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Unique continuation for first-order systems with integrable coefficients and applications to elasticity and plasticity

Continuation unique pour des systèmes du premier ordre avec des coefficients intégrables et applications à l'élasticité et à la plasticité

Johannes Lankeit, Patrizio Neff, Dirk Pauly

Fakultät für Mathematik, Universität Duisburg-Essen, Thea-Leymann-Straße 9, 45127 Essen, Germany

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ABSTRACT

Let $\Omega \subset \mathbb{R}^N$ be a Lipschitz domain and Γ be a relatively open and non-empty subset of its boundary $\partial\Omega$. We show that the solution to the linear first-order system:

$$\nabla\zeta = G\zeta, \quad \zeta|_\Gamma = 0, \tag{1}$$

vanishes if $G \in \mathsf{L}^1(\Omega; \mathbb{R}^{(N\times N)\times N})$ and $\zeta \in \mathsf{W}^{1,1}(\Omega; \mathbb{R}^N)$. In particular, square-integrable solutions ζ of (1) with $G \in \mathsf{L}^1 \cap \mathsf{L}^2(\Omega; \mathbb{R}^{(N\times N)\times N})$ vanish. As a consequence, we prove that:

$$|||\cdot||| : \mathsf{C}^\infty_\circ(\Omega, \Gamma; \mathbb{R}^3) \to [0, \infty), \qquad u \mapsto \|\operatorname{sym}(\nabla u P^{-1})\|_{\mathsf{L}^2(\Omega)}$$

is a norm if $P \in \mathsf{L}^\infty(\Omega; \mathbb{R}^{3\times3})$ with $\operatorname{Curl} P \in \mathsf{L}^p(\Omega; \mathbb{R}^{3\times3})$, $\operatorname{Curl} P^{-1} \in \mathsf{L}^q(\Omega; \mathbb{R}^{3\times3})$ for some $p, q > 1$ with $1/p + 1/q = 1$ as well as $\det P \geqslant c^+ > 0$. We also give a new and different proof for the so-called 'infinitesimal rigid displacement lemma' in curvilinear coordinates: Let $\Phi \in \mathsf{H}^1(\Omega; \mathbb{R}^3)$, $\Omega \subset \mathbb{R}^3$, satisfy $\operatorname{sym}(\nabla\Phi^\top \nabla\Psi) = 0$ for some $\Psi \in \mathsf{W}^{1,\infty}(\Omega; \mathbb{R}^3) \cap \mathsf{H}^2(\Omega; \mathbb{R}^3)$ with $\det \nabla\Psi \geqslant c^+ > 0$. Then there exists a constant translation vector $a \in \mathbb{R}^3$ and a constant skew-symmetric matrix $A \in \mathfrak{so}(3)$, such that $\Phi = A\Psi + a$.

RÉSUMÉ

Soit $\Omega \subset \mathbb{R}^N$ un domaine et $\emptyset \neq \Gamma \subset \partial\Omega$ un sous-ensemble relativement ouvert de sa frontière $\partial\Omega$, supposée lipschitzienne. Nous démontrons que la solution du système linéaire du premier ordre :

$$\nabla\zeta = G\zeta, \quad \zeta|_\Gamma = 0, \tag{1}$$

s'annule si $G \in \mathsf{L}^1(\Omega; \mathbb{R}^{(N\times N)\times N})$ et $\zeta \in \mathsf{W}^{1,1}(\Omega; \mathbb{R}^N)$. En particulier, les solutions de carré intégrable de (1) avec $G \in \mathsf{L}^1 \cap \mathsf{L}^2(\Omega; \mathbb{R}^{(N\times N)\times N})$ s'annulent. Comme conséquence, nous prouvons que :

$$|||\cdot||| : \mathsf{C}^\infty_\circ(\Omega, \Gamma; \mathbb{R}^3) \to [0, \infty), \quad u \mapsto \|\operatorname{sym}(\nabla u P^{-1})\|_{\mathsf{L}^2(\Omega)}$$

est une norme lorsque $P \in \mathsf{L}^\infty(\Omega; \mathbb{R}^{3\times3})$ avec $\operatorname{Curl} P \in \mathsf{L}^p(\Omega; \mathbb{R}^{3\times3})$, $\operatorname{Curl} P^{-1} \in \mathsf{L}^q(\Omega; \mathbb{R}^{3\times3})$ pour $p, q > 1$, $1/p + 1/q = 1$, et $\det P \geqslant c^+ > 0$. Nous présentons aussi une nouvelle démonstration du lemme du déplacement rigide infinitésimal en coordonnées curvilignes :

E-mail addresses: johannes.lankeit@uni-due.de (J. Lankeit), patrizio.neff@uni-due.de (P. Neff), dirk.pauly@uni-due.de (D. Pauly).

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si $\Phi \in \mathrm{H}^1(\Omega;\mathbb{R}^3)$ satisfait $\operatorname{sym}(\nabla\Phi^\top\nabla\Psi)=0$ pour certain $\Psi \in \mathrm{W}^{1,\infty}(\Omega;\mathbb{R}^3)\cap \mathrm{H}^2(\Omega;\mathbb{R}^3)$, avec $\det\nabla\Psi \geqslant c^+ > 0$, il existe des constantes $a\in\mathbb{R}^3$ et $A\in\mathfrak{so}(3)$ telles que $\Phi = A\Psi + a$.

1. Introduction

Consider the linear first-order system of partial differential equations:

$$\nabla\zeta = G\,\zeta, \quad \zeta|_\Gamma = 0. \tag{2}$$

Obviously, one solution is $\zeta = 0$. But is this solution unique? The answer is not as obvious as it may seem; consider for example in dimension $N := 1$, $G(t) := 1/t$ in the domain $\Omega := (0,1)$ with $\Gamma := \{0\} \subset \partial\Omega$. Then $\zeta := \mathrm{id} \neq 0$ solves (2). However, in the latter example, the solution becomes unique if $G \in \mathrm{L}^1(\Omega)$, which is easily deduced from Gronwall's lemma. Here we can see that the integrability condition on the coefficient G is relevant; for a precise formulation of the result, see Section 2. The uniqueness of the solution to (2) makes:

$$|||u||| := \left\|\operatorname{sym}\left(\nabla u P^{-1}\right)\right\|_{\mathrm{L}^2(\Omega)} \tag{3}$$

a norm on

$$\mathrm{C}^\infty_\circ\left(\Omega,\Gamma;\mathbb{R}^3\right) := \left\{u \in \mathrm{C}^\infty\left(\overline{\Omega};\mathbb{R}^3\right):\ \operatorname{dist}(\operatorname{supp} u,\Gamma) > 0\right\},$$

where

$$\mathrm{C}^\infty\left(\overline{\Omega};\mathbb{R}^3\right) := \left\{u|_\Omega:\ u \in \mathrm{C}^\infty_\circ\left(\mathbb{R}^3;\mathbb{R}^3\right)\right\},$$

for $P \in \mathrm{L}^\infty(\Omega;\mathbb{R}^{3\times3})$ with $\det P \geqslant c^+ > 0$, if in addition $\operatorname{Curl} P \in \mathrm{L}^p(\Omega;\mathbb{R}^{3\times3})$, $\operatorname{Curl} P^{-1} \in \mathrm{L}^q(\Omega;\mathbb{R}^{3\times3})$ for some $p,q > 1$ and $1/q + 1/p = 1$. Here the Curl of a matrix field is defined as the row-wise standard curl in $\mathbb{R}^3$.

The question whether an expression of the form (3) is a norm arises when trying to generalize Korn's first inequality to hold for non-constant coefficients, i.e.,

$$\exists c > 0 \quad \forall u \in \mathrm{H}^1_\circ\left(\Omega,\Gamma;\mathbb{R}^3\right) \quad \left\|\operatorname{sym}\left(\nabla u P^{-1}\right)\right\|_{\mathrm{L}^2(\Omega)} \geqslant c\|u\|_{\mathrm{H}^1(\Omega)}, \tag{4}$$

which was first done for P, P^{-1}, $\operatorname{Curl} P \in \mathrm{C}^1(\overline{\Omega};\mathbb{R}^{3\times3})$ by Neff in [7], see also [16]. Here $\mathrm{H}^1_\circ(\Omega,\Gamma;\mathbb{R}^3)$ denotes the closure of $\mathrm{C}^\infty_\circ(\Omega,\Gamma;\mathbb{R}^3)$ in $\mathrm{H}^1(\Omega;\mathbb{R}^3)$. The classical Korn's first inequality is obtained for P being the identity matrix, see [3,5,7,13,14]. The inequality (4) has been proved in [16] to hold for continuous P^{-1}, whereas it can be violated for $P^{-1} \in \mathrm{L}^\infty(\Omega)$ or $P^{-1} \in \mathrm{SO}(3)$ a.e. Each one of the counterexamples given by Pompe in [15–17] uses the fact that for such P, an expression of the form of $|||\cdot|||$ is not a norm (it has a nontrivial kernel) on the spaces of functions considered. Quadratic forms of the type (4) arise in applications to geometrically exact models of shells, plates and membranes, in micromorphic and Cosserat type models and in plasticity, [8–11].

The so-called 'infinitesimal rigid displacement lemma in curvilinear coordinates', a version of which can be found in [1] and which is important for linear elasticity in curvilinear coordinates (see also [2,4]) states the following: if $\Omega \subset \mathbb{R}^N$ is a bounded domain, $\Psi \in \mathrm{W}^{1,\infty}(\Omega;\mathbb{R}^N)$ satisfying $\det\nabla\Psi \geqslant c^+ > 0$ a.e. and $\Phi \in \mathrm{H}^1(\Omega;\mathbb{R}^N)$ with $\operatorname{sym}(\nabla\Phi^\top\nabla\Psi) = 0$ a.e., then on a dense open subset Ω' of Ω, there exist locally constant mappings $a : \Omega' \to \mathbb{R}^N$ and $A : \Omega' \to \mathfrak{so}(N)$ such that locally $\Phi = A\Psi + a$. If Ω is Lipschitz, then the terms 'locally' can be dropped. In their proof [1], the authors apply the chain rule to $\Theta := \Phi \circ \Psi^{-1}$ and use the observation that the conditions $\operatorname{sym}(\nabla\Phi^\top\nabla\Psi) = 0$ and $\operatorname{sym}(\nabla\Phi(\nabla\Psi)^{-1}) = 0$ are equivalent by a clever conjugation with $(\nabla\Psi)^{-1}$, that is:

$$(\nabla\Psi)^{-\top}\operatorname{sym}\left(\nabla\Phi^\top\nabla\Psi\right)(\nabla\Psi)^{-1} = \operatorname{sym}\left(\nabla\Phi(\nabla\Psi)^{-1}\right) = \operatorname{sym}\left(\nabla\left(\Phi\circ\Psi^{-1}\right)\right)\circ\Psi, \tag{5}$$

together with the classical infinitesimal rigid displacement lemma applied on Θ, defined on the domain $\Psi(\Omega)$. If a boundary condition $\Phi = 0$ on a relatively open subset of the boundary is added to this lemma, one obtains $\Phi = 0$ (cf. [2, 1.7-3(b)]).

The main part of our proof that $|||\cdot|||$ is a norm consists in obtaining $u = 0$ from $\operatorname{sym}(\nabla u P^{-1}) = 0$. By taking $P = \nabla\Psi$ to be a gradient, we present another proof of the infinitesimal rigid displacement lemma in dimension $N = 3$, which yields $\Phi = A\Psi + a$ with $A \in \mathfrak{so}(N)$, $a \in \mathbb{R}^N$. We need slightly more regularity than in [1], however. The key tool for obtaining our results is Neff's formula for the Curl of the product of two matrices, the first of which is skew-symmetric (see [7]).

2. Results

Let us first note that by ∇ we denote not only the gradient of a scalar-valued function, but also (as an usual gradient row-wise) the derivative or Jacobian of a vector field. The Curl of a matrix is to be taken row-wise as a usual curl for vector fields.

Theorem 2.1 *(Unique continuation). Let $\Omega \subset \mathbb{R}^N$, $N \in \mathbb{N}$, be a Lipschitz domain, Γ be a relatively open and non-empty subset of $\partial\Omega$ and $G \in \mathsf{L}^1(\Omega; \mathbb{R}^{(N\times N)\times N})$. If $\zeta \in \mathsf{W}^{1,1}(\Omega; \mathbb{R}^N)$ solves*

$$\nabla\zeta = G\,\zeta, \quad \zeta|_\Gamma = 0,$$

then $\zeta = 0$.

The differential equation itself cannot guarantee that a weak solution $\zeta \in \mathsf{L}^1(\Omega, \mathbb{R}^N)$ necessarily belongs to $\mathsf{W}^{1,1}(\Omega; \mathbb{R}^N)$. But this can be ensured by requiring higher integrability of G and ζ, since for bounded domains, e.g., the conditions $G \in \mathsf{L}^2(\Omega)$ and $\zeta \in \mathsf{L}^2(\Omega)$ imply $\nabla\zeta \in \mathsf{L}^1(\Omega)$ and hence $\zeta \in \mathsf{W}^{1,1}(\Omega)$; then an application of the theorem ensures that $\zeta = 0$. Thus we obtain the uniqueness of $\mathsf{L}^2(\Omega)$-solutions if the coefficients G are square-integrable. Of course, the same holds if $\zeta \in \mathsf{L}^p(\Omega)$ for arbitrary $p \geqslant 1$. Then G at least needs to be an $\mathsf{L}^q(\Omega)$-function, where $1/p + 1/q = 1$.

Theorem 2.2 *(Norm). Let $\Omega \subset \mathbb{R}^3$ be a Lipschitz domain, $\emptyset \neq \Gamma \subset \partial\Omega$ be relatively open, $P \in \mathsf{L}^\infty(\Omega; \mathbb{R}^{3\times3})$ with $\det P \geqslant c^+ > 0$, $\operatorname{Curl} P \in \mathsf{L}^p(\Omega; \mathbb{R}^{3\times3})$, $\operatorname{Curl} P^{-1} \in \mathsf{L}^q(\Omega; \mathbb{R}^{3\times3})$ for some $p, q > 1$ with $1/p + 1/q = 1$. Then*

$$|\!|\!|\cdot|\!|\!| : \mathsf{C}^\infty_\circ(\Omega, \Gamma; \mathbb{R}^3) \to [0, \infty), \qquad u \mapsto \left\|\operatorname{sym}(\nabla u P^{-1})\right\|_{\mathsf{L}^2(\Omega)} \tag{6}$$

defines a norm.

Remark 2.3. In the case of $p = q = 2$ and for $P \in \mathrm{SO}(3)$ a.e., $\operatorname{Curl} P^{-1} \in \mathsf{L}^2(\Omega)$ is no additional condition, since then $\operatorname{Curl} P \in \mathsf{L}^2(\Omega) \Leftrightarrow \operatorname{Curl} P^{-1} \in \mathsf{L}^2(\Omega)$. (Note that if $P \in \mathrm{SO}(3)$ a.e., then P, $\operatorname{Curl} P \in \mathsf{L}^p(\Omega)$ is equivalent to $P \in \mathsf{W}^{1,p}(\Omega)$, cf. [12].)

Conjecture 2.4. *Theorem 2.2 holds for $P \in \mathsf{L}^\infty(\Omega)$ with $\operatorname{Curl} P \in \mathsf{L}^p(\Omega)$ and $\det P \geqslant c^+ > 0$ for some $p > 1$ or even $p \geqslant 1$.*

Remark 2.5. Since the norms $|\!|\!|\cdot|\!|\!|$ and $\|\cdot\|_{\mathsf{H}^1(\Omega)}$ are not shown to be equivalent, it is not clear whether the spaces $\mathsf{H}^1_\circ(\Omega, \Gamma) = \overline{\mathsf{C}^\infty_\circ(\Omega, \Gamma)}^{\|\cdot\|_{\mathsf{H}^1(\Omega)}}$ and $\overline{\mathsf{C}^\infty_\circ(\Omega, \Gamma)}^{|\!|\!|\cdot|\!|\!|}$ coincide. However, by [16], these norms are equivalent if $P \in \mathsf{C}^0(\overline{\Omega})$ with $\det P \geqslant c^+ > 0$.

Conjecture 2.6. *The norms are equivalent if $P \in \mathsf{L}^\infty(\Omega)$ with $\operatorname{Curl} P \in \mathsf{L}^p(\Omega)$ and $\det P \geqslant c^+ > 0$ for some $p > 1$ or even $p \geqslant 1$.*

Theorem 2.7 *(Infinitesimal rigid displacement lemma). Let $\Omega \subset \mathbb{R}^3$ be a Lipschitz domain. Moreover, let $\Phi \in \mathsf{W}^{1,p}(\Omega; \mathbb{R}^3)$ and $\Psi \in \mathsf{W}^{1,\infty}(\Omega; \mathbb{R}^3) \cap \mathsf{W}^{2,q}(\Omega; \mathbb{R}^3)$ with $\det \nabla\Psi \geqslant c^+ > 0$ a.e. and $p, q > 1$, $1/p + 1/q = 1$. If*

$$\operatorname{sym}(\nabla\Phi^\top \nabla\Psi) = 0$$

then there exist $a \in \mathbb{R}^3$ and a constant skew-symmetric matrix $A \in \mathfrak{so}(3)$, such that $\Phi = A\Psi + a$.

3. Sketch of proofs

An application of Gronwall's inequality yields Theorem 2.1 for Ω being an interval. The case where Ω is a cube, and Γ a face of it can be reduced to this situation, ensuring also the unique continuation property for (1). For a general Lipschitz domain Ω we use a transformation of Γ and a neighborhood thereof onto such a cube. The proofs of Theorem 2.2 and Theorem 2.7 both rely heavily on the formula for the Curl of a product of two matrices, which reads (see [7]):

$$\operatorname{Curl}(XY) = \operatorname{mat} L_Y(\operatorname{vec}\nabla \operatorname{axl} X) + X \operatorname{Curl} Y, \quad \det L_Y = -2(\det Y)^3$$

in the special case of a skew-symmetric X. There $\operatorname{mat}: \mathbb{R}^9 \to \mathbb{R}^{3\times3}$, $\operatorname{vec}: \mathbb{R}^{3\times3} \to \mathbb{R}^9$, $\operatorname{axl}: \mathfrak{so}(3) \to \mathbb{R}^3$ and $L: \mathbb{R}^{3\times3} \to \mathbb{R}^{9\times9}$ denote usual identification operators. It is used to reduce Theorem 2.2 to (1) where an application of Theorem 2.1 yields the definiteness. This formula is also used to prove the infinitesimal rigid displacement lemma: with a suitable approximation of $A = \nabla\Phi(\nabla\Psi)^{-1}$, it is possible to show that the weak partial derivatives of A vanish. The details can be found in [6].

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