A unified approach for mixed formulations of elliptic problems with applications in structural mechanics

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Outline

EXAMPLE: linear elasticity

$$-\operatorname{Div}\sigma = \boldsymbol{f} \quad \text{and} \quad \sigma = \mathbb{C}\,\varepsilon(\boldsymbol{u}) \qquad \text{in } \Omega \subset \mathbb{R}^3,$$
$$\boldsymbol{u} = 0 \quad \text{on } \Gamma = \partial\Omega$$

stress tensor σ, displacement $\boldsymbol{u}$, force density $\boldsymbol{f}$.

$$\mathbb{C}\,\varepsilon = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-2\nu)\,\varepsilon + (1-\nu)\ \operatorname{tr}(\varepsilon)\,\boldsymbol{I}\right]$$

Young's modulus E, Poisson ratio ν

$$\varepsilon(\boldsymbol{u}) \equiv \operatorname{sym}\operatorname{Grad}\boldsymbol{u} \equiv \frac{1}{2}\left(\operatorname{Grad}\boldsymbol{u} + (\operatorname{Grad}\boldsymbol{u})^\top\right)$$

strain tensor $\varepsilon(\boldsymbol{u})$

primal variational formulation (principle of virtual work):

Find $\boldsymbol{u} \in \mathbf{H}_0^1(\Omega)$ such that

$$\left(\mathbb{C}\,\varepsilon(\boldsymbol{u}), \varepsilon(\boldsymbol{u})\right)_{\boldsymbol{\mathcal{L}}^2(\Omega)} = \left(\boldsymbol{f}, \boldsymbol{v}\right)_{\mathbf{L}^2(\Omega)} \;\left(\;\equiv F(\boldsymbol{v})\right) \quad \forall\, \boldsymbol{v} \in \mathbf{H}_0^1(\Omega)$$

stress as a **new variable**

$$\sigma = \mathbb{C}\varepsilon(\boldsymbol{u}) \in \boldsymbol{\mathcal{L}}^2(\Omega, \mathbb{S})$$

equivalent mixed formulation (Hellinger-Reissner principle):

Find $\sigma \in \boldsymbol{\mathcal{L}}^2(\Omega, \mathbb{S})$ and $\boldsymbol{u} \in \mathbf{H}_0^1(\Omega)$ such that

$$\begin{aligned} \left(\mathbb{C}^{-1}\sigma, \tau\right)_{\boldsymbol{\mathcal{L}}^2(\Omega)} \; - \left(\varepsilon(\boldsymbol{u}), \tau\right)_{\boldsymbol{\mathcal{L}}^2(\Omega)} &= 0 && \forall\, \tau \in \boldsymbol{\mathcal{L}}^2(\Omega, \mathbb{S}), \\ - \left(\sigma, \varepsilon(\boldsymbol{v})\right)_{\boldsymbol{\mathcal{L}}^2(\Omega)} &= -F(\boldsymbol{v}) && \forall\, \boldsymbol{v} \in \mathbf{H}_0^1(\Omega). \end{aligned}$$

original mixed formulation:

Find $\sigma \in \mathcal{L}^2(\Omega, \mathbb{S})$ and $\boldsymbol{u} \in \mathbf{H}_0^1(\Omega)$ such that

$$\begin{aligned} \left(\mathbb{C}^{-1}\sigma, \tau\right)_{\mathcal{L}^2(\Omega)} \quad - \left(\operatorname{Grad} \boldsymbol{u}, \tau\right)_{\mathcal{L}^2(\Omega)} &= 0 \qquad \forall\, \tau \in \mathcal{L}^2(\Omega, \mathbb{S}), \\ - \left(\operatorname{Grad} \boldsymbol{v}, \sigma\right)_{\mathcal{L}^2(\Omega)} &= -F(\boldsymbol{v}) \quad \forall\, \boldsymbol{v} \in \mathbf{H}_0^1(\Omega). \end{aligned}$$

new mixed formulation:

Find $\sigma \in \mathcal{H}(\operatorname{Div}, \Omega, \mathbb{S})$ and $\boldsymbol{u} \in \mathbf{L}^2(\Omega)$ such that

$$\begin{aligned} \left(\mathbb{C}^{-1}\sigma, \tau\right)_{\mathcal{L}^2(\Omega)} + \left(\operatorname{Div} \tau, \boldsymbol{u}\right)_{\mathbf{L}^2(\Omega)} &= 0 \qquad \forall\, \tau \in \mathcal{H}(\operatorname{Div}, \Omega, \mathbb{S}), \\ \left(\operatorname{Div} \sigma, \boldsymbol{v}\right)_{\mathbf{L}^2(\Omega)} &= -F(\boldsymbol{v}) \quad \forall\, \boldsymbol{v} \in \mathbf{L}^2(\Omega). \end{aligned}$$

Introduction

transition from the first to the second mixed problem:

$$-\big(\,\mathrm{Grad}\,\boldsymbol{v},\tau\big)_{\mathcal{L}^2(\Omega)} = \big(\,\mathrm{Div}\,\tau,\boldsymbol{v}\big)_{\mathbf{L}^2(\Omega)} \quad \text{for all } \tau\in\mathcal{H}(\mathrm{Div},\Omega),\ \boldsymbol{v}\in\mathbf{H}^1_0(\Omega).$$

The associated linear operators

$$\langle B\,\boldsymbol{v},\tau\rangle = -\big(\,\mathrm{Grad}\,\boldsymbol{v},\tau\big)_{\mathcal{L}^2(\Omega)} \quad \text{and} \quad \langle B^*\tau,\boldsymbol{v}\rangle = \big(\,\mathrm{Div}\,\tau,\boldsymbol{v}\big)_{\mathbf{L}^2(\Omega)}.$$

B is considered as a densely defined linear operators:

$$B\colon\ \mathbf{H}^1_0(\Omega) = \mathrm{dom}(B) \subset \mathbf{L}^2(\Omega) \longrightarrow [\mathcal{L}^2(\Omega,\mathbb{S})]^*.$$

Then B^* is its adjoint

$$B^*\colon\ \mathrm{dom}(B^*) = \mathcal{H}(\mathrm{Div},\Omega,\mathbb{S}) \subset \mathcal{L}^2(\Omega,\mathbb{S}) \longrightarrow [\mathbf{L}^2(\Omega)]^*$$

Note that B and B^* are unbounded operators.

The abstract framework

Let X and Y be two Hilbert spaces and let

$$T\colon \operatorname{dom}(T) \subset X \longrightarrow Y^*$$

be a linear operator, whose **domain of definition**

$$\operatorname{dom}(T) \quad \text{is dense in} \quad X.$$

T is called a **densely defined** linear operator.

The **domain of definition** $\operatorname{dom}(T^*)$ **of the adjoint operator**

$$T^*\colon \operatorname{dom}(T^*) \subset Y \longrightarrow X^*$$

is the **maximal subspace** of Y, where the identity

$$\langle T^* y, x\rangle = \langle Tx, y\rangle \quad \text{for all } x \in \operatorname{dom}(T),\ y \in \operatorname{dom}(T^*).$$

makes sense. More directly,

$$\operatorname{dom}(T^*) = \{y \in Y\colon |\langle Tx, y\rangle| \lesssim \|x\|_X \text{ for all } x \in \operatorname{dom}(T)\}.$$

- The adjoint T^* of a densely defined linear operator is **well-defined.**
- The adjoint operator T^* is **closed**, i.e.,
  - the graph of T^* is a closed subset of $Y \times X^*$, or, equivalently,
  - $\mathrm{dom}(T^*)$ **is a Hilbert space w.r.t. the graph norm** $\|.\|_{T^*}$, given by

$$\|y\|_{T^*}^2 = \|y\|_Y^2 + \|T^* y\|_{X^*}^2.$$

The abstract framework

Saddle point problem: Find $u \in V$ and $p \in \mathrm{dom}(B)$ such that

$$\begin{aligned} a(u,v) + \langle Bp, v\rangle &= \langle f, v\rangle \quad \text{for all } v \in V \\ \langle Bq, u\rangle - c(p,q) &= \langle g, q\rangle \quad \text{for all } q \in \mathrm{dom}(B), \end{aligned}$$

with a densely defined linear operator

$$B\colon \mathrm{dom}(B) \subset Q \longrightarrow V^*$$

Assumptions:

1. $(V, \|.\|_V)$, $(\mathrm{dom}(B), \|.\|_{\mathrm{dom}(B)})$, $(Q, \|.\|_Q)$ are Hilbert spaces with

   $$\|q\|_Q \le c_B \|q\|_{\mathrm{dom}(B)} \quad \text{for all } q \in \mathrm{dom}(B)$$

   for some constant c_B;
2. $f \in V^*$, $g \in Q^*$;
3. the bilinear form a is symmetric, non-negative, and bounded on V;
4. the bilinear form c is symmetric, non-negative, and bounded on Q.

first saddle point problem:

Find $u \in V$ and $p \in \mathrm{dom}(B)$ such that

$$\begin{aligned} a(u,v) + \langle Bp, v\rangle &= \langle f, v\rangle \quad \text{for all } v \in V \\ \langle Bq, u\rangle - c(p,q) &= \langle g, q\rangle \quad \text{for all } q \in \mathrm{dom}(B), \end{aligned}$$

with a densely defined linear operator

$$B\colon \mathrm{dom}(B) \subset Q \longrightarrow V^*$$

second saddle point problem:

Find $u \in \mathrm{dom}(B^*)$ and $p \in Q$ such that

$$\begin{aligned} a(u,v) \quad + \langle B^* v, p\rangle &= \langle f, v\rangle \quad \text{for all } v \in \mathrm{dom}(B^*) \\ \langle B^* u, q\rangle - c(p,q) \quad &= \langle g, q\rangle \quad \text{for all } q \in Q, \end{aligned}$$

with the adjoint

$$B^*\colon \mathrm{dom}(B^*) \subset V \longrightarrow Q^*$$

The abstract framework

associated linear operators:

$$\mathcal{A}_1 : X_1 \longrightarrow X_1^*, \quad X_1 = V \times \mathrm{dom}(B)$$

with

$$\left\langle \mathcal{A}_1 \begin{bmatrix} u \\ p \end{bmatrix}, \begin{bmatrix} v \\ q \end{bmatrix} \right\rangle = a(u,v) + \langle Bp, v \rangle + \langle Bq, u \rangle - c(p,q)$$

and

$$\mathcal{A}_2 : X_2 \longrightarrow X_2^*, \quad X_2 = \mathrm{dom}(B^*) \times Q$$

with

$$\left\langle \mathcal{A}_2 \begin{bmatrix} u \\ p \end{bmatrix}, \begin{bmatrix} v \\ q \end{bmatrix} \right\rangle = a(u,v) + \langle B^* v, p \rangle + \langle B^* u, q \rangle - c(p,q)$$

The abstract framework

Lemma

Let $(u, p) \in V \times \mathrm{dom}(B)$ be a ***solution to the first saddle point problem****. Then $u \in \mathrm{dom}(B^*)$ and (u, p) is also a* ***solution to the second saddle point problem****.*

Theorem

If the linear operator $\mathcal{A}_1$ is an ***isomorphism*** *from $X_1 = V \times \mathrm{dom}(B)$ to its dual, then the linear operator $\mathcal{A}_2$ is an* ***isomorphism*** *from $X_2 = \mathrm{dom}(B^*) \times V$ to its dual and*

$$\|\mathcal{A}_2\|_{L(X_2, X_2^*)} \leq \overline{c}, \quad \|\mathcal{A}_2^{-1}\|_{L(X_2^*, X_2)} \leq \frac{1}{\underline{c}}$$

with ***constants*** *$\underline{c}, \overline{c}$* ***that depend only on*** *$\|\mathcal{A}_1\|_{L(X_1, X_1^*)}$, $\|\mathcal{A}_1^{-1}\|_{L(X_1^*, X_1)}$, c_B,* ***and*** *$\|c\|_Q$.*

The abstract framework

EXAMPLE: Hellinger-Reissner principle (revisited):

Find $\sigma \in \boldsymbol{\mathcal{L}}^2(\Omega, \mathbb{S})$ and $\boldsymbol{u} \in \mathbf{H}_0^1(\Omega)$ such that

$$\begin{aligned} \big(\mathbb{C}^{-1}\sigma, \tau\big)_{\boldsymbol{\mathcal{L}}^2(\Omega)} \quad - \big(\,\mathrm{Grad}\,\boldsymbol{u}, \tau\big)_{\boldsymbol{\mathcal{L}}^2(\Omega)} &= 0 \qquad \forall\, \tau \in \boldsymbol{\mathcal{L}}^2(\Omega, \mathbb{S}), \\ -\big(\,\mathrm{Grad}\,\boldsymbol{v}, \sigma\big)_{\boldsymbol{\mathcal{L}}^2(\Omega)} \qquad\qquad &= -F(\boldsymbol{v}) \quad \forall\, \boldsymbol{v} \in \mathbf{H}_0^1(\Omega). \end{aligned}$$

associated linear operator B:

$$\big\langle B\boldsymbol{v}, \tau \big\rangle = -\big(\,\mathrm{Grad}\,\boldsymbol{v}, \tau\big)_{\boldsymbol{\mathcal{L}}^2(\Omega)}$$

as a densely defined linear operator

$$\begin{aligned} B\colon \mathbf{H}_0^1(\Omega) \subset \mathbf{L}^2(\Omega) &\longrightarrow \big[\boldsymbol{\mathcal{L}}^2(\Omega, \mathbb{S})\big]^* \quad \checkmark \\ B\colon \mathbf{H}_0^1(\Omega) \subset \mathbf{H}_0(\mathrm{curl}, \Omega) &\longrightarrow \big[\boldsymbol{\mathcal{L}}^2(\Omega, \mathbb{S})\big]^* \\ B\colon \mathbf{H}_0^1(\Omega) \subset \mathbf{H}_0(\mathrm{div}, \Omega) &\longrightarrow \big[\boldsymbol{\mathcal{L}}^2(\Omega, \mathbb{S})\big]^* \end{aligned}$$

second option

$$B\colon \mathbf{H}_0^1(\Omega) \subset \mathbf{H}_0(\mathrm{curl},\Omega) \longrightarrow \left[\boldsymbol{\mathcal{L}}^2(\Omega,\mathbb{S})\right]^*, \quad \left\langle B\boldsymbol{v},\tau\right\rangle = -\left(\,\mathrm{Grad}\,\boldsymbol{v},\tau\right)_{\boldsymbol{\mathcal{L}}^2(\Omega)}$$

Then

$$\begin{aligned}\mathrm{dom}(B^*) &= \left\{\tau \in \boldsymbol{\mathcal{L}}^2(\Omega,\mathbb{S})\colon \left|\left(\tau,\mathrm{Grad}\,\boldsymbol{v}\right)_{\boldsymbol{\mathcal{L}}^2(\Omega)}\right| \lesssim \|\boldsymbol{v}\|_{\mathbf{H}(\mathrm{curl},\Omega)}, \forall\ \boldsymbol{v} \in \mathbf{H}_0^1(\Omega)\right\}\\ &= \left\{\tau \in \boldsymbol{\mathcal{L}}^2(\Omega,\mathbb{S})\colon \left|\left(\tau,\nabla^2\varphi\right)_{\boldsymbol{\mathcal{L}}^2(\Omega)}\right| \lesssim \|\varphi\|_{H^1(\Omega)}\ \forall\ \varphi \in H_0^2(\Omega)\right\}\\ &= \left\{\tau \in \boldsymbol{\mathcal{L}}^2(\Omega,\mathbb{S})\colon\ \mathrm{divDiv}\,\tau \in H^{-1}(\Omega)\right\} \equiv \Sigma, \quad B^* \equiv \mathrm{Div}_\Sigma\end{aligned}$$

new mixed formulation:

Find $\sigma \in \Sigma$ and $\boldsymbol{u} \in \mathbf{H}_0(\mathrm{curl},\Omega)$ such that

$$\begin{aligned}\left(\mathbb{C}^{-1}\sigma,\tau\right)_{\boldsymbol{\mathcal{L}}^2(\Omega)} + \left\langle\,\mathrm{Div}_\Sigma\,\tau,\boldsymbol{u}\right\rangle &= 0 && \forall\ \tau \in \Sigma,\\ \left\langle\,\mathrm{Div}_\Sigma\,\sigma,\boldsymbol{v}\right\rangle &= -F(\boldsymbol{v}) && \forall\ \boldsymbol{v} \in \mathbf{H}_0(\mathrm{curl},\Omega)\end{aligned}$$

$\Sigma = \underline{\underline{H}}(divdiv)$: Pechstein/Schöberl (2011,2018).

third option

$$B\colon \mathbf{H}^1_0(\Omega) \subset \mathbf{H}_0(\mathrm{div},\Omega) \longrightarrow \big[\boldsymbol{\mathcal{L}}^2(\Omega)\big]^*, \quad \big\langle B\boldsymbol{v},\tau\big\rangle = -\big(\,\mathrm{Grad}\,\boldsymbol{v},\tau\big)_{\boldsymbol{\mathcal{L}}^2(\Omega)}$$

Then

$$\begin{aligned}\mathrm{dom}(B^*) &= \Big\{\tau \in \boldsymbol{\mathcal{L}}^2(\Omega,\mathbb{S})\colon \Big|\big(\tau,\mathrm{Grad}\,\boldsymbol{v}\big)_{\boldsymbol{\mathcal{L}}^2(\Omega)}\Big| \lesssim \|\boldsymbol{v}\|_{\mathbf{H}(\mathrm{div},\Omega)}\ \forall\ \boldsymbol{v} \in \mathbf{H}^1_0(\Omega)\Big\}\\ &= \Big\{\tau \in \boldsymbol{\mathcal{L}}^2(\Omega,\mathbb{S})\colon \Big|\big(\tau,\mathrm{Grad}(\mathrm{curl}\,\phi)\big)_{\boldsymbol{\mathcal{L}}^2(\Omega)}\Big| \lesssim \|\phi\|_{\mathbf{H}^1(\Omega)}\ \forall\ \phi \in \mathbf{H}^1_0(\mathrm{curl},\Omega)\Big\}\\ &= \Big\{\tau \in \boldsymbol{\mathcal{L}}^2(\Omega,\mathbb{S})\colon\ \mathrm{curl}\,\mathrm{Div}\,\tau \in \mathbf{H}^{-1}(\Omega)\Big\} \equiv \widetilde{\Sigma}, \quad B^* \equiv \mathrm{Div}_{\widetilde{\Sigma}}\end{aligned}$$

new mixed formulation:
Find $\sigma \in \widetilde{\Sigma}$ and $\boldsymbol{u} \in \mathbf{H}_0(\mathrm{div},\Omega)$ such that

$$\begin{aligned}\big(\mathbb{C}^{-1}\sigma,\tau\big)_{\boldsymbol{\mathcal{L}}^2(\Omega)} + \big\langle\,\mathrm{Div}_{\widetilde{\Sigma}}\,\tau,\boldsymbol{u}\big\rangle &= 0 && \forall\ \tau \in \widetilde{\Sigma},\\ \big\langle\,\mathrm{Div}_{\widetilde{\Sigma}}\,\sigma,\boldsymbol{v}\big\rangle &= -F(\boldsymbol{v}) && \forall\ \boldsymbol{v} \in \mathbf{H}_0(\mathrm{div},\Omega).\end{aligned}$$

$\widetilde{\Sigma} = \underline{\underline{H}}(\mathit{curldiv})$: Gopalakrishan/Lederer/Schöberl (2018,2019)

The Reissner-Mindlin plate bending model

We consider a plate with **mid-surface** $\Omega \subset \mathbb{R}^2$ and **thickness** t.

The **Reissner-Mindlin model** for a clamped plate:

Find $(u, \theta) \in \mathrm{H}_0^1(\Omega) \times \mathbf{H}_0^1(\Omega)$ such that

$$\left(\mathbb{D}\,\varepsilon(\theta), \varepsilon(\eta)\right)_{\mathcal{L}^2(\Omega)} + t^{-2}\left(\boldsymbol{G}\,(\operatorname{grad} u - \theta), \operatorname{grad} v - \eta\right)_{\mathbf{L}^2(\Omega)} = F(v, \eta)$$

for all $(v, \eta) \in \mathrm{H}_0^1(\Omega) \times \mathbf{H}_0^1(\Omega)$.

u **deflection**, θ **rotation of normals** to the mid-surface, $F(v, \eta)$ **load**,

$$\mathbb{D}\,\varepsilon = \frac{E}{12(1-\nu^2)}\left[(1-\nu)\,\varepsilon + \nu\ \mathrm{tr}(\varepsilon)\,\boldsymbol{I}\right], \quad \boldsymbol{G} = \frac{k_s\,E}{2(1+\nu)}\,\boldsymbol{I}.$$

Kirchhoff's model for a clamped plate: $\theta = \operatorname{grad} u$.

Find $u \in \mathrm{H}_0^2(\Omega)$ such that

$$\left(\mathbb{D}\,\nabla^2 u, \nabla^2 v\right)_{\mathcal{L}^2(\Omega)} = F(v, \operatorname{grad} v) \quad \forall\, v \in \mathrm{H}_0^2(\Omega)$$

The Reissner-Mindlin plate bending model

Standard finite element methods suffer from **locking effects.**

One possible remedy: replace θ by the **shear strain vector**

$$\gamma = t^{-1}(\operatorname{grad} u - \theta).$$

or, equivalently,

$$\theta = \operatorname{grad} u - t\,\gamma.$$

Problem: appropriate function space for γ resp. (u, γ)

Beirão da Veiga et.al. (2015): $(u, \gamma) \in \mathcal{X}$ with

$$\mathrm{H}_0^2(\Omega) \times \mathbf{H}_0^1(\Omega) \subset \mathcal{X} \subset \mathrm{H}_0^1(\Omega) \times \mathbf{L}^2(\Omega)$$

We present here a **different approach** with

$$\theta \in \mathbf{H}_0(\mathrm{rot}, \Omega), \quad \text{then} \quad \gamma \in \mathbf{H}_0(\mathrm{rot}, \Omega),$$

where

$$\operatorname{rot} \phi = \partial_1 \phi_2 - \partial_2 \phi_1.$$

The Reissner-Mindlin plate bending model

bending moment tensor (scaled by t^{-3}) as a **new variable**

$$\boldsymbol{M} = -\mathbb{D}\,\varepsilon(\theta),$$

equivalent mixed formulation:

Find $(\boldsymbol{M}, \theta, u) \in \boldsymbol{\mathcal{L}}^2(\Omega, \mathbb{S}) \times \mathbf{H}^1_0(\Omega) \times \mathrm{H}^1_0(\Omega)$ such that

$$\begin{aligned}
\left(\mathbb{D}^{-1}\boldsymbol{M}, \boldsymbol{K}\right)_{\boldsymbol{\mathcal{L}}^2(\Omega)} + \left(\varepsilon(\theta), \boldsymbol{K}\right)_{\boldsymbol{\mathcal{L}}^2(\Omega)} &= 0, \\
\left(\boldsymbol{M}, \varepsilon(\eta)\right)_{\boldsymbol{\mathcal{L}}^2(\Omega)} \quad - t^{-2}\left(\boldsymbol{G}\,(\operatorname{grad} u - \theta), \operatorname{grad} v - \eta\right)_{\mathbf{L}^2(\Omega)} &= -F(v, \eta)
\end{aligned}$$

for all $(\boldsymbol{K}, \eta, v) \in \boldsymbol{\mathcal{L}}^2(\Omega, \mathbb{S}) \times \mathbf{H}^1_0(\Omega) \times \mathrm{H}^1_0(\Omega)$.

this first mixed formulation is well-posed with respect to the norm

$$\|(\boldsymbol{M}, \theta, u)\| = \left[\|\boldsymbol{M}\|^2_{\boldsymbol{\mathcal{L}}^2(\Omega)} + \|u\|^2_{\mathrm{H}^1(\Omega)} + \|\theta\|^2_{\mathbf{H}^1(\Omega)} + t^{-2}\,\|\theta - \operatorname{grad} u\|^2_{\mathbf{L}^2(\Omega)}\right]^{1/2}$$

with uniform bounds independent of t.

The Reissner-Mindlin plate bending model

associated linear operator B:

$$B\colon \mathbf{H}_0^1(\Omega) \subset \mathbf{H}_0(\mathrm{rot},\Omega) \longrightarrow \left[\boldsymbol{\mathcal{L}}^2(\Omega,\mathbb{S})\right]^*, \quad \langle B\eta, \boldsymbol{K}\rangle = \left(\mathrm{Grad}\,\eta, \boldsymbol{K}\right)_{\boldsymbol{\mathcal{L}}^2(\Omega)}$$

adjoint operator:

$$B^*\colon\ \mathrm{dom}(B^*) \subset \boldsymbol{\mathcal{L}}^2(\Omega,\mathbb{S}) \longrightarrow \left[\mathbf{H}_0(\mathrm{rot},\Omega)\right]^*$$

with

$$\mathrm{dom}(B^*) = \left\{\boldsymbol{K} \in \boldsymbol{\mathcal{L}}^2(\Omega,\mathbb{S})\colon\ \mathrm{divDiv}_{\mathcal{M}}\,\boldsymbol{K} \in \mathrm{H}^{-1}(\Omega)\right\} \equiv \mathcal{M}, \quad B^* \equiv -\,\mathrm{Div}_{\mathcal{M}}$$

second mixed formulation:

Find $(\boldsymbol{M},\theta,u) \in \mathcal{M} \times \mathbf{H}_0(\mathrm{rot},\Omega) \times \mathrm{H}_0^1(\Omega)$ such that

$$\begin{aligned} \left(\mathbb{D}^{-1}\boldsymbol{M},\boldsymbol{K}\right)_{\boldsymbol{\mathcal{L}}^2(\Omega)} - \left\langle\, \mathrm{Div}_{\mathcal{M}}\,\boldsymbol{K},\theta\right\rangle \qquad &= 0 \quad ,\\ -\left\langle\, \mathrm{Div}_{\mathcal{M}}\,\boldsymbol{M},\boldsymbol{v}\right\rangle - t^{-2}\left(\boldsymbol{G}\,(\mathrm{grad}\,u-\theta),\mathrm{grad}\,v-\eta\right)_{\mathbf{L}^2(\Omega)} &= -F(v,\eta) \end{aligned}$$

for all $(\boldsymbol{K},\eta,v) \in \mathcal{M} \times \mathbf{H}_0(\mathrm{rot},\Omega) \times \mathrm{H}_0^1(\Omega)$. Pechstein/Schöberl (2017).

The Reissner-Mindlin plate bending model

change of variables:

$$\theta = \operatorname{grad} u - t\,\gamma \quad \text{with} \quad u \in \mathrm{H}^1_0(\Omega), \quad \gamma \in \mathbf{H}_0(\mathrm{rot}, \Omega)$$

decomposition of the bending moment:

$$\boldsymbol{M} = p\,\boldsymbol{I} + \boldsymbol{M}_0 \quad \text{with} \quad p \in \mathrm{H}^1_0(\Omega), \quad \operatorname{divDiv}_{\mathcal{M}} \boldsymbol{M}_0 = 0$$

Ω is simply connected:

$$\boldsymbol{M}_0 = \operatorname{symCurl} \phi, \quad \phi \in \mathbf{H}^1(\Omega) \quad \text{with} \quad \operatorname{Curl} \phi = \begin{bmatrix} \partial_2 \phi_1 & -\partial_1 \phi_1 \\ \partial_2 \phi_2 & -\partial_1 \phi_2 \end{bmatrix}$$

Hence

$$\boldsymbol{M} = p\,\boldsymbol{I} + \operatorname{symCurl} \phi \quad \text{with} \quad p \in \mathrm{H}^1_0(\Omega), \quad \phi \in \mathbf{H}^1(\Omega)$$

The Reissner-Mindlin plate bending model

Find $(\boldsymbol{M}, \theta, u) \in \mathcal{M} \times \mathbf{H}_0(\mathrm{rot}, \Omega) \times \mathrm{H}_0^1(\Omega)$ such that

$$\begin{aligned}
\left(\mathbb{D}^{-1}\boldsymbol{M}, \boldsymbol{K}\right)_{\mathcal{L}^2(\Omega)} - \left\langle \mathrm{Div}_{\mathcal{M}}\, \boldsymbol{K}, \theta \right\rangle \quad &= 0 \quad , \\
-\left\langle \mathrm{Div}_{\mathcal{M}}\, \boldsymbol{M}, \eta \right\rangle - t^{-2}\left(\boldsymbol{G}\,(\mathrm{grad}\, u - \theta), (\mathrm{grad}\, v - \eta)\right)_{\mathbf{L}^2(\Omega)} &= -F(v, \eta)
\end{aligned}$$

for all $(\boldsymbol{K}, \eta, v) \in \mathcal{M} \times \mathbf{H}_0(\mathrm{rot}, \Omega) \times \mathrm{H}_0^1(\Omega)$.

Find $(p, \phi, \gamma, u) \in \mathrm{H}_0^1(\Omega) \times \mathbf{H}^1(\Omega) \times \mathbf{H}_0(\mathrm{rot}, \Omega) \times \mathrm{H}_0^1(\Omega)$ such that

$$\begin{aligned}
\left(\mathbb{D}^{-1}(p\,\boldsymbol{I} + \mathrm{symCurl}\,\phi), (q\,\boldsymbol{I})\right)_{\mathcal{L}^2(\Omega)} - \left(\mathrm{grad}\, q, (\mathrm{grad}\, u - t\,\gamma)\right)_{\mathbf{L}^2(\Omega)} &= 0 \quad , \\
\left(\mathbb{D}^{-1}(p\,\boldsymbol{I} + \mathrm{symCurl}\,\phi), (\mathrm{symCurl}\,\psi)\right)_{\mathcal{L}^2(\Omega)} + \frac{t}{2}\left(\mathrm{div}\,\psi, \mathrm{rot}\,\gamma\right)_{\mathrm{L}^2(\Omega)} &= 0 \quad , \\
t\left(\mathrm{grad}\, p, \zeta\right)_{\mathbf{L}^2(\Omega)} + \frac{t}{2}\left(\mathrm{div}\,\phi, \mathrm{rot}\,\zeta\right)_{\mathrm{L}^2(\Omega)} - \left(\boldsymbol{G}\gamma, \zeta\right)_{\mathbf{L}^2(\Omega)} &= t\, F(0, \zeta) \\
-\left(\mathrm{grad}\, p, \mathrm{grad}\, v\right)_{\mathbf{L}^2(\Omega)} &= -F(v, \mathrm{grad}\, v)
\end{aligned}$$

for all $(q, \psi, \zeta, v) \in \mathrm{H}_0^1(\Omega) \times \mathbf{H}^1(\Omega) \times \mathbf{H}_0(\mathrm{rot}, \Omega) \times \mathrm{H}_0^1(\Omega)$.

The Reissner-Mindlin plate bending model

three (consecutively to solve) second-order problems:

Find $p \in H^1_0(\Omega)$ such that

$$\left(\operatorname{grad} p, \operatorname{grad} v\right)_{\mathbf{L}^2(\Omega)} = F(v, \operatorname{grad} v) \quad \forall\, v \in H^1_0(\Omega).$$

Find $(\boldsymbol{\phi}, \boldsymbol{\gamma}) \in \mathbf{H}^1(\Omega) \times \mathbf{H}_0(\operatorname{rot}, \Omega)$ such that

$$\begin{aligned}
\left(\mathbb{D}^{-1} \operatorname{symCurl} \boldsymbol{\phi}, \operatorname{symCurl} \boldsymbol{\psi}\right)_{\boldsymbol{\mathcal{L}}^2(\Omega)} + \frac{t}{2}\left(\operatorname{div} \boldsymbol{\psi}, \operatorname{rot} \boldsymbol{\gamma}\right)_{L^2(\Omega)} &= -\left(p, \operatorname{tr}(\mathbb{D}^{-1} \operatorname{symCurl} \boldsymbol{\psi})\right)_{L^2(\Omega)},\\
\frac{t}{2}\left(\operatorname{div} \boldsymbol{\phi}, \operatorname{rot} \boldsymbol{\zeta}\right)_{L^2(\Omega)} - \left(\boldsymbol{G}\boldsymbol{\gamma}, \boldsymbol{\zeta}\right)_{\mathbf{L}^2(\Omega)} &= t\, F(0, \boldsymbol{\zeta}) - t\left(\operatorname{grad} p, \boldsymbol{\zeta}\right)_{\mathbf{L}^2(\Omega)}
\end{aligned}$$

for all $(\boldsymbol{\psi}, \boldsymbol{\zeta}) \in \mathbf{H}^1(\Omega) \times \mathbf{H}_0(\operatorname{rot}, \Omega)$.

Find $u \in H^1_0(\Omega)$ such that

$$\left(\operatorname{grad} u, \operatorname{grad} q\right)_{\mathbf{L}^2(\Omega)} = \int_\Omega \left[\operatorname{tr}\left(\mathbb{D}^{-1}(p\,\boldsymbol{I} + \operatorname{symCurl} \boldsymbol{\phi})\right) q + t\,\boldsymbol{\gamma} \cdot \operatorname{grad} q\right] dx \quad \forall\, q \in H^1_0(\Omega).$$

Amara et.al. (2002), Gallistl (2017), Krendl/Rafetseder/Z. (2016), Rafetseder/Z. (2018)

The Reissner-Mindlin plate bending model

Find $(\phi, \gamma) \in \mathbf{H}^1(\Omega) \times \mathbf{H}_0(\mathrm{rot}, \Omega)$ such that

$$\begin{aligned}
\left(\mathbb{D}^{-1}\,\mathrm{symCurl}\,\phi, \mathrm{symCurl}\,\psi\right)_{\mathcal{L}^2(\Omega)} + \frac{t}{2}\left(\mathrm{div}\,\psi, \mathrm{rot}\,\gamma\right)_{\mathrm{L}^2(\Omega)} &= -\left(p, \mathrm{tr}(\mathbb{D}^{-1}\,\mathrm{symCurl}\,\psi)\right)_{\mathrm{L}^2(\Omega)}, \\
\frac{t}{2}\left(\mathrm{div}\,\phi, \mathrm{rot}\,\zeta\right)_{\mathrm{L}^2(\Omega)} - \left(\boldsymbol{G}\gamma, \zeta\right)_{\mathbf{L}^2(\Omega)} &= t\,F(0, \zeta) - t\left(\mathrm{grad}\,p, \zeta\right)_{\mathbf{L}^2(\Omega)}
\end{aligned}$$

for all $(\psi, \zeta) \in \mathbf{H}^1(\Omega) \times \mathbf{H}_0(\mathrm{rot}, \Omega)$.

locking-free finite element spaces:

$$\phi_h \subset \Phi_h \subset \mathbf{H}_0^1(\Omega), \quad \gamma_h \subset \Gamma_h \subset \mathbf{H}_0(\mathrm{rot}, \Omega)$$

with

$$\mathrm{rot}\,\Gamma_h \subset \mathrm{div}\,\Phi_h$$

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