Universität Bielefeld

Existence of flows for linear Fokker-Planck-Kolmogorov equations and its connection to well-posedness

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This talk is based on my recent paper [1] on the arXiv:

M. Rehmeier, *Existence of flows for linear Fokker-Planck-Kolmogorov equations and its connection to well-posedness*; https://arxiv.org/abs/1904.04756 (April 2019).

From SDEs to linear FPK-equations

For $T > 0$, consider the stochastic differential equation in $\mathbb{R}^d$:

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t,\ t \in [0, T] \qquad \text{(SDE)}$$

i.e. $b = (b_1, \dots, b_d) : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d, \sigma = (\sigma_{ij})_{i,j \le d} : [0, T] \times \mathbb{R}^d \to \mathrm{Sym}^d_+$ measurable, $W = (W_t)_{\in[0,T]}$ $\mathbb{R}^d$-valued Brownian motion.
$[\sigma \equiv id_{\mathbb{R}^d} \implies$ (SDE) is "ODE perturbed by randomly drawn Brownian paths"$]$

Let $X = (X_t)_{t\in[0,T]}$ be a probabilistic weak solution (continuous!) on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a Brownian motion W (i.e. $t \mapsto X_t(\omega)$ solves (SDE) for $\mathbb{P}$-a.a. $\omega \in \Omega$) with $\mathbb{P} \circ X_0^{-1} =: \gamma$ and let

$$p_t := \mathbb{P} \circ X_t^{-1} \text{(probability measure on } \mathbb{R}^d \text{ for each } t \in [0, T])$$

be the corresponding curve of marginals.

Note: Continuity of $t \mapsto X_t(\omega) \implies$ curve of measures $t \mapsto p_t$ is continuous in $\mathcal{P}_d$ (Borel probability measures on $\mathbb{R}^d$ with topology of weak convergence, Polish!).

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By the time-dependent Itô formula ("fundamental theorem of calculus for stochastic processes")

$$\frac{d}{dt}\int_{\mathbb{R}^d} f(t,x)dp_t(x) \overset{(*)}{=} \int_{\mathbb{R}^d} \partial_t f(t,x) + L_t f(t,x)dp_t(x)$$

for all $f \in C_c^\infty((0,T)\times\mathbb{R}^d)$, where
$L_t f(t,x) := \sum_{i=1}^d b_i(t,x)\partial_i f(t,x) + \sum_{i,j=1}^d \frac{1}{2}a_{ij}(t,x)\partial_i\partial_j f(t,x)$ ("generator" or "Kolmogorov operator" of (SDE)) and $a_{ij} := (\sigma\sigma^T)_{ij}$.

Integrating $(*)$ w.r.t. Lebesgue measure dt on $[0,T]$:

$$\int_0^T 1\cdot\frac{d}{dt}\int_{\mathbb{R}^d} f(t,x)dp_t(x)dt = \int_0^T\int_{\mathbb{R}^d} \partial_t f(t,x) + L_t f(t,x)dp_t(x)dt.$$

By integration by parts ($f(0,\cdot)\equiv 0\equiv f(T,\cdot)$!):

$$0 = \int_0^T\int_{\mathbb{R}^d} \partial_t f(t,x) + L_t f(t,x)dp_t(x)dt.$$

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(i) $t \mapsto p_t$ is weakly continuous

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(iii) $p_0 = \gamma$.

In general, such curves of probability measures are called solutions to the Cauchy problem of the linear Fokker-Planck-Kolmogorov equation (w.r.t. b_i, a_{ij}) with i.c. (s,ν), formally

$$\begin{cases} \partial_t \mu_t & = \partial_i\partial_j(\frac{1}{2}a_{ij}\mu_t) - \partial_i(b_i\mu_t) \\ \mu_s & = \nu, \end{cases} \tag{FPK}$$

with $(s,\nu) \in [0,T]\times\mathcal{P}_d =$ initial condition ("curve of probability measures starts in ν at time s")

Thus: FPK-equations are second-order PDEs for (probability) measures and we look for weak solutions in the sense of integrating against sufficiently many test functions.

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Definition

For measurable coefficients $b_i, a_{ij} : [s,T] \times \mathbb{R}^d \to \mathbb{R}, i,j \leq d$, *set* $L_t f(t,x) := \sum_{i=1}^d b_i(t,x)\partial_i f(t,x) + \sum_{i,j=1}^d a_{ij}(t,x)\partial_i\partial_j f(t,x)$. *A weakly continuous curve* $(\mu_t)_{t\in[s,T]}$ *in* $\mathcal{P}_d$ *is called* ***(continuous) solution to (FPK)*** *with i.c.* $(s,\nu) \in [0,T] \times \mathcal{P}_d$, *if*

1. *"coefficients are integrable against* $(\mu_t)_{t\in[s,T]}d_t$*"*
2. $\int_s^T \int_{\mathbb{R}^d} \partial_t f(t,x) + L_t f(t,x) d\mu_t(x)dt = 0 \ \forall \ f \in C_c^\infty((s,T) \times \mathbb{R}^d)$
3. $\mu_s = \nu$.

We denote the set of all such solutions by $FP(s,\nu)$.

Remarks:

(i) First point irrelevant for us: Will only consider globally bounded coefficients.

(iii) Seen above: Marginals of solution to (SDE) solve the corresponding FPK-eq.!

(iv) Results on existence, uniqueness, properties of solutions: See [2] by Bogachev, Krylov, Röckner, Shaposhnikov.

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Well-known: If (SDE) has unique weak probabilistic solution for any initial time and distribution (e.g. for Lipschitz and sublinear growth of coefficients, but also under much weaker conditions), then these solutions form a **Markov process**.

In particular, setting

$X^{s,\nu} = (X_t^{s,\nu})_{t\in[s,T]} =$ solution with i.c. (s,ν) and $p_t^{s,\nu} :=$ distribution of $X_t^{s,\nu}$,

the following *flow equations* are satisfied for the marginals of the solutions:

$$p_t^{s,\nu} = p_t^{r,p_r^{s,\nu}} \,\forall\, 0 \le s \le r \le t \le T, \nu \text{ as above.} \tag{CK}$$

(**Chapman-Kolmogorov equations** in probability theory).

Why FPK-equations? If (SDE) not probabilistically weakly well-posed, then solutions (if existing) are not necessarily Markov. Since we know: Curve of marginals of any solution to (SDE) solves corresponding FPK-eq. and (CK) involves only marginals, it could be hopeful to check Chapman-Kolmogorov equations on level of FPK-equation.

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In particular, the following aspects are interesting:

1. Assume (SDE) has several probabilistic weak solutions. Then, as explained above, the corresponding FPK-equation has several solutions and from the SDE-approach we do not know whether the flow equations (the Chapman-Kolmogorov equations) as above hold.
   Question: Can we choose a *Markovian selection* on the level of the FPK-eq., i.e. pick a solution curve $\gamma^{s,\nu} = (\gamma_t^{s,\nu})_{t\in[s,T]} \in FP(s,\nu)$ for each initial condition (s,ν) such that the collection of all such $\gamma^{s,\nu}$ fulfills (CK)?
2. In particular, this question is interesting in situations where existence of solutions to (SDE) is a priori not known, while existence of solutions to (FPK) can be shown!

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   Question: Can we choose a *Markovian selection* on the level of the FPK-eq., i.e. pick a solution curve $\gamma^{s,\nu} = (\gamma_t^{s,\nu})_{t\in[s,T]} \in FP(s,\nu)$ for each initial condition (s,ν) such that the collection of all such $\gamma^{s,\nu}$ fulfills (CK)?
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2. In particular, this question is interesting in situations where existence of solutions to (SDE) is a priori not known, while existence of solutions to (FPK) can be shown!

Recall: $FP(s,\nu)$ = set of all weakly continuous solutions to (FPK) with i.c. (s,ν).

Definition

Let $\gamma^{s,\nu} = (\gamma_t^{s,\nu})_{t\in[s,T]} \in FP(s,\nu)$ for every $(s,\nu) \in [0,T]\times\mathcal{P}_d$. We say that the family $(\gamma^{s,\nu})_{(s,\nu)\in[0,T]\times\mathcal{P}_d}$ has the flow property, if for every $0 \le s \le r \le t \le T$ and $\nu \in \mathcal{P}_d$ we have

$$\gamma_t^{s,\nu} = \gamma_t^{r,\gamma_r^{s,\nu}}. \tag{1}$$

Clear: If the FPK-eq. is well-posed, then the unique family of solutions has the flow property. Hence, interesting cases arise when several solutions exist. Our main theorem is:

Theorem 1

Let b_i, a_{ij} be globally bounded and x-continuous and assume $FP(s,\nu) \neq \emptyset$ for every (s,ν). Then there exists a family $(\gamma^{s,\nu})_{(s,\nu)\in[0,T]\times\mathcal{P}_d}$, which has the flow property.

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Important aspect of the proof:

$$\text{Compactness of } FP(s,\nu) \subseteq C([s,T],\mathcal{P}_d) \tag{2}$$

For reasons of time we do not discuss this. From now on: Assume compactness of $FP(s,\nu) \subseteq C([s,T],\mathcal{P}_d)$ for each $(s,\nu) \in [0,T] \times \mathcal{P}_d$ is settled.

Notation: $\mathbb{Q}_s^T := \mathbb{Q} \cap [s,T]$ for $0 \leq s \leq T$,
$J_s : C([s,T],\mathcal{P}_d) \to \mathcal{P}_d^{\mathbb{Q}_s^T}$ (weak prod. top.!), $J_s((\mu_t)_{t\in[s,T]}) := (\mu_q)_{q\in\mathbb{Q}_s^T}$.
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Note: $FP(s,\nu) \subseteq C([s,T],\mathcal{P}_d) \implies \mu = (\mu_t)_{t\in[s,T]} \in FP(s,\nu)$ determined by values on $\mathbb{Q}_s^T \implies$ suffices to select element in $J_s(FP(s,\nu))$.
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$$G_1^{s,\nu} : J_s\big(FP(s,\nu)\big) \to \mathbb{R},\ (\mu_q)_{q\in\mathbb{Q}_s^T} \mapsto \int f_1 d\mu_{q_1},$$

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Let all assumptions of Theorem 1 be in force. Then the following are equivalent:

(i) *The FPK-eq. is well-posed among weakly continuous probability solutions, i.e.* $\big|FP(s,\nu)\big| = 1$ *for all* $(s,\nu) \in [0,T] \times \mathcal{P}_d$.

(ii) *There exists exactly one family of solutions* $(\mu^{s,\nu})_{(s,\nu)\in[0,T]\times\mathcal{P}_d}$ *with the flow-property.*

Scheme of proof: With Theorem 1 in mind, $(i) \implies (ii)$ is immediate. Concerning $(ii) \implies (i)$, assume by contradiction there is $(\bar{s},\bar{\nu})$ such that $|FP(\bar{s},\bar{\nu})| \geq 2$, i.e. there are $\mu^1,\mu^2 \in FP(\bar{s},\bar{\nu})$ with $\mu^1_{\bar{q}} \neq \mu^2_{\bar{q}}$ for some $\bar{q} \in \mathbb{Q}^T_{\bar{s}}$. Assume: have constructed a flow family, which includes μ^1 (hence not μ^2). Now we perform the selection method of the proof of Theorem 1 again, but with a different enumeration of $\{f_i\}_{i\in I} \times \mathbb{Q}^T_{\bar{s}}$, for which (f_1,q_1) fulfills

$$\int f_1 d\mu^2_{q_1} > \int f_1 d\mu^1_{q_1}.$$

Clearly, the thus selected flow will not contain μ^1 and hence differs from the orginial one. □

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Concerning $(ii) \implies (i)$, assume by contradiction there is $(\bar{s},\bar{\nu})$ such that $|FP(\bar{s},\bar{\nu})| \geq 2$, i.e. there are $\mu^1, \mu^2 \in FP(\bar{s},\bar{\nu})$ with $\mu^1_{\bar{q}} \neq \mu^2_{\bar{q}}$ for some $\bar{q} \in \mathbb{Q}^T_{\bar{s}}$. Assume: have constructed a flow family, which includes μ^1 (hence not μ^2). Now we perform the selection method of the proof of Theorem 1 again, but with a different enumeration of $\{f_i\}_{i\in I} \times \mathbb{Q}^T_{\bar{s}}$, for which (f_1, q_1) fulfills

$$\int f_1 d\mu^2_{q_1} > \int f_1 d\mu^1_{q_1}.$$

Clearly, the thus selected flow will not contain μ^1 and hence differs from the orginial one. □

Theorem 2

Let all assumptions of Theorem 1 be in force. Then the following are equivalent:

(i) *The FPK-eq. is well-posed among weakly continuous probability solutions, i.e.* $\big|FP(s,\nu)\big| = 1$ *for all* $(s,\nu) \in [0,T] \times \mathcal{P}_d$*.*

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Thank you for your attention!