The Regularity Problem for Milnor's Infinite-Dimensional Lie Groups

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Lie Groups and Bastiani's Differential Calculus

- Lie group is infinite-dimensional manifold with smooth group structure.
- Infinite-dimensional manifold M is Hausdorff topological space, covered by charts that map open subsets of M homeomorphically to open subsets of a fixed Hausdorff locally convex vector space (modeling space), such that coordinate changes are **smooth:**
  - $f : F \supseteq U \to E$ differentiable if directional derivative exist:

$$(D_v f)(x) := \lim_{t \to 0} \frac{1}{h} \cdot (f(x + h \cdot v) - f(x)) \in E \qquad \forall\, x \in U,\ v \in E.$$

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  - f of class C^1 if differentiable with

    $$\mathrm{d}f\colon U \times F \ni (x, v) \mapsto (D_v f)(x) \in E$$

    continuous w.r.t. $\times$-topology (HLCVS) $\implies$ $\mathrm{d}_x f \equiv \mathrm{d}f(x, \cdot)$ linear.

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  - f of class C^2 *iff* of class C^1, with (of class C^n – inductively)

    $$\mathrm{d}[\mathrm{d}f]\colon \widetilde{U} \times \widetilde{F} \to E$$

    of class C^1 for $\widetilde{U} \equiv U \times F \subseteq \widetilde{F} \equiv F \times F$.

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  - Alternatively, f of class C^n if ($\mathrm{d}^p f(x, \ldots)$ – multilinear)

    $$\mathrm{d}^p f\colon (x; v_p, \ldots, v_1) \mapsto (D_{v_p}(D_{v_{p-1}}(\ldots(D_{v_1}(f))\ldots)))(x)$$

    defined for all $x \in U$, $v_1, \ldots, v_p \in F$ and continuous, for $1 \le p \le n$.

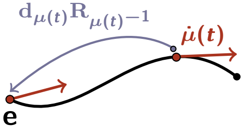
Overview

In 1983 Milnor introduced regularity as tool to extend elementary results in Lie theory to infinite dimensions. (integr. of Lie algebra homomorphisms)

Roughly speaking: G is C^k-regular ($k \in \mathbb{N} \sqcup \{\mathrm{lip}, \infty\}$) if

$$\phi = \mathrm{d}_\mu \mathrm{R}_{\mu^{-1}}(\dot{\mu}) \qquad \text{for} \qquad \phi \in C^k([0,1], \mathfrak{g})$$

has solution $\mu \equiv \mathrm{Evol}(\phi) \in C_*^{k+1}([0,1], G)$ with smooth dependence on ϕ.



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Regularity Problem: When is Lie group C^k-regular ? $(k \equiv \infty)$

Further Motivation:

- $\exp(X) = \mathrm{Evol}(\phi_X)(1)$ with $\phi_X \colon [0,1] \ni t \mapsto X \in \mathfrak{g}$ constant.
- Let (P, M) be principal G-bundle, $\omega \in \Omega^1(P, \mathfrak{g})$, $s \colon M \supseteq U \to P$ local section, and $\gamma \in C^1([0,1], U)$. Then,

  $$\widetilde{\gamma}(t) := (s \circ \gamma)(t) \cdot \mathrm{Evol}(-(s^*\omega)(\dot{\gamma}))(t) \qquad \forall\, t \in [0,1]$$

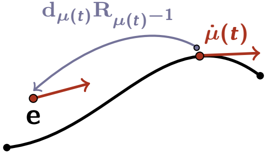
  horizontal lift of γ in $s(\gamma(0))$, i.e., $\widetilde{\gamma}(1) = \mathcal{P}^\omega_\gamma(s(0))$ (holonomy).

The Evolution Map

G – fixed Lie group with Lie algebra $\mathfrak{g}$
$\mathfrak{P}$ – continuous seminorms on $\mathfrak{g}$

Right Logarithmic Derivative:

$$\delta^r\colon C^1(K,G) \to \mathfrak{D} \subseteq C^0(K,\mathfrak{g}), \qquad \mu \mapsto \mathrm{d}_\mu \mathrm{R}_{\mu^{-1}}(\dot{\mu}).$$



Elementary Properties:

$$\delta^r(\mu \cdot g) = \delta^r(\mu) \qquad \text{and} \qquad \delta^r(\mu|_{K'}) = \delta^r(\mu)|_{K'}$$

$$\delta^r(\mu \cdot \nu) = \delta^r(\mu) + \mathrm{Ad}_\mu(\delta^r(\nu))$$

$$\delta^r(\mu \circ \varrho) = \dot{\varrho} \cdot (\delta^r(\mu) \circ \varrho)$$

for all $\mu, \nu \in C^1(K,G)$, $g \in G$, $K' \subseteq K$, and $\varrho\colon K'' \to K$ of class C^1.

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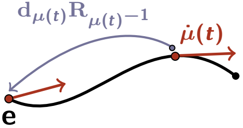

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Evolution Map:

$$\mathrm{Evol}\colon \mathfrak{D} \to C^1_*(K,G), \qquad \phi \equiv \delta^r(\mu) \mapsto \mu \qquad (\delta^r(\mu) = \phi)$$

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Product Integral:

$$\textstyle\int_s^t \phi := \mathrm{Evol}\big(\phi|_{[s,t]}\big)(t) \in G \qquad \forall\, [s,t] \subseteq \mathrm{dom}[\phi],\ \ \phi \in \mathfrak{D}.$$

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Elementary Properties: $\qquad (\int \phi \equiv \int_r^{r'} \phi)$

$$\textstyle\int_r^{r'} \phi = \int_{t_n}^{r'} \phi \cdot \ldots \cdot \int_r^{t_1} \phi \qquad \text{splitting}$$

$$\textstyle\int \phi \cdot \int \psi = \int \phi + \mathrm{Ad}_{\int_r^\bullet \phi}(\psi) \qquad \text{product}$$

$$\textstyle\int \phi = \int \dot{\varrho} \cdot (\phi \circ \varrho) \qquad \text{substitution}$$

for all $\phi, \psi \in \mathfrak{D} \cap C^0([r,r'],\mathfrak{g})$, and $\varrho\colon [\ell,\ell'] \to [r,r']$ pos. of class C^1.

Glueing with Bump Functions

$(*)\ \int \phi = \int_{t_{n-1}}^{1} \phi \cdot \ldots \cdot \int_0^{t_1} \phi$

$(**)\ \int \phi = \int \dot{\varrho} \cdot (\phi \circ \varrho)$

Interval $[a, b]$ given, then find

$$\varrho\colon [a, b] \to [a, b] \quad \text{smooth with} \quad \varrho(a) = a,\ \varrho(b) = b,$$

such that $\dot{\varrho}\colon [a, b] \to [0, 2]$ is bump function, i.e.,

$$\dot{\varrho}|_{(a,b)} \neq 0 \qquad \text{and} \qquad \dot{\varrho}^{(\ell)}(a) = 0 = \dot{\varrho}^{(\ell)}(b) \quad \forall\, \ell \in \mathbb{N}.$$

Thus, given $\phi \in \mathfrak{D} \cap C^k([a, b], \mathfrak{g})$, we have

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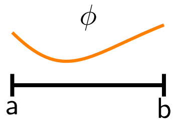

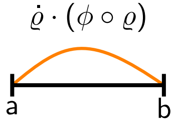

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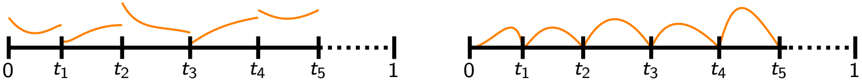
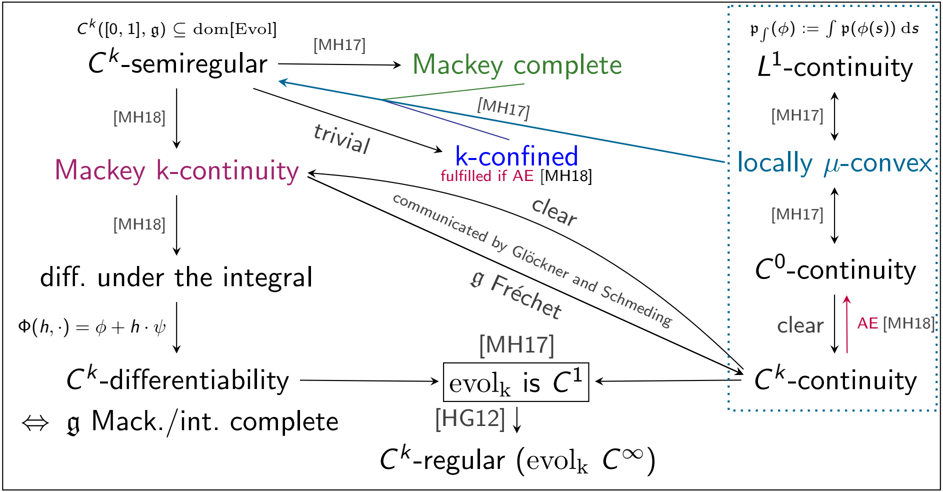


$$\int \phi \overset{(*)}{=} \int \dot{\varrho}_{n-1} \cdot (\phi_{n-1} \circ \varrho_{n-1}) \cdot \ldots \cdot \int \dot{\varrho}_0 \cdot (\phi_0 \circ \varrho_0) \overset{(**)}{=} \int \phi_{n-1} \cdot \ldots \cdot \int \phi_0$$

$$\mathfrak{q}_\infty(\phi) \leq 2 \cdot \max(\mathfrak{q}_\infty(\phi_0), \ldots, \mathfrak{q}_\infty(\phi_{n-1})) \qquad \forall\, \mathfrak{q} \in \mathfrak{P}$$

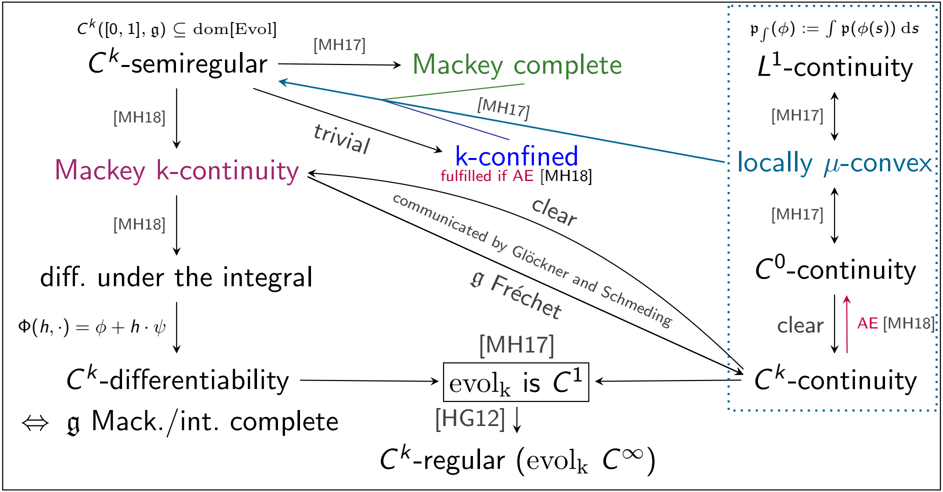


locally μ-convex: To each $\mathfrak{p} \in \mathfrak{P}$, there exists $\mathfrak{q} \in \mathfrak{P}$ with

$$(\mathfrak{p} \circ \Xi)\big(\Xi^{-1}(X_1) \cdot \ldots \cdot \Xi^{-1}(X_n)\big) \leq \mathfrak{q}(X_1) + \ldots + \mathfrak{q}(X_n)$$

if $\mathfrak{q}(X_1) + \ldots + \mathfrak{q}(X_n) \leq 1$; for all $n \geq 1$.

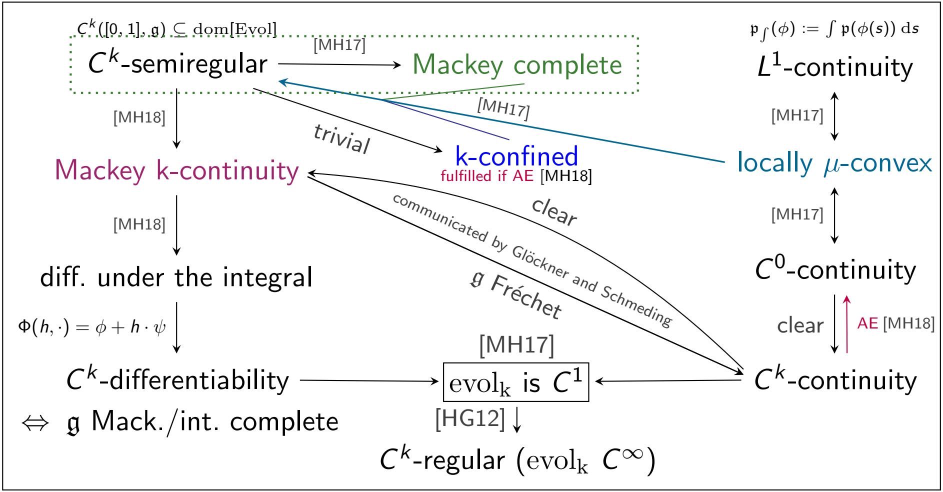
Introduced in [HG12]



AE (asymptotic estimate): To each $\mathfrak{v} \in \mathfrak{P}$, there exists $\mathfrak{v} \leq \mathfrak{w} \in \mathfrak{P}$, such that

$$\mathfrak{v}([X_1,[X_2,[\dots[X_n,Y]\dots]]]) \leq \mathfrak{w}(X_1)\cdot \ldots \cdot \mathfrak{w}(X_n)\cdot \mathfrak{w}(Y)$$

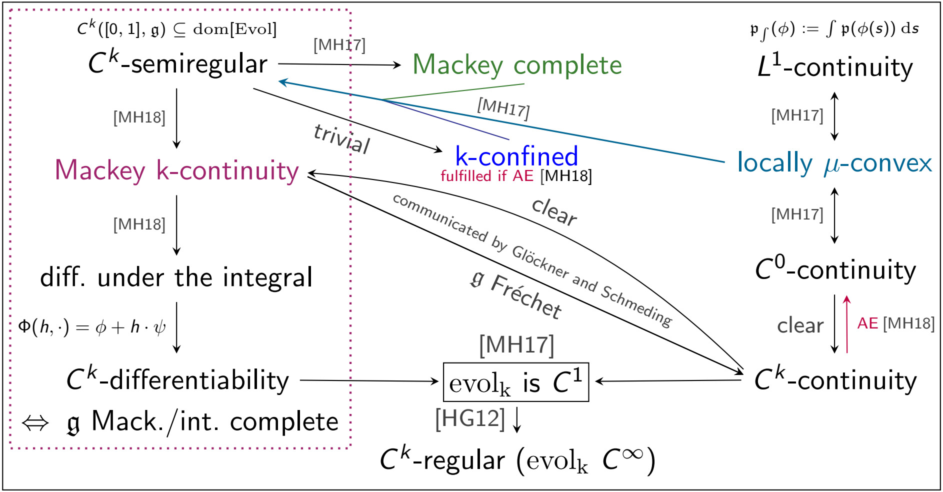
for $X_1,\dots,X_n, Y \in \mathfrak{g}$, $n \geq 1$. (Banach, abelian, nilpotent)



M. c.: Each Mackey-Cauchy sequence in G converges; i.e., each $\{g_n\}_{n\in\mathbb{N}} \subseteq G$ with

$$(\mathfrak{p}\circ\Xi)(g_m^{-1}\cdot g_n) \leq \mathfrak{c}_\mathfrak{p}\cdot\lambda_{m,n} \qquad \forall\, m,n\in\mathbb{N},\ \mathfrak{p}\in\mathfrak{P}$$

for $\{\mathfrak{c}_\mathfrak{p}\}_{\mathfrak{p}\in\mathfrak{P}} \subseteq \mathbb{R}_{\geq 0}$, $\mathbb{R}_{>0} \supseteq \{\lambda_{m,n}\}_{(m,n)\in\mathbb{N}\times\mathbb{N}} \to 0$.



M. k-c.: $\underbrace{\{\phi_n\}_{n\in\mathbb{N}} \rightharpoonup_{\mathfrak{m}.\mathrm{k}} \phi} \implies \lim_n^\infty \int_0^\bullet \phi_n = \int_0^\bullet \phi \qquad (\int_0^\bullet \phi_n \in \int_0^\bullet \phi \cdot U;\ n \geq N_U)$

Mackey-like convergence in the C^k-topology – e.g., for $k\in\mathbb{N}$:

$$\mathfrak{p}^k_\infty(\phi - \phi_n) \leq \mathfrak{c}_\mathfrak{p} \cdot \lambda_n \qquad \forall\, n \in \mathbb{N},\ \mathfrak{p}\in\mathfrak{P}$$

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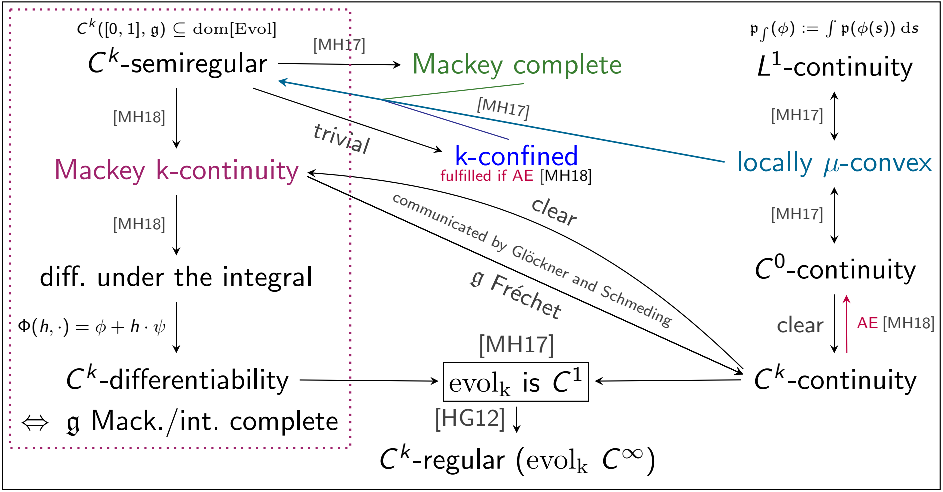
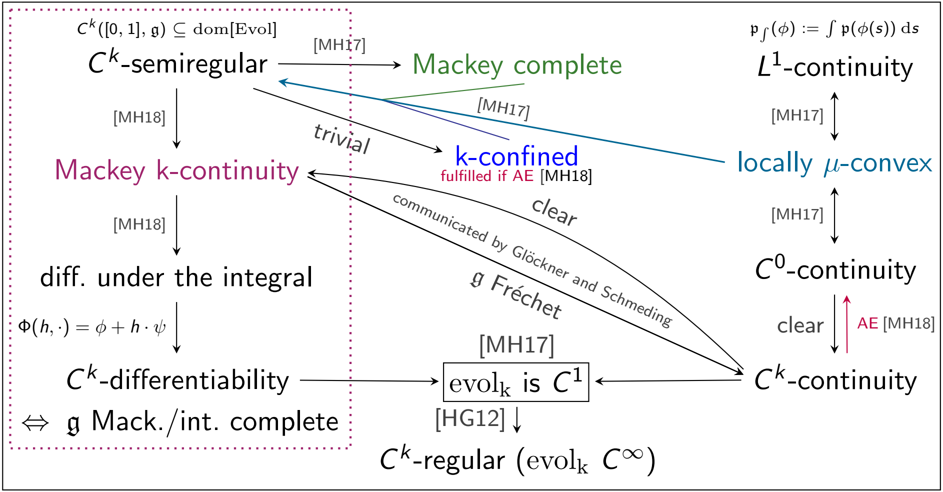


$$\frac{\mathrm{d}}{\mathrm{d}h}\Big|_{h=0} \int \Phi(h,\cdot) = \overline{\mathrm{d}_e \mathrm{L}_{\int \Phi(0,\cdot)}}\Big(\int \mathrm{Ad}_{[\int_0^s \Phi(0,\cdot)]^{-1}}(\partial_1\Phi(0,s))\,\mathrm{d}s\Big) \in \overline{T_{\int \Phi(0,\cdot)} G}$$

$\Phi\colon (-\varepsilon,\varepsilon)\times[0,1]\to\mathfrak{g}$ with $\Phi(h,\cdot)\in C^k([0,1],\mathfrak{g})$ and $\partial_1\Phi(0,\cdot)\in C^k([0,1],\mathfrak{g})$, such that to $\mathfrak{p}\in\mathfrak{P}$, $\mathrm{s}\preceq k$: $\exists\ L_{\mathfrak{p},\mathrm{s}}\geq 0<\varepsilon_{\mathfrak{p},\mathrm{s}}\leq\varepsilon$ with

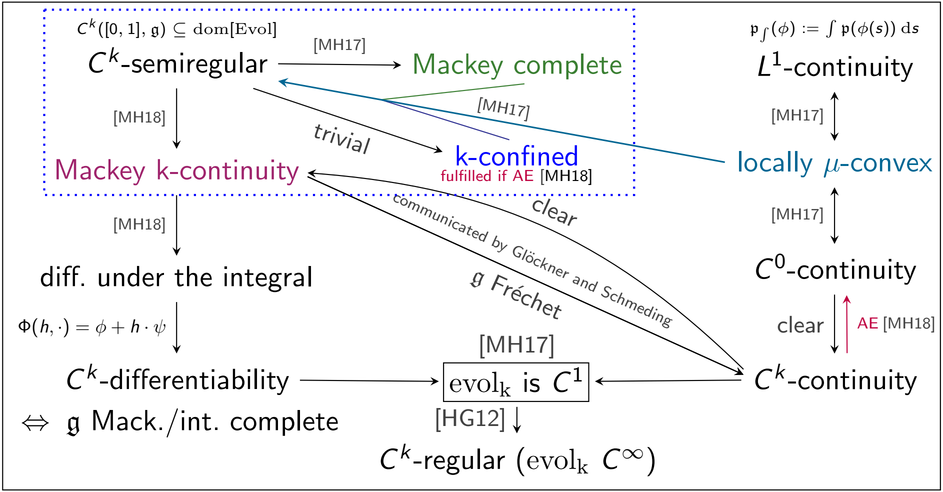
$$1/|h|\cdot\mathfrak{p}^{\mathrm{s}}_\infty(\Phi(h,\cdot)-\Phi(0,\cdot))\leq L_{\mathfrak{p},\mathrm{s}} \qquad \forall\, 0<|h|\leq\varepsilon_{\mathfrak{p},\mathrm{s}}.$$

Duhamel $\partial_h \exp(X+h\cdot Y)$: $\Phi(h,\cdot)=\phi_X+h\cdot\phi_Y$



$$\mathrm{d}_\phi \mathrm{evol_k}(\psi) = \mathrm{d}_e \mathrm{L}_{\int\phi}\Big(\int \mathrm{Ad}_{[\int_0^s \phi]^{-1}}(\psi(s))\,\mathrm{d}s\Big) \qquad \mathrm{evol_k} := \int\Big|_{C^k([0,1],\mathfrak{g})}$$

1. $\mathrm{d}_\phi\mathrm{evol_k}\colon C^k([0,1],\mathfrak{g}) \to T_{\int\phi}G$ is linear and C^0-continuous.
2. $\lim_{(n,\alpha)} \mathrm{d}_{\phi_n}\mathrm{evol_k}(\psi_\alpha) = \mathrm{d}_\phi\mathrm{evol_k}(\psi)$ holds for each
   - sequence $\{\phi_n\}_{m\in\mathbb{N}} \rightharpoonup_{\mathfrak{m}.\mathrm{k}} \phi$,
   - net $\{\psi_\alpha\}_{n\in\mathbb{N}} \rightharpoonup_{\mathfrak{n}.0} \psi$.



k-c: Can approximate $\phi \in C^k([0,1],\mathfrak{g})$ by M.-C. sequence $\{\phi_n\}_{n\in\mathbb{N}} \subseteq \mathfrak{D}\mathrm{P}([0,1],\mathfrak{g})$, such that to each $\mathfrak{q} \in \mathfrak{P}$, there exists $\mathfrak{m} \in \mathfrak{P}$ with

$$\mathfrak{q} \circ \mathrm{Ad}_{[\int_0^\bullet \phi_n]^{-1}} \leq \mathfrak{m} \qquad \forall\, n \in \mathbb{N}.$$

Let $\mu(t) := \lim_n \int_0^t \phi_n$ for each $t \in [0,1]$; and verify (ptw.) convergence, as well as the solution property $\delta^r(\mu) = \phi$. $\quad \left[\int_0^t \phi_n\right]^{-1}\left[\int_0^t \phi_m\right] = \int_0^t \mathrm{Ad}_{[\int_0^\bullet \phi_n]^{-1}}(\phi_n - \phi_m)$

Thank you for your Attention !

Strong Trotter Property

G has strong Trotter property *iff* for each $\mu \in C^1_*([0,1],G)$ with $\dot{\mu}(0) \in \mathrm{dom}[\exp]$, we have

$$\lim_n \mu(\tau/n)^n = \exp(\tau \cdot \dot{\mu}(0)) \qquad \forall\, \tau \in [0,\ell] \qquad (*)$$

uniformly for each $\ell > 0$.

Proposition [MH18]:

1. If G is sequentially 0-continuous, then G has strong Trotter property.
2. If G is Mackey 0-continuous, then $(*)$ holds for each $\mu \in C^1_*([0,1],G)$ with $\dot{\mu}(0) \in \mathrm{dom}[\exp]$ and $\delta^r(\mu) \in C^{\mathrm{lip}}([0,1],\mathfrak{g})$.

   Fulfilled, e.g., if G is C^0-semiregular, and μ is of class C^2.

Mackey Completeness

Mackey $\cong$ "uniform in $\mathfrak{P}$"

Theorem [MH17]: If G is C^∞-semiregular, then G is **Mackey complete**.

MC: Each Mackey-Cauchy sequence in G converges; i.e., each sequence $\{g_n\}_{n\in\mathbb{N}} \subseteq G$ with

$$(\mathfrak{p}\circ\Xi)(g_m^{-1}\cdot g_n) \leq \mathfrak{c}_\mathfrak{p}\cdot\lambda_{m,n} \qquad \forall\, m,n\in\mathbb{N},\ \ \mathfrak{p}\in\mathfrak{P}$$

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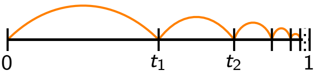
Sketch of the Proof:

- Pass to sufficiently fast decreasing subsequence of $\{g_n\}_{n\in\mathbb{N}}$.
- Use bump functions to construct from the "differences" $g_n^{-1}g_{n-1}$ some $\phi\in C^\infty([0,1],\mathfrak{g})$ with $\phi(1)=0$ and

$$\textstyle\int\phi|_{[t_n,t_{n+1}]} = g_n^{-1}g_{n-1} \qquad \text{for} \qquad t_0 := 0,\ t_n := \sum_{k=1}^n 2^{-k} \qquad \text{thus,}$$

$$\begin{aligned}(\textstyle\int\phi\cdot g_0^{-1})^{-1} &= \lim_n\,(\textstyle\int_0^{t_{n+1}}\phi\cdot g_0^{-1})^{-1}\\ &= \lim_n\,(g_n^{-1}g_{n-1}\cdot\ldots\cdot g_1^{-1}g_0\cdot g_0^{-1})^{-1}\\ &= \lim_n g_n.\end{aligned}$$



Mackey Completeness

Theorem [MH17]: If G is C^∞-semiregular, then G is **Mackey complete**.

MC: Each Mackey-Cauchy sequence in G converges; i.e., each sequence $\{g_n\}_{n\in\mathbb{N}} \subseteq G$ with

$$(\mathfrak{p} \circ \Xi)(g_m^{-1} \cdot g_n) \leq \mathfrak{c}_\mathfrak{p} \cdot \lambda_{m,n} \qquad \forall\, m, n \in \mathbb{N},\ \ \mathfrak{p} \in \mathfrak{P}$$

for $\{\mathfrak{c}_\mathfrak{p}\}_{\mathfrak{p}\in\mathfrak{P}} \subseteq \mathbb{R}_{\geq 0}$, $\mathbb{R}_{>0} \supseteq \{\lambda_{m,n}\}_{(m,n)\in\mathbb{N}\times\mathbb{N}} \to 0$.

Theorem [MH17]: If G locally μ-convex, then C^k-semiregular for $k \in \mathbb{N}_{\geq 1} \sqcup \{\mathrm{lip}, \infty\}$ if and only if G Mackey complete and **k-confined**.

k-c: Can approximate $\phi \in C^k([0,1], \mathfrak{g})$ by C^0-M.-C. sequence $\{\phi_n\}_{n\in\mathbb{N}} \subseteq \mathfrak{D}\mathrm{P}([0,1], \mathfrak{g})$, such that to each $\mathfrak{q} \in \mathfrak{P}$, there exists $\mathfrak{m} \in \mathfrak{P}$ with

$$\mathfrak{q} \circ \mathrm{Ad}_{[\int_0^\bullet \phi_n]^{-1}} \ \leq\ \mathfrak{m} \qquad \forall\, n \in \mathbb{N}. \qquad (*)$$

Automatically k-c: if $(\mathfrak{g}, [\cdot,\cdot])$ asymptotic estimate; hence, **[MH18]**

If G AE : $\quad C^k$-regular for $k > 0 \quad \Longleftrightarrow \quad C^\infty$-regular.

Mackey Completeness

Mackey $\cong$ "uniform in $\mathfrak{P}$"

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Idea of the Proof: Let $\int_0^t \phi := \lim_n \int_0^t \phi_n$ for each $t \in [0,1]$; and verify (ptw.) convergence, as well as the solution property $\delta^r(\int_0^\bullet \phi) = \phi$.

$$\big[\textstyle\int_0^t \phi_n\big]^{-1}\big[\int_0^t \phi_m\big] = \int_0^t \mathrm{Ad}_{[\int_0^\bullet \phi_n]^{-1}}(\phi_n - \phi_m) \quad \overset{(*)}{\Longrightarrow} \quad \{\int_0^t \phi_n\}_{n\in\mathbb{N}} \text{ is MC.}$$

Mackey Continuity

$\mathfrak{p}^k_\infty(\phi) = \max(0 \leq p \leq k \mid \mathfrak{p}_\infty(\phi^{(p)}))$

Theorem [MH18]: If G is C^k-semiregular for $k \in \mathbb{N} \sqcup \{\mathrm{lip}, \infty\}$, then G is **Mackey k-continuous**.

M. k-c.: $\underbrace{\{\phi_n\}_{n\in\mathbb{N}} \rightharpoonup_{\mathfrak{m}.\mathrm{k}} \phi}_{(*)} \quad \Longrightarrow \quad \underbrace{\lim_n^\infty \int_0^\bullet \phi_n = \int_0^\bullet \phi}_{(**)}$

(∗) Mackey-like convergence in the C^k-topology – e.g., for $k \in \mathbb{N}$ that

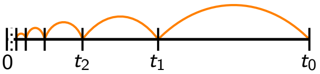
$$\mathfrak{p}^k_\infty(\phi - \phi_n) \leq \mathfrak{c}_\mathfrak{p} \cdot \lambda_n \qquad \forall\, n \in \mathbb{N},\ \mathfrak{p} \in \mathfrak{P}$$

for $\{\mathfrak{c}_\mathfrak{p}\}_{\mathfrak{p}\in\mathfrak{P}} \subseteq \mathbb{R}_{\geq 0}$, $\mathbb{R}_{>0} \supseteq \{\lambda_n\}_{n\in\mathbb{N}} \to 0$.

(∗∗) Uniform convergence: Given $e \in U \subseteq G$ neighbourhood, there exists $N_U \in \mathbb{N}$ with

$$\int_0^t \phi_n \in \int_0^t \phi \cdot U \qquad \forall\, t \in [0,1],\ n \geq N_U.$$

Mackey Continuity



Theorem [MH18]: If G is C^k-semiregular for $k \in \mathbb{N} \sqcup \{\mathrm{lip}, \infty\}$, then G is **Mackey k-continuous**.

Sketch of the Proof: (indirect argument)

- If claim is wrong, there exists $e \in U \subseteq G$ open and $\{\phi_n\}_{n\in\mathbb{N}} \rightharpoonup_{\mathfrak{m}.\mathrm{k}} 0$ with $\int_0^\bullet \phi_n \nsubseteq U$ for all $n \in \mathbb{N}$.
- Use bump functions $\rho_n \equiv \dot{\varrho}_n$ to construct $\phi \in C^k([0,1], \mathfrak{g})$ with $\phi(0) = 0$ and

$$\int_{t_{n+1}}^{t} \phi|_{[t_{n+1}, t_n]} = \int_0^{\varrho_n(t)} \phi_n \qquad \text{for} \qquad t_0 := 1,\ t_n := 1 - \sum_{k=1}^{n} 2^{-k}.$$

- Choose $e \in V \subseteq G$ open with $V \cdot V^{-1} \subseteq U$, as well as $0 < s \leq 1$ with $\int_0^t \phi \in V$ for $0 \leq t \leq s$.
- Obtain contradiction by

$$\int_0^{\varrho_n(t)} \phi_n = \int_{t_{n+1}}^{t} \phi|_{[t_{n+1}, t_n]} = [\int_0^t \phi] \cdot [\int_0^{t_{n+1}} \phi]^{-1} \in V \cdot V^{-1} \subseteq U,$$

for each $t \in [t_{n+1}, t_n]$ with $t_n \leq s$.