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On Korn's first inequality for mixed tangential and normal boundary conditions on bounded Lipschitz domains in $\mathbb{R}^N$

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Abstract We prove that for bounded Lipschitz domains in $\mathbb{R}^N$ Korn's first inequality holds for vector fields satisfying homogeneous mixed tangential and normal boundary conditions.

Keywords Korn inequality · Tangential and normal boundary conditions · Boltzmann equation

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1 Introduction

Recently, motivated by [3,4] and inspired by the ideas and techniques presented in [9–11] for estimating the Maxwell constants, we have shown in [2] that Korn's first inequality, i.e.,

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$$|\nabla v|_{\mathsf{L}^2(\Omega)} \le c_{\mathsf{k}} \,|\operatorname{sym} \nabla v|_{\mathsf{L}^2(\Omega)}, \tag{1}$$

holds with $c_{\mathsf{k}} = \sqrt{2}$ for all vector fields $v \in \mathsf{H}^1(\Omega)$ satisfying (possibly mixed) homogeneous normal or homogenous tangential boundary conditions and for all globally Lipschitz and piecewise $\mathsf{C}^{1,1}$-domains $\Omega \subset \mathbb{R}^N$, $N \ge 2$, with concave boundary parts. In this contribution, we extend (1) to any bounded (strong) Lipschitz domain $\Omega \subset \mathbb{R}^N$, $N \ge 2$. As pointed out in [4], this Korn inequality has an important application in statistical physics, more precisely in the study of relaxation to equilibrium of rarefied gases modeled by Boltzmann's equation.

2 Preliminaries

We will utilize the notations from [2]. Throughout this paper and unless otherwise explicitly stated, let $\Omega \subset \mathbb{R}^N$, $N \ge 2$, be a bounded domain with strong Lipschitz boundary $\Gamma := \partial\,\Omega$, i.e., locally Γ can be represented as a graph of a Lipschitz function. As in [2], we introduce the standard scalar valued Lebesgue and Sobolev spaces by $\mathsf{L}^2(\Omega)$ and $\mathsf{H}^1(\Omega)$ as well as

$$\mathring{\mathsf{H}}^1(\Omega) := \overline{\mathring{\mathsf{C}}^\infty(\Omega)}^{\mathsf{H}^1(\Omega)},$$

respectively, where $\mathring{\mathsf{C}}^\infty(\Omega)$ denotes the test functions yielding the usual Sobolev space $\mathring{\mathsf{H}}^1(\Omega)$ with zero boundary traces. These definitions extend component-wise to vector or matrix, or more general tensor fields and we will use the same notations for these spaces. Moreover, we will consistently denote functions by u and vector fields by v. We define the vector valued H^1-Sobolev space $\mathring{\mathsf{H}}^1_{\mathsf{t}}(\Omega)$ resp. $\mathring{\mathsf{H}}^1_{\mathsf{n}}(\Omega)$ as closure in $\mathsf{H}^1(\Omega)$ of the set of test vector fields

$$\begin{aligned}\mathring{\mathsf{C}}^\infty_{\mathsf{t}}(\Omega) &:= \big\{v|_\Omega \,:\, v \in \mathring{\mathsf{C}}^\infty(\mathbb{R}^N),\ v_{\mathsf{t}} = 0\big\},\\ \mathring{\mathsf{C}}^\infty_{\mathsf{n}}(\Omega) &:= \big\{v|_\Omega \,:\, v \in \mathring{\mathsf{C}}^\infty(\mathbb{R}^N),\ v_{\mathsf{n}} = 0\big\},\end{aligned} \tag{2}$$

respectively, generalizing homogeneous tangential resp. normal boundary conditions. Here, ν denotes the a.e. defined outer unit normal at Γ giving a.e. the normal resp. tangential component

$$v_{\mathsf{n}} := \nu \cdot v|_\Gamma, \quad v_{\mathsf{t}} := v|_\Gamma - v_{\mathsf{n}}\nu$$

of v on Γ. We assume additionally that Γ is decomposed into two relatively open subsets Γ_{t} and $\Gamma_{\mathsf{n}} := \Gamma \setminus \overline{\Gamma_{\mathsf{t}}}$ and introduce the vector valued H^1-Sobolev space of mixed boundary conditions $\mathring{\mathsf{H}}^1_{\mathsf{t,n}}(\Omega)$ as closure in $\mathsf{H}^1(\Omega)$ of the set of test vector fields

$$\mathring{\mathsf{C}}^\infty_{\mathsf{t,n}}(\Omega) := \big\{v|_\Omega \,:\, v \in \mathring{\mathsf{C}}^\infty(\mathbb{R}^N),\ v_{\mathsf{t}}|_{\Gamma_{\mathsf{t}}} = 0,\ v_{\mathsf{n}}|_{\Gamma_{\mathsf{n}}} = 0\big\}. \tag{3}$$

2.1 Korn's second inequality

It is well known that Korn's second inequality can easily be proved by a simple H^{-1}-argument using Nečas inequality. Let us illustrate a simple and short proof: In the sense of distributions we have e.g. for all vector fields $v \in \mathsf{L}^2(\Omega)$ that the components of $\nabla\nabla v$[1] consist only of components of $\nabla \operatorname{sym} \nabla v$, i.e.,

$$\forall\, i, j, k = 1, \ldots, N \qquad \partial_i \partial_j v_k = \partial_i \operatorname{sym}_{j,k} \nabla v + \partial_j \operatorname{sym}_{i,k} \nabla v - \partial_k \operatorname{sym}_{i,j} \nabla v, \tag{4}$$

where $\operatorname{sym}_{j,k} T := (\operatorname{sym} T)_{j,k}$. By e.g. [12, 1.1.3 Lemma] we have (for scalar functions) the Nečas estimate

$$\exists\, c > 0 \quad \forall\, u \in \mathsf{L}^2(\Omega) \quad c\,|u|_{\mathsf{L}^2(\Omega)} \leq |\nabla u|_{\mathsf{H}^{-1}(\Omega)} + |u|_{\mathsf{H}^{-1}(\Omega)} \leq (\sqrt{N}+1)|u|_{\mathsf{L}^2(\Omega)}, \tag{5}$$

where $\mathsf{H}^{-1}(\Omega) := \big(\overset{\circ}{\mathsf{H}}{}^1(\Omega)\big)'$ and e.g. by using the full $\mathsf{H}^1(\Omega)$-norm

$$|u|_{\mathsf{H}^{-1}(\Omega)} := \sup_{0 \neq \varphi \in \overset{\circ}{\mathsf{H}}{}^1(\Omega)} \frac{\langle u, \varphi\rangle_{\mathsf{L}^2(\Omega)}}{|\varphi|_{\mathsf{H}^1(\Omega)}}, \qquad |\nabla u|_{\mathsf{H}^{-1}(\Omega)} := \sup_{0 \neq \phi \in \overset{\circ}{\mathsf{H}}{}^1(\Omega)} \frac{\langle u, \operatorname{div} \phi\rangle_{\mathsf{L}^2(\Omega)}}{|\phi|_{\mathsf{H}^1(\Omega)}}.$$

For the original results of (5) see the works of Nečas, e.g. [7,8], from the 1960s.

Remark 1 Nečas' estimate (5) can be refined to

$$\begin{aligned} &\exists\, c > 0 \quad \forall\, u \in \mathsf{L}^2_0(\Omega) := \{u \in \mathsf{L}^2(\Omega) \,:\, \langle u, 1\rangle_{\mathsf{L}^2(\Omega)} = 0\} \\ &c\,|u|_{\mathsf{L}^2(\Omega)} \leq |\nabla u|_{\mathsf{H}^{-1}(\Omega)} \leq \sqrt{N}|u|_{\mathsf{L}^2(\Omega)}. \end{aligned} \tag{6}$$

The best constant $c > 0$ in (6) is also called inf-sup- or LBB-constant as by using the $\mathsf{H}^1(\Omega)$-half norm

$$c = \inf_{0 \neq u \in \mathsf{L}^2_0(\Omega)} \frac{|\nabla u|_{\mathsf{H}^{-1}(\Omega)}}{|u|_{\mathsf{L}^2(\Omega)}} = \inf_{0 \neq u \in \mathsf{L}^2_0(\Omega)} \sup_{v \in \overset{\circ}{\mathsf{H}}{}^1(\Omega)} \frac{\langle u, \operatorname{div} v\rangle_{\mathsf{L}^2(\Omega)}}{|u|_{\mathsf{L}^2(\Omega)} |\nabla v|_{\mathsf{L}^2(\Omega)}} = c_{\mathsf{LBB}}.$$

We note that the LBB-constant can be bounded from below by the inverse of the continuity constant c_A of the H^1-potential operator (often called Bogovskii operator) $A : \mathsf{L}^2_0(\Omega) \to \overset{\circ}{\mathsf{H}}{}^1(\Omega)$ with $\operatorname{div} Au = u$, i.e.,

$$\forall\, u \in \mathsf{L}^2_0(\Omega) \quad |\nabla Au|_{\mathsf{L}^2(\Omega)} \leq c_A |u|_{\mathsf{L}^2(\Omega)}.$$

[1] We denote by ∇v the transpose of the Jacobian of v and by $\nabla\nabla v$ the tensor of second derivatives of v.

This follows directly by setting $v := Au$ (note that $\nabla Au \neq 0$ for $0 \neq u \in \mathsf{L}^2_0(\Omega)$) and

$$c_{\mathsf{LBB}} \geq \inf_{0 \neq u \in \mathsf{L}^2_0(\Omega)} \frac{|u|^2_{\mathsf{L}^2(\Omega)}}{|u|_{\mathsf{L}^2(\Omega)} |\nabla Au|_{\mathsf{L}^2(\Omega)}} \geq \frac{1}{c_A}.$$

We immediately get:

Theorem 2 (Korn's second inequality) *There exists* $c > 0$ *such that for all* $v \in \mathsf{H}^1(\Omega)$

$$|\nabla v|_{\mathsf{L}^2(\Omega)} \leq c\big(|\operatorname{sym} \nabla v|_{\mathsf{L}^2(\Omega)} + |v|_{\mathsf{L}^2(\Omega)}\big).$$

Proof Let $v \in \mathsf{H}^1(\Omega)$. Combining (4) and (5) we estimate

$$\begin{aligned} |\nabla v|_{\mathsf{L}^2(\Omega)} &\leq c\left(|\nabla \nabla v|_{\mathsf{H}^{-1}(\Omega)} + |\nabla v|_{\mathsf{H}^{-1}(\Omega)}\right) \\ &\leq c\left(|\nabla \operatorname{sym} \nabla v|_{\mathsf{H}^{-1}(\Omega)} + |\nabla v|_{\mathsf{H}^{-1}(\Omega)}\right) \leq c\left(|\operatorname{sym} \nabla v|_{\mathsf{L}^2(\Omega)} + |v|_{\mathsf{L}^2(\Omega)}\right), \end{aligned}$$

showing the stated result. □

By standard mollification we see that the restrictions of $\mathring{\mathsf{C}}^\infty(\mathbb{R}^N)$-vector fields to Ω are dense in

$$\mathsf{S}(\Omega) := \{v \in \mathsf{L}^2(\Omega) \ : \ \operatorname{sym} \nabla v \in \mathsf{L}^2(\Omega)\},$$

even if Ω just has the segment property. Especially $\mathsf{H}^1(\Omega)$ is dense in $\mathsf{S}(\Omega)$. This shows immediately:

Theorem 3 (H^1-regularity) *It holds* $\mathsf{S}(\Omega) = \mathsf{H}^1(\Omega)$.

Proof Let $v \in \mathsf{S}(\Omega)$. By density, there exists a sequence $(v_n) \subset \mathsf{H}^1(\Omega)$ converging to v in $\mathsf{S}(\Omega)$. By Theorem 2 (v_n) is a Cauchy sequence in $\mathsf{H}^1(\Omega)$ converging to v, yielding $v \in \mathsf{H}^1(\Omega)$. □

Remark 4 The latter arguments show, that for any domain allowing for Nečas' estimate (5) Korn's second inequality Theorem 2 holds. In these domains we have also the H^1-regularity Theorem 3, provided that the segment property holds.

Remark 5 (5) is well known to hold also in the $\mathsf{L}^q/\mathsf{W}^{-1,q}$-setting for $1 < q < \infty$. As (4) and the mollification techniques are available for general q, it follows that Theorem 2 and Theorem 3 immediately extend to the $\mathsf{L}^q/\mathsf{W}^{1,q}/\mathsf{S}^q$-setting for all $1 < q < \infty$.

2.2 Poincaré inequality for elasticity

To apply standard solution theories for linear elasticity, such as Fredholm's alternative for bounded domains or Eidus' limiting absorption principle [5] for exterior domains, it is most important to ensure for bounded domains the compact embedding

$$\mathsf{S}(\Omega) \hookrightarrow \mathsf{L}^2(\Omega). \tag{7}$$

As long as Korn's second inequality, i.e., the continuous embedding $\mathsf{S}(\Omega) \hookrightarrow \mathsf{H}^1(\Omega)$, holds true, the compact embedding (7) follows immediately by Rellich's selection theorem, i.e., the compact embedding $\mathsf{H}^1(\Omega) \hookrightarrow \mathsf{L}^2(\Omega)$. As shown in [13], there are bounded irregular domains, more precisely bounded domains with the p-cusp property (Hölder boundaries), see [14, Definition 3] or [13, Definition 2], with $1 < p < 2$, for which Korn's second inequality fails and so the embedding $\mathsf{S}(\Omega) \subset \mathsf{H}^1(\Omega)$ by the closed graph theorem[2], but the important compact embedding (7) remains valid. More precisely, by [13, Theorem 2] the compact embedding (7) holds for bounded domains having the p-cusp property with $1 \leq p < 2$[3], and (7) implies immediately a Poincaré type inequality for elasticity by a standard indirect argument. For this we define

$$\mathsf{S}_0(\Omega) := \{v \in \mathsf{S}(\Omega) \,:\, \operatorname{sym} \nabla v = 0\} = \{v \in \mathsf{L}^2(\Omega) \,:\, \operatorname{sym} \nabla v = 0\}.$$

It is well known that even for any domain Ω

$$\mathsf{S}_0(\Omega) = \mathcal{R}$$

holds, where $\mathcal{R} := \{Sx + a \,:\, S \in \mathfrak{so} \wedge a \in \mathbb{R}^N\}$ is the space rigid motions and $\mathfrak{so} = \mathfrak{so}(N)$ the vector space of constant skew-symmetric matrices. This follows easily for $v \in \mathsf{S}_0(\Omega)$ by approximating Ω by smooth domains Ω_n, in each of which $v_n := v|_{\Omega_n}$ equals the same rigid motion $r \in \mathcal{R}$.

Theorem 6 (Poincaré inequality for elasticity) *Let Ω be bounded and possess the p-cusp property with some $1 \leq p < 2$. Then there exists $c > 0$ such that for all $v \in \mathsf{S}(\Omega) \cap \mathcal{R}^\perp$*

$$|v|_{\mathsf{L}^2(\Omega)} \leq c\, |\operatorname{sym} \nabla v|_{\mathsf{L}^2(\Omega)}.$$

Equivalently, for all $v \in \mathsf{S}(\Omega)$

$$|v - r_v|_{\mathsf{L}^2(\Omega)} \leq c\, |\operatorname{sym} \nabla v|_{\mathsf{L}^2(\Omega)}, \qquad r_v := \pi_{\mathcal{R}} v.$$

[2] The identity mapping $\mathrm{id}_{\mathsf{S}} : \mathsf{S}(\Omega) \to \mathsf{H}^1(\Omega)$ is continuous, if and only if id_{S} is closed, if and only if $\mathsf{S}(\Omega) \subset \mathsf{H}^1(\Omega)$.

[3] For $p = 1$ the 1-cusp property equals the strict cone property, which itself holds for strong Lipschitz domains.

Here and throughout the paper, we denote orthogonality in $\mathsf{L}^2(\Omega)$ by $\perp$. Moreover, $\pi_{\mathcal{R}}$ denotes the $\mathsf{L}^2(\Omega)$-orthogonal projector onto the rigid motions $\mathcal{R}$.

Proof If the assertion was wrong, there exists a sequence $(v_n) \subset \mathsf{S}(\Omega) \cap \mathcal{R}^\perp$ with $|v_n|_{\mathsf{L}^2(\Omega)} = 1$ and $|\operatorname{sym} \nabla v_n|_{\mathsf{L}^2(\Omega)} \to 0$. By (7) we can assume without loss of generality that (v_n) converges in $\mathsf{L}^2(\Omega)$ to some $v \in \mathsf{L}^2(\Omega)$. But then $v \in \mathsf{S}_0(\Omega) \cap \mathcal{R}^\perp = \{0\}$, in contradiction to $1 = |v_n|_{\mathsf{L}^2(\Omega)} \to |v|_{\mathsf{L}^2(\Omega)} = 0$. □

Under the assumptions of Theorem 6, the variational static linear elasticity problem, for $f \in \mathsf{L}^2(\Omega)$ find $v \in \mathsf{S}(\Omega) \cap \mathcal{R}^\perp$ such that

$$\forall \phi \in \mathsf{S}(\Omega) \cap \mathcal{R}^\perp \qquad \langle \operatorname{sym} \nabla v, \operatorname{sym} \nabla \phi \rangle_{\mathsf{L}^2(\Omega)} = \langle f, \phi \rangle_{\mathsf{L}^2(\Omega)},$$

is uniquely solvable with continuous resp. compact inverse $\mathsf{L}^2(\Omega) \to \mathsf{S}(\Omega)$ resp. $\mathsf{L}^2(\Omega) \to \mathsf{L}^2(\Omega)$, which shows that Fredholm's alternative holds for the corresponding reduced operators.

3 Korn's first inequality

By Rellich's selection theorem, Theorem 2 and an indirect argument we can easily prove:

Theorem 7 (Korn's first inequality without boundary conditions) *There exists $c > 0$ such that for all $v \in \mathsf{H}^1(\Omega)$ with $\nabla v \perp \mathfrak{so}$*

$$|\nabla v|_{\mathsf{L}^2(\Omega)} \leq c\, |\operatorname{sym} \nabla v|_{\mathsf{L}^2(\Omega)}. \tag{8}$$

Equivalently for all $v \in \mathsf{H}^1(\Omega)$

$$|\nabla v - S_v|_{\mathsf{L}^2(\Omega)} \leq c\, |\operatorname{sym} \nabla v|_{\mathsf{L}^2(\Omega)}, \qquad S_v := \frac{1}{|\Omega|} \operatorname{skw} \int_\Omega \nabla v.$$

Here, $S_v = \pi_{\mathfrak{so}} \nabla v$ is the $\mathsf{L}^2(\Omega)$-orthogonal projection of ∇v onto $\mathfrak{so}$.

Proof The equivalence is clear by the orthogonal projection.[4] If (8) was wrong, there exists a sequence $(v_n) \subset \mathsf{H}^1(\Omega)$ with $\nabla v_n \perp \mathfrak{so}$ and $|\nabla v_n|_{\mathsf{L}^2(\Omega)} = 1$ and

[4] We can also compute it by hand: For $v \in \mathsf{H}^1(\Omega)$ with $\nabla v \perp \mathfrak{so}$ we see

$$|S_v|^2 = \frac{1}{|\Omega|} \langle \operatorname{skw} \int_\Omega \nabla v, S_v \rangle = \frac{1}{|\Omega|} \langle \nabla v, S_v \rangle_{\mathsf{L}^2(\Omega)} = 0$$

since $S_v \in \mathfrak{so}$. For $v \in \mathsf{H}^1(\Omega)$ and $T \in \mathfrak{so}$ we have

$$\langle \nabla v - S_v, T \rangle_{\mathsf{L}^2(\Omega)} = \int_\Omega \langle \operatorname{skw} \nabla v, T \rangle - \langle S_v, T \rangle_{\mathsf{L}^2(\Omega)} = \langle \int_\Omega \operatorname{skw} \nabla v, T \rangle - |\Omega| \langle S_v, T \rangle = 0,$$

implying $v + s_v \in \mathsf{H}^1(\Omega)$ with $\nabla(v + s_v) = (\nabla v - S_v) \perp \mathfrak{so}$ and $\operatorname{sym} \nabla(v + s_v) = \operatorname{sym}(\nabla v - S_v) = \operatorname{sym} \nabla v$, where $s_v(x) := S_v x$.

$|\operatorname{sym}\nabla v_n|_{\mathsf{L}^2(\Omega)} \to 0$. Without loss of generality we can assume $v_n \perp \mathbb{R}^N$. By Poincare's inequality (v_n) is bounded in $\mathsf{H}^1(\Omega)$. Thus, by Rellich's selection theorem we can assume without loss of generality that (v_n) converges in $\mathsf{L}^2(\Omega)$ to some $v \in \mathsf{L}^2(\Omega)$. By Theorem 2 (v_n) is a Cauchy sequence in $\mathsf{H}^1(\Omega)$. Therefore (v_n) converges in $\mathsf{H}^1(\Omega)$ to $v \in \mathsf{H}^1(\Omega) \cap (\mathbb{R}^N)^\perp$ with $\operatorname{sym}\nabla v = 0$ and $\nabla v \perp \mathfrak{so}$. But then ∇v is even constant and belongs to $\mathfrak{so}$. Hence $\nabla v = 0$[5] in contradiction to $1 = |\nabla v_n|_{\mathsf{L}^2(\Omega)} \to |\nabla v|_{\mathsf{L}^2(\Omega)} = 0$. □

Using Poincare's inequality we immediately obtain:

Corollary 8 (Korn's first inequality without boundary conditions) *There exists $c > 0$ such that for all $v \in \mathsf{H}^1(\Omega) \cap (\mathbb{R}^N)^\perp$ with $\nabla v \perp \mathfrak{so}$*

$$|v|_{\mathsf{H}^1(\Omega)} \leq c\,|\operatorname{sym}\nabla v|_{\mathsf{L}^2(\Omega)}.$$

In order to prove Korn's first inequality in $\mathring{\mathsf{H}}^1_{\mathsf{t},\mathsf{n}}(\Omega)$ we need a Poincaré type estimate on this space. It should be noted that in general mixed boundary conditions are not sufficient to rule out a kernel of the gradient operator. For example, consider the cube $\Omega := (0,1)^3 \subset \mathbb{R}^3$ with Γ_t being the union of the top and bottom together with the constant vector field $r(x) := (0, 0, 1)^t$. Then $r \in \mathring{\mathsf{H}}^1_{\mathsf{t},\mathsf{n}}(\Omega)$. On this account, such constant vector fields have to be excluded separately.

Lemma 9 (Poincaré inequality with tangential or normal boundary conditions) *There exists $c > 0$ such that for all $v \in \mathring{\mathsf{H}}^1_{\mathsf{t},\mathsf{n}}(\Omega) \cap \big(\mathring{\mathsf{H}}^1_{\mathsf{t},\mathsf{n}}(\Omega) \cap \mathbb{R}^N\big)^\perp$*

$$|v|_{\mathsf{L}^2(\Omega)} \leq c\,|\nabla v|_{\mathsf{L}^2(\Omega)}.$$

Proof If the assertion was wrong, there exists some sequence $(v_n) \subset \mathring{\mathsf{H}}^1_{\mathsf{t},\mathsf{n}}(\Omega) \cap \big(\mathring{\mathsf{H}}^1_{\mathsf{t},\mathsf{n}}(\Omega) \cap \mathbb{R}^N\big)^\perp$ with $|v_n|_{\mathsf{L}^2(\Omega)} = 1$ and $|\nabla v_n|_{\mathsf{L}^2(\Omega)} \to 0$. Thus, by Rellich's selection theorem we can assume without loss of generality that (v_n) converges in $\mathsf{L}^2(\Omega)$ to some $v \in \mathsf{L}^2(\Omega)$. Hence, (v_n) is a Cauchy sequence in $\mathsf{H}^1(\Omega)$ and converges in $\mathsf{H}^1(\Omega)$ to $v \in \mathring{\mathsf{H}}^1_{\mathsf{t},\mathsf{n}}(\Omega) \cap \big(\mathring{\mathsf{H}}^1_{\mathsf{t},\mathsf{n}}(\Omega) \cap \mathbb{R}^N\big)^\perp$ with $\nabla v = 0$. Therefore, v is a constant in $\mathring{\mathsf{H}}^1_{\mathsf{t},\mathsf{n}}(\Omega) \cap \mathbb{R}^N$ and must vanish in contradiction to $1 = |v_n|_{\mathsf{L}^2(\Omega)} \to |v|_{\mathsf{L}^2(\Omega)} = 0$. □

As an easy consequence we get

Corollary 10 $\nabla \mathring{\mathsf{H}}^1_{\mathsf{t},\mathsf{n}}(\Omega)$ *is a closed subspace of* $\mathsf{L}^2(\Omega)$.

[5] We note that even $v \in \mathbb{R}^N$ holds and thus $v = 0$.

Proof Let $(v_n) \subset \overset{\circ}{\mathsf{H}}^1_{\mathsf{t,n}}(\Omega)$ such that $\nabla v_n \to G \in \mathsf{L}^2(\Omega)$ in $\mathsf{L}^2(\Omega)$. Without loss of generality we can assume $(v_n) \subset \overset{\circ}{\mathsf{H}}^1_{\mathsf{t,n}}(\Omega) \cap \big(\overset{\circ}{\mathsf{H}}^1_{\mathsf{t,n}}(\Omega) \cap \mathbb{R}^N\big)^\perp$, otherwise we replace v_n by

$$\tilde{v}_n := v_n - \pi_{\overset{\circ}{\mathsf{H}}^1_{\mathsf{t,n}}(\Omega)\cap\mathbb{R}^N} v_n \in \overset{\circ}{\mathsf{H}}^1_{\mathsf{t,n}}(\Omega) \cap \big(\overset{\circ}{\mathsf{H}}^1_{\mathsf{t,n}}(\Omega) \cap \mathbb{R}^N\big)^\perp,$$

where $\pi_{\overset{\circ}{\mathsf{H}}^1_{\mathsf{t,n}}(\Omega)\cap\mathbb{R}^N}$ is the orthogonal projector onto $\overset{\circ}{\mathsf{H}}^1_{\mathsf{t,n}}(\Omega) \cap \mathbb{R}^N$. Because of Lemma 9 (v_n) is a Cauchy sequence in $\overset{\circ}{\mathsf{H}}^1_{\mathsf{t,n}}(\Omega)$, which converges in $\mathsf{H}^1(\Omega)$ to $v \in \overset{\circ}{\mathsf{H}}^1_{\mathsf{t,n}}(\Omega)$. Hence, $G \leftarrow \nabla v_n \to \nabla v \in \nabla \overset{\circ}{\mathsf{H}}^1_{\mathsf{t,n}}(\Omega)$. □

To exclude the kernel of the sym ∇-operator on $\overset{\circ}{\mathsf{H}}^1_{\mathsf{t,n}}(\Omega)$, we define

$$\mathcal{K} := \{\nabla v \,:\, v \in \overset{\circ}{\mathsf{H}}^1_{\mathsf{t,n}}(\Omega),\ \operatorname{sym} \nabla v = 0\} = \nabla \big(\mathcal{R} \cap \overset{\circ}{\mathsf{H}}^1_{\mathsf{t,n}}(\Omega)\big) = \mathfrak{so} \cap \nabla \overset{\circ}{\mathsf{H}}^1_{\mathsf{t,n}}(\Omega).$$

Theorem 11 (Korn's first inequality with tangential or normal boundary conditions) *There exists $c > 0$ such that for all $v \in \overset{\circ}{\mathsf{H}}^1_{\mathsf{t,n}}(\Omega)$ with $\nabla v \perp \mathcal{K}$*

$$|\nabla v|_{\mathsf{L}^2(\Omega)} \leq c\, |\operatorname{sym} \nabla v|_{\mathsf{L}^2(\Omega)}. \tag{9}$$

Equivalently, for all $v \in \overset{\circ}{\mathsf{H}}^1_{\mathsf{t,n}}(\Omega)$

$$|\nabla v - \pi_{\mathcal{K}} \nabla v|_{\mathsf{L}^2(\Omega)} \leq c\, |\operatorname{sym} \nabla v|_{\mathsf{L}^2(\Omega)}.$$

Here, $\pi_{\mathcal{K}}$ denotes the $\mathsf{L}^2(\Omega)$-orthogonal projector onto $\mathcal{K}$.

Proof Equivalence is again clear by the orthogonal projection. If (9) was wrong, there exists a sequence $(v_n) \subset \overset{\circ}{\mathsf{H}}^1_{\mathsf{t,n}}(\Omega)$ with $\nabla v_n \perp \mathcal{K}$ and $|\nabla v_n|_{\mathsf{L}^2(\Omega)} = 1$ and $|\operatorname{sym} \nabla v_n|_{\mathsf{L}^2(\Omega)} \to 0$. Without loss of generality we can assume $(v_n) \subset \overset{\circ}{\mathsf{H}}^1_{\mathsf{t,n}}(\Omega) \cap \big(\overset{\circ}{\mathsf{H}}^1_{\mathsf{t,n}}(\Omega) \cap \mathbb{R}^N\big)^\perp$. By Lemma 9 (v_n) is bounded in $\mathsf{H}^1(\Omega)$, and thus, using Rellich's selection theorem, we can assume without loss of generality that (v_n) converges in $\mathsf{L}^2(\Omega)$ to some $v \in \mathsf{L}^2(\Omega)$. By Theorem 2 (v_n) is a Cauchy sequence in $\mathsf{H}^1(\Omega)$. Therefore, (v_n) converges in $\mathsf{H}^1(\Omega)$ to $v \in \overset{\circ}{\mathsf{H}}^1_{\mathsf{t,n}}(\Omega)$ with $\operatorname{sym} \nabla v = 0$ and $\nabla v \perp \mathcal{K}$. But then, ∇v is even a constant in $\mathfrak{so}$, i.e., $\nabla v \in \mathcal{K}$, in contradiction to $1 = |\nabla v_n|_{\mathsf{L}^2(\Omega)} \to |\nabla v|_{\mathsf{L}^2(\Omega)} = 0$. □

Remark 12 Similar to Remark 5, all the results from Theorems 7 to 11 extend to the $\mathsf{L}^q/\mathsf{W}^{1,q}$-setting for all $1 < q < \infty$ with the obvious modifications. The same holds true for all results presented in the subsequent sections.

3.1 Discussing the set $\mathcal{K}$

In this section we shall discuss which combinations of domains Ω and boundary parts Γ_{t} allow for a non-constant rigid motion $r \in \overset{\circ}{\mathsf{H}}{}^1_{\mathsf{t,n}}(\Omega) \cap \mathcal{R}$, i.e., $\mathcal{K} \neq \{0\}$. We start with the case $\Gamma_{\mathsf{t}} = \Gamma$, i.e, with the full tangential boundary condition.

Theorem 13 *If* $\Gamma_{\mathsf{t}} = \Gamma$, *then* $\mathcal{K} = \{0\}$ *and there exists a constant* $c > 0$ *such that for all* $v \in \overset{\circ}{\mathsf{H}}{}^1_{\mathsf{t}}(\Omega)$

$$|\nabla v|_{\mathsf{L}^2(\Omega)} \leq c\, |\operatorname{sym} \nabla v|_{\mathsf{L}^2}.$$

Proof We give a proof by contradiction. Assume $r \in \mathcal{R} \cap \overset{\circ}{\mathsf{H}}{}^1_{\mathsf{t}}(\Omega)$ and $r \neq 0$. Let us define the null space $\mathcal{N}_r := \{x \in \mathbb{R}^N : r(x) = 0\}$. Then $\mathcal{N}_r$ is an empty set or an affine plane in $\mathbb{R}^N$ with dimension $d_{\mathcal{N}_r} \leq N - 2$. We recall that ν is the outer unit normal at Γ defined a.e. on Γ w.r.t. the $(N-1)$-dimensional Lebesgue measure. Since r is normal on Γ, we conclude for almost all $x \in \Gamma \setminus \mathcal{N}_r$

$$\nu(x) = \pm \frac{r(x)}{|r(x)|}. \tag{10}$$

Because Ω is locally on one side of the boundary Γ, the unit normal ν cannot change sign in (10) in any connected component of $\Gamma \setminus \mathcal{N}_r$. But since $d_{\mathcal{N}_r} \leq N-2$, it follows that $\Gamma \setminus \mathcal{N}_r$ is connected, and w.l.o.g.

$$\nu(x) = \frac{r(x)}{|r(x)|} \qquad \text{for almost all } x \in \Gamma \setminus \mathcal{N}_r. \tag{11}$$

As $\Gamma \cap \mathcal{N}_r$ has measure zero, we can replace $\Gamma \setminus \mathcal{N}_r$ by Γ in (11). With Gauß' theorem we conclude

$$0 = \int_\Omega \operatorname{div} r = \int_\Gamma \nu \cdot r = \int_\Gamma |r| > 0,$$

a contradiction. □

Next we turn to the full normal boundary condition, i.e. $\Gamma_{\mathsf{t}} = \emptyset$. In [3] it is proved that for smooth bounded domains $\Omega \subset \mathbb{R}^N$ Korn's first inequality holds for all $v \in \overset{\circ}{\mathsf{H}}{}^1_{\mathsf{n}}(\Omega)$, i.e. $\mathcal{K} = \{0\}$, if and only if Ω is not axisymmetric. Furthermore an explicit upper bound on the constant is given.[6] In that contribution and here axisymmetry is defined as follows.

Definition 14 Ω is called axisymmetric if there is a non-trivial rigid motion $r \in \mathcal{R}$ tangential to the boundary Γ of Ω, i.e. $0 \neq r \in \overset{\circ}{\mathsf{H}}{}^1_{\mathsf{n}}(\Omega)$.

[6] In [3] a C^1-boundary is assumed, but it seems that for the proof of [3, Lemma 4] actually a C^2-boundary is needed in order to guaranty H^1-regularity of $\nabla \phi$, where ϕ is the solution of [3, (14)].

In a more elementary and canonical approach in $\mathbb{R}^3$ a domain is called axisymmetric w.r.t. an axis a if it is a body of rotation around this axis. In order to show that in $\mathbb{R}^3$ both concepts coincide for bounded Lipschitz domains, we make use of the invariance of a Lipschitz boundary under the flow of a tangential vector field.

Proposition 15 *Let $\Omega \subset \mathbb{R}^N$ be a (not necessarily bounded) domain with a (strong) Lipschitz boundary Γ and $r : \mathbb{R}^N \to \mathbb{R}^N$ a locally Lipschitz continuous vector field that is tangential on Γ a.e. w.r.t. the $(N-1)$-dimensional Lebesgue measure on Γ. Let $p \in \Gamma$ and let $t \mapsto \gamma(t)$ the maximal solution of the ordinary differential equation*

$$\dot{\gamma} = r(\gamma), \quad \gamma(0) = p \tag{12}$$

existing on the interval I_p. Then for all $t \in I_p$

$$\gamma(t) \in \Gamma. \tag{13}$$

This proposition is a variant of Nagumo's invariance theorem, see [1, Theorem 2, p. 180], c.f. also [6], where the tangential condition on r is defined in terms of a so called 'Bouligand contingent cone'. As we need this statement for a Lipschitz boundary we give a self-contained proof in the Appendix.

The next lemma states that for bounded domains in $\mathbb{R}^3$ both definitions of axisymmetry coincide. An elementary proof is provided in the appendix.

Lemma 16 *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain.*

(i) Assume $\sigma, b \in \mathbb{R}^3$, $|\sigma| = 1$ and let $g = \{\lambda\sigma + b : \lambda \in \mathbb{R}\}$. Assume that Ω is axisymmetric w.r.t. the axis g. Then the vector field r with $r(x) := \sigma \wedge (x - b)$ is a rigid motion, which is tangential at Γ, i.e. $r \in \mathcal{R} \cap \overset{\circ}{\mathsf{H}}{}^1_{\mathsf{n}}(\Omega)$.

(ii) Let $r \in \mathcal{R} \cap \overset{\circ}{\mathsf{H}}{}^1_{\mathsf{n}}(\Omega)$, $r(x) = \omega\sigma \wedge x + b$ for all $x \in \mathbb{R}^3$ with $\sigma, b \in \mathbb{R}^3$, $|\sigma| = 1$ and $\omega \in \mathbb{R}$. Then $\omega \neq 0$, $\langle b, \sigma\rangle = 0$, and Ω is axisymmetric w.r.t. the axis $g = \left\{\lambda\sigma + \frac{1}{w}\sigma \wedge b : \lambda \in \mathbb{R}\right\}$.

Remark 17 There are rigid motions tangential to the boundary of some unbounded domains in $\mathbb{R}^3$, which do not exhibit any axis of symmetry. Consider, for example, a domain Ω built from a plane square which simultaneously is lifted along and rotated around the axis perpendicular to it, e.g.

$$\Omega := \left\{\left(x_1\cos(t) - x_2\sin(t), x_1\sin(t) + x_2\cos(t), t\right)^t : |x_1| + |x_2| < 1,\ t \in \mathbb{R}\right\}.$$

Then $r(x) := (-x_2, x_1, 1)^t$ is tangential to Γ.

Using Definition 14, Korn's first inequality for normal boundary conditions is more or less obvious.

Theorem 18 *Let $\Gamma_{\mathsf{t}} = \emptyset$. Then Korn's first inequality holds for all $v \in \overset{\circ}{\mathsf{H}}{}^1_{\mathsf{n}}(\Omega)$, if and only if $\mathcal{K} = \{0\}$, if and only if Ω is not axisymmetric.*

Proof The first 'if and only if' is just the assertion of Theorem 11. For the second 'if and only if' according to the definition of axisymmetry the only remaining issue is to prove that there is no constant vector field tangential to a bounded Lipschitz domain (in that case we would have a non-trivial rigid motion, which gives no contribution to $\mathcal{K}$). Assume that a constant vector $0 \neq a \in \mathbb{R}^N$ tangential to Γ exists, i.e. $a \in \overset{\circ}{\mathsf{H}}{}^1_{\mathsf{n}}(\Omega)$, and let $\hat{x} \in \Gamma$. Then according to Proposition 15 the unbounded curve $t \mapsto \hat{x} + ta$ would remain in Γ, which contradicts the boundedness of Ω. □

Remark 19 The latter proof shows that a bounded domain is axisymmetric if and only if there is a non-constant rigid motion tangential to the boundary.

For mixed boundary conditions there are domains of rather special type with $\mathcal{K} \neq \{0\}$. Consider, for example, a half cylinder

$$\Omega := \left\{ x \in \mathbb{R}^3 \,:\, x_1 > 0,\ x_1^2 + x_2^2 < 1,\ 0 < x_3 < 1 \right\},$$

or more generally, the domain

$$\Omega := \left\{ (r\cos\phi, r\sin\phi, x_3)^t \,:\, \phi_1 < \phi < \phi_2,\ 0 < x_3 < 1,\ 0 < r < h(x_3) \right\}$$

with $\Gamma_{\mathsf{t}} := \Gamma \cap \left\{ (r\cos\phi_{1/2}, r\sin\phi_{1/2}, x_3)^t \,:\, 0 \leq r,\ 0 < x_3 < 1 \right\}$ and for some positive Lipschitz function $h : \mathbb{R} \to \mathbb{R}$ and some $-\pi < \phi_1 < \phi_2 < \pi$. Define $r(x) := (-x_2, x_1, 0)^t$. Then r is a rigid motion and $r \in \overset{\circ}{\mathsf{H}}{}^1_{\mathsf{t,n}}(\Omega)$. In the next theorem we will show that in $\mathbb{R}^3$ all bounded domains Ω with $\mathcal{K} \neq \{0\}$ are compositions of subdomains of this kind.

Theorem 20 *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain and let $\emptyset \neq \Gamma_{\mathsf{t}} \neq \Gamma$. Assume that there is a non-constant rigid motion $r \in \mathcal{R} \cap \overset{\circ}{\mathsf{H}}{}^1_{\mathsf{t,n}}(\Omega)$, $r(x) = \omega\,\sigma \wedge x + b$ for all $x \in \mathbb{R}^3$ with $\omega \in \mathbb{R}$ and $|\sigma| = 1$. Define $g_r \subset \mathbb{R}^3$ by $g_r := \left\{ \lambda\,\sigma + \frac{1}{\omega}\sigma \wedge b \,:\, \lambda \in \mathbb{R} \right\}$. Then $\langle \sigma, b \rangle = 0$, Γ_{t} is a subset of a union of affine planes, where each of these planes contains g_r. Every connected component of Γ_{n} is a subset of a surface which is axisymmetric w.r.t. g_r.*

By this theorem the aforementioned cube, i.e. $\Omega = (0,1)^3 \subset \mathbb{R}^3$ with Γ_{t} being the union of the top and bottom faces, has a trivial kernel $\mathcal{K} = \{0\}$, which means Korn's first inequality Theorem 11 holds on $\overset{\circ}{\mathsf{H}}{}^1_{\mathsf{t,n}}(\Omega)$, while Poincaré's inequality Lemma 9 only holds on $\overset{\circ}{\mathsf{H}}{}^1_{\mathsf{t,n}}(\Omega) \cap \left((0,0,1)^t \right)^{\perp}$.

Proof First we note that the scalar-product $\langle \sigma, b \rangle$ is independent of the chosen Cartesian coordinates, i.e. if we choose another positively oriented Euclidian coordinate system (y_1, y_2, y_3) and represent the vector field r by means of the y-coordinates, then there exist vectors $\sigma_y, b_y \in \mathbb{R}^3$ with $|\sigma_y| = 1$ and $r(y) = \omega\,\sigma_y \wedge y + b_y$ for all $y \in \mathbb{R}^3$. Furthermore $\langle \sigma_y, b_y \rangle = \langle \sigma, b \rangle$. In the same way the representation of the axis g_r associated to r is independent of the Cartesian coordinates chosen; in y-coordinates we have $g_r = \left\{ \lambda\,\sigma_y + \frac{1}{\omega}\sigma_y \wedge b_y \,:\, \lambda \in \mathbb{R} \right\}$.

Suppose $r \in \mathcal{R} \cap \overset{\circ}{\mathsf{H}}{}^1_{\mathsf{t},\mathsf{n}}(\Omega)$ and that r is not constant. We fix some $p \in \Gamma_{\mathsf{t}}$ together with a neighborhood $U \subset \mathbb{R}^3$ of p, an open subset $V \subset \mathbb{R}^2$, Euclidian coordinates $(x_1, x_2, x_3) = (x', x_3)$ and a Lipschitz map $h : V \subset \mathbb{R}^2 \to \mathbb{R}$, such that for all $x \in U$ we have $x = (x', x^3) \in \Gamma_{\mathsf{t}}$ if and only if $x^3 = h(x')$. Since r is normal and by Rademacher's theorem, we have

$$r(x', h(x')) = f(x')\big(\nabla_{x'} h(x'), -1\big)^t \tag{14}$$

with some function $f : V \subset \mathbb{R}^2 \to \mathbb{R}$ a.e. in V.

In x-coordinates r can be represented by $r(x) = \omega\sigma \wedge x + b$ with some $b, \sigma \in \mathbb{R}^3$, $|\sigma| = 1$ and $0 \neq \omega \in \mathbb{R}$. From (14) we conclude

$$b_1 + \omega\sigma_2 h(x') - \omega\sigma_3 x_2 = f(x')\, \partial_1 h(x'), \tag{15}$$

$$b_2 + \omega\sigma_3 x_1 - \omega\sigma_1 h(x') = f(x')\, \partial_2 h(x'), \tag{16}$$

$$b_3 + \omega\sigma_1 x_2 - \omega\sigma_2 x_1 = -f(x'). \tag{17}$$

We differentiate (in the sense of distributions) (15) w.r.t. x_2 and (16) w.r.t. x_1, compute the difference as well as the sum of the resulting equations, and conclude using (17)

$$\sigma_3 = \sigma_1\, \partial_1 h + \sigma_2\, \partial_2 h, \tag{18}$$

$$0 = f\, \partial_1 \partial_2 h. \tag{19}$$

Differentiating (15) w.r.t. x_1 and (16) w.r.t. x_2 yields

$$f\, \partial_1^2 h = f\, \partial_2^2 h = 0. \tag{20}$$

Now we multiply (15) by σ_1, (16) by σ_2, equate the resulting equations for $\sigma_1\sigma_2 h$, use (17), (18), and obtain

$$0 = \langle b, \sigma \rangle. \tag{21}$$

From (19), (20) we conclude that $\nabla_{x'} h$ is constant on connected components of $V \cap \{f \neq 0\}$. Therefore, h is an affine function on each part and continuous on the whole of V. Note that $\{f = 0\}$ is a subset of the line $\mathcal{N}_{\sigma,b} := \big\{x' \in \mathbb{R}^2 \,:\, b_3 + \omega\sigma_1 x_2 - \omega\sigma_2 x_1 = 0\big\}$. Now we extend the affine function from one connected component of $V \cap \{f \neq 0\}$ to $\mathbb{R}^2$ and call the resulting affine function $\tilde{h}$. Because of (18) the plane $\mathcal{E}_{\tilde{h}} := \big\{(x', \tilde{h}(x')) \,:\, x' \in \mathbb{R}^2\big\}$ is collinear to g_r. Recalling $\langle \sigma, b \rangle = 0$, it is straightforward to check that g_r is the affine kernel of r. Now we use this fact together with the collinearity of $\mathcal{E}_{\tilde{h}}$ and g_r in order to prove $g_r \subset \mathcal{E}_{\tilde{h}}$. It is sufficient to show that $\mathcal{E}_{\tilde{h}} \cap \{r = 0\}$ is not void. But in view of (17) and (14) this is obvious.

Now let $p \in \Gamma_{\mathsf{n}}$. Since $\langle \sigma, b \rangle = 0$, the solutions γ of $\dot{\gamma} = r(\gamma)$ are circles, contained in planes perpendicular to g_r and with centers on g_r (See also the computations in the proof of Lemma 16). Hence, applying Proposition 15, every connected component is a subset of some hyper surface being axisymmetric w.r.t. g_r. □

Appendix

Proof of Proposition 15 Clearly, it is sufficient to prove the invariance locally. Since Γ is Lipschitz, after rotation there is a neighborhood $U = V \times I$ of p with $V \subset \mathbb{R}^{N-1}$, $I \subset \mathbb{R}$, orthonormal coordinates $(x^1, \ldots, x^N) = (x', x^N) \in V \times I$, a point $x'_0 \in V$ and a Lipschitz continuous function $h : V \to I$ such that $p = (x'_0, h(x'_0))$, and for all $x \in U$ we have $x \in \Gamma$ iff $x^N = h(x')$. By Rademacher's theorem h is differentiable a.e. with respect to the $(N-1)$-dimensional Lebesgue measure on V, and $\nabla_{x'} h \in \mathsf{L}^\infty(V)$. Furthermore, the set of the $N-1$ vectors

$$t_1(x') := \big(1, 0, \ldots, 0, \partial_1 h(x')\big)^t, \ \ldots \ , t_{N-1}(x') := \big(0, \ldots, 0, 1, \partial_{N-1} h(x')\big)^t$$

gives a basis of the tangential space of Γ in the point $(x', h(x'))$ for almost all $x' \in V$. Therefore, on $\Gamma \cap U$ we have two representations of the vector field r, one representation in the coordinate vectors of $x^1, \ldots, x^N$ holding on the whole of U,

$$r(x) = r_U(x) = \big(r_U^1(x), \ldots, r_U^N(x)\big)^t,$$

and the functions r_U^i, $i = 1, \ldots, N$, are Lipschitz continuous functions on U. On the other hand, for almost all $x' \in V$

$$r(x', h(x')) = r_V^1(x') t_1(x') + \cdots + r_V^{N-1}(x') t_{N-1}(x').$$

We define $r_V := (r_V^1, \ldots, r_V^{N-1})^t$. Comparison yields a.e. on V and for all $i = 1, \ldots, N-1$

$$r_U^i(x', h(x')) = r_V^i(x'). \tag{22}$$

Hence, r_V is Lipschitz continuous on V. Furthermore,

$$\begin{aligned} r_U^N(x', h(x')) &= r_V^1(x')\, \partial_1 h(x') + \cdots + r_V^{N-1}(x')\, \partial_{N-1} h(x') \\ &= r_V(x') \cdot \nabla_{x'} h(x') \end{aligned} \tag{23}$$

holds for almost all $x' \in V$. Since h is Lipschitz on V and r_U^N is Lipschitz on U, $r_V \cdot \nabla_{x'} h$ is also Lipschitz on V. Now we define the flow of r_V: For $x' \in V$ we set $\psi(\,\cdot, x')$ as the solution of the ordinary differential equation

$$\dot{\psi}(t, x') = r_V\big(\psi(t, x')\big), \quad \psi(0, x') = x'. \tag{24}$$

Since r_V is Lipschitz on V, we can restrict the flow such that for some $\epsilon > 0$ and some neighborhood $\bar{V} \subset V$ of x'_0 the solution ψ is Lipschitz continuous on $(-\epsilon, \epsilon) \times \bar{V}$. Next we lift up this flow to Γ and define

$$\gamma_V(t) := \big(\psi(t, x'_0), h(\psi(t, x'_0))\big)^t.$$

By definition $\gamma_V(0) = p$ and $\gamma_V(t) \in \Gamma$ for all $t \in (-\epsilon, \epsilon)$.

In the next step we have to prove that γ_V is also a solution of (12) on $(-\epsilon, \epsilon)$. With regard to (22) it only remains to prove that the mapping $t \mapsto h(\psi(t, x_0'))$ is classically differentiable with derivative $\partial_t \big(h(\psi(t, x_0'))\big) = r_U^N\big(\psi(t, x_0'), h(\psi(t, x_0'))\big)$. We denote the l-dimensional Lebesgue measure by $\mathcal{L}^l$. For all $t \in (-\epsilon, \epsilon)$ it holds that $\psi(t, \cdot)$ is a bi-Lipschitz homeomorphism with inverse Lipschitz transformation $\psi(t, \cdot)^{-1} = \psi(-t, \cdot)$. Therefore, if $\mathcal{L}^{N-1}(\psi(t, \cdot)(\tilde{V})) = 0$ for some set $\tilde{V} \subset \bar{V}$, then also $\mathcal{L}^{N-1}(\tilde{V}) = 0$, because $\tilde{V} = \psi(-t, \cdot)\big(\psi(t, \cdot)(\tilde{V})\big)$. Fix a measurable set $V_0 \subset V$ such that $\mathcal{L}^{N-1}(V_0) = 0$ and h is classically differentiable for every $x' \in V \setminus V_0$. Let us define

$$W_0 := \big\{(t, x) \in (-\epsilon, \epsilon) \times \bar{V} \;:\; \psi(t, x) \in V_0\big\}.$$

Then W_0 is measurable and using Tonelli's and Fubini's theorems and the change of variable formula we obtain

$$\mathcal{L}^N(W_0) = \int_{(-\epsilon,\epsilon)\times\bar{V}} \mathbf{1}_{W_0} \le c \int_{(-\epsilon,\epsilon)} \int_{V_0} 1 = 0.$$

Therefore, and since ψ is differentiable w.r.t. t everywhere, we have by using (23)

$$\partial_t\, h(\psi(t, x')) = \nabla h(\psi(t, x')) \cdot \partial_t\, \psi(t, x') = r_U^N\big(\psi(t, x'), h(\psi(t, x'))\big) \tag{25}$$

for almost all $(t, x') \in (-\epsilon, \epsilon) \times \bar{V}$. Consequently this formula holds in the distributional sense. Because $h \circ \psi$ is continuous and its distributional derivative w.r.t. t is also continuous, it is also differentiable w.r.t. t in the classical sense. This can be seen as follows: We define

$$v(t, x') := h(\psi(0, x')) + \int_0^t r^N\big(\psi(\tau, x'), h(\psi(\tau, x'))\big)\, \mathrm{d}\tau.$$

The vector field v is classically differentiable w.r.t. t and $\partial_t\, v(t, x') = r_U^N\big(\psi(t, x'), h(\psi(t, x'))\big)$ holds for all $(t, x') \in (-\epsilon, \epsilon) \times \bar{V}$. Furthermore, for all $\phi \in \mathring{\mathsf{C}}^\infty\big((-\epsilon, \epsilon) \times \bar{V}\big)$

$$\int_{(-\epsilon,\epsilon)\times\bar{V}} (v - h \circ \psi)\, \partial_t\, \phi = 0.$$

This yields $h \circ \psi(t, x') = v(t, x') + w(x')$. Since for all $x' \in \bar{V}$ we have $h \circ \psi(0, x') = v(0, x')$, we finally conclude $w = 0$ on $\bar{V}$ and hence $v = h \circ \psi$. □

Proof of Lemma 16 For (i) we choose $\sigma_1, \sigma_2 \in \mathbb{R}^3$ such that the set $\{\sigma_1, \sigma_2, \sigma\}$ gives a positively oriented orthonormal basis of $\mathbb{R}^3$. Let $x \in \Gamma$ and define $d := \operatorname{dist}(g, x)$. Since Ω is axisymmetric w.r.t. g, for all $t \in \mathbb{R}$

$$\gamma(t) := \langle x, \sigma\rangle\sigma + \big(\langle b, \sigma_1\rangle + d\cos(t)\big)\sigma_1 + \big(\langle b, \sigma_2\rangle + d\sin(t)\big)\sigma_2 \in \Gamma.$$

Therefore, $\dot{\gamma}(t)$ is a tangential vector at Γ located in x. On the other hand

$$\begin{aligned} r(x) &= \sigma \wedge (x-b) = \sigma_2\langle x-b,\sigma_1\rangle - \sigma_1\langle x-b,\sigma_2\rangle \\ &= \sigma_2\big\langle\big(\langle b,\sigma_1\rangle + d\cos(t)\big)\sigma_1 - b,\sigma_1\big\rangle - \sigma_1\big\langle\big(\langle b,\sigma_2\rangle + d\sin(t)\big)\sigma_2 - b,\sigma_2\big\rangle \\ &= \sigma_2 d\cos(t) - \sigma_1 d\sin(t) = \dot{\gamma}(t), \end{aligned}$$

which yields $r \in \overset{\circ}{\mathsf{H}}{}^1_{\mathsf{n}}(\Omega) \cap \mathcal{R}$.

No we turn to the proof of (ii). If $\omega = 0$ then $x(t) = x_0 + tb$ remains in Γ for all t if $x_0 \in \Gamma$ (Proposition 15) and Ω would be unbounded. Therefore, we have $\omega \neq 0$. We choose again $\sigma_1, \sigma_2 \in \mathbb{R}^3$ such that the set $\{\sigma_1, \sigma_2, \sigma\}$ gives an orthonormal basis of $\mathbb{R}^3$ with positive orientation. The solution of the ordinary differential equation system

$$\begin{aligned} \dot{s}_1 &= -\omega s_2 + \langle b,\sigma_1\rangle, & \dot{s}_2 &= \omega s_1 + \langle b,\sigma_2\rangle, & \dot{s}_3 &= \langle b,\sigma\rangle, \\ s_1(0) &= \langle \hat{x},\sigma_1\rangle, & s_2(0) &= \langle \hat{x},\sigma_2\rangle, & s_3(0) &= \langle \hat{x},\sigma\rangle \end{aligned}$$

is given by

$$\begin{aligned} s_1(t) &= c_1\cos(\omega t) - c_2\sin(\omega t) - \frac{1}{\omega}\langle b,\sigma_2\rangle, \\ s_2(t) &= c_1\sin(\omega t) + c_2\cos(\omega t) + \frac{1}{\omega}\langle b,\sigma_1\rangle, \\ s_3(t) &= \langle \hat{x},\sigma\rangle + t\langle b,\sigma\rangle, \end{aligned}$$

where c_1 and c_2 are uniquely defined by the initial conditions on s_1 and s_2. Then

$$x(t) := s_1(t)\sigma_1 + s_2(t)\sigma_2 + s_3(t)\sigma$$

is the unique solution of

$$\dot{x} = r(x), \quad x(0) = \hat{x}.$$

Due to Proposition 15 and since $r \in \overset{\circ}{\mathsf{H}}{}^1_{\mathsf{n}}(\Omega)$, we have $x(t) \in \Gamma$ for all $t \in \mathbb{R}$. Because Ω is bounded, we conclude $\langle b,\sigma\rangle = 0$. Therefore, the trajectory $t \mapsto x(t)$ is a circle lying in a plane perpendicular to σ with center

$$-\frac{1}{\omega}\langle b,\sigma_2\rangle\sigma_1 + \frac{1}{\omega}\langle b,\sigma_1\rangle\sigma_2 + \langle \hat{x},\sigma\rangle\sigma.$$

Consequently, Ω is axisymmetric w.r.t. to g. □

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