Essen

$\sum a_n e^{-\lambda_n s}$ general Dirichlet series, where

- (a_n) Dirichlet coeff
- $\lambda = (\lambda_n)$ frequency, that is $0 \le \lambda_1 < \lambda_2 < \dots$, $\lambda_n \to +\infty$
- s complex variable

$$\sum a_n e^{-ns} \overset{z = e^{-s}}{\longleftrightarrow} \sum a_n z^n$$

Ex: $\lambda_n = n$:

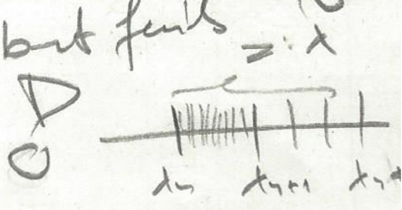
$\lambda_n = \log(n)$: $\sum a_n n^{-s}$

$\lambda \subset \{n + \sqrt{2}\, m \mid n, m \in \mathbb{Z}\} \subset \mathbb{R}$ dense

① A natural space of gDs:

$\mathcal{D}_\infty(\lambda) := \{ D = \sum a_n e^{-\lambda_n s} \mid D$ conv on $[\mathrm{Re} > 0]$ + limit function f is bounded there $\}$

$\|D\|_\infty := \sup_{[\mathrm{Re}>0]} |f(s)|$ Normed space, but fails to be complete.

D 

Sufficient conditions:

1) $\lambda = (\lambda_n)$ $\mathbb{Q}$-lin-independent. Then: $\mathcal{D}_\infty(\lambda) = \ell_1(\mathbb{N})$

Ex: $\mathcal{D}_\infty((n)) = H_\infty(\mathbb{D})$

2) $L(\lambda) := \lim_{n \to +\infty} \frac{\log(n)}{\lambda_n} = 0$

ORIGIN of (BC)

3) Bohr's condition: $\exists \ell > 0\ \forall \delta > 0\ \exists C\ \forall n: \lambda_{n+1} - \lambda_n \ge C e^{-(\ell+\delta)\lambda_n}$

Then: $\sum_{n=1}^{\infty} e^{-(\ell+\delta)\lambda_n} < \infty$. Ex: $\lambda = (\log(n))$ with $\ell = 1$.

New: 4) Landau's condition: $\forall \delta > 0\ \exists C > 0\ \forall n$: $\lambda_{n+1} - \lambda_n \ge C e^{-e^{\delta \lambda_n}}$

Ex: $\lambda = (\sqrt{\log n})$.

Q1: $\mathcal{D}_\infty((\log(\log n)))$ complete?

I

② Hardy spaces $H_\infty^\lambda(G)$:

Why? 1) $\forall \lambda$: $\mathcal{D}_\infty(\lambda) \overset{\text{isom.}}{\subset} H_\infty^\lambda(G)$

2) $\mathcal{D}_\infty(\lambda)$ complete $\Leftrightarrow \mathcal{D}_\infty(\lambda) = H_\infty^\lambda(G)$.

$\xrightarrow[\text{talk}]{\text{Aim of this}} \Uparrow$ $(LC) \ni \lambda$

Let G be a compact abelian group, $\beta : \mathbb{R} \to G$ homom.

Assume: β continuous & dense range

Then: $\hat{\beta} : \hat{G} \overset{\text{inj}}{\hookrightarrow} \hat{\mathbb{R}} = \mathbb{R}$ $= e^{-ix\cdot}$, that is,
$\gamma \mapsto \gamma \circ \beta = e^{-ix\cdot}$

$\forall \gamma \in \hat{G}\ \exists !\, x \in \mathbb{R} : \gamma \circ \beta = e^{-ix\cdot}$

Def: $h_x := \hat{\beta}^{-1}(x)$, if $x \in \operatorname{Im} \hat{\beta}$.

Then: $\hat{G} = \{ h_x \mid x \in \operatorname{Im}\hat{\beta} \} = \operatorname{Im}\hat{\beta} \subset \mathbb{R}$

Fix frequency λ.

Def: (G, β) λ-Dirichlet group $:\Leftrightarrow \lambda \subset \operatorname{Im}\hat{\beta}$

Def: $H_\infty^\lambda(G) := \{ f \in L_\infty(G) \mid \forall x : \hat{f}(h_x) \neq 0 \Rightarrow x \in \lambda \}$

Bohr transform: $\mathcal{B} : H_\infty^\lambda(G) \longrightarrow \mathcal{D}(\lambda)$
$f \longmapsto \sum \hat{f}(h_{\lambda_n}) e^{-\lambda_n s}$

Def: $\mathcal{H}_\infty(\lambda) := \mathcal{B}(H_\infty^\lambda(G))$, $\|D\|_\infty := \|f\|_\infty$, where $\mathcal{B}(f) = D$.

Thm: Def of $\mathcal{H}_\infty(\lambda)$ is independent of the chosen λ-DG (G, β).

II

③ Convergence of Fourier series

$f \in H^{\lambda}_{\infty}(G)$: $f \sim \sum \hat{f}(h_{\lambda_n}) h_{\lambda_n}$.

<u>Thm</u>: $\forall \lambda \; \forall\, 0 < r < 1 \; \forall N \; \forall f \in H^{\lambda}_{\infty}(G)$

$$\Big\| \sum_{n=1}^{N} \hat{f}(h_{\lambda_n}) h_{\lambda_n} \Big\|_{\infty} \leq \frac{C}{r} \left(\frac{\lambda_{N+1}}{\lambda_{N+1} - \lambda_N} \right)^{r} \| f \|_{\infty} .$$

The choice $r_N := e^{-\delta \lambda_N}$ and assuming (LC) gives

<u>Corollary</u>: Let $\lambda \in (LC)$. Then (G).

$\forall \delta > 0 \; \exists C \; \forall f \in H^{\lambda}_{\infty}(G)$:

$$\Big\| \sum_{n=1}^{N} \hat{f}(h_{\lambda_n}) h_{\lambda_n} \Big\|_{\infty} \leq e^{\delta \lambda_N} \cdot C(\delta) \| f \|_{\infty}$$

In part.: $D := \mathcal{B}(f)$ converges uniformly on $[\mathrm{Re} > \delta]$ for all $\delta > 0$, and $D \in \mathcal{D}_{\infty}(\lambda)$.

<u>Corollary</u>: $\lambda \in (LC) \Rightarrow \mathcal{H}_{\infty}(\lambda) = \mathcal{D}_{\infty}(\lambda)$.

($\mathcal{D}_{\infty}(\lambda) \subset \mathcal{H}_{\infty}(\lambda)$ always true) <u>recall</u>

<u>Ex.</u> $(\overline{\mathbb{R}}, \beta_{\overline{\mathbb{R}}})$?

PH