DE GRUYTER Comput. Methods Appl. Math. 2016; 16 (4):609–631

Research Article

Immanuel Anjam* and Dirk Pauly

Functional A Posteriori Error Control for Conforming Mixed Approximations of Coercive Problems with Lower Order Terms

DOI: 10.1515/cmam-2016-0016
Received December 29, 2015; revised March 16, 2016; accepted March 18, 2016

Abstract: The results of this contribution are derived in the framework of functional type a posteriori error estimates. The error is measured in a combined norm which takes into account both the primal and dual variables denoted by x and y, respectively. Our first main result is an error equality for all equations of the class $\mathrm{A}^*\mathrm{A}x + x = f$ or in mixed formulation $\mathrm{A}^*y + x = f$, $\mathrm{A}x = y$, where the exact solution (x, y) is in $D(\mathrm{A}) \times D(\mathrm{A}^*)$. Here A is a linear, densely defined and closed (usually a differential) operator and A^* its adjoint. In this paper we deal with very conforming mixed approximations, i.e., we assume that the approximation $(\tilde{x}, \tilde{y})$ belongs to $D(\mathrm{A}) \times D(\mathrm{A}^*)$. In order to obtain the exact global error value of this approximation one only needs the problem data and the mixed approximation itself, i.e., we have the *equality*

$$|x - \tilde{x}|^2 + |\mathrm{A}(x - \tilde{x})|^2 + |y - \tilde{y}|^2 + |\mathrm{A}^*(y - \tilde{y})|^2 = \mathcal{M}(\tilde{x}, \tilde{y}),$$

where $\mathcal{M}(\tilde{x}, \tilde{y}) := |f - \tilde{x} - \mathrm{A}^*\tilde{y}|^2 + |\tilde{y} - \mathrm{A}\tilde{x}|^2$ contains only known data. Our second main result is an error estimate for all equations of the class $\mathrm{A}^*\mathrm{A}x + ix = f$ or in mixed formulation $\mathrm{A}^*y + ix = f$, $\mathrm{A}x = y$, where i is the imaginary unit. For this problem we have the *two-sided estimate*

$$\frac{\sqrt{2}}{\sqrt{2}+1}\mathcal{M}_i(\tilde{x}, \tilde{y}) \le |x - \tilde{x}|^2 + |\mathrm{A}(x - \tilde{x})|^2 + |y - \tilde{y}|^2 + |\mathrm{A}^*(y - \tilde{y})|^2 \le \frac{\sqrt{2}}{\sqrt{2}-1}\mathcal{M}_i(\tilde{x}, \tilde{y}),$$

where $\mathcal{M}_i(\tilde{x}, \tilde{y}) := |f - i\tilde{x} - \mathrm{A}^*\tilde{y}|^2 + |\tilde{y} - \mathrm{A}\tilde{x}|^2$ contains only known data. We will point out a motivation for the study of the latter problems by time discretizations or time-harmonic ansatz of linear partial differential equations and we will present an extensive list of applications including the reaction-diffusion problem and the eddy current problem.

Keywords: Functional A Posteriori Error Estimates, Error Equalities, Mixed Formulations, Combined Norm

MSC 2010: 65N15

Dedicated to Sergey Igorevich Repin

1 Introduction

The results presented in this paper are based on the conception of functional type a posteriori error control. Often these type of estimates are valid for any conforming approximation and contain only global constants. In the case of the class of problems studied in this paper the results do not contain even global constants, just fixed numbers. For a detailed exposition see the books by Repin, Neittaanmäki, and Mali [7, 8, 12].

***Corresponding author: Immanuel Anjam:** Fakultät für Mathematik, Universität Duisburg-Essen, Campus Essen, Germany, e-mail: immanuel.anjam@uni-due.de
Dirk Pauly: Fakultät für Mathematik, Universität Duisburg-Essen, Campus Essen, Germany, e-mail: dirk.pauly@uni-due.de

In this paper we will consider only conforming approximations, and we will measure the error of our approximations in a combined norm, which includes the error of both the primal and the dual variable. This is especially useful for mixed methods where one calculates an approximation for both the primal and dual variables, see, e.g., the book of Brezzi and Fortin [3]. We call this approximation pair a mixed approximation. We note here that we consider more regular mixed approximations than in [3]. This regularity can always be achieved by post-processing techniques.

To the best of our knowledge functional a posteriori error estimates for combined norms were first exposed in the paper [14], where Repin, Sauter, and Smolianski present two-sided estimates bounding the error by the same quantity from below and from above aside from basic and global Poincaré type constants and some special numbers. They studied real-valued elliptic problems of the type $\mathrm{A}^*\alpha\mathrm{A}x = f$ given in mixed formulation $\mathrm{A}^*y = f$, $\alpha\mathrm{A}x = y$.

The first class of problems we study in the paper at hand is the linear equation

$$(\mathrm{A}^*\alpha_2\mathrm{A} + \alpha_1)x = f$$

presented in the mixed formulation

$$\mathrm{A}^*y + \alpha_1 x = f, \quad \alpha_2\mathrm{A}x = y, \tag{1.1}$$

where α_1, α_2 are linear, self-adjoint, and uniformly positive topological isomorphisms (continuous with continuous inverse) on two complex Hilbert spaces H_1 and H_2, and $\mathrm{A} : D(\mathrm{A}) \subset \mathsf{H}_1 \to \mathsf{H}_2$ is a linear, densely defined, and closed operator with adjoint operator $\mathrm{A}^* : D(\mathrm{A}^*) \subset \mathsf{H}_2 \to \mathsf{H}_1$. Throughout this paper we will refer to the class of problems represented by (1.1) as 'Case I' in section headings. Our first main result is Theorem 2.5 and it shortly reads as the functional a posteriori error equality

$$|x - \tilde{x}|^2_{\mathsf{H}_1,\alpha_1} + |\mathrm{A}(x - \tilde{x})|^2_{\mathsf{H}_2,\alpha_2} + |y - \tilde{y}|^2_{\mathsf{H}_2,\alpha_2^{-1}} + |\mathrm{A}^*(y - \tilde{y})|^2_{\mathsf{H}_1,\alpha_1^{-1}} = |f - \alpha_1\tilde{x} - \mathrm{A}^*\tilde{y}|^2_{\mathsf{H}_1,\alpha_1^{-1}} + |\tilde{y} - \alpha_2\mathrm{A}\tilde{x}|^2_{\mathsf{H}_2,\alpha_2^{-1}}$$

being valid for any conforming mixed approximation pair $(\tilde{x}, \tilde{y}) \in D(\mathrm{A}) \times D(\mathrm{A}^*)$ of the exact solution pair $(x, y) \in D(\mathrm{A}) \times D(\mathrm{A}^*)$. In the purely real case this result can also be derived as a special case of the very general result [8, (7.2.14)] in the context of the dual variational technique. However, we prove this result here by elementary methods in a general Hilbert space setting. Our results hold then also for the complex case. The equality for the purely real reaction-diffusion equation ($\mathrm{A} = \nabla$, $\mathrm{A}^* = -\operatorname{div}$), was found also by Cai and Zhang [4, Remark 6.12] and has been used for error indication of the primal variable.

The second class of problems we study in this paper is the linear equation

$$(\mathrm{A}^*\alpha_2\mathrm{A} + i\omega\alpha_1)x = f$$

presented in the mixed formulation

$$\mathrm{A}^*y + i\omega\alpha_1 x = f, \quad \alpha_2\mathrm{A}x = y, \tag{1.2}$$

where $\omega \in \mathbb{R} \setminus \{0\}$. Throughout this paper we will refer to the class of problems represented by (1.2) as 'Case II' in section headings. Our second main result is Theorem 2.13 and it shortly reads as the two-sided functional a posteriori error estimate

$$\begin{aligned}
&\frac{\sqrt{2}}{\sqrt{2}+1}\Big(|f - i\omega\alpha_1\tilde{x} - \mathrm{A}^*\tilde{y}|^2_{\mathsf{H}_1,(|\omega|\alpha_1)^{-1}} + |\tilde{y} - \alpha_2\mathrm{A}\tilde{x}|^2_{\mathsf{H}_2,\alpha_2^{-1}}\Big) \\
&\quad \le |x - \tilde{x}|^2_{\mathsf{H}_1,|\omega|\alpha_1} + |\mathrm{A}(x - \tilde{x})|^2_{\mathsf{H}_2,\alpha_2} + |y - \tilde{y}|^2_{\mathsf{H}_2,\alpha_2^{-1}} + |\mathrm{A}^*(y - \tilde{y})|^2_{\mathsf{H}_1,(|\omega|\alpha_1)^{-1}} \\
&\quad \le \frac{\sqrt{2}}{\sqrt{2}-1}\Big(|f - i\omega\alpha_1\tilde{x} - \mathrm{A}^*\tilde{y}|^2_{\mathsf{H}_1,(|\omega|\alpha_1)^{-1}} + |\tilde{y} - \alpha_2\mathrm{A}\tilde{x}|^2_{\mathsf{H}_2,\alpha_2^{-1}}\Big)
\end{aligned}$$

being valid for any conforming mixed approximation pair $(\tilde{x}, \tilde{y}) \in D(\mathrm{A}) \times D(\mathrm{A}^*)$ of the exact solution pair $(x, y) \in D(\mathrm{A}) \times D(\mathrm{A}^*)$. We note that the square root of the ratio of the upper bound and lower bound is always $1 + \sqrt{2} < 2.42$, so the estimate gives reliable information of the combined error value. To the best of our knowledge this result is new.

A motivation to study these problems comes from time-dependent PDEs. For many problems, if the time-derivative is discretized with 'finite differences', e.g., the backward Euler scheme, then on each time-step one solves a static problem of the type (1.1). On the other hand, many time-dependent problems, e.g., the eddy current problem, can be approximated by a series resp. sum of static complex-valued problems of the kind (1.2) by using multifrequency analysis, e.g., Fourier transformation. We elaborate on this in Section 2.4.

The paper is organized as follows. In Section 2 we derive our main results in an abstract Hilbert space setting. In Section 3 we show applications of the general results to several partial differential equations.

2 Results in the General Setting

In this section we derive our main results in an abstract Hilbert space setting, which allows for mixed boundary conditions as well as coefficients for the case, where the underlying problem is a PDE.

Let H_1 and H_2 be two complex Hilbert spaces with the inner products $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathsf{H}_1}$ and $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathsf{H}_2}$, respectively. The right-hand side f belongs to H_1. Let $\mathrm{A} : D(\mathrm{A}) \subset \mathsf{H}_1 \to \mathsf{H}_2$ be a densely defined and closed linear operator and $\mathrm{A}^* : D(\mathrm{A}^*) \subset \mathsf{H}_2 \to \mathsf{H}_1$ its adjoint. We note $\mathrm{A}^{**} = \mathrm{A}$ and

$$\langle \mathrm{A}\varphi, \psi\rangle_{\mathsf{H}_2} = \langle \varphi, \mathrm{A}^*\psi\rangle_{\mathsf{H}_1} \quad \text{for all } \varphi \in D(\mathrm{A}),\ \psi \in D(\mathrm{A}^*). \tag{2.1}$$

Equipped with the natural graph norms, $D(\mathrm{A})$ and $D(\mathrm{A}^*)$ are Hilbert spaces. Furthermore, we introduce two linear, self-adjoint, and positive topological isomorphisms $\alpha_1 : \mathsf{H}_1 \to \mathsf{H}_1$ and $\alpha_2 : \mathsf{H}_2 \to \mathsf{H}_2$. Especially, there exists a $c > 0$ such that

$$c^{-1}|\varphi|^2_{\mathsf{H}_1} \le \langle \alpha_1\varphi, \varphi\rangle_{\mathsf{H}_1} \le c|\varphi|^2_{\mathsf{H}_1} \quad \text{for all } \varphi \in \mathsf{H}_1,$$

and the corresponding holds for α_2. In case the underlying problem is a PDE, the operators α_1 and α_2 describe material properties, and are often called material coefficients, giving the constitutive laws.

For any inner product and corresponding norm we introduce weighted counterparts with sub-index notation. As an example, for elements from H_1 we define a new inner product $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathsf{H}_1,\alpha_1} := \langle \alpha_1\,\cdot\,,\,\cdot\,\rangle_{\mathsf{H}_1}$ and a new induced norm $|\cdot|_{\mathsf{H}_1,\alpha_1}$. Note that in Section 2.2 we slightly abuse this notation: We also utilize $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathsf{H}_1,\omega\alpha_1} = \langle \omega\alpha_1\,\cdot\,,\,\cdot\,\rangle_{\mathsf{H}_1}$, where $\omega \ne 0$ is possibly a negative real number. Clearly, this sesquilinear form neither defines an inner product nor a norm, if ω is negative.

2.1 Case I: Error Equality for Coefficients α_1 and α_2

Extending the sub-index notation, we define for $\varphi \in D(\mathrm{A})$ and $\psi \in D(\mathrm{A}^*)$ new weighted norms on $D(\mathrm{A})$, $D(\mathrm{A}^*)$ and on the product space $D(\mathrm{A}) \times D(\mathrm{A}^*)$ by

$$\begin{aligned}
|\varphi|^2_{D(\mathrm{A}),\alpha_1,\alpha_2} &:= |\varphi|^2_{\mathsf{H}_1,\alpha_1} + |\mathrm{A}\varphi|^2_{\mathsf{H}_2,\alpha_2},\\
|\psi|^2_{D(\mathrm{A}^*),\alpha_1^{-1},\alpha_2^{-1}} &:= |\psi|^2_{\mathsf{H}_2,\alpha_2^{-1}} + |\mathrm{A}^*\psi|^2_{\mathsf{H}_1,\alpha_1^{-1}},\\
\|(\varphi,\psi)\|^2 &:= |\varphi|^2_{D(\mathrm{A}),\alpha_1,\alpha_2} + |\psi|^2_{D(\mathrm{A}^*),\alpha_1^{-1},\alpha_2^{-1}}.
\end{aligned}$$

By the Lax–Milgram lemma (or by Riesz' representation theorem) we get immediately:

Lemma 2.1. *The (primal) variational problem*

$$\langle \mathrm{A}x, \mathrm{A}\varphi\rangle_{\mathsf{H}_2,\alpha_2} + \langle x, \varphi\rangle_{\mathsf{H}_1,\alpha_1} = \langle f, \varphi\rangle_{\mathsf{H}_1} \quad \textit{for all } \varphi \in D(\mathrm{A})$$

admits a unique solution $x \in D(\mathrm{A})$ satisfying $|x|_{D(\mathrm{A}),\alpha_1,\alpha_2} \le |f|_{\mathsf{H}_1,\alpha_1^{-1}}$. Moreover, $y_x := \alpha_2\mathrm{A}x$ belongs to $D(\mathrm{A}^*)$ and $\mathrm{A}^*y_x = f - \alpha_1 x$. Hence, the strong and mixed formulations

$$\mathrm{A}^*\alpha_2\mathrm{A}x + \alpha_1 x = f, \tag{2.2}$$

$$\mathrm{A}^*y_x + \alpha_1 x = f, \quad \alpha_2\mathrm{A}x = y_x \tag{2.3}$$

hold with $(x, y_x) \in D(\mathrm{A}) \times (D(\mathrm{A}^*) \cap \alpha_2 R(\mathrm{A}))$.

To get the dual problem, we multiply the first equation of (2.3) by $\mathrm{A}^*\psi$ with $\psi \in D(\mathrm{A}^*)$ taking the right weighted scalar product and use $y_x = \alpha_2 \mathrm{A}x \in D(\mathrm{A}^*)$. We obtain

$$\langle \mathrm{A}^* y_x, \mathrm{A}^*\psi\rangle_{\mathrm{H}_1,\alpha_1^{-1}} + \langle \alpha_1 x, \mathrm{A}^*\psi\rangle_{\mathrm{H}_1,\alpha_1^{-1}} = \langle f, \mathrm{A}^*\psi\rangle_{\mathrm{H}_1,\alpha_1^{-1}}.$$

Since $x \in D(\mathrm{A})$, we have

$$\langle \alpha_1 x, \mathrm{A}^*\psi\rangle_{\mathrm{H}_1,\alpha_1^{-1}} = \langle x, \mathrm{A}^*\psi\rangle_{\mathrm{H}_1} = \langle \mathrm{A}x, \psi\rangle_{\mathrm{H}_2} = \langle y_x, \psi\rangle_{\mathrm{H}_2,\alpha_2^{-1}}.$$

Again by the Lax–Milgram lemma, we get the following result.

Lemma 2.2. *The (dual) variational problem*

$$\langle \mathrm{A}^* y, \mathrm{A}^*\psi\rangle_{\mathrm{H}_1,\alpha_1^{-1}} + \langle y, \psi\rangle_{\mathrm{H}_2,\alpha_2^{-1}} = \langle f, \mathrm{A}^*\psi\rangle_{\mathrm{H}_1,\alpha_1^{-1}} \quad \text{for all } \psi \in D(\mathrm{A}^*) \tag{2.4}$$

admits a unique solution $y \in D(\mathrm{A}^*)$ satisfying $|y|_{D(\mathrm{A}^*),\alpha_1^{-1},\alpha_2^{-1}} \le |f|_{\mathrm{H}_1,\alpha_1^{-1}}$. Moreover, $y = y_x$ holds and thus y even belongs to $D(\mathrm{A}^*) \cap \alpha_2 R(\mathrm{A})$ with x and y_x from Lemma 2.1. Furthermore, $\alpha_1^{-1}(\mathrm{A}^* y - f) \in D(\mathrm{A})$ with $A\alpha_1^{-1}(\mathrm{A}^* y - f) = -\alpha_2^{-1} y$.

Proof. We just have to show that $y_x \in D(\mathrm{A}^*)$ solves (2.4). But this follows directly since, for all $\psi \in D(\mathrm{A}^*)$,

$$\begin{aligned}\langle \mathrm{A}^* y_x, \mathrm{A}^*\psi\rangle_{\mathrm{H}_1,\alpha_1^{-1}} &= -\langle x, \mathrm{A}^*\psi\rangle_{\mathrm{H}_1} + \langle f, \mathrm{A}^*\psi\rangle_{\mathrm{H}_1,\alpha_1^{-1}} \\ &= -\langle \mathrm{A}x, \psi\rangle_{\mathrm{H}_2} + \langle f, \mathrm{A}^*\psi\rangle_{\mathrm{H}_1,\alpha_1^{-1}} = -\langle y_x, \psi\rangle_{\mathrm{H}_2,\alpha_2^{-1}} + \langle f, \mathrm{A}^*\psi\rangle_{\mathrm{H}_1,\alpha_1^{-1}}.\end{aligned}$$

Hence $y_x = y$ and $\mathrm{A}^{**} = \mathrm{A}$ completes the proof. □

Remark 2.3. We know that $|x|_{D(\mathrm{A}),\alpha_1,\alpha_2} \le |f|_{\mathrm{H}_1,\alpha_1^{-1}}$ and $|y|_{D(\mathrm{A}^*),\alpha_1^{-1},\alpha_2^{-1}} \le |f|_{\mathrm{H}_1,\alpha_1^{-1}}$. It is indeed notable that

$$\|(x, y)\| = |f|_{\mathrm{H}_1,\alpha_1^{-1}}$$

holds, which follows immediately by $y = \alpha_2 \mathrm{A}x$ and

$$\begin{aligned}|f|^2_{\mathrm{H}_1,\alpha_1^{-1}} &= |\mathrm{A}^*\alpha_2\mathrm{A}x + \alpha_1 x|^2_{\mathrm{H}_1,\alpha_1^{-1}} \\ &= |\mathrm{A}^* y|^2_{\mathrm{H}_1,\alpha_1^{-1}} + |\alpha_1 x|^2_{\mathrm{H}_1,\alpha_1^{-1}} + 2\Re\underbrace{\langle \mathrm{A}^*\alpha_2\mathrm{A}x, \alpha_1 x\rangle_{\mathrm{H}_1,\alpha_1^{-1}}}_{=\langle \mathrm{A}^*\alpha_2\mathrm{A}x, x\rangle_{\mathrm{H}_1}} \\ &= |\mathrm{A}^* y|^2_{\mathrm{H}_1,\alpha_1^{-1}} + |x|^2_{\mathrm{H}_1,\alpha_1} + 2\underbrace{\Re\langle \alpha_2\mathrm{A}x, \mathrm{A}x\rangle_{\mathrm{H}_2}}_{=|\mathrm{A}x|^2_{\mathrm{H}_2,\alpha_2}} \\ &= \|(x, y)\|^2.\end{aligned}$$

Thus the solution operator

$$L : \mathrm{H}_1 \to D(\mathrm{A}) \times D(\mathrm{A}^*), \quad f \mapsto (x, y)$$

(equipped with the proper weighed norms) has norm $|L| = 1$, i.e., L is an isometry.

By the latter remark the combined norm on $D(\mathrm{A}) \times D(\mathrm{A}^*)$ yields an isometry. This motivates the usage of the combined norm also for error estimates. As it turns out, we even obtain error equalities. First we show that an error equality follows directly from the isometry property of Remark 2.3 if the approximation of the primal variable x is regular enough.

Theorem 2.4. *Let $(x, y) \in D(\mathrm{A}) \times D(\mathrm{A}^*)$ be the exact solution of (2.3). Let $\tilde{x} \in D(\mathrm{A})$ have additional regularity so that $\tilde{y} = \alpha_2\mathrm{A}\tilde{x} \in D(\mathrm{A}^*)$. Then, for the mixed approximation $(\tilde{x}, \tilde{y})$ we have*

$$\|(x, y) - (\tilde{x}, \tilde{y})\|^2 = \mathcal{I}(\tilde{x}, \tilde{y}) \tag{2.5}$$

and the normalized counterpart

$$\frac{\|(x, y) - (\tilde{x}, \tilde{y})\|^2}{\|(x, y)\|^2} = \frac{\mathcal{I}(\tilde{x}, \tilde{y})}{|f|^2_{\mathrm{H}_1,\alpha_1^{-1}}}, \tag{2.6}$$

where

$$\mathcal{I}(\tilde{x}, \tilde{y}) := |f - \alpha_1\tilde{x} - \mathrm{A}^*\tilde{y}|^2_{\mathrm{H}_1,\alpha_1^{-1}}.$$

Proof. Since $\tilde{x}$ is very regular, especially $\tilde{y} = \alpha_2 \mathrm{A}\tilde{x} \in D(\mathrm{A}^*)$, the pair $(\tilde{x}, \tilde{y})$ is the exact solution of the problem

$$\mathrm{A}^*\tilde{y} + \alpha_1\tilde{x} =: \tilde{f}, \quad \alpha_2\mathrm{A}\tilde{x} = \tilde{y},$$

i.e., we have $L(\tilde{f}) = (\tilde{x}, \tilde{y})$. Then (2.5) is given directly by Remark 2.3:

$$\|(x, y) - (\tilde{x}, \tilde{y})\|^2 = \|L(f - \tilde{f})\|^2 = |f - \tilde{f}|^2_{\mathrm{H}_1, \alpha_1^{-1}},$$

since L is linear. The estimate (2.6) follows by Remark 2.3 as well. □

Satisfying the high regularity property required in Theorem 2.4 may not be convenient for practical calculations. The next result, the first main result of the paper, holds for less regular approximations.

Theorem 2.5. *Let $(x, y), (\tilde{x}, \tilde{y}) \in D(\mathrm{A}) \times D(\mathrm{A}^*)$ be the exact solution of (2.3) and any conforming approximation, respectively. Then*

$$\|(x, y) - (\tilde{x}, \tilde{y})\|^2 = \mathcal{M}(\tilde{x}, \tilde{y}) \tag{2.7}$$

and the normalized counterpart

$$\frac{\|(x, y) - (\tilde{x}, \tilde{y})\|^2}{\|(x, y)\|^2} = \frac{\mathcal{M}(\tilde{x}, \tilde{y})}{|f|^2_{\mathrm{H}_1, \alpha_1^{-1}}} \tag{2.8}$$

hold, where

$$\mathcal{M}(\tilde{x}, \tilde{y}) := |f - \alpha_1\tilde{x} - \mathrm{A}^*\tilde{y}|^2_{\mathrm{H}_1, \alpha_1^{-1}} + |\tilde{y} - \alpha_2\mathrm{A}\tilde{x}|^2_{\mathrm{H}_2, \alpha_2^{-1}}.$$

Proof. Using (2.2) and inserting $0 = \alpha_2\mathrm{A}x - y$, we get by (2.1)

$$\begin{aligned}
\mathcal{M}(\tilde{x}, \tilde{y}) &= |\alpha_1 x - \alpha_1\tilde{x} + \mathrm{A}^*y - \mathrm{A}^*\tilde{y}|^2_{\mathrm{H}_1, \alpha_1^{-1}} + |\tilde{y} - y + \alpha_2\mathrm{A}x - \alpha_2\mathrm{A}\tilde{x}|^2_{\mathrm{H}_2, \alpha_2^{-1}} \\
&= |x - \tilde{x}|^2_{\mathrm{H}_1, \alpha_1} + |\mathrm{A}^*(y - \tilde{y})|^2_{\mathrm{H}_1, \alpha_1^{-1}} + 2\Re\langle \alpha_1(x - \tilde{x}), \mathrm{A}^*(y - \tilde{y})\rangle_{\mathrm{H}_1, \alpha_1^{-1}} \\
&\qquad + |\tilde{y} - y|^2_{\mathrm{H}_2, \alpha_2^{-1}} + |\mathrm{A}(x - \tilde{x})|^2_{\mathrm{H}_2, \alpha_2} + 2\Re\langle \tilde{y} - y, \alpha_2\mathrm{A}(x - \tilde{x})\rangle_{\mathrm{H}_2, \alpha_2^{-1}} \\
&= |x - \tilde{x}|^2_{D(\mathrm{A}), \alpha_1, \alpha_2} + |y - \tilde{y}|^2_{D(\mathrm{A}^*), \alpha_1^{-1}, \alpha_2^{-1}} + 2\Re\langle x - \tilde{x}, \mathrm{A}^*(y - \tilde{y})\rangle_{\mathrm{H}_1} - 2\Re\langle \mathrm{A}(x - \tilde{x}), y - \tilde{y}\rangle_{\mathrm{H}_2} \\
&= \|(x, y) - (\tilde{x}, \tilde{y})\|^2.
\end{aligned}$$

Equation (2.8) follows by the isometry property in Remark 2.3, completing the proof. □

We note that the isometry property, i.e., $\|(x, y)\| = |f|_{\mathrm{H}_1, \alpha_1^{-1}}$, can be seen by inserting $(\tilde{x}, \tilde{y}) = (0, 0)$ into (2.7) as well. The result of Theorem 2.4 can also be seen from Theorem 2.5.

Remark 2.6. In the purely real case, where the Hilbert spaces are over $\mathbb{R}$ and all objects are real valued, Theorem 2.5 can also be deduced as a special case of [8, (7.2.14)]. The equality for the purely real reaction-diffusion equation ($\mathrm{A} = \nabla$, $\mathrm{A}^* = -\operatorname{div}$), was found also by Cai and Zhang in [4, Remark 6.12].

Corollary 2.7. *Theorem 2.5 provides the well-known a posteriori error estimates for the primal and dual problems.*

(i) *For any $\tilde{x} \in D(\mathrm{A})$ it holds*

$$|x - \tilde{x}|^2_{D(\mathrm{A}), \alpha_1, \alpha_2} = \min_{\psi \in D(\mathrm{A}^*)} \mathcal{M}(\tilde{x}, \psi) = \mathcal{M}(\tilde{x}, y).$$

(ii) *For any $\tilde{y} \in D(\mathrm{A}^*)$ it holds*

$$|y - \tilde{y}|^2_{D(\mathrm{A}^*), \alpha_1^{-1}, \alpha_2^{-1}} = \min_{\varphi \in D(\mathrm{A})} \mathcal{M}(\varphi, \tilde{y}) = \mathcal{M}(x, \tilde{y}).$$

Proof. We just have to estimate

$$|x - \tilde{x}|^2_{D(\mathrm{A}), \alpha_1, \alpha_2} \leq \|(x, y) - (\tilde{x}, \tilde{y})\|^2 = \mathcal{M}(\tilde{x}, \tilde{y})$$

and note that the left-hand side does not depend on $\tilde{y} \in D(\mathrm{A}^*)$. Setting $\psi := \tilde{y} \in D(\mathrm{A}^*)$, we get

$$|x - \tilde{x}|^2_{D(\mathrm{A}), \alpha_1, \alpha_2} \leq \inf_{\psi \in D(\mathrm{A}^*)} \mathcal{M}(\tilde{x}, \psi).$$

But for $\psi = y \in D(\mathrm{A}^*)$ we see $\mathcal{M}(\tilde{x}, y) = |x - \tilde{x}|^2_{D(\mathrm{A}),\alpha_1,\alpha_2}$, which proves (i). Analogously, we estimate

$$|y - \tilde{y}|^2_{D(\mathrm{A}^*),\alpha_1^{-1},\alpha_2^{-1}} \le \|(x, y) - (\tilde{x}, \tilde{y})\|^2 = \mathcal{M}(\tilde{x}, \tilde{y})$$

and note that the left-hand side does not depend on $\tilde{x} \in D(\mathrm{A})$. Setting $\varphi := \tilde{x} \in D(\mathrm{A})$, we get

$$|y - \tilde{y}|^2_{D(\mathrm{A}^*),\alpha_1^{-1},\alpha_2^{-1}} \le \inf_{\varphi \in D(\mathrm{A})} \mathcal{M}(\varphi, \tilde{y}).$$

But for $\varphi = x \in D(\mathrm{A})$ we see $\mathcal{M}(x, \tilde{y}) = |y - \tilde{y}|^2_{D(\mathrm{A}^*),\alpha_1^{-1},\alpha_2^{-1}}$, which shows (ii). □

Remark 2.8. (i) Since $y \perp_{\alpha_2^{-1}} N(\mathrm{A}^*)$ by (2.4), we immediately get $y \in \alpha_2 \overline{R(\mathrm{A})}$ by the Helmholtz decomposition

$$\mathsf{H}_2 = N(\mathrm{A}^*) \oplus_{\alpha_2^{-1}} \alpha_2 \overline{R(\mathrm{A})}.$$

(ii) If $\alpha_1^{-1} f \in D(\mathrm{A})$, we have $z := \alpha_1^{-1} \mathrm{A}^* y \in D(\mathrm{A})$, and the strong and mixed formulations of (2.4) read

$$\mathrm{A}\alpha_1^{-1}\mathrm{A}^* y + \alpha_2^{-1} y = \mathrm{A}\alpha_1^{-1} f,$$
$$\mathrm{A}z + \alpha_2^{-1} y = \mathrm{A}\alpha_1^{-1} f, \qquad \alpha_1^{-1}\mathrm{A}^* y = z.$$

Then for all $\varphi \in D(\mathrm{A})$ we have

$$\begin{aligned}\langle \mathrm{A}z, \mathrm{A}\varphi \rangle_{\mathsf{H}_2,\alpha_2} + \langle z, \varphi \rangle_{\mathsf{H}_1,\alpha_1} &= -\langle y, \mathrm{A}\varphi \rangle_{\mathsf{H}_2} + \langle z, \varphi \rangle_{\mathsf{H}_1,\alpha_1} + \langle \mathrm{A}\alpha_1^{-1} f, \mathrm{A}\varphi \rangle_{\mathsf{H}_2,\alpha_2} \\ &= \langle \mathrm{A}\alpha_1^{-1} f, \mathrm{A}\varphi \rangle_{\mathsf{H}_2,\alpha_2}\end{aligned}$$

and hence $z \in (D(\mathrm{A}) \cap \alpha_1^{-1} R(\mathrm{A}^*)) \subset D(\mathrm{A})$ is the unique solution of this variational problem. Furthermore, $\alpha_2(\mathrm{A}z - \mathrm{A}\alpha_1^{-1} f) \in D(\mathrm{A}^*)$ and $\mathrm{A}^* \alpha_2(\mathrm{A}z - \mathrm{A}\alpha_1^{-1} f) = -\alpha_1 z$. If $\alpha_2 \mathrm{A}\alpha_1^{-1} f$ belongs to $D(\mathrm{A}^*)$, this yields $\alpha_2 \mathrm{A}z \in D(\mathrm{A}^*)$ and the strong equation

$$\mathrm{A}^* \alpha_2 \mathrm{A}z + \alpha_1 z = \mathrm{A}^* \alpha_2 \mathrm{A}\alpha_1^{-1} f.$$

2.2 Case II: Two-Sided Error Estimate for Coefficients $i\omega\alpha_1$ and α_2

In the following we assume $\omega \in \mathbb{R} \setminus \{0\}$. Using the sub-index notation, we define for $\varphi \in D(\mathrm{A})$ and $\psi \in D(\mathrm{A}^*)$ new weighted norms on $D(\mathrm{A})$, $D(\mathrm{A}^*)$ as well as on the product space $D(\mathrm{A}) \times D(\mathrm{A}^*)$ by

$$\begin{aligned}|\varphi|^2_{D(\mathrm{A}),|\omega|\alpha_1,\alpha_2} &= |\varphi|^2_{\mathsf{H}_1,|\omega|\alpha_1} + |\mathrm{A}\varphi|^2_{\mathsf{H}_2,\alpha_2}, \\ |\psi|^2_{D(\mathrm{A}^*),(|\omega|\alpha_1)^{-1},\alpha_2^{-1}} &= |\psi|^2_{\mathsf{H}_2,\alpha_2^{-1}} + |\mathrm{A}^*\psi|^2_{\mathsf{H}_1,(|\omega|\alpha_1)^{-1}}, \\ |||(\varphi, \psi)|||^2 &:= |\varphi|^2_{D(\mathrm{A}),|\omega|\alpha_1,\alpha_2} + |\psi|^2_{D(\mathrm{A}^*),(|\omega|\alpha_1)^{-1},\alpha_2^{-1}}.\end{aligned}$$

By the Lax–Milgram lemma we get immediately:

Lemma 2.9. *The (primal) variational problem*

$$\langle \mathrm{A}x, \mathrm{A}\varphi \rangle_{\mathsf{H}_2,\alpha_2} + i\langle x, \varphi \rangle_{\mathsf{H}_1,\omega\alpha_1} = \langle f, \varphi \rangle_{\mathsf{H}_1} \quad \text{for all } \varphi \in D(\mathrm{A}) \tag{2.9}$$

admits a unique solution $x \in D(\mathrm{A})$ satisfying $|x|_{D(\mathrm{A}),|\omega|\alpha_1,\alpha_2} \le \sqrt{2}|f|_{\mathsf{H}_1,(|\omega|\alpha_1)^{-1}}$. Moreover, $y_x := \alpha_2 \mathrm{A}x$ belongs to $D(\mathrm{A}^*)$ and $\mathrm{A}^* y_x = f - i\omega\alpha_1 x$. Hence, the strong and mixed formulations

$$\mathrm{A}^* \alpha_2 \mathrm{A}x + i\omega\alpha_1 x = f, \tag{2.10}$$
$$\mathrm{A}^* y_x + i\omega\alpha_1 x = f, \qquad \alpha_2 \mathrm{A}x = y_x \tag{2.11}$$

hold with $(x, y_x) \in D(\mathrm{A}) \times (D(\mathrm{A}^*) \cap \alpha_2 R(\mathrm{A}))$.

To get the dual problem, we multiply the first equation of (2.11) by $\mathrm{A}^*\psi$ with $\psi \in D(\mathrm{A}^*)$ taking the right weighted scalar product and use $y_x = \alpha_2 \mathrm{A}x \in D(\mathrm{A}^*)$. We obtain

$$\langle \mathrm{A}^* y_x, \mathrm{A}^*\psi \rangle_{\mathsf{H}_1,(\omega\alpha_1)^{-1}} + \langle i\omega\alpha_1 x, \mathrm{A}^*\psi \rangle_{\mathsf{H}_1,(\omega\alpha_1)^{-1}} = \langle f, \mathrm{A}^*\psi \rangle_{\mathsf{H}_1,(\omega\alpha_1)^{-1}}.$$

Since $x \in D(\mathrm{A})$, it holds

$$\langle i\omega\alpha_1 x, \mathrm{A}^*\psi\rangle_{\mathsf{H}_1,(\omega\alpha_1)^{-1}} = i\langle x, \mathrm{A}^*\psi\rangle_{\mathsf{H}_1} = i\langle \mathrm{A}x, \psi\rangle_{\mathsf{H}_2} = i\langle y_x, \psi\rangle_{\mathsf{H}_2,\alpha_2^{-1}},$$

and we get again by the Lax–Milgram lemma (see Lemma 2.2) the following result.

Lemma 2.10. *The (dual) variational problem*

$$\langle \mathrm{A}^*y, \mathrm{A}^*\psi\rangle_{\mathsf{H}_1,(\omega\alpha_1)^{-1}} + i\langle y, \psi\rangle_{\mathsf{H}_2,\alpha_2^{-1}} = \langle f, \mathrm{A}^*\psi\rangle_{\mathsf{H}_1,(\omega\alpha_1)^{-1}} \quad \text{for all } \psi \in D(\mathrm{A}^*) \tag{2.12}$$

admits a unique solution $y \in D(\mathrm{A}^*)$ satisfying $|y|_{D(\mathrm{A}^*),(|\omega|\alpha_1)^{-1},\alpha_2^{-1}} \le \sqrt{2}|f|_{\mathsf{H}_1,(|\omega|\alpha_1)^{-1}}$. Moreover, $y = y_x$ holds and thus y belongs to $D(\mathrm{A}^*) \cap \alpha_2 R(\mathrm{A})$ with x and y_x from Lemma 2.9. Furthermore, $(\omega\alpha_1)^{-1}(\mathrm{A}^*y - f) \in D(\mathrm{A})$ with $A(\omega\alpha_1)^{-1}(\mathrm{A}^*y - f) = -i\alpha_2^{-1}y$.

Remark 2.11. We know that

$$|x|_{D(\mathrm{A}),|\omega|\alpha_1,\alpha_2} \le \sqrt{2}|f|_{\mathsf{H}_1,(|\omega|\alpha_1)^{-1}} \quad \text{and} \quad |y|_{D(\mathrm{A}^*),(|\omega|\alpha_1)^{-1},\alpha_2^{-1}} \le \sqrt{2}|f|_{\mathsf{H}_1,(|\omega|\alpha_1)^{-1}}. \tag{2.13}$$

It is indeed notable that

$$|f|^2_{\mathsf{H}_1,(|\omega|\alpha_1)^{-1}} = |\mathrm{A}^*y|^2_{\mathsf{H}_1,(|\omega|\alpha_1)^{-1}} + |x|^2_{\mathsf{H}_1,|\omega|\alpha_1} \tag{2.14}$$

and

$$|f|_{\mathsf{H}_1,(|\omega|\alpha_1)^{-1}} \le |||(x,y)||| \le \sqrt{2}|f|_{\mathsf{H}_1,(|\omega|\alpha_1)^{-1}} \tag{2.15}$$

hold. The identity (2.14) follows immediately by $y = \alpha_2\mathrm{A}x$ and

$$\begin{aligned}
|f|^2_{\mathsf{H}_1,(|\omega|\alpha_1)^{-1}} &= |\mathrm{A}^*\alpha_2\mathrm{A}x + i\omega\alpha_1 x|^2_{\mathsf{H}_1,(|\omega|\alpha_1)^{-1}} \\
&= |\mathrm{A}^*y|^2_{\mathsf{H}_1,(|\omega|\alpha_1)^{-1}} + |i\omega\alpha_1 x|^2_{\mathsf{H}_1,(|\omega|\alpha_1)^{-1}} + 2\Re\underbrace{\langle \mathrm{A}^*\alpha_2\mathrm{A}x, i\omega\alpha_1 x\rangle_{\mathsf{H}_1,(|\omega|\alpha_1)^{-1}}}_{=-i\,\mathrm{sign}\,\omega\langle \mathrm{A}^*\alpha_2\mathrm{A}x,x\rangle_{\mathsf{H}_1}} \\
&= |\mathrm{A}^*y|^2_{\mathsf{H}_1,(|\omega|\alpha_1)^{-1}} + |x|^2_{\mathsf{H}_1,|\omega|\alpha_1} - 2\underbrace{\Re\big(i\,\mathrm{sign}\,\omega\langle \alpha_2\mathrm{A}x, \mathrm{A}x\rangle_{\mathsf{H}_2}\big)}_{=0}.
\end{aligned}$$

The lower bound in (2.15) follows from (2.14). The upper bound in (2.15) is seen as follows: First we take (2.9) with $\varphi = x$ and (2.12) with $\psi = y$, and obtain

$$\begin{aligned}
|\mathrm{A}x|^2_{\mathsf{H}_2,\alpha_2} + i\omega|x|^2_{\mathsf{H}_1,\alpha_1} &= \langle f, x\rangle_{\mathsf{H}_1}, \\
\omega^{-1}|\mathrm{A}^*y|^2_{\mathsf{H}_1,\alpha_1^{-1}} + i|y|^2_{\mathsf{H}_2,\alpha_2^{-1}} &= \langle f, \mathrm{A}^*y\rangle_{\mathsf{H}_1,(\omega\alpha_1)^{-1}}.
\end{aligned}$$

Taking the norm of both sides, we obtain

$$\begin{aligned}
|\mathrm{A}x|^4_{\mathsf{H}_2,\alpha_2} + |\omega|^2|x|^4_{\mathsf{H}_1,\alpha_1} &= |\langle f, x\rangle_{\mathsf{H}_1}|^2, \\
|\omega|^{-2}|\mathrm{A}^*y|^4_{\mathsf{H}_1,\alpha_1^{-1}} + |y|^4_{\mathsf{H}_2,\alpha_2^{-1}} &= |\langle f, \mathrm{A}^*y\rangle_{\mathsf{H}_1,(\omega\alpha_1)^{-1}}|^2,
\end{aligned}$$

showing

$$\begin{aligned}
\frac{1}{\sqrt{2}}|x|^2_{D(\mathrm{A}),|\omega|\alpha_1,\alpha_2} &\le |f|_{\mathsf{H}_1,(|\omega|\alpha_1)^{-1}}|x|_{\mathsf{H}_1,|\omega|\alpha_1}, \\
\frac{1}{\sqrt{2}}|y|^2_{D(\mathrm{A}^*),(|\omega|\alpha_1)^{-1},\alpha_2^{-1}} &\le |f|_{\mathsf{H}_1,(|\omega|\alpha_1)^{-1}}|\mathrm{A}^*y|_{\mathsf{H}_1,(|\omega|\alpha_1)^{-1}}.
\end{aligned}$$

From these inequalities we can derive the estimates (2.13) for x and y separately. Moreover, by summing up and (2.14), we get

$$\begin{aligned}
\frac{1}{\sqrt{2}}|||(x,y)|||^2 &\le |f|_{\mathsf{H}_1,(|\omega|\alpha_1)^{-1}}\big(|x|_{\mathsf{H}_1,|\omega|\alpha_1} + |\mathrm{A}^*y|_{\mathsf{H}_1,(|\omega|\alpha_1)^{-1}}\big) \\
&\le |f|_{\mathsf{H}_1,(|\omega|\alpha_1)^{-1}}\sqrt{2}\sqrt{|x|^2_{\mathsf{H}_1,|\omega|\alpha_1} + |\mathrm{A}^*y|^2_{\mathsf{H}_1,(|\omega|\alpha_1)^{-1}}} = \sqrt{2}|f|^2_{\mathsf{H}_1,(|\omega|\alpha_1)^{-1}}
\end{aligned}$$

and we have the upper bound in (2.15). Thus the norm of the solution operator

$$L_i : \mathsf{H}_1 \to D(\mathrm{A}) \times D(\mathrm{A}^*), \quad f \mapsto (x, y)$$

(equipped with the proper weighted norms) satisfies $1 \le |L_i| \le \sqrt{2}$. Hence L_i is 'almost' an isometry.

We also note that the upper bound in (2.15) is sharp: Let $\mathsf{H}_1 = \mathsf{H}_2$, $\mathrm{A} := \mathrm{A}^* := \mathrm{id}$, $\omega := 1$ and $\alpha_1 := \alpha_2 := 1$. Then $x = y$, $(1+i)x = f$ and $|||(x,y)|||^2 = 4|x|^2_{\mathsf{H}_1} = 2|f|^2_{\mathsf{H}_1}$.

The latter remark motivates the usage of the combined norm also for error estimates. First we show that a two-sided error estimate follows directly from Remark 2.11, if the approximation of the primal variable x is regular enough.

Theorem 2.12. *Let $(x, y) \in D(\mathrm{A}) \times D(\mathrm{A}^*)$ be the exact solution of (2.11). Let $\tilde{x} \in D(\mathrm{A})$ have additional regularity so that $\tilde{y} = \alpha_2 \mathrm{A}\tilde{x} \in D(\mathrm{A}^*)$. Then, for the mixed approximation $(\tilde{x}, \tilde{y})$ we have*

$$\mathcal{I}_i(\tilde{x}, \tilde{y}) \le |||(x, y) - (\tilde{x}, \tilde{y})|||^2 \le 2\mathcal{I}_i(\tilde{x}, \tilde{y}) \tag{2.16}$$

and the normalized counterpart

$$\frac{1}{2} \cdot \frac{\mathcal{I}_i(\tilde{x}, \tilde{y})}{|f|^2_{\mathsf{H}_1,(|\omega|\alpha_1)^{-1}}} \le \frac{|||(x, y) - (\tilde{x}, \tilde{y})|||^2}{|||(x, y)|||^2} \le 2\frac{\mathcal{I}_i(\tilde{x}, \tilde{y})}{|f|^2_{\mathsf{H}_1,(|\omega|\alpha_1)^{-1}}}, \tag{2.17}$$

where

$$\mathcal{I}_i(\tilde{x}, \tilde{y}) := |f - i\omega\alpha_1\tilde{x} - \mathrm{A}^*\tilde{y}|^2_{\mathsf{H}_1,(|\omega|\alpha_1)^{-1}}.$$

Proof. Since $\tilde{x}$ is very regular, especially $\tilde{y} = \alpha_2\mathrm{A}\tilde{x} \in D(\mathrm{A}^*)$, the pair $(\tilde{x}, \tilde{y})$ is the exact solution of the problem

$$\mathrm{A}^*\tilde{y} + i\omega\alpha_1\tilde{x} =: \tilde{f}, \quad \alpha_2\mathrm{A}\tilde{x} = \tilde{y},$$

i.e., we have $L_i(\tilde{f}) = (\tilde{x}, \tilde{y})$. Then (2.16) is given directly by Remark 2.11:

$$|f - \tilde{f}|^2_{\mathsf{H}_1,(|\omega|\alpha_1)^{-1}} \le |||(x, y) - (\tilde{x}, \tilde{y})|||^2 = |||L_i(f - \tilde{f})|||^2 \le 2|f - \tilde{f}|^2_{\mathsf{H}_1,(|\omega|\alpha_1)^{-1}}.$$

Estimate (2.17) follows by Remark 2.11 as well. □

The square root of the ratio of the bounds in (2.16) is always $\sqrt{2} < 1.42$. The square root of the ratio of the bounds in (2.17) is always 2. However, satisfying the high regularity property required in Theorem 2.12 may not be convenient for practical calculations. The next result, the second main result of the paper, holds for less regular approximations.

Theorem 2.13. *Let $(x, y), (\tilde{x}, \tilde{y}) \in D(\mathrm{A}) \times D(\mathrm{A}^*)$ be the exact solution of (2.11) and any conforming approximation, respectively. Then*

$$\frac{\sqrt{2}}{\sqrt{2}+1}\mathcal{M}_i(\tilde{x}, \tilde{y}) \le |||(x, y) - (\tilde{x}, \tilde{y})|||^2 \le \frac{\sqrt{2}}{\sqrt{2}-1}\mathcal{M}_i(\tilde{x}, \tilde{y}) \tag{2.18}$$

and the normalized counterpart

$$\frac{\sqrt{2}}{2(\sqrt{2}+1)} \cdot \frac{\mathcal{M}_i(\tilde{x}, \tilde{y})}{|f|^2_{\mathsf{H}_1,(|\omega|\alpha_1)^{-1}}} \le \frac{|||(x, y) - (\tilde{x}, \tilde{y})|||^2}{|||(x, y)|||^2} \le \frac{\sqrt{2}}{\sqrt{2}-1} \cdot \frac{\mathcal{M}_i(\tilde{x}, \tilde{y})}{|f|^2_{\mathsf{H}_1,(|\omega|\alpha_1)^{-1}}} \tag{2.19}$$

hold, where

$$\mathcal{M}_i(\tilde{x}, \tilde{y}) := |f - i\omega\alpha_1\tilde{x} - \mathrm{A}^*\tilde{y}|^2_{\mathsf{H}_1,(|\omega|\alpha_1)^{-1}} + |\tilde{y} - \alpha_2\mathrm{A}\tilde{x}|^2_{\mathsf{H}_2,\alpha_2^{-1}}.$$

Proof. Using (2.10) and inserting $0 = \alpha_2\mathrm{A}x - y$, we get

$$\begin{aligned}
\mathcal{M}_i(\tilde{x}, \tilde{y}) &= |i\omega\alpha_1 x - i\omega\alpha_1\tilde{x} + \mathrm{A}^*y - \mathrm{A}^*\tilde{y}|^2_{\mathsf{H}_1,(|\omega|\alpha_1)^{-1}} + |\tilde{y} - y + \alpha_2\mathrm{A}x - \alpha_2\mathrm{A}\tilde{x}|^2_{\mathsf{H}_2,\alpha_2^{-1}} \\
&= |x - \tilde{x}|^2_{\mathsf{H}_1,|\omega|\alpha_1} + |\mathrm{A}^*(y - \tilde{y})|^2_{\mathsf{H}_1,(|\omega|\alpha_1)^{-1}} + 2\Re\langle i\omega\alpha_1(x - \tilde{x}), \mathrm{A}^*(y - \tilde{y})\rangle_{\mathsf{H}_1,(|\omega|\alpha_1)^{-1}} \\
&\quad + |\tilde{y} - y|^2_{\mathsf{H}_2,\alpha_2^{-1}} + |\mathrm{A}(x - \tilde{x})|^2_{\mathsf{H}_2,\alpha_2} + 2\Re\langle \tilde{y} - y, \alpha_2\mathrm{A}(x - \tilde{x})\rangle_{\mathsf{H}_2,\alpha_2^{-1}} \\
&= |x - \tilde{x}|^2_{D(\mathrm{A}),|\omega|\alpha_1,\alpha_2} + |y - \tilde{y}|^2_{D(\mathrm{A}^*),(|\omega|\alpha_1)^{-1},\alpha_2^{-1}} \\
&\quad + 2\,\mathrm{sign}\,\omega\,\Re\langle i(x - \tilde{x}), \mathrm{A}^*(y - \tilde{y})\rangle_{\mathsf{H}_1} - 2\Re\langle \mathrm{A}(x - \tilde{x}), y - \tilde{y}\rangle_{\mathsf{H}_2}.
\end{aligned} \tag{2.20}$$

The last two terms in (2.20) can be written as (for brevity we use the notation $e := x - \tilde{x}$ and $h := y - \tilde{y}$)

$$\begin{aligned}
&2 \operatorname{sign} \omega\, \mathbb{R}(i\langle e, \mathrm{A}^* h\rangle_{\mathrm{H}_1}) - 2\mathbb{R}\langle \mathrm{A}e, h\rangle_{\mathrm{H}_2}\\
&\quad = -2 \operatorname{sign} \omega\, \mathbb{J}\langle e, \mathrm{A}^* h\rangle_{\mathrm{H}_1} - 2\mathbb{R}\langle \mathrm{A}e, h\rangle_{\mathrm{H}_2}\\
&\quad \geq -2|\mathbb{J}\langle e, \mathrm{A}^* h\rangle_{\mathrm{H}_1}| - 2|\mathbb{R}\langle \mathrm{A}e, h\rangle_{\mathrm{H}_2}|\\
&\quad = -\Big(\underbrace{|\mathbb{J}\langle e, \mathrm{A}^* h\rangle_{\mathrm{H}_1}| + |\mathbb{R}\langle e, \mathrm{A}^* h\rangle_{\mathrm{H}_1}|}_{\leq \sqrt{2}|\langle e, \mathrm{A}^* h\rangle_{\mathrm{H}_1}|} + \underbrace{|\mathbb{J}\langle \mathrm{A}e, h\rangle_{\mathrm{H}_2}| + |\mathbb{R}\langle \mathrm{A}e, h\rangle_{\mathrm{H}_2}|}_{\leq \sqrt{2}|\langle \mathrm{A}e, h\rangle_{\mathrm{H}_2}|}\Big)\\
&\quad \geq -\sqrt{2}(|\langle e, \mathrm{A}^* h\rangle_{\mathrm{H}_1}| + |\langle \mathrm{A}e, h\rangle_{\mathrm{H}_2}|)\\
&\quad \geq -\sqrt{2}(|e|_{\mathrm{H}_1,|\omega|\alpha_1} |\mathrm{A}^* h|_{\mathrm{H}_1,(|\omega|\alpha_1)^{-1}} + |\mathrm{A}e|_{\mathrm{H}_2,\alpha_2} |h|_{\mathrm{H}_2,\alpha_2^{-1}})\\
&\quad \geq -\sqrt{2}\Big(\frac{1}{2\delta}|e|^2_{\mathrm{H}_1,|\omega|\alpha_1} + \frac{\delta}{2}|\mathrm{A}^* h|^2_{\mathrm{H}_1,(|\omega|\alpha_1)^{-1}} + \frac{1}{2\delta}|\mathrm{A}e|^2_{\mathrm{H}_2,\alpha_2} + \frac{\delta}{2}|h|^2_{\mathrm{H}_2,\alpha_2^{-1}}\Big),
\end{aligned} \tag{2.21}$$

for all $\delta > 0$. One can repeat these calculations by estimating from above, and arrive at

$$\begin{aligned}
&2 \operatorname{sign} \omega\, \mathbb{R}(i\langle e, \mathrm{A}^* h\rangle_{\mathrm{H}_1}) - 2\mathbb{R}\langle \mathrm{A}e, h\rangle_{\mathrm{H}_2}\\
&\quad \leq \sqrt{2}\Big(\frac{1}{2\delta}|e|^2_{\mathrm{H}_1,|\omega|\alpha_1} + \frac{\delta}{2}|\mathrm{A}^* h|^2_{\mathrm{H}_1,(|\omega|\alpha_1)^{-1}} + \frac{1}{2\delta}|\mathrm{A}e|^2_{\mathrm{H}_2,\alpha_2} + \frac{\delta}{2}|h|^2_{\mathrm{H}_2,\alpha_2^{-1}}\Big).
\end{aligned} \tag{2.22}$$

Together (2.20)–(2.22) give

$$\Big(1 - \sqrt{2}\frac{1}{2\delta}\Big)|x - \tilde{x}|^2_{D(\mathrm{A}),|\omega|\alpha_1,\alpha_2} + \Big(1 - \sqrt{2}\frac{\delta}{2}\Big)|y - \tilde{y}|^2_{D(\mathrm{A}^*),(|\omega|\alpha_1)^{-1},\alpha_2^{-1}} \leq \mathcal{M}_i(\tilde{x}, \tilde{y}), \tag{2.23}$$

$$\Big(1 + \sqrt{2}\frac{1}{2\delta}\Big)|x - \tilde{x}|^2_{D(\mathrm{A}),|\omega|\alpha_1,\alpha_2} + \Big(1 + \sqrt{2}\frac{\delta}{2}\Big)|y - \tilde{y}|^2_{D(\mathrm{A}^*),(|\omega|\alpha_1)^{-1},\alpha_2^{-1}} \geq \mathcal{M}_i(\tilde{x}, \tilde{y}). \tag{2.24}$$

Estimate (2.18) follows by setting $\delta = 1$ in (2.23) and (2.24). Estimate (2.19) follows by property (2.15) in Remark 2.11, completing the proof. □

The square root of the ratio of the upper and lower bound in (2.18) is always $1 + \sqrt{2} < 2.42$. The square root of the ratio of the bounds of the normalized counterpart (2.19) is always $2 + \sqrt{2} < 3.42$. We can conclude that the bounds are close to each other and give reliable information of the error of a mixed approximation.

Theorem 2.14. *From the proof of Theorem 2.13 we can deduce the following a posteriori error estimates for the primal and dual problems.*

(i) *For any $\tilde{x} \in D(\mathrm{A})$ it holds*

$$|x - \tilde{x}|^2_{D(\mathrm{A}),|\omega|\alpha_1,\alpha_2} \leq 2\mathcal{M}_i(\tilde{x}, \psi) \quad \textit{for all } \psi \in D(\mathrm{A}^*).$$

(ii) *For any $\tilde{y} \in D(\mathrm{A}^*)$ it holds*

$$|y - \tilde{y}|^2_{D(\mathrm{A}^*),(|\omega|\alpha_1)^{-1},\alpha_2^{-1}} \leq 2\mathcal{M}_i(\varphi, \tilde{y}) \quad \textit{for all } \varphi \in D(\mathrm{A}).$$

Proof. Estimate (i) follows from (2.23) by setting $\delta = \sqrt{2}$, and (ii) from (2.23) by setting $\delta = 1/\sqrt{2}$. □

Remark 2.15. (i) Since $y \perp_{\alpha_2^{-1}} N(\mathrm{A}^*)$ by (2.12), we immediately get $y \in \alpha_2\overline{R(\mathrm{A})}$ by the Helmholtz decomposition

$$\mathrm{H}_2 = N(\mathrm{A}^*) \oplus_{\alpha_2^{-1}} \alpha_2\overline{R(\mathrm{A})}.$$

(ii) If $(\omega\alpha_1)^{-1}f \in D(\mathrm{A})$, we have $z := (\omega\alpha_1)^{-1}\mathrm{A}^* y \in D(\mathrm{A})$, and the strong and mixed formulations of (2.12) read

$$\begin{aligned}
\mathrm{A}(\omega\alpha_1)^{-1}\mathrm{A}^* y + i\alpha_2^{-1} y &= \mathrm{A}(\omega\alpha_1)^{-1} f,\\
\mathrm{A}z + i\alpha_2^{-1} y &= \mathrm{A}(\omega\alpha_1)^{-1} f, \quad (\omega\alpha_1)^{-1}\mathrm{A}^* y = z.
\end{aligned}$$

Then for all $\varphi \in D(\mathrm{A})$ we have

$$\begin{aligned}
\langle \mathrm{A}z, \mathrm{A}\varphi\rangle_{\mathrm{H}_2,\alpha_2} + i\langle z, \varphi\rangle_{\mathrm{H}_1,\omega\alpha_1} &= -i\langle y, \mathrm{A}\varphi\rangle_{\mathrm{H}_2} + i\langle z, \varphi\rangle_{\mathrm{H}_1,\omega\alpha_1} + \langle \mathrm{A}(\omega\alpha_1)^{-1} f, \mathrm{A}\varphi\rangle_{\mathrm{H}_2,\alpha_2}\\
&= \langle \mathrm{A}(\omega\alpha_1)^{-1} f, \mathrm{A}\varphi\rangle_{\mathrm{H}_2,\alpha_2}
\end{aligned}$$

and hence $z \in (D(\mathrm{A}) \cap \alpha_1^{-1} R(\mathrm{A}^*)) \subset D(\mathrm{A})$ is the unique solution of this variational problem. Furthermore, $\alpha_2(\mathrm{A}z - \mathrm{A}(\omega\alpha_1)^{-1} f) \in D(\mathrm{A}^*)$ and $\mathrm{A}^*\alpha_2(\mathrm{A}z - \mathrm{A}\alpha_1^{-1} f) = -i\omega\alpha_1 z$. If $\alpha_2 \mathrm{A}(\omega\alpha_1)^{-1} f$ belongs to $D(\mathrm{A}^*)$, this yields $\alpha_2 \mathrm{A}z \in D(\mathrm{A}^*)$ and the strong equation

$$\mathrm{A}^*\alpha_2 \mathrm{A}z + i\omega\alpha_1 z = \mathrm{A}^*\alpha_2 \mathrm{A}(\omega\alpha_1)^{-1} f.$$

2.3 Error Indication Properties for PDEs

In this section we assume that the underlying problem is a PDE such that A and A^* are differential operators and the Hilbert spaces are scalar-, vector-, or tensor-valued L^2-spaces, i.e., $\mathsf{H}_1 = \mathsf{L}^2(\Omega)$ and $\mathsf{H}_2 = \mathsf{L}^2(\Omega)$. Here $\Omega \subset \mathbb{R}^d$, $d \geq 1$, is a domain.

Let $\mathcal{T}$ denote a discretization of the domain Ω into a mesh of non-overlapping elements T. Note that we assume $\bigcup_{T\in\mathcal{T}} \overline{T} = \overline{\Omega}$, i.e., in particular that the boundary of Ω is exactly represented by the mesh. This is necessary in order to have conforming approximations in the first place: They must satisfy exactly the imposed boundary conditions.

Aside from global error values we are also interested in estimating the error distribution in the mesh $\mathcal{T}$. In the following we use the previously derived error equality and error estimate to define error indicators and study their properties.

Case I. We define the following error indicator based on the equality of Theorem 2.5:

$$\eta_T(\tilde{x}, \tilde{y}) := \sqrt{|f - \alpha_1\tilde{x} - \mathrm{A}^*\tilde{y}|^2_{\mathsf{L}^2(T),\alpha_1^{-1}} + |\tilde{y} - \alpha_2 \mathrm{A}\tilde{x}|^2_{\mathsf{L}^2(T),\alpha_2^{-1}}}.$$

The error indicator η_T will indicate the exact error distribution

$$e_T(\tilde{x}, \tilde{y}) := \sqrt{|x - \tilde{x}|^2_{\mathsf{L}^2(T),\alpha_1} + |\mathrm{A}(x - \tilde{x})|^2_{\mathsf{L}^2(T),\alpha_2} + |y - \tilde{y}|^2_{\mathsf{L}^2(T),\alpha_2^{-1}} + |\mathrm{A}^*(y - \tilde{y})|^2_{\mathsf{L}^2(T),\alpha_1^{-1}}}.$$

In the following we use

$$\eta := \sqrt{\sum_{T\in\mathcal{T}} \eta_T^2} \quad \text{and} \quad e := \sqrt{\sum_{T\in\mathcal{T}} e_T^2}.$$

The error indicator η should satisfy the following properties:

(i) The indicator η must satisfy the global relation $\underline{c}\,\eta \leq e \leq \overline{c}\,\eta$ with some constants $\underline{c} > 0$ and $\overline{c} > 0$. The constant $\underline{c}$ is often called the global efficiency constant, and $\overline{c}$ the global reliability constant. If $\overline{c}$ or an upper bound of it is known, the indicator can be used to provide a stopping criterion for adaptive computations.

(ii) The local indicator η_T must satisfy $c_T\eta_T \leq e_T$ in all elements T in $\mathcal{T}$ with some constants $c_T > 0$, which are often called the local efficiency constants. If c_T are of the same magnitude, the indicator is then appropriate for estimating the error distribution in the mesh, and can then be used for adaptive mesh-refinement.

It is desirable that the constants $\underline{c}$, $\overline{c}$ and c_T are not dependent on the problem data or the mesh. If the constants $\underline{c}$ and $\overline{c}$ are known, they give a good idea of the quality of the indicator η in a global context. It is also desirable that the local constants c_T are known for all elements T. The closer the values are to $\underline{c}$, the better.

Note that $\eta(\tilde{x}, \tilde{y}) = \mathcal{M}(\tilde{x}, \tilde{y})^{1/2} = e(\tilde{x}, \tilde{y})$, so according to Theorem 2.5 the first property is satisfied with constants $\underline{c} = \overline{c} = 1$. This is the best case possible.

We show the second property of local efficiency by using (2.2) and inserting $0 = \alpha_2 \mathrm{A}x - y$ into η_T:

$$\begin{aligned}\eta_T(\tilde{x}, \tilde{y})^2 &= |\alpha_1 x - \alpha_1\tilde{x} + \mathrm{A}^*y - \mathrm{A}^*\tilde{y}|^2_{\mathsf{L}^2(T),\alpha_1^{-1}} + |\tilde{y} - y + \alpha_2\mathrm{A}x - \alpha_2\mathrm{A}\tilde{x}|^2_{\mathsf{L}^2(T),\alpha_2^{-1}} \\ &\leq 2\Big(|x - \tilde{x}|^2_{\mathsf{L}^2(T),\alpha_1} + |\mathrm{A}^*(y - \tilde{y})|^2_{\mathsf{L}^2(T),\alpha_1^{-1}} + |\tilde{y} - y|^2_{\mathsf{L}^2(T),\alpha_2^{-1}} + |\mathrm{A}(x - \tilde{x})|^2_{\mathsf{L}^2(T),\alpha_2}\Big),\end{aligned}$$

which gives us

$$\frac{1}{\sqrt{2}}\eta_T(\tilde{x}, \tilde{y}) \leq e_T(\tilde{x}, \tilde{y}).$$

The indicator η then satisfies the second property with the constant $c_T = 1/\sqrt{2} > 0.7$ for all elements $T \in \mathcal{T}$. This constant is rather sharp, since $\underline{c} = \overline{c} = 1$. This means that η provides a good error indicator for guiding mesh-adaptive methods for mixed approximations.

Case II. We define the following error indicator based on the estimate of Theorem 2.13:

$$\eta_{i,T}(\tilde{x}, \tilde{y}) := \sqrt{|f - i\omega\alpha_1\tilde{x} - \mathrm{A}^*\tilde{y}|^2_{\mathrm{L}^2(T),(|\omega|\alpha_1)^{-1}} + |\tilde{y} - \alpha_2\mathrm{A}\tilde{x}|^2_{\mathrm{L}^2(T),\alpha_2^{-1}}}.$$

The error indicator $\eta_{i,T}$ will indicate the exact error distribution

$$e_{i,T}(\tilde{x}, \tilde{y}) := \sqrt{|x - \tilde{x}|^2_{\mathrm{L}^2(T),|\omega|\alpha_1} + |\mathrm{A}(x - \tilde{x})|^2_{\mathrm{L}^2(T),\alpha_2} + |y - \tilde{y}|^2_{\mathrm{L}^2(T),\alpha_2^{-1}} + |\mathrm{A}^*(y - \tilde{y})|^2_{\mathrm{L}^2(T),(|\omega|\alpha_1)^{-1}}}.$$

In the following we use

$$\eta_i := \sqrt{\sum_{T\in\mathcal{T}} \eta^2_{i,T}} \quad \text{and} \quad e_i := \sqrt{\sum_{T\in\mathcal{T}} e^2_{i,T}}.$$

Note that $\eta_i(\tilde{x}, \tilde{y}) = \mathcal{M}_i(\tilde{x}, \tilde{y})^{1/2}$, so according to Theorem 2.13 the first property is satisfied with constants

$$\underline{c} = \sqrt{\frac{\sqrt{2}}{\sqrt{2} + 1}} > 0.76 \quad \text{and} \quad \overline{c} = \sqrt{\frac{\sqrt{2}}{\sqrt{2} - 1}} < 1.85,$$

with ratio $1 + \sqrt{2} < 2.42$.

We show the second property of local efficiency by using (2.10) and inserting $0 = \alpha_2\mathrm{A}x - y$ into $\eta_{i,T}$:

$$\begin{aligned}\eta_{i,T}(\tilde{x}, \tilde{y})^2 &= |i\omega\alpha_1 x - i\omega\alpha_1\tilde{x} + \mathrm{A}^*y - \mathrm{A}^*\tilde{y}|^2_{\mathrm{L}^2(T),(|\omega|\alpha_1)^{-1}} + |\tilde{y} - y + \alpha_2\mathrm{A}x - \alpha_2\mathrm{A}\tilde{x}|^2_{\mathrm{L}^2(T),\alpha_2^{-1}}\\ &\leq 2\Big(|x - \tilde{x}|^2_{\mathrm{L}^2(T),|\omega|\alpha_1} + |\mathrm{A}^*(y - \tilde{y})|^2_{\mathrm{L}^2(T),(|\omega|\alpha_1)^{-1}} + |\tilde{y} - y|^2_{\mathrm{L}^2(T),\alpha_2^{-1}} + |\mathrm{A}(x - \tilde{x})|^2_{\mathrm{L}^2(T),\alpha_2}\Big),\end{aligned}$$

which gives us

$$\frac{1}{\sqrt{2}}\eta_{i,T}(\tilde{x}, \tilde{y}) \leq e_{i,T}(\tilde{x}, \tilde{y}).$$

The indicator η_i then satisfies the second property with the constant $c_T = 1/\sqrt{2} > 0.7$ for all elements $T \in \mathcal{T}$. This constant is again rather sharp, since $0.76 < \underline{c} < \overline{c} < 1.85$. This means that η_i provides a good error indicator for guiding mesh-adaptive methods for mixed approximations.

2.4 Motivation: Error Control for Time-Dependent PDEs

As mentioned in the introduction, a motivation to study a posteriori error estimation for the two classes of problems considered in this paper comes from time-dependent partial differential equations, more precisely from their time discretizations or from assuming that they are time-harmonic.

Case I. A main application of our error equality of Theorem 2.5 might be that equations of the type

$$\mathrm{A}^*\alpha_2\mathrm{A}x + \alpha_1 x = f \tag{2.25}$$

naturally occur in many types of time discretizations, e.g., for linear parabolic heat type equations or linear hyperbolic wave propagation type equations.

Let us consider the linear parabolic heat type equation

$$(\partial_t + \mathrm{A}^*\mathrm{A})x = f, \tag{2.26}$$

generalizing the most prominent example of the heat equation

$$(\partial_t - \Delta)u = (\partial_t - \operatorname{div}\nabla)u = g$$

with appropriate boundary and initial conditions. A standard implicit time discretization for (2.26) is, e.g., the backward Euler scheme, yielding

$$\delta_n^{-1}(x_n - x_{n-1}) + \mathrm{A}^*\mathrm{A}x_n = f_n, \quad \delta_n := t_n - t_{n-1},$$

and hence (2.25) is recovered by

$$\mathrm{A}^*\mathrm{A}x_n + \delta_n^{-1}x_n = \tilde{f}_n := f_n - \delta_n^{-1}x_{n-1}.$$

We note that our arguments extend to 'all' practically used time discretizations. Functional a posteriori error estimates for parabolic equations can be found, e.g., in [8, 12].

A large class of linear wave propagation models, like electromagnetics or acoustics, have the structure

$$(\partial_t\Lambda^{-1} + \mathrm{M})\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} g \\ h \end{bmatrix}, \quad \mathrm{M} = \begin{bmatrix} 0 & -\mathrm{A}^* \\ \mathrm{A} & 0 \end{bmatrix}, \quad \Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \tag{2.27}$$

or more explicit

$$\partial_t\lambda_1^{-1}x - \mathrm{A}^*y = g, \quad \partial_t\lambda_2^{-1}y + \mathrm{A}x = h \tag{2.28}$$

completed by appropriate initial conditions. Often the material is assumed to be time-independent, i.e., Λ does not depend on time. In this case $i\Lambda\mathrm{M}$ is self-adjoint in the proper Hilbert spaces and the solution theory follows immediately by the spectral theorem (variation in constant formula) or by semigroup theory. We note that formally the second-order wave equation

$$(\partial_t^2 - (\Lambda\mathrm{M})^2)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \tilde{g} \\ \tilde{h} \end{bmatrix} := (\partial_t - \Lambda\mathrm{M})\Lambda\begin{bmatrix} g \\ h \end{bmatrix}, \quad (\Lambda\mathrm{M})^2 = \begin{bmatrix} -\lambda_1\mathrm{A}^*\lambda_2\mathrm{A} & 0 \\ 0 & -\lambda_2\mathrm{A}\lambda_1\mathrm{A}^* \end{bmatrix}$$

holds, i.e., component-wise

$$(\partial_t^2 + \lambda_1\mathrm{A}^*\lambda_2\mathrm{A})x = \tilde{g}, \quad (\partial_t^2 + \lambda_2\mathrm{A}\lambda_1\mathrm{A}^*)y = \tilde{h}.$$

Hence the linear hyperbolic wave type equation

$$(\partial_t^2 + \mathrm{A}^*\mathrm{A})x = f \tag{2.29}$$

pops up, generalizing the most prominent example of the wave equation

$$(\partial_t^2 - \Delta)u = (\partial_t^2 - \operatorname{div}\nabla)u = j$$

with appropriate boundary and initial conditions. A standard implicit time discretization for (2.28) is, e.g., the backward Euler scheme, i.e.,

$$\delta_n^{-1}\lambda_1^{-1}(x_n - x_{n-1}) - \mathrm{A}^*y_n = g_n, \quad \delta_n^{-1}(y_n - y_{n-1}) + \lambda_2\mathrm{A}x_n = \lambda_2 h_n.$$

Hence, we obtain, e.g., for x_n,

$$\mathrm{A}^*\lambda_2\mathrm{A}x_n + \delta_n^{-2}\lambda_1^{-1}x_n = f_n := \mathrm{A}^*(\lambda_2 h_n + \delta_n^{-1}y_{n-1}) + \delta_n^{-2}\lambda_1^{-1}x_{n-1} + \delta_n^{-1}g_n$$

provided that $\lambda_2 h_n \in D(\mathrm{A}^*)$. Therefore (2.25) holds for x_n with, e.g., $\alpha_1 = \delta_n^{-2}\lambda_1^{-1}$ and $\alpha_2 = \lambda_2$. Of course, a similar equation holds for y_n as well. We note that our arguments extend to 'all' practically used time discretizations. Functional a posteriori error estimates for wave equations can be found in [11, 13].

Case II. A main application of our two-sided error estimate of Theorem 2.13 might be that equations of the type

$$\mathrm{A}^*\alpha_2\mathrm{A}x + i\omega\alpha_1 x = f \tag{2.30}$$

naturally occur for time-harmonic problems, e.g., for time-harmonic Maxwell equations. Maxwell's equations are hyperbolic and read

$$\begin{aligned} \partial_t D - \operatorname{rot} H &= J = j + \sigma E, & \operatorname{div} D &= \rho, & D &= \epsilon E, \\ \partial_t B + \operatorname{rot} E &= 0, & \operatorname{div} B &= 0, & B &= \mu H \end{aligned}$$

with appropriate boundary and initial conditions. These equations can be written in the style of (2.27) as

$$\left(\partial_t \begin{bmatrix} \epsilon & 0 \\ 0 & \mu \end{bmatrix} + \begin{bmatrix} 0 & -\operatorname{rot} \\ \operatorname{rot} & 0 \end{bmatrix} - \begin{bmatrix} \sigma & 0 \\ 0 & 0 \end{bmatrix}\right) \begin{bmatrix} E \\ H \end{bmatrix} = \begin{bmatrix} j \\ 0 \end{bmatrix}.$$

Let us assume that ϵ, μ and σ are independent of time. Then, formally, we have

$$\begin{aligned} \partial_t^2 \epsilon E &= \operatorname{rot} \mu^{-1} \partial_t B + \partial_t j + \partial_t \sigma E = -\operatorname{rot} \mu^{-1} \operatorname{rot} E + \partial_t J, \\ \partial_t^2 \mu H &= -\operatorname{rot} \epsilon^{-1} \partial_t D = -\operatorname{rot} \epsilon^{-1} \operatorname{rot} H - \operatorname{rot} \epsilon^{-1} J, \end{aligned}$$

i.e., we get the wave equations

$$(\partial_t^2 + \epsilon^{-1} \operatorname{rot} \mu^{-1} \operatorname{rot}) E = \partial_t \epsilon^{-1} J, \quad (\partial_t^2 + \mu^{-1} \operatorname{rot} \epsilon^{-1} \operatorname{rot}) H = -\mu^{-1} \operatorname{rot} \epsilon^{-1} J$$

as another example of (2.29). The eddy current model neglects time variations of the electric field, i.e., assumes $\partial_t D = \partial_t \epsilon E = 0$, and hence leads to the parabolic equation

$$\sigma \partial_t E = -\operatorname{rot} \mu^{-1} \partial_t B - F = \operatorname{rot} \mu^{-1} \operatorname{rot} E - F, \quad F := \partial_t j,$$

i.e.,

$$-\sigma \partial_t E + \operatorname{rot} \mu^{-1} \operatorname{rot} E = F.$$

A time-harmonic ansatz leads to

$$\operatorname{rot} \mu^{-1} \operatorname{rot} \tilde{E} + i\omega\sigma \tilde{E} = \tilde{F}$$

as a prominent example of (2.30) for which our results are stated in Section 3.2.

3 Applications

In this section we will discuss some standard applications. Let $\Omega \subset \mathbb{R}^d$, $d \geq 1$, be a bounded Lipschitz domain with boundary Γ. Moreover, let Γ_{D} be an open subset of Γ and $\Gamma_{\mathrm{N}} := \Gamma \setminus \overline{\Gamma_{\mathrm{D}}}$ its complement. We will denote by n the outward unit normal of the boundary Γ. We note that our results extend to unbounded domains without any changes.

We denote by $\langle \cdot\,, \cdot \rangle_{\mathsf{L}^2}$ and $|\cdot|_{\mathsf{L}^2}$ the inner product and the norm in L^2 for scalar-, vector- and matrix-valued functions. Throughout this section we will not indicate the dependence on Ω in our notations of the functional spaces.

For the first application, the reaction-diffusion problem, we repeat all the results of Section 2. For the rest of the applications we will repeat only the main results of Theorems 2.5 and 2.13 for the sake of brevity.

3.1 Reaction-Diffusion

We define the usual Sobolev spaces

$$\mathsf{H}^1 := \{\varphi \in \mathsf{L}^2 \mid \nabla\varphi \in \mathsf{L}^2\}, \quad \mathsf{D} := \{\psi \in \mathsf{L}^2 \mid \operatorname{div} \psi \in \mathsf{L}^2\},$$

and the spaces

$$\mathsf{H}^1_{\Gamma_{\mathrm{D}}} := \overline{\mathsf{C}^\infty_{\Gamma_{\mathrm{D}}}}^{\mathsf{H}^1}, \quad \mathsf{D}_{\Gamma_{\mathrm{N}}} := \overline{\mathsf{C}^\infty_{\Gamma_{\mathrm{N}}}}^{\mathsf{D}},$$

were $\mathsf{C}^\infty_{\Gamma_{\mathrm{D}}}$ resp. $\mathsf{C}^\infty_{\Gamma_{\mathrm{N}}}$ is the space of smooth test functions resp. vector fields having supports bounded away from Γ_{D} resp. Γ_{N}. These are Hilbert spaces equipped with the graph norms denoted by $|\cdot|_{\mathsf{H}^1}$, $|\cdot|_{\mathsf{D}}$, respectively. Table 1 shows the relation to the notation of Section 2. We note that indeed $D(\mathrm{A}^*) = \mathsf{D}_{\Gamma_{\mathrm{N}}}$ holds for Lipschitz domains, see, e.g., [2, 5]. Relation (2.1) reads now

$$\langle \nabla\varphi, \psi \rangle_{\mathsf{L}^2} = -\langle \varphi, \operatorname{div} \psi \rangle_{\mathsf{L}^2} \quad \text{for all } \varphi \in \mathsf{H}^1_{\Gamma_{\mathrm{D}}},\ \psi \in \mathsf{D}_{\Gamma_{\mathrm{N}}}.$$

α_1	α_2	A	A*	H_1	H_2	$D(A)$	$D(A^*)$
ρ	α	∇	$-\operatorname{div}$	L^2	L^2	$H^1_{\Gamma_D}$	D_{Γ_N}

Table 1. Relation to the notation of Section 2.

Case I. Find the scalar potential $u \in H^1$ such that

$$\begin{cases} -\operatorname{div}\alpha\nabla u + \rho\, u = f & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_D, \\ n\cdot\alpha\nabla u = 0 & \text{on } \Gamma_N. \end{cases} \tag{3.1}$$

The quadratic diffusion matrix $\alpha \in L^\infty$ is symmetric, real valued, and uniformly positive definite. The real-valued reaction coefficient $\rho \geq \rho_0 > 0$ belongs to L^∞ and the source f to L^2. The dual variable for this problem is the flux $p = \alpha\nabla u \in D$. The mixed formulation of (3.1) reads: Find $(u, p) \in H^1_{\Gamma_D} \times D_{\Gamma_N}$ such that

$$-\operatorname{div} p + \rho\, u = f, \quad \alpha\nabla u = p \quad \text{in } \Omega. \tag{3.2}$$

The primal and dual variational problems are: Find $(u, p) \in H^1_{\Gamma_D} \times D_{\Gamma_N}$ such that

$$\begin{aligned} \langle\nabla u, \nabla\varphi\rangle_{L^2,\alpha} + \langle u, \varphi\rangle_{L^2,\rho} &= \langle f, \varphi\rangle_{L^2} && \text{for all } \varphi \in H^1_{\Gamma_D}, \\ \langle\operatorname{div} p, \operatorname{div}\psi\rangle_{L^2,\rho^{-1}} + \langle p, \psi\rangle_{L^2,\alpha^{-1}} &= -\langle f, \operatorname{div}\psi\rangle_{L^2,\rho^{-1}} && \text{for all } \psi \in D_{\Gamma_N}. \end{aligned}$$

Considering the norms, we have

$$\begin{aligned} |u|^2_{H^1,\rho,\alpha} &= |u|^2_{L^2,\rho} + |\nabla u|^2_{L^2,\alpha}, \\ |p|^2_{D,\rho^{-1},\alpha^{-1}} &= |p|^2_{L^2,\alpha^{-1}} + |\operatorname{div} p|^2_{L^2,\rho^{-1}}, \\ \|(u,p)\|^2 &= |u|^2_{H^1,\rho,\alpha} + |p|^2_{D,\rho^{-1},\alpha^{-1}}. \end{aligned}$$

Now Remark 2.3, Theorem 2.4, Theorem 2.5, and Corollary 2.7 read:

Remark 3.1. We note $|u|_{H^1,\rho,\alpha} \leq |f|_{L^2,\rho^{-1}}$ and $|p|_{D,\rho^{-1},\alpha^{-1}} \leq |f|_{L^2,\rho^{-1}}$ and indeed

$$\|(u,p)\| = |f|_{L^2,\rho^{-1}}.$$

The solution operator $L : L^2 \to H^1_{\Gamma_D} \times D_{\Gamma_N}$, $f \mapsto (u, p)$ is an isometry, i.e. $|L| = 1$.

Theorem 3.2. *Let* $(u, p) \in H^1_{\Gamma_D} \times D_{\Gamma_N}$ *be the exact solution of* (3.2). *Let* $\tilde{u} \in H^1_{\Gamma_D}$ *and* $\tilde{p} = \alpha\nabla\tilde{u} \in D_{\Gamma_N}$. *Then, for the mixed approximation* $(\tilde{u}, \tilde{p})$ *we have*

$$\|(u,p) - (\tilde{u},\tilde{p})\|^2 = \mathcal{I}_{rd}(\tilde{u},\tilde{p}), \qquad \frac{\|(u,p) - (\tilde{u},\tilde{p})\|^2}{\|(u,p)\|^2} = \frac{\mathcal{I}_{rd}(\tilde{u},\tilde{p})}{|f|^2_{L^2,\rho^{-1}}},$$

where $\mathcal{I}_{rd}(\tilde{u},\tilde{p}) = |f - \rho\,\tilde{u} + \operatorname{div}\tilde{p}|^2_{L^2,\rho^{-1}}$.

Theorem 3.3. *Let* $(u, p), (\tilde{u}, \tilde{p}) \in H^1_{\Gamma_D} \times D_{\Gamma_N}$ *be the exact solution of* (3.2) *and any approximation, respectively. Then*

$$\|(u,p) - (\tilde{u},\tilde{p})\|^2 = \mathcal{M}_{rd}(\tilde{u},\tilde{p}), \qquad \frac{\|(u,p) - (\tilde{u},\tilde{p})\|^2}{\|(u,p)\|^2} = \frac{\mathcal{M}_{rd}(\tilde{u},\tilde{p})}{|f|^2_{L^2,\rho^{-1}}}$$

hold, where $\mathcal{M}_{rd}(\tilde{u},\tilde{p}) = |f - \rho\,\tilde{u} + \operatorname{div}\tilde{p}|^2_{L^2,\rho^{-1}} + |\tilde{p} - \alpha\nabla\tilde{u}|^2_{L^2,\alpha^{-1}}$.

Corollary 3.4. *Theorem 3.3 provides the well-known a posteriori error estimates for the primal and dual problems.*

(i) *For any* $\tilde{u} \in H^1_{\Gamma_D}$ *it holds*

$$|u - \tilde{u}|^2_{H^1,\rho,\alpha} = \min_{\psi\in D_{\Gamma_N}} \mathcal{M}_{rd}(\tilde{u},\psi) = \mathcal{M}_{rd}(\tilde{u}, p).$$

(ii) *For any* $\tilde{p} \in \mathsf{D}_{\Gamma_N}$ *it holds*

$$|p - \tilde{p}|^2_{\mathsf{D},\rho^{-1},\alpha^{-1}} = \min_{\varphi \in \mathsf{H}^1_{\Gamma_D}} \mathcal{M}_{\mathrm{rd}}(\varphi, \tilde{p}) = \mathcal{M}_{\mathrm{rd}}(u, \tilde{p}).$$

Error indication properties of Section 2.3 hold as well:

Remark 3.5. Let $\mathcal{T}$ denote a discretization of the domain Ω into a mesh of non-overlapping elements T such as described in Section 2.3. We define the following error indicator using the functional of Theorem 3.3:

$$\eta_T(\tilde{u}, \tilde{p}) := \sqrt{|f - \rho\,\tilde{u} + \operatorname{div}\tilde{p}|^2_{\mathsf{L}^2(T),\rho^{-1}} + |\tilde{p} - \alpha\nabla\tilde{u}|^2_{\mathsf{L}^2(T),\alpha^{-1}}}, \quad \eta := \sqrt{\sum_{T\in\mathcal{T}} \eta_T^2}.$$

The error indicator η will indicate the exact error distribution

$$e_T(\tilde{x}, \tilde{y}) := \sqrt{|u - \tilde{u}|^2_{\mathsf{H}^1(T),\rho,\alpha} + |p - \tilde{p}|^2_{\mathsf{D}(T),\rho^{-1},\alpha^{-1}}}, \quad e := \sqrt{\sum_{T\in\mathcal{T}} e_T^2}.$$

As shown in Section 2.3, the global reliability constant, global efficiency constant, and the local efficiency constants are

$$\overline{c} = 1, \quad \underline{c} = 1, \quad c_T = \frac{1}{\sqrt{2}} > 0.7 \quad \text{for all } T \in \mathcal{T},$$

respectively.

Related results and numerical tests for exterior domains can be found in, e.g., [6, 9].

Case II. Find the scalar potential $u \in \mathsf{H}^1$ such that

$$\begin{cases} -\operatorname{div}\alpha\nabla u + i\omega\rho\,u = f & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_D, \\ n\cdot\alpha\nabla u = 0 & \text{on } \Gamma_N, \end{cases} \tag{3.3}$$

where α, ρ, and f are as before, and $\omega \in \mathbb{R}\setminus\{0\}$. The dual variable for this problem is the flux $p = \alpha\nabla u \in \mathsf{D}$. The mixed formulation of (3.3) reads: Find $(u, p) \in \mathsf{H}^1_{\Gamma_D} \times \mathsf{D}_{\Gamma_N}$ such that

$$-\operatorname{div} p + i\omega\rho\,u = f, \quad \alpha\nabla u = p \quad \text{in } \Omega. \tag{3.4}$$

Considering the norms, we have

$$\begin{aligned} |u|^2_{\mathsf{H}^1,|\omega|\rho,\alpha} &= |u|^2_{\mathsf{L}^2,|\omega|\rho} + |\nabla u|^2_{\mathsf{L}^2,\alpha}, \\ |p|^2_{\mathsf{D},(|\omega|\rho)^{-1},\alpha^{-1}} &= |p|^2_{\mathsf{L}^2,\alpha^{-1}} + |\operatorname{div} p|^2_{\mathsf{L}^2,(|\omega|\rho)^{-1}}, \\ |||(u,p)|||^2 &= |u|^2_{\mathsf{H}^1,|\omega|\rho,\alpha} + |p|^2_{\mathsf{D},(|\omega|\rho)^{-1},\alpha^{-1}}. \end{aligned}$$

The primal and dual variational problems are: Find $(u, p) \in \mathsf{H}^1_{\Gamma_D} \times \mathsf{D}_{\Gamma_N}$ such that

$$\begin{aligned} \langle\nabla u, \nabla\varphi\rangle_{\mathsf{L}^2,\alpha} + i\langle u, \varphi\rangle_{\mathsf{L}^2,\omega\rho} &= \langle f, \varphi\rangle_{\mathsf{L}^2} && \text{for all } \varphi \in \mathsf{H}^1_{\Gamma_D}, \\ \langle\operatorname{div} p, \operatorname{div}\psi\rangle_{\mathsf{L}^2,(\omega\rho)^{-1}} + i\langle p, \psi\rangle_{\mathsf{L}^2,\alpha^{-1}} &= -\langle f, \operatorname{div}\psi\rangle_{\mathsf{L}^2,(\omega\rho)^{-1}} && \text{for all } \psi \in \mathsf{D}_{\Gamma_N}. \end{aligned}$$

Now Remark 2.11, Theorem 2.12, Theorem 2.13, and Theorem 2.14 read:

Remark 3.6. We note $|u|_{\mathsf{H}^1,|\omega|\rho,\alpha} \le \sqrt{2}|f|_{\mathsf{L}^2,(|\omega|\rho)^{-1}}$ and $|p|_{\mathsf{D},(|\omega|\rho)^{-1},\alpha^{-1}} \le \sqrt{2}|f|_{\mathsf{L}^2,(|\omega|\rho)^{-1}}$ and indeed

$$|f|_{\mathsf{L}^2,(|\omega|\rho)^{-1}} \le |||(u,p)||| \le \sqrt{2}|f|_{\mathsf{L}^2,(|\omega|\rho)^{-1}}.$$

The norm of the solution operator $L_i : \mathsf{L}^2 \to \mathsf{H}^1_{\Gamma_D} \times \mathsf{D}_{\Gamma_N}, f \mapsto (u, p)$ then satisfies $1 \le |L_i| \le \sqrt{2}$.

Theorem 3.7. *Let* $(u, p) \in \mathsf{H}^1_{\Gamma_D} \times \mathsf{D}_{\Gamma_N}$ *be the exact solution of* (3.2). *Let* $\tilde{u} \in \mathsf{H}^1_{\Gamma_D}$ *and* $\tilde{p} = \alpha\nabla\tilde{u} \in \mathsf{D}_{\Gamma_N}$. *Then, for the mixed approximation* $(\tilde{u}, \tilde{p})$ *we have*

$$\mathcal{I}_{i,\mathrm{rd}}(\tilde{u}, \tilde{p}) \le |||(u,p) - (\tilde{u},\tilde{p})|||^2 \le 2\mathcal{I}_{i,\mathrm{rd}}(\tilde{u}, \tilde{p})$$

and

$$\frac{1}{2}\cdot\frac{\mathcal{I}_{i,\mathrm{rd}}(\tilde u,\tilde p)}{|f|^2_{\mathsf{L}^2,(|\omega|\rho)^{-1}}}\le\frac{|||(u,p)-(\tilde u,\tilde p)|||^2}{|||(u,p)|||^2}\le 2\frac{\mathcal{I}_{i,\mathrm{rd}}(\tilde u,\tilde p)}{|f|^2_{\mathsf{L}^2,(|\omega|\rho)^{-1}}},$$

where $\mathcal{I}_{i,\mathrm{rd}}(\tilde u,\tilde p)=|f-i\omega\rho\,\tilde u+\operatorname{div}\tilde p|^2_{\mathsf{L}^2,(|\omega|\rho)^{-1}}$.

Theorem 3.8. *Let* $(u,p),(\tilde u,\tilde p)\in\mathsf{H}^1_{\Gamma_\mathrm{D}}\times\mathsf{D}_{\Gamma_\mathrm{N}}$ *be the exact solution of* (3.4) *and any approximation, respectively. Then*

$$\frac{\sqrt2}{\sqrt2+1}\mathcal{M}_{i,\mathrm{rd}}(\tilde u,\tilde p)\le|||(u,p)-(\tilde u,\tilde p)|||^2\le\frac{\sqrt2}{\sqrt2-1}\mathcal{M}_{i,\mathrm{rd}}(\tilde u,\tilde p)$$

and

$$\frac{\sqrt2}{2(\sqrt2+1)}\cdot\frac{\mathcal{M}_{i,\mathrm{rd}}(\tilde u,\tilde p)}{|f|^2_{\mathsf{L}^2,(|\omega|\rho)^{-1}}}\le\frac{|||(u,p)-(\tilde u,\tilde p)|||^2}{|||(u,p)|||^2}\le\frac{\sqrt2}{\sqrt2-1}\cdot\frac{\mathcal{M}_{i,\mathrm{rd}}(\tilde u,\tilde p)}{|f|^2_{\mathsf{L}^2,(|\omega|\rho)^{-1}}}$$

hold, where $\mathcal{M}_{i,\mathrm{rd}}(\tilde u,\tilde p)=|f-i\omega\rho\,\tilde u+\operatorname{div}\tilde p|^2_{\mathsf{L}^2,(|\omega|\rho)^{-1}}+|\tilde p-\alpha\nabla\tilde u|^2_{\mathsf{L}^2,\alpha^{-1}}$.

Theorem 3.9. *We have the following a posteriori error estimates for the primal and dual problems.*

(i) *For any* $\tilde u\in\mathsf{H}^1_{\Gamma_\mathrm{D}}$ *it holds*

$$|u-\tilde u|^2_{\mathsf{H}^1,|\omega|\rho,\alpha}\le 2\mathcal{M}_{i,\mathrm{rd}}(\tilde u,\psi)\quad\text{for all }\psi\in\mathsf{D}_{\Gamma_\mathrm{N}}.$$

(ii) *For any* $\tilde p\in\mathsf{D}_{\Gamma_\mathrm{N}}$ *it holds*

$$|p-\tilde p|^2_{\mathsf{D},(|\omega|\rho)^{-1},\alpha^{-1}}\le 2\mathcal{M}_{i,\mathrm{rd}}(\varphi,\tilde p)\quad\text{for all }\varphi\in\mathsf{H}^1_{\Gamma_\mathrm{D}}.$$

The error indication properties of Section 2.3 hold as well:

Remark 3.10. Let $\mathcal{T}$ denote a discretization of the domain Ω into a mesh of non-overlapping elements T such as described in Section 2.3. We define the following error indicator using the functional of Theorem 3.8:

$$\eta_{i,T}(\tilde u,\tilde p):=\sqrt{|f-i\omega\rho\,\tilde u+\operatorname{div}\tilde p|^2_{\mathsf{L}^2(T),(|\omega|\rho)^{-1}}+|\tilde p-\alpha\nabla\tilde u|^2_{\mathsf{L}^2(T),\alpha^{-1}}},\quad\eta_i:=\sqrt{\sum_{T\in\mathcal{T}}\eta^2_{i,T}}.$$

The error indicator η_i will indicate the exact error distribution

$$e_{i,T}(\tilde x,\tilde y):=\sqrt{|u-\tilde u|^2_{\mathsf{H}^1(T),|\omega|\rho,\alpha}+|p-\tilde p|^2_{\mathsf{D}(T),(|\omega|\rho)^{-1},\alpha^{-1}}},\quad e_i:=\sqrt{\sum_{T\in\mathcal{T}}e^2_{i,T}}.$$

As shown in Section 2.3, the global reliability constant, global efficiency constant, and the local efficiency constants are

$$\overline{c}=\sqrt{\frac{\sqrt2}{\sqrt2-1}}<1.85,\quad \underline{c}=\sqrt{\frac{\sqrt2}{\sqrt2+1}}>0.76,\quad c_T=\frac{1}{\sqrt2}>0.7\quad\text{for all }T\in\mathcal{T},$$

respectively.

3.2 Maxwell Type Problems (3D)

Let $d=3$. We need the Sobolev spaces

$$\mathsf{R}:=\{\Phi\in\mathsf{L}^2\mid\operatorname{rot}\Phi\in\mathsf{L}^2\},\quad\mathsf{R}_{\Gamma_\mathrm{D}}:=\overline{\mathsf{C}^\infty_{\Gamma_\mathrm{D}}}^{\mathsf{R}},\quad\mathsf{R}_{\Gamma_\mathrm{N}}:=\overline{\mathsf{C}^\infty_{\Gamma_\mathrm{N}}}^{\mathsf{R}}.$$

Table 2 shows the relation to the notation of Section 2. We note that indeed $D(\mathrm{A}^*)=\mathsf{R}_{\Gamma_\mathrm{N}}$ holds for Lipschitz domains, see, e.g., [2, 5]. Relation (2.1) reads now

$$\langle\operatorname{rot}\Phi,\Psi\rangle_{\mathsf{L}^2}=\langle\Phi,\operatorname{rot}\Psi\rangle_{\mathsf{L}^2}\quad\text{for all }\Phi\in\mathsf{R}_{\Gamma_\mathrm{D}},\ \Psi\in\mathsf{R}_{\Gamma_\mathrm{N}}.$$

α_1	α_2	A	A*	H_1	H_2	D(A)	D(A*)
ϵ, σ	μ^{-1}	rot	rot	L^2	L^2	R_{Γ_D}	R_{Γ_N}

Table 2. Relation to the notation of Section 2.

Case I: A Maxwell Type Problem. The problem reads: Find the electric field $E \in R$ such that

$$\begin{cases} \operatorname{rot}\mu^{-1}\operatorname{rot}E + \epsilon E = J & \text{in } \Omega, \\ n \times E = 0 & \text{on } \Gamma_D, \\ n \times \mu^{-1}\operatorname{rot}E = 0 & \text{on } \Gamma_N. \end{cases} \tag{3.5}$$

We assume that the magnetic permeability μ and the electric permittivity ϵ are symmetric, real-valued, and uniformly positive definite matrices from L^∞. The electric current J belongs to L^2. The dual variable for this problem is the magnetic field $H = \mu^{-1}\operatorname{rot}E \in R$. The mixed formulation of (3.5) reads as follows: Find $(E, H) \in R_{\Gamma_D} \times R_{\Gamma_N}$ such that

$$\operatorname{rot}H + \epsilon E = J, \quad \mu^{-1}\operatorname{rot}E = H \quad \text{in } \Omega. \tag{3.6}$$

Considering the norms, we have

$$\begin{aligned} |E|^2_{R,\epsilon,\mu^{-1}} &= |E|^2_{L^2,\epsilon} + |\operatorname{rot}E|^2_{L^2,\mu^{-1}}, \\ |H|^2_{R,\epsilon^{-1},\mu} &= |H|^2_{L^2,\mu} + |\operatorname{rot}H|^2_{L^2,\epsilon^{-1}}, \\ \|(E,H)\|^2 &= |E|^2_{R,\epsilon,\mu^{-1}} + |H|^2_{R,\epsilon^{-1},\mu}. \end{aligned}$$

Now Theorem 2.5 reads:

Theorem 3.11. *Let* $(E, H), (\tilde{E}, \tilde{H}) \in R_{\Gamma_D} \times R_{\Gamma_N}$ *be the exact solution of* (3.6) *and any approximation, respectively. Then*

$$\|(E,H) - (\tilde{E},\tilde{H})\|^2 = \mathcal{M}_{ec}(\tilde{E},\tilde{H}), \qquad \frac{\|(E,H) - (\tilde{E},\tilde{H})\|^2}{\|(E,H)\|^2} = \frac{\mathcal{M}_{ec}(\tilde{E},\tilde{H})}{|J|^2_{L^2,\epsilon^{-1}}}$$

hold, where $\mathcal{M}_{ec}(\tilde{E},\tilde{H}) = |J - \epsilon\tilde{E} - \operatorname{rot}\tilde{H}|^2_{L^2,\epsilon^{-1}} + |\tilde{H} - \mu^{-1}\operatorname{rot}\tilde{E}|^2_{L^2,\mu}$.

Earlier results for eddy current and static Maxwell problems can be found in [1, 10].

Case II: Eddy-Current. The problem reads: Find the electric field $E \in R$ such that

$$\begin{cases} \operatorname{rot}\mu^{-1}\operatorname{rot}E + i\omega\sigma E = J & \text{in } \Omega, \\ n \times E = 0 & \text{on } \Gamma_D, \\ n \times \mu^{-1}\operatorname{rot}E = 0 & \text{on } \Gamma_N, \end{cases} \tag{3.7}$$

where μ and J are as before, the conductivity σ is a symmetric, real-valued, and uniformly positive definite matrix from L^∞, and $\omega \in \mathbb{R} \setminus \{0\}$. The dual variable for this problem is the magnetic field $H = \mu^{-1}\operatorname{rot}E \in R$. The mixed formulation of (3.7) reads: Find $(E, H) \in R_{\Gamma_D} \times R_{\Gamma_N}$ such that

$$\operatorname{rot}H + i\omega\sigma E = J, \quad \mu^{-1}\operatorname{rot}E = H \quad \text{in } \Omega. \tag{3.8}$$

Considering the norms, we have

$$\begin{aligned} |E|^2_{R,|\omega|\sigma,\mu^{-1}} &= |E|^2_{L^2,|\omega|\sigma} + |\operatorname{rot}E|^2_{L^2,\mu^{-1}}, \\ |H|^2_{R,(|\omega|\sigma)^{-1},\mu} &= |H|^2_{L^2,\mu} + |\operatorname{rot}H|^2_{L^2,(|\omega|\sigma)^{-1}}, \\ |||(E,H)|||^2 &= |E|^2_{R,|\omega|\sigma,\mu^{-1}} + |H|^2_{R,(|\omega|\sigma)^{-1},\mu}. \end{aligned}$$

Now Theorem 2.13 reads:

Theorem 3.12. *Let* $(E, H), (\tilde{E}, \tilde{H}) \in \mathsf{R}_{\Gamma_D} \times \mathsf{R}_{\Gamma_N}$ *be the exact solution of* (3.8) *and any approximation, respectively. Then*

$$\frac{\sqrt{2}}{\sqrt{2}+1}\mathcal{M}_{i,\mathrm{ec}}(\tilde{E}, \tilde{H}) \le |||(E, H) - (\tilde{E}, \tilde{H})|||^2 \le \frac{\sqrt{2}}{\sqrt{2}-1}\mathcal{M}_{i,\mathrm{ec}}(\tilde{E}, \tilde{H})$$

and

$$\frac{\sqrt{2}}{2(\sqrt{2}+1)} \cdot \frac{\mathcal{M}_{i,\mathrm{ec}}(\tilde{E}, \tilde{H})}{|J|^2_{\mathsf{L}^2,(|\omega|\sigma)^{-1}}} \le \frac{|||(E, H) - (\tilde{E}, \tilde{H})|||^2}{|||(E, H)|||^2} \le \frac{\sqrt{2}}{\sqrt{2}-1} \cdot \frac{\mathcal{M}_{i,\mathrm{ec}}(\tilde{E}, \tilde{H})}{|J|^2_{\mathsf{L}^2,(|\omega|\sigma)^{-1}}}$$

hold, where $\mathcal{M}_{i,\mathrm{ec}}(\tilde{E}, \tilde{H}) = |J - i\omega\sigma\tilde{E} - \operatorname{rot}\tilde{H}|^2_{\mathsf{L}^2,(|\omega|\sigma)^{-1}} + |\tilde{H} - \mu^{-1}\operatorname{rot}\tilde{E}|^2_{\mathsf{L}^2,\mu}$.

3.3 Maxwell Type Problems (2D)

Let $d = 2$. In the following we simply indicate the changes compared to the previous section. First, we have to understand the double rot as $\nabla^\perp$ rot, where

$$\operatorname{rot} E := \operatorname{div} QE = \partial_1 E_2 - \partial_2 E_1, \quad \nabla^\perp H := Q\nabla H = \begin{bmatrix} \partial_2 H \\ -\partial_1 H \end{bmatrix}, \quad Q := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

and $E \in \mathsf{R}$ is a vector field and $H \in \mathsf{H}^1$ a scalar function. In the literature, the operator $\nabla^\perp$ is often called co-gradient or vector rotation $\vec{\operatorname{rot}}$ as well. Also μ is scalar. Table 3 shows the relation to the notation of Section 2. Relation (2.1) reads now

$$\langle \operatorname{rot}\Phi, \psi\rangle_{\mathsf{L}^2} = \langle \Phi, \nabla^\perp\psi\rangle_{\mathsf{L}^2} \quad \text{for all } \Phi \in \mathsf{R}_{\Gamma_D}, \ \psi \in \mathsf{H}^1_{\Gamma_N}.$$

α_1	α_2	A	A*	H_1	H_2	$D(\mathsf{A})$	$D(\mathsf{A}^*)$
ϵ, σ	μ^{-1}	rot	$\nabla^\perp$	L^2	L^2	R_{Γ_D}	$\mathsf{H}^1_{\Gamma_N}$

Table 3. Relation to the notation of Section 2.

Case I: A Maxwell Type Problem. Now (3.5) reads: Find the electric field $E \in \mathsf{R}$ such that

$$\begin{cases} \nabla^\perp\mu^{-1}\operatorname{rot} E + \epsilon E = J & \text{in } \Omega, \\ n \times E = 0 & \text{on } \Gamma_D, \\ \mu^{-1}\operatorname{rot} E = 0 & \text{on } \Gamma_N. \end{cases}$$

The mixed formulation of the problem is: Find $(E, H) \in \mathsf{R}_{\Gamma_D} \times \mathsf{H}^1_{\Gamma_N}$ such that

$$\nabla^\perp H + \epsilon E = J, \quad \mu^{-1}\operatorname{rot} E = H \quad \text{in } \Omega. \tag{3.9}$$

The norm for H is

$$|H|^2_{\mathsf{H}^1,\epsilon^{-1},\mu} = |H|^2_{\mathsf{L}^2,\mu} + |\nabla^\perp H|^2_{\mathsf{L}^2,\epsilon^{-1}}.$$

Now Theorem 3.11 (and thus Theorem 2.5) reads:

Theorem 3.13. *Let* $(E, H), (\tilde{E}, \tilde{H}) \in \mathsf{R}_{\Gamma_D} \times \mathsf{H}^1_{\Gamma_N}$ *be the exact solution of* (3.9) *and any approximation, respectively. Then*

$$\|(E, H) - (\tilde{E}, \tilde{H})\|^2 = \mathcal{M}_{\mathrm{ec}}(\tilde{E}, \tilde{H}), \qquad \frac{\|(E, H) - (\tilde{E}, \tilde{H})\|^2}{\|(E, H)\|^2} = \frac{\mathcal{M}_{\mathrm{ec}}(\tilde{E}, \tilde{H})}{|J|^2_{\mathsf{L}^2,\epsilon^{-1}}}$$

hold, where $\mathcal{M}_{\mathrm{ec}}(\tilde{E}, \tilde{H}) = |J - \epsilon\tilde{E} - \nabla^\perp\tilde{H}|^2_{\mathsf{L}^2,\epsilon^{-1}} + |\tilde{H} - \mu^{-1}\operatorname{rot}\tilde{E}|^2_{\mathsf{L}^2,\mu}$.

Case II: Eddy-Current. Now (3.7) reads: Find the electric field $E \in \mathsf{R}$ such that

$$\begin{cases} \nabla^{\perp}\mu^{-1} \operatorname{rot} E + i\omega\sigma E = J & \text{in } \Omega, \\ n \times E = 0 & \text{on } \Gamma_{\mathrm{D}}, \\ \mu^{-1} \operatorname{rot} E = 0 & \text{on } \Gamma_{\mathrm{N}}. \end{cases}$$

The mixed formulation of the problem is: Find $(E, H) \in \mathsf{R}_{\Gamma_{\mathrm{D}}} \times \mathsf{H}^1_{\Gamma_{\mathrm{N}}}$ such that

$$\nabla^{\perp} H + i\omega\sigma E = J, \quad \mu^{-1} \operatorname{rot} E = H \quad \text{in } \Omega. \tag{3.10}$$

The norm for H is

$$|H|^2_{\mathsf{H}^1,(|\omega|\sigma)^{-1},\mu} = |H|^2_{\mathsf{L}^2,\mu} + |\nabla^{\perp} H|^2_{\mathsf{L}^2,(|\omega|\sigma)^{-1}}.$$

Now Theorem 3.12 (and thus Theorem 2.13) reads:

Theorem 3.14. *Let $(E, H), (\tilde{E}, \tilde{H}) \in \mathsf{R}_{\Gamma_{\mathrm{D}}} \times \mathsf{H}^1_{\Gamma_{\mathrm{N}}}$ be the exact solution of (3.10) and any approximation, respectively. Then*

$$\frac{\sqrt{2}}{\sqrt{2}+1}\mathcal{M}_{i,\mathrm{ec}}(\tilde{E}, \tilde{H}) \le |||(E, H) - (\tilde{E}, \tilde{H})|||^2 \le \frac{\sqrt{2}}{\sqrt{2}-1}\mathcal{M}_{i,\mathrm{ec}}(\tilde{E}, \tilde{H})$$

and

$$\frac{\sqrt{2}}{2(\sqrt{2}+1)} \cdot \frac{\mathcal{M}_{i,\mathrm{ec}}(\tilde{E}, \tilde{H})}{|J|^2_{\mathsf{L}^2,(|\omega|\sigma)^{-1}}} \le \frac{|||(E, H) - (\tilde{E}, \tilde{H})|||^2}{|||(E, H)|||^2} \le \frac{\sqrt{2}}{\sqrt{2}-1} \cdot \frac{\mathcal{M}_{i,\mathrm{ec}}(\tilde{E}, \tilde{H})}{|J|^2_{\mathsf{L}^2,(|\omega|\sigma)^{-1}}}$$

hold, where $\mathcal{M}_{i,\mathrm{ec}}(\tilde{E}, \tilde{H}) = |J - i\omega\sigma\tilde{E} - \nabla^{\perp}\tilde{H}|^2_{\mathsf{L}^2,(|\omega|\sigma)^{-1}} + |\tilde{H} - \mu^{-1} \operatorname{rot} \tilde{E}|^2_{\mathsf{L}^2,\mu}$.

3.4 Linear Elasticity Type Problems

We will need ∇_{s}, which is the symmetric part of the gradient

$$\nabla_{\mathrm{s}} u := \operatorname{sym} \nabla u = \frac{1}{2}(\nabla u + (\nabla u)^{\top}),$$

where ∇u is understood as the Jacobian of the vector field u and $^{\top}$ denotes the transpose. $\nabla_{\mathrm{s}} u$, often denoted by $\epsilon(u)$, is also called the infinitesimal strain tensor. For a tensor σ the notation $\sigma \in \mathsf{D}$ and the application of Div to σ are to be understood row-wise as the usual divergence div. Moreover, we define

$$\operatorname{Div}_{\mathrm{s}} \sigma := \operatorname{Div} \operatorname{sym} \sigma.$$

Table 4 shows the relation to the notation of Section 2. The notation $\sigma \in \operatorname{sym}^{-1} \mathsf{D}_{\Gamma_{\mathrm{N}}}$ means $\operatorname{sym} \sigma \in \mathsf{D}_{\Gamma_{\mathrm{N}}}$. More precisely, $\psi \in D(\mathrm{A}^*)$ if and only if

$$\langle \nabla_{\mathrm{s}}\varphi, \psi \rangle_{\mathsf{L}^2} = \langle \varphi, \mathrm{A}^*\psi \rangle_{\mathsf{L}^2} \quad \text{for all } \varphi \in D(\mathrm{A}) = \mathsf{H}^1_{\Gamma_{\mathrm{D}}}.$$

Since $\langle \nabla_{\mathrm{s}}\varphi, \psi \rangle_{\mathsf{L}^2} = \langle \nabla\varphi, \operatorname{sym} \psi \rangle_{\mathsf{L}^2}$, we see that this holds if and only if $\operatorname{sym} \psi \in \mathsf{D}_{\Gamma_{\mathrm{N}}}$ and $\mathrm{A}^*\psi = -\operatorname{Div} \operatorname{sym} \psi$. Equation (2.1) turns into

$$\langle \nabla_{\mathrm{s}}\varphi, \psi \rangle_{\mathsf{L}^2} = -\langle \varphi, \operatorname{Div}_{\mathrm{s}} \psi \rangle_{\mathsf{L}^2} \quad \text{for all } \varphi \in \mathsf{H}^1_{\Gamma_{\mathrm{D}}},\ \psi \in \operatorname{sym}^{-1} \mathsf{D}_{\Gamma_{\mathrm{N}}}.$$

α_1	α_2	A	A*	H_1	H_2	$D(\mathrm{A})$	$D(\mathrm{A}^*)$
ρ	Λ	∇_{s}	$-\operatorname{Div}_{\mathrm{s}}$	L^2	L^2	$\mathsf{H}^1_{\Gamma_{\mathrm{D}}}$	$\operatorname{sym}^{-1} \mathsf{D}_{\Gamma_{\mathrm{N}}}$

Table 4. Relation to the notation of Section 2.

Case I. Find the displacement vector field $u \in \mathsf{H}^1$ such that

$$\begin{cases} -\operatorname{Div} \Lambda \nabla_s u + \rho\, u = f & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_D, \\ \Lambda \nabla_s u \cdot n = 0 & \text{on } \Gamma_N. \end{cases} \tag{3.11}$$

The fourth-order stiffness tensor of elastic moduli $\Lambda \in \mathsf{L}^\infty$, mapping symmetric matrices to symmetric matrices point-wise, and the second-order tensor (quadratic matrix) of reaction ρ are assumed to be symmetric, real valued, and uniformly positive definite. The vector field f (body force) belongs to L^2 and the dual variable for this problem is the Cauchy stress tensor $\sigma = \Lambda \nabla_s u \in \mathsf{D}$. Note that σ is indeed symmetric. We note that the first equation in (3.11) can also be written as

$$-\operatorname{Div}_s \Lambda \nabla_s u + \rho\, u = f.$$

The mixed formulation of (3.11) reads: Find $(u, \sigma) \in \mathsf{H}^1_{\Gamma_D} \times \mathsf{D}_{\Gamma_N}$ such that

$$-\operatorname{Div} \sigma + \rho\, u = f, \quad \Lambda \nabla_s u = \sigma \quad \text{in } \Omega. \tag{3.12}$$

For the norms we have

$$\begin{aligned} |u|^2_{\mathsf{H}^1,\rho,\Lambda} &= |u|^2_{\mathsf{L}^2,\rho} + |\nabla_s u|^2_{\mathsf{L}^2,\Lambda}, \\ |\sigma|^2_{\operatorname{sym}^{-1}\mathsf{D},\rho^{-1},\Lambda^{-1}} &= |\sigma|^2_{\mathsf{L}^2,\Lambda^{-1}} + |\operatorname{Div}_s \sigma|^2_{\mathsf{L}^2,\rho^{-1}}, \\ \|(u,\sigma)\|^2 &= |u|^2_{\mathsf{H}^1,\rho,\Lambda} + |\sigma|^2_{\operatorname{sym}^{-1}\mathsf{D},\rho^{-1},\Lambda^{-1}}. \end{aligned}$$

Now Theorem 2.5 reads:

Theorem 3.15. *Let $(u, \sigma), (\tilde{u}, \tilde{\sigma}) \in \mathsf{H}^1_{\Gamma_D} \times \operatorname{sym}^{-1} \mathsf{D}_{\Gamma_N}$ be the exact solution of (3.12) and any approximation, respectively. Then*

$$\|(u,\sigma) - (\tilde{u},\tilde{\sigma})\|^2 = \mathcal{M}_{\mathrm{le}}(\tilde{u},\tilde{\sigma}), \qquad \frac{\|(u,\sigma) - (\tilde{u},\tilde{\sigma})\|^2}{\|(u,\sigma)\|^2} = \frac{\mathcal{M}_{\mathrm{le}}(\tilde{u},\tilde{\sigma})}{|f|^2_{\mathsf{L}^2,\rho^{-1}}}$$

hold, where $\mathcal{M}_{\mathrm{le}}(\tilde{u},\tilde{\sigma}) = |f - \rho\,\tilde{u} + \operatorname{Div}_s \tilde{\sigma}|^2_{\mathsf{L}^2,\rho^{-1}} + |\tilde{\sigma} - \Lambda \nabla_s \tilde{u}|^2_{\mathsf{L}^2,\Lambda^{-1}}$.

Moreover, since the tensor σ is symmetric, the above results hold for all pairs $(\tilde{u}, \tilde{\sigma}) \in \mathsf{H}^1_{\Gamma_D} \times \mathsf{D}_{\Gamma_N}$ with symmetric tensor $\tilde{\sigma}$, and the functional simplifies to $\mathcal{M}_{\mathrm{le}}(\tilde{u},\tilde{\sigma}) = |f - \rho\,\tilde{u} + \operatorname{Div} \tilde{\sigma}|^2_{\mathsf{L}^2,\rho^{-1}} + |\tilde{\sigma} - \Lambda \nabla_s \tilde{u}|^2_{\mathsf{L}^2,\Lambda^{-1}}$.

Case II. Find the displacement vector field $u \in \mathsf{H}^1$ such that

$$\begin{cases} -\operatorname{Div} \Lambda \nabla_s u + i\omega\rho\, u = f & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_D, \\ \Lambda \nabla_s u \cdot n = 0 & \text{on } \Gamma_N, \end{cases} \tag{3.13}$$

where Λ, ρ, and f are as before, and $\omega \in \mathbb{R} \setminus \{0\}$. The dual variable for this problem is the Cauchy stress tensor $\sigma = \Lambda \nabla_s u \in \mathsf{D}$. We note again that σ is symmetric, and that the first equation of (3.13) can also be written as

$$-\operatorname{Div}_s \Lambda \nabla_s u + i\omega\rho\, u = f.$$

The mixed formulation of (3.13) reads: Find $(u, \sigma) \in \mathsf{H}^1_{\Gamma_D} \times \mathsf{D}_{\Gamma_N}$ such that

$$-\operatorname{Div} \sigma + i\omega\rho\, u = f, \quad \Lambda \nabla_s u = \sigma \quad \text{in } \Omega. \tag{3.14}$$

For the norms we have

$$\begin{aligned} |u|^2_{\mathsf{H}^1,|\omega|\rho,\Lambda} &= |u|^2_{\mathsf{L}^2,|\omega|\rho} + |\nabla_s u|^2_{\mathsf{L}^2,\Lambda}, \\ |\sigma|^2_{\operatorname{sym}^{-1}\mathsf{D},(|\omega|\rho)^{-1},\Lambda^{-1}} &= |\sigma|^2_{\mathsf{L}^2,\Lambda^{-1}} + |\operatorname{Div}_s \sigma|^2_{\mathsf{L}^2,(|\omega|\rho)^{-1}}, \\ |\!|\!|(u,\sigma)|\!|\!|^2 &= |u|^2_{\mathsf{H}^1,|\omega|\rho,\Lambda} + |\sigma|^2_{\operatorname{sym}^{-1}\mathsf{D},(|\omega|\rho)^{-1},\Lambda^{-1}}. \end{aligned}$$

Now Theorem 2.13 reads:

Theorem 3.16. *Let $(u, \sigma), (\tilde{u}, \tilde{\sigma}) \in \mathsf{H}^1_{\Gamma_D} \times \operatorname{sym}^{-1} \mathsf{D}_{\Gamma_N}$ be the exact solution of (3.14) and any approximation, respectively. Then*

$$\frac{\sqrt{2}}{\sqrt{2}+1}\mathcal{M}_{i,\mathrm{le}}(\tilde{u},\tilde{\sigma}) \leq |||(u,\sigma)-(\tilde{u},\tilde{\sigma})|||^2 \leq \frac{\sqrt{2}}{\sqrt{2}-1}\mathcal{M}_{i,\mathrm{le}}(\tilde{u},\tilde{\sigma})$$

and

$$\frac{\sqrt{2}}{2(\sqrt{2}+1)} \cdot \frac{\mathcal{M}_{i,\mathrm{le}}(\tilde{u},\tilde{\sigma})}{|f|^2_{\mathsf{L}^2,(|\omega|\rho)^{-1}}} \leq \frac{|||(u,\sigma)-(\tilde{u},\tilde{\sigma})|||^2}{|||(u,\sigma)|||^2} \leq \frac{\sqrt{2}}{\sqrt{2}-1} \cdot \frac{\mathcal{M}_{i,\mathrm{le}}(\tilde{u},\tilde{\sigma})}{|f|^2_{\mathsf{L}^2,(|\omega|\rho)^{-1}}}$$

hold, where $\mathcal{M}_{i,\mathrm{le}}(\tilde{u},\tilde{\sigma}) = |f - i\omega\rho\,\tilde{u} + \operatorname{Div}_{\mathsf{s}}\tilde{\sigma}|^2_{\mathsf{L}^2,(|\omega|\rho)^{-1}} + |\tilde{\sigma} - \Lambda\nabla_{\mathsf{s}}\tilde{u}|^2_{\mathsf{L}^2,\Lambda^{-1}}$.

Moreover, since the tensor σ is symmetric, the above results hold for all pairs $(\tilde{u},\tilde{\sigma}) \in \mathsf{H}^1_{\Gamma_D} \times \mathsf{D}_{\Gamma_N}$ with symmetric $\tilde{\sigma}$, and the functional simplifies to $\mathcal{M}_{i,\mathrm{le}}(\tilde{u},\tilde{\sigma}) = |f - i\omega\rho\,\tilde{u} + \operatorname{Div}\tilde{\sigma}|^2_{\mathsf{L}^2,(|\omega|\rho)^{-1}} + |\tilde{\sigma} - \Lambda\nabla_{\mathsf{s}}\tilde{u}|^2_{\mathsf{L}^2,\Lambda^{-1}}$.

3.5 Different Boundary Conditions and Other Problems

We note that the (non-normalized) error equalities and error estimates hold without change with non-homogeneous boundary conditions. Also Robin boundary conditions can be treated (see Appendix A).

It is clear that the list of applications of our theory is much longer. For example:

- generalized reaction-diffusion, linear acoustics and electromagnetics on Riemannian manifolds
$$-\delta \mathrm{d} + 1, \quad -\delta \mathrm{d} + i,$$
were d and δ denote the exterior and co-derivative, respectively;
- the fourth-order problem
$$\operatorname{div}\operatorname{Div}\nabla\nabla + 1, \quad \operatorname{div}\operatorname{Div}\nabla\nabla + i;$$
- the biharmonic problem
$$\Delta\Delta + 1, \quad \Delta\Delta + i;$$
- certain generalized Stokes and Oseen type problems.

A Inhomogeneous and More Boundary Conditions

We will demonstrate that our results also hold for Robin type boundary conditions, which means that our results are true for many commonly used boundary conditions. Moreover, we emphasize that we can also handle inhomogeneous boundary conditions. Since it is clear that this method works in the general setting for both Cases I and II, we will demonstrate it here just for a simple reaction-diffusion type model problem belonging to the class of Case I. Let Ω be as in the latter section and now the boundary Γ be decomposed into three disjoint parts Γ_D, Γ_N and Γ_R.

The model problem is: Find the scalar potential $u \in \mathsf{H}^1$ such that

$$\begin{cases} -\operatorname{div}\nabla u + u = f & \text{in } \Omega, \\ u = g_1 & \text{on } \Gamma_D, \\ n \cdot \nabla u = g_2 & \text{on } \Gamma_N, \\ n \cdot \nabla u + \gamma u = g_3 & \text{on } \Gamma_R \end{cases}$$

hold. Hence, on Γ_D, Γ_N and Γ_R we impose Dirichlet, Neumann and Robin type boundary conditions, respectively. In the Robin boundary condition, we assume that the coefficient $\gamma \geq \gamma_0 > 0$ belongs to L^∞. The dual variable for this problem is the flux $p := \nabla u \in \mathsf{D}$. Furthermore, as long as $\Gamma_R \neq \emptyset$ and to avoid tricky discussions about traces and the corresponding $\mathsf{H}^{-1/2}$-spaces of Γ, Γ_D, Γ_N, and Γ_R, which can be quite complicated, we assume for simplicity that $u \in \mathsf{H}^2$. Then, $p \in \mathsf{H}^1$ and all g_i belong to L^2 even to $\mathsf{H}^{1/2}$ of Γ. For the norms we simply have

$$\|(u,p)\|^2 = |u|^2_{\mathsf{H}^1} + |p|^2_{\mathsf{D}}.$$

Theorem A.1. *For any approximation pair $(\tilde{u},\tilde{p}) \in \mathsf{H}^2 \times \mathsf{H}^1$ with $u-\tilde{u} \in \mathsf{H}^1_{\Gamma_\mathrm{D}}$ and $p-\tilde{p} \in \mathsf{D}_{\Gamma_\mathrm{N}}$ as well as $n\cdot(p-\tilde{p}) + \gamma(u-\tilde{u}) = 0$ on Γ_R it holds*

$$\|(u,p)-(\tilde{u},\tilde{p})\|^2 + 2|u-\tilde{u}|^2_{\mathsf{L}^2(\Gamma_\mathrm{R}),\gamma} = \mathcal{M}(\tilde{u},\tilde{p})$$

with $\mathcal{M}(\tilde{u},\tilde{p}) := |f-\tilde{u}+\operatorname{div}\tilde{p}|^2_{\mathsf{L}^2} + |\tilde{p}-\nabla\tilde{u}|^2_{\mathsf{L}^2}$. Moreover, $|u-\tilde{u}|_{\mathsf{L}^2(\Gamma_\mathrm{R}),\gamma} = |n\cdot(p-\tilde{p})|_{\mathsf{L}^2(\Gamma_\mathrm{R}),\gamma^{-1}}$.

Proof. Following the proof of Theorem 2.5, we have

$$\mathcal{M}(\tilde{u},\tilde{p}) = \underbrace{|u-\tilde{u}|^2_{\mathsf{H}^1} + |p-\tilde{p}|^2_{\mathsf{D}}}_{=\|(u,p)-(\tilde{u},\tilde{p})\|^2} + 2\Re\langle u-\tilde{u}, \operatorname{div}(\tilde{p}-p)\rangle_{\mathsf{L}^2} + 2\Re\langle \nabla(u-\tilde{u}), \tilde{p}-p\rangle_{\mathsf{L}^2}.$$

Moreover, since $n\cdot(\tilde{p}-p)$ and $u-\tilde{u}$ belong to $\mathsf{L}^2(\Gamma)$, we have

$$\begin{aligned}\langle \nabla(u-\tilde{u}), \tilde{p}-p\rangle_{\mathsf{L}^2} + \langle u-\tilde{u}, \operatorname{div}(\tilde{p}-p)\rangle_{\mathsf{L}^2} &= \langle n\cdot(\tilde{p}-p), u-\tilde{u}\rangle_{\mathsf{L}^2(\Gamma)}\\ &= \langle n\cdot(\tilde{p}-p), u-\tilde{u}\rangle_{\mathsf{L}^2(\Gamma_\mathrm{R})}\\ &= \langle \gamma(u-\tilde{u}), u-\tilde{u}\rangle_{\mathsf{L}^2(\Gamma_\mathrm{R})}.\end{aligned}$$

As $\langle \gamma(u-\tilde{u}), u-\tilde{u}\rangle_{\mathsf{L}^2(\Gamma_\mathrm{R})} = \langle \gamma^{-1} n\cdot(p-\tilde{p}), n\cdot(p-\tilde{p})\rangle_{\mathsf{L}^2(\Gamma_\mathrm{R})}$, we get the assertion. □

Remark A.2. If all $g_i = 0$, we can set $(\tilde{u},\tilde{p}) = (0,0)$ and get

$$\|(u,p)\|^2 + 2|u|^2_{\mathsf{L}^2(\Gamma_\mathrm{R}),\gamma} = |f|^2_{\mathsf{L}^2},$$

which follows also by

$$\begin{aligned}|f|^2_{\mathsf{L}^2} &= |\operatorname{div} p|^2_{\mathsf{L}^2} + |u|^2_{\mathsf{L}^2} - 2\Re\langle \operatorname{div}\nabla u, u\rangle_{\mathsf{L}^2}\\ &= |\operatorname{div} p|^2_{\mathsf{L}^2} + |u|^2_{\mathsf{L}^2} + 2|\nabla u|_{\mathsf{L}^2} - 2\Re\langle n\cdot\nabla u, u\rangle_{\mathsf{L}^2(\Gamma)}\\ &= |\operatorname{div} p|^2_{\mathsf{L}^2} + |u|^2_{\mathsf{L}^2} + 2|\nabla u|_{\mathsf{L}^2} - 2\Re\underbrace{\langle n\cdot\nabla u, u\rangle_{\mathsf{L}^2(\Gamma_\mathrm{R})}}_{=-|u|^2_{\mathsf{L}^2(\Gamma_\mathrm{R}),\gamma}}.\end{aligned}$$

Thus, in this case the assertion of Theorem A.1 has a normalized counterpart as well.

If $\Gamma_\mathrm{R} = \emptyset$, we have a pure mixed Dirichlet and Neumann boundary.

Theorem A.3. *Let $\Gamma_\mathrm{R} = \emptyset$. For any approximation $(\tilde{u},\tilde{p}) \in \mathsf{H}^1 \times \mathsf{D}$ with $u-\tilde{u} \in \mathsf{H}^1_{\Gamma_\mathrm{D}}$ and $p-\tilde{p} \in \mathsf{D}_{\Gamma_\mathrm{N}}$ we have*

$$\|(u,p)-(\tilde{u},\tilde{p})\|^2 = \mathcal{M}(\tilde{u},\tilde{p}).$$

Corollary A.4. *Let $\Gamma_\mathrm{R} = \emptyset$. Theorem A.3 provides the well-known a posteriori error estimates for the primal and dual problems.*

(i) *For any $\tilde{u} \in \mathsf{H}^1$ with $u-\tilde{u} \in \mathsf{H}^1_{\Gamma_\mathrm{D}}$ it holds*

$$|u-\tilde{u}|^2_{\mathsf{H}^1} = \min_{\substack{\psi\in\mathsf{D}\\ p-\psi\in\mathsf{D}_{\Gamma_\mathrm{N}}}} \mathcal{M}(\tilde{u},\psi) = \mathcal{M}(\tilde{u},p).$$

(ii) *For any $\tilde{p} \in \mathsf{D}$ with $p-\tilde{p} \in \mathsf{D}_{\Gamma_\mathrm{N}}$ it holds*

$$|p-\tilde{p}|^2_{\mathsf{D}} = \min_{\substack{\varphi\in\mathsf{H}^1\\ u-\varphi\in\mathsf{H}^1_{\Gamma_\mathrm{D}}}} \mathcal{M}(\varphi,\tilde{p}) = \mathcal{M}(u,\tilde{p}).$$

Funding: The first author thanks the Emil Aaltonen Foundation for support.

References

[1] I. Anjam, O. Mali, A. Muzalevskiy, P. Neittaanmäki and S. Repin, A posteriori error estimates for a Maxwell type problem, *Russian J. Numer. Anal. Math. Modelling* **24** (2009), no. 5, 395–408.

[2] S. Bauer, D. Pauly and M. Schomburg, The Maxwell compactness property in bounded weak Lipschitz domains with mixed boundary conditions, preprint (2015), http://arxiv.org/abs/1511.06697.

[3] F. Brezzi and M. Fortin, *Mixed and Hybrid Finite Element Methods*, Springer, New York, 1991.

[4] Z. Cai and S. Zhang, Flux recovery and a posteriori error estimators: Conforming elements for scalar elliptic equations, *SIAM J. Numer. Anal.* **48** (2010), no. 2, 578–602.

[5] F. Jochmann, A compactness result for vector fields with divergence and curl in $L^q(\Omega)$ involving mixed boundary conditions, *Appl. Anal.* **66** (1997), 189–203.

[6] O. Mali, A. Muzalevskiy and D. Pauly, Conforming and non-conforming functional a posteriori error estimates for elliptic boundary value problems in exterior domains: Theory and numerical tests, *Russian J. Numer. Anal. Math. Modelling* **28** (2013), no. 6, 577–596.

[7] O. Mali, P. Neittaanmäki and S. Repin, *Accuracy Verification Methods, Theory and Algorithms*, Springer, Dordrecht, 2014.

[8] P. Neittaanmäki and S. Repin, *Reliable Methods for Computer Simulation, Error Control and a Posteriori Estimates*, Elsevier, New York, 2004.

[9] D. Pauly and S. Repin, Functional a posteriori error estimates for elliptic problems in exterior domains, *J. Math. Sci. (N. Y.)* **162** (2009), no. 3, 393–406.

[10] D. Pauly and S. Repin, Two-sided a posteriori error bounds for electro-magneto static problems, *J. Math. Sci. (N. Y.)* **166** (2010), no. 1, 53–62.

[11] D. Pauly, S. Repin and T. Rossi, Estimates for deviations from exact solutions of Maxwell's initial boundary value problem, *Ann. Acad. Sci. Fenn. Math.* **36** (2011), no. 2, 661–676.

[12] S. Repin, *A Posteriori Estimates for Partial Differential Equations*, Radon Ser. Comput. Appl. Math. 4, Walter de Gruyter, Berlin, 2008.

[13] S. Repin, Estimates of deviations from exact solutions of initial boundary value problems for the wave equation, *J. Math. Sci. (N. Y.)* **159** (2009), no. 2, 229–240.

[14] S. Repin, S. Sauter and A. Smolianski, Two-sided a posteriori error estimates for mixed formulations of elliptic problems, *SIAM J. Numer. Anal.* **45** (2007), no. 3, 928–945.