On Abstract Friedrichs Systems and Some of their Applications.

41. Nordwestdeutsches Funktionalanalysis-Kolloquium an der U D-E in Essen

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Introduction

Key idea: exponential weight function $t \mapsto \exp(-\rho\, t)$, $\rho \in \mathbb{R}$, generates a weighted L^2-space $H_{\rho,0}(\mathbb{R}, X)$ (inner product $\langle \cdot | \cdot \rangle_{\rho,0,0}$, norm: $|\cdot|_{\rho,0,0}$), X a **real** Hilbert space,

$$(\varphi, \psi) \mapsto \int_{\mathbb{R}} \langle \varphi(t) | \psi(t) \rangle_H \exp(-2\rho\, t)\, dt.$$

Time-differentiation ∂_t as a closed operator in $H_{\rho,0}(\mathbb{R}, X)$ induced by

$$\begin{aligned} \mathring{C}_1(\mathbb{R}, H) \subseteq H_{\rho,0}(\mathbb{R}, H) &\to H_{\rho,0}(\mathbb{R}, H), \\ \varphi &\mapsto \varphi'. \end{aligned}$$

Introduction

Time-differentiation ∂_t is a *normal* operator in $H_{\rho,0}(\mathbb{R},X)$

$$\partial_t = \mathfrak{sym}(\partial_t) + \mathfrak{skew}(\partial_t) = \frac{1}{2}(\partial_t + \partial_t^*) + \frac{1}{2}(\partial_t - \partial_t^*)$$

with $\mathfrak{sym}(\partial_t)$ self-adjoint and , $\mathfrak{skew}(\partial_t)$ skew-selfadjoint and commuting resolvents:

$$\mathfrak{sym}(\partial_t) = \rho.$$

For $\rho \in \mathbb{R} \setminus \{0\}$: continuous invertibility of ∂_t. For $\rho \in]0,\infty[$:

$$\mathfrak{sym}(\partial_t) = \rho > 0.$$

Dynamic abstract Friedrichs system (1954,1958): For A skew-selfadjoint in a real Hilbert space H

$$\boxed{\left(\partial_t M_0 + M_1 + A\right) U = F}$$

$$\begin{aligned}\partial_t M_0 + M_1 + A &= \\ &= \left(\rho M_0 + \mathfrak{sym}\left(M_1\right)\right) + \left(\left(\partial_t - \rho\right) M_0 + \mathfrak{skew}\left(M_1\right) + A\right) \\ &= E_0 + \mathscr{A}.\end{aligned}$$

E_0 symmetric strictly positive definite, $\mathscr{A}$ skew-selfadjoint in $H_{\rho,0}\left(\mathbb{R}, H\right)$. W.l.o.g. $E_0 = 1$, since we have the congruence

$$\sqrt{E_0}\left(1 + \sqrt{E_0^{-1}}\mathscr{A}\sqrt{E_0^{-1}}\right)\sqrt{E_0} = E_0 + \mathscr{A},$$

and note that

$$\sqrt{E_0^{-1}}\mathscr{A}\sqrt{E_0^{-1}}$$

remains skew-selfadjoint. Such dynamic abstract Friedrichs systems are of interest in the following. Indeed, our core topic focuses on the **skew-selfadjointness of the operator** A as the center-piece of abstract Friedrichs systems.

The Time Derivative as a Normal Operator

Fourier-Laplace transform: unitary extension of

$$\mathring{C}_\infty(\mathbb{R},X) \subseteq H_{\rho,0}(\mathbb{R},X) \to H_{0,0}(\mathbb{R},X) = L^2(\mathbb{R},X)$$
$$\varphi \mapsto \mathscr{L}_\rho \varphi$$

with $\mathscr{L}_\rho\varphi(x) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \exp(-\mathrm{i}\,x\,t)\exp(-\rho\,t)\,\varphi(t)\,dt,\ x \in \mathbb{R}.$
is spectral representation for $\mathfrak{Im}\,\partial_t$:

$$\mathfrak{Im}\,\partial_t = \frac{1}{\mathrm{i}}\mathfrak{skew}\,\partial_t = \mathscr{L}_\rho^{-1}\mathbf{m}_0\,\mathscr{L}_\rho$$

and so

$$\partial_t = \mathscr{L}_\rho^{-1}(\mathrm{i}\,\mathbf{m}_0 + \rho)\,\mathscr{L}_\rho.$$

Here $\mathbf{m}_0$ is the selfadjoint multiplication-by-argument operator in $L^2(\mathbb{R},X)$:

$$(\mathbf{m}_0\varphi)(x) = x\varphi(x)$$

for $x \in \mathbb{R}$ and $\varphi \in \mathring{C}_\infty(\mathbb{R},X)$.

Material Law Operator:

$$\mathscr{M} = M\left(\partial_t^{-1}\right).$$

It is
$$M\left(\partial_t^{-1}\right) := \mathscr{L}_\rho^{-1}\, M\left(\frac{1}{\mathrm{i}\,\mathrm{m}_0 + \rho}\right) \mathscr{L}_\rho,$$

where
$$M\left(\frac{1}{\mathrm{i}\,\mathrm{m}_0 + \rho}\right)\Phi := \left(\omega \mapsto M\left(\frac{1}{\mathrm{i}\,\omega + \rho}\right)\Phi(\omega)\right)$$

for $\Phi \in \mathring{C}_\infty(\mathbb{R}, X)$.

Here $(M(z))_{z\in B_{\mathbb{C}}(r,r)}$ is a uniformly bounded, holomorphic family of linear operators in H with $r \geq \frac{1}{2\rho} > 0$. The operator $M\left(\partial_t^{-1}\right)$ will be referred to as the **material law operator**. The operator-valued function M will be referred to as the **material law function**.

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Basic Solution Theory $H_{\rho,0}(\mathbb{R}, H)$

Evolutionary Problem:

$$\overline{\left(\partial_t M\left(\partial_t^{-1}\right) + A\right)} U = F$$

When is $\left(\partial_t M\left(\partial_t^{-1}\right) + A\right)$ (and its adjoint) strictly positive definite in $H_{\rho,0}(\mathbb{R}, H)$ (for all sufficiently large $\rho \in \mathbb{R}_{>0}$)?

Assumptions(S):

- A **skew-selfadjoint** in H (lifted to $H_{\rho,0}(\mathbb{R}, H)$),
- $z \mapsto M(z)$ (values in $L(H, H)$), for simplicity analytic at 0.
- $M(0) \geq 0$ **selfadjoint**, $\rho M(0) + \mathfrak{sym}(M'(0)) \geq c_0 > 0$ **(strictly positive definite)** for ρ **sufficiently large.**

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Theorem

*Let M and A satisfy **Assumptions (S)**. Then we have for all sufficiently large $\rho \in \mathbb{R}_{>0}$ that for every $f \in H_{\rho,0}(\mathbb{R}, H)$ there is a unique solution $U \in H_{\rho,0}(\mathbb{R}, H)$ of the problem*

$$\overline{\left(\partial_t M\left(\partial_t^{-1}\right) + A\right)} U = f .$$

The solution operator $\left(\overline{\partial_t M\left(\partial_t^{-1}\right) + A}\right)^{-1}$ is continuous and causal on $H_{\rho,0}(\mathbb{R}, H)$.

Causal? For every $a \in \mathbb{R}$ we have:

If $F \in H_{\rho,0}(\mathbb{R}, H)$ vanishes on the time interval $]-\infty, a[$, then so does $\left(\overline{\partial_t M\left(\partial_t^{-1}\right) + A}\right)^{-1} F$.

The Basic Solution Theorem

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An Illustrative Example

Frequently,

$$A = \begin{pmatrix} 0 & -G^* \\ G & 0 \end{pmatrix},$$

where G is a closed densely defined linear operator.

We recall that we will here consider only simple material laws

$$M\left(\partial_t^{-1}\right) = M(0) + \partial_t^{-1} M'(0),$$

i.e. on the case associated with abstract Friedrichs systems:

$$\boxed{\left(\partial_t M(0) + M'(0) + A\right) U = F.}$$

Consider a material law with

$$M\left(\partial_t^{-1}\right) = \begin{pmatrix} \varepsilon_1 & 0 \\ 0 & \varepsilon_2 \end{pmatrix} + \partial_t^{-1} \begin{pmatrix} (1-\varepsilon_1) & 0 \\ 0 & (1-\varepsilon_2) \end{pmatrix}, \; \varepsilon_1, \varepsilon_2 \in \{0,1\}.$$

- $\varepsilon_1 = 1, \varepsilon_2 = 1$: $\begin{pmatrix} \partial_t & -G^* \\ G & \partial_t \end{pmatrix} \sim \begin{pmatrix} \partial_t^2 + G^*G & 0 \\ G & \partial_t \end{pmatrix}$ by a formal row operation ("hyperbolic").
- $\varepsilon_1 = 1, \varepsilon_2 = 0$: $\begin{pmatrix} \partial_t & -G^* \\ G & 1 \end{pmatrix} \sim \begin{pmatrix} \partial_t + G^*G & 0 \\ G & 1 \end{pmatrix}$ by a formal row operation ("parabolic"). Note that $\varepsilon_1 = 0, \varepsilon_2 = 1$ is analogous.
- $\varepsilon_1 = 0, \varepsilon_2 = 0$: $\begin{pmatrix} 1 & -G^* \\ G & 1 \end{pmatrix} \sim \begin{pmatrix} 1 + G^*G & 0 \\ G & 1 \end{pmatrix}$ by a formal row operation ("elliptic").

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Four Tools for Establishing Skew-Selfadjointness
Tool 1: Abstract grad-div Systems.

For abstract grad – div systems the spatial operator A is still of the form

$$A = \begin{pmatrix} 0 & -G^* \\ G & 0 \end{pmatrix},$$

but here

$$G = \begin{pmatrix} G_1 \\ \vdots \\ G_n \end{pmatrix} : D(G) \subseteq H_0 \to H_1 \oplus \cdots \oplus H_n$$

(in the standard case of grad – div systems $G_k = \mathring{\partial}_k$ or $G_k = \partial_k$ but in general G_k need **not** necessarily be **closable**). Thus, the range space is a ***direct sum of real Hilbert spaces***.

Application: Acoustics with Damping Boundary Constraints

$$\text{acoustics: } A = \begin{pmatrix} 0 & -\left(\mathring{\operatorname{div}}\right)^* \\ \mathring{\operatorname{div}} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \operatorname{grad} \\ \mathring{\operatorname{div}} & 0 \end{pmatrix}$$

$$\left(\partial_t \begin{pmatrix} \rho_* & 0 \\ 0 & \kappa^{-1} \end{pmatrix} + A\right)\begin{pmatrix} v \\ p \end{pmatrix} = \begin{pmatrix} 0 \\ f \end{pmatrix}$$

where

$$\kappa = \rho_* c_*^2 \text{ bulk modulus, } \rho_* \text{ mass density, } c_* \text{ speed of sound.}$$

We expand this to

$$\left(\partial_t \begin{pmatrix} \rho_* & \begin{pmatrix} 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} \kappa^{-1} & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix} + \begin{pmatrix} 0 & \begin{pmatrix} 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & \beta_* \end{pmatrix} \end{pmatrix} + \widetilde{A}\right)\begin{pmatrix} v \\ \begin{pmatrix} p \\ \tau \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0 \\ \begin{pmatrix} f \\ h \end{pmatrix} \end{pmatrix}.$$

Application: Acoustics with Damping Boundary Constraints

Here

$$\widetilde{A} = \begin{pmatrix} 0 & -\begin{pmatrix} \operatorname{div} \\ \delta_{\mathrm{div},\partial\Omega} \end{pmatrix}^{*} \\ \begin{pmatrix} \operatorname{div} \\ \delta_{\mathrm{div},\partial\Omega} \end{pmatrix} & 0 \end{pmatrix}$$

with

$$\delta_{\mathrm{div},\partial\Omega} f = n^{\top} f \in L^{2}(\partial\Omega),$$

$$\left(\delta_{\mathrm{div},\partial\Omega} f\right)(\varphi) = \int_{\partial\Omega} \varphi\, n^{\top} f \,\mathrm{Vol}_{\partial\Omega} = \left\langle \varphi | n^{\top} f \right\rangle_{L^{2}(\partial\Omega)}.$$

We note

$$\begin{pmatrix} \mathring{\operatorname{div}} \\ 0 \end{pmatrix} \subseteq \begin{pmatrix} \operatorname{div} \\ \delta_{\mathrm{div},\partial\Omega} \end{pmatrix}$$

and so

$$\begin{pmatrix} \mathring{\operatorname{grad}} & -\delta^{*}_{\mathrm{div},\partial\Omega} \end{pmatrix} \subseteq -\begin{pmatrix} \operatorname{div} \\ \delta_{\mathrm{div},\partial\Omega} \end{pmatrix}^{*} \subseteq \begin{pmatrix} \operatorname{grad} & 0 \end{pmatrix}.$$

Application: Acoustics with Damping Boundary Constraints

What does it mean if $\begin{pmatrix} v \\ \tau \end{pmatrix} \in \operatorname{dom}\left(\begin{pmatrix} \operatorname{div} \\ \delta_{\mathrm{div},\partial\Omega} \end{pmatrix}^{*}\right)$?

From

$$\begin{pmatrix} \operatorname{div} \\ \delta_{\mathrm{div},\partial\Omega} \end{pmatrix}^{*} \subseteq \begin{pmatrix} -\operatorname{grad} & 0 \end{pmatrix}$$

we have

$$\langle \operatorname{grad} p | w \rangle = \left\langle \begin{pmatrix} \operatorname{grad} & 0 \end{pmatrix} \begin{pmatrix} p \\ \tau \end{pmatrix} \middle| w \right\rangle = \left\langle -\begin{pmatrix} \operatorname{div} \\ \delta_{\mathrm{div},\partial\Omega} \end{pmatrix}^{*} \begin{pmatrix} p \\ \tau \end{pmatrix} \middle| w \right\rangle$$

$$= -\left\langle \begin{pmatrix} p \\ \tau \end{pmatrix} \middle| \begin{pmatrix} \operatorname{div} \\ \delta_{\mathrm{div},\partial\Omega} \end{pmatrix} w \right\rangle = -\langle p | \operatorname{div} w \rangle - \langle \tau | \delta_{\mathrm{div},\partial\Omega} w \rangle$$

that is

$$\int_{\Omega} (\operatorname{grad} p)\, w \; \mathrm{Vol}_{\Omega} + \int_{\Omega} p\, (\operatorname{div} w)\; \mathrm{Vol}_{\Omega} = \int_{\partial\Omega} \tau \left(n^{\top} w \right) \mathrm{Vol}_{\partial\Omega}.$$

Application: Acoustics with Damping Boundary Constraints

On the other hand, we have

$$\int_\Omega (\operatorname{grad} p)\, w \ \mathrm{Vol}_\Omega + \int_\Omega p\,(\operatorname{div} w)\ \mathrm{Vol}_\Omega = \int_{\partial\Omega} p\left(n^\top w\right) \mathrm{Vol}_{\partial\Omega}$$

and so by comparison $\tau = -p$. Thus, we found

$$\begin{pmatrix} p \\ \tau \end{pmatrix} \in \operatorname{dom}\left(\begin{pmatrix} \operatorname{div} \\ \delta_{\mathrm{div},\partial\Omega} \end{pmatrix}^*\right) \Longrightarrow \begin{pmatrix} p \\ -p \end{pmatrix} \in \operatorname{dom}(\operatorname{grad}) \oplus L^2(\partial\Omega).$$

We formally read off as the last equation

$$-\beta_* p + n^\top v = h \text{ on } \partial\Omega,$$

yielding (a dynamic Robin type boundary condition)

$$n^\top \rho_*^{-1} \operatorname{grad} p + \beta_* \partial_t p = \partial_t h \text{ on } \partial\Omega.$$

Tool 2: The Mother-Descendant Mechanism

Theorem

Let $G : D(C) \subseteq H_0 \to H_1$ be a closed densely defined linear operator, H_k, $k = 0, 1$, real Hilbert spaces. If $B_0 : H_0 \to X_0$ is a continuous linear mapping, X_0 real Hilbert space such that

$$GB_0^* \text{ densely defined.}$$

Then $\overline{\begin{pmatrix} B_0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -G^* \\ G & 0 \end{pmatrix}} \begin{pmatrix} B_0^* & 0 \\ 0 & 1 \end{pmatrix}$ *is skew-selfadjoint.*

"Mother" and"descendant". Can be repeated!

Tool 2: The Mother-Descendant Mechanism

Dirichlet boundary condition $G = \mathring{\text{grad}}$ acting on tensor fields of all ranks:

$$A := \begin{pmatrix} 0 & -G^* \\ G & 0 \end{pmatrix} = \begin{pmatrix} 0 & \text{div} \\ \mathring{\text{grad}} & 0 \end{pmatrix}.$$

Initial boundary value problems of classical mathematical physics can be produced from this particular “mother” operator A by choosing suitable projections for constructing “descendants”.

In Nowacki's non-symmetric elasticity we are dealing with a skew-selfadjoint spatial operator of the form

$$\begin{pmatrix} 0 & -\operatorname{div}_2 \\ -\mathring{\operatorname{grad}}_1 & 0 \end{pmatrix} \text{ on } L^2\left(\Omega, \mathbb{R}^3\right) \oplus L^2\left(\Omega, \mathbb{R}^{3\times 3}\right).$$

Here, with

$$\widetilde{\mathrm{Op}}\, W := \left(\mathrm{Op}\, W^\top\right)^\top$$

where Op denotes a matrix PDE operator, we have

$$\operatorname{grad}_1 v := \widetilde{\nabla} v = \left(\nabla\, v^\top\right)^\top \text{ (Jacobian of } v\text{)},$$

$$\operatorname{div}_2 T := \widetilde{\nabla^\top}\, T = \left(\nabla^\top\, T^\top\right)^\top.$$

As descendants we obtain

$$\overline{\begin{pmatrix} 1 & 0 \\ 0 & \iota_{\mathfrak{sym}}^* \end{pmatrix} \begin{pmatrix} 0 & -\operatorname{div}_2 \\ -\mathring{\operatorname{grad}}_1 & 0 \end{pmatrix}} \begin{pmatrix} 1 & 0 \\ 0 & \iota_{\mathfrak{sym}} \end{pmatrix} = \begin{pmatrix} 0 & -\operatorname{Div} \\ -\mathring{\operatorname{Grad}} & 0 \end{pmatrix}$$

i.e. classical symmetric elasticity, and classical electrodynamics

$$\overline{\begin{pmatrix} 1 & 0 \\ 0 & -I_0^* \iota_{\mathfrak{skew}}^* \end{pmatrix} \begin{pmatrix} 0 & -\operatorname{div}_2 \\ -\mathring{\operatorname{grad}}_1 & 0 \end{pmatrix}} \begin{pmatrix} 1 & 0 \\ 0 & -\iota_{\mathfrak{skew}} I_0 \end{pmatrix} = \begin{pmatrix} 0 & \operatorname{curl} \\ \mathring{\operatorname{curl}} & 0 \end{pmatrix}$$

where $\operatorname{curl} = \nabla \times = \begin{pmatrix} 0 & -\partial_3 & \partial_2 \\ \partial_3 & 0 & -\partial_1 \\ -\partial_2 & \partial_1 & 0 \end{pmatrix}$, $I_0 : \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} \mapsto \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -\alpha_3 & \alpha_2 \\ \alpha_3 & 0 & -\alpha_1 \\ -\alpha_2 & \alpha_1 & 0 \end{pmatrix}$
and $\iota_{\mathfrak{skew}}^* T = \frac{1}{2}\left(T - T^\top\right) = \frac{1}{2}\left(T - T^*\right)$.

Also acoustics can be obtained as a descendant from non-symmetric elasticity via

$$\overline{\begin{pmatrix} 1 & 0 \\ 0 & -\mathrm{trace} \end{pmatrix} \begin{pmatrix} 0 & -\operatorname{div}_2 \\ -\mathring{\operatorname{grad}}_1 & 0 \end{pmatrix}} \begin{pmatrix} 1 & 0 \\ 0 & -\mathrm{trace}^* \end{pmatrix} = \begin{pmatrix} 0 & \operatorname{grad}_0 \\ \mathring{\operatorname{div}}_1 & 0 \end{pmatrix}.$$

Here trace denotes the standard matrix trace and its adjoint evaluates simply to

$$\mathrm{trace}^* p = \begin{pmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{pmatrix} = p\, \mathbb{1}_{3\times 3}.$$

Tool 3: Coupling of Different Physical Phenomena

Block-diagonal operator matrix:

$$A = \begin{pmatrix} A_0 & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & A_n \end{pmatrix}$$

skew-selfadjoint in $H = \bigoplus_{k=0,\ldots,n} H_k$, if diagonal block entries $A_k : D(A_k) \subseteq H_k \to H_k, k = 0, \ldots, n$, are skew-selfadjoint.

Proper coupling: M contains off-diagonal block entries

$$M\left(\partial_t^{-1}\right) := \begin{pmatrix} M_{00}\left(\partial_t^{-1}\right) & \cdots & M_{0n}\left(\partial_t^{-1}\right) \\ \vdots & \ddots & \vdots \\ M_{n0}\left(\partial_t^{-1}\right) & \cdots & M_{nn}\left(\partial_t^{-1}\right) \end{pmatrix}.$$

Coupling elasticity and acoustics

$$\partial_t \begin{pmatrix} \kappa & 0 & 0 & 0 \\ 0 & \nu_1 & 0 & 0 \\ 0 & 0 & \nu_2 & 0 \\ 0 & 0 & 0 & C^{-1} \end{pmatrix} + \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & d & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + A$$

with

$$A = \begin{pmatrix} \begin{pmatrix} 0 & \operatorname{grad}_0 \\ \mathring{\operatorname{div}}_1 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & -\operatorname{Div} \\ -\mathring{\operatorname{Grad}} & 0 \end{pmatrix} \end{pmatrix}.$$

projection onto $\ker(\partial_3) = L^2(\Omega_0)$ assuming $\Omega := \Omega_0 \times \mathbb{T} \subseteq \mathbb{R}^2 \times \mathbb{T} =: M$ (instead of $M = \mathbb{R}^3$) we can reduce this by one spatial dimension (mother-descendant mechanism) to a (1+2)-dimensional evolutionary problem.

The resulting evolutionary equation looks the same, but now it has to be interpreted (by dropping zero rows and columns in A and adapting the material law) in $L^2\left(\Omega_0,\mathbb{R}^2\right)\oplus L^2\left(\Omega_0,\mathbb{R}\right)\oplus L^2\left(\Omega_0,\mathbb{R}^2\right)\oplus L^2_2\left(\Omega_0,\mathfrak{sym}\left[\mathbb{R}^{2\times 2}\right]\right)$ with $\Omega_0\subseteq\mathbb{R}^2$.

This is the Reissner-Mindlin plate system commonly used in engineering models.

Remark:(Kirchhoff-Love plate)

Letting $\kappa=0$ and $v_2=0$ (in consequence destroying well-posedness for associated initial boundary value problems) and eliminating the first and third unknowns and equations and then eliminating the stress yields for isotropic homogeneous media

$$\partial_t^2 v_1\eta+d\partial_t\eta+(2\kappa+\lambda)\Delta^2\eta=\partial_t f.$$

Tool 4: Weak = Strong

Transmutator

$$[L, C, R] := LC - CR$$

assumed to be defined on dom (C). The commutator

$$[L, C] := [L, C, L]$$
$$[C, L] := -[L, C]$$

is a special case.

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Let A_k, $k = 1, 2$, be closed densely defined operators from H to K, $\operatorname{dom}(A_1 + A_2) = \operatorname{dom}(A_1) \cap \operatorname{dom}(A_2)$ dense in H.

Theorem

Let $(L_\varepsilon)_{\varepsilon\in]0,1[}$, $(R_\varepsilon)_{\varepsilon\in]0,1[}$ be bounded families of continuous linear mappings in K and H, respectively, and $[L_\varepsilon, A_1 + A_2, R_\varepsilon]$ defined on $\operatorname{dom}(A_1) \cap \operatorname{dom}(A_2)$ such that $\overline{[L_\varepsilon, A_1 + A_2, R_\varepsilon]} \in \mathscr{L}(H, K)$. Moreover,

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Then

$$(A_1 + A_2)^* = \overline{A_1^* + A_2^*}.$$

Corollary

Let A_1, A_2 be skew-selfadjoint, then under the assumptions of the previous theorem we have

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Application: (acoustics in moving media) assuming that

$$\mathfrak{sym}\left(\alpha\partial_3\begin{pmatrix}\rho_* & 0\\ 0 & \kappa^{-1}\end{pmatrix}\right)$$ is continuous

$$\partial_t\begin{pmatrix}\rho_* & 0\\ 0 & \kappa^{-1}\end{pmatrix}+\alpha\partial_3\begin{pmatrix}\rho_* & 0\\ 0 & \kappa^{-1}\end{pmatrix}+\begin{pmatrix}0 & \operatorname{grad}\\ \overset{\circ}{\operatorname{div}} & 0\end{pmatrix}=$$

$$=\partial_t\begin{pmatrix}\rho_* & 0\\ 0 & \kappa^{-1}\end{pmatrix}+\frac{1}{2}\mathfrak{sym}\left(\alpha\partial_3\begin{pmatrix}\rho_* & 0\\ 0 & \kappa^{-1}\end{pmatrix}\right)+A_1+A_2$$

with

$$A_1=\mathfrak{skew}\overline{\left(\partial_3\alpha\begin{pmatrix}\rho_* & 0\\ 0 & \kappa^{-1}\end{pmatrix}\right)},\text{(skew-selfadjoint for suitable } \alpha,\Omega)$$

$$A_2=\begin{pmatrix}0 & \operatorname{grad}\\ \overset{\circ}{\operatorname{div}} & 0\end{pmatrix},$$

$$R_\varepsilon=L_\varepsilon=(1+\varepsilon\partial_3)^{-1}.$$

The End

Thank You for Your Attention!