{\circ}{\operatorname{curl}}_{\Omega\setminus\overline{\Omega_c}}\right)$$

and so

$$\Phi \in \mathcal{H}_{D,\Omega\setminus\overline{\Omega_c}}.$$

This yields the converse inclusion. □

The latter lemma particularly implies

$$\begin{aligned} H_0^{\perp_{L^2(\Omega)}} &= N\left(\overset{\circ}{\operatorname{curl}}_{\Omega\setminus\overline{\Omega_c}}\right) \\ &= R\left(\overset{\circ}{\operatorname{grad}}_{\Omega\setminus\overline{\Omega_c}}\right) \oplus \mathcal{H}_{D,\Omega\setminus\overline{\Omega_c}} \\ &= R\left(\overset{\diamond}{\operatorname{grad}}_{\Omega\setminus\overline{\Omega_c}}\right), \end{aligned}$$

where we have used Lemma 4.14 for the first equality. Since, according to the projection theorem, the canonical embedding

$$\begin{aligned} \begin{pmatrix} \iota_{H_0} & \iota_{H_0^\perp} \end{pmatrix} : H_0 \oplus H_0^{\perp_{L^2(\Omega)}} &\to L^2(\Omega, \mathbb{R}^3) \\ \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} &\mapsto x_0 + x_1 \end{aligned}$$

is unitary we have its adjoint

$$\begin{pmatrix} \iota_{H_0}^* \\ \iota_{H_0^\perp}^* \end{pmatrix} : L^2\left(\Omega, \mathbb{R}^3\right) \to H_0 \oplus H_0^{\perp_{L^2(\Omega)}}$$

as the inverse. Thus, we may consider equivalently

$$W \begin{pmatrix} \begin{pmatrix} \partial_0 \sigma + \operatorname{curl} \mathring{\operatorname{curl}} & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 \\ \overset{\diamond}{\operatorname{grad}}_{\Omega \setminus \overline{\Omega_c}} \end{pmatrix} \\ \begin{pmatrix} 0 & \overset{\diamond}{\operatorname{div}}_{\Omega \setminus \overline{\Omega_c}} \end{pmatrix} & 0 \end{pmatrix} W^* =$$

$$= \begin{pmatrix} \partial_0 \sigma + \operatorname{curl} \mathring{\operatorname{curl}} & \overset{\diamond}{\operatorname{grad}}_{\Omega \setminus \overline{\Omega_c}} \\ \overset{\diamond}{\operatorname{div}}_{\Omega \setminus \overline{\Omega_c}} & 0 \end{pmatrix}$$

now on $L^2\left(\Omega, \mathbb{R}^3\right) \oplus L^2\left(\Omega \setminus \overline{\Omega_c}, \mathbb{R}\right)$ as underlying Hilbert space with the unitary map

$$W = \begin{pmatrix} \begin{pmatrix} \iota_{H_0} & \iota_{H_0^\perp} \end{pmatrix} & 0 \\ \begin{pmatrix} 0_{H_0} & 0_{H_0^\perp} \end{pmatrix} & 1 \end{pmatrix}.$$

Thus, we are led to discuss equations of the form

$$\begin{pmatrix} \partial_0 \sigma + \operatorname{curl} \mathring{\operatorname{curl}} & \overset{\diamond}{\operatorname{grad}}_{\Omega \setminus \overline{\Omega_c}} \\ \overset{\diamond}{\operatorname{div}}_{\Omega \setminus \overline{\Omega_c}} & 0 \end{pmatrix} \begin{pmatrix} E \\ p \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}.$$

From this “saddle point formulation” we can recover E as the solution of

$$\partial_0 \sigma E + \operatorname{curl} \mathring{\operatorname{curl}} E = \iota_{H_0}^* f. \tag{32}$$

Indeed, we have the following result.

Theorem 5.4. *Assume Ω to be open and bounded with connected boundary. Moreover, let the Assumptions 4.3, 4.6, 4.8, and 5.1 be in effect. Then the (closure of the) operator*

$$\begin{pmatrix} \begin{pmatrix} \partial_0 \sigma + \operatorname{curl} \mathring{\operatorname{curl}} & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 \\ \overset{\diamond}{\operatorname{grad}}_{\Omega \setminus \overline{\Omega_c}} \end{pmatrix} \\ \begin{pmatrix} 0 & \overset{\diamond}{\operatorname{div}}_{\Omega \setminus \overline{\Omega_c}} \end{pmatrix} & 0 \end{pmatrix}$$

is continuously invertible in $H_{\varrho,0}(\mathbb{R}, H_0 \oplus H_0^\perp \oplus L^2\left(\Omega \setminus \overline{\Omega_c}, \mathbb{R}\right))$ for sufficiently large $\varrho > 0$.

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Proof. Note that since Ω is open and bounded, we infer by Poincaré's inequality that $R\left(\mathring{\operatorname{grad}}\right)$ is closed. This implies that $R\left(\overset{\diamond}{\operatorname{grad}}_{\Omega\setminus\overline{\Omega_c}}\right)$ is closed as well as the range $R\left(\overset{\diamond}{\operatorname{div}}_{\Omega\setminus\overline{\Omega_c}}\right)$ of its adjoint $-\overset{\diamond}{\operatorname{div}}_{\Omega\setminus\overline{\Omega_c}}$. This makes

$$\begin{pmatrix} 0 & \overset{\diamond}{\operatorname{grad}}_{\Omega\setminus\overline{\Omega_c}} \\ \overset{\diamond}{\operatorname{div}}_{\Omega\setminus\overline{\Omega_c}} & 0 \end{pmatrix}$$

continuously invertible on $R\left(\overset{\diamond}{\operatorname{grad}}_{\Omega\setminus\overline{\Omega_c}}\right) \oplus R\left(\overset{\diamond}{\operatorname{div}}_{\Omega\setminus\overline{\Omega_c}}\right)$. Moreover, it is a consequence of the above lemma that

$$R\left(\overset{\diamond}{\operatorname{grad}}_{\Omega\setminus\overline{\Omega_c}}\right) = H_0^{\perp_{L^2(\Omega)}}.$$

Furthermore, since $\mathring{\operatorname{grad}}$ is injective, we infer that

$$N\left(\overset{\diamond}{\operatorname{grad}}_{\Omega\setminus\overline{\Omega_c}}\right) = \{0\},$$

which, thus, implies that

$$R\left(\overset{\diamond}{\operatorname{div}}_{\Omega\setminus\overline{\Omega_c}}\right) = L^2\left(\Omega\setminus\overline{\Omega_c}, \mathbb{R}\right).$$

Hence, we infer the claim of the theorem by the well-posedness result from Theorem 4.21. □

The solution (E, p) of the extended system now yields indeed a solution E of the pre-Maxwell system (32). If $f \in H_0$ we have of course $f = P_{H0}f$ and $p = 0$.

Remark 5.5. For numerical purposes approximations of the equation $\overset{\diamond}{\operatorname{div}}_{\Omega\setminus\overline{\Omega_c}}E = 0$ would be based on its 'weak' form

$$\left\langle \overset{\diamond}{\operatorname{grad}}_{\Omega\setminus\overline{\Omega_c}}\psi | E \right\rangle_{L^2(\Omega\setminus\overline{\Omega_c},\mathbb{R}^3)} = 0,$$

so that E could be approximated in suitable finite-dimensional subspaces of $D\left(\mathring{\operatorname{curl}}\right)$.

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6. Justification of the pre-Maxwell system

We conclude our considerations with a justification of the pre-Maxwell system; that is, the degenerate eddy current problem,[6] as an approximation of Maxwell's system (including the displacement current). The system of Maxwell's equations reads as

$$\begin{aligned}\partial_0 \varepsilon \mathrm{E} + \sigma \mathrm{E} - \operatorname{curl} \mathrm{H} &= -\mathrm{J},\\ \partial_0 \mu \mathrm{H} + \mathring{\operatorname{curl}} \mathrm{E} &= \mathrm{K},\end{aligned}$$

where K denotes a magnetic source term (perhaps induced by initial data for H) and $\varepsilon \in]0, \infty[$. Throughout, let $\varrho \geq 1$. The question is if and in which sense do the solutions converge to the solutions of the degenerate eddy current problem as ε tends to 0. For this transition we restrict our attention to current densities J in the correct subspace for the limit problem $\varepsilon = 0$; that is,

$$J \in H_{\varrho,0}(\mathbb{R}, H_0).$$

Again, as before, we shall assume that Ω is open, bounded with connected boundary. Furthermore, we shall assume throughout that the Assumptions 4.3, 4.6, 4.8 are in effect. We shall furthermore note that a standard application of Theorem 2.1 leads to

$$\widetilde{S}_\varepsilon := \left(\overline{\left(\partial_0 \begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} + \begin{pmatrix} \sigma & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -\operatorname{curl} \\ \mathring{\operatorname{curl}} & 0 \end{pmatrix}\right)}\right)^{-1} \in L(H_{\varrho,k}(\mathbb{R}; L^2(\Omega, \mathbb{R}^6)))$$

for *every* $\varrho > 0$ and $k \in \mathbb{Z}$. Here and in the following we use $|\cdot|_{\varrho,k,0}$ as the notation for the norm corresponding to the Hilbert space inner product induced by $\langle \cdot | \cdot \rangle_{\varrho,k,0} := \langle \partial_0^k \cdot | \partial_0^k \cdot \rangle_{\varrho,0,0}$. $H_{\varrho,k}\left(\mathbb{R}, L^2(\Omega, \mathbb{R}^6)\right)$ denotes the Hilbert space obtained by completion of $D(\partial_0^k)$. We denote

$$S_0 := \left(\overline{\partial_0 \sigma + \operatorname{curl} \mu^{-1} \mathring{\operatorname{curl}}}\right)^{-1} \in L(H_{\varrho,0}(\mathbb{R}, H_0), H_{\varrho,0}(\mathbb{R}, D(\mathring{\operatorname{curl}}))$$

for some fixed sufficiently large $\varrho > 0$. Furthermore, we define for all $\varepsilon > 0$

$$S_\varepsilon := \pi_1 \widetilde{S}_\varepsilon,$$

where $\pi_1(\mathrm{E}, \mathrm{H}) = \mathrm{E}$ reads off the first three components of a 6-component vector field. Assuming

$$\operatorname{curl} \mu^{-1} \partial_0^{-1} \mathrm{K} \in H_{\varrho,k}\left(\mathbb{R}, L^2\left(\Omega, \mathbb{R}^3\right)\right)$$

[6] For the non-degenerate eddy current problem this has been given in the current functional analytical setting in [21,42] in both the autonomous and non-autonomous cases, respectively.

the simple substitution

$$\mathrm{H} = \mu^{-1}\partial_0^{-1}\mathrm{K} - \mu^{-1}\mathring{\mathrm{curl}}\partial_0^{-1}\mathrm{E}$$

leads to $S_\varepsilon(\mathrm{J}, \mathrm{K}) = \mathrm{E}$ being the unique solution of

$$\partial_0\varepsilon\mathrm{E} + \sigma\mathrm{E} + \mathrm{curl}\,\mu^{-1}\mathring{\mathrm{curl}}\partial_0^{-1}\mathrm{E} = -\mathrm{J} + \mathrm{curl}\,\mu^{-1}\partial_0^{-1}\mathrm{K}.$$

By a slight abuse of notation, we shall view S_ε as a mapping from $H_{\varrho,0}(\mathbb{R}; L^2(\Omega, \mathbb{R}^3))$ into itself. Thus, instead of $S_\varepsilon(\mathrm{J}, \mathrm{K})$ we shall write $S_\varepsilon(-\mathrm{J} + \mathrm{curl}\,\mu^{-1}\partial_0^{-1}\mathrm{K})$. This provides a second order formulation, which we actually can compare with the degenerate equation. Due to the particular structure of the right-hand side, we furthermore remark here that $f = -\mathrm{J} + \mathrm{curl}\,\mu^{-1}\partial_0^{-1}\mathrm{K}$ takes values in H_0 if and only if J does. The main result of this section reads as follows.

Theorem 6.1. *For all $k \in \mathbb{Z}$ and $f \in H_{\varrho,k}(\mathbb{R}; H_0)$ we have*

$$|S_\varepsilon f - S_0 f|_{\varrho,k-2,0} \to 0$$

as $\varepsilon \to 0$.

Before proving this result, we provide the following auxiliary result.

Lemma 6.2. *For all $k \in \mathbb{Z}$, we have*

$$\sup_{\varepsilon>0} \|S_\varepsilon\|_{H_{\varrho,k}(\mathbb{R},H_0)\to H_{\varrho,k-2}(\mathbb{R},H_0)} < \infty.$$

Proof. Let $f \in H_{\varrho,k+1}(\mathbb{R}, H_0)$, $\varepsilon > 0$. Then $\mathrm{E} = S_\varepsilon f$ satisfies

$$\partial_0\varepsilon\mathrm{E} + \sigma\mathrm{E} + \mathrm{curl}\,\mu^{-1}\mathring{\mathrm{curl}}\partial_0^{-1}\mathrm{E} = f.$$

We shall now separate this equation into the parts in H_0 and $H_0^\perp$ separately. Denoting $\begin{pmatrix} \mathrm{E}_0 \\ \mathrm{E}_1 \end{pmatrix} = \begin{pmatrix} \iota_{H_0}^*\mathrm{E} \\ \iota_{H_0^\perp}^*\mathrm{E} \end{pmatrix}$, we obtain

$$\begin{aligned} \partial_0\varepsilon E_0 + \sigma E_0 + \mathrm{curl}\,\mu^{-1}\mathring{\mathrm{curl}}\partial_0^{-1}E_0 &= \iota_{H_0}^* f, \\ \partial_0\varepsilon E_1 &= 0, \end{aligned}$$

where we have used that $f \in H_0$. By the second equation we have

$$\partial_0\varepsilon\mathrm{E}_1 = 0$$

and thus, continuous invertibility of ∂_0 implies $\mathrm{E}_1 = 0$. Testing the equation for E_0 with E_0, we deduce

$$\begin{aligned}\varrho\left|\varepsilon^{1/2}\mathrm{E}_0\right|^2_{\varrho,k,0}+\left|\sigma^{1/2}\mathrm{E}_0\right|^2_{\varrho,k,0}+\left\langle \mathring{\mathrm{curl}}\mathrm{E}_0|\mu^{-1}\partial_0^{-1}\mathring{\mathrm{curl}}\mathrm{E}_0\right\rangle_{\varrho,k,0}\\ =\langle \mathrm{E}_0|f\rangle_{\varrho,k,0}\leq |\mathrm{E}_0|_{\varrho,k-1,0}\,|f|_{\varrho,k+1,0}\,.\end{aligned}$$

Using

$$\left|\sigma^{1/2}\mathrm{E}_0\right|_{\varrho,k-1,0}\leq\frac{1}{\varrho}\left|\sigma^{1/2}\mathrm{E}_0\right|_{\varrho,k,0},$$

and

$$\begin{aligned}\left\langle \mathring{\mathrm{curl}}\mathrm{E}_0|\mu^{-1}\partial_0^{-1}\mathring{\mathrm{curl}}\mathrm{E}_0\right\rangle_{\varrho,k,0}&=\left\langle \partial_0\partial_0^{-1}\mathring{\mathrm{curl}}\mathrm{E}_0|\mu^{-1}\partial_0^{-1}\mathring{\mathrm{curl}}\mathrm{E}_0\right\rangle_{\varrho,k,0}\\ &=\varrho\left|\partial_0^{-1}\mu^{-1/2}\mathring{\mathrm{curl}}\mathrm{E}_0\right|^2_{\varrho,k,0}\\ &=\varrho\left|\mu^{-1/2}\mathring{\mathrm{curl}}\mathrm{E}_0\right|^2_{\varrho,k-1,0}\end{aligned}$$

we infer

$$\varrho^2\left|\sigma^{1/2}\mathrm{E}_0\right|^2_{\varrho,k-1,0}+\varrho\left|\mu^{-1/2}\mathring{\mathrm{curl}}\mathrm{E}_0\right|^2_{\varrho,k-1,0}\leq|E_0|_{\varrho,k-1,0}|f|_{\varrho,k+1,0}.$$

On the other hand we know by (31) that

$$\left|\sigma^{1/2}\mathrm{E}_0\right|^2_{\varrho,k-1,0}+\left|\mu^{-1/2}\mathring{\mathrm{curl}}\mathrm{E}_0\right|^2_{\varrho,k-1,0}\geq c_0\,|\mathrm{E}_0|^2_{\varrho,k-1,0}$$

for some $c_0\in\,]0,\infty[$. Thus, as $\varrho\geq 1$ we have

$$c_0\,|\mathrm{E}_0|^2_{\varrho,k-1,0}\leq|\mathrm{E}_0|_{\varrho,k-1,0}\,|f|_{\varrho,k+1,0}\,.$$

Consequently, we have the uniform estimate

$$c_0\,|\mathrm{E}_0|_{\varrho,k-1,0}\leq|f|_{\varrho,k+1,0}\,,$$

which yields

$$\sup_{\varepsilon>0}\|S_\varepsilon\|_{H_{\varrho,k}(\mathbb{R},H_0)\to H_{\varrho,k-2}(\mathbb{R},H_0)}=\sup_{\varepsilon>0}\|S_\varepsilon\|_{H_{\varrho,k+1}(\mathbb{R},H_0)\to H_{\varrho,k-1}(\mathbb{R},H_0)}\leq\frac{1}{c_0}.\qquad\square$$

Proof of Theorem 6.1. For $\varepsilon>0$ and $f\in H_{\varrho,k+1}(\mathbb{R},H_0)$ we find

$$\begin{aligned}S_\varepsilon f-S_0f&=S_\varepsilon\left(S_0^{-1}-S_\varepsilon^{-1}\right)S_0f\\ &=S_\varepsilon\varepsilon\partial_0S_0f\end{aligned}$$

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and so

$$
\begin{aligned}
&|S_\varepsilon f - S_0 f|_{\varrho,k-2,0} \qquad (33)\\
&= |S_\varepsilon \varepsilon \partial_0 S_0 f|_{\varrho,k-2,0}\\
&\leq \|S_\varepsilon\|_{H_{\varrho,k}(\mathbb{R},H_0)\to H_{\varrho,k-2}(\mathbb{R},H_0)} |\varepsilon \partial_0 S_0 f|_{\varrho,k,0}\\
&\leq \varepsilon \|S_\varepsilon\|_{H_{\varrho,k}(\mathbb{R},H_0)\to H_{\varrho,k-2}(\mathbb{R},H_0)} |S_0 \partial_0 f|_{\varrho,k,0}\\
&\leq \varepsilon \|S_\varepsilon\|_{H_{\varrho,k}(\mathbb{R},H_0)\to H_{\varrho,k-2}(\mathbb{R},H_0)} \|S_0\|_{H_{\varrho,k}(\mathbb{R},H_0)\to H_{\varrho,k}(\mathbb{R},H_0)} |\partial_0 f|_{\varrho,k,0}\\
&\leq \varepsilon \|S_\varepsilon\|_{H_{\varrho,k}(\mathbb{R},H_0)\to H_{\varrho,k-2}(\mathbb{R},H_0)} \|S_0\|_{H_{\varrho,k-1}(\mathbb{R},H_0)\to H_{\varrho,k-1}(\mathbb{R},H_0)} |f|_{\varrho,k+1,0}\,.
\end{aligned}
$$

By Lemma 6.2, we deduce that

$$|S_\varepsilon f - S_0 f|_{\varrho,k-2,0} \overset{\varepsilon\to 0}{\to} 0$$

for every $f \in H_{\varrho,k+1}(\mathbb{R}, H_0)$. By density of $H_{\varrho,k+1}(\mathbb{R}, H_0)$ in $H_{\varrho,k}(\mathbb{R}, H_0)$ and uniform boundedness of $(S_\varepsilon)_{\varepsilon\geq 0}$ it follows that

$$|S_\varepsilon f - S_0 f|_{\varrho,k-2,0} \overset{\varepsilon\to 0}{\to} 0$$

for all $f \in H_{\varrho,k}(\mathbb{R}, H_0)$, which is the desired convergence result. □

Remark 6.3. The justification of the eddy-current model as the low electric permittivity limit of the classical Maxwell system is performed in [32, Theorem 2.5] with a focus on the frequency domain for fixed frequency. The quantitative estimate is of the order $O(\varepsilon)$ as $\varepsilon \to 0$. The estimates and derivations described in the proof above provide the same quantitative nature for fixed frequency (see the estimates in (33)). Since the above result covers the full time line (and thus all frequencies) *simultaneously* some time regularity loss has to be expected if one wants to keep the same order of ε. Indeed, also in [32, proof of Theorem 2.5] the frequency dependence of the quantitative estimate suggests a (time) regularity loss if one wants to keep the derived quantitative estimate for the full space-time problem (note the ω^2 in [32, proof of Theorem 2.5]). Furthermore, this effect has been observed in the context of Maxwell's equations in [21,42]. A similar observation can be made for approximations in quantitative homogenisation theory: Whereas for fixed frequencies one obtains optimal quantitative estimates [9,10], the estimates for the full space-time problem experience a loss of derivatives if one wants to retain the same quantitative behaviour, see [44,13]. It is possible to accommodate for this regularity loss with an analogue of Littlewood–Paley type spaces, see [9].

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