L^p-theory for Schrödinger systems

Abdelaziz Rhandi
(University of Salerno)

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Problem:

Here we are concerned with the generation of C_0-semigroups on $L^p(\mathbb{R}^d;\mathbb{C}^m)$, $1<p<\infty$, associated with

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$$\mathcal{L}u := \Delta_Q u + Vu,$$

where $\Delta_Q u = \left[\operatorname{div}(Q\nabla u_k)\right]_{k=1,\ldots,m}$ and $V:\mathbb{R}^d \to \mathbb{R}^{m\times m}$ be a matrix-valued function with entries in $L^\infty_{\mathrm{loc}}(\mathbb{R}^d)$.

Motivations:

- Time-dependent Born–Openheimer theory: See for example Betz-Goddard-Teufel, Proc. R. Soc. A. 2009.

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- Study of Nash equilibria to stochastic differential games: Cf. Addona, Angiuli, Lorenzi, Tessitore, ESAIM Control. Optim. Calc. Var. 2017.

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Given scalar-valued functions $v, w : \mathbb{R}^d \to \mathbb{R}$, and consider the matrix potential

$$V(x) = \begin{pmatrix} w(x) & -v(x) \\ v(x) & w(x) \end{pmatrix} = w(x) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + v(x) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

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Diagonalizing the latter matrix via the matrix $P = \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}$, we see that $\Delta_Q + V$ is similar, via P, to the diagonal operator

$$\begin{pmatrix} \Delta_Q & 0 \\ 0 & \Delta_Q \end{pmatrix} + \begin{pmatrix} w(x) + iv(x) & 0 \\ 0 & w(x) - iv(x) \end{pmatrix}.$$

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- Positivity, ultracontractivity and Gaussian estimates;
- Compactness of the resolvent.

Assumptions:

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(a) $Q : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ be *Lipschitz cont.* s.t. $q_{ij} = q_{ji}$, $\forall i, j \in \{1, \dots, d\}$,

$$\eta_1|\xi|^2 \le \Re\langle Q(x)\xi, \xi\rangle \le \eta_2|\xi|^2$$

(b) $V : \mathbb{R}^d \to \mathbb{R}^{m\times m}$ s.t. $v_{ij} \in L^\infty_{loc}(\mathbb{R}^d)$,

$$\Re\langle V(x)\xi, \xi\rangle \le 0, \quad \forall x \in \mathbb{R}^d, \, \xi \in \mathbb{C}^m.$$

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A direct computation shows that $\lambda\phi - \mathcal{L}\phi = f$ does not admit solutions in the maximal domain $D_{p,\max}(\mathcal{L}) = \{\phi \in L^p(\mathbb{R};\mathbb{R}^2) : \mathcal{L}\phi \in L^p(\mathbb{R};\mathbb{R}^2)\}$ for any $\lambda > 0$ and $f \in L^p(\mathbb{R};\mathbb{R}^2)$.

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For $u = (u_1, \ldots, u_m) \in H^1_{loc}(\mathbb{R}^d; \mathbb{R}^m)$ s.t.
$\Delta_Q u_j \in L^1_{loc}(\mathbb{R}^d)$, $j = 1, \ldots, d$, the following hold:

$$\Delta_Q |u| \geq 1\!\!\mathrm{I}_{\{u \neq 0\}} \frac{1}{|u|} \sum_{j=1}^{m} u_j \Delta_Q u_j$$

in the sense of distributions.

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- $L_2 u = \mathcal{L}u$, $u \in D(L_2) := \{u \in H^1(\mathbb{R}^d,\mathbb{C}^m) : \mathcal{L}u \in L^2(\mathbb{R}^d,\mathbb{C}^m)\}$ generates a C_0-semigroup on $L^2(\mathbb{R}^d,\mathbb{C}^m)$.

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- $C_c^\infty(\mathbb{R}^d,\mathbb{R}^m)$ is a core for L.
- Trotter-Kato product formula:

$$T(t)f = \lim_{n\to\infty}\left[e^{\frac{t}{n}\Delta_Q}e^{\frac{t}{n}V}\right]^n f, \quad t>0,\ f\in L^2(\mathbb{R}^d;\mathbb{C}^m).$$

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- $T(\cdot)$ can be extrapolated to a C_0-semigroup $T_p(\cdot)$ on $L^p(\mathbb{R}^d;\mathbb{C}^m)$, $1 \le p < \infty$.

Further properties:

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(4) $D(L_p) = \{u\in L^p(\mathbb{R}^d;\mathbb{C}^m)\cap W^{2,p}_{loc}(\mathbb{R}^d;\mathbb{C}^m) : \mathcal{L}u \in L^p(\mathbb{R}^d;\mathbb{C}^m)\} =: D_{p,\max}(\mathcal{L})$, $1<p<\infty$.

Analyticity and positivity of $T(\cdot)$:

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Consider $\mathcal{U}_\sigma f(x) = f(x-\sigma)$, $x \in \mathbb{R}$, $f \in L^p(\mathbb{R})$, for arbitrary fixed $\sigma \in \mathbb{R}$. So,

$$\mathcal{U}_{-\sigma} B_\pm \mathcal{U}_\sigma = B \mp i\sigma I.$$

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Thus, the semigroups generated by $B_\pm$ are not analytic.

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- $T_p(\cdot) \geq 0 \Leftrightarrow v_{kl}(x) \geq 0,\ k \neq l$ for a.e. $x \in \mathbb{R}^d$:
  This follows from the positive minimum principle.

Example: Analytic semigroups

consider

$$V(x) = \begin{pmatrix} -(1+|x|^r) & -x \\ x & -(1+|x|^r) \end{pmatrix} = \begin{pmatrix} 0 & -x \\ x & 0 \end{pmatrix} - (1+|x|^r)I_2,$$

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where $r \geq 1$. For $\xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \in \mathbb{C}^2$ one has

$$\langle V(x)\xi, \xi \rangle = -(1+|x|^r)(\xi_1^2 + \xi_2^2) + x(\xi_1 \bar{\xi}_2 - \bar{\xi}_1 \xi_2).$$

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Hence,

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Thus the semigroup is analytic.

Ultracontractivity:

$T_0(\cdot)$ the semigroup on $L^p(\mathbb{R}^d)$ generated by the scalar operator $\Delta_Q = \mathrm{div}(Q\nabla\cdot)$, with domain $W^{2,p}(\mathbb{R}^d)$.

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$$\begin{aligned}\frac{1}{2}\partial_t|u|^2 = \langle \partial_t u, u\rangle &= \sum_{k=1}^m \mathrm{div}(Q\nabla u_k)u_k + \langle Vu,u\rangle\\ &\le \frac{1}{2}\Delta_Q|u|^2.\end{aligned}$$

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Hence, $w(t,\cdot) \le w(0,\cdot)$. □

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Lemma 8 $\Rightarrow$

- $$\|T(t)f\|_\infty \leq Mt^{-\frac{d}{2}}\|f\|_1, \quad t>0\, f\in L^1(\mathbb{R}^d;\mathbb{C}^m).$$

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- $\exists K(t,\cdot,\cdot)\in L^\infty(\mathbb{R}^d\times\mathbb{R}^d;\mathbb{R}^{m\times m})$ s.t.

$$(T(t)f)(x)=\int_{\mathbb{R}^d}K(t,x,y)f(y)dy,\ \ t>0,\ x\in\mathbb{R}^d, f\in L^p(\mathbb{R}^d;\mathbb{C}^m).$$

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- $$|k_{ij}(t,x,y)|\leq C_1 t^{-\frac{d}{2}}\exp\{-C_2\frac{|x-y|^2}{4t}\},\ x,y\in\mathbb{R}^d,\ t>0.$$

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- $\sigma(L_p)$ is independent of p and consists of eigenvalues only.

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$$S_p(t)u = \begin{pmatrix} e^{t\Delta_p} f \\ e^{t\Delta_p} f \end{pmatrix}, \ \ t > 0, \ f \in L^p(\mathbb{R}^d).$$

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For $f \in W^{2,p}(\mathbb{R}^d;\mathbb{C}^m), \; u = (f, f) \in D(L_p)$ and $L_p u = (\Delta_p f, \Delta_p f)$. So,

$$S_p(t)u = \begin{pmatrix} e^{t\Delta_p} f \\ e^{t\Delta_p} f \end{pmatrix}, \; t > 0, \; f \in L^p(\mathbb{R}^d).$$

Thus,

Compactness:

Compactness may fail even if all entries in the potential V are unbounded near ∞.

Example 4:

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Thus, $S_p(\cdot)$ cannot be compact.

- $$D(L_p) = W^{2,p}(\mathbb{R}^d, \mathbb{C}^m) \cap D(V)?$$

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- (Maximal inequality)

  $$c_1(\|Vf\|_p + \|f\|_{W^{2,p}(\mathbb{R}^d,\mathbb{C}^m)}) \leq \|L_p f\| \leq c_2(\|Vf\|_p + \|f\|_{W^{2,p}(\mathbb{R}^d,\mathbb{C}^m)})?$$

  $f \in D(L_p)$, $1 < p < \infty$.

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- $\langle V(x)\xi, \xi\rangle \leq 0, \quad \forall x \in \mathbb{R}^d,\, \xi \in \mathbb{R}^m$,
- $x \mapsto D_j V(x)(-V(x))^{-\alpha}$ is uniformly bounded in $\mathbb{R}^d$ for some $\alpha \in [0, 1/2)$.

Example:

Consider $d = 1$, $m = 2$. Choosing $r \in [1, 2)$,

$$V(x) = \begin{pmatrix} 0 & 1 + |x|^r \\ -(1 + |x|^r) & 0 \end{pmatrix} = (1 + |x|^r) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \ x \in \mathbb{R}.$$

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Moreover,

$$D_x V(x) \cdot (-V(x))^{-\alpha} = r|x|^{r-2}(1 + |x|^r)^{-\alpha} x \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^{-\alpha}.$$

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Now, if we pick $\alpha \in (\frac{r-1}{r}, \frac{1}{2})$, then $r - 1 - \alpha r < 0$, so that $x \mapsto D_x V(x) \cdot (-V(x))^{-\alpha}$ is bounded.

Maximal inequalities results:

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Theorem 1:

Under the above assumptions, $-(\Delta_Q + V)$, defined on $W^{2,p}(\mathbb{R}^d, \mathbb{C}^m) \cap D(V_p)$, is closed, densely defined and sectorial. So, Lumer-Phillips Theorem implies $L_p = \Delta_Q + V$ with domain $W^{2,p}(\mathbb{R}^d, \mathbb{C}^m) \cap D(V_p)$ generates a C_0-semigroup of contractions on $L^p(\mathbb{R}^d, \mathbb{R}^m)$, $1 < p < \infty$.

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$$C_{\lambda,\mu}f := (-A_p)(\lambda - A_p)^{-1}[(-A_p)^{-1}(\mu - V_p)^{-1} - (\mu - V_p)^{-1}(-A_p)^{-1}]f,$$

$\forall f \in L^p(\mathbb{R}^d, \mathbb{C}^m)$, $\lambda \in \Sigma_{\pi-\theta_{A_p}}$, $\mu \in \Sigma_{\pi-\theta_{V_p}}$.

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Use a noncommutative version of the Dore-Venni perturbation theorem, see Monniaux-Prüss, TAM. 1997, to conclude.

More general potentials:

Consider $0 < v \in W^{1,\infty}_{loc}(\mathbb{R}^d)$, B_p the multiplication by v in $L^p(\mathbb{R}^d, \mathbb{C}^m)$. Let $v_\varepsilon = v(1 + \varepsilon v)^{-1}$, $\varepsilon > 0$. Denote by $B_{p,\varepsilon}$ the multiplication by v_ε (the Yosida approximation of B_p).

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Theorem 2:

$L_p - sB_p$ with domain $D(L_p) \cap D(B_p)$ for suitable $s > 0$ generates a contraction C_0-semigroup on $L^p(\mathbb{R}^d, \mathbb{C}^m)$, under the condition

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To show such result we use a perturbation theorem due to Okazawa, Japan J. Math. 1996.

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- For $r \in [1,2)$, $\alpha \geq 1$,

$$\begin{pmatrix} \Delta_Q & 0 \\ 0 & \Delta_Q \end{pmatrix} + \begin{pmatrix} |x|^\alpha & |x|^r \\ -|x|^r & |x|^\alpha \end{pmatrix}.$$

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Many thanks