On polynomial and exponential decay of eigen-solutions to exterior boundary value problems for the generalized time-harmonic Maxwell system

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Abstract. We prove polynomial and exponential decay at infinity of eigen-vectors of partial differential operators related to radiation problems for time-harmonic generalized Maxwell systems in an exterior domain $\Omega \subset \mathbb{R}^N$, $N \geqslant 1$, with non-smooth inhomogeneous, anisotropic coefficients converging near infinity with a rate $r^{-\tau}$, $\tau > 1$, towards the identity. As a canonical application we show that the corresponding eigen-values do not accumulate and that by means of Eidus' limiting absorption principle a Fredholm alternative holds true.

Keywords: Maxwell's equations, exterior boundary value problems, radiating solutions, polynomial and exponential decay of eigen-solutions, variable coefficients, electro-magnetic theory

1. Introduction

To establish a solution theory for time-harmonic boundary value problems in exterior domains it is now well known that Eidus' limiting absorption principle [3] is a major tool. For this, one crucial step is to show that there are no point eigen-values, or at least that possible point eigen-values do not accumulate and that the corresponding eigen-spaces are finite dimensional. The absence of non-vanishing eigen-vectors can be proved by a general pattern, which was suggested by Vogelsang [20,21] and Eidus [5] and consists of the following partial results:

Step 1: eigen-solutions decay polynomially,
Step 2: eigen-solutions decay exponentially,
Step 3: eigen-solutions have compact support,
Step 4: eigen-solutions vanish.

These results are well known, for instance, for Helmholtz' equation including perturbations. See [1,5, 7,8,18,19,21] and the literature cited there. In the case of time-harmonic Maxwell's equations, Steps 1 and 2 have been shown by Eidus [5] and Step 4, the unique continuation property, is an old result due to Leis [10–12]. The pattern was just recently completed in a sufficient manner by Bauer [2], who could prove the last remaining Step 3. All these results are known for C^2-coefficients with proper decay at

infinity except of Step 1, which could have been proved even for L^∞-coefficients by Picard, Weck and Witsch [17].

In the paper at hand we address Steps 1 and 2 for a generalized time-harmonic Maxwell problem formulated in the language of alternating differential forms. To show Step 1 we follow closely the arguments of Picard, Weck and Witsch [17] and Step 2 will be proved by the methods of Eidus [5]. We note that Steps 3 and 4 are still open problems in our general case. The only known result for Step 4 is the case of scalar-valued C^2-coefficients.

We consider an exterior domain, i.e., a connected open set with compact complement, $\Omega \subset \mathbb{R}^N$, $N \geqslant 1$, as a N-dimensional Riemannian manifold with compact boundary and the generalized time-harmonic Maxwell equations with real frequency $\omega \neq 0$

$$\delta H + \mathrm{i}\omega\varepsilon E = -\mathrm{i}\varepsilon F, \qquad \mathrm{d}E + \mathrm{i}\omega\mu H = -\mathrm{i}\mu G \quad \text{in } \Omega, \tag{1.1}$$

$$\iota^* E = 0 \qquad\qquad\qquad\qquad\qquad \text{in } \partial\Omega, \tag{1.2}$$

together with the corresponding radiation condition

$$(-1)^{qN} * \mathrm{d}r \wedge *H + E, \qquad \mathrm{d}r \wedge E + H \quad \text{decay at infinity.} \tag{1.3}$$

Here E, F and H, G are differential forms of rank q (q-forms) and $q + 1$ (($q + 1$)-forms), respectively, and d resp. $\delta = (-1)^{qN} * \mathrm{d} *$ is the exterior differential resp. co-differential, the latter acting on ($q + 1$)-forms. By $*$ we denote as usual Hodge's star operator and by $\wedge$ the exterior product. $\iota : \partial\Omega \hookrightarrow \overline{\Omega}$ is the natural embedding of the boundary and ι^* is the pull-back of ι, i.e., the tangential trace operator. We intend to model non-smooth, inhomogeneous and anisotropic media by linear transformations ε and μ on q- and ($q + 1$)-forms, respectively.

For sake of brevity of notation we introduce the ordered pairs of q-forms and ($q + 1$)-forms

$$u := (E, H), \qquad f := (F, G)$$

and the formal operator matrices

$$M := \begin{bmatrix} 0 & \delta \\ \mathrm{d} & 0 \end{bmatrix}, \qquad S := \begin{bmatrix} 0 & T \\ R & 0 \end{bmatrix}, \qquad \Lambda := \begin{bmatrix} \varepsilon & 0 \\ 0 & \mu \end{bmatrix}, \qquad M_\Lambda := \mathrm{i}\Lambda^{-1}M,$$

where $R := \mathrm{d}r\wedge$ and $T := (-1)^{qN} * R *$, and write our problem (1.1)–(1.3) more compactly as

$$(M_\Lambda - \omega)u = f \quad \text{in } \Omega,$$

$$\iota^* E = 0 \qquad\quad \text{in } \partial\Omega, \tag{1.4}$$

$$(S + 1)u \text{ decays at infinity.}$$

For the system (1.4) we will show polynomial and exponential decay of eigen-forms. For the polynomial decay we can admit L^∞-coefficients ε, μ, while we need C^2-coefficients to prove exponential decay. In both cases the coefficients must converge at infinity with a rate $r^{-\tau}$, $\tau > 1$, towards homogeneous and isotropic coefficients.

The main tool to handle irregular coefficients is a decomposition lemma, which allows us to prove the polynomial decay of eigen-forms by reduction to the similar result known for the scalar Helmholtz

equation. The keys to this decomposition lemma are weighted Hodge–Helmholtz decompositions, i.e., decompositions into irrotational and solenoidal forms, in the whole space case and a well known procedure to decouple the electric and magnetic form by discussing a second order elliptic system. To illustrate this calculation let us look at (1.4) in the homogeneous case. Applying $M_{\mathrm{Id}} + \omega$ yields

$$\left(M^2 + \omega^2\right)u = 0. \tag{1.5}$$

By (1.4) E is solenoidal, i.e., $\delta E = 0$, and G irrotational, i.e., $\mathrm{d}\,H = 0$, since $\delta\delta = 0$, $\mathrm{d}\,\mathrm{d} = 0$. From $\Delta = \mathrm{d}\,\delta + \delta\,\mathrm{d}$, where the Laplacian acts on each Euclidean component, we get the identity $M^2 u = (\delta\,\mathrm{d}E, \mathrm{d}\,\delta H) = \Delta u$ and finally (1.5) turns to the component-wise Helmholtz equation

$$\left(\Delta + \omega^2\right)u = 0. \tag{1.6}$$

The polynomial decay of eigen-forms together with an a priori estimate for the solutions corresponding to non-real frequencies is sufficient to prove a Fredholm alternative for (1.4) utilizing the limiting absorption principle introduced by Eidus [3]. Moreover, we get at most finite dimensional eigen-spaces for possible eigen-values but these cannot accumulate.

2. Definitions and preliminaries

For later purpose let us fix $r_0 > 0$, such that $\mathbb{R}^N \setminus \Omega \subset B_{r_0}$, where B_θ denotes the open ball of radius θ centered at the origin. We also define the exterior of the closed ball $\check{B}_\theta := \mathbb{R}^N \setminus \overline{B_\theta}$ and the sphere S_θ, both of radius θ.

Using the weight function

$$\rho := \left(1 + r^2\right)^{1/2}, \qquad r(x) := x,$$

we introduce for $m \in \mathbb{N}_0$ and $s \in \mathbb{R}$ the weighted scalar Sobolev spaces

$$\mathsf{H}^m_s(\Omega) := \left\{\psi \in \mathsf{L}^2_{\mathrm{loc}}(\Omega):\ \rho^s \partial^\alpha \psi \in \mathsf{L}^2(\Omega), \forall |\alpha| \leqslant m\right\}.$$

In Ω we have a canonical global chart, the identity, and thus, Ω becomes naturally a N-dimensional smooth Riemannian manifold with Cartesian coordinates $\{x_1, \ldots, x_N\}$. For alternating differential forms of rank $q \in \mathbb{Z}$ (q-forms) we define component-wise partial derivatives $\partial^\alpha \Phi = (\partial^\alpha \Phi_I)\,\mathrm{d}x^I$, if $\Phi = \Phi_I\,\mathrm{d}x^I$ (sum convention!), where I are ordered multi-indices of length q. Then, for $m \in \mathbb{N}_0$ and $s \in \mathbb{R}$ we define weighted Sobolev spaces $\mathsf{H}^{m,q}_s(\Omega)$ of q-forms as well. Equipped with their natural scalar products all these spaces become Hilbert spaces. For $m = 0$ we also utilize the notation $\mathsf{L}^{2,q}_s(\Omega) := \mathsf{H}^{0,q}_s(\Omega)$ and for pairs of forms we introduce product spaces like

$$\mathsf{L}^{2,q,q+1}_s(\Omega) := \mathsf{L}^{2,q}_s(\Omega) \times \mathsf{L}^{2,q+1}_s(\Omega).$$

In the special case $s = 0$ we neglect the index 0 and we have in $\mathsf{L}^{2,q}(\Omega) = \mathsf{H}^{0,q}(\Omega) = \mathsf{H}^{0,q}_0(\Omega)$ the scalar product

$$\langle\Phi, \Psi\rangle_{\mathsf{L}^{2,q}(\Omega)} = \int_\Omega \Phi \wedge *\overline{\Psi} = \int_\Omega *\langle\Phi, \Psi\rangle_q = \int_\Omega \langle\Phi, \Psi\rangle_q\,\mathrm{d}\lambda = \int_\Omega \Phi_I \overline{\Psi}_I\,\mathrm{d}\lambda$$

for $\Phi = \Phi_I \, \mathrm{d}x^I$, $\Psi = \Psi_I \, \mathrm{d}x^I \in \mathsf{L}^{2,q}(\Omega)$. Here λ denotes Lebesgue's measure and $\langle \cdot, \cdot \rangle_q$ the point-wise scalar product on q-forms.

By Stokes' theorem and the product rule the exterior derivative and co-derivative are formally skew-adjoint to each other, i.e.,

$$\langle \mathrm{d}\,\Phi, \Psi \rangle_{\mathsf{L}^{2,q+1}(\Omega)} = -\langle \Phi, \delta\Psi \rangle_{\mathsf{L}^{2,q}(\Omega)} \quad \forall (\Phi, \Psi) \in \overset{\circ}{\mathsf{C}}{}^{\infty,q,q+1}(\Omega),$$

which gives rise to weak definitions of d and δ. Here we denote the vector space of all smooth q-forms with compact support in Ω by $\overset{\circ}{\mathsf{C}}{}^{\infty,q}(\Omega)$. We note that still $\mathrm{d}\,\mathrm{d} = 0$, $\delta\delta = 0$ and $\mathrm{d}\,\delta + \delta\,\mathrm{d} = \Delta$ hold true in the weak sense. Furthermore, for $s \in \mathbb{R}$ we introduce some special weighted Sobolev spaces suited for Maxwell's equations

$$\mathsf{D}_s^q(\Omega) := \big\{ \Phi \in \mathsf{L}_s^{2,q}(\Omega) \colon \mathrm{d}\,\Phi \in \mathsf{L}_s^{2,q+1}(\Omega) \big\},$$
$$\Delta_s^q(\Omega) := \big\{ \Phi \in \mathsf{L}_s^{2,q}(\Omega) \colon \delta\Phi \in \mathsf{L}_s^{2,q-1}(\Omega) \big\}.$$

Equipped with their natural graph norms these are Hilbert spaces as well. To generalize the homogeneous boundary condition we introduce $\overset{\circ}{\mathsf{D}}{}_s^q(\Omega)$ as the closure of $\overset{\circ}{\mathsf{C}}{}^{\infty,q}(\Omega)$ in the norm of $\mathsf{D}_s^q(\Omega)$. Utilizing Stokes' theorem we see that in fact the homogeneous boundary condition $\iota^* E = 0$ is generalized in $\overset{\circ}{\mathsf{D}}{}_s^q(\Omega)$. The spaces $\mathsf{D}_s^q(\Omega)$, $\Delta_s^q(\Omega)$ and even $\overset{\circ}{\mathsf{D}}{}_s^q(\Omega)$ are invariant under multiplication with bounded smooth functions φ. A subscript 0 at the lower left corner indicates vanishing exterior derivative resp. co-derivative, for instance, $_0\overset{\circ}{\mathsf{D}}{}_s^q(\Omega) = \{ \Phi \in \overset{\circ}{\mathsf{D}}{}_s^q(\Omega) \colon \mathrm{d}\,\Phi = 0 \}$. The indices loc and vox refer as usual to local integrability and compact supports, respectively. If the whole space $\Omega = \mathbb{R}^N$ is under consideration, we omit the dependence on the domain and write simply, for example, $_0\mathsf{D}_s^q := {}_0\mathsf{D}_s^q(\mathbb{R}^N)$. Moreover, for weighted Sobolev spaces V_t, $t \in \mathbb{R}$, we define

$$\mathsf{V}_{<s} := \bigcap_{t<s} \mathsf{V}_t, \qquad \mathsf{V}_{>s} := \bigcup_{t>s} \mathsf{V}_t, \quad s \in \mathbb{R}.$$

Now let us introduce the properties of our transformations ε, μ, Λ:

Definition 2.1. Let $\tau \geqslant 0$. We call a transformation ε τ-admissible, if:

(i) $\varepsilon(x)$ is a linear transformation on q-forms for all $x \in \Omega$;
(ii) ε possesses $\mathsf{L}^\infty(\Omega)$-coefficients, i.e., the matrix representation of ε corresponding to the canonical basis (and then for every chart basis) has $\mathsf{L}^\infty(\Omega)$-entries;
(iii) ε is symmetric, i.e.,

$$\forall \Phi, \Psi \in \mathsf{L}^{2,q}(\Omega) \quad \langle \varepsilon\Phi, \Psi \rangle_{\mathsf{L}^{2,q}(\Omega)} = \langle \Phi, \varepsilon\Psi \rangle_{\mathsf{L}^{2,q}(\Omega)};$$

(iv) ε is uniformly positive definite, i.e.,

$$\exists c > 0 \; \forall \Phi \in \mathsf{L}^{2,q}(\Omega) \quad \langle \varepsilon\Phi, \Phi \rangle_{\mathsf{L}^{2,q}(\Omega)} \geqslant c \|\Phi\|_{\mathsf{L}^{2,q}(\Omega)}^2;$$

(v) ε is asymptotically the identity, i.e., $\varepsilon = \varepsilon_0 \,\mathrm{Id} + \hat{\varepsilon}$ with $\varepsilon_0 \in \mathbb{R}_+$ and $\hat{\varepsilon} = \mathcal{O}(r^{-\tau})$ as $r \to \infty$.

Moreover, for $n \in \mathbb{N}_0$ we call ε τ-C^n-admissible, if ε is τ-admissible and

(vi) $\hat{\varepsilon} \in \mathsf{C}^n(\check{B}_{r_0})$ with bounded derivatives, which means that the matrix representation of $\hat{\varepsilon}$ corresponding to the canonical basis (and then for every chart basis) has $\mathsf{C}^n(\check{B}_{r_0})$-entries and all derivatives are bounded.

We call τ the order of decay of the perturbation $\hat{\varepsilon}$.

We remark that by a transformation $\tilde{x} := \alpha x$, $\tilde{H} := \beta H$ we may assume without loss of generality $\varepsilon_0 = \mu_0 = 1$ throughout this paper.

Finally, we note that the multiplication operators R, T and S are related to the differential operators d, δ, M through the following formulas:

$$C_{D,\varphi(r)} = \varphi'(r)X, \qquad \mathcal{F}D = \mathrm{i}rX\mathcal{F}, \qquad D\mathcal{F} = -\mathrm{i}\mathcal{F}rX.$$

Here $(D, X) \in \{(\mathrm{d}, R), (\delta, T), (M, S)\}$ and $C_{A,B}$ denotes the commutator of two operators A, B, i.e., $C_{D,\varphi(r)} = D\varphi(r) - \varphi(r)D$, where φ is a smooth function on $\mathbb{R}$. Furthermore, $\mathcal{F}$ denotes the Fourier transformation on q-forms in $\mathbb{R}^N$ (component-wise in Euclidean coordinates), which is an unitary mapping on $\mathsf{L}^{2,q}$.

3. A decomposition lemma

The following decomposition lemma is essential and allows us to transfer results known from Helmholtz' equation to Maxwell's equations without any further regularity assumptions. To use results from Weck and Witsch [23] we set

$$\mathbb{I} := \{n + N/2, 1 - n - N/2 \colon n \in \mathbb{N}_0\}.$$

Throughout this section, let Λ be τ-admissible with order of decay

$$\tau \geqslant 0.$$

Furthermore, let K be a compact subset of $\mathbb{C} \setminus \{0\}$, $\omega \in K$ and $t, s \in \mathbb{R}$ with

$$0 \leqslant s \in \mathbb{R} \setminus \mathbb{I}, \qquad t \leqslant s \leqslant t + \tau$$

as well as $\theta \geqslant r_0$ and $\varphi := \eta(r/\theta)$, where $\eta \in \mathsf{C}^\infty(\mathbb{R})$ is a cut-off function with

$$\operatorname{supp} \varphi \subset [1, \infty), \quad \varphi|_{[2,\infty)} = 1.$$

For $u \in \mathsf{D}_t^q(\Omega) \times \Delta_t^{q+1}(\Omega)$ we set

$$f := (M_\Lambda - \omega)u, \qquad \hat{f} := (C_{M,\varphi} - \mathrm{i}\omega\varphi\hat{\Lambda})u - \mathrm{i}\varphi\Lambda f.$$

By decomposing $\hat{f}$ according to [23], Theorem 4, we define

$$\hat{f} =: f_{\mathrm{d},\delta} + f_{\delta,\mathrm{d}} + f_{\mathrm{S}} \in \left({}_0\mathsf{D}_s^q \times {}_0\Delta_s^{q+1}\right) \dotplus \left({}_0\Delta_s^q \times {}_0\mathsf{D}_s^{q+1}\right) \dotplus \mathcal{S}_s^{q,q+1}, \qquad \tilde{f} := f_{\delta,\mathrm{d}} + \frac{\mathrm{i}}{\omega}Mf_{\mathrm{S}}$$

and

$$u_{d,\delta} := -\frac{i}{\omega}(f_{d,\delta} + f_S), \qquad\qquad u_{\delta,d} := -\frac{i}{\omega}(f_{\delta,d} - M\varphi u),$$

$$u_{\mathcal{F}} := \mathcal{F}^{-1}\big((1 + r^2)^{-1}(1 - irS)\mathcal{F}\tilde{f}\big), \qquad u_\Delta := u_{\delta,d} - u_{\mathcal{F}}$$

provided that $\hat{f} \in \mathsf{L}_s^{2,q,q+1}$.

Lemma 3.1. *Let* $u \in \mathsf{D}_t^q(\Omega) \times \Delta_t^{q+1}(\Omega)$ *with* $f \in \mathsf{L}_s^{2,q,q+1}(\Omega)$*. Then* $\hat{f} \in \mathsf{L}_s^{2,q,q+1}$ *and* $\tilde{f} \in {}_0\Delta_s^q \times {}_0\mathsf{D}_s^{q+1}$*. Moreover, u is decomposed into*

$$u = (1 - \varphi)u + \varphi u, \quad \varphi u = u_{d,\delta} + u_{\delta,d}, \quad u_{\delta,d} = u_{\mathcal{F}} + u_\Delta,$$

i.e.,

$$u = (1 - \varphi)u + u_{d,\delta} + u_{\mathcal{F}} + u_\Delta,$$

and

$$(1 - \varphi)u \in \mathsf{D}_{\mathrm{vox}}^q(\Omega) \times \Delta_{\mathrm{vox}}^{q+1}(\Omega), \qquad u_{\delta,d} \in \mathsf{H}_t^{1,q,q+1} \cap \big({}_0\Delta_t^q \times {}_0\mathsf{D}_t^{q+1}\big),$$

$$\varphi u \in \mathsf{D}_t^q \times \Delta_t^{q+1}, \qquad\qquad u_{\mathcal{F}} \in \mathsf{H}_s^{1,q,q+1} \cap \big({}_0\Delta_s^q \times {}_0\mathsf{D}_s^{q+1}\big),$$

$$u_{d,\delta} \in \mathsf{D}_s^q \times \Delta_s^{q+1}, \qquad\qquad u_\Delta \in \mathsf{H}_t^{2,q,q+1} \cap \big({}_0\Delta_t^q \times {}_0\mathsf{D}_t^{q+1}\big)$$

hold. These forms solve

$$(M + i\omega)\varphi u = \hat{f}, \qquad (M + i\omega)u_{\delta,d} = \tilde{f} = (M + 1)u_{\mathcal{F}}, \qquad (M + i\omega)u_\Delta = (1 - i\omega)u_{\mathcal{F}}$$

and

$$(\Delta + \omega^2)u_\Delta = (1 - i\omega)\tilde{f} - (1 + \omega^2)u_{\mathcal{F}}.$$

Furthermore, there exists a constant $c > 0$ *independent of* u*,* f *or* $\omega, \lambda \in K$*, such that for all* $\tilde{t} \leqslant t$ *the estimates*

$$\|\tilde{f}\|_{\mathsf{L}_s^{2,q,q+1}} \leqslant c\|\hat{f}\|_{\mathsf{L}_s^{2,q,q+1}},$$

$$\|\hat{f}\|_{\mathsf{L}_s^{2,q,q+1}} \leqslant c\big(\|f\|_{\mathsf{L}_s^{2,q,q+1}(\Omega)} + \|u\|_{\mathsf{L}_{s-\tau}^{2,q,q+1}(\Omega)}\big),$$

$$\big\|(1 - \varphi)u\big\|_{\mathsf{D}_{\tilde{t}}^q(\Omega) \times \Delta_{\tilde{t}}^{q+1}(\Omega)} \leqslant c\big(\|f\|_{\mathsf{L}_s^{2,q,q+1}(\Omega)} + \|u\|_{\mathsf{L}_{s-\tau}^{2,q,q+1}(\Omega)}\big),$$

$$\|u_{d,\delta}\|_{\mathsf{D}_s^q \times \Delta_s^{q+1}} \leqslant c\|\hat{f}\|_{\mathsf{L}_s^{2,q,q+1}},$$

$$\|u_{\mathcal{F}}\|_{\mathsf{H}_s^{1,q,q+1}} \leqslant c\|\tilde{f}\|_{\mathsf{L}_s^{2,q,q+1}},$$

$$\|u_\Delta\|_{\mathsf{H}_{\tilde{t}}^{2,q,q+1}} \leqslant c\big(\|u_\Delta\|_{\mathsf{L}_{\tilde{t}}^{2,q,q+1}} + \|u_{\mathcal{F}}\|_{\mathsf{H}_{\tilde{t}}^{1,q,q+1}}\big)$$

and

$$\|u\|_{\mathsf{D}_{\tilde t}^q(\Omega)\times\Delta_{\tilde t}^{q+1}(\Omega)} \leqslant c(\|f\|_{\mathsf{L}_s^{2,q,q+1}(\Omega)} + \|u\|_{\mathsf{L}_{s-\tau}^{2,q,q+1}(\Omega)} + \|u_\Delta\|_{\mathsf{L}_{\tilde t}^{2,q,q+1}}),$$

$$\|(\Delta+\omega^2)u_\Delta\|_{\mathsf{L}_s^{2,q,q+1}} \leqslant c(\|f\|_{\mathsf{L}_s^{2,q,q+1}(\Omega)} + \|u\|_{\mathsf{L}_{s-\tau}^{2,q,q+1}(\Omega)}),$$

$$\|(M-\mathrm{i}\lambda S)u\|_{\mathsf{L}_{\tilde t}^{2,q,q+1}(\Omega)} \leqslant c(\|f\|_{\mathsf{L}_s^{2,q,q+1}(\Omega)} + \|u\|_{\mathsf{L}_{s-\tau}^{2,q,q+1}(\Omega)} + \|(M-\mathrm{i}\lambda S)u_\Delta\|_{\mathsf{L}_{\tilde t}^{2,q,q+1}})$$

hold.

Proof. Obviously, $(M_\Lambda - \omega)u = f \in \mathsf{L}_s^{2,q,q+1}(\Omega)$ is equivalent to

$$(M + \mathrm{i}\omega\Lambda)u = -\mathrm{i}\Lambda f \in \mathsf{L}_s^{2,q,q+1}(\Omega).$$

Since $\varphi u \in \mathsf{D}_t^q \times \Delta_t^{q+1}$ we have

$$M\varphi u = \varphi M u + C_{M,\varphi}u = -\mathrm{i}\omega\varphi\Lambda u - \mathrm{i}\varphi\Lambda f + C_{M,\varphi}u$$

and thus

$$(M + \mathrm{i}\omega)\varphi u = \hat f \in \mathsf{L}_s^{2,q,q+1}, \tag{3.1}$$

since $C_{M,\varphi} = \eta'(r/\theta)\theta^{-1}S$ is compactly supported and $t + \tau \geqslant s$. We rewrite (3.1) in the form

$$\mathrm{i}\omega\varphi u = -M\varphi u + f_{\delta,\mathrm{d}} + f_{\mathrm{d},\delta} + f_S$$

and note

$$u_{\delta,\mathrm{d}} = -\frac{\mathrm{i}}{\omega}(f_{\delta,\mathrm{d}} - M\varphi u) \in \big(\mathsf{D}_t^q \cap {}_0\Delta_t^q\big) \times \big(\Delta_t^{q+1} \cap {}_0\mathsf{D}_t^{q+1}\big) \subset \mathsf{H}_t^{1,q,q+1},$$

$$u_{\mathrm{d},\delta} = -\frac{\mathrm{i}}{\omega}(f_{\mathrm{d},\delta} + f_S) \in \mathsf{D}_s^q \times \Delta_s^{q+1}$$

with $\varphi u = u_{\delta,\mathrm{d}} + u_{\mathrm{d},\delta}$ and by regularity, e.g., [9], Lemma 4.2(i). (For $s < N/2$ we even have $f_S = 0$.) Moreover, $u_{\delta,\mathrm{d}}$ solves

$$M u_{\delta,\mathrm{d}} = M\varphi u - M u_{\mathrm{d},\delta} = -\mathrm{i}\omega u_{\delta,\mathrm{d}} + f_{\delta,\mathrm{d}} + \frac{\mathrm{i}}{\omega}M f_S,$$

i.e., $(M + \mathrm{i}\omega)u_{\delta,\mathrm{d}} = \tilde f \in {}_0\Delta_s^q \times {}_0\mathsf{D}_s^{q+1}$. Now, to define $(M+1)^{-1}\tilde f$ by the Fourier transformation we put

$$u_{\mathcal{F}} = \mathcal{F}^{-1}\big((1+r^2)^{-1}(1-\mathrm{i}rS)\mathcal{F}\tilde f\big).$$

Then, $u_{\mathcal{F}} \in \mathsf{L}^{2,q,q+1}$ as well as $\mathcal{F}u_{\mathcal{F}} \in \mathsf{L}_1^{2,q,q+1}$ are implied by $\mathcal{F}\tilde f \in \mathsf{L}^{2,q,q+1}$. Hence,

$$u_{\mathcal{F}} \in \mathsf{H}^{1,q,q+1}.$$

From $\tilde{f} \in \mathsf{L}_s^{2,q,q+1}$ we get by definition $\mathcal{F}\tilde{f} \in \mathsf{H}^{s,q,q+1}$. The components of $\mathcal{F}u_{\mathcal{F}}$ arise from those of $\mathcal{F}\tilde{f}$ by multiplication with bounded C^∞-functions. Thus, also

$$\mathcal{F}u_{\mathcal{F}} \in \mathsf{H}^{s,q,q+1}$$

follows; see, e.g., Wloka [25], p. 71, Lemma 3.2. Again, by definition, $u_{\mathcal{F}} \in \mathsf{L}_s^{2,q,q+1}$ and we obtain the estimate

$$\|u_{\mathcal{F}}\|_{\mathsf{L}_s^{2,q,q+1}} \leqslant c\|\tilde{f}\|_{\mathsf{L}_s^{2,q,q+1}}.$$

Since $M\mathcal{F}^{-1} = \mathrm{i}\mathcal{F}^{-1}rS$ we compute

$$(1 + r^2)\mathcal{F}(M+1)u_{\mathcal{F}} = (1 + \mathrm{i}rS)(1 - \mathrm{i}rS)\mathcal{F}\tilde{f} = (1 + r^2S^2)\mathcal{F}\tilde{f}.$$

$\delta\tilde{F} = 0$ and $\mathrm{d}\,\tilde{G} = 0$ imply $T\mathcal{F}\tilde{F} = 0$ and $R\mathcal{F}\tilde{G} = 0$, respectively, and therefore, using $RT + TR = 1$

$$S^2\mathcal{F}\tilde{f} = \begin{bmatrix} TR & 0 \\ 0 & RT \end{bmatrix} \mathcal{F}\tilde{f} = \mathcal{F}\tilde{f}$$

holds. Hence,

$$\mathcal{F}(M+1)u_{\mathcal{F}} = \mathcal{F}\tilde{f}, \quad \text{i.e. } (M+1)u_{\mathcal{F}} = \tilde{f}.$$

Besides, we have $u_{\mathcal{F}} \in (\mathsf{D}_s^q \cap {_0\Delta_s^q}) \times (\Delta_s^{q+1} \cap {_0\mathsf{D}_s^{q+1}})$ and thus,

$$u_{\mathcal{F}} \in \mathsf{H}_s^{1,q,q+1} \cap \left({_0\Delta_s^q} \times {_0\mathsf{D}_s^{q+1}}\right)$$

again by regularity. Considering

$$u_\Delta = u_{\delta,\mathrm{d}} - u_{\mathcal{F}} \in \mathsf{H}_t^{1,q,q+1} \cap \left({_0\Delta_t^q} \times {_0\mathsf{D}_t^{q+1}}\right)$$

we calculate

$$(M + \mathrm{i}\omega)u_\Delta = (1 - \mathrm{i}\omega)u_{\mathcal{F}}.$$

Once more by regularity we even obtain $u_\Delta \in \mathsf{H}_t^{2,q,q+1} \cap ({_0\Delta_t^q} \times {_0\mathsf{D}_t^{q+1}})$ and we compute

$$\begin{aligned}
(\Delta + \omega^2)u_\Delta &= (M - \mathrm{i}\omega)(M + \mathrm{i}\omega)u_\Delta = (1 - \mathrm{i}\omega)(M - \mathrm{i}\omega)u_{\mathcal{F}} \\
&= -(1 + \omega^2)u_{\mathcal{F}} + (1 - \mathrm{i}\omega)\tilde{f}.
\end{aligned}$$

Finally, we achieve the asserted estimates from the regularity result and the continuity of the projections in $\mathsf{L}_s^{2,q}$ onto ${_0\mathsf{D}_s^q}$, ${_0\Delta_s^q}$ resp. $\mathcal{S}_s^q$ mentioning that

$$\|Mf_{\mathcal{S}}\| \leqslant c\|f_{\mathcal{S}}\|$$

holds in any norm since $\mathcal{S}_s^{q,q+1}$ is finite dimensional and M linear. $\quad\square$

4. Polynomial decay

First, we need a trivial but useful technical lemma.

Lemma 4.1. *For all $t, \tilde{t} \in \mathbb{R}$ with $\tilde{t} < t$ and all $\vartheta > 0$ there exist constants $c, \theta > 0$, such that*

$$\|\psi\|_{\mathsf{L}^2_{\tilde{t}}(\Omega)} \leqslant c\|\psi\|_{\mathsf{L}^2(\Omega \cap B_\theta)} + \vartheta\|\psi\|_{\mathsf{L}^2_t(\Omega)}$$

holds for all $\psi \in \mathsf{L}^2_t(\Omega)$.

Proof. For sufficient large $\theta > 0$ we get from $\tilde{t} - t < 0$

$$\|\psi\|^2_{\mathsf{L}^2_{\tilde{t}}(\Omega)} = \|\rho^{\tilde{t}}\psi\|^2_{\mathsf{L}^2(\Omega \cap B_\theta)} + \|\rho^{\tilde{t}-t}\psi\|^2_{\mathsf{L}^2_t(\check{B}_\theta)} \leqslant c_{\Omega,\tilde{t},\theta}\|\psi\|^2_{\mathsf{L}^2(\Omega \cap B_\theta)} + \left(1 + \theta^2\right)^{\tilde{t}-t}\|\psi\|^2_{\mathsf{L}^2_t(\check{B}_\theta)}.$$

Thus, $\lim_{\theta \to \infty}(1 + \theta^2)^{\tilde{t}-t} = 0$ completes the proof. $\square$

Our decomposition lemma implies the following theorem.

Theorem 4.2. *Let Λ be τ-admissible with $\tau > 1$. Moreover, let $I \subset \mathbb{R}_+$ be a closed interval and $\omega \in I$ as well as $1/2 < s \in \mathbb{R} \setminus \mathbb{I}$. If*

$$u \in \mathsf{D}^q_{>-1/2}(\Omega) \times \Delta^{q+1}_{>-1/2}(\Omega)$$

is a solution of Maxwell's equation

$$(M_\Lambda - \omega)u =: f \in \mathsf{L}^{2,q,q+1}_s(\Omega),$$

then $u \in \mathsf{D}^q_{s-1}(\Omega) \times \Delta^{q+1}_{s-1}(\Omega)$ and there exist constants $c, \theta > 0$ independent of u, f or ω, such that

$$\|u\|_{\mathsf{D}^q_{s-1}(\Omega) \times \Delta^{q+1}_{s-1}(\Omega)} \leqslant c\big(\|f\|_{\mathsf{L}^{2,q,q+1}_s(\Omega)} + \|u\|_{\mathsf{L}^{2,q,q+1}(\Omega \cap B_\theta)}\big).$$

Proof. Let $t > -1/2$ and $u \in \mathsf{D}^q_t(\Omega) \times \Delta^{q+1}_t(\Omega)$ with $t < s - 1$. Without loss of generality we may assume $t + 1 < s < t + \tau$. Otherwise, we replace t and s by $t_k := t + k\alpha$ and $s_k := t + 1 + (k+1)\alpha \leqslant s$, $k = 0, \ldots$, with $\alpha := (\tau - 1)/2 > 0$ and obtain the assertions after finitely many α-steps.

Decomposing u by Lemma 3.1 we get solutions $u_\Delta \in \mathsf{H}^{2,q,q+1}_t$ of Helmholtz' equation in $\mathbb{R}^N$

$$\left(\Delta + \omega^2\right)u_\Delta =: f_\Delta \in {}_0\Delta^q_s \times {}_0\mathsf{D}^{q+1}_s.$$

A component-wise application of [24], Lemma 5, yields $u_\Delta \in \mathsf{H}^{2,q,q+1}_{s-1}$ and with a constant $c > 0$ independent of u_Δ, f_Δ or ω we have

$$\|u_\Delta\|_{\mathsf{H}^{2,q,q+1}_{s-1}} \leqslant c\big(\|f_\Delta\|_{\mathsf{L}^{2,q,q+1}_s} + \|u_\Delta\|_{\mathsf{L}^{2,q,q+1}_{s-2}}\big).$$

Furthermore, from Lemma 3.1 we have $u \in \Delta_{s-1}^{q}(\Omega) \times \Delta_{s-1}^{q+1}(\Omega)$ and the estimate

$$
\begin{aligned}
\|u\|_{\mathsf{D}_{s-1}^{q}(\Omega) \times \Delta_{s-1}^{q+1}(\Omega)} &\leqslant c\big(\|u_\Delta\|_{\mathsf{L}_{s-1}^{2,q,q+1}} + \|f\|_{\mathsf{L}_s^{2,q,q+1}(\Omega)} + \|u\|_{\mathsf{L}_{s-\tau}^{2,q,q+1}(\Omega)}\big) \\
&\leqslant c\big(\|f_\Delta\|_{\mathsf{L}_s^{2,q,q+1}} + \|f\|_{\mathsf{L}_s^{2,q,q+1}(\Omega)} + \|u\|_{\mathsf{L}_{s-\tau}^{2,q,q+1}(\Omega)}\big) \\
&\leqslant c\big(\|f\|_{\mathsf{L}_s^{2,q,q+1}(\Omega)} + \|u\|_{\mathsf{L}_{s-\tau}^{2,q,q+1}(\Omega)}\big),
\end{aligned}
$$

where we assumed without loss of generality $\tau < 2$. Since $s - \tau < s - 1$ the assertion follows now by Lemma 4.1. $\quad\square$

Remark 4.3. Let the assumptions of Theorem 4.2 be satisfied. If $f \in \mathsf{L}_s^{2,q,q+1}(\Omega)$ for all $s \in \mathbb{R}$, we get

$$
u \in \mathsf{D}_s^{q}(\Omega) \times \Delta_s^{q+1}(\Omega)
$$

for all $s \in \mathbb{R}$. This holds, for instance, if f is exponentially decaying or even compactly supported.

5. Exponential decay

Using the 'partial integration' technique introduced by Eidus [5] for the classical Maxwell equations we will prove the following theorem.

Theorem 5.1. *Let $\omega \in \mathbb{R} \setminus \{0\}$ and Λ be τ-C^2-admissible with $\tau > 1$. If*

$$
u \in \mathsf{D}_{>-1/2}^{q}(\Omega) \times \Delta_{>-1/2}^{q+1}(\Omega)
$$

is a solution of Maxwell's equation

$$
(M_\Lambda - \omega)u =: f \in \mathrm{e}^{tr}\mathsf{H}^{2,q,q+1}(\check{B}_{r_0})
$$

for all $t \in \mathbb{R}$, then

$$
\mathrm{e}^{tr}u \in \big(\mathsf{D}^{q}(\Omega) \times \Delta^{q+1}(\Omega)\big) \cap \mathsf{H}^{2,q,q+1}(\check{B}_{r_0+1})
$$

holds for all $t \in \mathbb{R}$. The assertion holds in particular if f is compactly supported.

Proof. The idea of the proof is to estimate the exponential series. For this, we need some technical preliminaries. For all $s \in \mathbb{R}$ we have

$$
(M + \mathrm{i}\omega\Lambda)u = -\mathrm{i}\Lambda f \in \mathsf{L}_s^{2,q,q+1}(\Omega) \cap \mathsf{H}_s^{2,q,q+1}(\check{B}_{r_0}).
$$

Hence, by Theorem 4.2 and Remark 4.3 u belongs to $\mathsf{D}_s^{q}(\Omega) \times \Delta_s^{q+1}(\Omega)$ for all $s \in \mathbb{R}$. Therefore, inner regularity, e.g., a combination of a standard cut-off technique together with [9], Lemma 4.2(i), yields $u \in \mathsf{H}_s^{2,q,q+1}(\check{B}_{r_0+\theta})$ for all $s \in \mathbb{R}$ and all positive θ. Thus,

$$
(M + \mathrm{i}\omega)u = -\mathrm{i}\Lambda f - \mathrm{i}\omega\hat{\Lambda}u =: \tilde{f} \in \mathsf{L}_s^{2,q,q+1}(\Omega) \cap \mathsf{H}_s^{2,q,q+1}(\check{B}_{r_0+\theta})
$$

for all $\theta > 0$ and all $s \in \mathbb{R}$. Consequently, applying δ, d and $(M - \mathrm{i}\omega)$ to the latter equation we can compute in $\check{B}_{r_0}$

$$\mathrm{i}\omega\delta E = \delta\tilde{F}, \qquad \mathrm{i}\omega\,\mathrm{d}H = \mathrm{d}\tilde{G} \tag{5.1}$$

and

$$(M^2 + \omega^2)u = (M - \mathrm{i}\omega)\tilde{f}.$$

By defining $\square := \Delta - M^2$ we note

$$M^2 = \begin{bmatrix} \delta\,\mathrm{d} & 0 \\ 0 & \mathrm{d}\,\delta \end{bmatrix}, \qquad \square = \begin{bmatrix} \mathrm{d}\,\delta & 0 \\ 0 & \delta\,\mathrm{d} \end{bmatrix}, \qquad M^2 + \square = \Delta.$$

Therefore, we achieve

$$(\Delta + \omega^2)u = \left(M - \mathrm{i}\omega - \frac{\mathrm{i}}{\omega}\square \right)\tilde{f} =: \hat{f},$$

which is the equation we intend to work with. Now, we multiply this equation and all forms by r^m with some $m \in \mathbb{R}$ and indicate the resulting forms by an index m. We note that all occurring forms are well defined elements of $\mathsf{H}^{2,q+1}(\check{B}_{r_0+\theta})$ for all $\theta > 0$ and all $m \in \mathbb{R}$. Using Lemma B.1 we obtain in $\check{B}_{r_0}$

$$\left(\Delta + \omega^2 + \frac{m(m+2-N)}{r^2} \right)u_m - 2\frac{m}{r}\partial_r u_m$$

$$= \hat{f}_m = \left(M - \mathrm{i}\omega - \frac{\mathrm{i}}{\omega}\square \right)\tilde{f}_m + \left(-C_{M,r^m} + \frac{\mathrm{i}}{\omega}C_{\square,r^m} \right)\tilde{f}$$

$$= -\mathrm{i}\left(\mathrm{i}M + \omega + \frac{1}{\omega}\square - \frac{1}{\omega}\frac{m}{r}\left(\mathrm{i}\omega S + \mathrm{sw}\,MS + \mathrm{sw}\,SM - \frac{m+1}{r}\mathrm{sw}\,S^2 \right) \right)\tilde{f}_m. \tag{5.2}$$

With η from Lemma 3.1 we define the cut-off function $\varphi_\theta := \eta(r - \theta + 1)$ for all $\theta > r_0 + 1$. Then,

$$\operatorname{supp}\varphi_\theta \subset \check{B}_\theta, \qquad \operatorname{supp}\nabla\varphi_\theta, \operatorname{supp}(1 - \varphi_\theta) \cap \check{B}_\theta \subset \check{B}_\theta \cap B_{\theta+1}.$$

Without loss of generality let any form be real-valued. We multiply (5.2) by $\varphi_\theta r^p u_m$ with $p \in \mathbb{R}$ resp. $\varphi_\theta r \partial_r u_m$ and integrate over $\mathbb{R}^N$. We achieve

$$\left\langle \hat{f}_m, \varphi_\theta r^p u_m \right\rangle_{\mathsf{L}^{2,q,q+1}}$$

$$= \left\langle \left(\Delta + \omega^2 + \frac{m(m+2-N)}{r^2} \right)u_m - 2\frac{m}{r}\partial_r u_m, \varphi_\theta r^p u_m \right\rangle_{\mathsf{L}^{2,q,q+1}}$$

$$= \int_{\mathbb{R}^N} \Delta u_m \varphi_\theta r^p u_m + \omega^2 \varphi_\theta r^p |u_m|^2_{q,q+1}$$

$$\qquad + m(m+2-N)\varphi_\theta r^{p-2}|u_m|^2_{q,q+1} - m\varphi_\theta r^{p-1}\partial_r |u_m|^2_{q,q+1}\,\mathrm{d}\lambda \tag{5.3}$$

resp.

$$\langle \hat{f}_m, \varphi_\theta r \partial_r u_m \rangle_{\mathsf{L}^{2,q,q+1}}$$

$$= \left\langle \left(\Delta + \omega^2 + \frac{m(m+2-N)}{r^2} \right) u_m - 2\frac{m}{r}\partial_r u_m, \varphi_\theta r \partial_r u_m \right\rangle_{\mathsf{L}^{2,q,q+1}}$$

$$= \int_{\mathbb{R}^N} \Delta u_m \varphi_\theta r \partial_r u_m + \frac{\omega^2}{2}\varphi_\theta r \partial_r |u_m|^2_{q,q+1}$$

$$+ \frac{m}{2}\frac{m+2-N}{r}\varphi_\theta \partial_r |u_m|^2_{q,q+1} - 2m\varphi_\theta |\partial_r u_m|^2_{q,q+1} \, \mathrm{d}\lambda. \tag{5.4}$$

By partial integration we get from (5.3) resp. (5.4)

$$\left| \int_{\mathbb{R}^N} \varphi_\theta r^p \left(\left(\omega^2 + \frac{\gamma_{m,p,N}}{r^2} \right) |u_m|^2_{q,q+1} - \sum_{|\alpha|=1} |\partial^\alpha u_m|^2_{q,q+1} \right) \mathrm{d}\lambda \right|$$

$$\leqslant \left| \langle \hat{f}_m, \varphi_\theta r^p u_m \rangle_{\mathsf{L}^{2,q,q+1}} \right| + cm(\theta+1)^{2m+p} \tag{5.5}$$

resp.

$$\left| \int_{\mathbb{R}^N} \varphi_\theta \left(\left(N\omega^2 + \frac{\tilde{\gamma}_{m,N}}{r^2} \right) |u_m|^2_{q,q+1} - (N-2)\sum_{|\alpha|=1} |\partial^\alpha u_m|^2_{q,q+1} + 4m|\partial_r u_m|^2_{q,q+1} \right) \mathrm{d}\lambda \right|$$

$$\leqslant \left| \langle \hat{f}_m, \varphi_\theta r \partial_r u_m \rangle_{\mathsf{L}^{2,q,q+1}} \right| + cm^2(\theta+1)^{2m+1}, \tag{5.6}$$

where

$$\gamma_{m,p,N} := m(m+p) + p(p+N-2)/2, \qquad \tilde{\gamma}_{m,N} := m(m+2-N)(N-2)$$

and c is a generic constant independent of m and θ. Now, we multiply (5.5) for $p = 0$ by $N - 2$ and add (5.5) and (5.6) in a suitable way. We obtain

$$\int_{\mathbb{R}^N} \varphi_\theta \left(\left(2\omega^2 - \frac{m(N-2)^2}{r^2} \right) |u_m|^2_{q,q+1} + 4m|\partial_r u_m|^2_{q,q+1} \right) \mathrm{d}\lambda$$

$$\leqslant c \left(\left| \langle \hat{f}_m, \varphi_\theta u_m \rangle_{\mathsf{L}^{2,q,q+1}} \right| + \left| \langle \hat{f}_m, \varphi_\theta r \partial_r u_m \rangle_{\mathsf{L}^{2,q,q+1}} \right| + m^2(\theta+1)^{2m+1} \right),$$

which yields

$$\int_{\mathbb{R}^N} \varphi_\theta |u_m|^2_{q,q+1} \, \mathrm{d}\lambda \leqslant (N-2)^2 \int_{\mathbb{R}^N} \varphi_\theta \frac{1}{r}\frac{m}{r} |u_m|^2_{q,q+1} \, \mathrm{d}\lambda + cm^2(\theta+1)^{2m+1}$$

$$+ c \left(\left| \langle \hat{f}_m, \varphi_\theta u_m \rangle_{\mathsf{L}^{2,q,q+1}} \right| + \left| \langle \hat{f}_m, \varphi_\theta r \partial_r u_m \rangle_{\mathsf{L}^{2,q,q+1}} \right| \right). \tag{5.7}$$

To take care of the right-hand side we prove two lemmas.

Lemma 5.2. *For all $p \in \mathbb{R}$ there exists a constant $c > 0$, such that for all $\psi \in \mathsf{H}^2_{p/2}(\check{B}_{r_0+1})$ and all $\theta > r_0 + 1$*

$$\sum_{|\alpha|=2} \int_{\mathbb{R}^N} \varphi_\theta r^p |\partial^\alpha \psi|^2_{q,q+1} \, \mathrm{d}\lambda \leqslant c \int_{\check{B}(\theta)} r^p \Big(\sum_{|\alpha|=1} |\partial^\alpha \psi|^2_{q,q+1} + |\Delta \psi|^2_{q,q+1} \Big) \, \mathrm{d}\lambda.$$

Proof. By a cut-off technique we may assume $\psi \in \overset{\circ}{\mathsf{H}}{}^2_{p/2}(\check{B}_{r_0+1})$. Hence, it suffices to prove the assertion only for $\psi \in \overset{\circ}{\mathsf{C}}{}^{\infty,q}(\check{B}_{r_0+1})$ by continuity. But for those ψ the result is simply shown by several partial integrations. $\square$

Lemma 5.3. *Let $p, \tilde{p}, \hat{p} \in \mathbb{R}$ with $\tilde{p} > p$ and $\hat{p} > 2$. There are a constant $c > 0$ and a non-negative function $\varkappa$ tending to zero at infinity, such that for all $m \in \mathbb{R}$ and $\sigma > 0$, $\theta > r_0 + 1$ the following estimates hold*:

(i) $\quad |\langle \hat{f}_m, \varphi_\theta r^p u_m \rangle_{\mathsf{L}^{2,q,q+1}}| \leqslant \varkappa(\theta) \Big(m(\theta+1)^{2m+p} + \int_{\mathbb{R}^N} \varphi_\theta r^{\tilde{p}} |f_m|^2_{q,q+1} \, \mathrm{d}\lambda$

$$+ \int_{\mathbb{R}^N} \varphi_\theta r^p \Big(\frac{m^4}{r^4} |u_m|^2_{q,q+1} + \sum_{|\alpha|\leqslant 2} |\partial^\alpha u_m|^2_{q,q+1} \Big) \mathrm{d}\lambda \Big)$$

(ii) $\quad |\langle \hat{f}_m, \varphi_\theta r^p u_m \rangle_{\mathsf{L}^{2,q,q+1}}| \leqslant \varkappa(\theta) \Big(m(\theta+1)^{2m+p} + \sum_{|\alpha|\leqslant 1} \int_{\mathbb{R}^N} \varphi_\theta r^{\tilde{p}} |\partial^\alpha f_m|^2_{q,q+1} \, \mathrm{d}\lambda$

$$+ \int_{\mathbb{R}^N} \varphi_\theta r^p \Big(\frac{m^4}{r^4} |u_m|^2_{q,q+1} + \sum_{|\alpha|\leqslant 1} |\partial^\alpha u_m|^2_{q,q+1} \Big) \mathrm{d}\lambda \Big)$$

$$+ c \int_{\mathbb{R}^N} \varphi_\theta r^p \Big(\frac{1}{\sigma} |u_m|^2_{q,q+1} + \sigma \sum_{|\alpha|=1} |\partial^\alpha u_m|^2_{q,q+1} \Big) \mathrm{d}\lambda$$

(iii) $\quad |\langle \hat{f}_m, \varphi_\theta r \partial_r u_m \rangle_{\mathsf{L}^{2,q,q+1}}|$

$$\leqslant \varkappa(\theta) \Big(m^2(\theta+1)^{2m+1} + \sum_{|\alpha|\leqslant 2} \int_{\mathbb{R}^N} \varphi_\theta r^{\hat{p}} |\partial^\alpha f_m|^2_{q,q+1} \, \mathrm{d}\lambda$$

$$+ \int_{\mathbb{R}^N} \varphi_\theta \Big(\sum_{|\alpha|\leqslant 1} \frac{m^4}{r^4} |\partial^\alpha u_m|^2_{q,q+1} + \sum_{|\alpha|\leqslant 2} |\partial^\alpha u_m|^2_{q,q+1} \Big) \mathrm{d}\lambda \Big).$$

Proof. By several partial integrations we remove all derivatives from $\hat{\Lambda} u_m$ resp. f_m and use the decay of $\hat{\Lambda}$ resp. the integrability of f_m. This yields (i). In (ii) there are no longer second derivatives allowed on u_m. Hence, we have to insert a σ into the estimate since we do not demand any decay properties of the derivatives of $\hat{\Lambda}$. To prove (iii) we have to handle one challenging term, i.e.,

$$\langle \Box \hat{\Lambda} u_m, \varphi_\rho r \partial_r u_m \rangle_{\mathsf{L}^{2,q,q+1}} = -\langle \delta \hat{\varepsilon} E_m, \delta(\varphi_\rho r \partial_r E_m) \rangle_{\mathsf{L}^{2,q-1}} - \langle \mathrm{d}\,\hat{\mu} H_m, \mathrm{d}(\varphi_\rho r \partial_r H_m) \rangle_{\mathsf{L}^{2,q+2}}.$$

The part of the first summand causing the biggest difficulties is

$$\langle \delta\hat{\varepsilon}E_m, \varphi_\rho r \partial_r \delta E_m \rangle_{\mathsf{L}^{2,q-1}}$$

because we cannot integrate by parts anymore since E is only twice weakly differentiable. But with (5.1) and Lemma B.1(ii) we can substitute

$$\delta E_m = \frac{m}{r}T E_m - \frac{1}{\omega}\left(\delta - \frac{m}{r}T\right)\varepsilon F_m - \left(\delta - \frac{m}{r}T\right)\hat{\varepsilon}E_m.$$

Thus, the most challenging term reads now

$$\langle \delta\hat{\varepsilon}E_m, \varphi_\rho r \partial_r \delta\hat{\varepsilon}E_m \rangle_{\mathsf{L}^{2,q-1}} = \frac{1}{2}\int_{\mathbb{R}^N} \varphi_\rho r \partial_r |\delta\hat{\varepsilon}E_m|^2_{q-1}\, \mathrm{d}\lambda$$

and can be handled easily by two partial integrations. □

We proceed with the proof of the theorem by further estimating (5.7) using Lemma 5.3(i) and (iii).

$$\int_{\mathbb{R}^N} \varphi_\theta |u_m|^2_{q,q+1}\, \mathrm{d}\lambda \leqslant \varkappa(\theta)\bigg(\sum_{|\alpha|\leqslant 1}\int_{\mathbb{R}^N}\varphi_\theta\left(1 + \frac{m^4}{r^4}\right)|\partial^\alpha u_m|^2_{q,q+1}\, \mathrm{d}\lambda$$

$$+ \sum_{|\alpha|=2}\int_{\mathbb{R}^N}\varphi_\theta|\partial^\alpha u_m|^2_{q,q+1}\, \mathrm{d}\lambda$$

$$+ \sum_{|\alpha|\leqslant 2}\int_{\mathbb{R}^N}\varphi_\theta r^{\hat{p}}|\partial^\alpha f_m|^2_{q,q+1}\, \mathrm{d}\lambda\bigg) + cm^2(\theta + 1)^{2m+1}. \tag{5.8}$$

Utilizing Lemma 5.2 we estimate the second derivatives of u_m by the first ones and Δu_m. Then we substitute Δu_m with (5.2) and get for sufficient large θ

$$\sum_{|\alpha|=2}\int_{\mathbb{R}^N}\varphi_\theta|\partial^\alpha u_m|^2_{q,q+1}\, \mathrm{d}\lambda \leqslant c\bigg(\sum_{|\alpha|\leqslant 1}\int_{\mathbb{R}^N}\varphi_\theta\left(1 + \frac{m^4}{r^4}\right)|\partial^\alpha u_m|^2_{q,q+1}\, \mathrm{d}\lambda$$

$$+ \sum_{|\alpha|\leqslant 2}\int_{\mathbb{R}^N}\varphi_\theta\left(1 + \frac{m^4}{r^4}\right)|\partial^\alpha f_m|^2_{q,q+1}\, \mathrm{d}\lambda\bigg) + cm^4(\theta + 1)^{2m}.$$

Now we insert this estimate into (5.8) and obtain

$$\int_{\mathbb{R}^N} \varphi_\theta |u_m|^2_{q,q+1}\, \mathrm{d}\lambda \leqslant \varkappa(\theta)\bigg(\sum_{|\alpha|\leqslant 1}\int_{\mathbb{R}^N}\varphi_\theta\left(1 + \frac{m^4}{r^4}\right)|\partial^\alpha u_m|^2_{q,q+1}\, \mathrm{d}\lambda$$

$$+ \sum_{|\alpha|\leqslant 2}\int_{\mathbb{R}^N}\varphi_\theta r^{\hat{p}}\left(1 + \frac{m^4}{r^4}\right)|\partial^\alpha f_m|^2_{q,q+1}\, \mathrm{d}\lambda\bigg) + cm^4(\theta + 1)^{2m+1}.$$

$$\tag{5.9}$$

For all $p \in \mathbb{R}$ (5.5) yields the estimate

$$\sum_{|\alpha|=1} \int_{\mathbb{R}^N} \varphi_\theta r^p |\partial^\alpha u_m|^2_{q,q+1} \, \mathrm{d}\lambda \leqslant c \int_{\mathbb{R}^N} \varphi_\theta r^p \left(1 + \frac{m^2}{r^2}\right) |u_m|^2_{q,q+1} \, \mathrm{d}\lambda + |\langle \hat{f}_m, \varphi_\theta r^p u_m \rangle_{\mathsf{L}^{2,q,q+1}}|$$
$$+ cm(\theta + 1)^{2m+p},$$

such that by Lemma 5.3 (ii) for sufficient small σ and large θ

$$\sum_{|\alpha|=1} \int_{\mathbb{R}^N} \varphi_\theta r^p |\partial^\alpha u_m|^2_{q,q+1} \, \mathrm{d}\lambda \leqslant c \int_{\mathbb{R}^N} \varphi_\theta r^p \left(1 + \frac{m^4}{r^4}\right) |u_m|^2_{q,q+1} \, \mathrm{d}\lambda$$
$$+ c \sum_{|\alpha|\leqslant 1} \int_{\mathbb{R}^N} \varphi_\theta r^{\tilde{p}} |\partial^\alpha f_m|^2_{q,q+1} \, \mathrm{d}\lambda + cm(\theta + 1)^{2m+p}$$

follows. Now, we plug the latter estimate for $p = 0$ and $p = -4$ into (5.9) obtaining

$$\int_{\mathbb{R}^N} \varphi_\theta |u_m|^2_{q,q+1} \, \mathrm{d}\lambda \leqslant \varkappa(\theta) \left(\int_{\mathbb{R}^N} \varphi_\theta \left(1 + \frac{m^8}{r^8}\right) |u_m|^2_{q,q+1} \, \mathrm{d}\lambda + \sum_{|\alpha|\leqslant 2} \int_{\mathbb{R}^N} \varphi_\theta m^4 r^{\hat{p}} |\partial^\alpha f_m|^2_{q,q+1} \, \mathrm{d}\lambda \right)$$
$$+ cm^5(\theta + 1)^{2m+1},$$

where we assume without loss of generality $\tilde{p} \leqslant \hat{p}$. Therefore, for sufficient large θ

$$\int_{\mathbb{R}^N} \varphi_\theta |u_m|^2_{q,q+1} \, \mathrm{d}\lambda \leqslant \varkappa(\theta) \left(\int_{\mathbb{R}^N} \varphi_\theta \frac{m^8}{r^8} |u_m|^2_{q,q+1} \, \mathrm{d}\lambda + \sum_{|\alpha|\leqslant 2} \int_{\mathbb{R}^N} \varphi_\theta m^4 r^{\hat{p}} |\partial^\alpha f_m|^2_{q,q+1} \, \mathrm{d}\lambda \right)$$
$$+ cm^5(\theta + 1)^{2m+1},$$

i.e.

$$\int_{\check{B}_\theta} |u_m|^2_{q,q+1} \, \mathrm{d}\lambda$$
$$\leqslant \varkappa(\theta) \left(\int_{\check{B}_\theta} \frac{m^8}{r^8} |u_m|^2_{q,q+1} \, \mathrm{d}\lambda + \sum_{|\alpha|\leqslant 2} \int_{\check{B}_\theta} m^4 r^{\hat{p}} |\partial^\alpha f_m|^2_{q,q+1} \, \mathrm{d}\lambda \right) + cm^5(\theta + 1)^{2m+1}.$$

Setting $k := 2m$ we finally get for all $k \in \mathbb{R}$ and sufficient large θ

$$\int_{\check{B}_\theta} r^k |u|^2_{q,q+1} \, \mathrm{d}\lambda$$
$$\leqslant \varkappa(\theta) k^8 \left(\int_{\check{B}_\theta} r^{k-8} |u|^2_{q,q+1} \, \mathrm{d}\lambda + \sum_{|\alpha|\leqslant 2} \int_{\check{B}_\theta} r^{k+\hat{p}} |\partial^\alpha f|^2_{q,q+1} \, \mathrm{d}\lambda \right) + ck^5(\theta + 1)^{k+1}. \tag{5.10}$$

We are ready to prove the exponential decay. Let $t \in \mathbb{R}_+$ and θ large enough, such that (5.10) is holds. Then, we have for all natural numbers $8 \leqslant K_1 \leqslant K_2$

$$
\sum_{k=K_1}^{K_2} \frac{t^k}{k!} \int_{\check{B}_\theta} r^k |u|^2_{q,q+1} \, d\lambda
$$

$$
\leqslant \varkappa(\theta) \sum_{k=K_1}^{K_2} \frac{t^k}{k!} k^8 \int_{\check{B}_\theta} r^{k-8} |u|^2_{q,q+1} \, d\lambda
$$

$$
+ \varkappa(\theta) \sum_{k=K_1}^{K_2} \frac{t^k}{k!} \sum_{|\alpha|\leqslant 2} \int_{\check{B}_\theta} \underbrace{k^8 r^{k+\hat{p}}}_{\leqslant 8^k r^k r^{\hat{p}}} |\partial^\alpha f|^2_{q,q+1} \, d\lambda + c \sum_{k=K_1}^{K_2} \frac{t^k}{k!} \underbrace{k^5 (\theta+1)^{k+1}}_{\leqslant (5(\theta+1))^k (\theta+1)}
$$

$$
\leqslant \varkappa(\theta) \sum_{k=K_1}^{K_2} \frac{t^k}{(k-8)!} \int_{\check{B}_\theta} r^{k-8} |u|^2_{q,q+1} \, d\lambda
$$

$$
+ \varkappa(\theta) \sum_{|\alpha|\leqslant 2} \int_{\check{B}_\theta} e^{8tr} r^{\hat{p}} |\partial^\alpha f|^2_{q,q+1} \, d\lambda + c e^{5t(\theta+1)} (\theta+1)
$$

$$
\leqslant \varkappa(\theta) \sum_{k=K_1-8}^{K_2} \frac{t^{k+8}}{k!} \int_{\check{B}_\theta} r^k |u|^2_{q,q+1} \, d\lambda + \varkappa(\theta) + c e^{6t(\theta+1)}.
$$

Now, let θ be so large, such that $\varkappa(\theta) t^8 \leqslant 1/2$. Then,

$$
\sum_{k=K_1}^{K_2} \frac{t^k}{k!} \int_{\check{B}_\theta} r^k |u|^2_{q,q+1} \, d\lambda \leqslant \varkappa(\theta) \sum_{k=0}^{K_1-1} \frac{t^{k+8}}{k!} \int_{\check{B}_\theta} r^k |u|^2_{q,q+1} \, d\lambda + c.
$$

Since the right-hand side is independent of K_2 we obtain by the monotone convergence theorem for $K_2 \to \infty$

$$
\int_{\check{B}_\theta} e^{tr} |u|^2_{q,q+1} \, d\lambda < \infty,
$$

i.e., $e^{tr} u \in \mathsf{L}^{2,q,q+1}(\Omega)$ for all $t \in \mathbb{R}$. The differential equation yields

$$
e^{tr} M u \in \mathsf{L}^{2,q,q+1}(\Omega), \qquad e^{tr}(\delta \varepsilon E, d \mu H) \in \mathsf{L}^{2,q-1,q+1}(\check{B}_{r_0+\theta})
$$

for all $t \in \mathbb{R}$ and all $\theta > 0$. Consequently, by

$$
M e^{tr} u = e^{tr} M u + t e^{tr} S u,
$$

$$
\delta \varepsilon e^{tr} E = e^{tr} \delta \varepsilon E + t e^{tr} T \varepsilon E, \qquad d \mu e^{tr} H = e^{tr} d \mu H + t e^{tr} R \mu H
$$

and inner regularity (see the beginning of the proof) we achieve $e^{tr} u \in \mathsf{H}^{1,q,q+1}(\check{B}_{r_0+\theta})$ for all $t \in \mathbb{R}$ and all $\theta > 0$. Repeating this argument yields finally the same assertion for H^2. $\quad\square$

6. Solution theory for time-harmonic Maxwell equations

As a canonical application we intend to present a solution theory for the radiation problem (1.4). We will follow in close lines the first part of [17]. It can be seen easily that

$$\mathrm{d}:\mathring{\mathsf{D}}^q(\Omega) \subset \mathsf{L}^{2,q}(\Omega) \to \mathsf{L}^{2,q+1}(\Omega), \qquad \delta:\Delta^{q+1}(\Omega) \subset \mathsf{L}^{2,q+1}(\Omega) \to \mathsf{L}^{2,q}(\Omega)$$

are skew-adjoint to each other. Consequently, for 0-admissible coefficients Λ

$$\mathcal{M}:D(\mathcal{M}):=\mathring{\mathsf{D}}^q(\Omega) \times \Delta^{q+1}(\Omega) \subset {}_\Lambda\mathsf{L}^{2,q,q+1}(\Omega) \to {}_\Lambda\mathsf{L}^{2,q,q+1}(\Omega), \qquad \mathcal{M}u := M_\Lambda u$$

is self-adjoint. Here, ${}_\Lambda\mathsf{L}^{2,q,q+1}(\Omega) := \mathsf{L}^{2,q,q+1}(\Omega)$ equipped with the scalar product

$$(u, v) \mapsto \langle \Lambda u, v \rangle_{\mathsf{L}^{2,q,q+1}(\Omega)}.$$

Furthermore, we will denote the kernel and the image of $\mathcal{M}$ by $N(\mathcal{M})$ and $I(\mathcal{M})$, respectively.

Definition 6.1. Let $\omega \in \mathbb{C} \setminus \mathbb{R}$ and $f \in \mathsf{L}^{2,q,q+1}_{\mathrm{loc}}(\Omega)$. Then we say that u solves $\mathsf{Max}(\Lambda, \omega, f)$ if $u \in D(\mathcal{M})$ and $(M_\Lambda - \omega)u = f$.

The self-adjointness of $\mathcal{M}$ yields the unique solvability of the problem $\mathsf{Max}(\Lambda, \omega, f)$ for non-real frequencies $\omega \in \mathbb{C} \setminus \mathbb{R}$ and right-hand sides $f \in \mathsf{L}^{2,q,q+1}(\Omega)$. We denote the continuous solution operator by $\mathcal{L}_\omega$. Since the spectrum of $\mathcal{M}$ is contained in the real axis we expect from well known facts about Helmholtz' equation that we have to work in weighted L^2-spaces and utilize radiating solutions to get a proper solution theory for real frequencies.

Definition 6.2. Let $\omega \in \mathbb{R} \setminus \{0\}$ and $f \in \mathsf{L}^{2,q,q+1}_{\mathrm{loc}}(\Omega)$. Then we say that u solves $\mathsf{Max}(\Lambda, \omega, f)$ if

$$u \in \mathring{\mathsf{D}}^q_{<-1/2}(\Omega) \times \Delta^{q+1}_{<-1/2}(\Omega), \qquad (S+1)u \in \mathsf{L}^{2,q,q+1}_{>-1/2}(\Omega), \qquad (M_\Lambda - \omega)u = f.$$

The second constraint will be called 'Maxwell incoming radiation condition' or simply 'radiation condition'.

We will establish a solution theory using Eidus' limiting absorption principle. The key tool for the application of this principle is an a priori estimate, which ensures the uniform continuity of $\mathcal{L}_\omega$ operating in proper Hilbert spaces even up to the real axis.

6.1. An a priori estimate

Lemma 6.3. *Let $I \subset \mathbb{R}\setminus\{0\}$ be a compact interval and $s, -t > 1/2$. Furthermore, let Λ be τ-admissible with $\tau > 1$. Then, there exist constants $c, \theta > 0$ and a $\hat{t} > -1/2$, such that for all $\omega \in \mathbb{C}_+$, which means that ω has got non-negative imaginary part, with $\omega^2 = \lambda^2 + \mathrm{i}\sigma\lambda$, $\lambda \in I$, $\sigma \in (0, 1]$ and $f \in \mathsf{L}^{2,q,q+1}_s(\Omega)$ as well as $u := \mathcal{L}_\omega f$ the estimate*

$$\|u\|_{\mathsf{D}^q_t(\Omega) \times \Delta^{q+1}_t(\Omega)} + \|(S+1)u\|_{\mathsf{L}^{2,q,q+1}_{\hat{t}}(\Omega)} \leqslant c \big(\|f\|_{\mathsf{L}^{2,q,q+1}_s(\Omega)} + \|u\|_{\mathsf{L}^{2,q,q+1}(\Omega \cap B_\theta)}\big)$$

holds true.

Proof. Without loss of generality let $s \in (1/2, 1)$. We note $(1/2, 1) \cap \mathbb{I} = \emptyset$. Decomposing f and u using Lemma 3.1 with $s = s$, $t = 0$ we have $f_{\mathsf{S}} = 0$ since $s < N/2$ and obtain $u_{\Delta} \in \mathsf{H}^{2,q,q+1}$ satisfying

$$(\Delta + \omega^2)u_{\Delta} = -(1 + \omega^2)u_{\mathcal{F}} + (1 - \mathrm{i}\omega)\tilde{f} =: f_{\Delta} \in \mathsf{L}_s^{2,q,q+1}.$$

The self-adjointness of $\Delta : \mathsf{H}^{2,q,q+1} \subset \mathsf{L}^{2,q,q+1} \to \mathsf{L}^{2,q,q+1}$ yields $(\Delta + \omega^2)^{-1}f_{\Delta} = u_{\Delta}$. Applying [24], Lemma 7, which is a well-known a priori estimate for the scalar Helmholtz equation in $\mathbb{R}^N$; see also Ikebe and Saito [6] or Vogelsang [20, Section 2], component-wise to u_{Δ} and using Lemma 3.1 with $M(\mathrm{e}^{-\mathrm{i}\lambda r}u_{\Delta}) = \mathrm{e}^{-\mathrm{i}\lambda r}(M - \mathrm{i}\lambda S)u_{\Delta}$ we get the estimate

$$\|u_{\Delta}\|_{\mathsf{L}_t^{2,q,q+1}} + \|(M - \mathrm{i}\lambda S)u_{\Delta}\|_{\mathsf{L}_{s-1}^{2,q,q+1}}$$

$$\leqslant c\bigg(\|u_{\Delta}\|_{\mathsf{L}_t^{2,q,q+1}} + \sum_{|\alpha|=1}\big\|\partial^{\alpha}\big(\mathrm{e}^{-\mathrm{i}\lambda r}u_{\Delta}\big)\big\|_{\mathsf{L}_{s-1}^{2,q,q+1}}\bigg)$$

$$\leqslant c\|f_{\Delta}\|_{\mathsf{L}_s^{2,q,q+1}} \leqslant c\big(\|f\|_{\mathsf{L}_s^{2,q,q+1}(\Omega)} + \|u\|_{\mathsf{L}_{s-\tau}^{2,q,q+1}(\Omega)}\big), \tag{6.1}$$

which holds uniformly in u_{Δ}, f_{Δ} and λ. But actually we are interested in estimating another term, which is $\|(M - \mathrm{i}\omega S)u_{\Delta}\|_{\mathsf{L}_{s-1}^{2,q,q+1}}$. This needs an additional argument. The standard resolvent estimate yields

$$\sigma|\lambda|\|u_{\Delta}\|_{\mathsf{L}^{2,q,q+1}} \leqslant \|f_{\Delta}\|_{\mathsf{L}^{2,q,q+1}} \tag{6.2}$$

and we have $|\omega + \lambda| \geqslant |\lambda|$ since $|\operatorname{Re}\omega| \geqslant |\lambda|\sqrt{2}/2$ and $\omega \in \mathbb{C}_+$. Thus, by (6.2) and

$$\omega - \lambda = \frac{\omega^2 - \lambda^2}{\omega + \lambda} = \frac{\mathrm{i}\sigma\lambda}{\omega + \lambda}$$

we achieve uniformly in ω

$$\|(M - \mathrm{i}\omega S)u_{\Delta}\|_{\mathsf{L}_{s-1}^{2,q,q+1}} \leqslant \|(M - \mathrm{i}\lambda S)u_{\Delta}\|_{\mathsf{L}_{s-1}^{2,q,q+1}} + c|\omega - \lambda|\|u_{\Delta}\|_{\mathsf{L}_{s-1}^{2,q,q+1}}$$

$$\leqslant \|(M - \mathrm{i}\lambda S)u_{\Delta}\|_{\mathsf{L}_{s-1}^{2,q,q+1}} + \frac{c}{|\lambda|}\|f_{\Delta}\|_{\mathsf{L}^{2,q,q+1}}.$$

A combination of the latter estimate with (6.1) and Lemma 3.1 yield

$$\|u\|_{\mathsf{D}_t^q(\Omega) \times \Delta_t^{q+1}(\Omega)} + \|(M - \mathrm{i}\omega S)u\|_{\mathsf{L}_{s-1}^{2,q,q+1}(\Omega)}$$

$$\leqslant c\big(\|u_{\Delta}\|_{\mathsf{L}_t^{2,q,q+1}} + \|(M - \mathrm{i}\omega S)u_{\Delta}\|_{\mathsf{L}_{s-1}^{2,q,q+1}} + \|f\|_{\mathsf{L}_s^{2,q,q+1}(\Omega)} + \|u\|_{\mathsf{L}_{s-\tau}^{2,q,q+1}(\Omega)}\big)$$

$$\leqslant c\big(\|f\|_{\mathsf{L}_s^{2,q,q+1}(\Omega)} + \|u\|_{\mathsf{L}_{s-\tau}^{2,q,q+1}(\Omega)}\big)$$

uniformly in u, f and ω. Using $(M - \mathrm{i}\omega S)u = -\mathrm{i}\omega(S + 1 + \hat{\Lambda})u + f$ we finally arrive at

$$\|u\|_{\mathsf{D}_t^q(\Omega) \times \Delta_t^{q+1}(\Omega)} + \|(S + 1)u\|_{\mathsf{L}_{s-1}^{2,q,q+1}(\Omega)} \leqslant c\big(\|f\|_{\mathsf{L}_s^{2,q,q+1}(\Omega)} + \|u\|_{\mathsf{L}_{s-\tau}^{2,q,q+1}(\Omega)}\big).$$

Because of the monotone dependence of the weighted L^2-norms on the weights we may assume t near to $-1/2$ and s near to $1/2$, such that $1 < s - t < \tau$ holds. Then Lemma 4.1 completes the proof. $\quad\square$

6.2. *Fredholm theory using the limiting absorption principle*

We prove three more technical lemmas.

Lemma 6.4. *Let $\alpha, \beta \in \mathbb{R}$ with $0 \leqslant \alpha < \beta$ and $\mathbb{R}^N \setminus \Omega \subset B_\alpha$. Moreover, for some $t \in \mathbb{R}$ let $E \in \mathring{\mathsf{D}}_t^q(\Omega)$, $H \in \Delta_t^{q+1}(\Omega)$ and $\varphi \in \mathsf{C}^0([\alpha, \beta], \mathbb{C})$. Then with*

$$\psi : [0, \beta] \to \mathbb{C},$$

$$\sigma \mapsto \int_{\max\{\alpha, \sigma\}}^{\beta} \varphi(s) \, \mathrm{d}s$$

and $\Phi := \varphi \circ r, \Psi := \psi \circ r$

$$\langle \Phi R E, H \rangle_{\mathsf{L}^{2,q+1}(\check{B}_\alpha \cap B_\beta)} = \langle \Psi \mathrm{d}E, H \rangle_{\mathsf{L}^{2,q+1}(\Omega \cap B_\beta)} + \langle \Psi E, \delta H \rangle_{\mathsf{L}^{2,q}(\Omega \cap B_\beta)}.$$

Proof. Assume $E \in \mathring{\mathsf{C}}^{\infty,q}(\Omega)$, $H \in \mathsf{C}^{\infty,q+1}(\Omega)$. With $\gamma := \int_\alpha^\beta \varphi(s) \, \mathrm{d}s$ we have $\psi|_{[0,\alpha]} = \gamma$, $\psi(\beta) = 0$ and $\psi \in \mathsf{C}^1((\alpha, \beta), \mathbb{C})$ with $\psi' = -\varphi$. By Stokes' theorem we compute

$$\langle \Psi \mathrm{d}E, H \rangle_{\mathsf{L}^{2,q+1}(\Omega \cap B_\beta)} + \langle \Psi E, \delta H \rangle_{\mathsf{L}^{2,q}(\Omega \cap B_\beta)}$$

$$= \gamma \langle \mathrm{d}E, H \rangle_{\mathsf{L}^{2,q+1}(\Omega \cap B_\alpha)} + \gamma \langle E, \delta H \rangle_{\mathsf{L}^{2,q}(\Omega \cap B_\alpha)}$$

$$\quad + \langle \Psi \mathrm{d}E, H \rangle_{\mathsf{L}^{2,q+1}(\check{B}_\alpha \cap B_\beta)} + \langle \Psi E, \delta H \rangle_{\mathsf{L}^{2,q}(\check{B}_\alpha \cap B_\beta)}$$

$$= \gamma \int_{S_\alpha} \iota_\alpha^*(E \wedge *\bar{H}) + \langle \Phi R E, H \rangle_{\mathsf{L}^{2,q+1}(\check{B}_\alpha \cap B_\beta)}$$

$$\quad - \gamma \int_{S_\alpha} \iota_\alpha^*(E \wedge *\bar{H}) + \psi(\beta) \int_{S_\beta} \iota_\beta^*(E \wedge *\overline{H}),$$

where $\iota_\theta : S_\theta \to \mathbb{R}^N$ denotes the natural embedding. With the help of mollifiers we get the desired formula for all $H \in \Delta_t^{q+1}(\Omega)$. Since $\mathring{\mathsf{C}}^{\infty,q}(\Omega)$ is dense in $\mathring{\mathsf{D}}_t^q(\Omega)$ the assertion holds as stated. $\quad\square$

By the same approximation technique we obtain the rule of partial integration for weighted forms.

Lemma 6.5. *Let $t, s \in \mathbb{R}$ and $E \in \mathring{\mathsf{D}}_t^q(\Omega)$, $H \in \Delta_s^{q+1}(\Omega)$ as well as $\varphi_\theta := 1 - \eta(\cdot/\theta)$, $\Phi_\theta := \varphi_\theta \circ r$ with some $\theta > r_0$. Then*

$$\langle \Phi_\theta \, \mathrm{d}E, H \rangle_{\mathsf{L}^{2,q+1}(\Omega)} + \langle \Phi_\theta E, \delta H \rangle_{\mathsf{L}^{2,q}(\Omega)} = -\langle \varphi_\theta'(r) R E, H \rangle_{\mathsf{L}^{2,q+1}(\Omega)}$$

holds. Additionally, if $t + s \geqslant 0$ then

$$\langle \mathrm{d}E, H \rangle_{\mathsf{L}^{2,q+1}(\Omega)} + \langle E, \delta H \rangle_{\mathsf{L}^{2,q}(\Omega)} = 0.$$

Remark 6.6. If $u \in \mathring{\mathsf{D}}_t^q(\Omega) \times \Delta_t^{q+1}(\Omega)$ and $v \in \mathring{\mathsf{D}}_s^q(\Omega) \times \Delta_s^{q+1}(\Omega)$ with $t + s \geqslant 0$ then

$$\langle Mu, v \rangle_{\mathsf{L}^{2,q,q+1}(\Omega)} + \langle u, Mv \rangle_{\mathsf{L}^{2,q,q+1}(\Omega)} = 0.$$

Since $\mathcal{M}$ is self-adjoint we have the well-known Hodge–Helmholtz decomposition.

Lemma 6.7.

$$\mathsf{L}^{2,q,q+1}(\Omega) = {}_\Lambda \mathsf{L}^{2,q,q+1}(\Omega) = N(\mathcal{M}) \oplus_\Lambda \overline{I(\mathcal{M})}$$
$$= \left({}_0\mathring{\mathsf{D}}^q(\Omega) \times {}_0\Delta^{q+1}(\Omega) \right) \oplus_\Lambda \Lambda^{-1}\left(\overline{\delta\Delta^{q+1}(\Omega)} \times \overline{\mathrm{d}\,\mathring{\mathsf{D}}^q(\Omega)} \right).$$

Here $\oplus_\Lambda$ denotes the orthogonal sum in ${}_\Lambda \mathsf{L}^{2,q,q+1}(\Omega)$ and the closures are taken in the respective L^2-spaces.

Another essential ingredient of the solution theory generating convergence in the limiting absorption argument is the so-called Maxwell local compactness property MLCP, i.e., the embeddings

$$\mathring{\mathsf{D}}_s^q(\Omega) \cap \Delta_s^q(\Omega) \hookrightarrow \mathsf{L}_t^{2,q}(\Omega)$$

have to be compact for all $t < s$ and all q. For a detailed analysis of this property of $\partial\Omega$ we refer to [17, 22] and [13–16] as well as the papers cited there.

Definition 6.8. For $\omega \in \mathbb{C} \setminus \{0\}$ we define

$$P := \{ \omega \in \mathbb{C} \setminus \{0\} \colon \mathsf{Max}(\Lambda, \omega, 0) \text{ has a nontrivial solution} \},$$
$$N_\omega := \{ u \colon u \text{ is a solution of } \mathsf{Max}(\Lambda, \omega, 0) \}.$$

We remark $P \subset \mathbb{R} \setminus \{0\}$ or $N_\omega = N(\mathcal{M} - \omega) = \{0\}$ if $\omega \in \mathbb{C} \setminus \mathbb{R}$. We are ready to prove the main result of this section.

Theorem 6.9. *Let $\omega \in \mathbb{R} \setminus \{0\}$ and Λ be τ-admissible with $\tau > 1$.*

(i) *Eigen-solutions decay polynomially, i.e., for all $t \in \mathbb{R}$*

$$N_\omega = N(\mathcal{M} - \omega) \subset \left(\mathring{\mathsf{D}}_t^q(\Omega) \cap \varepsilon^{-1}{}_0\Delta_t^q(\Omega) \right) \times \left(\Delta_t^{q+1}(\Omega) \cap \mu^{-1}{}_0\mathring{\mathsf{D}}_t^{q+1}(\Omega) \right).$$

Additionally, let Ω possess the MLCP. Then

(ii) *N_ω is finite dimensional;*

(iii) *P has no accumulation point in $\mathbb{R} \setminus \{0\}$;*

(iv) *for every $f \in \mathsf{L}_{>1/2}^{2,q,q+1}(\Omega)$ there exists a solution u of the problem $\mathsf{Max}(\Lambda, \omega, f)$, if and only if $f \perp_\Lambda N_\omega$, i.e.,*

$$\forall v \in N_\omega \quad \langle \Lambda f, v \rangle_{\mathsf{L}^{2,q,q+1}(\Omega)} = 0. \tag{6.3}$$

The solution can be chosen, such that $u \perp_\Lambda N_\omega$, i.e.,

$$\forall v \in N_\omega \quad \langle \Lambda u, v \rangle_{\mathsf{L}^{2,q,q+1}(\Omega)} = 0. \tag{6.4}$$

By this condition u is uniquely determined and the solution operator $\mathcal{L}_\omega f := u$ is continuous. More precisely, $\mathcal{L}_\omega$ maps $\mathsf{L}_s^{2,q,q+1}(\Omega) \cap N_\omega^{\perp_\Lambda}$ to $(\overset{\circ}{\mathsf{D}}_t^q(\Omega) \times \Delta_t^{q+1}(\Omega)) \cap N_\omega^{\perp_\Lambda}$ continuously for all $s, -t > 1/2$. Here we denote the orthogonality corresponding to the $_\Lambda \mathsf{L}^{2,q,q+1}(\Omega)$-scalar product by $\perp_\Lambda$.

Proof. We follow the proof of [17], Theorem 2.10. To show (i), i.e., the polynomial decay of any eigen-solution u, we only have to prove

$$u \in \mathsf{L}_{>-1/2}^{2,q,q+1}(\Omega)$$

because of Theorem 4.2, Remark 4.3, the equation $Mu = -\mathrm{i}\omega \Lambda u$ and the inclusions

$$\overline{\mathrm{d}\,\overset{\circ}{\mathsf{D}}{}^q(\Omega)} \subset {}_0\overset{\circ}{\mathsf{D}}{}^{q+1}(\Omega), \qquad \overline{\delta \Delta^{q+1}(\Omega)} \subset {}_0\Delta^q(\Omega). \tag{6.5}$$

Using the second part of the radiation condition we obtain some $t > -1/2$, such that

$$\lim_{\beta \to \infty} \|RE + H\|_{\mathsf{L}_t^{2,q+1}(\check{B}_{r_0} \cap B_\beta)} < \infty$$

holds true. We calculate

$$\|RE + H\|_{\mathsf{L}_t^{2,q+1}(\check{B}_{r_0} \cap B_\beta)}^2$$

$$= \|RE\|_{\mathsf{L}_t^{2,q+1}(\check{B}_{r_0} \cap B_\beta)}^2 + \|H\|_{\mathsf{L}_t^{2,q+1}(\check{B}_{r_0} \cap B_\beta)}^2 + 2\operatorname{Re}\langle \Phi RE, H \rangle_{\mathsf{L}^{2,q+1}(\check{B}_{r_0} \cap B_\beta)}$$

with $\Phi := \rho^{2t}$. Lemma 6.4, the differential equation and the symmetry of ε, μ yield

$$\langle \Phi RE, H \rangle_{\mathsf{L}^{2,q+1}(\check{B}_{r_0} \cap B_\beta)} = \langle \Psi \mathrm{d}E, H \rangle_{\mathsf{L}^{2,q+1}(\Omega \cap B_\beta)} + \langle \Psi E, \delta H \rangle_{\mathsf{L}^{2,q}(\Omega \cap B_\beta)}$$

$$= -\mathrm{i}\omega \underbrace{\langle \Psi \mu H, H \rangle_{\mathsf{L}^{2,q+1}(\Omega \cap B_\beta)}}_{\in \mathbb{R}} + \mathrm{i}\omega \underbrace{\langle \Psi E, \varepsilon E \rangle_{\mathsf{L}^{2,q}(\Omega \cap B_\beta)}}_{\in \mathbb{R}}.$$

Thus, $\operatorname{Re}\langle \Phi RE, H \rangle_{\mathsf{L}^{2,q+1}(\check{B}_{r_0} \cap B_\beta)} = 0$ and hence, by means of the monotone convergence theorem we have $H \in \mathsf{L}_t^{2,q+1}(\Omega)$ for $\beta \to \infty$. Finally we get $E \in \mathsf{L}_{>-1/2}^{2,q}(\Omega)$ using the first part of the radiation condition.

If the assertions (ii) or (iii) would be wrong then there would exist a sequence of eigen-values $(\omega_\ell)_{\ell \in \mathbb{N}} \subset \mathbb{R} \setminus \{0\}$ tending to ω and a sequence of eigen-forms $(u_\ell)_{\ell \in \mathbb{N}} \subset N_{\omega_\ell}$, such that $(u_\ell)_{\ell \in \mathbb{N}}$ is an ortho-normal system with respect to the $_\Lambda \mathsf{L}^{2,q,q+1}(\Omega)$-scalar product. As an ortho-normal system $(u_\ell)_{\ell \in \mathbb{N}}$ converges in $\mathsf{L}^{2,q,q+1}(\Omega)$ weakly to zero. Moreover, by the differential equation $(u_\ell)_{\ell \in \mathbb{N}}$ is bounded in

$$\left(\overset{\circ}{\mathsf{D}}{}^q(\Omega) \cap \varepsilon^{-1}{}_0\Delta^q(\Omega)\right) \times \left(\Delta^{q+1}(\Omega) \cap \mu^{-1}{}_0\overset{\circ}{\mathsf{D}}{}^{q+1}(\Omega)\right).$$

Hence, from the MLCP we can extract a subsequence $(u_{\pi\ell})_{\ell\in\mathbb{N}}$, where $\pi:\mathbb{N}\to\mathbb{N}$ is strictly monotone, converging for all $t < 0$ in $\mathsf{L}_t^{2,q,q+1}(\Omega)$ to 0. The latter is due to the weak convergence. For $1 \leqslant s \in \mathbb{R}\setminus\mathbb{I}$ Theorem 4.2 yields uniformly in $(u_{\pi\ell})_{\ell\in\mathbb{N}}$ and $(\omega_{\pi\ell})_{\ell\in\mathbb{N}}$ the estimate

$$1 = \langle \Lambda u_{\pi\ell}, u_{\pi\ell}\rangle_{\mathsf{L}^{2,q,q+1}(\Omega)} \leqslant c\|u_{\pi\ell}\|_{\mathsf{L}^{2,q,q+1}(\Omega)}^2 \leqslant c\|u_{\pi\ell}\|_{\mathsf{L}_{s-1}^{2,q,q+1}(\Omega)}^2$$

$$\leqslant c\|u_{\pi\ell}\|_{\mathsf{L}^{2,q,q+1}(\Omega\cap B_\theta)}^2 \xrightarrow{\ell\to\infty} 0,$$

which is a contradiction.

We prove (iv): first of all (6.3) is necessary since we get for all eigen-forms $v \in N_\omega$ by their polynomial decay and Remark 6.6

$$\langle \Lambda f, v\rangle_{\mathsf{L}^{2,q,q+1}(\Omega)} = \mathrm{i}\langle (M + \mathrm{i}\omega\Lambda)u, v\rangle_{\mathsf{L}^{2,q,q+1}(\Omega)} = -\mathrm{i}\langle u, \underbrace{(M + \mathrm{i}\omega\Lambda)v}_{=0}\rangle_{\mathsf{L}^{2,q,q+1}(\Omega)} = 0.$$

To show existence we now use Eidus' famous principle of limiting absorption. For that purpose let $f \in \mathsf{L}_{>1/2}^{2,q,q+1}(\Omega)$ with (6.3). Moreover, let $(\sigma_\ell)_{\ell\in\mathbb{N}}$ be a positive sequence tending to zero and $(f_\ell)_{\ell\in\mathbb{N}} \subset \mathsf{L}_s^{2,q,q+1}(\Omega)$ with some $s > 1/2$ be a sequence satisfying $f_\ell \perp_\Lambda N_\omega$, such that $(f_\ell)_{\ell\in\mathbb{N}}$ converges to f in $\mathsf{L}_s^{2,q,q+1}(\Omega)$ as ℓ tends to infinity. Defining non-real frequencies in the upper half plane $\omega_\ell \in \mathbb{C}_+ \setminus \mathbb{R}$ with $\omega_\ell^2 = \omega^2 + \mathrm{i}\sigma_\ell\omega$ and $\omega_\ell \to \omega$ we obtain L^2-solutions

$$u_\ell := \mathcal{L}_{\omega_\ell}f_\ell \in D(\mathcal{M}) = \mathring{\mathsf{D}}^q(\Omega) \times \Delta^{q+1}(\Omega)$$

solving the Maxwell problem $\mathsf{Max}(\Lambda, \omega_\ell, f_\ell)$, i.e.,

$$(M_\Lambda - \omega_\ell)u_\ell = f_\ell. \tag{6.6}$$

Applying the orthogonal Hodge–Helmholtz decomposition we decompose u_ℓ and f_ℓ according to Lemma 6.7

$$u_\ell = u_\ell^N + u_\ell^I, \qquad f_\ell = f_\ell^N + f_\ell^I \in N(\mathcal{M})\bigoplus\nolimits_\Lambda \overline{I(\mathcal{M})}.$$

Therefore,

$$(M_\Lambda - \omega_\ell)u_\ell^I - f_\ell^I = \omega_\ell u_\ell^N + f_\ell^N \in N(\mathcal{M}) \cap \overline{I(\mathcal{M})} = \{0\}.$$

As orthogonal projections the sequences of forms $(f_\ell^N)_{\ell\in\mathbb{N}}$, $(f_\ell^I)_{\ell\in\mathbb{N}}$ converge in $\mathsf{L}^{2,q,q+1}(\Omega)$. Hence, so does $(u_\ell^N)_{\ell\in\mathbb{N}}$. Let us assume the boundedness of $(u_\ell^I)_{\ell\in\mathbb{N}}$ or

$$\forall t < -1/2 \; \exists c > 0 \; \forall \ell \in \mathbb{N} \quad \|u_\ell\|_{\mathsf{L}_t^{2,q,q+1}(\Omega)} \leqslant c \tag{6.7}$$

for a moment. At the end of the proof we will show by contradiction that in fact (6.7) holds. Let t' be such a t with (6.7). Then, $(u_\ell^I)_{\ell\in\mathbb{N}}$ is bounded in $\mathsf{L}_{t'}^{2,q,q+1}(\Omega)$ and by the differential equation, i.e.,

$$(M_\Lambda - \omega_\ell)u_\ell^I = f_\ell^I,$$

and by (6.5) even in $(\mathring{\mathsf{D}}_{t'}^q(\Omega) \cap \varepsilon^{-1}{}_0\Delta_{t'}^q(\Omega)) \times (\Delta_{t'}^{q+1}(\Omega) \cap \mu^{-1}{}_0\mathring{\mathsf{D}}_{t'}^{q+1}(\Omega))$. Hence, the MLCP yields for an arbitrary $\tilde{t} < t'$ a subsequence $(u_{\pi\ell}^I)_{\ell \in \mathbb{N}}$ converging in $\mathsf{L}_{\tilde{t}}^{2,q,q+1}(\Omega)$ and even in $\mathring{\mathsf{D}}_{\tilde{t}}^q(\Omega) \times \Delta_{\tilde{t}}^{q+1}(\Omega)$ by the differential equation. Therefore, the entire sequence $(u_{\pi\ell})_{\ell \in \mathbb{N}}$ converges in $\mathring{\mathsf{D}}_{\tilde{t}}^q(\Omega) \times \Delta_{\tilde{t}}^{q+1}(\Omega)$ to, let us say,

$$u \in \mathring{\mathsf{D}}_{\tilde{t}}^q(\Omega) \times \Delta_{\tilde{t}}^{q+1}(\Omega),$$

which solves

$$(M_\Lambda - \omega)u = f.$$

With the polynomial decay of eigen-solutions and Remark 6.6 we compute for all eigen-forms $v \in N_\omega$ and all $\ell \in \mathbb{N}$

$$0 = \langle \Lambda f_{\pi\ell}, v \rangle_{\mathsf{L}^{2,q,q+1}(\Omega)} = -\mathrm{i}\langle u_{\pi\ell}, (M + \mathrm{i}\omega_{\pi\ell}\Lambda)v \rangle_{\mathsf{L}^{2,q,q+1}(\Omega)} = \underbrace{(\overline{\omega}_{\pi\ell} - \omega)}_{\neq 0}\langle \Lambda u_{\pi\ell}, v \rangle_{\mathsf{L}^{2,q,q+1}(\Omega)}.$$

Consequently, $\langle \Lambda u_{\pi\ell}, v \rangle_{\mathsf{L}^{2,q,q+1}(\Omega)} = 0$. Since $\langle \cdot, \Lambda v \rangle_{\mathsf{L}^{2,q,q+1}(\Omega)}$ is a continuous linear functional on $\mathsf{L}_{\tilde{t}}^{2,q,q+1}(\Omega)$ for all $v \in N_\omega$ we obtain

$$\forall v \in N_\omega \quad \langle \Lambda u, v \rangle_{\mathsf{L}^{2,q,q+1}(\Omega)} = 0. \tag{6.8}$$

Now, we pick some $t < -1/2$. Then, we get by Lemma 6.3 constants $\hat{t} > -1/2$ and $c, \theta > 0$, such that by the monotone convergence theorem

$$\|u\|_{\mathsf{D}_t^q(\Omega) \times \Delta_t^{q+1}(\Omega)} + \|(S+1)u\|_{\mathsf{L}_{\hat{t}}^{2,q,q+1}(\Omega)} \leqslant c\big(\|f\|_{\mathsf{L}_s^{2,q,q+1}(\Omega)} + \|u\|_{\mathsf{L}^{2,q,q+1}(\Omega \cap B_\theta)}\big)$$

holds. Therefore, u is an element of $\mathring{\mathsf{D}}_{<-1/2}^q(\Omega) \times \Delta_{<-1/2}^{q+1}(\Omega)$ and satisfies the radiation condition, i.e., $(S+1)u \in \mathsf{L}_{>-1/2}^{2,q,q+1}(\Omega)$. In other words, u solves $\mathsf{Max}(\Lambda, \omega, f)$.

We note that this proves the principle of limiting absorption to hold. The choice $f_\ell := f$ for all $\ell \in \mathbb{N}$ yields the existence of a solution of $\mathsf{Max}(\Lambda, \omega, f)$ and this one is unique because of (6.8).

Moreover, for $-t, s > 1/2$ the solution operator $\mathcal{L}_\omega$ maps $D_s(\mathcal{L}_\omega)$ to $I_t(\mathcal{L}_\omega)$, where

$$D_s(\mathcal{L}_\omega) := \mathsf{L}_s^{2,q,q+1}(\Omega) \cap N_\omega^{\perp_\Lambda}, \qquad I_t(\mathcal{L}_\omega) := \big(\mathring{\mathsf{D}}_t^q(\Omega) \times \Delta_t^{q+1}(\Omega)\big) \cap N_\omega^{\perp_\Lambda},$$

continuously. This follows by the closed graph theorem because $D_s(\mathcal{L}_\omega)$ and $I_t(\mathcal{L}_\omega)$ are Hilbert spaces by the polynomial decay of eigen-solutions and $\mathcal{L}_\omega$ is closed. The latter assertion is a consequence of Lemma 6.3 and the monotone convergence theorem.

Finally, it remains to contradict the contrary assumption to (6.7). To this end, let $t < -1/2$ and $(u_\ell)_{\ell \in \mathbb{N}} \subset \mathring{\mathsf{D}}_t^q(\Omega) \times \Delta_t^{q+1}(\Omega)$ be a sequence with $\|u_\ell\|_{\mathsf{L}_t^{2,q,q+1}(\Omega)} \to \infty$. Defining the normalized forms

$$\tilde{u}_\ell := \|u_\ell\|_{\mathsf{L}_t^{2,q,q+1}(\Omega)}^{-1} u_\ell, \qquad \tilde{f}_\ell := \|u_\ell\|_{\mathsf{L}_t^{2,q,q+1}(\Omega)}^{-1} f_\ell$$

we have $\|\tilde{u}_\ell\|_{\mathsf{L}_t^{2,q,q+1}(\Omega)} = 1$ for all $\ell \in \mathbb{N}$ and $\|\tilde{f}_\ell\|_{\mathsf{L}_s^{2,q,q+1}(\Omega)} \to 0$. Moreover, the equation

$$(M_\Lambda - \omega_\ell)\tilde{u}_\ell = \tilde{f}_\ell$$

holds. Following the arguments above we obtain a subsequence $(\tilde{u}_{\pi\ell})_{\ell\in\mathbb{N}}$ converging in $\mathsf{L}_{\tilde{t}}^{2,q,q+1}(\Omega)$ with $\tilde{t} < t$ towards $\tilde{u} \in N_\omega^{\perp_\Lambda}$, which solves $\mathsf{Max}(\Lambda, \omega, 0)$. Hence, $\tilde{u} = 0$ and Lemma 6.3 yields constants $c, \theta > 0$ independent of $\sigma_{\pi\ell}$, $\tilde{f}_{\pi\ell}$ or $\tilde{u}_{\pi\ell}$, such that

$$1 = \|\tilde{u}_{\pi\ell}\|_{\mathsf{L}_t^{2,q,q+1}(\Omega)} \leqslant c(\underbrace{\|\tilde{f}_{\pi\ell}\|_{\mathsf{L}_s^{2,q,q+1}(\Omega)}}_{\to 0} + \underbrace{\|\tilde{u}_{\pi\ell}\|_{\mathsf{L}^{2,q,q+1}(\Omega\cap B_\theta)}}_{\to 0})$$

holds true; a contradiction. □

The polynomial decay of eigen-solutions proved above and Theorem 5.1 yield the following corollary.

Corollary 6.10. *Let* $\omega \in \mathbb{R} \setminus \{0\}$ *and* Λ *be* τ-C^2-*admissible with* $\tau > 1$. *Then, any eigen-solution* $u \in N_\omega$ *decays exponentially, i.e.,*

$$\mathrm{e}^{tr}u \in (\mathring{\mathsf{D}}^q(\Omega) \cap \varepsilon^{-1}\Delta^q(\Omega)) \times (\Delta^{q+1}(\Omega) \cap \mu^{-1}\mathring{\mathsf{D}}^{q+1}(\Omega)) \cap \mathsf{H}^{2,q,q+1}(\check{B}_{r_0+1})$$

holds for all $t \in \mathbb{R}$.

Remark 6.11. The polynomial resp. exponential decay of eigen-solutions holds for arbitrary exterior domains Ω, i.e., Ω does not need to have the MLCP.

Remark 6.12. If the medium is homogeneous and isotropic in the exterior of some ball in $\mathbb{R}^N$, this is, $\mathrm{supp}\,\hat{\Lambda} \cup (\mathbb{R}^N \setminus \Omega) \subset B_\theta$ for some $\theta > 0$, then

$$u = 0 \quad \text{in } \check{B}_\theta$$

for all $\omega \in \mathbb{R} \setminus \{0\}$ and $u \in N_\omega$. Because in this case u solves component-wise Helmholtz' equation $(\Delta + \omega^2)u = 0$ in $\check{B}_\theta$ and therefore, by Rellich's estimate [18] or [12], p. 59, must vanish in $\check{B}_\theta$. If the principle of unique continuation holds for our Maxwell system under consideration then even

$$N_\omega = \{0\}.$$

Remark 6.13. We note that P even has no accumulation point in $\mathbb{R}$, i.e., P does not accumulate in 0 as well, provided that Λ is τ-C^1-admissible with $\tau > (N + 1)/2$ (and decaying derivatives as well). This has already been proved in [13], Lemma 5.2(i).

Using the a priori estimate of the limiting absorption principle and some indirect arguments followed by the 'trivial' decomposition of $\mathsf{L}_s^{2,q}(\Omega)$ from [15], Lemma 4.1, we are able to prove stronger estimates for the solution operator $\mathcal{L}_\omega$ as the ones given in Theorem 6.9 (iv). We only note the results here. For this, let Ω possess the MLCP and let $s, -t > 1/2$ as well as K be a compact subset of $\mathbb{C}_+ \setminus \{0\}$. Furthermore, let Λ be τ-admissible with some $\tau > 1$.

Lemma 6.14. *There exist constants $c, \theta > 0$ and some $\hat{t} > -1/2$, such that*

$$\left\| \mathcal{L}_\omega f \right\|_{\mathsf{D}_t^q(\Omega) \times \Delta_t^{q+1}(\Omega)} + \left\| (S+1)\mathcal{L}_\omega f \right\|_{\mathsf{L}_{\hat{t}}^{2,q,q+1}(\Omega)} \leqslant c \big(\|f\|_{\mathsf{L}_s^{2,q,q+1}(\Omega)} + \|\mathcal{L}_\omega f\|_{\mathsf{L}^{2,q,q+1}(\Omega \cap B_\theta)} \big)$$

holds for all $\omega \in K$ and $f \in \mathsf{L}_s^{2,q,q+1}(\Omega) \cap N_\omega^{\perp_\Lambda}$.

Corollary 6.15. *Let $K \cap P = \emptyset$. Then there exist constants $c > 0$ and $\hat{t} > -1/2$, such that the estimate*

$$\left\| \mathcal{L}_\omega f \right\|_{\mathsf{D}_t^q(\Omega) \times \Delta_t^{q+1}(\Omega)} + \left\| (S+1)\mathcal{L}_\omega f \right\|_{\mathsf{L}_{\hat{t}}^{2,q,q+1}(\Omega)} \leqslant c \|f\|_{\mathsf{L}_s^{2,q,q+1}(\Omega)}$$

holds true for all $\omega \in K$ and $f \in \mathsf{L}_s^{2,q,q+1}(\Omega)$. In particular, the solution operator $\mathcal{L}_\omega$ mapping $\mathsf{L}_s^{2,q,q+1}(\Omega)$ to $\mathring{\mathsf{D}}_t^q(\Omega) \times \Delta_t^{q+1}(\Omega)$ is equi-continuous with respect to $\omega \in K$.

Theorem 6.16. *Let $K \cap P = \emptyset$. Then, the mapping*

$$\mathcal{L} : K \to \mathsf{B}\big(\mathsf{L}_s^{2,q,q+1}(\Omega), \mathring{\mathsf{D}}_t^q(\Omega) \times \Delta_t^{q+1}(\Omega) \big),$$

$$\omega \mapsto \mathcal{L}_\omega$$

is uniformly continuous. Here we denote the set of bounded linear operators from some normed space X to some normed space Y by $\mathsf{B}(X, Y)$.

Acknowledgements

Appendix A: Classical equations

We want to point out briefly, which classical equations are covered by our generalized approach. Since the relation between the differential form calculus and the classical vector calculus is very well known we directly translate our equations (1.4), i.e., in the longer version (1.1)–(1.3), into terms of vector analysis.

$q = 0$: E, F are scalar functions and H, G are vector fields. We have the equations of linear acoustics with Dirichlet boundary condition in first order form

$$\operatorname{div} H + \mathrm{i}\omega\varepsilon E = -\mathrm{i}\varepsilon F, \qquad \operatorname{grad} E + \mathrm{i}\omega\mu H = -\mathrm{i}\mu G \quad \text{in } \Omega,$$

$$E = 0 \hspace{8cm} \text{on } \partial\Omega$$

and $\xi \cdot H + E, E\xi + H$ decay at infinity, where $\xi(x) := x/r(x)$. Using the differential equation for H, i.e.,

$$-\mathrm{i}\omega(\xi \cdot H + E) = \underbrace{\xi \cdot \operatorname{grad} E}_{=\partial_r E} - \mathrm{i}\omega E + \cdots,$$

we see that we get Sommerfeld's classical radiation condition, i.e., $(\partial_r - \mathrm{i}\omega)E$ decays at infinity. $q = N - 1$: E, F are vector fields and H, G are scalar functions. We have the equations of linear acoustics with Neumann boundary condition in first order form

$$\operatorname{grad} H + \mathrm{i}\omega\varepsilon E = -\mathrm{i}\varepsilon F, \qquad \operatorname{div} E + \mathrm{i}\omega\mu H = -\mathrm{i}\mu G \quad \text{in } \Omega,$$

$$\nu \cdot E = 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{on } \partial\Omega$$

and $H\xi + E$, $\xi \cdot E + H$ decay at infinity. Here ν is the unit normal vector pointing outwards. Now, using the differential equation for E we get again Sommerfeld's radiation condition. Note that by the differential equation the homogeneous boundary condition is equivalent to the inhomogeneous Neumann boundary condition

$$\partial_\nu H = -\mathrm{i}\omega\nu \cdot \hat{\varepsilon}(E - f) - \mathrm{i}\nu \cdot F \quad \text{on } \partial\Omega.$$

$q = N$: E, F are scalar functions and H, G vanish. We have trivial equations

$$\omega E = -F \text{ in } \Omega, \quad E \text{ decays at infinity.}$$

$q = 1$, $N = 3$: E, F, H, G are all vector fields. We have the classical Maxwell equations with homogeneous electric boundary condition, i.e., $\partial\Omega$ is a perfect conductor,

$$-\operatorname{curl} H + \mathrm{i}\omega\varepsilon E = -\mathrm{i}\varepsilon F, \qquad \operatorname{curl} E + \mathrm{i}\omega\mu H = -\mathrm{i}\mu G \quad \text{in } \Omega,$$

$$\nu \times E = 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{on } \partial\Omega$$

and $-\xi \times H + E$, $\xi \times E + H$ decay at infinity. The latter are the classical Silver–Müller radiation conditions for Maxwell's equations.

Appendix B: Some calculations

Let tr and sw denote the formal trace and the formal swapping operator, respectively, i.e.,

$$\operatorname{tr} \begin{bmatrix} A & a \\ b & B \end{bmatrix} = A + B, \qquad \operatorname{sw} \begin{bmatrix} A & a \\ b & B \end{bmatrix} = \begin{bmatrix} B & a \\ b & A \end{bmatrix}.$$

We note

$$S^2 = \begin{bmatrix} TR & 0 \\ 0 & RT \end{bmatrix}, \qquad 1 - S^2 = \begin{bmatrix} RT & 0 \\ 0 & TR \end{bmatrix}$$

and formally $1 - S^2 = \operatorname{sw} S^2$ and $\operatorname{tr} S^2 = \operatorname{tr} \operatorname{sw} S^2 = RT + TR = 1$. Moreover,

$$SM = \begin{bmatrix} T\mathrm{d} & 0 \\ 0 & R\delta \end{bmatrix}, \qquad MS = \begin{bmatrix} \delta R & 0 \\ 0 & \mathrm{d}T \end{bmatrix}$$

and formally $\operatorname{tr} SM = T\mathrm{d} + R\delta$, $\operatorname{tr} MS = \mathrm{d}T + \delta R$. By a straight forward computation we obtain the following lemma.

Lemma B.1. *Let $m \in \mathbb{R}$.*

(i) $\quad C_{\partial_n, r^m} = \dfrac{m}{r} r^{-1} \operatorname{Id}_n r^m, \qquad C_{\Delta, r^m} = \dfrac{m}{r} \left(2\partial_r - \dfrac{m+2-N}{r} \right) r^m,$

(ii) $\quad C_{\mathrm{d}, r^m} = \dfrac{m}{r} R r^m, \qquad C_{\delta, r^m} = \dfrac{m}{r} T r^m, \qquad C_{M, r^m} = \dfrac{m}{r} S r^m,$

(iii) $\quad C_{\delta\,\mathrm{d}, r^m} = \dfrac{m}{r} \left(\delta R + T\mathrm{d} - \dfrac{m+1}{r} TR \right) r^m, \qquad C_{\mathrm{d}\,\delta, r^m} = \dfrac{m}{r} \left(\mathrm{d} T + R\delta - \dfrac{m+1}{r} RT \right) r^m,$

(iv) $\quad C_{M^2, r^m} = \dfrac{m}{r} \left(MS + SM - \dfrac{m+1}{r} S^2 \right) r^m,$

(v) $\quad C_{\square, r^m} = \dfrac{m}{r} \left(\operatorname{sw} MS + \operatorname{sw} SM - \dfrac{m+1}{r} \operatorname{sw} S^2 \right) r^m,$

(vi) $\quad C_{\Delta, r^m} = \dfrac{m}{r} \left(\operatorname{tr} MS + \operatorname{tr} SM - \dfrac{m+1}{r} \right) r^m.$

Looking at the two formulas for C_{Δ, r^m} we get the relation

$$\mathrm{d} T + \delta R + T\mathrm{d} + R\delta = \operatorname{tr}(SM + MS) = \dfrac{(N-1)}{r} + 2\partial_r.$$

Here ∂_r denotes component-wise radial differentiation with respect to Cartesian coordinates.

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